On inverse scattering in electromagnetic field in classical relativistic mechanics at high energies

Alexandre Jollivet

On inverse scattering in electromagnetic field in classical relativistic mechanics at high energies

Alexandre Jollivet

Abstract. We consider the multidimensional Newton-Einstein equation in static electromagnetic field

$$\dot{p} = F(x,\dot{x}),\ F(x,\dot{x}) = -\nabla V(x) + \frac{1}{c}B(x)\dot{x},$$
$$p = \frac{\dot{x}}{\sqrt{1-\frac{|\dot{x}|^2}{c^2}}},\ \dot{p} = \frac{dp}{dt},\ \dot{x} = \frac{dx}{dt},\ x \in C^1(\mathbb{R},\mathbb{R}^d), \qquad (*)$$

where $V \in C^2(\mathbb{R}^d,\mathbb{R})$, $B(x)$ is the $d\times d$ real antisymmetric matrix with elements $B_{i,k}(x) = \frac{\partial}{\partial x_i}\mathbf{A}_k(x) - \frac{\partial}{\partial x_k}\mathbf{A}_i(x)$, and $|\partial_x^j \mathbf{A}_i(x)| + |\partial_x^j V(x)| \le \beta_{|j|}(1+|x|)^{-(\alpha+|j|)}$ for $x \in \mathbb{R}^d$, $|j| \le 2$, $i = 1..d$ and some $\alpha > 1$. We give estimates and asymptotics for scattering solutions and scattering data for the equation $(*)$ for the case of small angle scattering. We show that at high energies the velocity valued component of the scattering operator uniquely determines the X-ray transforms $P\nabla V$ and $PB_{i,k}$ for $i,k = 1..d$, $i \ne k$. Applying results on inversion of the X-ray transform P we obtain that for $d \ge 2$ the velocity valued component of the scattering operator at high energies uniquely determines (V,B). In addition we show that our high energy asymptotics found for the configuration valued component of the scattering operator doesn't determine uniquely V when $d \ge 2$ and B when $d = 2$ but that it uniquely determines B when $d \ge 3$.

1 Introduction

1.1 *The Newton-Einstein equation.*
Consider the multidimensional Newton-Einstein equation in static electromagnetic field

$$\dot{p} = F(x,\dot{x}),\ F(x,\dot{x}) = -\nabla V(x) + \frac{1}{c}B(x)\dot{x},$$
$$p = \frac{\dot{x}}{\sqrt{1-\frac{|\dot{x}|^2}{c^2}}},\ \dot{p} = \frac{dp}{dt},\ \dot{x} = \frac{dx}{dt},\ x \in C^1(\mathbb{R},\mathbb{R}^d), \qquad (1.1)$$

where $V \in C^2(\mathbb{R}^d,\mathbb{R})$, $B(x)$ is the $d\times d$ real antisymmetric matrix with elements $B_{i,k}(x) = \frac{\partial}{\partial x_i}\mathbf{A}_k(x) - \frac{\partial}{\partial x_k}\mathbf{A}_i(x)$, $\mathbf{A} = (\mathbf{A}_1,\ldots,\mathbf{A}_d) \in C^2(\mathbb{R}^d,\mathbb{R}^d)$ and

$$|\partial_x^j \mathbf{A}_i(x)| + |\partial_x^j V(x)| \le \beta_{|j|}(1+|x|)^{-(\alpha+|j|)} \qquad (1.2)$$

for $x \in \mathbb{R}^d$, $|j| \le 2$, $i = 1..d$ and some $\alpha > 1$ (here j is the multiindex $j \in (\mathbb{N} \cup \{0\})^d, |j| = \sum_{n=1}^d j_n$ and $\beta_{|j|}$ are positive real constants and $B(x)\dot{x} = \Big(\sum_{l=1}^d B_{1,l}(x)\dot{x}_l, \ldots, \ \sum_{l=1}^d B_{d,l}(x)\dot{x}_l\Big)$). The equation (1.1) is an equation for $x = x(t)$ and is the equation of motion in $\mathbb{R}^d$ of a relativistic particle of mass $m = 1$ and charge $e = 1$ in an external electromagnetic field described by V and $\mathbf{A}$ (see [E] and, for example, Section 17 of [LL2]). In this equation x is the position of the particle, p is its impulse, F is the force acting on the particle, t is the time and c is the speed of light.

For the equation (1.1) the energy

$$E = c^2\sqrt{1 + \frac{|p(t)|^2}{c^2}} + V(x(t))$$

is an integral of motion. Note that the energy E does not depend on $\mathbf{A}$ because the magnetic force $(1/c)B(x)\dot{x}$ is orthogonal to the velocity $\dot{x}$ of the particle.

1.2 *Yajima's results.*

Yajima [Y] studied in dimension 3 (without loss of generality for the case of dimension $d \ge 2$) the direct scattering of relativistic particle in an external electromagnetic field described by four vector $(V(x), \mathbf{A}(x))$ where the scalar potential V and the vector potential $\mathbf{A}$ are both rapidly decreasing. We recall results of Yajima [Y] in our case. We denote by B_c the euclidean open ball whose radius is c and whose centre is 0.

Under the conditions (1.2), the following is valid (see [Y]): for any $(v_-, x_-) \in B_c \times \mathbb{R}^d$, $v_- \ne 0$, the equation (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^d)$ such that

$$x(t) = v_-t + x_- + y_-(t), \tag{1.3}$$

where $\dot{y}_-(t) \to 0$, $y_-(t) \to 0$, as $t \to -\infty$; in addition for almost any $(v_-, x_-) \in B_c \times \mathbb{R}^d$, $v_- \ne 0$,

$$x(t) = v_+t + x_+ + y_+(t), \tag{1.4}$$

where $v_+ \ne 0$, $|v_+| < c$, $v_+ = a(v_-, x_-)$, $x_+ = b(v_-, x_-)$, $\dot{y}_+(t) \to 0$, $y_+(t) \to 0$, as $t \to +\infty$.

The map $S : B_c \times \mathbb{R}^d \to B_c \times \mathbb{R}^d$ given by the formulas

$$v_+ = a(v_-, x_-),\ x_+ = b(v_-, x_-) \tag{1.5}$$

is called the scattering map for the equation (1.1). The functions $a(v_-, x_-)$, $b(v_-, x_-)$ are called the scattering data for the equation (1.1).

By $\mathcal{D}(S)$ we denote the domain of definition of S; by $\mathcal{R}(S)$ we denote the range of S (by definition, if $(v_-, x_-) \in \mathcal{D}(S)$, then $v_- \ne 0$ and $a(v_-, x_-) \ne 0$).

Under the conditions (1.2), the map S has the following simple properties (see [Y]): $\mathcal{D}(S)$ is an open subset of $B_c \times \mathbb{R}^d$ and $\mathrm{Mes}((B_c \times \mathbb{R}^d)\backslash\mathcal{D}(S)) = 0$ for the Lebesgue measure on $B_c \times \mathbb{R}^d$ induced by the Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}^d$; the map $S : \mathcal{D}(S) \to \mathcal{R}(S)$ is continuous and preserves the element of volume, $a(v_-, x_-)^2 = v_-^2$.

1.3 *A representation of the scattering data.*
If $V(x) \equiv 0$ and $B(x) \equiv 0$, then $a(v_-, x_-) = v_-$, $b(v_-, x_-) = x_-$, $(v_-, x_-) \in B_c \times \mathbb{R}^d$, $v_- \neq 0$. Therefore for $a(v_-, x_-)$, $b(v_-, x_-)$ we will use the following representation

$$\begin{aligned} a(v_-, x_-) =& v_- + a_{sc}(v_-, x_-) \\ b(v_-, x_-) =& x_- + b_{sc}(v_-, x_-) \end{aligned} \qquad (v_-, x_-) \in \mathcal{D}(S). \tag{1.6}$$

We will use the fact that, under the conditions (1.2), the map S is uniquely determined by its restriction to $\mathcal{M}(S) = \mathcal{D}(S) \cap \mathcal{M}$, where

$$\mathcal{M} = \{(v_-, x_-) \in B_c \times \mathbb{R}^d | v_- \neq 0, v_- x_- = 0\}.$$

This observation is completely similar to the related observation of [No1], [Jo] and is based on the fact that if $x(t)$ satisfies (1.1), then $x(t + t_0)$ also satisfies (1.1) for any $t_0 \in \mathbb{R}$.

1.4 *X-ray transform.*
Consider

$$T\mathbb{S}^{d-1} = \{(\theta, x) | \theta \in \mathbb{S}^{d-1}, x \in \mathbb{R}^d, \theta x = 0\},$$

where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$.

Consider the X-ray transform P which maps each function f with the properties

$$f \in C(\mathbb{R}^d, \mathbb{R}^m), \ |f(x)| = O(|x|^{-\beta}), \text{ as } |x| \to \infty, \text{ for some } \beta > 1$$

into a function $Pf \in C(T\mathbb{S}^{d-1}, \mathbb{R}^m)$, where Pf is defined by

$$Pf(\theta, x) = \int_{-\infty}^{+\infty} f(t\theta + x) dt, \ (\theta, x) \in T\mathbb{S}^{d-1}.$$

Concerning the theory of the X-ray transform, the reader is referred to [GGG], [Na], [No1].

1.5 *Main results of the work.*
The main results of the present work consist in the small angle scattering estimates for the scattering data a_{sc} and b_{sc} (and scattering solutions) for the equation (1.1) and in application of these asymptotics and estimates to inverse scattering for the equation (1.1) at high energies. Our main results include, in particular, Theorem 1.1, Propositions 1.1, 1.2, formulated below in this subsection and Theorems 3.1, 3.2 given in Section 3.

