# On inverse problems for the multidimensional relativistic Newton equation at fixed energy

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**Abstract.** In this paper, we consider inverse scattering and inverse boundary value problems at sufficiently large and fixed energy for the multidimensional relativistic Newton equation with an external potential $V$, $V \in C^2$. Using known results, we obtain, in particular, theorems of uniqueness.

## 1. Introduction

1.1 *Relativistic Newton equation.* Consider the Newton equation in the relativistic case (that is the Newton-Einstein equation) in an open subset $\Omega$ of $\mathbb{R}^n$, $n \geq 2$,

$$\dot{p} = -\nabla V(x),$$

(1.1)
$$p = \frac{\dot{x}}{\sqrt{1 - \frac{|\dot{x}|^2}{c^2}}}, \ \dot{p} = \frac{dp}{dt}, \ \dot{x} = \frac{dx}{dt},$$

where $V \in C^2(\bar{\Omega}, \mathbb{R})$ (i.e. there exists $\tilde{V} \in C^2(\mathbb{R}^n, \mathbb{R})$ such that $\tilde{V}$ restricted to $\bar{\Omega}$ is equal to $V$) and $x = x(t)$ is a $C^1$ function with values in $\Omega$.

By $\|V\|_{C^2}$ we denote the supremum of the set $\{|\partial_x^j V(x)| \mid x \in \Omega, \ j = (j_1, .., j_n) \in (\mathbb{N} \cup \{0\})^n, \sum_{i=1}^n j_i \leq 2\}$.

The equation (1.1) is the equation of motion of a relativistic particle of mass $m = 1$ in an external scalar potential $V$ (see [E] and, for example, Section 17 of [LL]). The potential $V$ can be, for example, an electric potential or a gravitational potential. In this equation $x$ is the position of the particle, $p$ is its impulse, $t$ is the time and $c$ is the speed of light.

For the equation (1.1) the energy

$$E = c^2\sqrt{1 + \frac{|p(t)|^2}{c^2}} + V(x(t))$$

is an integral of motion. We denote by $B_c$ the euclidean open ball whose radius is c and whose centre is 0.

In this paper we consider the equation (1.1) in two situations. We study equation (1.1) when

(1.2a) $\Omega = D$ where $D$ is a bounded strictly convex open subset of $\mathbb{R}^n, n \geq 2$, with $C^2$ boundary.

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And we study equation (1.1) when

(1.2b) $\qquad\qquad \Omega = \mathbb{R}^n$ and $|\partial_x^j V(x)| \leq \beta_{|j|}(1+|x|)^{-\alpha-|j|}, \; x \in \mathbb{R}^n,$

for $|j| \leq 2$ and some $\alpha > 1$ (here $j$ is the multiindex $j \in (\mathbb{N} \cup \{0\})^n$, $|j| = \sum_{i=1}^n j_i$ and $\beta_{|j|}$ are positive real constants).

For the equation (1.1) under condition (1.2a), we consider boundary data. For equation (1.1) under condition (1.2b), we consider scattering data.

1.2 *Boundary data.* For the equation (1.1) under condition (1.2a), one can prove that at sufficiently large energy $E$ (i.e. $E > E(\|V\|_{C^2}, D)$), the solutions $x$ of energy $E$ have the following properties (see Subsections 3.1, 3.2 and 3.3 of Section 3):

for each solution $x(t)$ there are $t_1, t_2 \in \mathbb{R}, t_1 < t_2$, such that

(1.3) $\quad x \in C^3([t_1, t_2], \mathbb{R}^n), x(t_1), x(t_2) \in \partial D, x(t) \in D$ for $t \in ]t_1, t_2[,$

$\quad x(s_1) \neq x(s_2)$ for $s_1, s_2 \in [t_1, t_2], s_1 \neq s_2;$

(1.4) for any two distinct points $q_0, q \in \bar{D}$, there is one and only one solution $x(t) = x(t, E, q_0, q)$ such that $x(0) = q_0, x(s) = q$ for some $s > 0.$

Let $(q_0, q)$ be two distinct points of $\partial D$. By $s(E, q_0, q)$ we denote the time at which $x(t, E, q_0, q)$ reaches $q$. By $k(E, q_0, q)$ we denote the velocity vector $\dot{x}(s(E, q_0, q), E, q_0, q)$. We consider $k(E, q_0, q)$, $q_0, q \in \partial D, q_0 \neq q$, as the boundary value data.

1.3 *Scattering data.* For the equation (1.1) under condition (1.2b), the following is valid (see [Y]): for any $(v_-, x_-) \in B_c \times \mathbb{R}^n$, $v_- \neq 0$, the equation (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ such that

(1.5) $\qquad\qquad\qquad x(t) = v_- t + x_- + y_-(t),$

where $\dot{y}_-(t) \to 0$, $y_-(t) \to 0$, as $t \to -\infty$; in addition for almost any $(v_-, x_-) \in B_c \times \mathbb{R}^n$, $v_- \neq 0$,

(1.6) $\qquad\qquad\qquad x(t) = v_+ t + x_+ + y_+(t),$

where $v_+ \neq 0$, $|v_+| < c$, $v_+ = a(v_-, x_-)$, $x_+ = b(v_-, x_-)$, $\dot{y}_+(t) \to 0$, $y_+(t) \to 0$, as $t \to +\infty$.

