Accuracy on eigenvalues for a Schrodinger operator with a degenerate potential in the semi-classical limit

Abderemane Morame, Francoise Truc

Accuracy on eigenvalues for a Schrödinger operator with a degenerate potential in the semi-classical limit

Abderemane MORAME[1] **and Françoise TRUC**[2]
[1] *Université de Nantes, Faculté des Sciences, Dpt. Mathématiques, UMR 6629 du CNRS, B.P. 99208, 44322 Nantes Cedex 3, (FRANCE), E.Mail: morame@math.univ-nantes.fr*
[2] *Université de Grenoble I, Institut Fourier, UMR 5582 CNRS-UJF, B.P. 74, 38402 St Martin d'Hères Cedex, (France), E.Mail: Francoise.Truc@ujf-grenoble.fr*
[1]

Abstract

We consider a semi-classical Schrödinger operator $-h^2\Delta + V$ with a degenerate potential V(x,y) =f(x) g(y) .
g is assumed to be a homogeneous positive function of m variables and f is a strictly positive function of n variables, with a strict minimum. We give sharp asymptotic behaviour of low eigenvalues bounded by some power of the parameter h, by improving Born-Oppenheimer approximation.

1 Introduction

In our paper [MoTr] we have considered the Schrödinger operator on $L^2(\mathbb{R}^n_x \times \mathbb{R}^m_y)$

$$\widehat{H}_h \;=\; h^2 D_x^2 \;+\; h^2 D_y^2 \;+\; f(x)g(y) \tag{1.1}$$

with $g \in C^\infty(\mathbb{R}^m \setminus \{0\})$ homogeneous of degree $a > 0$,

$$g(\mu y) \;=\; \mu^a g(y) \;>\; 0\;, \quad \forall\, \mu > 0 \text{ and } \forall\, y \in \mathbb{R}^m \setminus \{0\}\;. \tag{1.2}$$

[1] *Keywords* : eigenvalues, semi-classical asymptotics , Born-Oppenheimer approximation.
Mathematical Classification : 35P20.

$h > 0$ is a semiclassical parameter we assume to be small.

We have investigated the asymptotic behavior of the number of eigenvalues less then λ of $\widehat{H}_h$,

$$N(\lambda,\ \widehat{H}_h) = tr(\chi_{]-\infty,\lambda[}(\widehat{H}_h)\ =\ \sum_{\lambda_k(\widehat{H}_h)<\lambda} 1\ . \tag{1.3}$$

($tr(P)$ denotes the trace of the operator P) .
If P is a self-adjoint operator on a Hilbert space $\mathcal{H}$, we denote respectively by $sp(P)$, $sp_{ess}(P)$ and $sp_d(P)$ the spectrum, the essential spectrum and the discret spectrum of P .

When $-\infty < \inf\ sp(P) < \inf\ sp_{ess}(P)$, we denote by $(\lambda_k(P))_{k>0}$ the increasing sequence of eigenvalues of P , repeated according to their multiplicity:

$$sp_d(P) \bigcap]-\infty,\ \inf\ sp_{ess}(P)[= \{\lambda_k(P)\}\ . \tag{1.4}$$

In this paper we are interested in a sharp estimate for some eigenvalues of $\widehat{H}_h$. We make the following assumptions on the other multiplicative part of the potential:

$$\begin{array}{l} f\ \in\ C^\infty(\mathbb{R}^n),\ \forall\alpha\in\mathbb{N}^n,\ (|f(x)|+1)^{-1}\partial_x^\alpha f(x)\ \in L^\infty(\mathbb{R}^n) \\ 0\ <\ f(0)\ =\ \inf_{x\in\mathbb{R}^n}\ f(x) \\ f(0)\ <\ \liminf_{|x|\to\infty}\ f(x)\ =\ f(\infty) \\ \partial^2 f(0)\ >\ 0 \end{array} \tag{1.5}$$

$\partial^2 f(a)$ denotes the hessian matrix:

$$\partial^2 f(a)\ =\ \left(\frac{\partial^2}{\partial x_i\partial x_j}f(a)\right)_{1\le i,j\le n}\ .$$

By dividing $\widehat{H}_h$ by $f(0)$, we can change the parameter h and assume that

$$f(0) = 1\ . \tag{1.6}$$

Let us define : $\hbar\ =\ h^{2/(2+a)}$ and change y in $y\hbar$; we can use the homogeneity of g (1.2) to get :

$$sp\ (\widehat{H}_h)\ =\ \hbar^a\ sp\ (\widehat{H}^\hbar)\ , \tag{1.7}$$

with $\widehat{H}^\hbar \ = \ \hbar^2 D_x^2 \ + \ D_y^2 \ + \ f(x)g(y) \ = \ \hbar^2 D_x^2 \ + \ Q(x,y,D_y)$:

$$Q(x,y,D_y) \ = \ D_y^2 \ + \ f(x)g(y) \ .$$

Let us denote the increasing sequence of eigenvalues of $D_y^2 \ + \ g(y)$, (on $L^2(\mathbb{R}^m)$) , by $(\mu_j)_{j>0}$.
The associated eigenfunctions will be denoted by $(\varphi_j)_j$:

$$\begin{array}{l} D_y^2\varphi_j(y) \ + \ g(y)\varphi_j(y) \ = \ \mu_j\varphi_j(y) \\ \langle \varphi_j \mid \varphi_k \rangle \ = \ \delta_{jk} \end{array} \tag{1.8}$$

and $(\varphi_j)_j$ is a Hilbert base of $L^2(\mathbb{R}^m)$.

By homogeneity (1.2) the eigenvalues of $Q_x(y,D_y) = \ D_y^2 \ + \ f(x)g(y)$, on $L^2(\mathbb{R}^m)$) , for a fixed x, are given by the sequence $(\lambda_j(x))_{j>0}$, where : $\lambda_j(x) = \mu_j \ f^{2/(2+a)}(x)$.
So as in [MoTr] we get :

$$\widehat{H}^\hbar \ \geq \ \left[\ \hbar^2 D_x^2 \ + \ \mu_1 f^{2/(2+a)}(x) \ \right] \ . \tag{1.9}$$

This estimate is sharp as we will see below.

