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Abstract. We consider an inverse spectral problem for a class of singular AKNS operators $H_a, a \in \mathbb{N}$ with an explicit singularity. We construct for each $a \in \mathbb{N}$, a standard map $\lambda^a \times \kappa^a$ with spectral data λ^a and some norming constant κ^a. For $a = 0$, $\lambda^a \times \kappa^a$ was known to be a local coordinate system on $L^2_{\mathbb{R}}(0,1) \times L^2_{\mathbb{R}}(0,1)$. Using adapted transformation operators, we extend this result to any non-negative integer a, give a description of isospectral sets and we obtain a Borg-Levinson type theorem.

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1. Introduction

The Schrödinger operator $\mathcal{H} = -\Delta + q(\|x\|)$ with a radial potential q, acting on the unit ball of $\mathbb{R}^3$, through a decomposition via spherical harmonics (see [19], p. $160-161$), is unitary equivalent to a collection of singular differential operators $\mathcal{H}_a(q)$, $a \in \mathbb{N}$ acting on $L^2_{\mathbb{R}}(0, 1)$, with Dirichlet boundary conditions, defined by

$$\mathcal{H}_a(y)(x) := \left(-\frac{d^2}{dx^2} + \frac{a(a+1)}{x^2} + q(x)\right) y(x) = \lambda y(x), \quad x \in [0, 1], \lambda \in \mathbb{C}.$$

With this splitting, it makes sense to study inverse spectral problems not for $\mathcal{H}$ itself but for each $\mathcal{H}_a$.

The inverse spectral problem for these operator is the construction for each $a \in \mathbb{N}$, of a regular coordinate system $\lambda^a \times \kappa^a$ for potentials $q \in L^2_{\mathbb{R}}(0, 1)$ where λ^a represent the spectrum of $\mathcal{H}_a$ and κ^a are convenient complementary data (regularity means stability of the inverse spectral problem).

This question is not new and has been answered: Borg [6] and Levinson [15] first, proved that $\lambda^0 \times \kappa^0$ was one-to-one on $L^2_{\mathbb{R}}(0, 1)$; then Pöschel and Trubowitz [18] completed this result obtaining $\lambda^0 \times \kappa^0$ as a global real-analytic coordinate system on $L^2_{\mathbb{R}}(0, 1)$. Guillot and Ralston [13] extended their results to $\lambda^1 \times \kappa^1$, passing through the singularity inside the equation. Next Zhornitskaya and Serov [24], and Carlson [7], proved that for all real $a \geq -1/2$, $\lambda^a \times \kappa^a$ is one-to-one on $L^2_{\mathbb{R}}(0, 1)$. Finally, the author [21] completed theses works proving that for all $a \in \mathbb{N}$ the map $\lambda^a \times \kappa^a$ was a local (hence global) diffeomorphism on $L^2_{\mathbb{R}}(0, 1)$.

Then, it is natural and interesting to wonder if these kind of results can be found for an other physical equation: the Dirac equation. Hence, as the radial Schrödinger operator, the Dirac operator with a radial electric potential acting on the unit ball of $\mathbb{R}^3$ is decomposed (see for instance [23]) into a collection of operators H_a defined on $[0, 1]$ by

$$H_a(V)Y(x) := \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{d}{dx} + \begin{bmatrix} 0 & -\frac{a}{x} \\ -\frac{a}{x} & 0 \end{bmatrix} + V(x)\right) Y(x) = \lambda Y(x), \tag{1}$$

where $Y = (Y_1, Y_2)$, $\lambda \in \mathbb{C}$ and

$$V(x) = \begin{bmatrix} q(x) + m & 0 \\ 0 & q(x) - m \end{bmatrix}, m \in \mathbb{R};$$

with general boundary conditions

$$Y_2(0) = 0; \quad Y(1) \cdot u_\beta = 0 \qquad u_\beta = \begin{bmatrix} \sin\beta \\ \cos\beta \end{bmatrix}, \quad \beta \in \mathbb{R}. \tag{2}$$

Written this way, the Dirac operator seems to be unadapted in view of inverse spectral problems. Indeed for $a = 0$, as raised by Levitan and Sargsjan in [16](Chap. 7) and pointed out more generally by Clark and Gesztesy in [8] (section 6), the existence of a gauge transformation on the potential V leaving the spectrum invariant leads to choose a normal form for the problem, namely, the AKNS system, obtained from (1) considering potentials V of the following shape:

$$V(x) = \begin{bmatrix} -q(x) & p(x) \\ p(x) & q(x) \end{bmatrix}, \quad (p, q) \in L^2_{\mathbb{R}}(0, 1) \times L^2_{\mathbb{R}}(0, 1). \tag{3}$$

Moreover, there are some clues showing that the inverse spectral problem is kind of degenerated: for instance, the Ambarzumian type theorem obtained by Kiss [14] who proves that for all $m \neq 0$ and $q \in \mathcal{C}([0, 1]; \mathbb{R})$, if $H_0(V)$ has the same eigenvalues as $H_0(0)$ then $q = 0$. An other reason, to turn to the AKNS operator, is it similarity with the Schrödinger operator as figured out the papers of Grébert and Guillot [11] and Amour and Guillot [3]. And finally, technical difficulties arise when computing asymptotics for solutions of the Dirac equation, see remark page 9.

Our purpose is the stability of the inverse spectral problem for H_a ((1)-(2)-(3)). For this, we construct for each $a \in \mathbb{N}$, a spectral map $\lambda^a \times \kappa^a$ for potentials V with spectral data λ^a and some norming constant κ^a. The framework is the work of Grébert and Guillot [11] for the regular operator ($a = 0$). They constructed a local coordinate system $\lambda^0 \times \kappa^0$ on $L^2_{\mathbb{R}}(0, 1) \times L^2_{\mathbb{R}}(0, 1)$ and proved it is global on $H^j_{\mathbb{R}}(0, 1) \times H^j_{\mathbb{R}}(0, 1)$ for $j = 1, 2$. With the singularity, interesting problem arise and add supplementary difficulties, especially when we study the invertibility of the Fréchet derivative of $\lambda^a \times \kappa^a$. For this, we use some transformation operators who, roughly speaking, reduce the singularity.

Our result is that for all $a \in \mathbb{N}$, $\lambda^a \times \kappa^a$ is a local diffeomorphism on $L^2_{\mathbb{R}}(0, 1) \times L^2_{\mathbb{R}}(0, 1)$ and one-to-one on $H^1_{\mathbb{R}}(0, 1) \times H^1_{\mathbb{R}}(0, 1)$. Moreover, we locally describe sets of isospectral potentials as smooth submanifolds of $L^2_{\mathbb{R}}(0, 1) \times L^2_{\mathbb{R}}(0, 1)$ with explicitly tangent and normal spaces.

2. The direct spectral problem

We will omit proofs which are nearly repetitions of the regular case (for details see [22]).

2.1. Solutions Properties

In this section, V is any 2×2 matrix with $L^2_{\mathbb{C}}(0, 1)$ coefficients. A fundamental system of solutions for (1) when $V = 0$ is given by

$$R(x, \lambda) = \frac{1}{\lambda^a} \begin{bmatrix} j_{a-1}(\lambda x) \\ -j_a(\lambda x) \end{bmatrix}, \quad S(x, \lambda) = \lambda^a \begin{bmatrix} -\eta_{a-1}(\lambda x) \\ \eta_a(\lambda x) \end{bmatrix},$$

where j_a and η_a are spherical Bessel functions (see section 4.1). These functions are called fundamental since their wronskian is equal to 1. From their behavior near $x = 0$, $R(x, \lambda)$ is called the regular solution, it is analytic on $[0, 1] \times \mathbb{C}$; $S(x, \lambda)$ is called the singular solution, it is analytic on $(0, 1] \times \mathbb{C}$.

Following Blancarte, Grébert and Weder [5], we construct solutions for (1) by a Picard's iteration method from R and S.

Let $\mathcal{R}$ and $\tilde{\mathcal{S}}$ be defined by

$$\mathcal{R}(x,\lambda,V)=\sum_{k\geq 0}R_k(x,\lambda,V),\quad \tilde{\mathcal{S}}(x,\lambda,V)=\sum_{k\geq 0}S_k(x,\lambda,V)$$

with

$$\begin{cases} R_0(x,\lambda,V)=R(x,\lambda), \\ R_{k+1}(x,\lambda,V)=\displaystyle\int_0^x \mathcal{G}(x,t,\lambda)V(t)R_k(t,\lambda,V)dt, \quad k\in\mathbb{N}; \end{cases} \tag{4}$$

$$\begin{cases} S_0(x,\lambda,V)=S(x,\lambda), \\ S_{k+1}(x,\lambda,V)=-\displaystyle\int_x^1 \mathcal{G}(x,t,\lambda)V(t)S_k(t,\lambda,V)dt, \quad k\in\mathbb{N}. \end{cases} \tag{5}$$

$\mathcal{G}$ is called Green function and is given by (see [4])

$$\mathcal{G}(x,t,\lambda)=S(x,\lambda)R(t,\lambda)^\top-R(x,\lambda)S(t,\lambda)^\top. \tag{6}$$

This construction is justified with the following

Lemma 2.1 *Series defined by (4), respectively by (5), uniformly converge on bounded sets of* $[0,1]\times\mathbb{C}\times(L^2_{\mathbb{C}}(0,1))^4$*, respectively of* $(0,1]\times\mathbb{C}\times(L^2_{\mathbb{C}}(0,1))^4$*, towards solutions of (1). Moreover, they satisfy the integral equations*

$$\mathcal{R}(x,\lambda,V)=R(x,\lambda)+\int_0^x \mathcal{G}(x,t,\lambda)V(t)\mathcal{R}(t,\lambda,V)dt,$$
$$\tilde{\mathcal{S}}(x,\lambda,V)=S(x,\lambda)-\int_x^1 \mathcal{G}(x,t,\lambda)V(t)\tilde{\mathcal{S}}(t,\lambda,V)dt,$$

and the estimates

$$|\mathcal{R}(x,\lambda,V)|\leq Ce^{|\operatorname{Im}\lambda|x}\left(\frac{x}{1+|\lambda|x}\right)^a,$$
$$\left|\tilde{\mathcal{S}}(x,\lambda,V)\right|\leq Ce^{|\operatorname{Im}\lambda|(1-x)}\left(\frac{1+|\lambda|x}{x}\right)^a,$$

with C *uniform on bounded sets of* $(L^2_{\mathbb{C}}(0,1))^4$.

Proof. We give it for $\mathcal{R}$, it is similar for $\tilde{\mathcal{S}}$. Estimate (A.2) for Bessel functions gives

$$|R(x,\lambda)|\leq Ce^{|\operatorname{Im}\lambda|x}\left(\frac{x}{1+|\lambda|x}\right)^a. \tag{7}$$

Iterative relation (4) leads to

$$R_1(x,\lambda,V)=\int_0^x \mathcal{G}(x,t,\lambda)V(t)R(t,\lambda)dt, \tag{8}$$

which, combining (7) and the Green function estimates (A.4), is bounded by

$$|R_1(x,\lambda,V)|\leq C^2e^{|\operatorname{Im}\lambda|x}\left(\frac{x}{1+|\lambda|x}\right)^a\int_0^x|V(t)|dt,$$

By successive iterations and recurrence, for all positive integer n, we get

$$|R_n(x,\lambda,V)|\leq \frac{C^{n+1}}{n!}e^{|\operatorname{Im}\lambda|x}\left(\frac{x}{1+|\lambda|x}\right)^a\left(\int_0^x|V(t)|dt\right)^n.$$

This proves uniform convergence on bounded sets of $[0,1]\times\mathbb{C}\times(L^2_{\mathbb{C}}(0,1))^4$ for $\mathcal{R}$ and the estimate. Integral equation follows from (4). □

This uniform convergence gives us the following

Proposition 2.1 (Analyticity of solutions)

(a) For all $x\in[0,1]$, $\mathcal{R}(x,\lambda,V)$ *is analytic on* $\mathbb{C}\times(L^2_{\mathbb{C}}(0,1))^4$. *Moreover, it is real valued on* $\mathbb{R}\times(L^2_{\mathbb{R}}(0,1))^4$.

(b) The map $\mathcal{R}:(\lambda,V)\mapsto\mathcal{R}(\cdot,\lambda,V)$ *is analytic from* $\mathbb{C}\times(L^2_{\mathbb{C}}(0,1))^4$ *to* $H^1([0,1],\mathbb{C}^2)$.

(c) For all $x\in(0,1]$, $\tilde{\mathcal{S}}(x,\lambda,V)$ *is analytic on* $\mathbb{C}\times(L^2_{\mathbb{C}}(0,1))^4$ *and real valued on* $\mathbb{R}\times(L^2_{\mathbb{R}}(0,1))^4$.

Let $\mathcal{W}(\lambda,V)$ be the wronskian of $\mathcal{R}$ and $\tilde{\mathcal{S}}$, defined by:

$$\mathcal{W}(\lambda,V):=\mathcal{W}\left(\mathcal{R}(x,\lambda,V),\tilde{\mathcal{S}}(x,\lambda,V)\right)=\det\left(\mathcal{R}(x,\lambda,V),\tilde{\mathcal{S}}(x,\lambda,V)\right).$$

Recall that $\mathcal{W}(\lambda,V)$ is independent of x. We follow the construction of a similar solution by Guillot and Ralston in [13]: $\mathcal{W}(\lambda,V)$ is not equal to 1. However, as we will see further, for $|\lambda|$ large enough, $\mathcal{W}$ doesn't vanishes (see Theorem 3.2). Thus we may define the so-called singular solution by

$$\mathcal{S}(x,\lambda,V)=\frac{\tilde{\mathcal{S}}(x,\lambda,V)}{\mathcal{W}(\lambda,V)},\quad x\in(0,1].$$

Regularity of $\mathcal{R}$ leads to existence of derivatives, obtained following [18]:

Proposition 2.2 *For all* $v\in(L^2_{\mathbb{C}}(0,1))^4$, *we have*

$$[d_V\mathcal{R}(x,\lambda,V)](v)=\int_0^x\tilde{\mathcal{G}}(x,t,\lambda,V)v(t)\mathcal{R}(t,\lambda,V)dt, \tag{9}$$

$$\frac{\partial\mathcal{R}}{\partial\lambda}(x,\lambda,V)=-[d_V\mathcal{R}(x,\lambda,V)](\mathrm{Id}), \tag{10}$$

where

$$\tilde{\mathcal{G}}(x,t,\lambda,V)=\mathcal{S}(x,\lambda,V)\mathcal{R}(t,\lambda,V)^\top-\mathcal{R}(x,\lambda,V)\mathcal{S}(t,\lambda,V)^\top.$$

Notations 1 *For simplicity, we name the components of solutions by*

$$\mathcal{R}(x,\lambda,p,q)=\begin{bmatrix}Y_1(x,\lambda,p,q)\\Z_1(x,\lambda,p,q)\end{bmatrix},\ \mathcal{S}(x,\lambda,p,q)=\begin{bmatrix}Y_2(x,\lambda,p,q)\\Z_2(x,\lambda,p,q)\end{bmatrix}$$

and we introduce the following quantities

$$\begin{aligned}a(x,\lambda,p,q)&=-\left[Y_1(x,\lambda,p,q)Z_2(x,\lambda,p,q)+Z_1(x,\lambda,p,q)Y_2(x,\lambda,p,q)\right],\\ b(x,\lambda,p,q)&=\quad\left[Y_1(x,\lambda,p,q)Y_2(x,\lambda,p,q)-Z_1(x,\lambda,p,q)Z_2(x,\lambda,p,q)\right].\end{aligned}$$

Now precise derivative expressions for AKNS potentials defined by (3). First, we define $L^2_{\mathbb{C}}(0,1)$-gradients for multiple variable functions.