For an energy $E > c^2$, the map $S_E : \mathbb{S}_E \times \mathbb{R}^n \to \mathbb{S}_E \times \mathbb{R}^n$ (where $\mathbb{S}_E = \{v \in B_c \mid |v| = c\sqrt{1 - \left(\frac{c^2}{E}\right)^2}\}$) given by the formulas

(1.7) $\qquad\qquad\qquad v_+ = a(v_-, x_-), \; x_+ = b(v_-, x_-),$

is called the scattering map at fixed energy $E$ for the equation (1.1) under condition (1.2b). By $\mathcal{D}(S_E)$ we denote the domain of definition of $S_E$. The data $a(v_-, x_-)$, $b(v_-, x_-)$ for $(v_-, x_-) \in \mathcal{D}(S_E)$ are called the scattering data at fixed energy $E$ for the equation (1.1) under condition (1.2b).

1.4 *Inverse scattering and boundary value problems.* In the present paper, we consider the following inverse boundary value problem at fixed energy for the equation (1.1) under condition (1.2a):

$$\textbf{Problem } 1 : \text{given } k(E, q_0, q) \text{ for all } (q_0, q) \in \partial D \times \partial D,$$
$$q_0 \neq q, \text{ at fixed sufficiently large energy } E, \text{ find } V.$$

The main results of the present work include the following theorem of uniqueness for Problem 1.

**Theorem 1.1.** *At fixed $E > E(\|V\|_{C^2}, D)$, the boundary data $k(E, q, q_0)$, $(q_0, q) \in \partial D \times \partial D$, $q_0 \neq q$, uniquely determine $V$.*

Theorem 1.1 follows from a reduction of Problem 1 to the problem of determining an isotropic Riemannian metric from its hodograph and from Theorem 3.1 (see Section 3).

In the present paper, we also consider the following inverse scattering problem at fixed energy for the equation (1.1) under condition (1.2b):

$$\textbf{Problem } 2 : \text{ given } S_E \text{ at fixed energy } E, \text{ find } V.$$

The main results of the present work include the following theorem of uniqueness for Problem 2.

**Theorem 1.2.** *Let $\lambda \in \mathbb{R}^+$ and let $D$ be a bounded strictly convex open subset of $\mathbb{R}^n$ with $C^2$ boundary. Let $V_1, V_2 \in C_0^2(\mathbb{R}^n, \mathbb{R})$, $\max(\|V_1\|_{C^2}, \|V_2\|_{C^2}) \leq \lambda$, and $\mathrm{supp}(V_1) \cup \mathrm{supp}(V_2) \subseteq D$. Let $S_E^i$ be the scattering map at fixed energy $E$ subordinate to $V_i$ for $i = 1, 2$. There exists a nonnegative real constant $E(\lambda, D)$ such that for any $E > E(\lambda, D)$, $V_1 \equiv V_2$ if and only if $S_E^1 \equiv S_E^2$.*

Theorem 1.2 follows from Theorem 1.1 and Proposition 2.1.

**Remark 1.1.** Note that for $V \in C_0^2(\mathbb{R}^n, \mathbb{R})$, if $E < c^2 + \sup\{V(x) \mid x \in \mathbb{R}^n\}$ then $S_E$ does not determine uniquely $V$.

Note also that reducing Problem 1 to the problem of determining an isotropic Riemannian metric from its hodograph, one can give also stability estimates for Problem 1 under the assumptions of Theorem 1.1.

1.5 *Historical remarks.* An inverse boundary value problem at fixed energy and at high energies was studied in [GN] for the multidimensional nonrelativistic Newton equation in a bounded open strictly convex domain. In [GN] results of uniqueness and stability for the inverse boundary value problem at fixed energy are derived

from results for the problem of determining an isotropic Riemannian metric from its hodograph (for this geometrical problem, see [MR], [B] and [BG]).

Novikov [N2] studied inverse scattering for nonrelativistic multidimensional Newton equation. Novikov [N2] gave, in particular, a connection between the inverse scattering problem at fixed energy and Gerver-Nadirashvili's inverse boundary value problem at fixed energy. Theorem 1.2 of the present work is a generalization of theorem 5.2 of [N2] to the relativistic case.

Inverse scattering at high energies for the relativistic multidimensional Newton equation was studied by the author (see [J1], [J2]).

As regards analogs of Theorems 1.1, 1.2 and Proposition 2.1 for nonrelativistic quantum mechanics see [N1], [NSU], [N3] and further references therein. As regards an analog of Theorem 1.2 for relativistic quantum mechanics see [I]. As regards results given in the literature on inverse scattering in quantum mechanics at high energy limit see references given in [J2].

1.6 *Structure of the paper.* The paper is organized as follows. In Section 2, we give some properties of boundary data and scattering data and we connect the inverse scattering problem at fixed energy to the inverse boundary value problem at fixed energy. In Section 3, we obtain a theorem of uniqueness and stability (Theorem 3.1) for the inverse boundary value problem. This theorem is a generalization to relativistic case of theorem 4 of [GN].

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## 2. Scattering data and boundary value data.

2.1 *Properties of the boundary value data.* Let $D$ be a bounded strictly convex open subset of $\mathbb{R}^n$, $n \geq 2$, with $C^2$ boundary.