Then using the same kind of estimate as (1.9), one can see that

$$\inf \ sp_{ess}(\widehat{H}^\hbar) \ \geq \ \mu_1 f^{2/(2+a)}(\infty) \ . \tag{1.10}$$

We are in the Born-Oppenheimer approximation situation described by A. Martinez in [Ma] : the "effective " potential is given by $\lambda_1(x) = \mu_1 \ f^{2/(2+a)}(x)$, the first eigenvalue of Q_x, and the assumptions on f ensure that this potential admits one unique and nondegenerate well $U = \{0\}$, with minimal value equal to μ_1. Hence we can apply theorem 4.1 of [Ma] and get :

Theorem 1.1 *Under the above assumptions, for any arbitrary $C > 0$, there exists $h_0 > 0$ such that, if $0 < \hbar < h_0$, the operator $(\widehat{H}^\hbar)$ admits a finite number of eigenvalues $E_k(\hbar)$ in $[\mu_1, \mu_1 + C\hbar]$, equal to the number of the eigenvalues e_k of $D_x^2 \ + \ \frac{\mu_1}{2+a} < \ \partial^2 f(0) \ x, \ x \ >$ in $[0, +C]$ such that :*

$$E_k(\hbar) = \lambda_k(\widehat{H}^\hbar) \ = \ \lambda_k\left(\hbar^2 D_x^2 + \mu_1 f^{2/(2+a)}(x)\right) \ + \ \mathbf{O}(\hbar^2) \ . \tag{1.11}$$

More precisely $E_k(\hbar) = \ \lambda_k(\widehat{H}^\hbar)$ has an asymptotic expansion

$$E_k(\hbar) \ \sim \ \mu_1 \ + \ \hbar \ (\ e_k \ + \ \sum_{j\geq 1} \alpha_{kj}\hbar^{j/2} \ ). \tag{1.12}$$

If $E_k(\hbar)$ *is asymptotically non degenerated, then there exists a quasimode*

$$\phi_k^\hbar(x,y) \ \sim \ \hbar^{-m_k} e^{-\psi(x)/\hbar} \sum_{j\geq 0} \hbar^{j/2} a_{kj}(x,y) \ , \tag{1.13}$$

satisfying

$$\begin{array}{l} C_0^{-1} \leq \|\hbar^{-m_k} e^{-\psi(x)/\hbar} a_{k0}(x,y)\| \leq C_0 \\ \|\hbar^{-m_k} e^{-\psi(x)/\hbar} a_{kj}(x,y)\| \leq C_j \\ \| \left(\widehat{H}^\hbar - \mu_1 - \ \hbar e_k - \sum_{1\leq j\leq J} \alpha_{kj} \hbar^{j/2}\right) \\ \hbar^{-m_k} e^{-\psi(x)/\hbar} \sum_{0\leq j\leq J} \hbar^{j/2} a_{kj}(x,y)\| \ \leq \ C_J \hbar^{(J+1)/2} \end{array} \tag{1.14}$$

The formula (1.12) implies

$$\lambda_k(\widehat{H}^\hbar) \ = \ \mu_1 + \hbar\lambda_k \left(D_x^2 + \frac{\mu_1}{2+a} < \ \partial^2 f(0) \ x \ , \ x \ > \right) + \mathbf{O}(\hbar^{3/2}) \ , \tag{1.15}$$

and when $k = 1$, one can improve $\mathbf{O}(\hbar^{3/2})$ into $\mathbf{O}(\hbar^2)$. The function ψ is defined by : $\psi(x) \ = \ d(x,0)$, where d denotes the Agmon distance related to the degenerate metric $\mu_1 \ f^{2/(2+a)}(x) dx^2$.

2 Lower energies

We are interested now with the lower energies of $\widehat{H}^\hbar$. Let us make the change of variables

$$(x, \ y) \ \rightarrow \ (x, \ f^{1/(2+a)}(x) y) \ . \tag{2.1}$$

The Jacobian of this diffeomorphism is $f^{m/(2+a)}(x)$, so we perform the change of test functions : $u \ \rightarrow \ f^{-m/(4+2a)}(x) u$, to get a unitary transformation.

Thus we get that

$$sp \ (\widehat{H}^\hbar) \ = \ sp \ (\widetilde{H}^\hbar) \tag{2.2}$$

where $\widetilde{H}^\hbar$ is the self-adjoint operator on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$ given by

$$\widetilde{H}^\hbar \ = \ \hbar^2 L^\star(x,y,D_x,D_y) L(x,y,D_x,D_y) \ + \ f^{2/(2+a)}(x) \left(D_y^2 + g(y)\right) \ , \tag{2.3}$$

with

$$L(x,y,D_x,D_y) \ = \ D_x \ + \ \frac{1}{(2+a)f(x)} [(y D_y) \ - \ i\frac{m}{2}] \nabla f(x) \ .$$

We decompose $\widetilde{H}^\hbar$ in four parts :

$$\begin{array}{l} \widetilde{H}^\hbar \ = \ \hbar^2 D_x^2 \ + \ f^{2/(2+a)}(x) \left(D_y^2 + g(y)\right) \\ +\hbar^2 \frac{2}{(2+a)f(x)}(\nabla f(x) D_x)(y D_y) \\ +i\hbar^2 \frac{1}{(2+a)f^2(x)} \left(|\nabla f(x)|^2 - f(x)\Delta f(x)\right) [(yD_y) \ - \ i\frac{m}{2}] \\ + \ \hbar^2 \frac{1}{(2+a)^2 f^2(x)} |\nabla f(x)|^2 [(yD_y)^2 \ + \ \frac{m^2}{4}] \end{array} \tag{2.4}$$

Our goal is to prove that the only significant role up to order 2 in $\hbar$ will be played by the first operator, namely : $\widetilde{H}_1^\hbar \ = \ \hbar^2 D_x^2 + f^{2/(2+a)}(x) \left(D_y^2 + g(y)\right)$.