Definition 2.1 *Let H be an Hilbert space. For a continuously differentiable complex valued map $f:(p,q)\mapsto f(p,q)$, the $L^2_{\mathbb{C}}(0,1)$-gradient with respect to (p,q) is the vector valued function*

$$\nabla_{p,q} f = \left(\frac{\partial f}{\partial p}, \frac{\partial f}{\partial q}\right)$$

where $\dfrac{\partial f}{\partial p}$, resp. $\dfrac{\partial f}{\partial q}$ is the Riesz representant of the partial differential $D_p f$, resp. $D_q f$ defined by

$$d_{p,q} f(v_1,v_2) = D_p f(v_1) + D_q f(v_2), \quad (v_1,v_2)\in H\times H.$$

Remark. If f is valued in $\mathbb{C}^n$, this notation is understood component by component.

Corollary 2.1 (AKNS Gradients) *For all $(p,q)\in L^2_{\mathbb{C}}(0,1)$, we have*

$$\left[\frac{\partial \mathcal{R}}{\partial p}(x,\lambda,p,q)\right](t) = \mathbb{1}_{[0,x]}(t)\Big[\mathcal{S}(x,\lambda,p,q)\,[2Y_1(t,\lambda,p,q)Z_1(t,\lambda,p,q)] + \mathcal{R}(x,\lambda,p,q)a(t,\lambda,p,q)\Big], \tag{11}$$

$$\left[\frac{\partial \mathcal{R}}{\partial q}(x,\lambda,p,q)\right](t) = \mathbb{1}_{[0,x]}(t)\Big[\mathcal{S}(x,\lambda,p,q)\,\big[Z_1(t,\lambda,p,q)^2 - Y_1(t,\lambda,p,q)^2\big] + \mathcal{R}(x,\lambda,p,q)b(t,\lambda,p,q)\Big], \tag{12}$$

$$\left[\frac{\partial \mathcal{R}}{\partial \lambda}(x,\lambda,p,q)\right] = \int_0^x \Big[-\mathcal{S}(x,\lambda,p,q)\,\big[Y_1(t,\lambda,p,q)^2 + Z_1(t,\lambda,p,q)^2\big] + \mathcal{R}(x,\lambda,p,q)\,[Y_1(t,\lambda,p,q)Y_2(t,\lambda,p,q) + Z_1(t,\lambda,p,q)Z_2(t,\lambda,p,q)]\Big]dt. \tag{13}$$

2.2. Spectra

Condition at $x=0$ selects a solution collinear to $\mathcal{R}$, condition at $x=1$ reduces spectrum to an eigenvalues-sequence. To this end, we set:

Notations 2 *Let $D(\lambda,V)$ be defined by:*

$$D(\lambda,V) = \mathcal{R}(1,\lambda,V)\cdot u_\beta. \tag{14}$$

Moreover, for all $u=(a,b)\in\mathbb{C}^2$, we define $u^\perp$ by

$$(a,b)^\perp = (b,-a). \tag{15}$$

Proposition 2.3 *D is analytic in λ and V. The roots of $\lambda\mapsto D(\lambda,V)$ are exactly the eigenvalues for (1)-(2). Moreover, if V is real-valued, they are all simple.*

Proof. Analyticity of D comes from $\mathcal{R}$. Since $\{\mathcal{R},\mathcal{S}\}$ is a basis for the solutions of (1), the identification between eigenvalues and roots of $\lambda\mapsto D(\lambda,V)$ follows.
Now suppose V is real-valued et let λ_0 be an eigenvalue of the problem. Simplicity lies on

$$\|\mathcal{R}(\cdot,\lambda_0,V)\|^2_{L^2_{\mathbb{R}}(0,1)} = -(\mathcal{R}(1,\lambda_0,V)\cdot {u_\beta}^\perp)\frac{\partial D}{\partial\lambda}(\lambda_0,V). \tag{16}$$

Indeed, from (9) and (10) we have

$$\frac{\partial D}{\partial \lambda}(\lambda_0, V) = -\left(\mathcal{S}(1,\lambda_0,V)\cdot u_\beta\right)\|\mathcal{R}(\cdot,\lambda_0,V)\|^2_{L^2_{\mathbb{R}}(0,1)}.$$

Then, rewriting the wronskian of $\mathcal{R}(1,\lambda_0,V)$ and $\mathcal{S}(1,\lambda_0,V)$ in the orthonormal basis $\{u_\beta, u_\beta{}^\perp\}$, we obtain $\left(\mathcal{R}(1,\lambda_0,V)\cdot u_\beta{}^\perp\right)\left(\mathcal{S}(1,\lambda_0,V)\cdot u_\beta\right) = 1$. □

From now, V is defined by (3), corresponding to an AKNS operator.

2.3. $H^1_{\mathbb{C}}(0,1)$-*estimates*

In order to obtain accurate asymptotics, we add some regularity on potentials. We use this roundabout method not because of the singularity a/x in the equation, but because of the AKNS operator itself. Indeed, contrary to the Schrödinger operator, there is no explicit decreasing for the Green function $\mathcal{G}$ with respect to λ; so we have to force it allowing some derivation. For the regular case ($a=0$), see for instance [11].

Theorem 2.1 *For* $(p,q)\in\left(H^1_{\mathbb{C}}(0,1)\right)^2$, *we have*

$$\left|\mathcal{R}(x,\lambda,p,q) - R(x,\lambda)\right| \le C\|V\|_{H^1_{\mathbb{C}}(0,1)}\left[\frac{x}{1+|\lambda x|}\right]^{a+1}\ln\left[2+|\lambda x|\right]e^{|\mathrm{Im}\,\lambda|x + C\|V\|_2}, \qquad (17)$$

uniformly on $[0,1]\times\mathbb{C}\times(H^1_{\mathbb{C}}(0,1)\times H^1_{\mathbb{C}}(0,1))$, *where* $\|V\|^2_{H^1_{\mathbb{C}}(0,1)} = \|p\|^2_{H^1_{\mathbb{C}}(0,1)} + \|q\|^2_{H^1_{\mathbb{C}}(0,1)}$.

Proof. From relation (4) at $k=1$ and (6), we have

$$\begin{aligned}
R_1(x,\lambda,p,q) &= S_0(x,\lambda)\int_0^x R_0(t,\lambda)^\top V(t) R_0(t,\lambda)dt \\
&\quad - R_0(x,\lambda)\int_0^x S_0(t,\lambda)^\top V(t) R_0(t,\lambda)dt \\
&= S_0(x,\lambda)\int_0^x \left[q(t)\left(R_0^2(t,\lambda)^2 - R_0^1(t,\lambda)^2\right) + 2p(t)R_0^1(t,\lambda)R_0^2(t,\lambda)\right]dt \\
&\quad - R_0(x,\lambda)\int_0^x \left[q(t)\left(S_0^2(t,\lambda)R_0^2(t,\lambda) - S_0^1(t,\lambda)R_0^1(t,\lambda)\right)\right. \\
&\qquad\qquad \left. + p(t)\left(S_0^1(t,\lambda)R_0^2(t,\lambda) + S_0^2(t,\lambda)R_0^1(t,\lambda)\right)\right]dt.
\end{aligned}$$

We can write $R_1(x,\lambda,p,q) = \lambda^{-a}\left[X(q)+Y(p)\right]$, where

$$\begin{aligned}
X(q) &= \begin{bmatrix} -\eta_{a-1}(\lambda x) \\ \eta_a(\lambda x)\end{bmatrix}\int_0^x \left\{[j_a(\lambda t)]^2 - [j_{a-1}(\lambda t)]^2\right\}q(t)dt \\
&\quad + \begin{bmatrix} j_{a-1}(\lambda x) \\ -j_a(\lambda x)\end{bmatrix}\int_0^x \left[\eta_a(\lambda t)\,j_a(\lambda t) - \eta_{a-1}(\lambda t)\,j_{a-1}(\lambda t)\right]q(t)dt, \\
Y(p) &= \begin{bmatrix} \eta_{a-1}(\lambda x) \\ -\eta_a(\lambda x)\end{bmatrix}\int_0^x \left[2 j_{a-1}(\lambda t)\,j_a(\lambda t)\right]p(t)dt \\
&\quad - \begin{bmatrix} j_{a-1}(\lambda x) \\ -j_a(\lambda x)\end{bmatrix}\int_0^x \left[\eta_{a-1}(\lambda t)\,j_a(\lambda t) + \eta_a(\lambda t)\,j_{a-1}(\lambda t)\right]p(t)dt.
\end{aligned}$$

Estimation for $X(q)$**:**

Integrating by parts, we get

$$X(q) = \frac{1}{\lambda}\begin{bmatrix} 0 \\ -j_a(\lambda x) \end{bmatrix} q(x) + \frac{1}{\lambda}\int_0^x \begin{bmatrix} -\eta_{a-1}(\lambda x)\, j_{a-1}(\lambda t) + j_{a-1}(\lambda x)\,\eta_{a-1}(\lambda t) \\ \eta_a(\lambda x)\, j_{a-1}(\lambda t) - j_a(\lambda x)\,\eta_{a-1}(\lambda t) \end{bmatrix} j_a(\lambda t)\, q'(t)dt.$$

Estimates (A.2),(A.4) and Sobolev inequality $\|q\|_\infty \leq C\|q\|_{H^1_{\mathbb{C}}(0,1)}$ give

$$|X(q)| \leq \frac{C}{|\lambda|}\left(\frac{|\lambda x|}{1+|\lambda x|}\right)^{a+1} e^{|\mathrm{Im}\,\lambda|x}\|q\|_{H^1_{\mathbb{C}}(0,1)}. \tag{18}$$

Estimation for $Y(p)$**:**

With notations from lemmas Appendix A.1 and Appendix A.2, integration by parts gives :

$$Y(p) = \begin{bmatrix} \eta_{a-1}(\lambda x) \\ -\eta_a(\lambda x) \end{bmatrix}\left(\left[\frac{1}{\lambda}F_1(\lambda t)p(t)\right]_0^x - \frac{1}{\lambda}\int_0^x F_1(\lambda t)p'(t)dt\right) - \begin{bmatrix} j_{a-1}(\lambda x) \\ -j_a(\lambda x) \end{bmatrix}\left(\left[\frac{1}{\lambda}F_2(\lambda t)p(t)\right]_0^x - \frac{1}{\lambda}\int_0^x F_2(\lambda t)p'(t)dt\right).$$

When $|\lambda x| \leq$**1.** Estimations A.2, A.3 and part *(i)* from lemmas Appendix A.1 and Appendix A.2 lead to

$$|Y(p)| \leq \frac{C}{|\lambda|}\left(\frac{|\lambda x|}{1+|\lambda x|}\right)^{a+1}\|p\|_{H^1_{\mathbb{C}}(0,1)}e^{|\mathrm{Im}\,\lambda|x}.$$

When $|\lambda x| \geq$**1.** Now, we only consider $Y(p)$ second component, the proof is similar for the first one. Terms to estimate contain :

$$g(x,t) := \eta_a(\lambda x)\, F_1(\lambda t) - j_a(\lambda x)\, F_2(\lambda t), \quad 0 \leq t \leq x.$$

If $|\lambda t| \leq$**1.** As for $|\lambda x| \leq 1$, we get $|g(x,t)| \leq 2C\left(\frac{|\lambda x|}{1+|\lambda x|}\right)^{a+1} e^{|\mathrm{Im}\,\lambda|x}$.

If $|\lambda t| \geq$**1.** Using points *(ii)* from lemmas Appendix A.1 and Appendix A.2, expressions (A.6) and (A.7), it follows :

$$\begin{aligned} g(x,t) = {} & \eta_a(\lambda x)r_a(\lambda t) \\ & -a\left[\cos\left(\lambda x - \frac{a\pi}{2}\right)\mathrm{ci}(2\lambda t) + \sin\left(\lambda x - \frac{a\pi}{2}\right)\mathrm{Si}(2\lambda t)\right]P_a(\lambda x) \\ & +a\left[\sin\left(\lambda x - \frac{a\pi}{2}\right)\mathrm{ci}(2\lambda t) - \cos\left(\lambda x - \frac{a\pi}{2}\right)\mathrm{Si}(2\lambda t)\right]I_a(\lambda x) \\ & +\left(P_a(\lambda x)p_a(\lambda t) - I_a(\lambda x)q_a(\lambda t)\right)\cos\left[\lambda(x-2t) - \frac{a\pi}{2}\right] \\ & -\left(P_a(\lambda x)q_a(\lambda t) + I_a(\lambda x)p_a(\lambda t)\right)\sin\left[\lambda(x-2t) - \frac{a\pi}{2}\right]. \end{aligned}$$

(To lighten, the polynomial variable X is replaced by $1/X$.) First term is bounded by $Ce^{|\mathrm{Im}\,\lambda|x}$ thanks to (A.3). The last two terms are uniformly bounded by $Ce^{|\mathrm{Im}\,\lambda|(x-2t)}$ on the considered area. Now remains the following expression

$$h(x,t) := \cos\left(\lambda x - \frac{a\pi}{2}\right)\mathrm{ci}(2\lambda t) + \sin\left(\lambda x - \frac{a\pi}{2}\right)\mathrm{Si}(2\lambda t).$$

According to [1] and [2], we have

$$\mathrm{ci}(z) = -\gamma - \frac{\log(z^2)}{2} + \frac{\sin z}{z}\left(1+\mathcal{O}_1\left(\frac{1}{z^2}\right)\right) - \frac{\cos z}{z^2}\left(1+\mathcal{O}_2\left(\frac{1}{z^2}\right)\right),$$

$$\mathrm{Si}(z) = \frac{\pi\sqrt{z^2}}{2z} - \frac{\cos z}{z}\left(1+\mathcal{O}_1\left(\frac{1}{z^2}\right)\right) - \frac{\sin z}{z^2}\left(1+\mathcal{O}_2\left(\frac{1}{z^2}\right)\right).$$