At fixed sufficiently large $E$ (i.e. $E > E(\|V\|_{C^2}, D) \geq c^2 + \sup_{x \in \bar{D}} V(x)$) solutions $x(t)$ of the equation (1.1) under condition (1.2a) have the following properties (see Subsections 3.1, 3.2 and 3.3 of Section 3):

for each solution $x(t)$ there are $t_1, t_2 \in \mathbb{R}, t_1 < t_2$, such that

$x \in C^3([t_1, t_2], \mathbb{R}^n), x(t_1), x(t_2) \in \partial D, x(t) \in D$ for $t \in ]t_1, t_2[$,

(2.1)　　$x(s_1) \neq x(s_2)$ for $s_1, s_2 \in [t_1, t_2], s_1 \neq s_2, \dot{x}(t_1)N(x(t_1)) < 0$

and $\dot{x}(t_2)N(x(t_2)) > 0$, where $N(x(t_i))$ is the unit outward

normal vector of $\partial D$ at $x(t_i)$ for $i = 1, 2$;

for any two points $q_0, q \in \bar{D}, q \neq q_0$, there is one and only one solution

(2.2)　$x(t) = x(t, E, q_0, q)$ such that $x(0) = q_0, x(s) = q$ for some $s > 0$;

$\dot{x}(0, E, q_0, q) \in C^1((\bar{D} \times \bar{D}) \backslash \bar{G}, \mathbb{R}^n)$, where $\bar{G}$ is the diagonal in $\bar{D} \times \bar{D}$,

(where by "$\dot{x}(0, E, q_0, q) \in C^1((\bar{D} \times \bar{D}) \backslash \bar{G}, \mathbb{R}^n)$" we mean that there exists an open neighborhood $\Omega$ of $\bar{D}$ such that $\dot{x}(0, E, q_0, q)$ is the restriction to $(\bar{D} \times \bar{D}) \backslash \bar{G}$ of a function which belongs to $C^1((\Omega \times \Omega) \backslash \Delta)$ where $\Delta$ is the diagonal of $\Omega \times \Omega$). Let $E > E(\|V\|_{C^2}, D)$. Consider the solution $x(t, E, q_0, q)$ from (2.2) for $q_0, q \in \partial D$, $q_0 \neq q$. We remind that $s = s(E, q_0, q)$ is the root of the equation

$$x(s, E, q_0, q) = q, \quad s > 0,$$

and we remind that $k(E, q_0, q) = \dot{x}(s(E, q_0, q), E, q_0, q)$. We consider $k(E, q_0, q)$, $q_0, q \in \partial D$, $q_0 \neq q$ as the boundary value data.

Let $k_0(E, q_0, q) = \dot{x}(0, E, q_0, q)$. Note that

(2.3)
$$k_0(E, q_0, q) = - k(E, q, q_0),$$
$$|k_0(E, q_0, q)| = c \sqrt{1 - \left( \frac{E - V(q_0)}{c^2} \right)^{-2}},$$

for $E > E(\|V\|_{C^2}, D)$ and $(q, q_0) \in (\partial D \times \partial D) \backslash \partial G$.

2.2 *Boundary data for the non relativistic case.* If one considers the nonrelativistic Newton equation in $D$ instead of the equation (1.1) under condition (1.2a), one obtains the existence of a constant $E'(\|V\|_{C^2}, D)$ such that the solutions $x(t)$ of the nonrelativistic Newton equation with energy $E = \frac{1}{2}|\dot{x}(t)|^2 + V(x(t))$, $E > E'(\|V\|_{C^2}, D)$, also have properties (2.1) and (2.2) (see [GN]). Hence one can define the time $s'(E, q_0, q)$ and the vector $k'(E, q_0, q)$ for $E > E'(\|V\|_{C^2}, D)$, $(q_0, q) \in (\partial D \times \partial D) \backslash \partial G$, as were defined $s(E, q_0, q)$, $k(E, q_0, q)$ for $E > E(\|V\|_{C^2}, D)$, $(q_0, q) \in (\partial D \times \partial D) \backslash \partial G$. In [GN], $s'(E, q_0, q)$ and $k'(E, q_0, q)$ for $E > E'(\|V\|_{C^2}, D)$, $(q_0, q) \in (\partial D \times \partial D) \backslash \partial G$, were taken as boundary value data for the multidimensional nonrelativistic Newton equation and [GN] obtains, in particular, that $s'(E, q_0, q)$ given for all $E > E'(\|V\|_{C^2}, D)$, $(q_0, q) \in (\partial D \times \partial D) \backslash \partial G$ uniquely determines $V$ and that $k'(E, q_0, q)$ given for all $(q_0, q) \in (\partial D \times \partial D) \backslash \partial G$ uniquely determines $V$ on $\bar{D}$ at fixed energy $E > E'(\|V\|_{C^2}, D)$.

2.3 *Properties of the scattering operator.* For equation (1.1) under condition (1.2b), the following is valid (see [Y]): for any $(v_-, x_-) \in B_c \times \mathbb{R}^d$, $v_- \neq 0$, the equation (1.1) under condition (1.2b) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^d)$ such that

(2.4)
$$x(t) = v_- t + x_- + y_-(t),$$

where $\dot{y}_-(t) \to 0$, $y_-(t) \to 0$, as $t \to -\infty$; in addition for almost any $(v_-, x_-) \in B_c \times \mathbb{R}^d$, $v_- \neq 0$,

(2.5)
$$x(t) = v_+ t + x_+ + y_+(t),$$

where $v_+ \neq 0$, $|v_+| < c$, $v_+ = a(v_-, x_-)$, $x_+ = b(v_-, x_-)$, $\dot{y}_+(t) \to 0$, $y_+(t) \to 0$, as $t \to +\infty$.

The map $S : B_c \times \mathbb{R}^d \to B_c \times \mathbb{R}^d$ given by the formulas

(2.6) $$v_+ = a(v_-, x_-), \ x_+ = b(v_-, x_-)$$

is called the scattering map for the equation (1.1) under condition (1.2b). The functions $a(v_-, x_-)$, $b(v_-, x_-)$ are called the scattering data for the equation (1.1) under condition (1.2b).

By $\mathcal{D}(S)$ we denote the domain of definition of $S$; by $\mathcal{R}(S)$ we denote the range of $S$ (by definition, if $(v_-, x_-) \in \mathcal{D}(S)$, then $v_- \neq 0$ and $a(v_-, x_-) \neq 0$).