Let us denote by $\nu_{j,k}^\hbar$ the eigenvalues of the operator $\hbar^2 D_x^2 \, + \, \mu_j f^{2/(2+a)}(x)$ and by $\psi_{j,k}^\hbar$ the associated normalized eigenfunctions .

Let us consider the following test functions :

$$u_{j,k}^\hbar(x,y) = \psi_{j,k}^\hbar(x)\varphi_j(y) \ ,$$

where the φ_j's are the eigenfunctions defined in (1.8); we have immediately :

$$\widetilde{H}_1^\hbar(u_{j,k}^\hbar(x,y)) \ = \nu_{j,k}^\hbar u_{j,k}^\hbar(x,y) \ .$$

We will need the following lemma :

Lemma 2.1 . *For any integer N , there exists a positive constant C depending only on N such that for any $k \leq N$, the eigenfunction $\psi_{j,k}^\hbar$ satisfies the following inequalities : for any $\alpha \in \mathbb{N}^n$, $|\alpha| \leq 2$,*

$$\begin{array}{lcl} \| \ \hbar_j^{|\alpha|/2} \ |D_x^\alpha \ \psi_{j,k}^\hbar| \ \| & < & C \\ \| \ \left(\frac{\nabla f(x)}{f(x)}\right)^\alpha \ \psi_{j,k}^\hbar \ \| & < & \hbar_j^{|\alpha|/2} C \end{array} \tag{2.5}$$

with $\hbar_j \ = \ \hbar\mu_j^{-1/2}$.

Proof.

Let us recall that it is well known, (see [He-Sj1]), that

$$\forall \, k \ \leq \ N \, , \quad \mu_j^{-1}\nu_{j,k}^\hbar \ = \ 1 \ + \ \mathbf{O}(\hbar_j) \ .$$

It is clear also that

$$\left[\hbar_j^2 D_x^2 \ + \ f^{2/(2+a)}(x) \ - \ \mu_j^{-1}\nu_{j,k}^\hbar\right] \psi_{j,k}^\hbar(x) \ = \ 0 \ . \tag{2.6}$$

We shall need the following inequality, that we can derive easily from (2.6) and the Agmon estimate (see [He-Sj1]) : $\forall\, \varepsilon \in]0,1[$,

$$\begin{array}{c}\varepsilon \int \left[f^{2/(2+a)}(x) - \mu_j^{-1}\nu^{\hbar}_{j,k}\right]_+ e^{2(1-\varepsilon)^{1/2}d_{j,k}(x)/\hbar_j}|\psi^{\hbar}_{j,k}(x)|^2\, dx \;\leq \\ \int \left[f^{2/(2+a)}(x) - \mu_j^{-1}\nu^{\hbar}_{j,k}\right]_- |\psi^{\hbar}_{j,k}(x)|^2\, dx\ ,\end{array} \tag{2.7}$$

where $d_{j,k}$ is the Agmon distance associated to the metric $\left[f^{2/(2+a)}(x) - \mu_j^{-1}\nu^{\hbar}_{j,k}\right]_+ dx^2$.

Let us prove the lemma for $|\alpha| = 1$.

As $\displaystyle\int \left[\hbar_j^2|D_x\, \psi^{\hbar}_{j,k}(x)|^2 + (f^{2/(2+a)}(x) - \mu_j^{-1}\nu^{\hbar}_{j,k})|\psi^{\hbar}_{j,k}(x)|^2\right] dx \;=\; 0$,
$\mu_j^{-1}\nu^{\hbar}_{j,k} - 1 \;=\; \mathbf{O}(\hbar_j)$, and $f^{2/(2+a)}(x) - 1 \;>\; 0$,
we get that $\hbar_j \|\, |D_x\, \psi^{\hbar}_{j,k}(x)|\, \|^2 \;\leq\; C$.

Furthermore, we use that $C^{-1}|\nabla f(x)|^2 \;\leq\; f^{2/(2+a)}(x) - 1 \;\leq C|\nabla f(x)|^2$, for $|x| \;\leq\; C^{-1}$, the exponential decreasing (in $\hbar_j$) of $\psi^{\hbar}_{j,k}$ given by (2.7) and the boundness of $|\nabla f(x)|/f(x)$ to get

$$\|\frac{|\nabla f(x)|}{f(x)}\psi^{\hbar}_{j,k}(x)\|^2 \;\leq\; C\int [f^{2/(2+a)}(x) - 1]\, |\psi^{\hbar}_{j,k}(x)|^2\, dx \;\leq\; \hbar_j C\ .$$

Now we study the case $|\alpha| = 2$.

If $c_0 \in]0,1]$ is large enough and $|x| \in [\hbar_j^{1/2}c_0,\ 2c_0]$, then we have

$$|x|^2/C \;\leq\; f^{2/(2+a)}(x) - \mu_j^{-1}\nu^{\hbar}_{j,k} \;\leq\; C|x|^2 \tag{2.8}$$

Therefore there exists $C_1 > 1$ such that $C_1^{-1}|x|^2 \;\leq\; d_{j,k}(x) \;\leq\; C_1|x|^2$, and then

$$|x|^2 \;\leq\; \hbar_j C e^{d_{j,k}(x)/\hbar_j}\ . \tag{2.9}$$

Then the inequality : $C^{-1}|x| \leq\; |\nabla f(x)| \;\leq\; C|x|$. together with (2.8) , (2.9) and (2.7) entail that

$$\begin{aligned}\int_{|x|\geq C_0\hbar_j^{12}} \tfrac{|\nabla f(x)|^4}{f^4(x)}|\psi^{\hbar}_{j,k}(x)|^2\, dx & \\ &\leq\; \hbar_j C \int \left[f^{2/(2+a)}(x) - \mu_j^{-1}\nu^{\hbar}_{j,k}\right]_+ e^{d_{j,k}(x)/\hbar_j}|\psi^{\hbar}_{j,k}(x)|^2\, dx \\ &\leq\; \hbar_j C \int \left[f^{2/(2+a)}(x) - \mu_j^{-1}\nu^{\hbar}_{j,k}\right]_- |\psi^{\hbar}_{j,k}(x)|^2\, dx \\ &\leq\; \hbar_j^2 C\ .\end{aligned}$$