Thus, we get

$$\begin{aligned} h(x,t) = &-\left[\gamma + \frac{\log(2\lambda t)^2}{2}\right]\cos\left(\lambda x - \frac{a\pi}{2}\right) + \frac{\pi\sqrt{(2\lambda t)^2}}{2\lambda t}\sin\left(\lambda x - \frac{a\pi}{2}\right) \\ &-\frac{1}{2\lambda t}\left(1+\mathcal{O}_1\left(\frac{1}{(2\lambda t)^2}\right)\right)\sin\left[\lambda(x-2t) - \frac{a\pi}{2}\right] \\ &-\frac{1}{(2\lambda t)^2}\left(1+\mathcal{O}_2\left(\frac{1}{(2\lambda t)^2}\right)\right)\cos\left[\lambda(x-2t) - \frac{a\pi}{2}\right]. \end{aligned}$$

The last three terms are also uniformly controlled by $Ce^{|\mathrm{Im}\,\lambda|(x-2t)}$; the first one is bounded by $C\ln|\lambda t|e^{|\mathrm{Im}\,\lambda|x}$. Combining the above estimates, we obtain the following uniform estimate

$$|Y(p)| \leq \frac{C}{|\lambda|}\left(\frac{|\lambda x|}{1+|\lambda x|}\right)^{a+1}\ln\left[2+|\lambda|x\right]\|p\|_{H^1_{\mathbb{C}}(0,1)}e^{|\mathrm{Im}\,\lambda|x}. \tag{19}$$

Relations (18)-(19) and the concavity rule

$$\forall (x,y)\in\mathbb{R}^2, \qquad \frac{|x|+|y|}{2} \leq \sqrt{\frac{|x|^2+|y|^2}{2}},$$

imply

$$|R_1(x,\lambda,p,q)| \leq \frac{C}{|\lambda|^{a+1}}\left(\frac{|\lambda x|}{1+|\lambda x|}\right)^{a+1}\ln\left[2+|\lambda|x\right]\|V\|_{H^1_{\mathbb{C}}(0,1)}e^{|\mathrm{Im}\,\lambda|x}. \tag{20}$$

From this estimate, as in the proof of lemma 2.1, we deduce estimate (17). □

Remark. A similar computation for the Dirac operator is not easy, even if $a = 0$. Indeed, when we compute the term R_1, we do not only get a term, loosely speaking, in $\mathcal{O}(1/\lambda)$ but also in $\mathcal{O}(1)$. And when iterating this, we get at each time a new term $\mathcal{O}(1)$ and $\mathcal{O}(1/\lambda)$. A way through this problem is given in [22] using the latter gauge transformation to deduce, for any $a\in\mathbb{N}$, some partial results from AKNS to Dirac operator: spectrum, asymptotic expansion for eigenvalues and eigenvectors, Borg-Levinson theorem type...

2.4. $L^2_{\mathbb{C}}(0,1)$*-Estimates*

To transform $H^1_{\mathbb{C}}(0,1)$-estimates into $L^2_{\mathbb{C}}(0,1)$-estimates, we need an auxiliary lemma (for the regular case, see [3], [17] and [10]).

Lemma 2.2 *Let* $V_0 \in L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1)$*,* $r_0 \geq 0$*,* $\varepsilon \geq 0$ *and let* $V_\epsilon \in H^1_{\mathbb{C}}(0,1)\times H^1_{\mathbb{C}}(0,1)$ *such that* $\|V_0 - V_\varepsilon\|_2 < \varepsilon$*. Then, for all* $V \in L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1)$ *such that* $\|V - V_0\|_2 < r_0$

and for all $(x,\lambda)\in[0,1]\times\mathbb{C}^*$, *we have*

$$|\mathcal{R}(x,\lambda,p,q)-R(x,\lambda)|\le C\left(r_0+\varepsilon+\frac{\ln|\lambda|}{|\lambda|}\|V_\varepsilon\|_{H^1_{\mathbb{C}}(0,1)}\right)\times\left(\frac{x}{1+|\lambda x|}\right)^a e^{|\operatorname{Im}\lambda|x+C\|V\|_2}. \qquad (21)$$

Proof. Since $V_\epsilon\in H^1_{\mathbb{C}}(0,1)\times H^1_{\mathbb{C}}(0,1)$, estimate (20) obtained during the proof of Theorem 2.1 becomes

$$|R_1(x,\lambda,V_\varepsilon)|\le\frac{C}{|\lambda|^{a+1}}\left(\frac{|\lambda x|}{1+|\lambda x|}\right)^{a+1}\ln\left[2+|\lambda|x\right]\|V_\varepsilon\|_{H^1_{\mathbb{C}}(0,1)}e^{|\operatorname{Im}\lambda|x}.$$

Using $\|V_0-V_\varepsilon\|_2<\varepsilon$ and $\|V-V_0\|_2<r_0$ in (8), estimations (A.4) and (A.2) lead to

$$|R_1(x,\lambda,p,q)-R_1(x,\lambda,V_\varepsilon)|\le\frac{C}{|\lambda|^a}\left(\frac{|\lambda x|}{1+|\lambda x|}\right)^a(r_0+\varepsilon)e^{|\operatorname{Im}\lambda|x}.$$

Combining these two inequalities, we get

$$|R_1(x,\lambda,p,q)|\le C\left(\frac{x}{1+|\lambda x|}\right)^a\left(r_0+\varepsilon+\frac{\ln\left[2+|\lambda|x\right]}{|\lambda|}\|V_\varepsilon\|_{H^1_{\mathbb{C}}(0,1)}\right)e^{|\operatorname{Im}\lambda|x}.$$

Iterating this with (4), we deduce for every $n\in\mathbb{N}$

$$|R_{n+1}(x,\lambda,p,q)|\le\frac{C^{n+1}}{n!}\left(r_0+\varepsilon+\frac{\ln\left[2+|\lambda|x\right]}{|\lambda|}\|V_\varepsilon\|_{H^1_{\mathbb{C}}(0,1)}\right)\times\left(\frac{x}{1+|\lambda x|}\right)^a e^{|\operatorname{Im}\lambda|x}\left(\int_0^1|V(t)|dt\right)^n.$$

Then, summing up, estimation (21) follows. □

We now deduce the following

Proposition 2.4 *Let* $(p,q)\in\left(L^2_{\mathbb{C}}(0,1)\right)^2$, *we have uniformly on* $[0,1]$,

$$\mathcal{R}(x,\lambda,p,q)=R(x,\lambda)+o\left[\left(\frac{x}{1+|\lambda x|}\right)^a e^{|\operatorname{Im}\lambda|x}\right],\quad|\lambda|\to\infty. \qquad (22)$$

Proof. From Lemma 2.2 with $r_0=0$, given $\delta>0$ there exists $\lambda_\delta>0$ such that

$$|\mathcal{R}(x,\lambda,p,q)-R(x,\lambda)|\le\delta\left(\frac{x}{1+|\lambda x|}\right)^a e^{|\operatorname{Im}\lambda|x+C\|V\|_2},$$

for all λ such that $|\lambda|>\lambda_\delta$. □

2.5. Spectrum localization

Theorem 2.2 (Counting Lemma)
Let $(p_0,q_0)\in L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1)$, *there exist* $\varepsilon>0$ *and an integer* $N_0>0$ *such that for all* $(p,q)\in L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1)$ *with* $\|(p,q)-(p_0,q_0)\|_{L^2_{\mathbb{C}}(0,1)}<\varepsilon$, *the following statements hold:*

- *For all* $|n|>N_0$, $\lambda\mapsto D(\lambda,p,q)$ *has exactly one root in* $\left|\lambda-\left(n\pi+\frac{a\pi}{2}+\beta\right)\right|<\frac{\pi}{2}$,

- $\lambda \mapsto D(\lambda,p,q)$ *has exactly* $2N_0+1-a$ *root counted with multiplicity in* $\left|\lambda-\left(\frac{a\pi}{2}+\beta\right)\right|<\left(N_0+\frac{1}{2}\right)\pi$,
- $\lambda \mapsto D(\lambda,p,q)$ *has no root elsewhere.*

Proof. Let $\varepsilon>0$, from estimate (21) and using Lemma 2.2 notations, we have

$$|\mathcal{R}(1,\lambda,p,q)-R(1,\lambda)| \le \frac{Ce^{|\mathrm{Im}\,\lambda|+C\|V\|_2}}{|\lambda|^a}\left(2\varepsilon+\frac{\ln|\lambda|}{|\lambda|}\|V_\varepsilon\|_{H^1_{\mathbb{C}}(0,1)}\right).$$

Bessel functions relation (A.6) implies the following uniform estimate on $|\lambda|>1$

$$R(1,\lambda)=\frac{1}{\lambda^a}\begin{bmatrix}\cos\left(\lambda-\frac{a\pi}{2}\right)\\ -\sin\left(\lambda-\frac{a\pi}{2}\right)\end{bmatrix}+\mathcal{O}\left(\frac{e^{|\mathrm{Im}\,\lambda|}}{|\lambda|^{a+1}}\right), \tag{23}$$

which leads, together with the previous one, to

$$\left|\lambda^a D(\lambda,p,q)-\sin\left(\beta+\frac{a\pi}{2}-\lambda\right)\right| \le \left(2\varepsilon+\frac{\ln|\lambda|}{|\lambda|}\|V_\varepsilon\|_{H^1_{\mathbb{C}}(0,1)}+\frac{1}{|\lambda|}\right)Ce^{C\|V\|_2}e^{|\mathrm{Im}\,\lambda|}.$$

Now introduce the circles:

- for $n\in\mathbb{Z}$, γ_n is defined by

$$\left|\lambda-\left(n\pi+\frac{a\pi}{2}+\beta\right)\right|=\frac{\pi}{2}.$$

- for $n\in\mathbb{N}$, C_n is defined by

$$\left|\lambda-\left(\frac{a\pi}{2}+\beta\right)\right|=\left(n+\frac{1}{2}\right)\pi.$$

We choose $\varepsilon>0$ such that $Ce^{C\|V\|_2}2\varepsilon<\frac{1}{8}$. Moreover, on each circle we have

$$|\lambda|>\left(N_0+\frac{1}{2}\right)\pi-\left|\frac{a\pi}{2}+\beta\right|$$

and since the map $t\mapsto\frac{\ln t}{t}$ decreases on $]e,\infty[$, we can pick up $N_0>0$ such that

$$Ce^{C\|V\|_2}\frac{\ln|\lambda|}{|\lambda|}\|V_\varepsilon\|_{H^1_{\mathbb{C}}(0,1)}<\frac{1}{8}.$$

Thus, we get the following

$$\left|\lambda^a D(\lambda,p,q)-\sin\left(\beta+\frac{a\pi}{2}-\lambda\right)\right|<\frac{1}{4}\,e^{|\mathrm{Im}\,\lambda|}=\frac{1}{4}\,e^{|\mathrm{Im}\left(\lambda-\frac{a\pi}{2}-\beta\right)|}.$$

Using the following estimate for all $k\in\mathbb{Z}$ (see Lemma 2.1 in [18])

$$e^{|\mathrm{Im}\,z|}<4|\sin z| \quad \text{for} \quad |z-k\pi|\ge\frac{\pi}{4},$$

on the sets γ_n and C_{N_0}, with $z=\lambda-\frac{a\pi}{2}-\beta$, we obtain

$$\left|\lambda^a D(\lambda,p,q)-\sin\left(\beta+\frac{a\pi}{2}-\lambda\right)\right|<\left|\sin\left(\beta+\frac{a\pi}{2}-\lambda\right)\right|.$$

Now, the use of the Rouché Theorem let us conclude that the analytical functions $\lambda \mapsto \lambda^a D(\lambda,p,q)$ and $\lambda \mapsto \sin\left(\beta + \frac{a\pi}{2} - \lambda\right)$ have the same number of roots counted with multiplicity inside theses circles. To show there is no other elsewhere, we just have to consider an other circle C_N with $N > N_0$ and apply again the Rouché Theorem. □

Now, we can order eigenvalues: when $n > N_0$, $\lambda_{a,n}(p,q)$ is the eigenvalue surrounded by γ_n. Next, we order lexicographically the $2N_0 + 1 - a$ eigenvalues lying in C_{N_0}, in other words, for $k = a - N_0, \ldots, N_0 - 1$:

$$\operatorname{Re}\lambda_{a,k}(p,q) < \operatorname{Re}\lambda_{a,k+1}(p,q)$$

or

$$\operatorname{Re}\lambda_{a,k}(p,q) = \operatorname{Re}\lambda_{a,k+1}(p,q) \quad \text{and} \quad \operatorname{Im}\lambda_{a,k}(p,q) \leq \operatorname{Im}\lambda_{a,k+1}(p,q).$$

To continue the numbering, the eigenvalue included in γ_{-n}, for $n > N_0$, must be $\lambda_{a,-n+a}$. To put it directly, we say that for $n > N_0 - a$, $\lambda_{a,-n}$ is the eigenvalue surrounded by $\gamma_{-(n+a)}$.

The localization gives us the following locally uniform estimates on $L^2_{\mathbb{C}}(0,1) \times L^2_{\mathbb{C}}(0,1)$

$$\lambda_{a,n}(p,q) = \left(n + \frac{a}{2}\right)\pi + \beta + \mathcal{O}(1), \quad n \to \infty, \quad |\mathcal{O}(1)| \leq \frac{\pi}{2}, \tag{24}$$

$$\lambda_{a,-n}(p,q) = -\left(n + \frac{a}{2}\right)\pi + \beta + \mathcal{O}(1), \quad n \to \infty, \quad |\mathcal{O}(1)| \leq \frac{\pi}{2}. \tag{25}$$

Proposition 2.5 *Let* $(p,q) \in L^2_{\mathbb{C}}(0,1) \times L^2_{\mathbb{C}}(0,1)$.

$$\lambda_{a,n}(p,q) = \left(n + \operatorname{sgn}(n)\frac{a}{2}\right)\pi + \beta + o(1) \quad , \quad |n| \to +\infty. \tag{26}$$

Proof. Relation (22) at $x = 1$ and definition (14) give

$$D(\lambda,p,q) = R(1,\lambda) \cdot u_\beta + o\left(\frac{e^{|\operatorname{Im}\lambda|}}{|\lambda|^a}\right)$$

then, estimate (23) implies

$$D(\lambda,p,q) = \frac{1}{\lambda^a}\sin\left(\beta + \frac{a\pi}{2} - \lambda\right) + o\left(\frac{e^{|\operatorname{Im}\lambda|}}{|\lambda|^a}\right).$$

According to the counting lemma, we have

$$\lambda_{a,n}(p,q) = \left(n + \operatorname{sgn}(n)\frac{a}{2}\right)\pi + \beta + \mathcal{O}(1), \quad |n| \to \infty,$$

knowing that $|\mathcal{O}(1)| < \frac{\pi}{2}$. We evaluate $D(\lambda,p,q)$ at $\lambda = \lambda_{a,n}(p,q)$ and use the above estimates to get

$$0 = \frac{1}{{\lambda_{a,n}}^a}\sin\left(\mathcal{O}(1)\right) + o\left(\frac{1}{|\lambda_{a,n}|^a}\right).$$

By identification, we found the result. □

Remarks Theses results have to be compared with those in the regular case ($a = 0$):

- Asymptotics of solutions and eigenvalues localization for $L^2_{\mathbb{C}}(0,1)$-potentiels are only locally uniform. This is due to the operator by itself and not to the singularity. In [12] is given a pair of potentials with identical $L^2_{\mathbb{C}}(0,1)$-norm whose eigenvalues numbering (localization) are different.
- A new phenomenon, relative to the numbering, is this loss of a eigenvalues lying near 0. It may be seen as the analogue of the shift by $a/2$ in the eigenvalues asymptotics of the radial Schrödinger operator (see for instance [13] when $a = 1$ and [21] for general a).

