Geometry over composition algebras : projective geometry

Pierre-Emmanuel Chaput

Geometry over composition algebras : projective geometry

Pierre-Emmanuel Chaput
Pierre-Emmanuel.Chaput@math.univ-nantes.fr
Laboratoire de Mathématiques Jean Leray UMR 6629
2 rue de la Houssinière - BP 92208 - 44322 Nantes Cedex 3

23rd September 2005

Abstract

The purpose of this article is to introduce projective geometry over composition algebras : the equivalent of projective spaces and Grassmannians over them are defined. It will follow from this definition that the projective spaces are in correspondance with Jordan algebras and that the points of a projective space correspond to rank one matrices in the Jordan algebra. A second part thus studies properties of rank one matrices. Finally, subvarieties of projective spaces are discussed.

AMS mathematical classification : 14N99, 14L35, 14L40.
Key-words: composition algebras, projective spaces, Grassmannians, Jordan algebras.

Introduction

This paper initiates a wider study of geometry over composition algebras. The general philosophy of this study is to discuss to what extent classical algebraic geometry constructions generalize over composition algebras. Let k be a commutative field. Let $\mathbb{R}_k, \mathbb{C}_k, \mathbb{H}_k, \mathbb{O}_k$ denote the four split composition algebras [Jac 58].

The usual projective algebraic varieties over k are thought of varieties over $\mathbb{R}_k$, and I want to understand analogs for $\mathbb{C}_k, \mathbb{H}_k$ and even $\mathbb{O}_k$. For example, in this article, I study "projective spaces" over composition algebras.

In general, in the octonionic case, we find varieties homogeneous under an exceptional algebraic group. For example, the $\mathbb{O}_k$-generalization of the usual projective plane $\mathbb{P}^2_k$ is homogeneous under a group of type E_6 over k. The analogy between this somewhat mysterious E_6-variety and a well-understood projective plane allows one to understand better the geometry of this variety, as well as some representations of this group, which can be thought as $SL_3(\mathbb{O}_k)$. I plan to show that we can similarly think of E_7 as "$Sp_6(\mathbb{O}_k)$".

Along with this geometric and representation-theory motivation for studying varieties defined over composition algebras, there is an algebraic one. Namely, the varieties that we will meet in this context can be defined in terms of algebraic structures (such as Jordan algebras, structurable algebras, and exceptional Lie algebras). As I want to show, the products on these algebras correspond to maps defined naturally in terms of geometries over a composition algebra. This gives new insight on these algebras.

I thank Laurent Manivel for many discussions and Laurent Bonavero for comments on a previous version of this article. I also thank Nicolas Ressayre and Michel Brion for discussions relative to the fifth part.

This paper is organised as follows : the first section recalls well-known facts about composition algebras. A short geometric proof of the triality principle is given. In the second section, Grassmannians over composition algebras are defined : different definitions of $G_{\mathcal{A}}(r,n)$ as sets of $\mathcal{A}$-submodules of $\mathcal{A}^n$ with some properties are compared.

Any "projective space" $G_{\mathcal{A}}(1,n)$ is then seen to live in the projectivisation of a Jordan algebra : we have naturally, as algebraic varieties over k, $G_{\mathcal{A}}(1,n) \subset \mathbb{P}V$, where V is a Jordan algebra. Moreover, $G_{\mathcal{A}}(1,n)$ is the variety of rank one elements in the Jordan algebra. This is the topic of the third section : different possible definitions of "rank one" elements in a Jordan algebra are compared (see theorem 3.1), and the connection with the structure group of the Jordan algebra is described.

In the fourth section, I introduce the notion of "$\mathcal{A}$-subvarieties" of a projective space $G_{\mathcal{A}}(1,n)$. I show that there are very few of them.

Finally, section five deals with the octonionic projective plane. This plane is defined and its automorphism group is shown to be a simple group of type E_6 in all characteristics (see theorem 5.1). The projective plane over the octonions with *real* coefficients $\mathbb{O}_{\mathbb{R}}$ has been studied extensively [Tit 53, Fre 54]. Here, I consider the different case of the algebra $\mathbb{O}_k$ containing zero-divisors, and explain the new point of view of the generalized Veronese map (theorem 5.2).

Contents

1 Background on composition algebras

The split composition algebras have a model over $\mathbb{Z}$ [Jac 58] : the ring $\mathbb{H}_k$ is the ring of 2×2-matrices with integral entries. The norm of a matrix A is $Q(A) = \det A$ and the conjugate of $A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$ is the matrix $\overline{A} = \begin{pmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{pmatrix}$. The rings $\mathbb{C}_{\mathbb{Z}}$ and $\mathbb{R}_{\mathbb{Z}}$ are respectively the subrings of diagonal and homothetic matrices. The ring $\mathbb{O}_{\mathbb{Z}}$ may be constructed via Cayley's process : it is the ring of couples (A,B) of matrices with product $(A,B)*(C,D) = (AC - \overline{D}B, B\overline{C} + DA)$, conjugation $\overline{(A,B)} = (\overline{A}, -B)$ and norm $Q(A,B) = Q(A) + Q(B)$. Therefore, $\mathbb{R}_{\mathbb{Z}}$ and $\mathbb{C}_{\mathbb{Z}}$ are commutative. For k a field and $\mathcal{A} \in \{\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}\}$, we set $\mathcal{A}_k = \mathcal{A}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$. We note $\langle x,y\rangle = Q(x+y) - Q(x) - Q(y)$ and $\mathtt{Re}(z) = \langle z,1\rangle$ (therefore $\mathtt{Re}(1) = 2$).

Notation 1.1. *A composition algebra over a unital commutative ring R is one of the following algebras : $\mathbb{R}_R, \mathbb{C}_R, \mathbb{H}_R, \mathbb{O}_R$.*

In the sequel, many arguments will use the fact that the isotropic linear spaces for Q can be described using the algebra; namely, for $z \in \mathcal{A}$, we denote $L(z)$ (resp. $R(z)$) the image of the left (resp. right) multiplication by z in $\mathcal{A}$, denoted L_z (resp. R_z).

Proposition 1.1. *Let k be any field and let $\mathcal{A}$ be a composition algebra over k different from $\mathbb{R}_k$. Let $z, z_1, z_2 \in \mathcal{A} - \{0\}$ with $Q(z_1) = Q(z_2) = 0$.*

- *If $Q(z) \neq 0$, then $L(z) = R(z) = \mathcal{A}$.*
- *If $Q(z) = 0$, then $L(z)$ and $R(z)$ are maximal (ie of dimension $\dim_k \mathcal{A}/2$) isotropic linear subspaces of $\mathcal{A}$. They belong to different connected components of the variety of maximal isotropic subspaces.*
- *$z_2 \in L(z_1) \iff z_1 \in L(z_2) \iff \overline{z}_1 z_2 = 0$.*

For example, this proposition implies that if $0 \neq z_1, z_2 \in \mathbb{H}_k$ and $Q(z_1) = Q(z_2) = 0$, then the dimension of $L(z_1) \cap L(z_2)$ is either 2 (ie $L(z_1) = L(z_2)$) or 0, depending on the fact that $z_2 \in L(z_1)$ (or equivalently $z_1 \in L(z_2)$) or not.
<u>Proof :</u> Let $\alpha = \dim \mathcal{A}/2$. The composition algebras are alternative [Sch 66], which means that $\forall x, z \in \mathcal{A}, z(zx) = (zz)x$. Therefore, $\overline{z}(zx) = Q(z).x$, or $L_{\overline{z}} \circ L_z = Q(z).Id$.

Thus, $Q(z) \neq 0$ if and only if L_z is invertible if and only if R_z is. Moreover, the kernel of L_z are elements t such that R_t is not invertible; therefore it is included in the quadric $\{Q = 0\}$. Since $Q(zx) = Q(z).Q(x)$, if $Q(z) = 0$, then we have also $L(z) \subset \{Q = 0\}$. Since $\dim L(z) + \dim\ker L_z = \dim \mathcal{A} = 2\alpha$ and an isotropic subspace has maximal dimension α, it follows that $\dim L(z) = \dim\ker L_z = \alpha$. The rest of the proposition is easy. $\square$

The octonionic case is related to the triality principle. The following proposition was proved in [BS 60] using a description of $Spin_8$ involving octonions.

Proposition 1.2. *If $\mathcal{A} = \mathbb{O}_k$, the maps L and R induce isomorphisms from the projective 6-dimensional quadric defined by Q and the two connected components of the Grassmannian of maximal isotropic spaces. Let $x, y \in \mathbb{O}_k$ such that $Q(x) = Q(y) = 0$.*

- $\dim(L(x) \cap L(y)) \geq 2 \Longleftrightarrow \dim(R(x) \cap R(y)) \geq 2 \Longleftrightarrow \langle x, y\rangle = 0$.
- *If $\dim L(x) \cap L(y) = 2$, then $L(x) \cap L(y) = L_x[L(\overline{y})] = L_y[L(\overline{x})]$, and $R(x) \cap R(y) = R_x[R(\overline{y})] = R_y[R(\overline{x})]$.*
- $xy = 0 \Longleftrightarrow \dim L(x) \cap R(y) = 3$.

<u>Proof :</u> Let $G^{\pm}(4, \mathbb{O}_k)$ denote the two connected components of the variety of maximal isotropic subspaces of $\{Q = 0\}$ in $\mathbb{O}_k$.

First, let $x_0 = \left(\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}\right)$. We have $L(x_0) = \left(\begin{pmatrix}* & *\\0 & 0\end{pmatrix}, \begin{pmatrix}* & 0\\* & 0\end{pmatrix}\right)$ and $R(x_0) = \left(\begin{pmatrix}* & 0\\* & 0\end{pmatrix}, \begin{pmatrix}0 & *\\0 & *\end{pmatrix}\right)$. We thus have $L(x_0) \cap R(x_0) = k.x_0$. Since for any $x, z \in \{Q = 0\}$ with $x \neq 0$ and $z \neq 0$, $\dim(L(x) \cap R(z)) \in \{1, 3\}$, for generic $x, z \in \{Q = 0\}$, one has $\dim(L(x) \cap R(z)) = 1$, so $L(x) \cap R(z) = k.xz$.

Let $x \in \{Q = 0\}$ be such that for generic $z \in \{Q = 0\}$, $L(x) \cap R(z) = k.xz$. By the following lemma 1.1, if y is such that $L(x) = L(y)$, then there exists $\lambda \in k$ such that $L_x = \lambda.L_y$, and so $x = \lambda.y$.

Therefore, the fiber of $[L] : \mathbb{P}\{Q = 0\} \to G^+_Q(\mathbb{O}_k)$ over $L(x)$ is only $\{[x]\}$: $[L]$ is generically injective. It follows that it is surjective, and the same holds for $[R]$.

The previous argument is therefore valid for any $[x] \in \mathbb{P}\{Q = 0\}$, and $[L]$ and $[R]$ are injective. We will now show that $[L]$ is an isomorphism. If $x, y \in \{Q = 0\}$ and $y \in L(x)$, $[y] \neq [x]$, then $L(x) \cap L(y) = Vect(x, y)$. Therefore, given $L(x) = \Lambda$, the point $[x] \in \mathbb{P}\mathbb{O}_k$ may be constructed as the intersection of the projective lines $\Lambda \cap L(y)$, for $y \in \Lambda$. This describes the inverse of $[L]$, which is therefore algebraic.