It remains to estimate $\hbar_j^2\|D_x^\alpha \psi^{\hbar}_{j,k}(x)\|$ with $|\alpha| = 2$.
We use that $-\hbar_j^2\Delta\psi^{\hbar}_{j,k}(x) \;=\; [-f^{2/(2+a)}(x) + \mu_j^{-1}\nu^{\hbar}_{j,k}]\psi^{\hbar}_{j,k}(x)$,
and that we have proved that $\|[-f^{2/(2+a)}(x) + \mu_j^{-1}\nu^{\hbar}_{j,k}]\psi^{\hbar}_{j,k}(x)\| \;\leq\; \hbar_j C$;
so $\|D_x^\alpha\psi^{\hbar}_{j,k}(x)\| \;\leq\; C/\hbar_j$ if $|\alpha| = 2$.

We will need the following result.

Proposition 2.2 *Let* $V(y) \in C^\infty(\mathbb{R}^m)$ *such that*

$$\begin{array}{l} \exists\, s > 0\,,\ C_0 > 0\ s.t. \quad -C_0 + |y|^s/C_0 \ \leq\ V(y)\ \leq\ C_0(|y|^s+1) \\ \forall\, \alpha \in \mathbb{N}^m\,,\ (1+|y|^2)^{(s-|\alpha|)/2}\partial_y^\alpha V(y) \ \in\ L^\infty(\mathbb{R}^m)\,. \end{array} \tag{2.10}$$

If $u(y) \in L^2(\mathbb{R}^m)$ *and* $D_y^2 u(y) + V(y)u(y) \in S(\mathbb{R}^m)$ *,*
then $u \in S(\mathbb{R}^m)$ *. (* $S(\mathbb{R}^m)$ *is the Schwartz space).*

The proof comes from the fact that there exists a parametrix of $D_y^2 + V(y)$ in some class of pseudodifferential operator: see for the more general case in [Hor], or for this special case in Shubin book [Shu].

Theorem 2.3 .

Under the assumptions (1.2) and (1.5), for any fixed integer $N > 0$ *, there exists a positive constant* $h_0(N)$ *verifying : for any* $\hbar \in]0, h_0(N)[$*, for any* $k \leq N$ *and any* $j \leq N$ *such that*

$$\mu_j \ <\ \mu_1 f^{2/(2+a)}(\infty)\,,$$

there exists an eigenvalue $\lambda_{jk} \in sp_d\,(\widehat{H}^\hbar)$ *such that*

$$|\ \lambda_{jk} \ -\ \lambda_k\left(\hbar^2 D_x^2 + \mu_j f^{2/(2+a)}(x)\right)\ |\ \leq\ \hbar^2 C\,. \tag{2.11}$$

Consequently, when $k=1$ *, we have*

$$|\ \lambda_{j1} \ -\ \left[\mu_j \ +\ \hbar(\mu_j)^{1/2}\frac{tr((\partial^2 f(0))^{1/2})}{(2+a)^{1/2}}\right]\ |\ \leq\ \hbar^2 C\,. \tag{2.12}$$

Proof .

The first part of the theorem will follow if we prove that :

$$\|(\widehat{H}^\hbar \ -\ \widetilde{H}_1^\hbar)(u_{j,k}^\hbar(x,y))\ \| \ =\ \|(\widehat{H}^\hbar \ -\ \nu_{j,k}^\hbar)u_{j,k}^\hbar(x,y)\| \ =\ \mathbf{O}(\hbar^2)\,.$$

Let us consider a function $\chi \in C^\infty(\mathbb{R})$ such that
$\chi(t) = 1$ if $|t| \leq 1/2$ and
$\chi(t) = 0$ if $|t| > 1$.
Then $(D_y^2 + g(y))(1-\chi(|y|))\varphi_j(y) \in S(\mathbb{R}^m)$,
and Proposition 2.2 shows that $(1-\chi(|y|))\varphi_j(y) \in S(\mathbb{R}^m)$.
As $D_y^2\varphi_j(y) = (\mu_j - g(y))\varphi_j(y)$, we get that

$$\forall\, k \in \mathbb{N}\,, \quad (1+|y|)^k[|\varphi_j(y)|^2 + |D_y\varphi_j(y)|^2 + |D_y^2\varphi_j(y)|^2] \ \in\ L^1(\mathbb{R}^m)\,. \tag{2.13}$$

The quantity $(\widehat{H}^\hbar - \widetilde{H}_1^\hbar)(u_{j,k}^\hbar(x,y))$ is, by (2.4) , composed of 3 parts. According to Lemma 2.1 and the estimate (2.13), the two last parts are bounded in L^2-norm by $\hbar^2 C$, $(\mu_j \leq C)$.

To obtain a bound for the first part, we integrate by parts to get that

$$\|\frac{\nabla f(x)}{f(x)} D_x \psi_{j,k}^\hbar\|^2 \leq C \left[\|D_x^2 \psi_{j,k}^\hbar\| \times \|\frac{|\nabla f(x)|^2}{f^2(x)} \psi_{j,k}^\hbar\| + \|D_x \psi_{j,k}^\hbar\| \times \|\frac{|\nabla f(x)|}{f(x)} \psi_{j,k}^\hbar\| \right] ,$$

and then we use again Lemma 2.1. Thus : $\|\frac{\nabla f(x)}{f(x)} D_x \psi_{j,k}^\hbar\| \leq C$.

According to estimate (2.13) we have finally $\|\frac{\nabla f(x)}{f(x)} D_x (y D_y) u_{j,k}^\hbar\| \leq C$.