3. Spectral Data

From this point, (p,q) are real-valued. Thus, $(\lambda_{a,n}(p,q))_{n\in\mathbb{Z}}$ is a strictly increasing sequence of real numbers. We set some notations :

Notations 3 *We define*

$$\mathcal{R}_n(t,p,q) = \mathcal{R}(t,\lambda_{a,n}(p,q),p,q) \quad \text{and} \quad \mathcal{S}_n(t,p,q) = \mathcal{S}(t,\lambda_{a,n}(p,q),p,q).$$

Let $G_n(t,p,q)$ *be the normed eigenvector with respect to* $\lambda_{a,n}(V)$ *defined by*

$$G_n(t,p,q) = \frac{\mathcal{R}_n(t,p,q)}{\|\mathcal{R}_n(\cdot,p,q)\|_2}.$$

We also define

$$A_n(x,p,q) = (a_n(x,p,q), b_n(x,p,q))$$

where $a_n(x,p,q) = a(x,\lambda_{a,n}(p,q),p,q)$ *and* $b_n(x,p,q) = b(x,\lambda_{a,n}(p,q),p,q)$ *(*a *and* b *are given on page 5).*

3.1. Regularity, derivatives

Eigenvalues regularity and associated derivatives follows like in [11] and [18] as pictured by the next proposition.

Proposition 3.1 *For all* $n \in \mathbb{Z}$, $(p,q) \mapsto \lambda_{a,n}(p,q)$ *is a real-analytic map on* $L^2_{\mathbb{R}}(0,1) \times L^2_{\mathbb{R}}(0,1)$. *Its* $L^2_{\mathbb{R}}(0,1)$*-gradient is given by*

$$\nabla_{p,q}\lambda_{a,n} = \left(\frac{\partial\lambda_{a,n}}{\partial p}, \frac{\partial\lambda_{a,n}}{\partial q}\right) \text{with} \left\{ \begin{array}{l} \dfrac{\partial\lambda_{a,n}}{\partial p} = 2\,G_{n,1}(t,p,q)\,G_{n,2}(t,p,q), \\ \dfrac{\partial\lambda_{a,n}}{\partial q} = G_{n,2}(t,p,q)^2 - G_{n,1}(t,p,q)^2. \end{array}\right. \tag{27}$$

Like in [18], or simply following [11], we need more information to recover a complete parametrization of $\left(L^2_{\mathbb{R}}(0,1)\right)^2$. Boundary condition at $x = 1$ defining each eigenvalue is an orthogonality relation following one direction. It sounds reasonable that the knowledge of a similar data in a complementary (here orthogonal) direction is enough.

Definition 3.1 *For all* $n \in \mathbb{Z}$, *we call normalization constants the quantities*

$$\kappa_{a,n}(p,q) = \mathcal{R}_n(1,p,q) \cdot {u_\beta}^{\perp}. \tag{28}$$

Following [11], we get :

Proposition 3.2 *For all* $n \in \mathbb{Z}$, $(p,q) \mapsto \kappa_{a,n}(p,q)$ *is a real-analytic map on* $L^2_{\mathbb{R}}(0,1)\times L^2_{\mathbb{R}}(0,1)$. *Its* $L^2_{\mathbb{R}}(0,1)$*-gradient is given by*

$$\frac{\nabla_{p,q}\kappa_{a,n}}{\kappa_{a,n}} = A_n(x,p,q) + \Big\langle \mathcal{R}_n(\cdot,p,q), \mathcal{S}_n(\cdot,p,q)\Big\rangle \nabla_{p,q}\lambda_{a,n}(p,q). \tag{29}$$

Now, precise the behavior of theses normalization constants.

Proposition 3.3 *Let* $(p,q) \in L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1)$, *we have*

$$\kappa_{a,n}(p,q) = \frac{(-1)^n}{\left(\left[|n|+\frac{a}{2}\right]\pi\right)^a}(1+o(1)) = \frac{(-1)^n}{|n\pi|^a}(1+o(1)) \quad , \quad |n|\to+\infty. \tag{30}$$

Proof. Introducing (22) in the $\kappa_{a,n}$ definition leads to

$$\kappa_{a,n} = \frac{1}{{\lambda_{a,n}}^a}\Big(j_{a-1}\left(\lambda_{a,n}\right)\cos\beta + j_a\left(\lambda_{a,n}\right)\sin\beta + o(1)\Big).$$

Relation (A.6) implies

$$\kappa_{a,n} = \frac{1}{{\lambda_{a,n}}^a}\Big(\cos\Big(\lambda_{a,n} - \frac{a\pi}{2} - \beta\Big) + o(1)\Big).$$

Now, with (26), we get

$$\begin{aligned}\kappa_{a,n} &= \frac{1}{(n+\mathrm{sgn} n\frac{a}{2})^a\pi^a}\left(\cos\left(n\pi + (\mathrm{sgn}(n)-1)\frac{a\pi}{2}\right) + o(1)\right),\\ &= \frac{(-1)^n}{(n+\mathrm{sgn} n\frac{a}{2})^a\pi^a}\left(\cos\left[a\pi\frac{\mathrm{sgn}(n)-1}{2}\right] + o(1)\right).\end{aligned}$$

Setting the signum of n gives the result. □

3.2. Orthogonality relations

The following results, especially the corollary, confirm the choice of the additional data: we have added only complementary data. As in [11], we obtain

Proposition 3.4 *For all* $(j,k)\in\mathbb{Z}^2$, *we have*

(i) $\left\langle \nabla_{p,q}\lambda_{a,j}, \nabla_{p,q}{\lambda_{a,k}}^{\perp}\right\rangle = 0$,

(ii) $\left\langle A_j(\cdot,p,q), \nabla_{p,q}{\lambda_{a,k}}^{\perp}\right\rangle = \delta_{j,k}$,

(iii) $\Big\langle A_j(\cdot,p,q), A_k(\cdot,p,q)^{\perp}\Big\rangle = 0$.

Before giving the corollary, be more specific:

Definition 3.2 *A vector family* $(u_k)_{k\in\mathbb{Z}}$ *of an Hilbert space is called free or its elements are linearly independent if each element of the family is not in the closed span of the others. More precisely:*

$$\forall k\in\mathbb{Z}\,,\quad u_k \notin \overline{\mathrm{Span}\,\{u_j | j\in\mathbb{Z}, j\neq k\}}.$$

Corollary 3.1 *For all* $(j,k)\in\mathbb{Z}^2$*, we have*

(i) $\left\langle \nabla_{p,q}\kappa_{a,j}, \nabla_{p,q}\kappa_{a,k}{}^{\perp}\right\rangle = 0$,

(ii) $\left\langle \nabla_{p,q}\kappa_{a,j}, \nabla_{p,q}\lambda_{a,k}{}^{\perp}\right\rangle = \kappa_{a,j}(p,q)\delta_{j,k}$.

$(\nabla_{p,q}\lambda_{a,n})_{n\in\mathbb{Z}} \cup (\nabla_{p,q}\kappa_{a,n})_{n\in\mathbb{Z}}$ *is a free family in* $L^2_{\mathbb{R}}(0,1)\times L^2_{\mathbb{R}}(0,1)$.

3.3. The spectral map

Introduce the quantities $\widetilde{\lambda}_{a,n}(p,q)$ and $\widetilde{\kappa}_{a,n}(p,q)$ such that

$$\lambda_{a,n}(p,q) = \left(n + \operatorname{sgn}(n)\frac{a}{2}\right)\pi + \beta + \widetilde{\lambda}_{a,n}(p,q).$$

$$\kappa_{a,n}(p,q) = \frac{(-1)^n}{\left[\left(|n|+\frac{a}{2}\right)\pi\right]^a}\left(1+\widetilde{\kappa}_{a,n}(p,q)\right).$$

Now, with the estimates (26) and (30), we define the spectral map $\lambda^a\times\kappa^a : L^2_{\mathbb{R}}(0,1)\times L^2_{\mathbb{R}}(0,1)\to c_0(\mathbb{Z})\times c_0(\mathbb{Z})$ by

$$[\lambda^a\times\kappa^a](p,q) = \left((\widetilde{\lambda}_{a,n}(p,q))_{n\in\mathbb{Z}}, (\widetilde{\kappa}_{a,n}(p,q))_{n\in\mathbb{Z}}\right), \tag{31}$$

where $c_0(\mathbb{Z})$ is the space of sequences $(u_n)_{n\in\mathbb{Z}}$ which tend to 0 when $|n|\to\infty$.

Following [18] or [13], to obtain regularity of $\lambda^a\times\kappa^a$ from its components, some uniformity is needed. To this end, we introduce some transformation operators.

3.4. Transformations operators

Such operators were first introduced by Guillot and Ralston in [13] for the inverse spectral problem of the radial Schrödinger operator when $a=1$; then used and extended to any integer a by Rundell and Sacks in [20] and by the present author in [21].

We construct similar operator adapted to the AKNS operator. An important difference, excepted the matrix form, is a better structure of the converse operators compared to the Schrödinger operator. These operators turn to be adapted to the spectral data, since both vectors family corresponding to λ^a and κ^a are well transformed. The proofs of the following lemmas are similar to those in [20]. The main tool is the use of Bessel function's properties (for a detailed proof see [22]). Now, give some notations.

Notations 4 *For all* $n\in\mathbb{N}$*, let* U_n *and* V_n *be defined by*

$$U_n(x) = \begin{bmatrix} 0 \\ x^n \end{bmatrix} \quad \text{and} \quad V_n(x) = \begin{bmatrix} x^n \\ 0 \end{bmatrix} \quad x\in[0,1].$$

Lemma 3.1 *For all* $a\in\mathbb{N}$*, let*

$$\begin{array}{rccc} S_{a+1}: & L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1) & \longrightarrow & L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1) \\ & (p,q) & \longmapsto & \left(S_{a,1}[p]\,, S_{a,2}[q]\right) \end{array}$$

$$\begin{array}{lrcl} \text{with} & S_{a,1}[p](x) & = & p(x) - 2(2a+1)x^{2a}\displaystyle\int_x^1 \frac{p(t)}{t^{2a+1}}dt, \\ \text{and} & S_{a,2}[q](x) & = & q(x) - 2(2a+1)x^{2a+1}\displaystyle\int_x^1 \frac{q(t)}{t^{2a+2}}dt. \end{array}$$

Moreover, we set $S_0 := \mathrm{Id}_{L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1)}$. *We have the following properties:*

(i) The adjoint of S_{a+1} *is* ${S_{a+1}}^*[f,g] = \Big({S_{a,1}}^*[f]\,,{S_{a,2}}^*[g]\Big)$ *where*

$$ {S_{a,1}}^*[f](x) = f(x) - \frac{2(2a+1)}{x^{2a+1}}\int_0^x t^{2a} f(t)dt, $$
$$ {S_{a,2}}^*[g](x) = g(x) - \frac{2(2a+1)}{x^{2a+2}}\int_0^x t^{2a+1} g(t)dt. $$

(ii) The family $\{S_a\}$ *pairwise commutes:* $S_a S_b = S_b S_a$ *for all* $(a,b)\in\mathbb{N}^2$.

(iii) S_a *is bounded on* $L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1)$.

(iv) Let $N_{a+1} := \ker {S_{a+1}}^*$, *then* $N_{a+1} = \mathrm{Vect}(U_{2a}, V_{2a+1})$.

(v) S_{a+1} *is a linear isomorphism between* $L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1)$ *and* ${N_{a+1}}^{\perp}$. *Its inverse is the bounded operator on* $L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1)$ *defined by*

$$ A_{a+1}[f,g] := \Big({S_{a,2}}^*[f]\,,{S_{a,1}}^*[g]\Big). $$

(vi) Φ_a *and* Ψ_a *defined by*

$$ \Phi_a(x) = \begin{bmatrix} -2j_{a-1}(x)j_a(x) \\ j_a(x)^2 - j_{a-1}(x)^2 \end{bmatrix} $$

and

$$ \Psi_a(x) = \begin{bmatrix} -\eta_{a-1}(x)j_a(x) - \eta_a(x)j_{a-1}(x) \\ -\eta_{a-1}(x)j_{a-1}(x) + \eta_a(x)j_a(x) \end{bmatrix} $$

satisfy the relations

$$ \Phi_{a+1} = -{S_{a+1}}^*[\Phi_a] \quad \text{and} \quad \Psi_{a+1} = -{S_{a+1}}^*[\Psi_a]. $$

Lemma 3.2 *For all* $a\in\mathbb{N}$ *we define* T_a *by*

$$ T_a = (-1)^{a+1} S_a S_{a-1}\cdots S_1 \quad , \quad T_0 = -S_0. \tag{32} $$

Let $T_a[f,g] = \Big(T_a^1[f]\,,T_a^2[g]\Big)$, *then*

(i) T_a *is a bounded, one-to-one operator on* $L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1)$ *such that for all* $p,q\in L^2_{\mathbb{C}}(0,1)$ *and all* $\lambda\in\mathbb{C}$

$$ \int_0^1 \Phi_a(\lambda t)\cdot \begin{bmatrix} p(t) \\ q(t) \end{bmatrix} dt = \int_0^1 \begin{bmatrix} \sin(2\lambda t) \\ \cos(2\lambda t) \end{bmatrix} \cdot T_a[p,q](t)dt, \tag{33} $$

$$ \int_0^1 \Psi_a(\lambda t)\cdot \begin{bmatrix} p(t) \\ q(t) \end{bmatrix} dt = \int_0^1 \begin{bmatrix} \cos(2\lambda t) \\ -\sin(2\lambda t) \end{bmatrix} \cdot T_a[p,q](t)dt. \tag{34} $$

(ii) The adjoint of T_a, $T_a^*[f,g] = \Big(T_a^{1*}[f]\,,T_a^{2*}[g]\Big)$ *verifies*

$$ \Phi_a(\lambda x) = T_a^*\begin{bmatrix} \sin(2\lambda x) \\ \cos(2\lambda x) \end{bmatrix} \quad \text{and} \quad \Psi_a(\lambda x) = T_a^*\begin{bmatrix} \cos(2\lambda x) \\ -\sin(2\lambda x) \end{bmatrix} \tag{35} $$

and

$$ \mathrm{Ker}(T_a^*) = \bigoplus_{k=1}^{a} N_k. $$

(iii) T_a *defines a linear isomorphism between* $L^2_{\mathbb{C}}(0,1) \times L^2_{\mathbb{C}}(0,1)$ *and* $\left(\bigoplus_{k=1}^{a} N_k\right)^{\perp}$.