Theorem 1.1. *Let the conditions* (1.2) *be valid and* $(\theta, x) \in T\mathbb{S}^{d-1}$. *Let* r *be a positive constant such that* $0 < r \le 1$, $r < c/\sqrt{2}$. *Then*

$$\lim_{\substack{s\to c\\ s<c}} \frac{s}{\sqrt{1-\frac{s^2}{c^2}}} a_{sc}(s\theta, x) = \int_{-\infty}^{+\infty} F(\tau\theta + x, c\theta)d\tau, \tag{1.7a}$$

and, in addition,

$$\left| \int_{-\infty}^{+\infty} F(\tau\theta + x, s\theta)d\tau - \frac{s}{\sqrt{1-\frac{s^2}{c^2}}} a_{sc}(s\theta, x) \right| \le \frac{C_1}{\sqrt{1+\frac{s^2}{4(c^2-s^2)}}} \tag{1.7b}$$

for $s_1 < s < c$, *where* $C_1 = C_1(c, d, \beta_0, \beta_1, \beta_2, \alpha, |x|, r)$ *and* $s_1 = s_1(c, d, \beta_1, \beta_2, \alpha, |x|, r)$ *are defined in Section* 4 (*in subsection* 4.3);

$$\lim_{\substack{s\to c\\ s<c}} \frac{s^2}{\sqrt{1-\frac{s^2}{c^2}}} b_{sc}(s\theta, x) = \int_{-\infty}^{0}\int_{-\infty}^{\tau} F(\sigma\theta + x, c\theta)d\sigma d\tau \tag{1.8a}$$
$$- \int_{0}^{+\infty}\int_{\tau}^{+\infty} F(\sigma\theta + x, c\theta)d\sigma d\tau + PV(\theta, x)\theta,$$

and, in addition,

$$\left| \frac{b_{sc}(s\theta, x)}{\sqrt{1-\frac{s^2}{c^2}}} - \frac{1}{c^2}PV(\theta, x)\theta + \frac{1}{s^2}\int_{0}^{+\infty}\int_{\tau}^{+\infty} F(\sigma\theta + x, s\theta)d\sigma d\tau \right.$$
$$\left. - \frac{1}{s^2}\int_{-\infty}^{0}\int_{-\infty}^{\tau} F(\sigma\theta + x, s\theta)d\sigma d\tau \right| \le C_2\sqrt{1-\frac{s^2}{c^2}} \tag{1.8b}$$

for $s_2 < s < c$, *and where* $C_2 = C_2(c, d, \beta_0, \beta_1, \beta_2, \alpha, |x|, r)$ *and* $s_2 = s_2(c, d, \beta_1, \beta_2, \alpha, |x|, r)$ *are defined in Section* 4 (*in subsection* 4.3).

Theorem 1.1 follows from Theorem 3.1 and Theorem 3.2 given in Section 3.

Consider the vector-functions $w_1(V, \mathbf{A}, \theta, x)$ and $w_2(V, \mathbf{A}, \theta, x)$, $(\theta, x) \in T\mathbb{S}^{d-1}$, arising in the right-hand sides of (1.7a) and (1.8a):

$$w_1(V, \mathbf{A}, \theta, x) = \int_{-\infty}^{+\infty} F(\tau\theta + x, c\theta)d\tau \tag{1.9a}$$

$$w_2(V, \mathbf{A}, \theta, x) = \int_{-\infty}^{0}\int_{-\infty}^{\tau} F(\sigma\theta + x, c\theta)d\sigma d\tau \tag{1.9b}$$
$$- \int_{0}^{+\infty}\int_{\tau}^{+\infty} F(\sigma\theta + x, c\theta)d\sigma d\tau + PV(\theta, x)\theta.$$

Remark 1.1. Using, in particular, that B is antisymmetric one can see that the vectors $w_1(V,\mathbf{A},\theta,x)$, $w_2(V,\mathbf{A},\theta,x)$ are orthogonal to θ for any $(\theta,x)\in T\mathbb{S}^{d-1}$ and any potential $(V,\mathbf{A})$ satisfying (1.2).

Let $(V,\mathbf{A})$ satisfy the conditions (1.2). Define $\tilde{w_1}(V,\mathbf{A}):\mathbb{R}^d\backslash\{0\}\times\mathbb{R}^d\to\mathbb{R}^d$ by

$$\begin{aligned}\tilde{w_1}(V,\mathbf{A})(y,x)=&-|y|\int_{-\infty}^{+\infty}\nabla V(sy+x)ds+\int_{-\infty}^{+\infty}B(sy+x)yds\\ =&|y|w_1(V,\mathbf{A},\frac{y}{|y|},x-\frac{xy}{|y|^2}y),\end{aligned}\tag{1.10}$$

for $y\in\mathbb{R}^d\backslash\{0\}$, $x\in\mathbb{R}^d$. Under the conditions (1.2), $\tilde{w_1}(V,\mathbf{A})=(\tilde{w_1}(V,\mathbf{A})_1,..,\tilde{w_1}(V,\mathbf{A})_d)\in C^1(\mathbb{R}^d\backslash\{0\}\times\mathbb{R}^d,\mathbb{R}^d)$.

Consider the d-dimensional smooth manifolds

$$\mathcal{V}_{i,k}=\{(\theta,x)\in T\mathbb{S}^{d-1}|\theta_j=0,j=1\ldots d,j\neq i,j\neq k\},\tag{1.11}$$

for $i,k=1..d$, $i\neq k$.

Proposition 1.1. *Let* $(V,\mathbf{A})\in C^2(\mathbb{R}^d,\mathbb{R})\times C^2(\mathbb{R}^d,\mathbb{R}^d)$ *satisfy* (1.2). *Then* $w_1(V,\mathbf{A},\theta,x)$ *given for all* $(\theta,x)\in T\mathbb{S}^{d-1}$ *uniquely determines* (V,B) *and the following formulas are valid:*

$$P(\nabla V)(\theta,x)=-\frac{1}{2}(w_1(V,\mathbf{A},\theta,x)+w_1(V,\mathbf{A},-\theta,x)),\tag{1.12a}$$

$$\begin{aligned}P(B_{i,k})(\theta,x)=&\frac{1}{2}\Big[\frac{\partial}{\partial y_k}(\tilde{w_1}(V,\mathbf{A}))_i(y,x)+\frac{\partial}{\partial y_k}(\tilde{w_1}(V,\mathbf{A}))_i(-y,x)\\ &-\frac{\partial}{\partial y_i}(\tilde{w_1}(V,\mathbf{A}))_k(y,x)-\frac{\partial}{\partial y_i}(\tilde{w_1}(V,\mathbf{A}))_k(-y,x)\Big]_{|y=\theta},\end{aligned}\tag{1.12b}$$

for $(\theta,x)\in T\mathbb{S}^{d-1}$, $i,k=1..d$, $i\neq k$;

$$\begin{aligned}PB_{i,k}(\theta,x)=&\theta_k\frac{1}{2}(w_1(V,\mathbf{A},\theta,x)_i-w_1(V,\mathbf{A},-\theta,x)_i)\\ &-\theta_i\frac{1}{2}(w_1(V,\mathbf{A},\theta,x)_k-w_1(V,\mathbf{A},-\theta,x)_k)\end{aligned}\tag{1.12c}$$

for $(\theta,x)\in\mathcal{V}_{i,k}$, $i,k=1..d$, $i\neq k$.

Remark 1.2. Using the formulas (1.12a), (1.12c) and methods of reconstruction of f from Pf (see [GGG], [Na], [No1]), $B_{i,k}$ and V can be reconstructed from $w_1(V,\mathbf{A},\theta,x)$ given for all $(\theta,x)\in\mathcal{V}_{i,k}$, for $i,k=1..d$, $i\neq k$.

Proposition 1.2. *Let* $(V,\mathbf{A})\in C^2(\mathbb{R}^d,\mathbb{R})\times C^2(\mathbb{R}^d,\mathbb{R}^d)$ *satisfy* (1.2). *Then*

$w_2(V, \mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$ does not determine uniquely V. For $d = 2$, $w_2(V, \mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$ does not determine uniquely B. For $d \geq 3$, $w_2(V, \mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$ uniquely determines B.

In Section 5 (see Proposition 5.3) we give formulas ((5.17a) and (5.17b)) which show that for $d = 3$, the Fourier transform of the first derivatives of B can be reconstructed from $w_2(V, \mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$ and we give a formula ((5.17c)) which shows that for $d \geq 4$ the X-ray transform of B can be reconstructed from $w_2(V, \mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$.

Proposition 1.1 and Proposition 1.2 are proved in Section 5.

From (1.7a) and (1.12) and inversion formulas for the X-ray transform P for $d \geq 2$ (see [R], [GGG], [Na], [No1]) it follows that a_{sc} determines uniquely ∇V and B at high energies. Moreover for $d \geq 2$ methods of reconstruction of f from Pf (see [R], [GGG], [Na], [No1]) permit to reconstruct ∇V and B from the velocity valued component a of the scattering map at high energies. The formula (1.8a) and Proposition 1.2 show that the first term of the asymptotics of b_{sc} doesn't determine uniquely the potential V when $d \geq 2$ and B when $d = 2$ but that it uniquely determines B when $d \geq 3$. Note that F. Nicoleau paid our attention to the fact that, in addition of Proposition 1.2, the vector function $w_2(V, \theta, x), (\theta, x) \in T\mathbb{S}^{d-1}$, uniquely determines V modulo spherical symmetric potentials when $d \geq 2$, and that $w_2(V, \theta, x), (\theta, x) \in T\mathbb{S}^{d-1}$, uniquely determines B modulo spherical symmetric magnetic fields when $d = 2$.

Remark 1.3. The condition (1.2) in all results and estimates which appear in Introduction and in Sections 2, 3, 4 can be weakened to condition (4.11) given at the end of Section 4.

1.6 *Historical remarks.*

Note that inverse scattering for the classical multidimensional Newton equation was first studied by Novikov [No1] without magnetic field (the existence and uniqueness of the scattering states, asymptotic completness and scattering map for the classical Newton equation were studied by Simon [S]). Novikov proved two formulas which link scattering data at high energies to the X-ray transform of $-\nabla V$ and V. Following Novikov's framework [No1], the author generalized these two formulas to the relativistic case without magnetic field in [Jo]. We shall follow the same way to obtain Theorem 1.1 of the present work. Note also that for the classical multidimensional Newton equation in a bounded open strictly convex domain an inverse boundary value problem at high energies was first studied in [GN].