I have shown that $[L]$ and $[R]$ are isomorphisms. If dim $(L(x) \cap R(y)) = 3$, then $xy = 0$, because otherwise the rational map

$$\begin{array}{rcl} G^+(4, \mathbb{O}_k) \times G^-(4, \mathbb{O}_k) & \to & \mathbb{P}\mathbb{O}_k \\ (\Lambda^+, \Lambda^-) & \mapsto & \Lambda^+ \cap \Lambda^- \end{array}$$

would be defined at $(L(x), R(y))$. Fixing x, $\{y : \dim\ (L(x) \cap R(y)) = 3\}$ and $\{y : xy = 0\}$ are isomorphic with $\mathbb{P}^3_k$, so they are equal, proving the third point.

If $0 \neq z \in L(x) \cap L(y)$, and $\lambda, \mu \in k$, then $\overline{z}(\lambda x + \mu y) = 0$. Therefore, $Q(\lambda x + \mu y) = 0$ and $\langle x, y\rangle = 0$. Since the variety $Q_x = \{y \in \{Q = 0\} : \dim(L(x) \cap L(y)) \geq 2\}$ is a hyperplane section of the quadric $\{Q = 0\}$, we deduce $Q_x = \{y : Q(y) = 0 \text{ and } \langle x, y\rangle = 0\}$.

Finally, the identity $z(\overline{z}t) = Q(z)t$ yields $x(\overline{y}t) = \langle x, y\rangle t - y(\overline{x}t)$, which implies that $L_x(L(\overline{y})) = L_y(L(\overline{x})) = L(x) \cap L(y)$, if $\langle x, y\rangle = 0$. $\square$

Lemma 1.1. *Let V, W be k-vector spaces, $X \subset V$ a variety and $f, g : V \to W$ linear maps. Assume :*

- $\forall x \in X - \ker f, \exists \lambda \in k : g(x) = \lambda f(x)$.

- *No quadric of rank four vanish on X.*
- *f has rank at least 2.*

Then, $\exists \lambda \in k : g = \lambda f$.

Proof : Choose a basis of W and let $f_1, \ldots, f_n, g_1, \ldots, g_n$ be linear forms such that $f = (f_i), g = (g_i)$. Then the minors $f_i g_j - f_j g_i$ vanish along X, so they vanish on V. The lemma follows. $\square$

2 Grassmannians over composition algebras

2.1 Definition

An element in the Grassmannian of r-planes on $k(= \mathbb{R}_k)$ is a k-vector space of dimension r. But if $\dim_k \mathcal{A} \geq 2$, $\mathcal{A}$ is not a field, and this definition does not make sense anymore. However, we can define $G_{\mathcal{A}}(r,n)$ as the set of all free right $\mathcal{A}$-submodules of $\mathcal{A}^n$ of rank r, that is the set of submodules M of the form:

$$M := \{\sum_{t=1}^{r} v_t \lambda_t, \lambda_t \in \mathcal{A}\},$$

with dimension $\dim \mathcal{A}.r$ over k ((v_t) is a r-uplet of elements in $\mathcal{A}$). The freeness condition generalizes the fact that the zero vector in k^n has no image in $\mathbb{P}^{n-1}$.

Another possible definition, which gives a structure of closed variety, considers right $\mathcal{A}$-submodules of $\mathcal{A}^n$ of the right dimension:

$$\tilde{G}_{\mathcal{A}}(r,n) = \{E \subset \mathcal{A}^n : \dim_k E = \dim \mathcal{A}.r \text{ and } \forall \lambda \in \mathcal{A}, E.\lambda \subset E\}.$$

We shall now study properties of these two sets. If V is a k-vector space and r an integer, I denote $G(r,V)$ the Grassmannian of r-subspaces of V. Let $G(r,n)$ denote $G(r,k^n)$. I want to show the following propositions:

Proposition 2.1. *$G_{\mathbb{H}}(r,n) = \tilde{G}_{\mathbb{H}}(r,n) \subset G(4r, \mathbb{H}_k^n)$ is a smooth subvariety isomorphic with the usual Grassmannian $G(2r,2n)$.*

Proposition 2.2. *The subvariety $G_{\mathbb{C}}(r,n)$ of $G(2r,\mathbb{C}_k^n)$ is isomorphic with $G(r,n) \times G(r,n)$. Moreover, $\tilde{G}_{\mathbb{C}}(r,n)$ is the union of* $\min\{n+1-r, r+1\}$ *connected components which are irreducible, one of which is $G_{\mathbb{C}}(r,n)$.*

First, let us show the following lemma (If x is a real number, I denote $[x]^+$ the least integer greater or equal to x):

Lemma 2.1. *Let $E \subset \mathbb{H}_k^n$ such that $\forall \lambda \in \mathbb{H}_k, E.\lambda \subset E$. Then there exist $c = [\frac{\dim E}{4}]^+$ vectors $v_1, \ldots, v_c \in \mathbb{H}_k^n$ such that $E = \oplus\{v_i.\lambda, \lambda \in \mathbb{H}_k\}$.*

Remark : Assuming the result, let $1 \leq t \leq c$ be an integer and $(v_{t,u})_{1 \leq u \leq n}$ the coordinates of the vector v_t. The kernel of the map $\mathbb{H}_k \to \mathbb{H}_k^n, \lambda \mapsto v_t.\lambda$ is trivial if one of the $v_{t,u}$ is invertible, and is $\cap_j L(\overline{v}_{i,j})$ otherwise, by proposition 1.1. Thus, it has even dimension by proposition 1.1, and the rank theorem shows that $\{v_t.\lambda : \lambda \in \mathbb{H}_k\}$ has also even dimension. Thus, proposition 2.1 shows that the dimension of such an E is even.

<u>Proof :</u> A large part of this proof holds for $\mathbb{C}_k$; for the moment $\mathcal{A}$ stands for $\mathbb{C}_k$ or $\mathbb{H}_k$, and accordingly we set α equals 1 or 2. I will precise which argument needs $\mathcal{A} = \mathbb{H}_k$.

Let $\pi_t : \mathcal{A}^n \to \mathcal{A}$ be the projection on the t-th factor, and p_t the restriction of π_t to E. If T is a subset of $\{1, \dots, n\}$, let p_T denote the products of the p_t's for $t \in T$.

If there exists t such that p_t has maximal rank 2α, then, choosing a vector v in E such that $v_t = 1$, we get an isomorphism

$$\begin{array}{cccc} s & : & \mathbb{H}_k \oplus (E \cap \{x_t = 0\}) & \to & E \\ & & (\lambda, x) & \mapsto & v.\lambda + x, \end{array}$$

so that we are done by an inductive argument. We thus suppose that no projection p_t has maximal rank. Since $\mathtt{Im}(p_t)$ is preserved by right multiplication by $\mathcal{A}$, by proposition 1.1, it has rank 0 or α. In the first case, we can also use an inductive argument, therefore, we can suppose that any projection p_t has rank α.

This implies that for any couple $(t, u) \in \{1, \dots n\}^2$, $F = p_{\{t,u\}}(E)$ has dimension α or 2α. In fact, $p_{t|F}$ has rank α and its kernel is preserved by right multiplication. If t and u are such that $p_{\{u,t\}}$ has rank α, then $p_{u|p_{\{u,t\}}(E)}$ is bijective, so that if ι denotes $\{1, \cdots, n\} - \{i\}$, p_ι is injective, and again we conclude by an inductive argument.

It is therefore sufficient to consider the case where any projection $p_{\{t,u\}}$ has rank 2α. Let t and u be arbitrary. Since $F = p_{\{t,u\}}(E)$ is preserved by right multiplication, and since each projection has rank α, there exist $x, y \in \mathcal{A}$ such that $F \subset L(x) \times L(y)$; since $\dim F = 2\alpha$, we have equality.

The following argument works only for $\mathcal{A} = \mathbb{H}_k$. If $z \in L(y)$, then $L(z) = L(y)$, since we have $L(z) \subset L(y)$ by associativity. Moreover, since $\mathcal{A} = \mathbb{H}_k$, we can choose $z \in L(y)$ such that z and y are not proportional; this implies $R(z) \neq R(x)$. Thus, eventually replacing y by z, we can assume $R(y) \neq R(x)$. This implies by conjugation $L(\overline{y}) \neq L(\overline{x})$. Thus these spaces are supplementary and the mapping $\mathcal{A} \to F, \lambda \mapsto (x, y).\lambda$ is injective, proving that (x, y) is a generator of F. It is enough to consider a vector which projection under $p_{\{t,u\}}$ is (x, y). $\square$

<u>Proof of proposition 2.2:</u> Let $E \subset \mathbb{C}_k^n$ be preserved by right-multiplication. Let us consider the base $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ of $\mathbb{C}_k$. If $v = (v_1, \dots, v_n) = (v_1^+ e + v_1^- f, \dots, v_n^+ e + v_n^- f) \in E$, then $v^+ = (v_1^+ e, \dots, v_n^+ e) = v.e \in E$, and $v^- = (v_1^- f, \dots, v_n^- f) = v.f \in E$. If E^+ (resp. E^-) denotes $E \cap (\mathbb{C}_k^n.e)$ (resp. $E \cap (\mathbb{C}_k^n.f)$), then $E = E^+ \oplus E^-$.

If r^+ and r^- are integers between 0 and n with sum $2r$, let $\tilde{G}_{r^+}(r, n)$ denote the set of linear spaces of the form $E = E^+ \oplus E^-$, with $E^+ \subset \mathbb{C}_k^n.e, E^- \subset \mathbb{C}_k^n.f$ and $\dim E^\pm = r^\pm$. Such a linear space is preserved by muliplication by e and f, thus it is an element of $\tilde{G}_{\mathbb{C}}(r, n)$. The variety $G_{r^+}(r, n)$ is isomorphic to $G(r^+, n) \times G(r^-, n)$.

On the other hand, we have seen that $\tilde{G}_{\mathbb{C}}(r, n) = \cup \tilde{G}_{r^+}(r, n)$. To prove that the $\tilde{G}_{r^+}(r, n)$ are the connected components of $\tilde{G}_{\mathbb{C}}(r, n)$, I recall that for $d \in \mathbb{N}$, $\{E : \dim(E \cap (\mathbb{C}_k^n.e)) \geq d\}$ and $\{E : \dim(E \cap (\mathbb{C}_k^n.f)) \geq d\}$ are closed subsets of $G(2r, \mathbb{C}_k^n)$.

It remains to check that $\tilde{G}_r(r, n) = G_{\mathbb{C}}(r, n)$. If $(v_1^\pm, \dots, v_r^\pm)$ is a basis of $E^\pm$, then $(v_t^+ e + v_t^- f)_t$ is a family of vectors which generates E. $\square$

Proof of proposition 2.1: Lemma 2.1 says that $G_{\mathbb{H}}(r,n) = \tilde{G}_{\mathbb{H}}(r,n)$. In the same way as proposition 2.2, one proves that any $E \in G_{\mathbb{H}}(r,n)$ may be written as $E^+ \oplus E^-$, with $E^+ \subset \mathbb{H}_k^n.e$ and $E^- \subset \mathbb{H}_k^n.f$. Moreover, let $h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; right-multiplication with h is an involutive linear automorphism of E which exchanges E^+ and E^-. Therefore, they have the same dimension $2r$.