Its inverse is the bounded operator on $L^2_{\mathbb{C}}(0,1) \times L^2_{\mathbb{C}}(0,1)$ *defined by*

$$B_a[f,g] := \left(T_a^{2^*}[f]\,, T_a^{1^*}[g]\right).$$

3.5. Asymptotics upgrade

The following asymptotics are delicate to obtain since we want them to figure both asymptotic behavior with respect to n and singular behavior with respect to x. Transformation operator will help us to handle this difficulty.

First, give a tool ensuring us some uniformity with respect to potentials. It is a Riemann-Lebesgue type lemma:

Lemma 3.3 (Lemma A.1. in [3](See also [17]))

$$\left(\int_0^1 f(t)e^{2\imath\pi(k+\varepsilon_k)t}dt\right)_{k\in\mathbb{Z}} \in \ell^2_{\mathbb{C}}(\mathbb{Z})$$

uniformly with respect to $(f, (\varepsilon_k)_{k\in\mathbb{Z}})$ *on bounded sets of* $L^2_{\mathbb{C}}(0,1) \times \ell^\infty_{\mathbb{C}}(\mathbb{Z})$.

Give an useful writing shortcut:

Notations 5 *Let* $(f_n)_{n\in\mathbb{Z}}$ *a sequence of* $L^\infty_{\mathbb{C}}(0,1)$ *functions. The equality*

$$f_n(x) = \ell^2(n), \quad x \in [0,1], \quad n \in \mathbb{Z}$$

means

$$(\|f_n\|_\infty)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}(\mathbb{Z}).$$

Theorem 3.1 *Uniformly on* $[0, 1]$ *and locally uniformly on* $L^2_{\mathbb{C}}(0,1) \times L^2_{\mathbb{C}}(0,1)$ *we have the following estimate:*

$$\Big|\mathcal{R}(x, \lambda_{a,n}(p,q), p, q) - R(x, \lambda_{a,n}(p,q))\Big| \leq C\left[\frac{x}{1+|\lambda_{a,n}|x}\right]^a \ell^2(n), \quad |n| \to \infty, \quad (36)$$

and locally uniformly on $L^2_{\mathbb{C}}(0,1) \times L^2_{\mathbb{C}}(0,1)$, *we have*

$$\lambda_{a,n}(p,q) = \left(n + \operatorname{sgn}(n)\frac{a}{2}\right)\pi + \beta + \ell^2(n), \quad |n| \to \infty. \quad (37)$$

Proof. We first prove a similar estimate for $R_1(x, \lambda_{a,n}(p,q), p, q)$. For this, recall (see the proof of Theorem 2.1) that $R_1(x, \lambda, p, q) = \lambda^{-a}[X(q) + Y(p)]$. Thus notations from Lemma 3.1 give

$$R_1(x,\lambda,p,q) = \frac{1}{\lambda^a}\begin{bmatrix} -\eta_{a-1}(\lambda x) \\ \eta_a(\lambda x) \end{bmatrix}\int_0^1 \Phi_a(\lambda t)\cdot\begin{bmatrix} \mathbb{1}_{[0,x]}(t)p(t) \\ \mathbb{1}_{[0,x]}(t)q(t) \end{bmatrix}dt$$
$$+\frac{1}{\lambda^a}\begin{bmatrix} j_{a-1}(\lambda x) \\ -j_a(\lambda x) \end{bmatrix}\int_0^1 \Psi_a(\lambda t)\cdot\begin{bmatrix} \mathbb{1}_{[0,x]}(t)p(t) \\ \mathbb{1}_{[0,x]}(t)q(t) \end{bmatrix}dt.$$

Estimate (24) implies that $\lambda_{a,n} = n\pi + \varepsilon_n$ with $(\varepsilon_n)_n \in \ell^\infty_{\mathbb{C}}(\mathbb{Z})$. Then, lemmas 3.2 and 3.3 give uniformly on $[0,1]$ and locally uniformly on $L^2_{\mathbb{C}}(0,1) \times L^2_{\mathbb{C}}(0,1)$:

$$\left(\int_0^1 \Phi_a(\lambda_{a,n}t) \cdot \begin{bmatrix} \mathbb{1}_{[0,x]}(t)p(t) \\ \mathbb{1}_{[0,x]}(t)q(t) \end{bmatrix} dt\right)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{C}}(\mathbb{Z}),$$

$$\left(\int_0^1 \Psi_a(\lambda_{a,n}t) \cdot \begin{bmatrix} \mathbb{1}_{[0,x]}(t)p(t) \\ \mathbb{1}_{[0,x]}(t)q(t) \end{bmatrix} dt\right)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{C}}(\mathbb{Z}),$$

in other words

$$R_1(x,\lambda_{a,n}(p,q),p,q) = \frac{\ell^2(n)}{{\lambda_{a,n}}^a}\begin{bmatrix} -\eta_{a-1}(\lambda_{a,n}x) \\ \eta_a(\lambda_{a,n}x) \end{bmatrix} + \frac{\ell^2(n)}{{\lambda_{a,n}}^a}\begin{bmatrix} j_{a-1}(\lambda_{a,n}x) \\ -j_a(\lambda_{a,n}x) \end{bmatrix}.$$

From (A.2), we obtain

$$\left|\frac{\ell^2(n)}{{\lambda_{a,n}}^a}\begin{bmatrix} j_{a-1}(\lambda_{a,n}x) \\ -j_a(\lambda_{a,n}x) \end{bmatrix}\right| \le C\left(\frac{x}{1+|\lambda_{a,n}|x}\right)^a \ell^2(n). \tag{38}$$

For the first term in $R_1(x,\lambda_{a,n}(p,q),p,q)$ we split $[0,1]$ in two:

$|\lambda_{a,n}x| \ge 1$**:** Since uniformly on $[0,1]$,

$$\int_0^1 \Phi_a(\lambda_{a,n}t) \cdot \begin{bmatrix} \mathbb{1}_{[0,x]}(t)p(t) \\ \mathbb{1}_{[0,x]}(t)q(t) \end{bmatrix} dt = \ell^2(n)$$

and

$$1 = \frac{1+|\lambda_{a,n}|x}{1+|\lambda_{a,n}|x} \le \frac{2|\lambda_{a,n}|x}{1+|\lambda_{a,n}|x},$$

we get

$$\left|\int_0^1 \Phi_a(\lambda_{a,n}t) \cdot \begin{bmatrix} \mathbb{1}_{[0,x]}(t)p(t) \\ \mathbb{1}_{[0,x]}(t)q(t) \end{bmatrix} dt\right| \le \left(\frac{2|\lambda_{a,n}|x}{1+|\lambda_{a,n}|x}\right)^{2a} \ell^2(n).$$

$|\lambda_{a,n}x| \le 1$**:** Estimate (A.2) gives

$$\left|\int_0^1 \Phi_a(\lambda_{a,n}t) \cdot \begin{bmatrix} \mathbb{1}_{[0,x]}(t)p(t) \\ \mathbb{1}_{[0,x]}(t)q(t) \end{bmatrix} dt\right| \le C\left(\frac{|\lambda_{a,n}|x}{1+|\lambda_{a,n}|x}\right)^{2a} \int_0^x \begin{bmatrix} |p(t)| \\ |q(t)| \end{bmatrix} dt,$$

where $C>0$ is uniform in x and n, then

$$\left|\int_0^1 \Phi_a(\lambda_{a,n}t) \cdot \begin{bmatrix} \mathbb{1}_{[0,x]}(t)p(t) \\ \mathbb{1}_{[0,x]}(t)q(t) \end{bmatrix} dt\right| \le C\left(\frac{|\lambda_{a,n}|x}{1+|\lambda_{a,n}|x}\right)^{2a} \int_0^{|\lambda_{a,n}|^{-1}} \begin{bmatrix} |p(t)| \\ |q(t)| \end{bmatrix} dt.$$

Lemma Appendix A.3 gives the good bound.

Combining theses two estimates, we get uniformly on $[0,1]$ and locally uniformly on $L^2_{\mathbb{C}}(0,1) \times L^2_{\mathbb{C}}(0,1)$:

$$\left|\int_0^1 \Phi_a(\lambda_{a,n}t) \cdot \begin{bmatrix} \mathbb{1}_{[0,x]}(t)p(t) \\ \mathbb{1}_{[0,x]}(t)q(t) \end{bmatrix} dt\right| \le C'\left(\frac{|\lambda_{a,n}|x}{1+|\lambda_{a,n}|x}\right)^{2a} \ell^2(n). \tag{39}$$

Estimate (A.3) together with (38) and (39) gives

$$\left|R_1(x,\lambda_{a,n}(p,q),p,q)\right| \leq \left(\frac{x}{1+|\lambda_{a,n}|x}\right)^a \ell^2(n)$$

locally uniformly on $L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1)$ and uniformly on $[0,1]$.

With the recurrence relation and the estimation for $\mathcal{G}(x,t,\lambda)$, follows uniformly on $[0,1]$ and locally uniformly on $L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1)$:

$$\left|R_{k+1}(x,\lambda_{a,n}(p,q),p,q)\right| \leq \frac{C^k}{k!}\left(\int_0^x (|p(t)|+|q(t)|)\,dt\right)^k \left(\frac{x}{1+|\lambda_{a,n}|x}\right)^a \ell^2(n),$$

summing up, we get the result. Eigenvalues estimate is deduced directly from $\mathcal{R}(x,\lambda_{a,n}(p,q),p,q)$'s estimate and from (24)-(25). □

In a very similar way, we upgrade the control of the singular solution and doing it justify the choice and existence of the singular solution as announced in the first remark.

Theorem 3.2 *Let* $(p,q)\in L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1)$*, then uniformly on* $(0,1]$ *and locally uniformly on* $L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1)$ *we have:*

$$\left|\mathcal{S}(x,\lambda_{a,n}(p,q),p,q) - S(x,\lambda_{a,n}(p,q))\right| \leq C\left[\frac{1+|\lambda_{a,n}|x}{x}\right]^a \ell^2(n). \tag{40}$$

Proof. As for the regular solution, we obtain (see [22]) the uniform estimate in $x\in[0,1]$ and locally uniform on $L^2_{\mathbb{R}}(0,1)\times L^2_{\mathbb{R}}(0,1)$:

$$\tilde{\mathcal{S}}(x,\lambda_{a,n}(p,q),p,q) = S(x,\lambda) + \mathcal{O}\left(\left[\frac{1+|\lambda_{a,n}|x}{x}\right]^a\right)\ell^2(n).$$

Then, we get easily $\mathcal{W}(\lambda_{a,n}(p,q),p,q) = \mathcal{W}(\lambda_{a,n}(p,q),0)+\ell^2(n) = 1+\ell^2(n)$ and through $\mathcal{S}(x,\lambda,p,q) = \dfrac{\tilde{\mathcal{S}}(x,\lambda,p,q)}{\mathcal{W}(\lambda,p,q)}$, we reach the result. □

Straightforward calculations let us deduce the following estimations:

Corollary 3.2 *Uniformly on* $[0,1]$ *and locally uniformly on* $L^2_{\mathbb{R}}(0,1)\times L^2_{\mathbb{R}}(0,1)$*, when* $|n|\to\infty$*, we have*

$$\|\mathcal{R}_n(\cdot,p,q)\|^2 = \frac{1}{{\lambda_{a,n}}^{2a}}\left(1+\ell^2(n)\right), \tag{41}$$

$$\Big\langle \mathcal{R}_n(\cdot,p,q),\mathcal{S}_n(\cdot,p,q)\Big\rangle = \ell^2(n), \tag{42}$$

$$G_n(x,p,q) = \begin{bmatrix} j_{a-1}(\lambda_{a,n}x) \\ -j_a(\lambda_{a,n}x)\end{bmatrix} + \ell^2(n), \tag{43}$$

$$\nabla_{p,q}\lambda_{a,n}(p,q) = \Phi_a(\lambda_{a,n}x) + \ell^2(n), \tag{44}$$

$$\kappa_{a,n}(p,q) = \frac{(-1)^n}{\left[\left(|n|+\frac{a}{2}\right)\pi\right]^a}\left[1+\ell^2(n)\right] = \frac{(-1)^n}{|n\pi|^a}\left[1+\ell^2(n)\right], \tag{45}$$

$$A_n(x,p,q) = \Psi_a(\lambda_{a,n}x) + \ell^2(n), \tag{46}$$

$$\frac{\nabla_{p,q}\kappa_{a,n}(p,q)}{\kappa_{a,n}(p,q)} = \Psi_a(\lambda_{a,n}x) + \ell^2(n). \tag{47}$$

Now, the spectral map can be correctly defined by

$$\begin{array}{rccc} \lambda^a\times\kappa^a: & L^2_{\mathbb{R}}(0,1)\times L^2_{\mathbb{R}}(0,1) & \longrightarrow & \ell^2(\mathbb{Z})\times\ell^2(\mathbb{Z}) \\ & (p,q) & \longmapsto & \Big((\widetilde{\lambda}_{a,n}(p,q))_{n\in\mathbb{Z}},(\widetilde{\kappa}_{a,n}(p,q))_{n\in\mathbb{Z}}\Big), \end{array}$$

and, following [18] and [13], previous analyticity results and the local uniformity with respect to the potentials give us:

Theorem 3.3 *$\lambda^a\times\kappa^a$ is a real-analytic map on $L^2_{\mathbb{R}}(0,1)\times L^2_{\mathbb{R}}(0,1)$.*
Its Fréchet derivative is given by the linear map from $L^2_{\mathbb{R}}(0,1)\times L^2_{\mathbb{R}}(0,1)$ to $\ell^2(\mathbb{Z})\times\ell^2(\mathbb{Z})$:

$$d_{p,q}(\lambda^a\times\kappa^a)(v)=\Big(\left(\langle\nabla_{p,q}\lambda_{a,n},v\rangle\right)_{n\in\mathbb{Z}},\left(\langle\nabla_{p,q}\widetilde{\kappa}_{a,n},v\rangle\right)_{n\in\mathbb{Z}}\Big).$$

4. The inverse spectral problem

Now, give the main result

Theorem 4.1
$d_{p,q}(\lambda^a\times\kappa^a)$ is an isomorphism between $L^2_{\mathbb{R}}(0,1)\times L^2_{\mathbb{R}}(0,1)$ and $\ell^2(\mathbb{Z})\times\ell^2(\mathbb{Z})$.