To our knowledge the inverse scattering problem for a particle in electromagnetic field in classical and classical relativistic mechanics was not considered in the literature for the case of nonzero magnetic field B before the present article (concerning results given in the literature on this problem for $B \equiv 0$ see [No1], [Jo] and references therein). However, in quantim mechanics the inverse scattering problem for a particle in electromagnetic field with $B \not\equiv 0$ was considered, in particular, in [HN], [ER1], [I], [Ju], [ER2], [Ni], [A], [Ha] (concerning results given in the literature on this problem for $B \equiv 0$ see, in addition, [F], [EW], [No2] and references given in [No2]).

1.7 *Structure of the paper.*
Further, our paper is organized as follows. In Section 2 we transform the differential equation (1.1) with initial conditions (1.3) into a system of integral equations which takes the form $(y_-, \dot{y}_-) = A_{v_-,x_-}(y_-, \dot{y}_-)$. Then we study A_{v_-,x_-} on a suitable space and we give estimates and contraction estimates about A_{v_-,x_-} (Lemmas 2.1, 2.2, 2.3). In Section 3 we give estimates and asymptotics for the deflection $y_-(t)$ from (1.3) and for scattering data $a_{sc}(v_-, x_-)$, $b_{sc}(v_-, x_-)$ from (1.6) (Theorem 3.1 and Theorem 3.2). From these estimates and asymptotics the two formulas (1.7a) and (1.8a) will follow when the parameters c, β_m, α, d, $\hat{p}_-$, x_- are fixed and $|v_-|$ increases (where $\beta_{|j|}$, α, d are constants from (1.2), $\beta_m = \max(\beta_0, \beta_1, \beta_2)$; $\hat{p}_- = v_-/|v_-|$). In these cases $\sup|\theta(t)|$ decreases, where $\theta(t)$ denotes the angle between the vectors $\dot{x}(t) = v_- + \dot{y}_-(t)$ and v_-, and we deal with small angle scattering. Note that, under the conditions of Theorem 3.1, without additional assumptions, there is the estimate $\sup|\theta(t)| < \frac{1}{4}\pi$ and we deal with rather small angle scattering (concerning the term "small angle scattering" see [No1] and Section 20 of [LL1]). Theorem 1.1 follows from Theorem 3.1 and Theorem 3.2. In Section 4 we sketch the proof of Lemmas 2.1, 2.2, 2.3 and Theorem 3.2. Section 5 is devoted to Proofs of Proposition 1.1 and Proposition 1.2.

Acknowledgement. This work was fulfilled in the framework of Ph. D. thesis researchs under the direction of R.G. Novikov.

2 A contraction map

Let us transform the differential equation (1.1) in a system of integral equations. Consider the function $g : \mathbb{R}^d \to B_c$ defined by

$$g(x) = \frac{x}{\sqrt{1 + \frac{|x|^2}{c^2}}}$$

where $x \in \mathbb{R}^d$. One can see that g has, in particular, the following simple properties:

$$|g(x) - g(y)| \leq \sqrt{d}|x - y|, \text{ for } x, y \in \mathbb{R}^d, \tag{2.1}$$

g is an infinitely smooth diffeomorphism between $\mathbb{R}^d$ and B_c, its inverse is given by

$$\gamma(x) = \frac{x}{\sqrt{1 - \frac{|x|^2}{c^2}}}, x \in B_c.$$

Now, if x satisfies the differential equation (1.1) and the initial conditions (1.3), then x satisfies the system of integral equations

$$x(t) = v_- t + x_- + \int_{-\infty}^{t} \left[g\left(\gamma(v_-) + \int_{-\infty}^{\tau} F(x(s), \dot{x}(s)))ds \right) - v_- \right] d\tau, \quad (2.2a)$$

$$\dot{x}(t) = g(\gamma(v_-) + \int_{-\infty}^{t} F(x(s), \dot{x}(s))ds), \quad (2.2b)$$

where $F(x, \dot{x}) = -\nabla V(x) + \frac{1}{c} B(x)\dot{x},\ v_- \in B_c \backslash \{0\}$.

For $y_-(t)$ of (1.3) this system takes the form

$$(y_-(t), u_-(t)) = A_{v_-,x_-}(y_-, u_-)(t), \quad (2.3)$$

where $u_-(t) = \dot{y}_-(t)$ and

$$A_{v_-,x_-}(f, h)(t) = (A^1_{v_-,x_-}(f, h)(t), A^2_{v_-,x_-}(f, h)(t))$$

$$A^1_{v_-,x_-}(f, h)(t) = \int_{-\infty}^{t} \left[g(\gamma(v_-) + \int_{-\infty}^{\tau} F(v_- s + x_- + f(s), v_- + h(s))ds) - v_- \right] d\tau,$$

$$A^2_{v_-,x_-}(f, h)(t) = g(\gamma(v_-) + \int_{-\infty}^{t} F(v_- s + x_- + f(s), v_- + h(s))ds) - v_-,$$

for $v_- \in B_c \backslash \{0\}$.

From (2.3), (1.2) , (2.1) (applied on "x"$= \gamma(v_-) + \int_{-\infty}^{\tau} F(v_- s + x_- + y_-(s), v_- + \dot{y}_-(s))ds$ and "y"$= \gamma(v_-)$) and $y_-(t) \in C^1(\mathbb{R}, \mathbb{R}^d)$, $|y_-(t)| + |\dot{y}_-(t)| \to 0$, as $t \to -\infty$, it follows, in particular, that

$$\begin{aligned} (y_-(t), \dot{y}_-(t)) \in & C(\mathbb{R}, \mathbb{R}^d) \times C(\mathbb{R}, \mathbb{R}^d) \\ \text{and } |\dot{y}_-(t)| = O(|t|^{-\alpha}),\ & |y_-(t)| = O(|t|^{-\alpha+1}),\ \text{as } t \to -\infty, \end{aligned} \quad (2.4)$$

where $v_- \in B_c \backslash \{0\}$ and x_- are fixed.

Consider the complete metric space

$$M_{T,r} = \{(f, h) \in C(]-\infty, T], \mathbb{R}^d) \times C(]-\infty, T], \mathbb{R}^d) |\ \|(f, h)\|_T \leq r\},$$

$$\text{where } \|(f, h)\|_T = \max\left(\sup_{t \in]-\infty, T]} |h(t)|, \sup_{t \in]-\infty, T]} |f(t) - th(t)| \right) \quad (2.5)$$

(where for $T = +\infty$ we understand $]-\infty, T]$ as $]-\infty, +\infty[$). From (2.4) it follows that, at fixed $T < +\infty$,

$$(y_-(t), \dot{y}_-(t)) \in M_{T,r} \text{ for some } r \text{ depending on } y_-(t) \text{ and } T. \quad (2.6)$$

Let $z_1(c, d, \beta_1, \alpha, r_x, r)$ be defined as the root of the following equation

$$\frac{z_1}{\sqrt{1-\frac{z_1^2}{c^2}}} - \frac{2^{\alpha+5}\beta_1 d(2+r/c)}{\alpha(z_1/\sqrt{2}-r)(r_x/\sqrt{2}+1)^{\alpha}} = 0,\ z_1 \in]\sqrt{2}r, c[, \tag{2.7}$$

where r_x and r are some nonnegative numbers such that $0 < r \le 1,\ r < c/\sqrt{2}$.

Lemma 2.1. *Under the conditions* (1.2), *the following is valid: if* $(f, h) \in M_{T,r}$, $0 < r \le 1$, $r < c/\sqrt{2}$, $x_- \in \mathbb{R}^d$, $v_- \in B_c$, $|v_-| \ge z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $v_- x_- = 0$, *then*

$$\begin{aligned}\|A_{v_-,x_-}(f,h)\|_T \le &\rho_T(c, d, \beta_1, \alpha, |v_-|, |x_-|, r) \\ =&\frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}} \\ &\times \frac{2^{\alpha+2}d\sqrt{d}\beta_1(2+r/c)(|v_-|/\sqrt{2}+1-r)}{(\alpha-1)(|v_-|/\sqrt{2}-r)^2(1+|x_-|/\sqrt{2}-(|v_-|/\sqrt{2}-r)T)^{\alpha-1}}\end{aligned} \tag{2.8a}$$

for $T \le 0$,

$$\begin{aligned}\|A_{v_-,x_-}(f,h)\|_T \le &\rho(c, d, \beta_1, \alpha, |v_-|, |x_-|, r) \\ =&\frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}} \\ &\times \frac{2^{\alpha+3}d\sqrt{d}\beta_1(2+r/c)(|v_-|/\sqrt{2}+1-r)}{(\alpha-1)(|v_-|/\sqrt{2}-r)^2(1+|x_-|/\sqrt{2})^{\alpha-1}}\end{aligned} \tag{2.8b}$$

for $T \le +\infty$; *if* (f_1, h_1), $(f_2, h_2) \in M_{T,r}$, $0 < r \le 1$, $r < c/\sqrt{2}$, $|v_-| < c$, $v_- x_- = 0$, $|v_-| \ge z_1(c, d, \beta_1, \alpha, |x_-|, r)$, *then*

$$\begin{aligned}&\|A_{v_-,x_-}(f_2,h_2) - A_{v_-,x_-}(f_1,h_1)\|_T \\ &\qquad \le \lambda_T(c, d, \tilde{\beta}, \alpha, |v_-|, |x_-|, r)\|(f_2-f_1, h_2-h_1)\|_T,\end{aligned} \tag{2.9a}$$

$$\begin{aligned}\lambda_T(c, d, \tilde{\beta}, \alpha, |v_-|, |x_-|, r) =&\frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}} \\ &\times \frac{2^{\alpha+4}d^2\tilde{\beta}(1+\frac{1}{c})(\frac{|v_-|}{\sqrt{2}}+1-r)^2}{(\alpha-1)(\frac{|v_-|}{\sqrt{2}}-r)^3(1+\frac{|x_-|}{\sqrt{2}}-(\frac{|v_-|}{\sqrt{2}}-r)T)^{\alpha-1}}\end{aligned}$$

for $T \le 0$,

$$\|A_{v_-,x_-}(f_2,h_2) - A_{v_-,x_-}(f_1,h_1)\|_T \le \lambda(c, d, \tilde{\beta}, \alpha, |v_-|, |x_-|, r)\|(f_2-f_1, h_2-h_1)\|_T, \tag{2.9b}$$

$$\lambda(c,d,\tilde\beta,\alpha,|v_-|,|x_-|,r) = \frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}} \times \frac{2^{2\alpha+9}3d^3\tilde\beta(1+\tilde\beta)(1+1/c)^3(|v_-|/\sqrt{2}+1-r)^3}{(\alpha-1)(|v_-|/\sqrt{2}-r)^4(1+|x_-|/\sqrt{2})^{\alpha-1}}$$

for $T \le +\infty$, *where* $\tilde\beta = \max(\beta_1,\beta_2)$.