Moreover, it implies that giving E is equivalent to giving E^+, and thus the map $E \mapsto E^+$ is the desired isomorphism between $H_{\mathbb{H}}(r,n)$ and $G(2r,2n)$. $\square$

2.2 Duality

In this paragraph, I show an analog of the well-known fact that

$$G(r,V) \simeq G(\dim V - r, V^*).$$

Definition 2.1. *Let V and W be right (resp. left) $\mathcal{A}$-modules. A right-linear (resp. left-linear) map from V to W is a map $f: V \to W$ such that $\forall x,y \in V, \forall \lambda, \mu \in \mathcal{A}, f(x.\lambda + y.\mu) = f(x).\lambda + f(y).\mu$ (resp. $f(\lambda.x + \mu.y) = \lambda.f(x) + \mu.f(y)$).*
A right- (resp. left-)form on V is a right- (resp. left-)linear map from $\mathcal{A}^n$ to $\mathcal{A}$.

A map $f: \mathcal{A}^n \to \mathcal{A}^m$ is right-linear (resp. left linear) if and only if there exists a matrix $(a_{t,u})$ such that $f((x_u)) = (\sum_u a_{t,u}x_u)_t$ (resp. $f((x_u)) = (\sum_u x_t a_{t,u})_t$). Therefore, if V is a free right $\mathcal{A}$-module of rank n, so is the set of left-linear forms on V, which I will denote by V^*.

Generalizing the construction of the previous section, if V is a free right $\mathcal{A}$-module of rank n, let $G_{\mathcal{A}}(r,V)$ denote the algebraic variety parametrizing the free right $\mathcal{A}$-submodules of V of rank r. This is obviously a variety isomorphic with $G_{\mathcal{A}}(r,n)$. Moreover, we have the following :

Proposition 2.3. *Let V be a free right $\mathcal{A}$-module of rank n. There is a canonical isomorphism $G_{\mathcal{A}}(r,V) \simeq G_{\mathcal{A}}(n-r,V^*)$.*

Proof : If $Y \subset V$ is any set, then $Y^\perp := \{l \in V^* : \forall y \in Y, l(y) = 0\}$ is a k-linear subspace of V^*, preserved by right multiplication by $\mathcal{A}$. If Y is an element in $G_{\mathcal{A}}(r,n)$, then it is generated by r vectors, and a form vanishes on Y if and only if it vanishes on the generators. Therefore, $Y^\perp$ is of dimension at least $\dim_k \mathcal{A}.(n-r)$. Since the pairing $\mathcal{A}^n \times \mathcal{A}_r^{n*} : (x,l) \mapsto \langle 1, l(x) \rangle$ is perfect, the dimension of $Y^\perp$ is exactly $\dim_k \mathcal{A}.(n-r)$.

If $\mathcal{A} = \mathbb{C}_k$, then $\dim(Y^\perp \cap (\mathbb{C}_k^n.e)) = \dim(Y^\perp \cap (\mathbb{C}_k^n.f)) = n-r$. We thus have $Y^\perp \in G(n-r, \mathcal{A}_r^{n*})$. Since the map $L \mapsto L^\perp$ is an isomorphism, the proposition is proved. $\square$

3 Properties of rank one matrices and generalized Veronese map

This section is concerned with the particular case $r = 1$ of the previous section, namely, I want to study projective spaces. We will see that there is a closed connection between these spaces, Jordan algebras, and a particular map which generalizes alltogether the Veronese map and the quotient rational map $k^{n+1} \dashrightarrow \mathbb{P}^n$.

3.1 Background on quadratic Jordan algebras

The satisfactory notion of Jordan algebras over a unital commutative ring A which includes the characteristic two case is the notion of quadratic Jordan algebras. They are by definition the free A-modules V of finite type with a non-zero distinguished element denoted 1 and equipped with a map $R : V \to End(V)$ which satisfies the following five axioms [Jac 69, definition 3,p.1.9] :

- R is quadratic.
- $R(1) = Id_V$.
- $R(a) \circ R(b) \circ R(a) = R[R(a).b]$.
- $V_{a,b} \circ U_b = U_b \circ V_{b,a}$, if $V_{a,b}(x) = R(x+b).a - R(x).a - R(b).a$.
- The two last identites hold after any extension of the base ring A.

Let $\mathcal{A}_{\mathbb{Z}}$ be a composition algebra over $\mathbb{Z}$, let r be an integer, and consider the $\mathbb{Z}$-module $H_r(\mathcal{A}_{\mathbb{Z}})$ of hermitian matrices with entries in $\mathcal{A}_{\mathbb{Z}}$. If $\mathcal{A}_{\mathbb{Z}}$ is associative, set $R(A).B = ABA$, for any $A, B \in H_r(\mathcal{A}_{\mathbb{Z}})$. If $\mathcal{A}_{\mathbb{Z}} = \mathbb{O}_{\mathbb{Z}}$, then R can be defined as the unique quadratic map such that $R(A).B = ABA$ if all the coefficients of A and B belong to an associative subalgebra of $\mathbb{O}_{\mathbb{Z}}$.

By [Jac 69, theorem 5,p.1.45], if $\mathcal{A}_{\mathbb{Z}}$ is associative or $r \leq 3$, then the triple $(H_r(\mathcal{A}_{\mathbb{Z}}), Id, R)$ is a quadratic Jordan algebra over $\mathbb{Z}$. If k is any field, set $H_r(\mathcal{A}_k) = H_r(\mathcal{A}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} k$; it is a quadratic Jordan algebra over k. Refering to the "second structure theorem" [Jac 69, p.3.59], it is seen that these examples of quadratic Jordan algebras play a major role in the theory of Jordan algebras.

Now, let, for $a = 1, 2, 4$, V_a^n be the vector space defined by

$$V_a^n = \begin{cases} \mathcal{S}_n(k) & \text{if } a = 1 \\ \mathcal{M}_n(k) & \text{if } a = 2 \\ \mathcal{AS}_{2n}(k) & \text{if } a = 4, \end{cases}$$

where $\mathcal{S}, \mathcal{M}, \mathcal{AS}$ respectively stand for the set of symmetric, arbitrary, and skew-symmetric with zero diagonal entries matrices. We choose any invertible $I \in V_a^n$, and set

$$R(A).B = AI^{-1}BI^{-1}A. \tag{1}$$

It is well-known that the algebra V_a^n is isomorphic with $H_n(\mathcal{A})$ (if $\dim \mathcal{A} = a$); the following corollary 3.3 gives a geometric understanding of this isomorphism when $a = 4$. In the sequel, I will be chosen to be the usual identity matrix for $a = 1, 2$, and the bloc-diagonal matrix with non-vanishing coefficients $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ for $a = 4$.

We note $\mathbb{P}^n_{\mathcal{A}} := G_{\mathcal{A}}(1, n+1)$. First, we remark that $\mathbb{P}^{n-1}_{\mathbb{C}} \simeq \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \subset \mathbb{P}V_2^n$ and $\mathbb{P}^{n-1}_{\mathbb{H}} \simeq G(2, 2n) \subset \mathbb{P}V_4^n$ are naturally embedded in the projectivisations of quadratic Jordan algebras. The same holds for $\mathbb{P}^{n-1}_{\mathbb{R}}$, which embeds via the second Veronese embedding in $\mathbb{P}V_1^n$. As we shall see in the last section, the exceptional quadratic Jordan algebra $H_3(\mathbb{O}_k)$ corresponds to the octonionic projective plane. The aim of this section is to give several equivalent caracterizations of the elements in $\mathbb{P}V_a^n$ which correspond to points of $\mathbb{P}^{n-1}_{\mathcal{A}}$.

3.2 Definition of Jordan rank one matrices

I start with a pure Jordan-algebra definition whose geometrical meaning will become clearer in the next subsections. In $H_r(\mathcal{A}_k)$, set

$$\langle A, B\rangle = \sum_{1\leq i<j\leq r} \langle A_{i,j}B_{i,j}\rangle + \sum_i A_{i,i}B_{i,i} \quad \text{and} \quad \mathrm{tr}(A) = \sum_i A_{i,i} = \langle 1, A\rangle.$$

Definition 3.1. *Let $V = H_r(\mathcal{A}_k)$ and $0 \neq A \in V$. We will say that the Jordan rank of A is one if $\forall B \in V, R(A)B = \langle A, B\rangle A$.*

Remark : Since tr and $\langle ., .\rangle$ can be defined in any quadratic Jordan algebra, the above definition makes sense not only in $H_r(\mathcal{A}_k)$, but in any (quadratic) Jordan algebra.

Each element $B \in V$ determines $\dim V$ quadratic equations given by the coordinates of $R(A)B - \langle A, B\rangle A$. It is clear that a quadratic Jordan algebra isomorphism induces an isomorphism of varieties of rank one elements.

Notation 3.2. *Let $\mathcal{Q}_2$ denote the space of quadrics generated, for all $B \in V$, by the coordinates of the equation on A: $R(A)B - \langle A, B\rangle A = 0$.*

Before explaining what Jordan rank one elements are in V_a^n, I want to show that this definition is well-behaved with respect to a "big" algebraic group.

The structure group of a Jordan algebra V, denoted $Str(V)$ is defined as the group of $g \in GL(V)$ such that $\forall B \in V, R(g.A) = g \circ R(A) \circ {}^t g$, the transposition being taken with respect to the scalar product $\langle ., .\rangle$. This definition may seem rather abstract to a reader not used to Jordan algebras, so I recall that the connected component of $Str(V_a^n)$ is $GL_n, (GL_n \times GL_n)/k^*$, or $GL(2n)$, according to $a = 1, 2, 4$, where k^* is diagonaly embedded in $GL_n \times GL_n$ and the actions on V_a^n are the natural ones.

Lemma 3.1. *The algebraic variety of rank one elements is preserved by $Str(V)$, as well as the vector space of quadrics $\mathcal{Q}_2$.*

Therefore, a straightforward computation shows that the set of Jordan rank one matrices is the closed orbit of $Str(V_a^n)$ in $\mathbb{P}V_a^n$, namely the set of (usual) rank one matrices if $a = 1$ or 2 and the set of rank 2 matrices if $a = 4$. Moreover, the equations $\mathcal{Q}_2$ are, respectively, the two by two minors, and the Plücker equations of the Grassmannian. Recalling propositions 2.1 and 2.2, the variety of rank one elements is thus $\mathbb{P}_{\mathcal{A}}^{n-1} \subset \mathbb{P}V_a^n$.

Proof : Let $f \in \mathcal{Q}_2$; we can assume that there exist $B, C \in V$ such that $f(A) = \langle R(A).B - \langle A, B\rangle A, C\rangle$. From the definition of $Str(V)$, it follows that $\forall g \in Str(V)$,

$$\begin{aligned}
(g^{-1}.f)(A) &= \langle R(g.A).B - \langle gA, B\rangle gA, C\rangle \\
&= \langle g[R(A).({}^t gB)] - \langle A, {}^t gB\rangle gA, C\rangle \\
&= \langle R(A).({}^t gB) - \langle A, {}^t gB\rangle A, {}^t gC\rangle.
\end{aligned}$$

Therefore this is a quadric in the vector space $\mathcal{Q}_2$. Thus this vector space, and the variety it defines, are preserved by $Str(V)$. □

3.3 Jordan rank and generalized Veronese maps

In this paragraph, I give an analog of the map $k^{n+1} \dashrightarrow \mathbb{P}^n$ for $\mathbb{P}_{\mathcal{A}}^n$ in terms of a map which generalizes the usual Veronese map in the case $\mathcal{A} = \mathbb{R}_k$.