Proof. In view of the relation

$$\nabla_{p,q}\widetilde{\kappa}_{a,n}=(-1)^n\left[\left(|n|+\frac{a}{2}\right)\pi\right]^a\nabla_{p,q}\kappa_{a,n},$$

corollary 3.1 implies that $(\nabla_{p,q}\lambda_{a,n})_{n\in\mathbb{Z}}\cup(\nabla_{p,q}\widetilde{\kappa}_{a,n})_{n\in\mathbb{Z}}$ is a free family in $L^2_{\mathbb{R}}(0,1)\times L^2_{\mathbb{R}}(0,1)$. Let define r_n and s_n by

$$r_n(x)=\nabla_{p,q}\lambda_{a,n}(x)-\Phi_a(\lambda_{a,n}x), \tag{48}$$
$$s_n(x)=\nabla_{p,q}\widetilde{\kappa}_{a,n}(x)-\Psi_a(\lambda_{a,n}x). \tag{49}$$

With lemma 3.2, we have for all $v\in L^2_{\mathbb{R}}(0,1)\times L^2_{\mathbb{R}}(0,1)$,

$$\left\langle\nabla_{(p,q)}\lambda_{a,n}(V),v\right\rangle=\int_0^1\left(\begin{bmatrix}\sin(2\lambda_{a,n}t)\\ \cos(2\lambda_{a,n}t)\end{bmatrix}+R_n(t)\right)\cdot T_a[v](t)dt, \tag{50}$$

$$\left\langle\nabla_{(p,q)}\widetilde{\kappa}_{a,n}(V),v\right\rangle=\int_0^1\left(\begin{bmatrix}\cos(2\lambda_{a,n}t)\\ -\sin(2\lambda_{a,n}t)\end{bmatrix}+S_n(t)\right)\cdot T_a[v](t)dt, \tag{51}$$

where $R_n=B_a^*[r_n]$ and $S_n=B_a^*[s_n]$. Introduce operator F defined by

$$F(w)=\left(\left\{\left\langle\begin{bmatrix}\sin(2\lambda_{a,n}t)\\ \cos(2\lambda_{a,n}t)\end{bmatrix}+R_n(t),w\right\rangle\right\}_{n\in\mathbb{Z}},\left\{\left\langle\begin{bmatrix}\cos(2\lambda_{a,n}t)\\ -\sin(2\lambda_{a,n}t)\end{bmatrix}+S_n(t),w\right\rangle\right\}_{n\in\mathbb{Z}}\right),$$

in order to get $d_{p,q}(\lambda^a\times\kappa^a)(v)=F\circ T_a[v]$. From lemma 3.2, T_a is a bijection between $L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1)$ and $\left(\bigoplus_{k=1}^a N_k\right)^\perp$. Thus, we have to prove that F is a bijection between $\left(\bigoplus_{k=1}^a N_k\right)^\perp$ and $\ell^2(\mathbb{Z})\times\ell^2(\mathbb{Z})$. To this end, we will show that the operator $\mathbf{F}$ sending functions in $L^2_{\mathbb{R}}(0,1) \times L^2_{\mathbb{R}}(0,1)$ into their Fourier coefficients (or, in other words, the scalar products) with respect to the family

$$\mathcal{F} = \left(\{U_{2k}\}_{k=0}^{a-1}, \left\{ \begin{bmatrix} \sin(2\lambda_{a,n}t) \\ \cos(2\lambda_{a,n}t) \end{bmatrix} + R_n(t) \right\}_{n\in\mathbb{Z}}, \right.$$
$$\left. \{V_{2k+1}\}_{k=0}^{a-1}, \left\{ \begin{bmatrix} \cos(2\lambda_{a,n}t) \\ -\sin(2\lambda_{a,n}t) \end{bmatrix} + S_n(t) \right\}_{n\in\mathbb{Z}} \right), \tag{52}$$

is a invertible map from $L^2_{\mathbb{R}}(0,1) \times L^2_{\mathbb{R}}(0,1)$ to $\ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$. For this, recall the following property (see [18]: Appendix D, theorem 3).

Lemma 4.1 *Let* $\{f_n\}_{n\in\mathbb{Z}}$ *be a free family of vectors in an Hilbert space* H *close to an orthonormal basis* $\{e_n\}_{n\in\mathbb{Z}}$ *of* H*, ie* $\sum \|f_n - e_n\|_2^2 < \infty$.
Then $\{f_n\}_{n\in\mathbb{Z}}$ *is a basis for* H *and the map* $\mathbf{F} : x \mapsto \{(f_n, x)\}_{n\in\mathbb{Z}}$ *is a linear isomorphism from* H *onto* $\ell^2(\mathbb{Z})$.

Estimates (44), (45) and (47) lead to $r_n = \ell^2(n)$ and $s_n = \ell^2(n)$. Boundedness of B_a^* thus gives $R_n = \ell^2(n)$ and $S_n = \ell^2(n)$ which, together with the orthogonal basis of $L^2_{\mathbb{R}}(0,1) \times L^2_{\mathbb{R}}(0,1)$

$$\mathcal{F}_0 = \left\{ \begin{bmatrix} \sin\left(2\left((n+\frac{a}{2})\pi+\beta\right)t\right) \\ \cos\left(2\left((n+\frac{a}{2})\pi+\beta\right)t\right) \end{bmatrix}, \begin{bmatrix} \cos\left(2\left((n+\frac{a}{2})\pi+\beta\right)t\right) \\ -\sin\left(2\left((n+\frac{a}{2})\pi+\beta\right)t\right) \end{bmatrix}, n\in\mathbb{Z} \right\}, \tag{53}$$

and a correct arrangement of each vectors family (see remark bellow), prove the closeness of $\mathcal{F}$ and $\mathcal{F}_0$. Lemma 4.2 gives the freedom of $\mathcal{F}$ and thus lemma 4.1 is applicable. □

Remark. At first sight, the "loss" of eigenvalues appeared in the counting lemma and the non-zero kernel of the transformation operator seem to be barriers to solve the inverse problem. In fact, it is not, it helps us to fit correctly vectors family $\mathcal{F}$ and $\mathcal{F}_0$. Be more specific: let $f^0_{n,1}$ and $f^0_{n,2}$ be defined by (53), in other words, we just write $\mathcal{F}_0 = \left\{f^0_{n,1}, f^0_{n,2}, n\in\mathbb{Z}\right\}$. For $\mathcal{F}$ we choose the following numbering: set $\mathcal{F} = \{f_{n,1}, f_{n,2}, n\in\mathbb{Z}\}$ where for any integer $n \geq 0$,

$$f_{n,1}(t) = \begin{bmatrix} \sin(2\lambda_{a,n}t) \\ \cos(2\lambda_{a,n}t) \end{bmatrix} + R_n(t), \quad f_{n,2}(t) = \begin{bmatrix} \cos(2\lambda_{a,n}t) \\ -\sin(2\lambda_{a,n}t) \end{bmatrix} + S_n(t),$$

for any integer n such that $n \in [\![-a,-1]\!]$,

$$f_{n,1} = U_{-2n-2}, \quad f_{n,2} = V_{-2n-1},$$

and for all integer n such that $n \leq -a-1$,

$$f_{n,1}(t) = \begin{bmatrix} \sin(2\lambda_{a,n+a}t) \\ \cos(2\lambda_{a,n+a}t) \end{bmatrix} + R_{n+a}(t),\ f_{n,2}(t) = \begin{bmatrix} \cos(2\lambda_{a,n+a}t) \\ -\sin(2\lambda_{a,n+a}t) \end{bmatrix} + S_{n+a}(t).$$

With this notation and using the eigenvalue estimate (37), for $j = 1,2$, $(f_{n,j})_n$ is asymptotically ℓ^2-close to $(f^0_{n,j})_n$ whenever $n \to \pm\infty$.

In order to prove the freedom of $\mathcal{F}$, give a little extension with the following

Proposition 4.1 *Let* $(E_{n,1},E_{n,2})_{n\in\mathbb{Z}}$ *be a free vector family in* $L^2_{\mathbb{R}}(0,1)\times L^2_{\mathbb{R}}(0,1)$ *satisfying the following properties:*

(i) Duality : there exists a bounded vector family $(F_{n,1},F_{n,2})_{n\in\mathbb{Z}}$ *in* $L^2_{\mathbb{R}}(0,1)\times L^2_{\mathbb{R}}(0,1)$, *such that*

$$\langle E_{n,j},F_{m,j}\rangle = 0, \quad (n,m)\in\mathbb{Z}^2, \quad j=1,2.$$
$$\langle E_{n,1},F_{m,2}\rangle = \langle E_{n,2},F_{m,1}\rangle = \delta_{n,m}, \quad \forall (n,m)\in\mathbb{Z}^2.$$

(ii) Asymptotics:

$$E_{n,1}=T_a^*\left(\begin{bmatrix}\sin(2\lambda_{a,n}t)\\ \cos(2\lambda_{a,n}t)\end{bmatrix}+e_{n,1}\right),\ E_{n,2}=T_a^*\left(\begin{bmatrix}\cos(2\lambda_{a,n}t)\\ -\sin(2\lambda_{a,n}t)\end{bmatrix}+e_{n,2}\right)$$

with $\left(\|e_{n,j}\|_{L^2_{\mathbb{R}}(0,1)\times L^2_{\mathbb{R}}(0,1)}\right)_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{R}}(\mathbb{Z})$, $j=1,2$.

(iii) Summability: for any $k\in[\![0,2a-1]\!]$, *there exists* $\omega\in\mathcal{C}_0^\infty([0,1],\mathbb{R}^2)$ *such that for all* $m\in[\![0,2a-1]\!]$, $\langle\omega,W_m\rangle=\delta_{k,m}$ *and*

$$(\langle\omega,e_{n,j}\rangle)_{n\in\mathbb{Z}}\in\ell^1_{\mathbb{R}}(\mathbb{Z}), \quad j=1,2.$$

Then, the following family is free in $L^2_{\mathbb{R}}(0,1)\times L^2_{\mathbb{R}}(0,1)$

$$\mathcal{F}=\left(\{U_{2k}\}_{k=0}^{a-1},\left\{\begin{bmatrix}\sin(2\lambda_{a,n}t)\\ \cos(2\lambda_{a,n}t)\end{bmatrix}+e_{n,1}(t)\right\}_{n\in\mathbb{Z}},\right.$$
$$\left.\{V_{2k+1}\}_{k=0}^{a-1},\left\{\begin{bmatrix}\cos(2\lambda_{a,n}t)\\ -\sin(2\lambda_{a,n}t)\end{bmatrix}+e_{n,2}(t)\right\}_{n\in\mathbb{Z}}\right).$$

Proof. Since T_a^* is bounded and $(E_{n,1},E_{n,2})_{n\in\mathbb{Z}}$ is free, condition *(ii)* implies the freedom of the following family

$$\left\{\begin{bmatrix}\sin(2\lambda_{a,n}t)\\ \cos(2\lambda_{a,n}t)\end{bmatrix}+e_{n,1}(t)\right\}_{n\in\mathbb{Z}}\cup\left\{\begin{bmatrix}\cos(2\lambda_{a,n}t)\\ -\sin(2\lambda_{a,n}t)\end{bmatrix}+e_{n,1}(t)\right\}_{n\in\mathbb{Z}}.$$

Let $k\in[\![0,2a-1]\!]$, we define W_k by $W_k=U_k$ if k is even and $W_k=V_k$ otherwise. Show that W_k is not in the closure of $\overline{\mathrm{Vect}\,(\mathcal{F}\setminus\{W_k\})}$. (Precisely, we should prove iteratively that $W_k\notin\overline{\mathrm{Span}\,\{\mathcal{F}\setminus\{W_j,j\in[\![k,2a-1]\!]\}\}}$, which is not necessary since it suffices to set $\alpha_m^{(j)}=0$ for $m\in[\![k,2a-1]\!]$ in the next expression.) For this, suppose the contrary: there exists a vector sequence defined for $j\in\mathbb{N}$ by

$$W_k^{(j)}(t)=\sum_{m\in[\![0,2a-1]\!],m\neq k}\alpha_m^{(j)}W_m(t)+\sum_{n\in[\![-N_j,N_j]\!]}a_n^{(j)}\left(\begin{bmatrix}\sin(2\lambda_{a,n}t)\\ \cos(2\lambda_{a,n}t)\end{bmatrix}+e_{n,1}(t)\right)$$
$$+\sum_{n\in[\![-N_j,N_j]\!]}b_n^{(j)}\left(\begin{bmatrix}\cos(2\lambda_{a,n}t)\\ -\sin(2\lambda_{a,n}t)\end{bmatrix}+e_{n,2}(t)\right),$$

with $N_j<\infty$, $\alpha_m^{(j)},a_n^{(j)},b_n^{(j)}\in\mathbb{R}$ such that $W_k^{(j)}\underset{j\to\infty}{\longrightarrow}W_k$ in $L^2_{\mathbb{R}}(0,1)\times L^2_{\mathbb{R}}(0,1)$. Recall that $T_a^*(W_m)=0$ for $m=0,\dots,2a-1$, thus the sequence

$$w^{(j)}:=T_a^*(W_k^{(j)})=\sum_{n\in[\![-N_j,N_j]\!]}a_n^{(j)}E_{n,1}+b_n^{(j)}E_{n,2}$$

converges towards 0 in $L^2_{\mathbb{R}}(0,1) \times L^2_{\mathbb{R}}(0,1)$ when $j \to \infty$, and point *(i)* leads to

$$a_n^{(j)} = \int_0^1 w^{(j)} \cdot F_{n,2}\, dt \underset{j\to\infty}{\longrightarrow} 0, \tag{54}$$

$$b_n^{(j)} = \int_0^1 w^{(j)} \cdot F_{n,1}\, dt \underset{j\to\infty}{\longrightarrow} 0. \tag{55}$$

and gives the uniform boundedness of $(a_n^{(j)})$ and $(b_n^{(j)})$ with respect to n and j.

Now consider $\omega \in \mathcal{C}_0^\infty([0,1],\mathbb{R}^2)$ as in *(iii)*. Its smoothness and support property imply that for all $N \in \mathbb{N}$,

$$\int_0^1 \omega(t) \cdot \begin{bmatrix} \sin(2\lambda_{a,n}t) \\ \cos(2\lambda_{a,n}t) \end{bmatrix} dt, \int_0^1 \omega(t) \cdot \begin{bmatrix} \cos(2\lambda_{a,n}t) \\ -\sin(2\lambda_{a,n}t) \end{bmatrix} dt = \mathcal{O}\left(\frac{1}{n^N}\right).$$

Thus, second part of *(iii)* shows the summability of

$$\left\{\left\langle \omega, t \mapsto \begin{bmatrix} \sin(2\lambda_{a,n}t) \\ \cos(2\lambda_{a,n}t) \end{bmatrix} + e_{n,1}(t) \right\rangle\right\}_{n\in\mathbb{Z}}$$

and

$$\left\{\left\langle \omega, t \mapsto \begin{bmatrix} \cos(2\lambda_{a,n}t) \\ -\sin(2\lambda_{a,n}t) \end{bmatrix} + e_{n,2}(t) \right\rangle\right\}_{n\in\mathbb{Z}}.$$

We complete the proof writing

$$\begin{aligned} \left\langle \omega, W_k^{(j)} \right\rangle = & \sum_{n\in[\![-N_j,N_j]\!]} a_n^{(j)} \left\langle \omega, \left(\begin{bmatrix} \sin(2\lambda_{a,n}t) \\ \cos(2\lambda_{a,n}t) \end{bmatrix} + e_{n,1}(t) \right) \right\rangle \\ & + \sum_{n\in[\![-N_j,N_j]\!]} b_n^{(j)} \left\langle \omega, \left(\begin{bmatrix} \cos(2\lambda_{a,n}t) \\ -\sin(2\lambda_{a,n}t) \end{bmatrix} + e_{n,2}(t) \right) \right\rangle, \end{aligned}$$

indeed, this shows, by dominated convergence, that

$$\left\langle \omega, W_k^{(j)} \right\rangle \underset{j\to\infty}{\longrightarrow} 0,$$

which is in contradiction with the definition of ω. So $\mathcal{F}$ is a free family. □

Lemma 4.2 *$\mathcal{F}$ is a free family in* $L^2_{\mathbb{R}}(0,1) \times L^2_{\mathbb{R}}(0,1)$.