The map $S$ has the following simple properties (see [Y]): for any $(v, x) \in B_c \times \mathbb{R}^d$, $(v, x) \in \mathcal{D}(S)$ if and only if $(-v, x) \in \mathcal{R}(S)$; $\mathcal{D}(S)$ is an open set of $B_c \times \mathbb{R}^d$ and $\mathrm{Mes}((B_c \times \mathbb{R}^d) \backslash \mathcal{D}(S)) = 0$ for the Lebesgue measure on $B_c \times \mathbb{R}^d$ induced by the Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}^d$; the map $S : \mathcal{D}(S) \to \mathcal{R}(S)$ is continuous and preserves the element of volume; for any $(v, x) \in \mathcal{D}(S)$, $a(v, x)^2 = v^2$.

The map $S$ restricted to

$$\Sigma_E = \{(v_-, x_-) \in B_c \times \mathbb{R}^d \mid |v_-| = c\sqrt{1 - \left(\frac{c^2}{E}\right)^2}\}$$

is the scattering operator at fixed energy $E$ and is denoted by $S_E$.

We will use the fact that the map $S$ is uniquely determined by its restriction to $\mathcal{M}(S) = \mathcal{D}(S) \cap \mathcal{M}$, where

$$\mathcal{M} = \{(v_-, x_-) \in B_c \times \mathbb{R}^d | v_- \neq 0, v_- x_- = 0\}.$$

This observation is completely similar to the related observation of [N2], [J1] and is based on the fact that if $x(t)$ satisfies (1.1), then $x(t + t_0)$ also satisfies (1.1) for any $t_0 \in \mathbb{R}$. In particular, the map S at fixed energy $E$ is uniquely determined by its restriction to $\mathcal{M}_E(S) = \mathcal{D}(S) \cap \mathcal{M}_E$, where $\mathcal{M}_E = \Sigma_E \cap \mathcal{M}$.

2.4 *Inverse scattering problem and inverse boundary value problem.* Assume that

(2.7) $$V \in C_0^2(\bar{D}, \mathbb{R}).$$

We consider equation (1.1) under condition (1.2a) and equation (1.1) under condition (1.2b). We shall connect the boundary value data $k(E, q, q_0)$ for $E > E(\|V\|_{C^2}, D)$ and $(q, q_0) \in (\partial D \times \partial D) \backslash \partial G$, to the scattering data $a, b$.

**Proposition 2.1.** *Let $E > E(\|V\|_{C^2}, D)$. Under condition (2.7), the following statement is valid: $s(E, q_0, q)$, $k(E, q_0, q)$ given for all $(q, q_0) \in (\partial D \times \partial D) \backslash \partial G$, are determined uniquely by the scattering data $a(v_-, x_-)$, $b(v_-, x_-)$ given for all $(v_-, x_-) \in \mathcal{M}_E(S)$. The converse statement holds: $s(E, q_0, q)$, $k(E, q_0, q)$ given for all $(q, q_0) \in (\partial D \times \partial D) \backslash \partial G$, determine uniquely the scattering data $a(v_-, x_-)$, $b(v_-, x_-)$ for all $(v_-, x_-) \in \mathcal{M}_E(S)$.*

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*Proof of Proposition* 2.1. First of all we introduce functions $\chi$, $\tau_-$ and $\tau_+$ dependent on $D$.

For $(v, x) \in \mathbb{R}^n \backslash \{0\} \times \mathbb{R}^n$, $\chi(v, x)$ denotes the nonnegative number of points contained in the intersection of $\partial D$ with the straight line parametrized by $\mathbb{R} \to \mathbb{R}^n, t \mapsto tv + x$. As $D$ is a strictly convex open subset of $\mathbb{R}^n$ with $C^2$ boundary, $\chi(v, x) \leq 2$ for all $v, x \in \mathbb{R}^n$, $v \neq 0$.

Let $(v, x) \in \mathbb{R}^n \backslash \{0\} \times \mathbb{R}^n$. Assume that $\chi(v, x) \geq 1$. The real $\tau_-(v, x)$ denotes the smallest real number $t$ such that $\tau_-(v, x)v + x \in \partial D$, and the real $\tau_+(v, x)$ denotes the greatest real number $t$ such that $\tau_+(v, x)v + x \in \partial D$ (if $\chi(v, x) = 1$ then $\tau_-(v, x) = \tau_+(v, x)$).

*Direct statement.* Let $(q_0, q) \in (\partial D \times \partial D) \backslash \partial G$. Under conditions (2.7) and from (2.1) and (2.2), it follows that there exists a unique $(v_-, x_-) \in \mathcal{M}_E(S)$ such that

$$\chi(v_-, x_-) = 2,$$
$$q_0 = x_- + \tau_-(v_-, x_-)v_-,$$
$$q = b(v_-, x_-) + \tau_+(a(v_-, x_-), b(v_-, x_-))a(v_-, x_-).$$

In addition, $s(E, q_0, q) = \tau_+(v_-, x_-) - \tau_-(v_-, x_-)$ and $k(E, q_0, q) = a(v_-, x_-)$.

*Converse statement.* Let $(v_-, x_-) \in \mathcal{M}_E(S)$. Under conditions (2.7), if $\chi(v_-, x_-) \leq 1$ then $(a(p_-, x_-), b(p_-, x_-)) = (p_-, x_-)$.