Following F.L. Zak [Zak 93, th. 4.9], let's consider the following map:

$$\begin{array}{cccc} \nu_2 : & \mathcal{A}^n & \dashrightarrow & \mathbb{P}H_n(\mathcal{A}) \\ & (z_i) & \mapsto & (z_i\overline{z_j}). \end{array}$$

As I will show, this map can be interpreted as the rational map $\mathcal{A}^n \dashrightarrow \mathbb{P}\mathcal{A}^{n-1}$:

Proposition 3.1. *Let $\lambda \in \mathcal{A}$ with $Q(\lambda) \neq 0$ and (z_i) such that $\nu_2(z_i)$ is well-defined. Then $\nu_2((z_t.\lambda))$ is also well-defined and equals $\nu_2((z_t))$.*

Proof : In fact, $((z_t\lambda)).(\overline{\lambda}\overline{z_u}) = Q(\lambda)z_t\overline{z_u}$. $\square$

Now, let us see that the image of this map is $\mathbb{P}^{n-1}_{\mathcal{A}}$:

Proposition 3.2. *The image of $\nu_2 : \mathcal{A}^n \dashrightarrow \mathbb{P}H_n(\mathcal{A})$ is the set of Jordan rank one elements.*

The image of this rational map is the set-theoretical one, namely the set of all the matrices which may be written as $\nu_2(z_t)$. The proposition shows that this set is closed.

Proof : First, a direct computation shows

$$A = (a_{t,u}) \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow ABA = (a_{t,1}a_{1,u})_{t,u}. \tag{2}$$

If $A = (z_t\overline{z_u})$ is in the image of ν_2, then for $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $ABA = Q(z_1)A$, thus is equal to $\langle A, B\rangle A$, by (2). We can also make a similar computation for $B = \begin{pmatrix} \begin{pmatrix} 0 & \overline{x} \\ x & 0 \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix}$. In fact, if $A = (z_t\overline{z_u})$, then $ABA = ([z_t\overline{z_1}][\overline{x}(z_2\overline{z_u})] + [z_t\overline{z_2}][x(z_1\overline{z_u})])_{t,u}$. Using associativity, this equals

$$(z_t\mathtt{Re}(\overline{z}_1\overline{x}z_2)\overline{z_u})_{t,u} = (z_t\mathtt{Re}(\overline{x}z_2z_1)\overline{z_u})_{t,u} = \langle A, B\rangle(z_t\overline{z_u})_{t,u}.$$

Using the permutations and linearity, A has Jordan rank one.

In the other way, if there exists a diagonal matrix B such that $\langle A, B\rangle \neq 0$, then if for example $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, we see from (2) that A is collinear with $\nu_2(\overline{a_{1,t}})$. If such a B does not exist, then for all diagonal B, $ABA = 0$. Thus, using (2)

$$\forall t, u, v,\ \overline{a_{v,t}}a_{v,u} = 0.$$

Then we can assume there exists B of the form $B = \begin{pmatrix} \begin{pmatrix} 0 & \overline{x} \\ x & 0 \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix}$ such that $\langle A, B\rangle \neq 0$; we deduce that A is proportional to $\nu_2(\overline{a_{1,t}} + \overline{a_{2,t}}x)$. $\square$

Let $X \subset \mathbb{P}H_n(\mathbb{H}_k)$ be the variety of rank one elements in the Jordan algebra $H_n(\mathbb{H}_k)$. Recall that $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{H}_k$. Any $z \in \mathbb{H}_k$ induces, by left multiplication, a linear morphism $R(e) \to R(e)$. Chosing the basis $\left\{e, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$ of $R(e)$, we can associate to z a two-by-two matrix $M(z)$ representing L_z. This map yields a map $\tilde{M} : H_n(\mathbb{H}_k) \to \mathcal{M}_{2n}$, given by $\tilde{M}((a_{t,u})_{t,u}) = (M(a_{t,u}))_{t,u}$. Recall that I is the $2n\times 2n$ bloc-diagonal matrix with diagonal entries $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let $\tilde{\phi}$ denote the map $H_n(\mathbb{H}_k) \to \mathcal{M}_{2n}, A \mapsto I.\tilde{M}(A)$. We also define, for $A \in H_n(\mathbb{H}_k)$, the map $L_A : \mathbb{H}_k^n \to \mathbb{H}_k^n$ defined, if $A = (a_{t,u})$, by $L_A(z_u) = (\sum_u a_{t,u} z_u)_t$. The following corollary gives a geometric understanding of the isomorphism $H_n(\mathbb{H}_k) \simeq V_4^n$.

Corollary 3.3. *The map*

$$\begin{array}{cccll} \phi & : & X \subset \mathbb{P}(H_n(\mathbb{H}_k)) & \to & G(2, k^{2n}) \subset \mathbb{P}(\Lambda^2 k^{2n}) \\ & & A & \mapsto & (\mathtt{Im} L_A) \cap R(e)^n \end{array}$$

is induced by the linear isomorphism $\tilde{\phi}$.

<u>Proof :</u> The proof is a computation left to the reader (the details are written in my thesis [Cha 03]). □

It is easily checked that, for the Jordan product on V_4^n given by (1), the map $\tilde{\phi}$ is also a Jordan algebra isomorphism. This should not come as a surprise, as the following proposition shows.

Proposition 3.4. *Let V_1 and V_2 be quadratic Jordan algebras isomorphic with some algebra $H_r(\mathcal{A}_k)$, I_1 and I_2 their units and $X_1 \subset \mathbb{P}V_1$ and $X_2 \subset \mathbb{P}V_2$ the corresponding varieties of rank one elements. Let $f : V_1 \to V_2$ be a linear map such that $f(I_1) = I_2$. The following conditions are equivalent:*

1. *f induces an isomorphism of varieties between X_1 and X_2.*
2. *f is a Jordan algebra isomorphism between V_1 and V_2.*

<u>Proof :</u> This result will not be used in the sequel, so this proof, which uses a little of Jordan algebra theory, could be skipped. Of course, (2) implies (1), since the varieties X_i are defined using only the Jordan algebra structure. Now, in the quadratic Jordan algebras I consider, there is a well-defined polynomial called the norm [Jac 63]. In V_a^n with $a = 1, 2$, it is the usual determinant of matrices and in V_a^n, it is the pfaffian. In the exceptional case $H_3(\mathbb{O}_k)$, the norm is defined by formula (3) in subsection 5.3. Moreover, I use the fact that the hypersurface defined by the norm is the closure of the set of sums of $r-1$ rank one elements. In fact, for the classical Jordan algebras, this is just an easy result of linear algebra, whereas for the exceptional algebra, it follows from proposition 5.2. Denoting $\det_1$ and $\det_2$ the norms of the Jordan algebras V_1 and V_2, we therefore have $\det_1(A_1) = \det_2[f(A_1)]$. Since the scalar product is the second logarithmic differential of the determinant at the identity ($\langle A, B\rangle = D_I^2 \log\det(A, B)$) [McC 65], it follows that $\langle A_1, B_1\rangle_1 = \langle f(A_1), f(B_1)\rangle_2$. Moreover, since the quadratic product itself is also the second logarithmic derivative of the determinant ($\langle R(A)^{-1}.B, C\rangle = -D_A^2 \log\det(B, C)$), we deduce that f is an algebra morphism. □

We now relate two other possible definitions of rank one matrices to the previous definition. In the case $n = 3$, the following proposition shows that Jordan rank one matrices are defined by minors (which is not the case in general).

Proposition 3.5. *A Hermitian matrix $(a_{t,u})_{1\leq t,u\leq 3}$ with value in $\mathcal{A}$ has Jordan rank one if and only if*

$$a_{1,1}a_{2,2}-Q(a_{1,2})=a_{1,1}a_{3,3}-Q(a_{1,3})=a_{2,2}a_{3,3}-Q(a_{2,3})=0 \text{ and}$$
$$a_{1,1}a_{2,3}-a_{2,1}a_{1,3}=a_{3,2}a_{2,1}-a_{3,1}a_{2,2}=a_{2,1}a_{3,3}-a_{2,3}a_{3,1}=0.$$

Proof : To prove this result, I use the corresponding one concerning the exceptional Jordan algebras (proposition 5.2, which proof is self-contained). If $(a_{t,u})$ has rank one, it equals $\nu_2((z_t))$ and these minors vanish. If these minors vanish, considering this matrix as a matrix with coefficients in $\mathbb{O}_k$, it has rank one by proposition 5.2 which means that $\forall B\in H_3(\mathbb{O}_k)$, $R(A)B=\langle A,B\rangle A$. Thus this equality holds for $B\in H_3(\mathcal{A})$, and A has rank one. □

Recall that for A a matrix of order n with coefficients in $\mathcal{A}$, we defined the map $L_A:\mathcal{A}^n\to\mathcal{A}^n$ by $L_A((z_u))=(\sum_u a_{t,u}z_u)_t$. This is a k-linear map. If $\mathcal{A}$ were a field, it would be clear that $\dim_k\mathcal{A}$ would divide $\dim_k \mathtt{Im}L_A$. Here, it is not the case, take for example $A=\begin{pmatrix} z & 0\\ 0 & 0\end{pmatrix}$ with z a zero divisor. However, this is true for Hermitian matrices:

Proposition 3.6. *Let $A\in H_n(\mathcal{A})$. Then $\dim_k\mathcal{A}$ divides $\dim_k\mathtt{Im}L_A$.*

Proof : Suppose first that $\mathcal{A}=\mathbb{H}_k$. Recall that I denote $e=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ and $f=\begin{pmatrix}0&0\\0&1\end{pmatrix}$. Under the ismoorphism of proposition 3.3, a matrix $A\in H_n(\mathbb{H}_k)$ identifies with $I.\tilde{M}(A)$, where I stands for the bloc diagonal matrix with entries $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, and $\tilde{M}(A)$ is the matrix of the restriction of L_A to $R(e)^n$. Since $I.\tilde{M}(A)$ is skew-symmetric, $\tilde{M}(A)$ has even rank, and since by associativity $R(f)^n=R(e)^n.\begin{pmatrix}0&1\\1&0\end{pmatrix}$, the rank of L_A is a multiple of 4.

Considering a matrix with entries in $\mathbb{C}_k$ as a matrix with entries in $\mathbb{H}_k$, we deduce the case $\mathcal{A}=\mathbb{C}_k$ from the case $\mathcal{A}=\mathbb{H}_k$, since for a matrix A with coefficients in $\mathbb{C}_k$, we have $L_A(\mathbb{H}_k^n)=L_A(\mathbb{C}_k^n)\oplus L_A(\mathbb{C}_k^n).\begin{pmatrix}0&1\\1&0\end{pmatrix}$. □

Proposition 3.7. *$A\in H_n(\mathcal{A})$ has Jordan rank one if and only if L_A has rank $\dim_k\mathcal{A}$.*

Proof : I thank Laurent Manivel for the simplification of the argument he suggested to me. Using the same argument as for the previous proposition, it is enough to consider the case when $\mathcal{A}=\mathbb{H}_k$.