3 Middle energies

We are going to refine the preceding results when $a \geq 2$ and $f(\infty) = \infty$. It is possible then to get sharp localization near the μ_j's for much higher values of j's. More precisely we prove :

Theorem 3.1 . *We assume (1.5) with* $f(\infty) = \infty$ *, (1.2) with* $a \geq 2$ *and with* $g \in C^\infty(\mathbb{R}^m)$.

Let us consider j *such that* $\mu_j \leq \hbar^{-2}$;
then for any integer N *, there exists a constant* C *depending only on* N *such that, for any* $k \leq N$ *, there exists an eigenvalue* $\lambda_{jk} \in sp_d\,(\widehat{H}^\hbar)$ *verifying*

$$|\, \lambda_{jk} - \lambda_k \left(\hbar^2 D_x^2 + \mu_j f^{2/(2+a)}(x)\right) | \leq C \mu_j \hbar^2 . \tag{3.1}$$

Consequently, when $k = 1$ *, we have*

$$|\, \lambda_{j1} - \left[\mu_j + \hbar(\mu_j)^{1/2} \frac{tr((\partial^2 f(0))^{1/2})}{(2+a)^{1/2}}\right] | \leq C \mu_j \hbar^2 . \tag{3.2}$$

Proof :

Let us define the class of symbols $S(p^s(y,\eta))$, $s \in \mathbb{R}$, with $p(y,\eta) = |\eta|^2 + g(y) + 1$.

$$q(y,\eta) \in S(p^s(y,\eta)) \quad \text{iff} \quad q(y,\eta) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m)$$

and for any α and $\beta \in \mathbb{N}^m$,

$$p^{-s}(y,\eta)(|\eta|+1)^{-|\alpha|}(|y|+1)^{-|\beta|} D_\eta^\alpha D_y^\beta q(y,\eta) \in L^\infty(\mathbb{R}^{2m}) .$$

For such a symbol $q(y,\eta) \in S(p^s(y,\eta)$, we define the operator Q on $S(\mathbb{R}^m)$:

$$Qf(y) \ = \ (2\pi)^{-m} \int_{\mathbb{R}^{2m}} q(\frac{y+z}{2},\eta) e^{i(y-z)\eta} f(z) dz d\eta \ .$$

We will say that $Q \in OPS(p^s(y,\eta))$.

It is well known, (see [Hor]) that $(D_y^2 + g(y))^s \in OPS(p^s(y,\eta))$.

As $a \geq 2$, we get that $yD_y \in OPS(p(y,\eta))$, and then that $yD_y(D_y^2+g(y))^{-1} \in OPS(1)$.

Therefore $yD_y(D_y^2+g(y))^{-1}$ and $(yD_y)^2(D_y^2+g(y))^{-2}$ are bounded operator on $L^2(\mathbb{R}^m)$, and we get as a consequence the following bound :

$$\mu_j^{-1}\|yD_y\varphi_j\| \ + \ \mu_j^{-2}\|(yD_y)^2\varphi_j\| \ \leq \ C \ . \tag{3.3}$$

As in the proof of Theorem 2.3, using (3.3) instead of (2.13), we get easily that

$$\|(\widehat{H}^\hbar - \widetilde{H}^\hbar)u_{j,k}^\hbar\| \ \leq \ C[\ \hbar^2\mu_j \ + \ \hbar^3\mu_j^{3/2}] \ \leq \ C\hbar^2\mu_j \ ,$$

and then Theorem 3.1 follows.

4 An application

We consider a Schrödinger operator on $L^2(\mathbb{R}_z^d)$ with $d \geq 2$,

$$P^h \ = \ -h^2\Delta \ + \ V(z) \tag{4.1}$$

with a real and regular potential $V(z)$ satisfying

$$\begin{array}{c} V \ \in \ C^\infty(\mathbb{R}^d \ ; \ [0,+\infty[) \\ \liminf_{|z|\to\infty} \ V(z) \ > \ 0 \\ \Gamma \ = \ V^{-1}(\{0\}) \quad \text{is a regular hypersurface.} \end{array} \tag{4.2}$$

By hypersurface, we mean a submanifold of codimension 1 . Moreover we assume that Γ is connected and that there exist $m \in \mathbb{N}^\star$ and $C_0 > 0$ such that for any z verifying $d(z,\ \Gamma) \ < \ C_0^{-1}$

$$C_0^{-1}d^{2m}(z,\ \Gamma) \ \leq \ V(z) \ \leq \ C_0\ d^{2m}(z,\ \Gamma) \tag{4.3}$$

($d(E,F)$ denotes the euclidian distance between E and F) .

We choose an orientation on Γ and a unit normal vector $N(s)$ on each $s \in \Gamma$, and then, we can define the function on Γ ,

$$f(s) \ = \ \frac{1}{(2m)!}\left(N(s)\frac{\partial}{\partial s}\right)^{2m} V(s) \ , \quad \forall\, s \ \in \ \Gamma \ . \tag{4.4}$$

Then by (4.2) and (4.6), $f(s) \ > \ 0$, $\forall\, s \ \in \ \Gamma$.

Finally we assume that the function f achieves its minimum on Γ on a finite number of discrete points:

$$\Sigma_0 \ = \ f^{-1}(\{\eta_0\}) \ = \ \{s_1, \ldots,\ s_{\ell_0}\} \ , \quad if \quad \eta_0 \ = \ \min_{s\in\Gamma} \ f(s) \ , \tag{4.5}$$

and the hessian of f at each point $s_j \ \in \ \Sigma_0$ is non degenerated:

$\exists\, \eta_1 \ > \ 0 \ \ s.t.$

$$\frac{1}{2}\langle d\,(\langle df \ ; \ w\rangle) \ ; \ w\rangle(s_j) \ \geq \ \eta_1 |w(s_j)|^2 \ , \quad \forall\, w \ \in \ T\Gamma \ , \ \forall\, s_j \ \in \ \Sigma_0 \ . \tag{4.6}$$

If $g \ = \ (g_{ij})$ is the riemannian metric on Γ , then $|w(s)| \ = \ (g(w(s), w(s)))^{1/2}$. The hessian of f at each $s_j \ \in \ \Sigma_0$, is the symmetric operator on $T_{s_j}\Gamma$, $Hess(f)_{s_j}$, associated to the two-bilinear form defined on $T_{s_j}\Gamma$ by :

$$(v,w) \ \in \ (T_{s_j}\Gamma)^2 \ \rightarrow \ \frac{1}{2}\langle d\,(\langle df \ ; \ \widetilde{v}\rangle) \ ; \ \widetilde{w}\rangle(s_j) \ , \tag{4.7}$$

$\forall\, (\widetilde{v}, \widetilde{w}) \ \in (T\Gamma)^2 \ \ s.t. \ \ (\widetilde{v}(s_j), \widetilde{w}(s_j)) \ = \ (v,w)$.