Note that

$$\max\left(\frac{\rho_T(c,d,\beta_1,\alpha,|v_-|,|x_-|,r)}{r}, \lambda_T(c,d,\tilde\beta,\alpha,|v_-|,|x_-|,r)\right)$$

$$\le \mu_T(c,d,\tilde\beta,\alpha,|v_-|,|x_-|,r) = \frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}} \times \frac{2^{\alpha+4}d^2\tilde\beta(1+1/c)(|v_-|/\sqrt{2}+1-r)^2}{r(\alpha-1)(|v_-|/\sqrt{2}-r)^3(1+|x_-|/\sqrt{2}-(|v_-|/\sqrt{2}-r)T)^{\alpha-1}} \tag{2.10a}$$

for $T \le 0$,

$$\max\left(\frac{\rho(c,d,\beta_1,\alpha,|v_-|,|x_-|,r)}{r}, \lambda(c,d,\tilde\beta,\alpha,|v_-|,|x_-|,r)\right)$$

$$\le \mu(c,d,\tilde\beta,\alpha,|v_-|,|x_-|,r) = \frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}} \times \frac{2^{2\alpha+9}3d^3\tilde\beta(1+\tilde\beta)(1+1/c)^3(|v_-|/\sqrt{2}+1-r)^3}{r(\alpha-1)(|v_-|/\sqrt{2}-r)^4(1+|x_-|/\sqrt{2})^{\alpha-1}} \tag{2.10b}$$

for $T \le +\infty$, where $\tilde\beta = \max(\beta_1,\beta_2)$, $0<r\le 1$, $r<c/\sqrt{2}$, $|v_-|<c$, $|v_-|\ge z_1$, $v_-x_-=0$.

From Lemma 2.1 and the estimates (2.10) we obtain the following result.

Corollary 2.1. *Under the conditions* (1.2), $0<r\le 1$, $r<c/\sqrt{2}$, $x_-\in\mathbb{R}^d$, $v_-\in B_c$, $|v_-|\ge z_1(c,d,\beta_1,\alpha,|x_-|,r)$, $v_-x_-=0$, *the following result is valid:*

if $\mu_T(c,d,\tilde\beta,\alpha,|v_-|,|x_-|,r)<1$, *then* A_{v_-,x_-} *is a contraction map in* $M_{T,r}$ *for* $T\le 0$;

if $\mu(c,d,\tilde\beta,\alpha,|v_-|,|x_-|,r)<1$, *then* A_{v_-,x_-} *is a contraction map in* $M_{T,r}$ *for* $T\le +\infty$.

Taking into account (2.6) and using Lemma 2.1, Corollary 2.1 and the lemma about the contraction maps we will study the solution $(y_-(t),u_-(t))$ of the equation (2.3) in $M_{T,r}$.

We will use also the following results.

Lemma 2.2. *Under the conditions* (1.2), $(f,h)\in M_{T,r},\ 0<r\le 1,\ r<c/\sqrt{2}$, $x_-\in\mathbb{R}^d$, $v_-\in B_c$, $|v_-|\ge z_1(c,d,\beta_1,\alpha,|x_-|,r),\ v_-x_-=0$, *the following is valid:*

$$\begin{aligned}|A^2_{v_-,x_-}(f,h)(t)|\le&\zeta_-(c,d,\beta_1,\alpha,|v_-|,|x_-|,r,t)\\ =&\frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}}\\ &\times\frac{d\sqrt{d}\beta_1 2^{\alpha+2}(2+r/c)}{\alpha(|v_-|/\sqrt{2}-r)(1+|x_-|/\sqrt{2}-(|v_-|/\sqrt{2}-r)t)^{\alpha}},\end{aligned}\tag{2.11}$$

$$\begin{aligned}|A^1_{v_-,x_-}(f,h)(t)|\le&\xi_-(c,d,\beta_1,\alpha,|v_-|,|x_-|,r,t)\\ =&\frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}}\\ &\times\frac{d\sqrt{d}\beta_1 2^{\alpha+2}(2+r/c)}{\alpha(\alpha-1)(\frac{|v_-|}{\sqrt{2}}-r)^2(1+\frac{|x_-|}{\sqrt{2}}-(\frac{|v_-|}{\sqrt{2}}-r)t)^{\alpha-1}},\end{aligned}\tag{2.12}$$

for $t\le T,\ T\le 0$;

$$A^1_{v_-,x_-}(f,h)(t)=k_{v_-,x_-}(f,h)t+l_{v_-,x_-}(f,h)+H_{v_-,x_-}(f,h)(t),\tag{2.13}$$

where

$$k_{v_-,x_-}(f,h)=g(\gamma(v_-)+\int_{-\infty}^{+\infty}F(v_-s+x_-+f(s),v_-+h(s))\,ds)-v_-,\tag{2.14a}$$

$$\begin{aligned}l_{v_-,x_-}(f,h)=&\int_{-\infty}^{0}\Big[g(\gamma(v_-)+\int_{-\infty}^{\tau}F(v_-s+x_-+f(s),v_-+h(s))\,ds)-v_-\Big]\,d\tau\\ &+\int_{0}^{+\infty}\Big[g(\gamma(v_-)+\int_{-\infty}^{\tau}F(v_-s+x_-+f(s),v_-+h(s))\,ds)\\ &-g(\gamma(v_-)+\int_{-\infty}^{+\infty}F(v_-s+x_-+f(s),v_-+h(s))\,ds)\Big]\,d\tau,\end{aligned}\tag{2.14b}$$

$$|k_{v_-,x_-}(f,h)|\le 2\zeta_-(c,d,\beta_1,\alpha,|v_-|,|x_-|,r,0),\tag{2.15a}$$
$$|l_{v_-,x_-}(f,h)|\le 2\xi_-(c,d,\beta_1,\alpha,|v_-|,|x_-|,r,0),\tag{2.15b}$$

$$\begin{aligned}|\dot H_{v_-,x_-}(f,h)(t)|\le&\zeta_+(c,d,\beta_1,\alpha,|v_-|,|x_-|,r,t)\\ =&\frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}\alpha(|v_-|/\sqrt{2}-r)}\end{aligned}\tag{2.16}$$

$$\times \frac{d\sqrt{d}\beta_1 2^{\alpha+2}(2+r/c)}{(1+|x_-|/\sqrt{2}+(|v_-|/\sqrt{2}-r)t)^{\alpha}},$$

$$|H_{v_-,x_-}(f,h)(t)| \le \xi_+(c,d,\beta_1,\alpha,|v_-|,|x_-|,r,t) \tag{2.17}$$

$$= \frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}\alpha(\alpha-1)(|v_-|/\sqrt{2}-r)^2}$$

$$\times \frac{d\sqrt{d}\beta_1 2^{\alpha+2}(2+r/c)}{(1+|x_-|/\sqrt{2}+(|v_-|/\sqrt{2}-r)t)^{(\alpha-1)}},$$

for $T = +\infty,\ t \ge 0$.

One can see that Lemma 2.2 gives, in particular, estimates and asymptotics for

$$A_{v_-,x_-}(f,h)(t) = (A^1_{v_-,x_-}(f,h)(t), A^2_{v_-,x_-}(f,h)(t)) \quad \text{as } t \to \pm\infty.$$

Lemma 2.3. *Let the conditions* (1.2) *be valid,* $(y_-(t), u_-(t)) \in M_{T,r}$ *be a solution of* (2.3), $T = +\infty$, $0 < r \le 1$, $r < c/\sqrt{2}$, $x_- \in \mathbb{R}^d$, $v_- \in B_c$, $|v_-| \ge z_1(c,d,\beta_1,\alpha,|x_-|,r)$, $v_- x_- = 0$, *then*

$$|k_{v_-,x_-}(y_-,u_-) - k_{v_-,x_-}(0,0)| \le \varepsilon'_a(c,d,\beta_1,\tilde{\beta},\alpha,|v_-|,|x_-|,r)$$

$$= \frac{d^2\tilde{\beta}(1+\frac{1}{c})2^{\alpha+5}(|v_-|/\sqrt{2}+1-r)}{\alpha(|v_-|/\sqrt{2}-r)^2(1+|x_-|/\sqrt{2})^{\alpha}} \tag{2.18a}$$

$$\times \frac{\rho(c,d,\beta_1,\alpha,|v_-|,|x_-|,r)}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}},$$

$$\left| \frac{k_{v_-,x_-}(y_-,u_-)}{\sqrt{1-\frac{|v_-|^2}{c^2}}} - \int_{-\infty}^{+\infty} F(x_- + v_- s, v_-)\, ds \right| \le \varepsilon_a(c,d,\beta_1,\tilde{\beta},\alpha,|v_-|,|x_-|,r) \tag{2.18b}$$

$$= \frac{2^{\alpha+5}d\sqrt{d}\tilde{\beta}(1+\frac{1}{c})(|v_-|/\sqrt{2}+1-r)}{\alpha(|v_-|/\sqrt{2}-r)^2(1+|x_-|/\sqrt{2})^{\alpha}}$$

$$\times \rho(c,d,\beta_1,\alpha,|v_-|,|x_-|,r),$$

$$|l_{v_-,x_-}(y_-,u_-) - l_{v_-,x_-}(0,0)| \le \varepsilon_b(c,d,\beta_1,\tilde{\beta},\alpha,|v_-|,|x_-|,r) \tag{2.18c}$$

$$= \frac{2^{2\alpha+9}d^3\tilde{\beta}(1+\tilde{\beta})3(1+1/c)^3(|v_-|/\sqrt{2}+1-r)^2}{(\alpha-1)(|v_-|/\sqrt{2}-r)^4(1+|x_-|/\sqrt{2})^{\alpha-1}}$$

$$\times \frac{\rho(c,d,\beta_1,\alpha,|v_-|,|x_-|,r)}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}},$$

where k_{v_-,x_-} *and* l_{v_-,x_-} *are defined in* (2.14) *and* ρ *is defined in* (2.8b).