Assume that $\chi(v_-, x_-) = 2$. Let

$$q_0 = x_- + \tau_-(v_-, x_-)v_-.$$

From (2.1) and (2.2) it follows that there is one and only one solution of the equation

(2.8) $$-k(E, q, q_0) = v_-, \ q \in \partial D, \ q \neq q_0.$$

We denote by $q(v_-, x_-)$ the unique solution of (2.8). Hence we obtain

$$a(v_-, x_-) = k(E, q_0, q(v_-, x_-)),$$
$$b(v_-, x_-) = q(v_-, x_-) - k(E, q_0, q(v_-, x_-))$$
$$\times (s(E, q_0, q(v_-, x_-)) + \tau_-(v_-, x_-)).$$

Proposition 2.1 is proved. $\square$

For a more complete discussion about connection between scattering data and boundary value data, see [N2] considering the non relativistic Newton equation.


## 3. Inverse boundary value problem.

In this Section, Problem 1 of Introduction is studied. Following [GN], we reduce the inverse boundary value problem to the problem of determining an isotropic Riemannian metric from its hodograph.

3.1 *Hamiltonian system.* Let $E > c^2 + \sup_{x \in \bar{D}} V(x)$. Take $\tilde{V} \in C^2(\mathbb{R}^n, \mathbb{R})$ such that $\tilde{V}_{|\bar{D}} \equiv V$. We shall still denote $\tilde{V}$ by $V$. Take an open neighborhood $\Omega$ of $\bar{D}$ such that $E > c^2 + \sup_{x \in \Omega} V(x)$. The equation (1.1) in $\Omega$ is the Euler-Lagrange equation for the Lagrangian $L$ defined by $L(\dot{x}, x) = -c^2 \sqrt{1 - \frac{\dot{x}^2}{c^2}} - V(x)$, $\dot{x} \in B_c$ and $x \in \Omega$. The Hamiltonian $H$ associated to the Lagrangian $L$ by Legendre's transform (with respect to $\dot{x}$) is $H(p, x) = c^2 \sqrt{1 + \frac{p^2}{c^2}} + V(x)$ where $p \in \mathbb{R}^n$ and $x \in \Omega$. Then equation (1.1) in $\Omega$ is equivalent to the Hamilton's equation

(3.1)
$$\dot{x} = \frac{\partial H}{\partial p}(p, x),$$
$$\dot{p} = -\frac{\partial H}{\partial x}(p, x).$$

3.2 *Maupertuis's principle.* In this subsection we apply the Maupertuis's principle to the Hamiltonian system (3.1).

Let $(p(t), x(t))$, $t \in [t_1, t_2]$, be a solution of (3.1). Let $\gamma(t) = (p(t), x(t), t)$, $t \in [t_1, t_2]$. Then

(3.2)
$\gamma$ is a critical point of the functional $J$ defined by
$$J(\gamma') = \int_{\gamma'} p dx - H(p, x) dt \text{ on the set of the } C^1 \text{ functions}$$
$\gamma' : [t_1, t_2] \to \mathbb{R} \times \Omega \times [t_1, t_2]$, $t \mapsto (p'(t), x'(t), t)$
with boundary conditions $x'(t_1) = x(t_1)$ and $x'(t_2) = x(t_2)$.

Let $\Sigma$ denote the $2n-1$-dimensional smooth manifold $\{(p, x) \in \mathbb{R}^n \times \Omega \mid H(p, x) = E\}$. From (3.2), it follows that

(3.3)
for any $(p(t), x(t))$, $t \in [t_1, t_2]$, solution of (3.1) with energy $E$
and for any strictly increasing $C^1$ function $\phi$ from some closed interval
$[t_-, t_+]$ of $\mathbb{R}$ onto $[t_1, t_2]$, the $C^1$ map $\bar{\gamma}$ defined by
$\bar{\gamma}(t) = (p(\phi(t)), x(\phi(t)))$, $t \in [t_-, t_+]$, is a critical point for the functional
$\bar{J}$ defined by $\bar{J}(\gamma') = \int_{\gamma'} p dx$ on the set of the $C^1$ functions
$\gamma' : [t_-, t_+] \to \Sigma$, $t \mapsto (p'(t), x'(t))$ with boundary conditions
$x'(t_-) = x(t_1)$ and $x'(t_+) = x(t_2)$.

Let $y \in C^2([t_1, t_2], \Omega)$ be such that $\dot{y}(t) \neq 0$, $t \in [t_1, t_2]$. Let $\phi_y$ be the strictly increasing $C^1$ function from $[t_1, t_+]$ ($t_+ > 0$) onto $[t_1, t_2]$ defined by $\phi_y(t_1) = t_1$ and $H(\frac{\partial L}{\partial \dot{x}}(\dot{\phi}(t)\dot{y}(\phi(t)), y(\phi(t))), y(\phi(t))) = E$, $t \in [t_1, t_+]$, i.e. $\phi_y$ is the function

which satisfies the ordinary differential equation $\dot{\phi}_y(t) = c^2 \frac{\sqrt{1-(\frac{E-V(y(\phi_y(t)))}{c^2})^{-2}}}{|\dot{y}(\phi_y(t))|}$,
$t \in [t_1, t_+]$, with initial datum $\phi_y(t_1) = t_1$. Let $\bar{\gamma}(t) = (\frac{\partial L}{\partial \dot{x}}(\dot{\phi}(\phi^{-1}(t))\dot{y}(t), y(t)),$
$y(t))$, $t \in [t_1, t_2]$. Then, $\bar{J}(\bar{\gamma}(t)) = \int_{t_1}^{t_+} \frac{|\dot{\phi}(t)\dot{y}(\phi(t))|}{\sqrt{1-\frac{\dot{y}(\phi(t))^2\dot{\phi}(t)^2}{c^2}}}|\dot{\phi}(t)\dot{y}(\phi(t))|dt$. Hence, using
that $H(\bar{\gamma}(t)) = E$, $t \in [t_1, t_2]$, we obtain that

$$
\bar{J}(\bar{\gamma}(t)) = \int_{t_1}^{t_+} r_{V,E}(y(\phi(t))|\dot{y}(\phi(t))|\dot{\phi}(t)dt
$$

(3.4)
$$
= \int_{t_1}^{t_2} r_{V,E}(y(t))|\dot{y}(t)|dt
$$

where $r_{V,E}(x) = c\sqrt{\left(\frac{E-V(x)}{c^2}\right)^2 - 1}$, $x \in \Omega$.