Since for $A=\nu_2(z_t)$, $\mathtt{Im}L_A\subset\{(z_t.\lambda),\lambda\in\mathcal{A}\}$ and since this rank is a multiple of $\dim\mathcal{A}$, we have equality and one implication is proved. For the reverse implication, we may by an inductive argument, left to the reader, suppose that A has order three. If $a_{1,1}\neq 0$, the hypothesis implies that all the columns of A are right-multiple of the first, which implies that A has rank one. Therefore, we may assume that the diagonal of A vanishes.

Moreover, if A has a vanishing row, since it is Hermitian, we in fact have to study a matrix of the form $\begin{pmatrix}0&\overline{z}\\ z&0\end{pmatrix}$, with $z\in\mathcal{A}$ such that $Q(z)=0$, and this matrix has rank one by proposition 3.5. We therefore assume that no row of A vanishes.

Let C_u denote the columns of A; I claim that all vector spaces $\{C_u.\lambda, \lambda \in \mathcal{A}\} \subset \mathcal{A}^3$ have dimension $\dim \mathcal{A}/2$. In fact, if $\{C_u.\lambda, \lambda \in \mathcal{A}\}$ has dimension $\dim \mathcal{A}$, then the hypothesis implies that all the columns of A belong to this vector space; since $a_{u,u} = 0$, A would have a vanishing row.

Let u be fixed. Since $\{C_u.\lambda, \lambda \in \mathcal{A}\}$ has dimension $\dim \mathcal{A}/2$, there exists a vector space K_u of dimension two such that $\forall t, \ker L_{a_{u,t}} \supset K_u$. Since K_u is of the form $L(z_u)$, we have $\forall t, a_{u,t} \in R(\overline{z}_u)$. Since A is Hermitian, $a_{u,t}$ therefore belongs to $L(z_t) \cap R(\overline{z}_u)$, so it is of the form $z_t.b_{t,u}.\overline{z}_u$. The proposition 3.5 shows that A has rank one. □

Finally, I would like to mention the following result, which makes a link between my definition of rank one and another definition that we find in the litterature [Har 90, p.290]:

Proposition 3.8. *Let $A \in H_3(\mathcal{A})$. Then A has Jordan rank one if and only if $A^2 = (trA).A$.*

<u>Proof :</u> Using definition 3.1 with $B = Id$, we see that if A has rank one, then $A^2 = (\mathrm{tr}A).A$ (even if we are in $H_n(\mathcal{A})$ with $n > 3$).

Conversely, a direct computation shows that $\mathrm{tr}L_A = (\dim \mathcal{A})(\mathrm{tr}A)$. To prove the proposition, we can assume $\mathcal{A} = \mathbb{H}_k$. If $\mathrm{tr}A = 0$, $A^2 = 0$, so $L_A^2 = 0$, and so L_A has rank at most 6. By propositions 3.6 and 3.7, A has Jordan rank one. If $\mathrm{tr}A = 1$, $A^2 = A$, so $L_A^2 = L_A$. So L_A has eigenvalue 1 with multiplicity 4 and eigenvalue 0 with multiplicity 8 ($\mathrm{tr}L_A = 4$). Therefore, the rank of L_A is four and proposition 3.7 applies. □

3.4 Summary of properties of rank one matrices

Let's summarize some of the results of the preceeding subsection:

Theorem 3.1. *Let V be a quadratic Jordan algebra isomorphic with $H_n(\mathcal{A})$ and $0 \neq A \in V$. The following conditions are equivalent :*

1. *A has Jordan rank one.*
2. *The class of A in $\mathbb{P}V$ belongs to the closed orbit of $Str(V)$.*
3. *For any isomorphism $\varphi : V \simeq \mathcal{S}_n(\mathbb{C})$ (resp. $\mathcal{M}_n(\mathbb{C})$, $\mathcal{AS}_{2n}(\mathbb{C})$), $\varphi(A)$ has minimal rank 1 (resp. 1,2).*
4. *For any isomorphism $\varphi : V \simeq H_n(\mathcal{A})$, $\varphi(A)$ is the Veronese image of a n-uple of elements in $\mathcal{A}$.*
5. *For any isomorphism $\varphi : V \simeq H_n(\mathcal{A})$, $L_{\varphi(A)}$ has rank $\dim \mathcal{A}$.*

Proposition 3.1 shows that the map $\nu_2 : \mathcal{A}^n \dashrightarrow \mathbb{P}^{n-1}_{\mathcal{A}}$ is exactly the analog of the map $k^n \dashrightarrow \mathbb{P}^{n-1}$. Let us understand better this map. In the case $\mathcal{A} = \mathbb{C}_k$, any vector $z \in (\mathbb{C}_k)^n$ can be written uniquely as $x + y$, with $x \in R(e)^n$ and $y \in R(f)^n$. If x and y don't vanish, then $\nu_2(z)$ identifies via $\mathbb{P}H_n(\mathbb{C}_k) \simeq \mathbb{P}V_2^n$ with $([x],[y]) \in \mathbb{P}R(e)^n \times \mathbb{P}R(f)^n \simeq \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$. Similarly, let $h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; a vector $z \in (\mathbb{H}_k)^n$ can be written uniquely as $x + y.h$, with $x, y \in R(e)^n$. The map $\mathbb{H}_k^n \dashrightarrow G(2, R(e)^n)$ corrresponding to ν_2 sends $z = (x + y.h), x, y \in R(e)^n$ on the line $(x, y) \in G(2, R(e)^n)$.

Any vector $z \in \mathcal{A}^n$ yields a k-linear map $\mathcal{A} \to \mathcal{A}^n, \lambda \mapsto z.\lambda$. Let $\mathcal{I}$ denote the closed subset of $\mathcal{A}^n$ where this map is not injective; we have:

Proposition 3.9. *The indeterminacy locus of the rational map $\nu_2 : \mathcal{A}^n \dashrightarrow \mathbb{P}\mathcal{A}^{n-1}$ is exactly $\mathcal{I}$. For $z \notin \mathcal{I}$, $\nu_2^{-1}[\nu_2(z)] = \{z.\lambda \in \mathcal{A} : Q(\lambda) \neq 0\} \simeq \{\lambda \in \mathcal{A} : Q(\lambda) \neq 0\}$.*

Therefore, we are as close as possible to the situation of a usual projective space. In fact, if we consider the usual map $\pi : k^{n+1} \dashrightarrow \mathbb{P}^n$ (case $\mathcal{A} = \mathbb{R}_k$), then $\pi(z)$ is defined if and only if $z \neq 0$, which is equivalent to $z \notin \mathcal{I}$. Moreover, if $\pi(z)$ is defined, then $\pi^{-1}[\pi(z)] = \{z.\lambda : \lambda \neq 0\}$.

Proof : If $(z_t) \in \mathcal{A}^n$ is such that $\forall t, u, z_t\overline{z}_u = 0$, then $\forall t, Q(z_t) = 0$, and if for example $z_1 \neq 0$, then $\forall t, z_t \in R(z_1)$. Therefore, $(z_t).\overline{z}_1 = 0$ and $(z_t) \in \mathcal{I}$. Thus this indeterminacy locus is included in $\mathcal{I}$.

In the case $\mathcal{A} = \mathbb{C}_k$, from the preceeding description of the map ν_2, it is clear that this map cannot extend to a point z in $R(e)$ or $R(f)$. Similarly, let $\mathcal{A} = \mathbb{H}_k$ and suppose $z \in \mathbb{H}_k^n$ is such that $\lambda \mapsto z.\lambda$ is not injective. If we write as before $z = x + y.h$, with $x, y \in R(e)$, we deduce that x and y are proportional, because $\dim\{z.\lambda, \lambda \in \mathbb{H}_k\} = 2$ implies $\dim(\{z.\lambda, \lambda \in \mathbb{H}_k\} \cap R(e)) = 1$ (proof of proposition 3.3). Therefore, again, we cannot extend the rational map ν_2 to z. ☐

As we will see in the final section, things are not so well behaved as far as octonions are concerned.

4 Subvarieties of projective spaces

In this section, we assume k has characteristic 0. I define and classify subvarieties of projective spaces: as we shall see, this classification is a little disappointing, because there are very few such subvarieties. I hope that further investigations will explain this fact.

If $X \subset \mathbb{P}\mathcal{A}^n$ is a subvariety, I denote $\tilde{X}$ the closure of its preimage by ν_2 (this preimage is a subset of $\mathcal{A}^{n+1}$).

From the theorem of Newlander and Nirenberg, it follows that a *real* subvariety of a usual complex variety is a complex subvariety if and only if each tangent space (which is a real subspace) is stable by the multiplication by complex numbers. Therefore, it seems natural to me to introduce the following

Definition 4.1. *A subvariety $X \subset \mathbb{P}\mathcal{A}^n$ is an $\mathcal{A}$-variety if the corresponding affine cone $\tilde{X} \subset \mathcal{A}^{n+1}$ has the property that $\forall x \in \tilde{X}, T_x\tilde{X} \subset \mathcal{A}^{n+1}$ is preserved by right multiplication by any element in $\mathcal{A}$.*

Since $\tilde{X}$ itself is preserved by right multiplication by $\mathcal{A}$, this condition is equivalent to the fact that $T_x\tilde{X} = T_{x.\lambda}\tilde{X}$.

Let us discuss what $\mathcal{A}$-varieties are.

Proposition 4.1. *Let $X \subset \mathbb{P}_{\mathbb{C}}^n = \mathbb{P}^n \times \mathbb{P}^n$ be a closed subvariety. The following conditions are equivalent:*

1. *X is a $\mathbb{C}_k$-variety.*

2. *There exist* $P_1, \dots, P_a \in \mathbb{C}_k[X_0, \dots, X_n]$ *such that*

$$x \in \tilde{X} \iff \forall 1 \leq t \leq a, P_t(x) = 0.$$

3. *There exist* $X_1, X_2 \subset \mathbb{P}^n$ *ordinary varieties such that*

$$X = X_1 \times X_2 \subset \mathbb{P}^n \times \mathbb{P}^n.$$

Proof : I will show that (1) and (2) are equivalent to (3). Let us show that (3) implies (1) and (2). If $X = X_1 \times X_2$, and if $\tilde{X}_i \subset k^{n+1}$ is the usual affine cone of X_i, then $\tilde{X} \simeq \tilde{X}_1 \oplus \tilde{X}_2$, under the isomorphism $(\mathbb{C}_k)^{n+1} = R(e)^{n+1} \oplus R(f)^{n+1} \simeq k^{n+1} \oplus k^{n+1}$. Let us denote by p_1 and p_2 the two projections on k^{n+1}, which correspond to the multiplications by e and f. We have that $T_{(x_1,x_2)}\tilde{X} = T_{x_1}X_1 \oplus T_{x_2}X_2$ is preserved by p_i. We thus have (1).