We sketch the proof of Lemmas 2.1, 2.2, 2.3 in Section 4.

3 Small angle scattering

Under the conditions (1.2), for any $(v_-, x_-) \in B_c \times \mathbb{R}^d,\ v_- \neq 0$, the equation (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^d)$ with the initial conditions (1.3). Consider the function $y_-(t)$ from (1.3). This function describes deflection from free motion.

Using Corollary 2.1 the lemma about contraction maps, and Lemmas 2.2 and 2.3 we obtain the following result.

Theorem 3.1. *Let the conditions* (1.2) *be valid,* $\mu(c, d, \tilde\beta, \alpha, |v_-|, |x_-|, r) < 1$, $\tilde\beta = \max(\beta_1, \beta_2)$, $0 < r \le 1$, $r < c/\sqrt{2}$, $x_- \in \mathbb{R}^d$, $v_- \in B_c$, $|v_-| \ge z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $v_- x_- = 0$, *where* μ *is defined by* (2.10b) *and* z_1 *is defined by* (2.7). *Then the deflection* $y_-(t)$ *has the following properties:*

$$(y_-, \dot y_-) \in M_{T,r},\ T = +\infty; \tag{3.1}$$

$$\begin{aligned} |\dot y_-(t)| \le& \zeta_-(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, t), && (3.2)\\ |y_-(t)| \le& \xi_-(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, t) \quad \text{for} \quad t \le 0; && (3.3)\\ y_-(t) =& a_{sc}(v_-, x_-)t + b_{sc}(v_-, x_-) + h(v_-, x_-, t), && (3.4)\end{aligned}$$

where

$$\left| a_{sc}(v_-, x_-) - \left[\frac{\gamma(v_-) + \int_{-\infty}^{+\infty} F(v_- s + x_-, v_-) ds}{\sqrt{1 + \frac{|\gamma(v_-) + \int_{-\infty}^{+\infty} F(v_- s + x_-, v_-) ds|^2}{c^2}}} - v_- \right] \right|$$
$$\le \varepsilon_a'(c, d, \beta_1, \tilde\beta, \alpha, |v_-|, |x_-|, r), \tag{3.5a}$$

$$\left| \frac{a_{sc}(v_-, x_-)}{\sqrt{1 - \frac{|v_-|^2}{c^2}}} - \int_{-\infty}^{+\infty} F(v_- s + x_-, v_-) ds \right| \le \varepsilon_a(c, d, \beta_1, \tilde\beta, \alpha, |v_-|, |x_-|, r), \tag{3.5b}$$

$$|b_{sc}(v_-, x_-) - l_{v_-, x_-}(0, 0)| \le \varepsilon_b(c, d, \beta_1, \tilde\beta, \alpha, |v_-|, |x_-|, r), \tag{3.5c}$$

$$\begin{aligned} |a_{sc}(v_-, x_-)| \le& 2\zeta_-(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, 0), && (3.6a)\\ |b_{sc}(v_-, x_-)| \le& 2\xi_-(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, 0), && (3.6b)\\ |\dot h(v_-, x_-, t)| \le& \zeta_+(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, t), && (3.7)\\ |h(v_-, x_-, t)| \le& \xi_+(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, t), && (3.8)\end{aligned}$$

for $t \ge 0$, *where* $l_{v_-, x_-}(0, 0)$ *(resp.* ε_a', ε_a, ε_b, ζ_-, ζ_+, ξ_- *and* ξ_+*) is defined in* (2.14b) (*resp.* (2.18a), (2.18b), (2.18c), (2.11), (2.16), (2.12) *and* (2.17)).

Let $z = z(c, d, \tilde{\beta}, \alpha, r_x, r)$ and $z_2 = z_2(c, d, \beta_1, \alpha, r_x)$ be defined as the roots of the following equations

$$\mu(c, d, \tilde{\beta}, \alpha, z, r_x, r) = 1,\ z \in]\sqrt{2}r, c[, \tag{3.9}$$

$$\frac{z_2}{\sqrt{1-\frac{z_2^2}{c^2}}} - \frac{32\beta_1 d}{\alpha(z_2/\sqrt{2})(1+r_x/\sqrt{2})^\alpha} = 0,\ z_2 \in]0, c[, \tag{3.10}$$

where μ is defined by (2.10b), r_x and r are some nonnegative numbers such that $0 < r \le 1,\ r < c/\sqrt{2}$, and where $\tilde{\beta} = \max(\beta_1, \beta_2)$.

We use the following observations.

(I) Let $0 < r \le 1, r < c/\sqrt{2},\ 0 \le \sigma$

$$\frac{s_1}{\sqrt{1-\frac{s_1^2}{c^2}}} - \frac{2^{\alpha+5}\beta_1 d(2+r/c)}{\alpha(s_1/\sqrt{2}-r)(\sigma/\sqrt{2}+1)^\alpha} > \frac{s_2}{\sqrt{1-\frac{s_2^2}{c^2}}} - \frac{2^{\alpha+5}\beta_1 d(2+r/c)}{\alpha(s_2/\sqrt{2}-r)(\sigma/\sqrt{2}+1)^\alpha}$$

for $\sqrt{2}r < s_2 < s_1 < c$.

(II) Let $0 < r \le 1, r < c/\sqrt{2},\ \sigma \in]\sqrt{2}r, c[$,

$$\frac{\sigma}{\sqrt{1-\frac{\sigma^2}{c^2}}} - \frac{2^{\alpha+5}\beta_1 d(2+r/c)}{\alpha(\sigma/\sqrt{2}-r)(s_1/\sqrt{2}+1)^\alpha} > \frac{\sigma}{\sqrt{1-\frac{\sigma^2}{c^2}}} - \frac{2^{\alpha+5}\beta_1 d(2+r/c)}{\alpha(\sigma/\sqrt{2}-r)(s_2/\sqrt{2}+1)^\alpha}$$

for $0 \le s_2 < s_1$.

(III) Let $0 < r \le 1,\ r < c/\sqrt{2}$, x some real nonnegative number, $\tilde{\beta} = \max(\beta_1, \beta_2)$ and $\sqrt{2}r < s < c$ then

$$\mu(c, d, \tilde{\beta}, \alpha, s, |x|, r) < 1 \Leftrightarrow s > z(c, d, \tilde{\beta}, \alpha, |x|, r).$$

Observations (I) and (II) imply that $z_1(c, d, \beta_1, \alpha, s_2, r) > z_1(c, d, \beta_1, \alpha, s_1, r)$ for $\sqrt{2}r < s_2 < s_1 < c$ when $c,\ \beta_1,\ \alpha,\ d,\ r$ are fixed.

Theorem 3.1 gives, in particular, estimates for the scattering process and asymptotics for the velocity valued component of the scattering map when c, β_1, β_2, α, d, $\hat{v}_-$, x_- are fixed (where $\hat{v}_- = v_-/|v_-|$) and $|v_-|$ increases or, e.g., c, β_1, β_2, α, d, v_-, $\hat{x}_-$ are fixed and $|x_-|$ increases. In these cases $\sup_{t\in\mathbb{R}} |\theta(t)|$ decreases, where $\theta(t)$ denotes the angle between the vectors $\dot{x}(t) = v_- + \dot{y}_-(t)$ and v_-, and we deal with small angle scattering. Note that already under the conditions of Theorem 3.1, without additional assumptions, there is the estimate $\sup_{t\in\mathbb{R}} |\theta(t)| < \frac{1}{4}\pi$ and we deal with a rather small angle scattering. Theorem 3.1 with (3.5c) will give the asymptotics of the configuration valued component $b(v_-, x_-)$ of the scattering map if we can study the asymptotics of $l_{v_-,x_-}(0, 0)$. This is the subject of Theorem 3.2.

Theorem 3.2. *Let c, d, β_0, β_1, α, $|x|$ be fixed. Then there exists a constant $C_{c,d,\beta_0,\beta_1,\alpha,|x|}$ such that*

$$\begin{aligned}\Bigg|\frac{l_{v,x}(0,0)}{\sqrt{1-\frac{|v|^2}{c^2}}}-\frac{1}{c^2}PV(\hat{v},x)\hat{v}+\frac{1}{|v|^2}\int_0^{+\infty}\int_\tau^{+\infty}F(\sigma\hat{v}+x,v)d\sigma d\tau\\ -\frac{1}{|v|^2}\int_{-\infty}^0\int_{-\infty}^\tau F(\sigma\hat{v}+x,v)d\sigma d\tau\Bigg| \leq & C_{c,d,\beta_0,\beta_1,\alpha,|x|}\sqrt{1-\frac{|v|^2}{c^2}}\end{aligned} \tag{3.11}$$

for any $v\in B_c$, $|v|\geq z_2(c,d,\beta_1,\alpha,|x|)$, $vx=0$, and where $\hat{v}=v/|v|$.