$Hess(f)_{s_j}$ has $d-1$ non negative eigenvalues

$$\rho_1^2(s_j) \leq \ \ldots \ \leq \ \rho_{d-1}^2(s_j) \ , \qquad (\ \rho_j(s_j) \ > \ 0) \ .$$

In local coordinates, those eigenvalues are the ones of the symmetric matrix

$$\frac{1}{2}G^{1/2}(s_j)\left(\frac{\partial^2}{\partial x_k \partial x_\ell} f(s_j)\right)_{1\leq k,\ell\leq d-1} G^{1/2}(s_j) \ , \quad (\ G(x) \ = \ (g_{k,\ell}(x))_{1\leq k,\ell \leq d-1} \ ) \ .$$

The eigenvalues $\rho_k^2(s_j)$ do not depend on the choice of coordinates. We denote

$$Tr^+(Hess(f(s_j))) \ = \ \sum_{\ell=1}^{d-1} \rho_\ell(s_j) \ . \tag{4.8}$$

We denote by $(\mu_j)_{j\geq 1}$ the increasing sequence of the eigenvalues of the operator $-\dfrac{d^2}{dt^2} \ + \ t^{2m}$ on $L^2(\mathbb{R})$,
and by $(\varphi_j(t)\ )_{j\geq 1}$ the associated orthonormal Hilbert base of eigenfunctions.

Theorem 4.1 *Under the above assumptions, for any* $N \in \mathbb{N}^\star$, *there exist* $h_0 \in]0,1]$ *and* $C_0 > 0$ *such that, if* $\mu_j << h^{-4m/(m+1)(2m+3)}$,
and if $\alpha \in \mathbb{N}^{d-1}$ *and* $|\alpha| \leq N$,
then $\forall\, s_\ell \in \Sigma_0$, $\exists\, \lambda^h_{j\ell\alpha} \in sp_d(P^h)$ *s.t.*

$$\left| \lambda^h_{j\ell\alpha} - h^{2m/(m+1)} \left[\eta_0^{1/(m+1)} \mu_j + h^{1/(m+1)} \mu_j^{1/2} \mathcal{A}_\ell(\alpha) \right] \right|$$

$$\leq h^2 \mu_j^{2+3/2m} C_0 \; ;$$

with $\mathcal{A}_\ell(\alpha) = \dfrac{1}{\eta_0^{m/(2m+2)} (m+1)^{1/2}} \left[2\alpha\rho(s_\ell) + Tr^+(Hess(f(s_\ell))) \right]$.
$(\alpha\rho(s_\ell) = \alpha_1\rho_1(s_\ell) + \ldots \alpha_{d-1}\rho_{d-1}(s_\ell)\ )$.

Proof :

Let $\mathcal{O}_0 \subset \mathbb{R}^d$ be an open neighbourhood of $s_l \in \Sigma_0$, such that there exists $\phi \in C^\infty(\mathcal{O}_0\,;\,\mathbb{R})$ satisfying

$$\begin{array}{c} \Gamma_0 = \Gamma \cap \mathcal{O}_0 = \{z \in \mathcal{O}_0\,;\; \phi(z) = 0\}\;; \\ |\nabla\phi(z)| = 1\,, \quad \forall\, z \in \mathcal{O}_0\,. \end{array} \tag{4.9}$$

After changing $\mathcal{O}_0$ into a smaller neighbourhood if necessary, we can find $\tau \in C^\infty(\mathcal{O}_0\,;\,\mathbb{R}^{d-1})$ such that $\tau(s_l) = 0$ and $\forall\, z \in \mathcal{O}_0$,

$$\begin{array}{c} \nabla\tau_j(z).\nabla\phi(z) = 0\,, \quad \forall j = 1,\ldots,d-1 \\ \mathrm{rank}\{\nabla\tau_1(z),\ldots,\nabla\tau_{d-1}(z)\} = d-1\,. \end{array} \tag{4.10}$$

Then $(x,y) = (x_1,\ldots,x_{d-1},y) = (\tau_1,\ldots,\tau_{d-1},\phi)$ are local coordinates in $\mathcal{O}_0$ such that

$$\begin{array}{c} \Delta = |\widetilde{g}|^{-1/2} \sum_{1\leq i,j\leq d-1} \partial_{x_i} \left(|\widetilde{g}|^{1/2}\, \widetilde{g}^{ij} \partial_{x_j} \right) + |\widetilde{g}|^{-1/2} \partial_y \left(|\widetilde{g}|^{1/2} \partial_y \right) \\ V = y^{2m} \widetilde{f}(x,y) \quad \text{with} \quad \widetilde{f} \in C^\infty(\mathcal{V}_0)\;; \end{array} \tag{4.11}$$

$\mathcal{V}_0$ is an open neighbourhood of zero in $\mathbb{R}^d$,
$\widetilde{g}^{ij}(x,y) = \widetilde{g}^{ji}(x,y) \in C^\infty(\mathcal{V}_0;\,\mathbb{R})$, $|\widetilde{g}|^{-1} = det\left(\widetilde{g}^{ij}(x,y)\right) > 0$.
$x = (x_1,\ldots,x_{d-1})$ are local coordinates on Γ_0
and the metric $g = (g_{ij})$ on Γ_0 is given by

$$(g_{ij}(x))_{1\leq i,j\leq d-1} = G(x)\,, \quad \text{with} \quad (G(x))^{-1} = \left(\widetilde{g}^{ij}(x,0)\right)_{1\leq i,j\leq d-1}\,.$$