Let (f_t) and (g_u) be defining equations of X_1 and X_2 in $\mathbb{P}^n$. The inclusion of algebras $k \subset \mathbb{C}_k$ induces an inclusion φ of $k[X_0, \dots, X_n]$ in $\mathbb{C}_k[X_0, \dots, X_n]$. If we set $P_t = e\varphi(f_t)$ and $Q_u = f\varphi(g_u)$, then it is clear that $\forall (a_v, b_v) \in \mathbb{C}^{n+1}$, $P_t[(a_v(e)+b_v(f))_v] = 0$ if and only if $f_t(a_v) = 0$, and similarly $Q_u[(a_v e+b_v f)_v] = 0$ if and only if $g_u(b_v) = 0$. Therefore, the polynomials (P_t, Q_u) define the variety $\tilde{X}_1 \oplus \tilde{X}_2$, and (2) is true. Note that we can reverse this computation : if (2) is true, then X is a product and we have (3).

The last thing to be proved is that (1) implies (3). If $T_{(x_1,x_2)}\tilde{X}$ is stable by the two projections p_1 and p_2, according to the more general following proposition, $\tilde{X}$ is the sum of two varieties.

Proposition 4.2. *Let* $V_t, 1 \leq t \leq a$ *be vector spaces over* k *and* $X \subset \bigoplus V_t =: V$ *an irreducible affine variety such that* $\forall (\lambda_t) \in k^a, (x_t) \in X \Rightarrow (\lambda_t.x_t) \in X$ *and* $\forall x \in X, T_xX$ *is the sum of its intersections with* $V_t \subset V$.

Then there exist irreducible affine varieties $X_i \subset V_i$ *such that* $X = \bigoplus X_i$.

Proof : By induction, we can assume that $a = 2$. Let us consider the restrictions $p_{1,2}$ to X of the projections on $V_{1,2}$. Let n_i be the generic dimension of $T_xX \cap V_i$ and $X_i = p_i(X)$. Let $x = (x_1, x_2) \in X$ be a smooth generic point. Then $T_xX = (T_xX \cap V_1) \oplus (T_xX \cap V_2)$ has dimension $n_1 + n_2$, so $\dim X = n_1 + n_2$. Moreover, the kernel of dp_1 is $T_xX \cap V_2$, of dimension n_2, and also the kernel of dp_2 has dimension n_1. It follows that $\dim X_i = n_i$. Thus $\forall x_2 \in X_2, p_2^{-1}(x_2)$ has dimension at least n_1. Since the restriction of p_1 to this preimage is an isomorphism on its image which is a subvariety of X_1, we deduce that $p_2^{-1}(x_2) = \{(x_1, x_2) : x_1 \in X_1\}$. Therefore $X = X_1 \oplus X_2$. □

We now consider the quaternionic case:

Proposition 4.3. *Let* $X \subset \mathbb{PH}_k^n = G(2, 2n+2)$ *be a closed subvariety. The following conditions are equivalent:*

1. X *is a* $\mathbb{H}_k$*-variety.*

2. *There exist* $l_1, \dots, l_a$ *right-linear forms such that* $x \in \tilde{X} \iff \forall 1 \leq t \leq a, l_t(x) = 0$.

3. *There exists a linear subspace* $L \subset k^{2n+2}$ *such that* $X = G(2, L) \subset G(2, k^{2n+2})$.

This proposition, though a little disappointing, does not really come as a surprise, at least from a heuristic point of view. In fact, due to the lack of commutativity of $\mathbb{H}_k$, the condition that a polynomial on $\mathbb{H}_k$ vanishes is well-defined on a quaternionic projective space only if this polynomial is linear.

Proof : If $X \subset G(2, k^{2n+2})$ is the subgrassmannian $G(2, L)$, then $\tilde{X} \subset (\mathbb{H}_k)^{n+1}$ is a sublinear space preserved by right mutiplication by $\mathbb{H}_k$ (in fact $L \subset R(e)^{n+1}$ is preserved by right multiplication by 1 and e, and so $L \oplus L.h$ is preserved by right multiplication by $\mathbb{H}_k$). Therefore (3) implies (1). By proposition 2.3, any linear space in $(\mathbb{H}_k)^n$ preserved by right-multiplication by $\mathbb{H}_k$ is given by linear equations, so (3) and (2) are equivalent.

To prove that (1) implies (3), I suggest two proves, one studying $\tilde{X}$ and one studying X. For the first proof, we note that $\tilde{X} \subset \mathbb{H}_k^{n+1}$ is stable by right multiplication by $\mathbb{C}_k \subset \mathbb{H}_k$ and all the tangent spaces $T_x\tilde{X}$ also. Therefore, by proposition 4.1, there exists $X_1 \subset R(e)^{n+1}$ and $X_2 \subset R(f)^{n+1}$ such that $\tilde{X} = X_1 \oplus X_2$. Since $\tilde{X}.h = \tilde{X}$, $X_2 = X_1.h$. Moreover, if $x_1, x_2 \in X_1$, then $(x_1, x_2.h) \in \tilde{X}$ and the fact that $T_{(x_1,x_2)}\tilde{X}$ is preserved by multiplication by h implies that $T_{x_2}X_2 = T_{x_1}X_1.h$. Thus $T_{x_1}X_1$ does not depend on x_1 and X_1 is a linear space.

For the second proof, we remark that if for $\tilde{x} \in \tilde{X} \cap (\mathbb{H}_k^n - \mathcal{I})$, $T_{\tilde{x}}\tilde{X}$ is a $\mathbb{H}_k$-linear space of dimension $4 \dim X$ corresponding to $M \in G(2 \dim X, R(e)^{n+1})$, then for $x = \nu_2(\tilde{x})$, $T_{[x]}X$ is included in (and thus equal to by dimension count) $L_x^* \otimes M/L_x$, where L_x stands for the linear space parametrized by $x \in G(2, R(e)^{n+1})$. This will imply that X is a subgrassmannian by the following more general proposition.

Proposition 4.4. *Let $r \geq 2$ and $X \subset G(r, V)$ a subvariety of a Grassmannian such that $\forall x \in X$ there exists a linear subspace M of V such that $T_xX = L_x^* \otimes (M/L_x) \subset L_x^* \otimes V/L_x = T_xG(r, V)$. Then there exists a linear subspace $L \subset V$ such that $X = G(r, L)$.*

Proof : The hypothesis implies that $\dim X$ is a multiple of r, so let d such that $\dim X = rd$. Consider the total space T of the restriction of the tautological bundle to X, and the natural projection $p : T \to V$. Let $L = p(T)$; if $0 \neq v \in p(T)$, then $p^{-1}(v) \simeq \{x \in X : v \in L_x\}$. Therefore, the tangent space to this fiber is the set of $\varphi \in L_x^* \otimes (V/L_x)$ such that $\varphi(v) = 0$. The hypothesis implies that this has dimension $r(d-1)$, thus $p(T)$ has dimension $rd+r-r(d-1) = r+d$. If $v \in p(T)$, let $X_v = \{x \in X : v \in L_x\}$, and let T_v be the restriction of T to X_v. Again, a fiber of the projection $p : T_v \to V$ has dimension $r(d-2)$ and thus $p(T_v)$ is also of dimension $r + d$. So, we have that $p(T_v) = p(T)$ and it follows that $p(T)$ is a linear space, and by dimension count $X = G(r, p(T))$. $\square$

5 The exceptional case

In this section, I give a study of the octonionic projective plane similar to that of the projective spaces over $\mathbb{R}_k, \mathbb{C}_k, \mathbb{H}_k$. First, I have to understand the structure group of the exceptional Jordan algebra. It has been known for very long that the exceptional Lie groups of types F_4 and E_6 can be defined in terms of Jordan algebras. Here however, I will describe the Chevalley group of type E_6 (over the integers) using the incidence geometry of the 27 lines on a smooth cubic surface. This idea comes from [Fau 01] and [Lur 01]. First, I consider a degree three polynomial, defined using the geometry of smooth cubic surfaces. I show that the group of elements preserving it is simple of type E_6 without using any results on Jordan algebras (theorem 5.1). Then I show that this polynomial is equivalent to the determinant of the exceptional Jordan algebra (proposition 5.1). Finally, using the known representation theory of $E_6(k)$, I describe the octonionic plane (theorem 5.2).

5.1 Preliminary facts on smooth cubic surfaces

Let $S \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth cubic surface and let $\mathcal{P}$ denote the set of lines in S. Let $\mathcal{L}$ denote the set of tritangent planes. If $p \in \mathcal{P}$ and $l \in \mathcal{L}$, I write $p \in l$ whenever the line lies in the plane. It is well known ([Har 77, section V.4.] or [DV 04, section 10]) that there are 27 lines on S and 45 tritangent planes, each of which containing three lines.

Following J.R. Faulkner [Fau 01], let us call a 3-grid a couple of triples of planes $[(l_1, l_2, l_3), (m_1, m_2, m_3)]$ such that the intersection of l_i and m_j is a line in S (the incidence relation of the 9 corresponding lines looks like a "3×3-grid").

5.2 Definition of the Chevalley group of type E_6

Following J.R. Faulkner, let $\theta : \mathcal{L} \to \{-1, 1\}$ be a function with the property that for any 3-grid $[(l_1, l_2, l_3), (m_1, m_2, m_3)]$ of $(\mathcal{P}, \mathcal{L})$, one has

$$\theta(l_1)\theta(l_2)\theta(l_3) + \theta(m_1)\theta(m_2)\theta(m_3) = 0$$

(theorem 5 in [Fau 01] exhibits such a function).

Let $V = \mathbb{Z}^{\mathcal{P}}$ and let α be the following form on this module:

$$\alpha(f) = \sum_{l\in\mathcal{L}} \theta(l) \prod_{p\in l} f(p).$$

We have the following:

Theorem 5.1. *The group-scheme of elements preserving α is isomorphic with the simply-connected Chevalley group of type E_6. Its projectivisation is the adjoint group. Moreover, if k is algebraically closed, then the closed orbit of $G(k)$ acting on $\mathbb{P}(V \otimes k)$ is the singular locus of the cubic hypersurface defined by α.*

Proof : Let G be this group-scheme. Let k be an infinite field; let us first show that $G(k)$ is split reductive of type E_6.

First of all, there is an explicit formula for α, computed as formula (7) in [Fau 01]: let V_1 be the $\mathbb{Z}$-module of 3×3-matrices with integer coefficients and $W = V_1 \oplus V_1 \oplus V_1$. Let β be the cubic form

$$\beta(A, B, C) = \det(A) + \det(B) + \det(C) - \operatorname{tr}(ABC).$$

Then (V, α) is isomorphic with (W, β).

I can exhibit a maximal torus in G: let $M, N, P \in SL_3(k)$. Then the action $(M, N, P).(A, B, C) = (MAN^{-1}, NBP^{-1}, PCM^{-1})$ defines an element in $G(k)$. Taking diagonal matrices, we thus have a torus $T \subset G$ of rank 6 (defined over $\mathbb{Z}$). Let us show that it is a maximal torus in G. Let $g \in G(k)$ be an element commuting with $T(k)$. Since g preserves the eigenlines of $T(k)$, it is of the form $(g.f)(p) = \lambda(g)f(p)$. Therefore, it preserves the three spaces $V_1 \oplus \{0\} \oplus \{0\}$, $\{0\} \oplus V_1 \oplus \{0\}$ and $\{0\} \oplus \{0\} \oplus V_1$. Since $\alpha(A, 0, 0) = \det(A)$, there exist M_1, N_1 such that $g.(A, 0, 0) = (M_1AN_1^{-1}, 0, 0)$. Moreover, since g

acts diagonaly, M_1 and N_1 are diagonal. Similarly, we prove that $g.(A,B,C) = (M_1AN_1^{-1}, N_2BP_1^{-1}, P_2CM_2^{-1})$. The fact that g preserves $\mathrm{tr}(ABC)$ implies $M_1 = M_2, N_1 = N_2$ and $P_1 = P_2$ and so $g \in T(k)$.