We sketch the proof of Theorem 3.2 in Section 4.

4 About the proof of Lemmas 2.1, 2.2, 2.3 and Theorems 3.2 and 1.1

The way we prove Lemmas 2.1, 2.2, 2.3 and Theorem 3.2 of the present work, is actually exactly the same as the way we prove lemmas 2.1, 2.2, 2.3 and theorem 3.2 of [Jo].

4.1 *Inequalities for F and g.*
Before sketching the proof of Lemmas 2.1, 2.2, 2.3 and Theorem 3.2, we shall give some estimates about the growth of g defined by

$$g(x)=\frac{x}{\sqrt{1+\frac{|x|^2}{c^2}}},\ x\in\mathbb{R}^d,$$

and we shall prove Lemma 4.1 given below.

We remind that g has the following simple properties (see [Jo]):

$$|\nabla g_i(x)|^2\leq\frac{1}{1+\frac{|x|^2}{c^2}}, \tag{4.1}$$

$$|g(x)-g(y)|\leq\sqrt{d}\sup_{\varepsilon\in[0,1]}\frac{1}{\sqrt{1+\frac{|\varepsilon x+(1-\varepsilon)y|^2}{c^2}}}|x-y|, \tag{4.2}$$

$$|\nabla g_i(x)-\nabla g_i(y)|\leq\frac{3\sqrt{d}}{c}\sup_{\varepsilon\in[0,1]}\frac{1}{1+\frac{|\varepsilon x+(1-\varepsilon)y|^2}{c^2}}|x-y|, \tag{4.3}$$

for $x,\ y\in\mathbb{R}^d,\ i=1..d$, and where $g=(g_1,..,g_d)$.

Lemma 4.1. *Under the conditions* (1.2), *the following estimates are valid:*

$$|F(x,y)|\leq 2d\beta_1(1+|x|)^{-(\alpha+1)}(1+\frac{1}{c}|y|)\ \textit{for}\ x,y\in\mathbb{R}^d, \tag{4.4}$$

$$|F(x,y)-F(x',y')| \le \frac{1}{c}2d\beta_1 \sup_{\varepsilon\in[0,1]}(1+|\varepsilon x+(1-\varepsilon)x'|)^{-(\alpha+1)}|y-y'| \tag{4.5}$$
$$+2d\sqrt{d}\beta_2 \sup_{\varepsilon\in[0,1]}(1+|\varepsilon y+(1-\varepsilon)y'|/c)(1+|\varepsilon x+(1-\varepsilon)x'|)^{-(\alpha+2)}$$
$$\times|x-x'|,$$

for $x,\ y,\ x',\ y' \in \mathbb{R}^d$.

Let $(f,h),\ (f_1,h_1),\ (f_2,h_2) \in M_{T,r},\ v_- \in B_c\backslash\{0\},\ v_-x_- = 0,\ |v_-| > \sqrt{2}r$, *then*

$$|f(s)| \le (1+|s|)\|(f,h)\|_T, \tag{4.6}$$
$$|h(s)| \le \|(f,h)\|_T, \tag{4.7}$$

for $s \le T$;

$$2(1+|x_-+v_-s+f(s)|) \ge (1+|x_-|/\sqrt{2}+(|v_-|/\sqrt{2}-r)|s|),\ \textit{for}\ s\le T, \tag{4.8}$$

$$\left|\int_{-\infty}^{t} F(v_-s+x_-+f(s),v_-+h(s))ds\right| \le \frac{\beta_1 d2^{\alpha+3}(2+r/c)}{\alpha(|v_-|/\sqrt{2}-r)(|x_-|/\sqrt{2}+1)^{\alpha}}, \tag{4.9}$$

$$\Bigg(1+\frac{1}{c^2}\Bigg|\gamma(v_-)+\varepsilon_1\int_{-\infty}^{t}F(v_-s+x_-+f_1(s),v_-+h_1(s))ds$$
$$+\varepsilon_2\int_{\sigma}^{\tau}F(v_-s+x_-+f_2(s),v_-+h_2(s))ds\Bigg|^2\Bigg)^{-\beta}$$
$$\le (1+\frac{|v_-|^2}{4(c^2-|v_-|^2)})^{-\beta}, \tag{4.10}$$

for $\tau, t \in]-\infty,T],\ \sigma\in[-\infty,\tau],\ \beta>0,\ -1\le\varepsilon_1\le 1,\ -1\le\varepsilon_2\le 1,\ (f_1,h_1),\ (f_2,h_2)\in M_{T,r}$ *and if* $|v_-|\ge z_1(c,d,\beta_1,\alpha,|x_-|,r)$, $|v_-|<c$, *where* γ *is defined by*

$$\gamma(v)=\frac{v}{\sqrt{1-|v|^2/c^2}},$$

for $v\in B_c$.

Proof of Lemma 4.1. The estimates (4.4) and (4.5) follows directly from the formula $F(x,y) = -\nabla V(x)+\frac{1}{c}B(x)y$ and $B(x)=[\frac{\partial}{\partial x_j}\mathbf{A}_k(x)-\frac{\partial}{\partial x_k}\mathbf{A}_j(x)]_{j,k=1..d}$ and the conditions (1.2). The inequalities (4.6), (4.7) and (4.8) follow from the definition of $M_{T,r}$. Using (4.4), (4.8), we obtain (4.9) and using (4.9) and the definition of $z_1(c,d,\beta_1,\alpha,|x_-|,r)$ we obtain (4.10). □

4.2 *Sketch of proofs of Lemmas 2.1, 2.2, 2.3 and Theorem 3.2.*

One can prove Lemmas 2.1, 2.2, 2.3 of the present work by repeating the proof of lemmas 2.1, 2.2, 2.3 of [Jo] and by making the following replacements. First the estimates given in lemmas 4.1, 4.3 of [Jo] are replaced by the estimates of Lemma 4.1 of the present work. Then, to prove Lemmas 2.1, 2.2, 2.3, we replace $\frac{d}{dt}A_{v_-,x_-}(f)$, $A_{v_-,x_-}(f)$ and $F(v_-s+x_-+f(s))$, for $f\in M_{T,r}$, in the proof of lemmas 2.1, 2.2, 2.3 of [Jo] by $A^2_{v_-,x_-}(f,h)$, $A^1_{v_-,x_-}(f,h)$ and $F(v_-s+x_-+f(s),v_-+h(s))$ for $(f,h)\in M_{T,r}$. $\square$

One can prove Theorem 3.2 by repeating the proof of theorem 3.2 of [Jo] and by making the following replacements. We replace the estimates given in lemmas 4.1, 4.3 of [Jo] by the estimates of Lemma 4.1 of our present work and we replace $F(\tau\theta+x)$ of the proof of theorem 3.2 of [Jo] by $F(\tau\theta+x,s\theta)$. $\square$

4.3 *Constants C_1, C_2, s_1, s_2 of Theorem 1.1.*
As it was mentioned already in Introduction, Theorem 1.1 follows from Theorem 3.1 and Theorem 3.2. In addition, constants C_1, C_2, s_1, s_2, which appear in Theorem 1.1, are given explicitly by

$$\begin{aligned}
s_1 =&\max(z(c,d,\tilde\beta,\alpha,|x|,r),z_1(c,d,\beta_1,\alpha,|x|,r)),\\
s_2 =&\max(z(c,d,\tilde\beta,\alpha,|x|,r),z_1(c,d,\beta_1,\alpha,|x|,r),z_2(c,d,\beta_1,\alpha,|x|)),\\
C_1 =&\frac{d^3\tilde\beta^2 2^{2\alpha+9}(1+\frac{1}{c})^2c(\frac{c}{\sqrt2}+1-r)^2}{\alpha(\alpha-1)(\frac{s_1}{\sqrt2}-r)^4(1+\frac{|x|}{\sqrt2})^{2\alpha-1}},\\
C_2 =&C_{c,d,\beta_0,\beta_1,\alpha,|x|}+\frac{4d^4\sqrt{d}\tilde\beta^2(1+\tilde\beta)2^{3\alpha+15}(1+1/c)^4(\frac{c}{\sqrt2}+1-r)^3}{(\alpha-1)^2(\frac{s_2}{\sqrt2}-r)^6(1+\frac{|x|}{\sqrt2})^{2\alpha-2}},
\end{aligned}$$

where $C_{c,d,\beta_0,\beta_1,\alpha,|x|}$ is the constant of Theorem 3.2 and z, z_1, z_2 are defined by (3.9), (2.7), (3.10) and where $\tilde\beta=\max(\beta_1,\beta_2)$.

4.4 *Weakened assumptions.*
Let $M_d(\mathbb{R})$ denote the space of $d\times d$ matrix with real elements. Let $V\in C^2(\mathbb{R}^d,\mathbb{R})$ so that:

$$|\partial_x^j V(x)|\le\beta_{|j|}(1+|x|)^{-(\alpha+|j|)}, \tag{4.11a}$$

for $|j|\le 2$ and some $\alpha>1$ (here j is the multiindex $j\in(\mathbb{N}\cup\{0\})^d$, $|j|=\sum_{n=1}^d j_n$ and $\beta_{|j|}$ are positive real constants). Let $B\in C^1(\mathbb{R}^d,M_d(\mathbb{R}))$ so that:

$B(x)$ is a $d\times d$ antisymmetric matrix with real elements $B_{m,n}(x)$, (4.11b)

$$\frac{\partial}{\partial x_i}B_{k,l}(x)+\frac{\partial}{\partial x_l}B_{i,k}(x)+\frac{\partial}{\partial x_k}B_{l,i}(x)=0, \tag{4.11c}$$

for $x\in\mathbb{R}^d$, for $i,k,l=1..d$;

$$|\partial_x^j B_{i,k}(x)|\le\beta_{|j|+1}(1+|x|)^{-(\alpha+|j|+1)}, \tag{4.11d}$$

for $i, k = 1..d$ and for $|j| \leq 1$.