If $w \in C_0^2(\mathcal{O}_0)$ then

$$\begin{array}{rcl} P^h w &=& \widehat{P}^h u \quad \text{with} \\ u &=& |\widetilde{g}|^{1/4} w \quad \text{and} \\ \widehat{P}^h &=& -h^2 \sum_{1\le i,j\le d-1} \partial_{x_i}\left(\widetilde{g}^{ij}\partial_{x_j}\right) \;-\; h^2\partial_y^2 \;+\; V \;+\; h^2 V_0 \,, \end{array} \tag{4.12}$$

for some $V_0 \in C^\infty(\mathcal{V}_0\,;\,\mathbb{R})$.

Let us write

$$V(x,y) \;=\; y^{2m} f(x) \;+\; y^{2m+1} f_1(x) \;+\; y^{2m+2}\widetilde{f}_2(x,y) \;: \tag{4.13}$$

$f(x) \;=\; \widetilde{f}(x,0)$ and $\widetilde{f}_2 \in C^\infty(\mathcal{V}_0)$.

We perform the change of variable (2.1) and the related unitary transformation,

$$(x,y) \;\to\; (x,t) \;=\; (x, f^{1/(2(m+1))}(x) y)\,, \quad u \;\to\; v \;=\; f^{-1/(4(m+1)} u\,,$$

to get that

$$\begin{array}{rcl} \widehat{P}^h u &=& \widehat{Q}^h v \qquad \text{with} \\ \widehat{Q}^h &=& Q_0^h \;+\; t^{2m+1} f_1^0(x) \;+\; h^2 R_0 \;+\; +h^2 t R_1 \;+\; t^{2m+2}\widetilde{f}_2^0 \;: \\ Q_0^h &=& -h^2 \sum_{1\le i,j\le d-1} \partial_{x_i}\left(g^{ij}\partial_{x_j}\right) \;+\; f^{1/(m+1)}(x)\left(-h^2\partial_t^2 \;+\; t^{2m}\right) \end{array} \tag{4.14}$$

and $\quad R_0 = t a(x,t)(\partial_x f(x)\partial_x)\partial_t + b(x,t) t\partial_t +$

$$\sum_{ij} b_{ij}(x,t)\partial_{x_i} f(x)\partial_{x_j} f(x)(t\partial_t)^2 + c(x,t)\,,$$

$R_1 \;=\; \displaystyle\sum_{1\le i,j\le d-1} \partial_{x_i}\left(\alpha_{ij}(x,t)\partial_{x_j}\right)$, all coefficients are regular in a neighbourhood of the zero in $\mathbb{R}^d$.

Let μ_j be as in the theorem 4.1. We define $h_j \;=\; h^{1/(m+1)}/\mu_j^{1/2}$.
Let $\mathcal{O}_0'$ be a bounded open neighbourhood of zero in $\mathbb{R}^{d-1}$ such that $\overline{\mathcal{O}_0'} \subset \mathcal{O}_0 \cap \{(x,0)\,;\, x\in\mathbb{R}^{d-1}\}$.
We consider the Dirichlet operator on $L^2(\mathcal{O}_0')$, $H_0^{h_j}$:

$$H_0^{h_j} \;=\; -\,h_j^2 \sum_{1\le k,\ell\le d-1} \partial_{x_k}\left(g^{k\ell}(x)\partial_{x_\ell}\right) \;+\; f^{1/(m+1)}(x)\,. \tag{4.15}$$

It is well known, (see for example [He1] or [He-Sj1], that for any $\alpha \in \mathbb{N}^{d-1}$ satisfying the assumptions of the theorem 4.1, one has:

$$\exists\, \lambda^h_{j,\alpha} \in sp\,(H_0^{h_j}) \quad \text{s.t.} \quad |\lambda^h_{j,\alpha} - [\eta_0^{1/(m+1)} + h_j \mathcal{A}_l(\alpha)| \leq h_j^2 C\ ;$$

$\mathcal{A}_l(\alpha)$ is defined in theorem 4.1 in relation with our $s_l \in \Sigma_0$.
C is a constant depending only on N . We will denote by $\psi^{h_j}_{j,\alpha}(x)$ any associated eigenfunction with a L^2-norm equal to 1 . Let $\chi_0 \in C^\infty(\mathbb{R})$ such that

$$\chi_0(t) = 1 \quad \text{if} \quad |t| \leq 1/2 \quad \text{and} \quad \chi(t) = 0 \quad \text{if} \quad |t| \geq 1\ .$$

We define the following function :

$$u^h_{j,\alpha}(x,t) = h^{-1/(2m+2)} \chi_0(t/\epsilon_0) \psi^{h_j}_{j,\alpha}(x) \left[\varphi_j(h^{-1/(m+1)}t) - h^{1/(m+1)} F^h_j, (x,t)\right]\ ,$$

with

$$F^h_j(x,t) = f^0_1(x) f^{-1/(m+1)}(x) \phi_j(h^{-1/(m+1)}t),$$

where $\phi_j \in S(\mathbb{R})$ is solution of :
$-\frac{d^2}{dt^2}\phi_j(t) + (t^{2m} - \mu_j)\phi_j(t) = t^{2m+1}\varphi_j(t)$,
and $\epsilon_0 \in]0,1]$ is a small enough constant, but independent of h and j .
ϕ_j exists because μ_j is a non-degenerated eigenvalue and the related eigenfunction φ_j (see 1.8) verifies $\int_{\mathbb{R}} t^{2m+1}\varphi_j^2(t)\, dt = 0$, since it is a real even or odd function.