I now show that the Weyl group of $G(k)$ is the Weyl group of type E_6. Let g be in the normalizer of $T(k)$. Then g permutes the eigenlines of $T(k)$; therefore it induces a bijection of $\mathcal{P}$. Since g preserves α, this bijection corresponds to a bijection of the incidence $(\mathcal{P},\mathcal{L})$. Since the isomorphism group of this geometry is $W(E_6)$ [Man 74, th. 23.9], we therefore have a map $W(G) \to W(E_6)$. We have already seen that this map is injective. Let us argue for its surjectivity.

Let w be an isomorphism of $(\mathcal{P},\mathcal{L})$.

Consider the function $\begin{array}{cccc} \psi & : & \mathcal{L} & \to & \{-1,1\} \\ & & l & \mapsto & \theta(l)/\theta(w.l) \end{array}$. It has the property that $\psi(l_1)\psi(l_2)\psi(l_3) = \psi(m_1)\psi(m_2)\psi(m_3)$ for any 3-grid $[(l_1,l_2,l_3),(m_1,m_2,m_3)]$. By lemma 4 in [Fau 01], there exists $x \in \{-1,1\}^{\mathcal{P}}$ such that $\psi(l) = \prod_{p\in l} x(p)$. Therefore, we can set $(g.f)(p) = x(p)g(w.p)$ to get an element $g \in N_G(T)$ which is equivalent to w modulo T.

From this it follows that $V \otimes k$ is an irreducible representation of $G(k)$. In fact, let $U \subset V \otimes k$ be any sub-representation. If U contains a vector different from 0, since k is infinite, U contains an eigenvector for $T \otimes k$. Using the above action of the Weyl group of $G(k)$, U contains all the eigenvectors and therefore equals $V \otimes k$. Thus, $G(k)$ is reductive.

Let $G(k)^0$ be the connected component of the identity element in $G(k)$. I have to show that the image of $N_{G(k)^0}(T)$ is also $W(E_6)$. Since this image is a normal subgroup of the group $W(E_6)$ which has a normal simple subgroup of index 2, it is enough to exhibit an odd element of this image. This is easy, considering the action $(A,B,C) \mapsto (MAN^{-1}, NBP^{-1}, PCM^{-1})$ with (M,N,P) in the connected variety $SL_3(k)^3$. Note that considering these elements of $G(k)$, one checks that $G(k)$ and $G(k)^0$ have the same center, namely $\{j.Id : j^3 = 1\}$.

I therefore have shown that $G(k)^0$ is a split reductive group of type E_6. To show that $G(k)$ is in fact connected, I first prove the result about the singular locus.

Assume that k is algebraically closed. Let $X' \subset \mathbb{P}(V \otimes k)$ denote the closed $G(k)^0$-orbit and let X be the singular locus of the hypersurface defined by α. Since X is a non-empty closed invariant subvariety of $\mathbb{P}(V \otimes k)$, we have $X \supset X'$. Let K and K' be the spaces of quadrics which vanish along X and X': $K \subset K'$. We have a $G(k)^0$-equivariant map $\varphi : (V \otimes k) \to K$ given by $v \mapsto (u \mapsto D_u\alpha(v))$. It is easy to check that φ is not identically 0. Since V is irreducible, by Schur's lemma, this is an injection and $\dim \varphi(V \otimes k) = 27$. Moreover, since $X' \subset \mathbb{P}(V \otimes k)$ is projectively normal [RR 85, theorem 1], the restriction map yields an exact sequence

$$0 \to K' \to Q(V \otimes k) \to H^0(X', \mathcal{O}(2)) \to 0.$$

($Q(V \otimes k)$ denotes the space of quadratic forms on $V \otimes k$). The dimensions of the vector spaces involved in this exact sequence are the same in positive characteristic as in zero characteristic (the middle dimension is obviously the same and the others could only eventually be larger, since X' can be realized as a flat scheme over $\mathbb{Z}$); therefore, $\dim K' = 27$. We thus have $\varphi(V) \subset K \subset K'$ and these vector spaces have dimension 27, so $\varphi(V) = K = K'$. Since the ideal of X' is generated by quadrics [Ram 87, th 3.8, p.86], $X = X'$.

Let P the stabilizor in $G(k)^0$ of a point in X. We have $X = G(k)^0/P$. It follows from [Dem 77, Théorème 1] that $G(k)^0 \to Aut(X)$ is surjective. Since X is the singular locus of $\{\alpha = 0\}$, we have exact sequences ($C(k)$ denotes the common center of $G(k)$ and $G(k)^0$)

$$1 \to C(k) \to G(k) \to Aut(X) \to 1$$
$$1 \to C(k) \to G(k)^0 \to Aut(X) \to 1.$$

Thus we have: $G(k) = G(k)^0$.

The center of $G(k)$ contains $\{j.Id : j^3 = 1\}$ which in any characteristic is a scheme of length three. Therefore $G(k)$ is simply-connected.

Hence for all algebraically closed fields k, $G(k)$ is the simply-connected simple group of type E_6. Moreover, it is proved in [Lur 01, Theorem 5.5.1] that the dimension of the Lie algebra of G over $\mathbb{Z}/p\mathbb{Z}$ is allways 78 (it does not depend on p). It follows therefore from the proof of [Har 77, proposition III 10.4] that G is smooth over $\mathbb{Z}$. From the uniqueness result [SGA 3, exposé XXIII, corollaire 5.4], the theorem follows. $\square$

5.3 E_6 and the exceptional Jordan algebra

The following result makes the link between the preceeding subsection and the rest of the article. Recall that the determinant of the exceptional Jordan algebra is defined by the equation [Jac 63, (18), p.37] :

$$\det \begin{pmatrix} r_1 & \overline{x}_3 & \overline{x}_2 \\ x_3 & r_2 & x_1 \\ x_2 & \overline{x_1} & r_3 \end{pmatrix} = r_1 r_2 r_3 + 2\langle x_1 x_2, x_3 \rangle - r_1 Q(x_1) - r_2 Q(x_2) - r_3 Q(x_3). \quad (3)$$

Proposition 5.1. *The previous form α is isomorphic with the determinant of the exceptional Jordan algebra* det.

Therefore, the group of elements preserving det is also the simple simply-connected group of type E_6.

<u>Proof :</u> If $a_{i,j}, b_{i,j}, c_{i,j},\ 1 \le i, j \le 3$, are integers, a courageous reader will check that the determinant of the hermitian matrix $H = \begin{pmatrix} b_{1,3} & \overline{x_3} & \overline{x_2} \\ x_3 & c_{3,1} & x_1 \\ x_2 & \overline{x_1} & -a_{1,1} \end{pmatrix}$ with

$$x_1 = \begin{pmatrix} a_{2,1} & -c_{3,3} \\ a_{3,1} & c_{3,2} \end{pmatrix} + \begin{pmatrix} b_{3,1} & -b_{2,1} \\ -b_{3,2} & b_{2,2} \end{pmatrix} e \ ;$$

$$x_2 = \begin{pmatrix} a_{1,2} & a_{1,3} \\ -b_{3,3} & b_{2,3} \end{pmatrix} + \begin{pmatrix} c_{2,2} & -c_{2,3} \\ -c_{1,2} & -c_{1,3} \end{pmatrix} e \ ;$$

$$x_3 = \begin{pmatrix} -a_{3,3} & a_{3,2} \\ a_{2,3} & -a_{2,2} \end{pmatrix} + \begin{pmatrix} b_{1,2} & c_{1,1} \\ b_{1,1} & c_{2,1} \end{pmatrix} e$$

is $\det A + \det B + \det C - \operatorname{tr}(ABC)$, if $A = (a_{i,j}), B = (b_{i,j})$ and $C = (c_{i,j})$. $\square$

Here is the explanation how I found this formula. A Schäfli's double-six is by definition [Har 77, p.403] a couple (E_i, F_i) of sextuples of lines in S such that E_i don't meet E_j if $i \neq j$, and E_i meets F_j if and only if $i \neq j$. Such double-sixes exist and label uniquely the other lines in S, since there is a unique line meeting E_i and F_j when $i \neq j$.

Now, let us say that a linear form $l \in \{a_{i,j}, b_{i,j}, c_{i,j}\}$ "meets" another linear form m if lm divides a monomial appearing in the expression of α. It is easily seen that $((a_{1,1}, a_{2,1}, a_{3,1}, b_{2,1}, b_{2,2}, b_{2,3}), (a_{1,2}, a_{2,2}, a_{3,2}, b_{1,1}, b_{1,2}, b_{1,3}))$ is a Schäfli's double-six for this incidence relation. Playing the same game with det, one sees that we can start filling a hermitian matrix of coordinates with the following forms :

$$\begin{pmatrix} b_{1,3} & \overline{x_3} & \overline{x_2} \\ x_3 = \begin{pmatrix} 0 & a_{3,2} \\ 0 & a_{2,2} \end{pmatrix} + \begin{pmatrix} b_{1,2} & 0 \\ b_{1,1} & 0 \end{pmatrix} e & c_{3,1} & x_1 = \begin{pmatrix} a_{2,1} & 0 \\ a_{3,1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & b_{2,1} \\ 0 & b_{2,2} \end{pmatrix} e \\ x_2 = \begin{pmatrix} a_{1,2} & 0 \\ 0 & b_{2,3} \end{pmatrix} & \overline{x_1} & a_{1,1} \end{pmatrix}.$$

Then, using the fact that a Schäfli's double-six labels all the lines, one can finish filling the above matrix. One gets the matrix H up to signs; the determinant of this matrix involves the 45 expected monomials, but with wrong signs. These signs may be corrected using the algorithm described in the proof of [Fau 01, lemma 4].

5.4 The octonionic projective space

Let $\mathcal{L}$ be the set of all the projective lines in the space $\mathbb{P}\{\mathtt{Re} = 0\} \subset \mathbb{P}(\mathbb{O}_k)$, on which the restriction of the octonionic product vanishes identically. Let X_0 be the set of matrices of the form $\begin{pmatrix} 0 & \overline{a} & \overline{b} \\ a & 0 & \overline{c} \\ b & c & 0 \end{pmatrix}$, with a, b, c octonions which generate a line in $\mathcal{L}$. Let X_1 be the set-theoretic image by ν_2 of triples of elements in $\mathbb{O}_k$ generating an associative subalgebra.