From (3.3) and (3.4), it follows that if $x(t)$, $t \in [t_1, t_2]$, is a solution of (1.1) in $\Omega$ with energy $E$, then $x(t)$ is a critical point of the functional $l(y) = \int_{t_1}^{t_2} r_{V,E}(y(t))|\dot{y}(t)|dt$ defined on the set of the functions $y \in C^1([t_1, t_2], \Omega)$ with boundary conditions $y(t_1) = x(t_1)$ and $y(t_2) = x(t_2)$ (Maupertuis's principle). As $l(y)$ is the Riemannian length of the curve parametrized by $y \in C^1([t_1, t_2], \Omega)$ for the Riemannian metric $r_{V,E}(x)|dx|$ in $\Omega$, one obtains that if $x(t)$, $t \in [t_1, t_2]$, is a solution of (1.1) with energy $E$, then $x(t)$ composed with its parametrization by arclength (for the Riemannian metric $r_{V,E}(x)|dx|$ in $\Omega$) gives a geodesic of the Riemannian metric $r_{V,E}(x)|dx|$ in $\Omega$.

For any solution $x : [0, t_+] \to \Omega$ of equation (1.1) in $\Omega$ with energy $E$, the parametrization by arclength of $x(t)$ is given by the strictly increasing $C^2$ function $\psi_x$ from $[0, t'_+]$ ($t'_+ > 0$) onto $[0, t_+]$ defined by the ordinary differential equation $\dot{\psi}_x(t) = \frac{\frac{E-V(x(\psi_x(t)))}{c^4}}{\left(\frac{E-V(x(\psi_x(t)))}{c^2}\right)^2 - 1}$ with initial datum $\psi_x(0) = 0$.

Applying Maupertuis's principle we obtained the following Lemma.

**Lemma 3.1.** *Under the assumption $E > c^2 + \sup_{x \in \Omega} V(x)$ the following statement is valid: for any solution $x : [0, t_+] \to \Omega$ of equation (1.1) in $\Omega$ with energy $E$, the map $y : [0, t'_+] \to \Omega$ defined by $y(t) = x(\psi_x(t))$, $t \in [0, t'_+]$, is a geodesic of the Riemannian metric $r_{V,E}(y)|dy|$ in $\Omega$ which satisfies $r_{V,E}(y)|\dot{y}| \equiv 1$.*

We obtain, in particular, that trajectories $\{x(t)\}$ of the multidimensional relativistic Newton equation in $\Omega$ with energy $E$ coincide with the geodesics of Riemannian metric $r_{V,E}(x)|dx|$ in $\Omega$ where $|dx|$ is the canonical euclidean metric on $\Omega$. (In connection with the Maupertuis's principle and analog of Lemma 3.1 for the Newton equation in the nonrelativistic case, see for example Section 45 of [A].)

3.3 *Simple metrics.* We recall the definition of a simple metric $g$ in a bounded open subset $U$ of $\mathbb{R}^n$ with $C^2$ boundary (denoted by $\partial U$) (see for example [SU]).

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Let $U$ be a bounded open subset of $\mathbb{R}^n$ with $C^2$ boundary (denoted by $\partial U$) and let $g$ be a $C^2$ Riemannian metric in $\bar{U}$. For $x \in \partial U$ the second fundamental form $\Pi$ (with respect to $g$) of the boundary at $x$ is defined on the tangent space $T_x(\partial U)$ of $\partial U$ at $x$ by the formula

$$\Pi(\zeta) = g_x(\nabla_\zeta N(x), \zeta)$$

where $\zeta \in T_x(\partial U)$ and $N(x)$ denotes the unit outward normal vector to the boundary at $x$ $(g_x(N(x), N(x)) = 1)$, and where $\nabla N$ denotes the covariant derivative of the vector field $N$ with respect to the Levi-Civita connection of the metric $g$.

We say that $g$ is *simple* in $\bar{U}$, if the second fundamental form is positive definite at every point $x \in \partial U$ and every two points $x, y \in \bar{U}$ are joint by an unique geodesic smoothly depending on $x$ and $y$. The latter means that the mapping $exp_x : exp_x^{-1}(\bar{U}) \subseteq T_x\bar{U} \to \bar{U}$ is a diffeomorphism for any $x \in \bar{U}$, where $exp_x(v)$ denotes the point which is reached at time 1 by the geodesic in $\bar{U}$ which starts at $x$ with the velocity $v$ at time 0 ($T_x\bar{U}$ denotes the tangent space of $\bar{U}$ at the point $x$).

As it was mentioned in [SU], if a Riemannian metric $g$ is close enough to a fixed simple metric $g_0$ in $C^2(\bar{U})$, then $g$ is also simple.

Here, as $D$ is assumed to be a bounded strictly convex open subset of $\mathbb{R}^n$ with $C^2$ boundary, it follows that the euclidean metric $|dx|$ is simple in $\bar{D}$. Hence, from the fact mentioned in [SU], it follows that there exists $E(\|V\|_{C^2}, D)$ such that for $E > E(\|V\|_{C^2}, D)$ the metric $\frac{cr_{V,E}(x)}{E}|dx| = c^2\sqrt{\left(\frac{1-\frac{V(x)}{E}}{c^2}\right)^2 - \frac{1}{E^2}}|dx|$ is also simple in $\bar{D}$.