Proof. Let us apply proposition 4.1. For this, we consider the following vectors

$$\begin{aligned} &E_{n,1} = \nabla_{p,q}\lambda_{a,n}, \quad E_{n,2} = \nabla_{p,q}\widetilde{\kappa}_{a,n}, \quad n \in \mathbb{Z}, \\ &F_{n,1} = \nabla_{p,q}\lambda_{a,n}{}^{\perp}, \quad F_{n,2} = -\nabla_{p,q}\widetilde{\kappa}_{a,n}{}^{\perp}, \quad n \in \mathbb{Z}. \end{aligned}$$

Results from section 3.2 show that $(E_{n,1}, E_{n,2})_{n\in\mathbb{Z}}$ are linearly independent and that condition *(i)* is verified.

Relations (50), (51) and estimates (44), (47) give us condition *(ii)* with

$$e_{n,1} = B_a{}^*[r_n], \quad e_{n,2} = B_a{}^*[s_n],$$

where r_n and s_n are defined by (48) and (49).

Now, condition *(iii)* is left to be proved.

First, there exists $\omega \in \mathcal{C}_0^\infty([0,1],\mathbb{R}^2)$ compactly supported in $[\delta,1]$ for some $\delta>0$, such that $m \in [\![0,2a-1]\!]$, $\langle \omega, W_m\rangle = \delta_{k,m}$. Second, from the definition of S_a^* given in lemma 3.1, $B_a[\omega]$ is in $\mathcal{C}^\infty([0,1],\mathbb{R}^2)$ and supported in $[\delta,1]$. We are now able to prove the summation properties.

Let $\varepsilon_n = (\varepsilon_n^1, \varepsilon_n^2)$ be defined by $\varepsilon_n(x,V) = \mathcal{R}_n(x,V) - R(x,\lambda_{a,n}(V))$ and plug it in $\nabla_{p,q}\lambda_{a,n}$ via (27). We get

$$\begin{aligned}2G_{n,1}(x,V)G_{n,2}(x,V) &= 2(R_1(x,\lambda_{a,n}) + \varepsilon_n^1)(R_2(x,\lambda_{a,n}) + \varepsilon_n^2)\|\mathcal{R}_n(\cdot,p,q)\|_2^{-2},\\ &= \Big(2R_1(x,\lambda_{a,n})R_2(x,\lambda_{a,n}) + 2R_1(x,\lambda_{a,n})\,\varepsilon_n^1 + 2R_2(x,\lambda_{a,n})\,\varepsilon_n^2\\ &\quad + \varepsilon_n^1\varepsilon_n^2\Big)\|\mathcal{R}_n(\cdot,p,q)\|_2^{-2}.\end{aligned}$$

From (36), we have

$$\left|\varepsilon_n^j(x)\right| \leq \left(\frac{x}{1+|\lambda_{a,n}|x}\right)^a \ell^2(n), \quad j=1,2.$$

Thus, using (41), we get

$$\begin{aligned}2G_{n,1}(x,V)G_{n,2}(x,V) = &-2j_a(\lambda_{a,n}x)j_{a-1}(\lambda_{a,n}x)(1+\ell^2(n))\\ &+ 2{\lambda_{a,n}}^a\Big(j_{a-1}(\lambda_{a,n}x)\,\varepsilon_n^2 - j_a(\lambda_{a,n}x)\,\varepsilon_n^1\Big) + \ell^1(n)\end{aligned}$$

and

$$\begin{aligned}G_{n,2}(x,V)^2 - G_{n,1}(x,V)^2 = &(j_a(\lambda_{a,n}x)^2 - j_{a-1}(\lambda_{a,n}x)^2)(1+\ell^2(n))\\ &+ 2{\lambda_{a,n}}^a\Big(-j_{a-1}(\lambda_{a,n}x)\,\varepsilon_n^1 - j_a(\lambda_{a,n}x)\,\varepsilon_n^2\Big) + \ell^1(n),\end{aligned}$$

then, we obtain uniformly for $x\in[0,1]$,

$$\begin{aligned}r_n(x,V) = 2{\lambda_{a,n}}^a&\Big[j_{a-1}(\lambda_{a,n}x)\,\varepsilon_n(x,V)^\perp - j_a(\lambda_{a,n}x)\,\varepsilon_n(x,V)\Big]\\ &+ \Phi_a(\lambda_{a,n}x)\ell^2(n) + \ell^1(n).\end{aligned}$$

With the uniform estimation on $[\delta,1]$, $j_a(\lambda_{a,n}x) = \sin\left(\lambda_{a,n}x - \frac{a\pi}{2}\right) + \mathcal{O}\left(\frac{1}{\lambda_{a,n}}\right)$, we get

$$\begin{aligned}\langle\omega, e_{n,1}\rangle &= \langle \omega, {B_a}^*[r_n]\rangle = \langle B_a[\omega], r_n\rangle\\ &= \int_0^1 \cos\left(\lambda_{a,n}t - \frac{a\pi}{2}\right) 2{\lambda_{a,n}}^a \varepsilon_n(t,V)^\perp\cdot B_a[\omega](t)dt\\ &\quad - \int_0^1 \sin\left(\lambda_{a,n}t - \frac{a\pi}{2}\right) 2{\lambda_{a,n}}^a\varepsilon_n(t,V)\cdot B_a[\omega](t)dt\\ &\quad + \left\langle \ell^2(n)B_a[\omega], \Phi_a(\lambda_{a,n}x)\right\rangle + \ell^1(n).\end{aligned}$$

Now, with lemma 3.3, notice that for all $f\in L^2_{\mathbb{R}}(0,1)$, we have uniformly on the bounded sets of $L^2_{\mathbb{R}}(0,1)$,

$$\left|\int_0^1 \cos(\lambda_{a,n}x)\, f(t)dt\right| = \|f\|_2 \left|\int_0^1 \cos(\lambda_{a,n}x)\,\frac{f(t)}{\|f\|_2}dt\right| \leq \|f\|_2\,\ell^2(n).$$

This leads for instance to

$$\begin{aligned}\left|\int_0^1 \sin\left(\lambda_{a,n}t-\frac{a\pi}{2}\right)2{\lambda_{a,n}}^a\varepsilon_n(t,V)\cdot B_a[\omega](t)dt\right| &\le 2\ell^2(n)\|{\lambda_{a,n}}^a\varepsilon_n\cdot B_a[w]\|_2,\\ &\le 2\ell^2(n)\ell^2(n)\|B_a[w]\|_2,\\ &\le \ell^1(n)\|B_a[w]\|_2.\end{aligned}$$

And with the transformation operator, we get $\langle \ell^2(n)B_a[\omega],\Phi_a(\lambda_{a,n}x)\rangle = \ell^1(n)$. Consequently, we have $\langle\omega,e_{n,1}\rangle=\ell^1(n)$.

Now let $\Sigma_n=(\Sigma_n^1,\Sigma_n^2)$ be defined by $\Sigma_n(x,V)=\mathcal{S}_n(x,V)-S(x,\lambda_{a,n})$. With (40), we have

$$\left|\Sigma_n^j(x)\right|\le\left(\frac{1+|\lambda_{a,n}|x}{x}\right)^a\ell^2(n),\quad j=1,2.$$

First, with the definition of $A_n(x,p,q)$ and relations (36) and (40), we have

$$\begin{aligned}A_n(x,p,q)=\Psi_a(\lambda_{a,n}x)&-{\lambda_{a,n}}^{-a}(j_{a-1}(\lambda_{a,n}x)\Sigma_n(x,V)^\perp-j_a(\lambda_{a,n}x)\Sigma_n(x,V))\\ &+{\lambda_{a,n}}^{a}(\eta_{a-1}(\lambda_{a,n}x)\varepsilon_n(x,V)^\perp-\eta_a(\lambda_{a,n}x)\varepsilon_n(x,V))+\ell^1(n),\end{aligned}$$

which leads, using (29) with (42), to

$$\begin{aligned}\frac{\nabla_{p,q}\kappa_{a,n}}{\kappa_{a,n}}&=\Psi_a(\lambda_{a,n}x)+\ell^2(n)\Psi_a(\lambda_{a,n}x)\\ &\quad-{\lambda_{a,n}}^{-a}\left(j_{a-1}(\lambda_{a,n}x)\Sigma_n(x,V)^\perp-j_a(\lambda_{a,n}x)\Sigma_n(x,V)\right)\\ &\quad+{\lambda_{a,n}}^{a}\left(\eta_{a-1}(\lambda_{a,n}x)\varepsilon_n(x,V)^\perp-\eta_a(\lambda_{a,n}x)\varepsilon_n(x,V)\right)+\ell^1(n).\end{aligned}$$

Then, we get

$$\begin{aligned}s_n(x)=\;&-{\lambda_{a,n}}^{-a}(j_{a-1}(\lambda_{a,n}x)\Sigma_n(x,V)^\perp-j_a(\lambda_{a,n}x)\Sigma_n(x,V))\\ &+{\lambda_{a,n}}^{a}(\eta_{a-1}(\lambda_{a,n}x)\varepsilon_n(x,V)^\perp-\eta_a(\lambda_{a,n}x)\varepsilon_n(x,V))\\ &+\ell^2(n)\Psi_a(\lambda_{a,n}x)+\ell^2(n)\Psi_a(\lambda_{a,n}x)+\ell^1(n).\end{aligned}$$

Now, with the same arguments as previously, using the transformation operator we find that

$$\{\langle\omega,e_{n,2}\rangle\}_{n\in\mathbb{Z}}\in\ell^1(\mathbb{Z}).$$

Thus, proposition 4.1 proves the result. □

We can go further in solving the inverse spectral problem. Indeed, we can give explicitly the inverse of the spectral map's differential. But first some notations:

Notations 6 *For all* $n\in\mathbb{Z}$, *we set*

$$X_{a,n}(p,q)=\frac{-{\nabla_{p,q}\kappa_{a,n}}^\perp}{\kappa_{a,n}(p,q)},\quad Y_{a,n}(p,q)=\frac{(-1)^n\,{\nabla_{p,q}\lambda_{a,n}}^\perp}{\left[\left(|n|+\frac{a}{2}\right)\pi\right]^a\kappa_{a,n}(p,q)}.$$

Notice that, according to estimations from corollary 3.2, we have

$$X_{a,n}(p,q)=-\Psi_a(\lambda_{a,n}x)^\perp+\ell^2(n),\;Y_{a,n}(p,q)=\Phi_a(\lambda_{a,n}x)^\perp+\ell^2(n).\quad(56)$$

Corollary 4.1 *$\lambda^a \times \kappa^a$ is a local real analytic diffeomorphism at every point in $L^2_{\mathbb{R}}(0,1) \times L^2_{\mathbb{R}}(0,1)$. Moreover, the inverse of $d_{p,q}(\lambda^a \times \kappa^a)$ is the linear map from $\ell^2_{\mathbb{R}}(\mathbb{Z}) \times \ell^2_{\mathbb{R}}(\mathbb{Z})$ onto $L^2_{\mathbb{R}}(0,1) \times L^2_{\mathbb{R}}(0,1)$ given by*

$$(d_{p,q}(\lambda^a \times \kappa^a))^{-1}(\xi,\eta) = \sum_{n\in\mathbb{Z}} \xi_n X_{a,n} + \sum_{n\in\mathbb{Z}} \eta_n Y_{a,n}.$$

Proof. First point comes directly from the theorem and the definition of a local diffeomorphism. Now consider $(\xi,\eta) \in \ell^2_{\mathbb{R}}(\mathbb{Z}) \times \ell^2_{\mathbb{R}}(\mathbb{Z})$ and let

$$u = \sum_{n\in\mathbb{Z}} \xi_n X_{a,n} + \sum_{n\in\mathbb{Z}} \eta_n Y_{a,n}.$$

Thanks to relation (35), the transformation operator lets us write estimations (56) in the following way

$$X_{a,n}(p,q) = B_a\left[\begin{bmatrix} \sin(2\lambda_{a,n}x) \\ \cos(2\lambda_{a,n}x) \end{bmatrix} + \ell^2(n)\right], Y_{a,n}(p,q) = B_a\left[\begin{bmatrix} \cos(2\lambda_{a,n}x) \\ -\sin(2\lambda_{a,n}x) \end{bmatrix} + \ell^2(n)\right].$$

Since B_a is bounded and ξ, η are in $\ell^2_{\mathbb{R}}(\mathbb{Z})$, the sum defining u exists in $L^2_{\mathbb{R}}(0,1)\times L^2_{\mathbb{R}}(0,1)$. Orthogonality relations from section 3.2 imply that for all $n \in \mathbb{Z}$

$$\langle \nabla_{p,q}\lambda_{a,n}, u\rangle = \xi_n \quad \text{et} \quad \langle \nabla_{p,q}\widetilde{\kappa}_{a,n}, u\rangle = \eta_n.$$

Thus we have $d_{p,q}(\lambda^a \times \kappa^a)(u) = (\xi,\eta)$, which proves the corollary. □

We finish the local inverse spectral problem with the description of isospectral sets. For $(p_0,q_0) \in L^2_{\mathbb{R}}(0,1) \times L^2_{\mathbb{R}}(0,1)$, we define the set of AKNS potentials with same spectrum as (p_0,q_0), called isospectral set of (p_0,q_0), by:

$$\text{Iso}(p_0,q_0,a) = \left\{(p,q) \in L^2_{\mathbb{R}}(0,1) \times L^2_{\mathbb{R}}(0,1) : \lambda^a(p,q) = \lambda^a(p_0,q_0)\right\}.$$

Theorem 4.2 *Let $(p_0,q_0) \in L^2_{\mathbb{R}}(0,1) \times L^2_{\mathbb{R}}(0,1)$, then*

(a) $\text{Iso}(p_0,q_0,a)$ *is a real analytic submanifold of $L^2_{\mathbb{R}}(0,1) \times L^2_{\mathbb{R}}(0,1)$.*