Using the similar estimates as in chapter 3 , one can get easily that

$$\mu_j^{-1}\|t\partial_t\varphi_j\|_{L^2(\mathbb{R})} + \mu_j^{-2}\|(t\partial_t)^2\varphi_j\|_{L^2(\mathbb{R})} \leq C$$

and $\forall\, k \in \mathbb{N}$, $\exists\, C_k > 0$ s.t. $\mu_j^{-k/2m}\|t^k\varphi_j\|_{L^2(\mathbb{R})} \leq C_k$.

It is well known that there exists $\epsilon_1 > 0$ s.t.

$|\mu_j - \mu_\ell| \geq \epsilon_1$, $\forall\, \ell \neq j$, then the inverse of $-\dfrac{d^2}{dt^2} + t^{2m} - \mu_j$ is $L^2(\mathbb{R})$-bounded by $1/\epsilon_1$, (on the orthogonal of φ_j) . So in the same way as in chapter 3 , we get also that

$$\mu_j^{-2-1/2m}\|t\partial_t\phi_j\|_{L^2(\mathbb{R})} + \mu_j^{-3-1/2m}\|(t\partial_t)^2\phi_j\|_{L^2(\mathbb{R})} \leq C$$

and $\forall\, k \in \mathbb{N}$, $\exists\, C_k > 0$ s.t. $\mu_j^{-1-(k+1)/2m}\|t^k\phi_j\|_{L^2(\mathbb{R})} \leq C_k$.
As in the proof of Theorem 3.1, we get easily that

$$\|[\widehat{Q}^h - \mu_j\lambda^h_{j,\alpha}]\chi_0(|x|/\epsilon_0)u^{h_j}_{j,\alpha}(x,t)\|_{L^2(\mathcal{O}_0)} \leq h^2\mu_j^{(4m+3)/2m}C$$

and
$| \, \|\chi_0(|x|/\epsilon_0) u^{h_j}_{j,\alpha}(x,t)\|_{L^2(\mathcal{O}_0)} \; - \; 1 \, | \; = \; \mathbf{O}(h^{1/(m+1)} \mu_j^{(2m+1)/2m}) \; = \; \circ(1)$.
So the theorem 4.1 follows easily.

Remark 4.2 *If in Theorem 4.1 we assume that* j *is also bounded by* N *, then, as in [He-Sj4], we can get a full asymptotic expansion*

$$\lambda^h_{j\ell\alpha} \; \sim \; h^{2m/(m+1)} \sum_{k=0}^{+\infty} c_{j\ell k\alpha} h^{k/(m+1)} \; ,$$

and for the related eigenfunction, a quasimode of the form

$$u^h_{j\ell\alpha}(x,t) \; \sim \; c(h) e^{-\psi(x)/h^{1/(m+1)}} \chi_0(t/\epsilon_0) \sum_{k=0}^{+\infty} h^{k/(2m+2)} a_{j\ell k\alpha}(x) \phi_{jk}(t/h^{1/(m+1)}) \; .$$

References

[Do] S. Dozias : Clustering for the spectrum of h-pseudodifferential operators with periodic flow on an energy surface. J. Funct. Anal. 145,(1997),p.296-311.

[He1] B. Helffer : Introduction to the semiclassical analysis for the Schrödinger operator and applications. *Springer lecture Notes in Math.*, n^0 1336 (1988).

[He-Ro-1] B. Helffer, D. Robert, Propriétés asymptotiques du spectre d'opérateurs pseudo-différentiels sur $\mathbb{R}^n$. Comm. in P.D.E., 7(7),(1982),p.795-882.

[He-Ro-2] B. Helffer, D. Robert, Comportement semi-classique du spectre des hamiltoniens quantiques hypoelliptiques. Annales ENS Pise IV, 9,(3),(1982),p.405-431.

[He-Sj1] B. Helffer, J. Sjoestrand, Multiple wells in the semi-classical limit. I. Comm. in P.D.E., 9, (4), (1984), p.337-408.

[He-Sj4] B. Helffer, J. Sjoestrand, Puits multiples en mecanique semi-classique. VI. Ann. Institut H. Poincaré, Phys. Theor., 46,(4),(1987),p.353-372.

[Hor] L. Hörmander, The Weyl calculus of pseudo-differential operators. Comm. Pure Appl. Math., 32, (1979), p.359-443.

[Ivr] V. Ivrii, Microlocal analysis and precise spectral asymptotic. Springer, Berlin 1998.

[Khu] D. Khuat-Duy, A semi-classical trace formula for Schroedinger operators in the case of critical energy level. J. Funct. Anal., 146,(2),(1997),p.299-351.

[Ma] A. Martinez, Développement asymptotiques et effet tunnel dans l'approximation de Born-Oppenheimer. Ann. Inst. Henri Poincaré, Vol. 49,(3),(1989),p.239-257.

[MoTr] A. Morame, F. Truc, Semiclassical Eigenvalue Asymptotics for a Schrödinger Operator with Degenerate Potential. Asymptotic Anal., 22(1), (2000), p. 39-49.

[Rob] D. Robert, Comportement asymptotique des valeurs propres d'opérateurs du type de Schrödinger à potentiel dégénéré. J. Math. Pures et Appl., 61,(1982),p.275-300.

[Roz] G. V. Rozenbljum, Asymptotics of the eigenvalues of the Schrödinger operator. Math. USSR Sbornik, (Eng. trans.), 22,(3),(1974),p.349-371.

[Sim] B. Simon, Nonclassical eigenvalue asymptotics. J. of Funct. Analysis, 53,(1983),p.84-98.

[Si1] B. Simon : Semi-classical analysis of low lying eigenvalues I. Ann. Inst. H. Poincaré, 38, (1983), p. 295-307.

[Sol] M. Z. Solomyak, Asymptotics of the spectrum of the Schrödinger operator with nonregular homogeneous potential. Math. USSR Sbornik, (Eng. trans.), 55,(1),(1986),p19-37.

[Shu] M.A. Shubin, Pseudodifferential operators and spectral theory. Springer-Verlag, Berlin 1987.

[Tru] F. Truc, Semi-classical asymptotics for magnetic bottles. Asymp. Analysis, 15,(1997),p.385-395.