Hence for $E > E(\|V\|_{C^2}, D)$ the metric $r_{V,E}(x)|dx|$ is simple in $\bar{D}$.

Then one can consider properties (2.1) and (2.2) as consequences of Lemma 3.1 and the fact that the metric $r_{V,E}(x)|dx|$ is simple in $\bar{D}$.

Let $l_{V,E}$ denote the distance on $\bar{D}$ induced by the Riemannian metric $r_{V,E}(x)|dx|$.

3.4 *Properties of $l_{V,E}$ at fixed and sufficiently large energy $E$.* Let $E > E(\|V\|_{C^2}, D)$. From properties (2.1) and (2.2) (or from the fact that $r_{V,E}(x)|dx|$ is simple), it follows that

(3.5) $\qquad\qquad l_{V,E} \in C(\bar{D} \times \bar{D}, \mathbb{R})$,

(3.6) $\qquad\qquad l_{V,E} \in C^2((\bar{D} \times \bar{D})\backslash \bar{G}, \mathbb{R})$,

(3.7) $\qquad\qquad \max(|\frac{\partial l_{V,E}}{\partial x_i}(\zeta, x)|, |\frac{\partial l_{V,E}}{\partial \zeta_i}(\zeta, x)|) \leq C_1$,

(3.8) $\qquad\qquad |\frac{\partial^2 l_{V,E}}{\partial \zeta_i \partial x_j}(\zeta, x)| \leq \frac{C_2}{|\zeta - x|}$,

for $(\zeta, x) \in (\bar{D} \times \bar{D})\backslash \bar{G}$, $\zeta = (\zeta_1, .., \zeta_d)$, $x = (x_1, .., x_d)$, and $i = 1..n$, $j = 1..n$, and where $C_1$ and $C_2$ depend on $V$ and $D$; the map $\nu_{V,E} : \partial D \times D \to \mathbb{S}^{n-1}$, defined

by

$$(3.9) \qquad \nu_{V,E}(\zeta, x) = \frac{-1}{r_{V,E}(x)} \left( \frac{\partial l_{V,E}}{\partial x_1}(\zeta, x), .., \frac{\partial l_{V,E}}{\partial x_n}(\zeta, x) \right)$$

has the following properties:

$$(3.10a) \qquad \nu_{V,E} \in C^1(\partial D \times D, \mathbb{S}^{n-1}),$$

(3.10b) 
$$\text{the map } \nu_{V,E,x} : \partial D \to \mathbb{S}^{n-1}, \ \zeta \to \nu_{V,E}(\zeta, x)$$
$$\text{is a } C^1 \text{ orientation preserving diffeomorphism from } \partial D \text{ onto } \mathbb{S}^{n-1}$$

for $x \in D$ (where we choose the canonical orientation of $\mathbb{S}^{n-1}$ and the orientation of $\partial D$ given by the canonical orientation of $\mathbb{R}^n$ and the unit outward normal vector),

$$(3.10c) \qquad \begin{aligned} \nu_{V,E}(\zeta, x) &= \frac{k_0(E, x, \zeta)}{|k_0(E, x, \zeta)|} \\ &= -\frac{k(E, \zeta, x)}{|k(E, \zeta, x)|}, \end{aligned}$$

for $(\zeta, x) \in \partial D \times D$. Note that from (2.3), (3.9) and (3.10c) one obtains

$$(3.11) \qquad \left( \frac{\partial l_{V,E}}{\partial x_1}(\zeta, x), .., \frac{\partial l_{V,E}}{\partial x_n}(\zeta, x) \right) = \frac{k(E, \zeta, x)}{\sqrt{1 - \frac{k(E,\zeta,x)^2}{c^2}}},$$

for $(\zeta, x) \in \partial D \times D$.

3.5 *Determination of an isotropic Riemannian metric.* We consider the following geometrical problem :

at fixed energy $E > E(\|V\|_{C^2}, D)$, does $l_{V,E}(\zeta, x)$, given for all $(\zeta, x) \in \partial D \times \partial D$, determine uniquely $r_{V,E}$ on $\bar{D}$ ?

Muhometov-Romanov [MR], Beylkin [B] and Bernstein-Gerver [BG] study the question of determining an isotropic Riemannian metric from its hodograph. Results in [B] and [BG] are obtained with smoothness conditions that are too strong so that one could apply these results to our problem. Therefore, for sake of consistency, we give results (Lemma 3.2 and Theorem 3.1) that already appear with stronger smoothness conditions in [B] and [BG].

We denote by $\omega_{0,V}$ the $n-1$ differential form on $\partial D \times D$ obtained in the following manner:

- for $x \in D$, let $\omega_{V,x}$ be the pull-back of $\omega_0$ by $\nu_{V,E,x}$ where $\omega_0$ denotes the canonical orientation form on $\mathbb{S}^{n-1}$ (i.e. $\omega_0(\zeta)(v_1, .., v_{n-1}) = \det(\zeta, v_1, .., v_{n-1})$, for $\zeta \in \mathbb{S}^{n-1}$ and $v_1, .., v_{n-1} \in T_\zeta \mathbb{S}^{n-1}$),

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- for $(\zeta, x) \in \partial D \times D$ and for any $(v_1, .., v_{n-1}) \in T_{(\zeta,x)}(\partial D \times D)$,

$$\omega_{0,V}(\zeta, x)(v_1, .., v_{n-1}) = \omega_{V,x}(\zeta)(\sigma'_{(\zeta,x)}(v_1), .., \sigma'_{(\zeta,x)}(v_{n-1})),$$

where $\sigma : \partial D \times D \to \partial D$, $(\zeta', x') \mapsto \zeta'$, and $\sigma'_{(\zeta,x)}$ denotes the derivative (linear part) of $\sigma$ at $(\zeta, x)$.