Let $\mathbf{A}$ be the transversal gauge of B, i.e.

$$\mathbf{A}(x) = -\int_0^1 sB(sx).xds. \tag{4.12}$$

Under the conditions (4.11b), (4.11c) and (4.11d), $\mathbf{A}$ satisfies

$$|\mathbf{A}(x)| \leq \beta(1+|x|)^{-1}, \tag{4.13a}$$

$$B_{i,k}(x) = \frac{\partial}{\partial x_i}\mathbf{A}_k(x) - \frac{\partial}{\partial x_k}\mathbf{A}_i(x). \tag{4.13b}$$

for $x \in \mathbb{R}^d$, $i, k = 1..d$ and some positive real constant β.

If we replace assumptions (1.2) by assumptions (4.11) given above, then the estimates (4.4) and (4.5) still hold. As a consequence, using also Remark 5.2, we obtain that assumptions (1.2) in all results and estimates which appear in Introduction and in Sections 2, 3 can be weakened to assumptions (4.11).

5 Proofs of Proposition 1.1 and Proposition 1.2

Let $\mathbf{A} \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ and

$$|\partial_x^j \mathbf{A}_i(x)| \leq \beta_{|j|}(1+|x|)^{-(\alpha+|j|)} \tag{5.1}$$

for $x \in \mathbb{R}^d$, $|j| \leq 2$, $i = 1..d$ and some $\alpha > 1$ (here j is the multiindex $j \in (\mathbb{N} \cup \{0\})^d, |j| = \sum_{n=1}^d j_n$ and $\beta_{|j|}$ are positive real constants). We define the magnetic field $B \in C^1(\mathbb{R}^d, \mathcal{M}_d(\mathbb{R}))$ by: $B(x)$ is the $d\times d$ real antisymmetric matrix with elements

$$B_{i,k}(x) = \frac{\partial}{\partial x_i}\mathbf{A}_k(x) - \frac{\partial}{\partial x_k}\mathbf{A}_i(x) \tag{5.2}$$

for $x \in \mathbb{R}^d$ (where $\mathcal{M}_d(\mathbb{R})$ denotes the space of $d \times d$ real matrix). For $\mathbf{A} \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ satisfying (5.1) and $(\theta, x) \in T\mathbb{S}^{d-1}$ we define the vectors $w_3(\mathbf{A}, \theta, x)$ and $w_4(\mathbf{A}, \theta, x)$:

$$w_3(\mathbf{A}, \theta, x) = \int_{-\infty}^{+\infty} B(x+\sigma\theta)\theta d\sigma, \tag{5.3a}$$

$$w_4(\mathbf{A}, \theta, x) = \int_{-\infty}^{0}\int_{-\infty}^{\tau} B(x+\sigma\theta)\theta d\sigma d\tau - \int_0^{+\infty}\int_\tau^{+\infty} B(x+\sigma\theta)\theta d\sigma d\tau, \tag{5.3b}$$

where B is defined by (5.2).

We also define a function $\tilde{w_3}(\mathbf{A}) : \mathbb{R}^d\backslash\{0\} \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$\tilde{w_3}(\mathbf{A})(y, x) = |y| w_3(\mathbf{A}, \frac{y}{|y|}, x - \frac{xy}{|y|^2}y), \tag{5.4}$$

for $x \in \mathbb{R}^d, y \in \mathbb{R}^d\backslash\{0\}$. From (5.1), (5.4) and (5.3a) it follows that

$$\tilde{w_3}(\mathbf{A})(y,x) = \int_{-\infty}^{+\infty} B(x+\sigma y)y d\sigma, \tag{5.5}$$

for $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d\backslash\{0\}$. From (5.1), it follows that $\tilde{w_3}(\mathbf{A}) = ((\tilde{w_3}(\mathbf{A}))_1, .., (\tilde{w_3}(\mathbf{A}))_d) \in C^1(\mathbb{R}^d\backslash\{0\} \times \mathbb{R}^d, \mathbb{R}^d)$.

To prove Proposition 1.1 we first prove the following result.

Proposition 5.1. *Let* $\mathbf{A} \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ *satisfy* (5.1) . *Then* $w_3(\mathbf{A}, \theta, x)$ *given for all* $(\theta, x) \in T\mathbb{S}^{d-1}$ *determines uniquely the magnetic field* B *defined by* (5.2) *and the following formulas are valid:*

$$PB_{i,k}(\theta,x) = \left(\frac{\partial}{\partial y_k}(\tilde{w_3}(\mathbf{A}))_i(y,x) - \frac{\partial}{\partial y_i}(\tilde{w_3}(\mathbf{A}))_k(y,x)\right)_{|y=\theta}, \tag{5.6a}$$

for $(\theta, x) \in T\mathbb{S}^{d-1},\ i,k = 1..d,\ i \neq k$;

$$PB_{i,k}(\theta,x) = \theta_k w_3(\mathbf{A},\theta,x)_i - \theta_i w_3(\mathbf{A},\theta,x)_k \tag{5.6b}$$

for $(\theta, x) \in \mathcal{V}_{i,k},\ i,k = 1..d,\ i \neq k$ *where* $\mathcal{V}_{i,k}$ *is the d-dimensional smooth manifold given by* (1.11).

Note that under different conditions on vector potentials $\mathbf{A}$, the question of the determination of B from w_3 was studied in [Ni], [Ju], [I]. However, to our knowledge the formulas (5.6) were not given in the literature.

Proof of Proposition 5.1. Under the conditions (5.1) and from (5.2) and (5.5) it follows that

$$\begin{aligned}\frac{\partial}{\partial y_k}(\tilde{w_3}(\mathbf{A}))_i(y,x) = &\int_{-\infty}^{+\infty}[\frac{\partial}{\partial x_i}\mathbf{A}_k(ty+x) - \frac{\partial}{\partial x_k}\mathbf{A}_i(ty+x)]dt \\ &+ \sum_{j=1}^d \int_{-\infty}^{+\infty} t[\frac{\partial^2}{\partial x_k \partial x_i}\mathbf{A}_j(ty+x) - \frac{\partial^2}{\partial x_k \partial x_j}\mathbf{A}_i(ty+x)]y_j dt\end{aligned} \tag{5.7}$$

for any $(y,x) \in \mathbb{R}^d\backslash\{0\} \times \mathbb{R}^d$ and $i,k = 1..d$. Let $i,k = 1..d$. From (5.7) it follows that

$$\begin{aligned}&\left(\frac{\partial}{\partial y_k}(\tilde{w_3}(\mathbf{A}))_i(y,x) - \frac{\partial}{\partial y_i}(\tilde{w_3}(\mathbf{A}))_k(y,x)\right)_{|y=\theta} \\ &= 2PB_{i,k}(\theta,x) + \int_{-\infty}^{+\infty} t\sum_{j=1}^d \frac{\partial}{\partial x_j}B_{i,k}(t\theta+x)\theta_j dt,\end{aligned} \tag{5.8}$$

for all $(\theta, x) \in T\mathbb{S}^{d-1}$, $\theta = (\theta_1, \ldots, \theta_d)$ and where P denotes the X-ray transform. Integrating by parts the integral of the right-hand side of (5.8), we obtain the formula (5.6a).

We recall that $w_3(\mathbf{A}, \theta, x)_i = \sum_{j=1}^{d} \int_{-\infty}^{+\infty} B_{i,j}(t\theta + x)\theta_j dt$, for $(\theta, x) \in T\mathbb{S}^{d-1}$, $i, k = 1..d$, $i \neq k$. Hence $\theta_k PB_{i,k}(\theta, x) = w_3(\mathbf{A}, \theta, x)_i$ for $(\theta, x) \in \mathcal{V}_{i,k}$, $i, k = 1..d$, $i \neq k$. This last formula implies (5.6b) ($\theta_i^2 + \theta_k^2 = 1$ for $(\theta, x) \in \mathcal{V}_{i,k}$, $\theta = (\theta_1, \ldots, \theta_d)$).

Then using results on inversion of the X-ray transform and using (5.6a) or (5.6b) and using (5.4) we obtain that $w_3(\mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$ uniquely determines the magnetic field B.

Proposition 5.1 is proved. □

Now we are ready to prove Proposition 1.1.

Let $(\theta, x) \in T\mathbb{S}^{d-1}$. We note that

$$\int_{-\infty}^{+\infty} B(\tau(-\theta) + x)(-\theta)d\tau = -\int_{-\infty}^{+\infty} B(\tau\theta + x)\theta d\tau \tag{5.9a}$$

and we remind that

$$P(\nabla V)(-\theta, x) = P(\nabla V)(\theta, x). \tag{5.9b}$$

Using (1.9a), (5.3a) and (5.9) we obtain the formula (1.12a) and the following formula

$$w_3(\mathbf{A}, \theta, x) = \frac{1}{2}(w_1(V, \mathbf{A}, \theta, x) - w_1(V, \mathbf{A}, -\theta, x)), \tag{5.10}$$

for $(\theta, x) \in T\mathbb{S}^{d-1}$. From (1.12a) and results on inversion of the X-ray transform, we obtain that $w_1(V, \mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$ uniquely determines ∇V and thus it uniquely determines V ($(V, \mathbf{A})$ satisfies (1.2)). From (5.10) and Proposition 5.1 it follows that $w_1(V, \mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$ uniquely determines B. In addition from (5.10) it follows that

$$\tilde{w_3}(\mathbf{A})(y, x) = \frac{1}{2}(\tilde{w_1}(V, \mathbf{A})(y, x) - \tilde{w_1}(V, \mathbf{A})(-y, x)),$$

for $y \in \mathbb{R}^d \backslash \{0\}$, $x \in \mathbb{R}^d$. Using this last formula and (5.6a) of Proposition 5.1 we obtain (1.12b). Using (5.10) and (5.6b), we obtain (1.12c).