Proposition 5.2. *Let $X \subset \mathbb{P}H_3(\mathbb{O}_k)$ be the variety of rank one elements in the exceptional Jordan algebra. Then*

- *$X = X_0 \amalg X_1$.*
- *X is the closed orbit of $Str(H_3(\mathbb{O}_k))$.*
- *X is the singular locus of the hypersurface $\{\det = 0\}$.*
- *X is defined by the following quadrics:*

$$a_{1,1}a_{2,2} = Q(a_{1,2}),\ a_{1,1}a_{3,3} = Q(a_{1,3}),\ a_{2,2}a_{3,3} = Q(a_{2,3})$$
$$a_{1,1}a_{2,3} = a_{2,1}a_{1,3},\ a_{3,2}a_{2,1} = a_{3,1}a_{2,2},\ a_{2,1}a_{3,3} = a_{2,3}a_{3,1}.$$

- *The hypersurface defined by the determinant is the closure of the set of sums of two rank one elements.*

Remark : This variety is not, according to [Zak 93, th.4.9, p.90], the image of all octonionic triples. In fact, the coefficients of a matrix in X belong to an

associative subalgebra of $\mathbb{O}_k$, which is not the case in general if we take the image of any triple. Moreover, all the elements of X are not images of ν_2. The claim of F.L. Zak, that $\nu_2(z.\lambda) = \nu_2(z)$ for any invertible octonion λ, is also wrong, due to the lack of associativity.
J. Roberts [Rob 88] has shown that the singular locus of the hypersurface defined by det is a Severi variety (cf [Zak 93, Cha 02] for the definition and study of Severi varieties).
If $A \in \mathbb{P}H_3(\mathbb{O}_\mathbb{C})$ is in fact in $\mathbb{P}H_3(\mathbb{O}_\mathbb{R})$ and annihilates the quadrics of the proposition, its diagonal cannot vanish; therefore it is easy to see that it is the image by ν_2 of a vector in $\mathbb{O}_\mathbb{R}^3$ with one coordinate equal to 1. We thus see the link with the set of matrices considered by Freudenthal or Tits [Fre 54, Tit 53].
Proof : We already know that the closed orbit is the singular locus of $\{\det = 0\}$. A direct computation using the explicit formula (3) shows that the equations of this locus are

$$\begin{aligned} a_{1,1}a_{2,2} = Q(a_{1,2}),\ a_{1,1}a_{3,3} = Q(a_{1,3}),\ a_{2,2}a_{3,3} = Q(a_{2,3}) \\ a_{1,1}a_{2,3} = a_{2,1}a_{1,3},\ a_{3,2}a_{2,1} = a_{3,1}a_{2,2},\ a_{2,1}a_{3,3} = a_{2,3}a_{3,1}. \end{aligned} \tag{4}$$

One can check that an element of rank one annihilates these quadrics; therefore X is the closed orbit.

Computing the number of roots of the parabolic subgroup stabilizing a highest weight vector, it is easily seen that $\dim X = 16$.

If X_1^0 is the image by ν_2 of vectors of the form $(1, z_1, z_2)$, it is clear that the preceeding quadrics vanish on X_1^0, therefore $\overline{X_1^0} \subset X$. Since $\dim X_1^0 = 16$, equality holds. Since $\mathbb{O}_k$ is an alternative algebra (meaning that every subalgebra generated by two elements is associative), all coefficients of a matrix in X' belong to an associative algebra. This explains why we consider images by ν_2 of triples of octonions generating an associative algebra.

If A and B belong to X, then det vanishes at order two on the line (AB) at the points A and B. Therefore, all the points of the line (AB) are with vanishing determinant. Now, if $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $T_A X \cap T_B X = \left\{ \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$. Therefore, the closure of the set of sums of two elements of X is at least an hypersurface in $H_3(\mathbb{O}_k)$. So, this is exactly the hypersurface defined by the determinant.

To finish the proof of the proposition, we have to understand the image X_1 of ν_2. Let $A \in X$. If A has non-vanishing first diagonal coefficient, we can suppose that this coefficient equals one, and then $A = \nu_2(1, a_{2,1}, a_{3,1})$ belongs to X_1^0 and thus X_1. The same holds for any matrix which diagonal does not vanish. If $\mathtt{Re}(a_{2,1}) \neq 0$, we can argue as in proposition 3.2 with the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
It is therefore sufficient to consider matrices of the form $\begin{pmatrix} 0 & \overline{a} & \overline{b} \\ a & 0 & \overline{c} \\ b & c & 0 \end{pmatrix}$ with $\mathtt{Re}(a) = \mathtt{Re}(b) = \mathtt{Re}(c) = 0$. In this case, the quadrics (4) show that $Q(a) = Q(b) = Q(c) = 0$ and $ab = ba = ac = ca = bc = cb = 0$ (since $\overline{a} = -a, \overline{b} = -b$

and $\overline{c} = -c$). Thus the octonions a, b, c are in a subalgebra where the product vanishes identically. This subalgebra has dimension at most 2 (in fact if a and b are not collinear, then since $\overline{a}c = \overline{b}c = 0$, $c \in L(a) \cap L(b) = Vect(a, b)$). If it has dimension 2, then $A \in X_0$. If it has dimension 1, then all coefficients of A are scalar multiple of some octonion with vanishing norm; since we can put this octonion in a subalgebra of $\mathbb{O}_k$ isomorphic with $\mathbb{H}_k$, proposition 3.2 shows that $A \in X_1$.

The last thing to prove is that X_0 and X_1 are disjoint.
Let $A = \begin{pmatrix} 0 & \overline{a} & \overline{b} \\ a & 0 & \overline{c} \\ b & c & 0 \end{pmatrix} \in X_0$, with a, b, c octonions as before, and suppose there exist z_1, z_2, z_3 such that $\nu_2(z_1, z_2, z_3) = A$, or:

$$\begin{cases} Q(z_1) = Q(z_2) = Q(z_3) = 0. \\ z_1\overline{z}_2 = a \\ z_1\overline{z}_3 = b \\ z_2\overline{z}_3 = c. \end{cases}$$

Suppose first that no element in $\{a, b, c\}$ is zero.
If a and b, b and c, and c and a are not collinear, we deduce from the system that $z_1 \in L(a) \cap L(b) = (a, b)$, and similarly $z_2, z_3 \in (a, b)$. We therefore have a contradiction, because this implies $z_1\overline{z}_2 = 0$.
If for example a and b are collinear, but not a and c, then, the system implies $z_2, z_3 \in L(a) \cap L(c) = (a, c)$. We therefore have $z_2\overline{z}_3 = 0$, a contradiction.

If for example $c = 0$ and $a, b \neq 0$, we still have $z_1 \in (a, b)$, but only $z_2 \in L(a)$ and $z_3 \in L(b)$. If we had $z_2 \in L(b)$, we would have $z_2 \in (a, b)$ and $a = z_1\overline{z}_2 = 0$, a contradiction. Thus, $z_2 \notin L(b)$ and so $bz_2 \neq 0$. This shows that $L(b) \cap R(z_2) = \langle bz_2 \rangle$ (proposition 1.2). Since $z_3 \in L(b) \cap R(z_2)$, it is propotionnal to bz_2. But in that case, since $z_2 \in L(a)$, $\overline{z}_3$ is in $R_b[R(a)] = R(a) \cap R(b) = (a, b)$ by proposition 1.2. This in turn would imply that $z_1\overline{z}_3 = 0$, and a contradiction. $\square$

References

[Cha 02] P.E. Chaput, *Severi varieties.* Math. Z. **240** (2002), no. 2, 451–459.

[Cha 03] P.E. Chaput, thesis, available at http://www.math.sciences.univ-nantes.fr/˜chaput

[DV 04] I.V. Dolgachev, A Verra, *Topics in classical algebraic geometry*, available at http://www.math.lsa.umich.edu/˜idolga/

[Fre 54] H. Freudenthal, *Beziehungen der E_7 und E_8 zur Oktavenebene*, Indagat. Math. **16** (1954) 218–230 and 636–638, **17** (1955) 151–157 and 277–285, **25** (1963) 457–487.

[SGA 3] Grothendick, A.,Demazure, M. Schémas en groupes. III: Structure des schémas en groupes réductifs. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Lecture Notes in Mathematics **153**.

[Dem 77] Demazure, M. *Automorphismes et déformations des variétés de Borel.* Invent. Math. **39** (1977), no. 2, 179–186.

[Fau 01] Faulkner, J.R. *Generalized quadrangles and cubic forms.* Comm. Algebra **29** (2001), no. 10, 4641–4653.

[Gar 01] Garibaldi, R.S. *Structurable algebras and groups of type E_6 and E_7.* J. Algebra **236** (2001), no. 2, 651–691.

[Har 90] F.R. Harvey, Spinors and calibrations. Perspectives in Mathematics, 9. Academic Press, Inc., Boston, MA, 1990.

[Har 77] Hartshorne, R. Algebraic geometry. Graduate Texts in Mathematics, **52**. Springer-Verlag, 1977.

[Jac 58] N. Jacobson, *Composition algebras and their automorphisms.* Rend. Circ. Mat. Palermo (2) **7** (1958), 55–80.

[Jac 59] N. Jacobson, *Some groups of transformations defined by Jordan algebras. I.* J. Reine Angew. Math. **201** (1959) 178–195.

[Jac 63] N. Jacobson, *Generic norm of an algebra.* Osaka Math. J. **15** (1963) 25–50.

[Jac 68] N. Jacobson, Structure and representations of Jordan algebras. Am. Math. Soc. Coll. Publ. vol XXXIX (1968).

[Jac 69] N. Jacobson, Lectures on quadratic Jordan algebras. Tata Institute of Fundamental Research Lectures on Mathematics, No. **45**. Tata Institute of Fundamental Research, Bombay, 1969

[Jac 85] N. Jacobson, *Some projective varieties defined by Jordan algebras.* J. Algebra **97** (1985), no. 2, 565–598.

[LM 99] J.M. Landsberg, L. Manivel, *The projective geometry of Freudenthal's magic square.* J. Algebra **239** (2001), no. 2, 477–512.

[Lur 01] Lurie, J. *On simply laced Lie algebras and their minuscule representations.* Comment. Math. Helv. **76** (2001), no. 3, 515–575.

[Man 74] Manin, Y.I. Cubic forms. Algebra, geometry, arithmetic. Second edition. North-Holland Mathematical Library, **4**.

[McC 65] K. McCrimmon, *Norms and non-commutative Jordan algebras.* Pacific J. Math **15**(1965) 925-956.

[RR 85] S. Ramanan, A. Ramanathan, *Projective normality of flag varieties and Schubert varieties.* Invent. Math. **79** (1985), no. 2, 217–224.

[Ram 87] A. Ramanathan, *Equations defining Schubert varieties and Frobenius splitting of diagonals.* Inst. Hautes Études Sci. Publ. Math. **65**, (1987), 61–90.

[Rob 88] J. Roberts, Projective embeddings of algebraic varieties. Monografías del Instituto de Matemáticas, 19. Universidad Nacional Autónoma de México, México, 1988.

[Sch 66] R.D. Schafer, An introduction to nonassociative algebras. Corrected reprint of the 1966 original. Dover Publications, Inc., New York, 1995

[BS 60] F. van der Blij, T.A. Springer, *Octaves and triality.* Nieuw Arch. Wisk. (3) **8** 1960 158–169.

[Tit 53] J. Tits, *Le plan projectif des octaves et les groupes de Lie exceptionnels.* Acad. Roy. Belgique. Bull. Cl. Sci. **39**(5), (1953). 309–329.

[Zak 93] F. L. Zak, Tangents and Secants of Algebraic Varieties. American Mathematical Society, Providence, RI (1993).