From smoothness of $\nu_{V,E}$, $\sigma$ and $\omega_0$, it follows that $\omega_{0,V}$ is a continuous $n-1$ form on $\partial D \times D$.

Now let $\lambda \in \mathbb{R}^+$ and $V_1, V_2 \in C^2(\bar{D}, \mathbb{R})$ such that $\max(\|V_1\|_{C^2}, \|V_2\|_{C^2}) \leq \lambda$. Let $E > E(\lambda, D)$.

Consider the differential forms $\Phi_0$ on $(\partial D \times \partial D) \backslash \bar{G}$ and $\Phi_1$ on $(\partial D \times \bar{D}) \backslash \bar{G}$ defined by

$$\Phi_0(\zeta, x) = -(-1)^{\frac{n(n+1)}{2}} d_x(l_{V_2,E} - l_{V_1,E})(\zeta, x) \wedge d_\zeta(l_{V_2,E} - l_{V_1,E})(\zeta, x)$$

(3.12)
$$\wedge \sum_{p+q=n-2} (dd_\zeta l_{V_1,E}(\zeta, x))^p \wedge (dd_\zeta l_{V_2,E}(\zeta, x))^q,$$

for $(\zeta, x) \in (\partial D \times \partial D) \backslash \bar{G}$, where $d = d_\zeta + d_x$,

$$\Phi_1(\zeta, x) = -(-1)^{\frac{n(n-1)}{2}} \left[ d_x l_{V_1,E}(\zeta, x) \wedge (dd_\zeta l_{V_1,E}(\zeta, x))^{n-1} \right.$$
$$+ d_x l_{V_2,E}(\zeta, x) \wedge (dd_\zeta l_{V_2,E}(\zeta, x))^{n-1} - d_x l_{V_1,E}(\zeta, x) \wedge (dd_\zeta l_{V_2,E}(\zeta, x))^{n-1}$$

(3.13)
$$\left. -d_x l_{V_2,E}(\zeta, x) \wedge (dd_\zeta l_{V_1,E}(\zeta, x))^{n-1} \right],$$

for $(\zeta, x) \in (\partial D \times \bar{D}) \backslash \bar{G}$, where $d = d_\zeta + d_x$.

From (3.6), (3.7) and (3.8), it follows that $\Phi_0$ is continuous on $(\partial D \times \partial D) \backslash \bar{G}$ and integrable on $\partial D \times \partial D$ and $\Phi_1$ is continuous on $(\partial D \times \bar{D}) \backslash \bar{G}$ and integrable on $\partial D \times \bar{D}$.

**Lemma 3.2.** Let $\lambda \in \mathbb{R}^+$ and $E > E(\lambda, D)$. Let $V_1, V_2 \in C^2(\bar{D}, \mathbb{R})$ such that $\max(\|V_1\|_{C^2}, \|V_2\|_{C^2}) \leq \lambda$. The following equalities are valid:

(3.14)
$$\int_{\partial D \times \partial D} \Phi_0 = \int_{\partial D \times \bar{D}} \Phi_1;$$

$$\frac{1}{(n-1)!} \Phi_1(\zeta, x) = (r_{V_1,E}(x)^n \omega_{0,V_1}(\zeta, x) + r_{V_2,E}(x)^n \omega_{0,V_2}(\zeta, x)$$

(3.15)
$$- \nabla_x l_{V_1,E}(\zeta, x) \nabla_x l_{V_2,E}(\zeta, x)$$
$$\times \left( r_{V_1,E}(x)^{n-2} \omega_{0,V_1}(\zeta, x) + r_{V_2,E}(x)^{n-2} \omega_{0,V_2}(\zeta, x) \right))$$
$$\wedge dx_1 \wedge .. \wedge dx_n,$$

for $(\zeta, x) \in \partial D \times D$, where $\nabla_x l_{V_i,E}(\zeta, x) = (\frac{\partial l_{V_i,E}}{\partial x_1}(\zeta, x), .., \frac{\partial l_{V_i,E}}{\partial x_n}(\zeta, x))$ for $(\zeta, x) \in \partial D \times D$ and $i = 1, 2$.

Equality (3.14) follows from regularization and Stokes' formula. Using Lemma 3.2, we obtain the following Theorem of uniqueness and stability.

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**Theorem 3.1.** *Let* $\lambda \in \mathbb{R}^+$ *and* $E > E(\lambda, D)$. *Let* $V_1, V_2 \in C^2(\bar{D}, \mathbb{R})$ *such that* $\max(\|V_1\|_{C^2}, \|V_2\|_{C^2}) \le \lambda$. *The following estimate is valid:*

$$\int_D \left( r_{V_1, E}(x) - r_{V_2, E}(x) \right) \left( r_{V_1, E}(x)^{n-1} - r_{V_2, E}(x)^{n-1} \right) dx \le$$

(3.16)
$$\frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}(n-1)!} \int_{\partial D \times \partial D} \Phi_0.$$

Note that for $V_1, V_2 \in C^3$ and $\partial D \in C^\infty$ Theorem 3.1 follows directly from the stability estimate of [B] and [BG] for the problem of determining an isotropic Riemannian metric from its hodographs. Similar Remarks are also valid for Lemma 3.2. For $V_1, V_2 \in C^2$ and $\partial D \in C^2$ an estimate similar to (3.16) follows directly from a stability estimate of [MR] for the problem of determining an isotropic Riemannian metric from its hodographs. As we have followed Gerver-Nadirashvili's framework [GN], we have chosen to extend the related results obtained namely in [BG] and [B] to the case of less smooth metrics.

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