(b) At every point (p,q) of $\text{Iso}(p_0,q_0,a)$*, the tangent space is*

$$T_{p,q}\text{Iso}(p_0,q_0,a) = \left\{\sum_{n\in\mathbb{Z}} \eta_n Y_{a,n}(p,q) \ : \eta \in \ell^2_{\mathbb{R}}(\mathbb{Z})\right\}$$

and the normal space is

$$N_{p,q}\text{Iso}(p_0,q_0,a) = \left\{\sum_{n\in\mathbb{Z}} \eta_n Y_{a,n}(p,q)^{\perp} \ : \eta \in \ell^2_{\mathbb{R}}(\mathbb{Z})\right\}.$$

Proof. Notice that the local real-analytic diffeomorphism $\lambda^a \times \kappa^a$ defines a chart at each point $(p,q) \in \text{Iso}(p_0,q_0,a)$, the definition of a submanifold gives point *(a)*.
Since $T_{p,q}\text{Iso}(p_0,q_0,a) = (d_{p,q}(\lambda^a \times \kappa^a))^{-1}(\{0_{\ell^2_{\mathbb{R}}(\mathbb{Z})}\} \times \ell^2_{\mathbb{R}}(\mathbb{Z}))$, corollary 4.1 gives the

expression of the tangent space. Now, the family $(Y_{a,n})_{n\in\mathbb{Z}}$ is free since $(\nabla_{p,q}\lambda_{a,n})_{n\in\mathbb{Z}}$ is. Moreover, it is orthogonal to $({Y_{a,n}}^{\perp})_{n\in\mathbb{Z}}$. Then we have the first inclusion

$$\left\{\sum_{n\in\mathbb{Z}} \eta_n Y_{a,n}(p,q)^{\perp} \ : \eta \in \ell^2_{\mathbb{R}}(\mathbb{Z})\right\} \subset N_{p,q}\mathrm{Iso}(p_0,q_0,a).$$

Now, every vector orthogonal to $({Y_{a,n}}^{\perp})_{n\in\mathbb{Z}}$ is orthogonal to the gradients $(\nabla_{p,q}\lambda_{a,n})_{n\in\mathbb{Z}}$, in other words, is in the kernel of $d_{p,q}\lambda^a$. Thus the second inclusion follows and so does point *(b)*. □

4.1. A Borg-Levinson theorem on $H^1_{\mathbb{R}}(0,1)\times H^1_{\mathbb{R}}(0,1)$

Theorem 4.3 $\lambda^a \times \kappa^a$ *is one-to-one on* $H^1_{\mathbb{R}}(0,1)\times H^1_{\mathbb{R}}(0,1)$.

As in the case of a radial Schrödinger operator (see for instance [7]), we introduce another solution to (1) with boundary condition at $x = 1$.

Lemma 4.3 *Let* $\rho(x,\lambda,V)$ *be the solution of (1) such that*

$$\rho(1,\lambda,V) = {u_\beta}^{\perp}. \tag{57}$$

Then ρ *verifies the following properties*

(i) For $V=(p,q)\in L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1)$ *and* $\delta>0$, *uniformly on* $[\delta,1]$,

$$\left|\rho(x,\lambda,V) - \begin{bmatrix} \cos\left(\lambda(1-x)-\beta\right) \\ \sin\left(\lambda(1-x)-\beta\right)\end{bmatrix}\right| \le K(x)e^{|\mathrm{Im}\,\lambda|(1-x)}$$

where $K(x) = \exp\left[\int_x^1 \left(|p(t)|+|q(t)|+\frac{a}{t}\right)dt\right]$.

(ii) For $V=(p,q)\in H^1\times H^1$ *and* $\delta>0$, *uniformly on* $[\delta,1]$,

$$\left|\rho(x,\lambda,V) - \begin{bmatrix} \cos\left(\lambda(1-x)-\beta\right) \\ \sin\left(\lambda(1-x)-\beta\right)\end{bmatrix}\right| \le C_a\frac{K(x)}{\lambda x}(\|V\|_{H^1}+1)e^{|\mathrm{Im}\,\lambda|(1-x)}$$

(iii) For all $x\in(0,1]$, $\rho(x,\lambda,V)$ *is analytic on* $\mathbb{C}\times L^2_{\mathbb{C}}(0,1)\times L^2_{\mathbb{C}}(0,1)$.

(iv) For $n\in\mathbb{Z}$ *and* $\lambda=\lambda_{a,n}(V)$, *we have*

$$\mathcal{R}_n(x,V) = \kappa_{a,n}(V)\rho(x,\lambda_{a,n}(V),V). \tag{58}$$

Lemma's proof.
Points *(i)*, *(ii)* et *(iii)* follow directly from a Picard iteration construction of ρ. Indeed, we define as in the regular case (see for instance [11]) ρ with

Now prove point *(iv)*: when $\lambda=\lambda_{a,n}(V)$, according to (2) and (57), $\rho(1,\lambda_{a,n}(V),V)$ and $\mathcal{R}(1,\lambda_{a,n}(V))$ are collinear. Then $\rho(x,\lambda_{a,n}(V),V)$ and $\mathcal{R}(x,\lambda_{a,n}(V))$ solutions of (1) with the same eigenvalue λ are also collinear, in other words there exists $C_n\in\mathbb{R}$ such that $\mathcal{R}_n(x,V)=C_n\,\rho(x,\lambda_{a,n}(V),V)$. Using again (2) then (57) and (28), we deduce that $\kappa_{a,n}(V)=C_n$. □

Proof of Theorem 4.3. Let $V, W \in L^2_{\mathbb{R}}(0,1) \times L^2_{\mathbb{R}}(0,1)$ such that $(\lambda^a \times \kappa^a)(V) = (\lambda^a \times \kappa^a)(W)$. For $u \in \mathbb{R}^2$, introduce the function

$$f(x,\lambda,V,W) = \frac{[\mathcal{R}(x,\lambda,V)\cdot u - \mathcal{R}(x,\lambda,W)\cdot u]\,[\rho(x,\lambda,V)\cdot u - \rho(x,\lambda,W)\cdot u]}{D(\lambda,V)}.$$

For all $x \in (0,1]$, $f : \lambda \mapsto f(x,\lambda,V,W)$ is a meromorphic function on $\mathbb{C}$ which has simple poles $\lambda_{a,n}(V)$, $n \in \mathbb{Z}$. From the simplicity of poles and since $f(\lambda) = h(\lambda)/g(\lambda)$, the residue of f at $\lambda_{a,n}(V)$ is

$$\mathrm{Res}(f, \lambda_{a,n}(V)) = \frac{h(\lambda_{a,n}(V))}{g'(\lambda_{a,n}(V))}.$$

Using that $\lambda_{a,n}(V) = \lambda_{a,n}(W)$ and $\kappa_{a,n}(V) = \kappa_{a,n}(W)$, together with relations (58) and (16), we obtain

$$\mathrm{Res}(f, \lambda_{a,n}(V)) = -\frac{[\mathcal{R}_n(x,V)\cdot u - \mathcal{R}_n(x,W)\cdot u]^2}{\|\mathcal{R}_n(\cdot,V)\|_2^2}.$$

To conclude, we make use of a complex analysis result

Lemma 4.4 (Lemma 3.2 [18]) *Let f be a meromorphic function on $\mathbb{C}$ such that*

$$\sup_{|\lambda| = r_n} |f(\lambda)| = o\left(\frac{1}{r_n}\right)$$

for an unbounded sequence of positive real numbers (r_n). Then, the sum of the residues of f is zero.

Let $N > 0$ be an integer and C_N be the circle defined by

$$\left|\lambda - \left(\frac{a\pi}{2} + \beta\right)\right| = \left(N + \frac{1}{2}\right)\pi.$$

Estimate $|\lambda f(x,\lambda,V,W)|$ on C_N. From (17) and (22) with the help of lemma (4.3), we have for N large enough

$$|\mathcal{R}(x,\lambda,V)\cdot u - \mathcal{R}(x,\lambda,W)\cdot u| \le C(\|V\|_{H^1} + \|W\|_{H^1}) e^{|\mathrm{Im}\,\lambda| x} \frac{\ln|\lambda|}{|\lambda|^{a+1}},$$

$$|\rho(x,\lambda,V)\cdot u - \rho(x,\lambda,W)\cdot u| \le \frac{K(x)}{|\lambda| x}(\|V\|_{H^1} + \|W\|_{H^1}) e^{|\mathrm{Im}\,\lambda|(1-x)},$$

$$|D(\lambda,V)| \ge |R(1,\lambda)\cdot u_\beta| - |(\mathcal{R}(1,\lambda,V) - R(1,\lambda))\cdot u_\beta| \ge \frac{C}{|\lambda|^a} e^{|\mathrm{Im}\,\lambda|}.$$

We deduce that uniformly for $x \in [\delta, 1]$ and $\lambda \in C_N$, $|\lambda f(\lambda,V,W)| \le C\dfrac{\ln|\lambda|}{|\lambda|}$. Thus, result from lemma 4.4 is valid for f. Since residues of f have the same sign, they are all zero. In conclusion, we have for all $n \in \mathbb{Z}$, $u \in \mathbb{R}^2$, $\delta \in (0,1]$ and $x \in [\delta, 1]$, $\mathcal{R}_n(x,V)\cdot u - \mathcal{R}_n(x,W)\cdot u = 0$. We can deduce, recalling continuousness of eigenvectors at $x = 0$, that for all $x \in [0,1]$ and all $n \in \mathbb{Z}$

$$\mathcal{R}_n(x,V) = \mathcal{R}_n(x,W).$$

Plug this in (1) to deduce that $V = W$ almost every where on $[0,1]$. □

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Appendix

Spherical Bessel functions j_a and η_a are defined through

$$j_a(z) = \sqrt{\frac{\pi z}{2}} J_{a+1/2}(z), \quad \eta_a(z) = (-1)^a \sqrt{\frac{\pi z}{2}} J_{-a-1/2}(z), \tag{A.1}$$

where J_ν is the first kind Bessel function of order ν (see [9] for precisions).

The following estimates can be found in [22].

- Uniform estimates on $\mathbb{C}$:

$$|j_a(z)| \le Ce^{|\operatorname{Im} z|} \left(\frac{|z|}{1+|z|}\right)^{a+1}, \tag{A.2}$$

$$|\eta_a(z)| \le Ce^{|\operatorname{Im} z|} \left(\frac{1+|z|}{|z|}\right)^{a}. \tag{A.3}$$

- Estimations for the Green function $G(x,t,\lambda)$ when $0 \le t \le x$:

$$|\mathcal{G}(x,t,\lambda)| \le Ce^{|\operatorname{Im}\lambda|(x-t)} \left(\frac{x}{1+|\lambda|x}\right)^a \left(\frac{1+|\lambda|t}{t}\right)^a. \tag{A.4}$$

- Estimations for the Green function $G(x,t,\lambda)$ when $0 \le x \le t \le 1$:

$$|\mathcal{G}(x,t,\lambda)| \le Ce^{|\operatorname{Im}\lambda|(t-x)} \left(\frac{1+|\lambda|x}{x}\right)^a \left(\frac{t}{1+|\lambda|t}\right)^a. \tag{A.5}$$

- Trigonometric expression ([9] formulas $(1-2)$ section 7.11 p.78),

$$j_a(z) = \sin\left(z - \frac{a\pi}{2}\right) P_a(z^{-1}) + \cos\left(z - \frac{a\pi}{2}\right) I_a(z^{-1}), \tag{A.6}$$

$$\eta_a(z) = \cos\left(z - \frac{a\pi}{2}\right) P_a(z^{-1}) - \sin\left(z - \frac{a\pi}{2}\right) I_a(z^{-1}) \tag{A.7}$$

where P_a and I_a are even, resp. odd, polynomials given by

$$P_a(z) = \sum_{m=0}^{\le a/2} (-1)^m (a+1/2, 2m)(2z)^{2m}, \qquad (P_a(0) = 1), \tag{A.8}$$

$$I_a(z) = \sum_{m=0}^{\le (a-1)/2} (-1)^m (a+1/2, 2m+1)(2z)^{2m+1}, \quad (I_a(0) = 0), \tag{A.9}$$

where $(\nu, m) = \dfrac{\Gamma(\nu+1/2+m)}{m!\Gamma(\nu+1/2-m)}$ is the Hankel symbol.

Appendix A.1. Technical lemmas

Lemma Appendix A.1 *Let* $f_1(z) = 2j_{a-1}(z)j_a(z)$. *Then* $F_1 = \int f_1(z)dz$ *such that* $F_1(0) = 0$ *verifies the properties*

(i) $|F_1(z)| \le C\left(\frac{|z|}{1+|z|}\right)^{2a+2}$ *for* $|z| \le 1$;

(ii) $F_1(z) = -a\mathrm{ci}(2z) + p_a\left(z^{-1}\right)\cos(2z) + q_a\left(z^{-1}\right)\sin(2z) + r_a\left(z^{-1}\right)$ *if* $z \neq 0$,

Where $\mathrm{ci}(z) = \int_0^z \frac{\cos t - 1}{t}\,dt$ *and* p_a, q_a, r_a *are resp. even, odd and even, polynomials.*

Lemma Appendix A.2 *Let* $f_2(z) = \eta_{a-1}(z)j_a(z) + \eta_a(z)j_{a-1}(z)$. *Then* $F_2 = \int f_2(z)dz$ *such that* $F_2(0) = 0$ *satisfies the properties*

(i) $|F_2(z)| \le C\frac{|z|}{1+|z|}$ *for* $|z| \le 1$;

(ii) $F_2(z) = a\mathrm{Si}(2z) - p_a\left(z^{-1}\right)\sin(2z) + q_a\left(z^{-1}\right)\cos(2z)$ *if* $z \neq 0$.

Where $\mathrm{Si}(z) = \int_0^z \frac{\sin t}{t}\,dt$ *and* p_a, q_a *are the previous polynomials.*

Appendix A.2. Calculation lemma

The following lemma is adapted from [7], its proof lies on some Hardy inequalities (for details see [7] and [22]). Together with the transformation operator, it is an essential tool for the computation of asymptotics for $L^2_{\mathbb{R}}(0,1)$ potentials.

Lemma Appendix A.3 (Carlson [7]) *Let* $f \in L^2_{\mathbb{C}}(0,1)$ *and* $(z_n)_{n\in\mathbb{N}}$ *a strictly positive real sequence such that*

$$z_0 > 0 \quad \text{and} \quad \exists (C_1, C_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*, \forall n \in \mathbb{N}, \quad C_1 \le z_{n+1} - z_n \le C_2.$$

Then, uniformly on bounded set in $L^2_{\mathbb{C}}(0,1)$,

$$\left(\int_0^{1/z_n} |f(t)|dt\right)_{n\in\mathbb{N}}, \left(\int_{1/z_n}^1 \left|\frac{f(t)}{z_n t}\right| dt\right)_{n\in\mathbb{N}} \in \ell^2_{\mathbb{R}}(\mathbb{N}).$$

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