Proposition 1.1 is proved. □

Let $i, k = 1..d$, $i \neq k$. To reconstruct $B_{i,k}$ from $w_1(V, \mathbf{A}, \theta, x)$ given for all $(\theta, x) \in \mathcal{V}_{i,k}$, we give the following scheme which is based on the formula (1.12c) ($\mathcal{V}_{i,k}$ is defined in (1.11)). The formula (1.12c) gives the value of all integrals of $B_{i,k}$ over any straight line of any two-dimensional affine plane Y whose tangent vector space is $Y_{i,k} = \{(x'_1, \ldots, x'_d) \in \mathbb{R}^d | x'_j = 0, j \neq i, j \neq k\}$. Now, to reconstruct $B_{i,k}$ at a point $x' \in \mathbb{R}^d$ we consider in $\mathbb{R}^d$ a two-dimensional plane Y containing x' and whose tangent vector space is $Y_{i,k}$. We interpret $T\mathbb{S}^{d-1}$ as the set of all rays in $\mathbb{R}^d$ and we consider in $T\mathbb{S}^{d-1}$ the subset $T\mathbb{S}^1(Y)$ which is the set of all rays lying in Y. Then we restrict $PB_{i,k}$ on $T\mathbb{S}^1(Y)$ and reconstruct $B_{i,k}(x')$ from these data using methods of reconstruction of f from Pf for $d=2$. (We can also use the formula (1.12b) for reconstruction of $B_{i,k}$ from $w_1(V,\mathbf{A},\theta,x)$ given for all $(\theta,x)\in T\mathbb{S}^{d-1}$.)

To prove Proposition 1.2 we will first prove Proposition 5.2 and Proposition 5.3 given below.

Let $\mathbf{A}\in C^2(\mathbb{R}^d,\mathbb{R}^d)$ satisfy (5.1). We define a function $\tilde{w_4}(\mathbf{A}):\mathbb{R}^d\backslash\{0\}\times\mathbb{R}^d\to\mathbb{R}^d$ by $\tilde{w_4}(\mathbf{A})(y,x)=|y|w_4(\mathbf{A},\frac{y}{|y|},x-\frac{x.y}{|y|^2}y)$ for $x\in\mathbb{R}^d,\ y\in\mathbb{R}^d\backslash\{0\}$. From (5.1) and (5.3b), it follows that

$$\tilde{w_4}(\mathbf{A})(y,x)=\int_{-\infty}^{\frac{-xy}{|y|^2}}\int_{-\infty}^{\tau}B(x+\sigma y)yd\sigma d\tau-\int_{\frac{-xy}{|y|^2}}^{+\infty}\int_{\tau}^{+\infty}B(x+\sigma y)yd\sigma d\tau, \quad (5.11)$$

and $\tilde{w_4}(\mathbf{A})=(\tilde{w_4}(\mathbf{A})_1,..,\tilde{w_4}(\mathbf{A})_d)\in C^1(\mathbb{R}^d\backslash\{0\}\times\mathbb{R}^d,\mathbb{R}^d)$.

Proposition 5.2. *Let $\mathbf{A}\in C^2(\mathbb{R}^d,\mathbb{R}^d)$ satisfy* (5.1), *then B defined by* (5.2) *satisfies:*

$$\sum_{j=1}^{d}\theta_j\left[\theta_k PB_{i,j}(\theta,x)-\theta_i PB_{k,j}(\theta,x)\right]-PB_{i,k}(\theta,x)= \quad (5.12a)$$
$$\frac{\partial}{\partial x_i}\tilde{w_4}(\mathbf{A})_k(\theta,x)-\frac{\partial}{\partial x_k}\tilde{w_4}(\mathbf{A})_i(\theta,x),$$

$$\sum_{j=1}^{d}\theta_j\left[\theta_k PB_{i,j,l}(\theta,x)-\theta_i PB_{k,j,l}(\theta,x)\right]-PB_{i,k,l}(\theta,x)=\tilde{w_4}(\mathbf{A})_{i,k,l}(\theta,x), \quad (5.12b)$$

for $(\theta,x)\in T\mathbb{S}^{d-1}$, $i,k,l=1..d$, where P denotes the X-ray transform and where $\tilde{w_4}(\mathbf{A})_{m,n,l}(\theta,x)=\frac{\partial}{\partial x_l}\left(\frac{\partial}{\partial x_m}\tilde{w_4}(\mathbf{A})_n-\frac{\partial}{\partial x_n}\tilde{w_4}(\mathbf{A})_m\right)(\theta,x)$, $B_{m,n,l}(x)=\frac{\partial}{\partial x_l}B_{m,n}(x)$, for $\theta\in\mathbb{S}^{d-1}$, $x\in\mathbb{R}^d$, $m,n=1..d$.

Remark 5.1. Let $d=3$, $l=1..d$. Formula (5.12b) gives in fact

$$\theta\star(-PB_{2,3,l}(\theta,x),PB_{1,3,l}(\theta,x),-PB_{1,2,l}(\theta,x))=$$
$$\theta\star(\tilde{w_4}(\mathbf{A})_{2,3,l}(\theta,x),-\tilde{w_4}(\mathbf{A})_{1,3,l}(\theta,x),\tilde{w_4}(\mathbf{A})_{1,2,l}(\theta,x)), \quad (5.13)$$

for any $(\theta,x)\in T\mathbb{S}^2$ where $\star$ denotes the usual scalar product on $\mathbb{R}^3$.

Proof of Proposition 5.2. Under the conditions (5.1), from (5.11) it follows that

$$\frac{\partial}{\partial x_k}\tilde{w_4}(\mathbf{A})_i(y,x)=-\frac{y_k}{|y|^2}\sum_{j=1}^{d}y_j\int_{-\infty}^{+\infty}B_{i,j}(\sigma y+x)d\sigma d\tau \quad (5.14)$$

$$+\sum_{j=1}^{d} y_j \left\{\int_{-\infty}^{-\frac{x\star y}{|y|^2}}\int_{-\infty}^{\tau}\frac{\partial}{\partial x_k}B_{i,j}(\sigma y+x)d\sigma d\tau\right.$$
$$\left.-\int_{-\frac{x\star y}{|y|^2}}^{+\infty}\int_{\tau}^{+\infty}\frac{\partial}{\partial x_k}B_{i,j}(\sigma y+x)d\sigma d\tau\right\}$$

for any $(y,x)\in\mathbb{R}^d\backslash\{0\}\times\mathbb{R}^d$ and $i,k=1..d$. Let $i,k,l=1..d$ be fixed. From (5.14) it follows that

$$\begin{aligned}&\frac{\partial}{\partial x_k}\tilde{w_4}(\mathbf{A})_i(\theta,x)-\frac{\partial}{\partial x_i}\tilde{w_4}(\mathbf{A})_k(\theta,x)\\ &=-\theta_k\sum_{j=1}^{d}\theta_j\int_{-\infty}^{+\infty}B_{i,j}(\sigma\theta+x)d\sigma d\tau+\theta_i\sum_{j=1}^{d}\theta_j\int_{-\infty}^{+\infty}B_{k,j}(\sigma\theta+x)d\sigma d\tau\\ &+\sum_{j=1}^{d}\theta_j\left\{\int_{-\infty}^{0}\int_{-\infty}^{\tau}[\frac{\partial}{\partial x_k}B_{i,j}(\sigma\theta+x)-\frac{\partial}{\partial x_i}B_{k,j}(\sigma\theta+x)]d\sigma d\tau\right.\\ &\left.-\int_{0}^{+\infty}\int_{\tau}^{+\infty}[\frac{\partial}{\partial x_k}B_{i,j}(\sigma\theta+x)-\frac{\partial}{\partial x_i}B_{k,j}(\sigma\theta+x)]d\sigma d\tau\right\}\end{aligned}\tag{5.15}$$

for $x\in\mathbb{R}^d,\ \theta\in\mathbb{S}^{d-1},\ \theta=(\theta_1,\ldots,\theta_d)$. From (5.1) and (5.2) it follows that

$$\frac{\partial}{\partial x_k}B_{i,j}(x)-\frac{\partial}{\partial x_i}B_{k,j}(x)=\frac{\partial}{\partial x_j}B_{i,k}(x),\ x\in\mathbb{R}^d,\ j=1..d.\tag{5.16}$$

Let $\theta\in\mathbb{S}^{d-1}$ be fixed. Using (5.15), (5.1) and (5.16) we obtain (5.12a). Under conditions (5.1), the function $h_{i,k,\theta}$ which is defined by $h_{i,k,\theta}(x)=\frac{\partial}{\partial x_k}\tilde{w_4}(\mathbf{A})_i(\theta,x)$ $-\frac{\partial}{\partial x_i}\tilde{w_4}(\mathbf{A})_k(\theta,x), x\in\mathbb{R}^d$, satisfies $h_{i,k,\theta}\in C^1(\mathbb{R}^d,\mathbb{R})$ and (5.12b) follows immediatly from (5.12a).

Proposition 5.2 is proved. □

Proposition 5.3. *Let* $\mathbf{A}\in C^2(\mathbb{R}^d,\mathbb{R}^d)$ *satisfy* (5.1).

i. *if* $d=2$ *then* $w_4(\mathbf{A},\theta,x)$ *given for all* $(\theta,x)\in T\mathbb{S}^{d-1}$ *does not determine uniquely the magnetic field* B *defined by* (5.2),

ii. *if* $d\geq 3$ *then* $w_4(\mathbf{A},\theta,x)$ *given for all* $(\theta,x)\in T\mathbb{S}^{d-1}$ *uniquely determines the magnetic field* B *defined by* (5.2). *In addition the following formulas are valid: if* $d=3$ *then*

$$(-\mathcal{F}B_{2,3,l}(0),\mathcal{F}B_{1,3,l}(0),-\mathcal{F}B_{1,2,l}(0))=\tag{5.17a}$$
$$(2\pi)^{-3/2}\sum_{j=1}^{3}\left[\theta^j\star\left(\int_{\Pi_{\theta^j}}\tilde{w_4}(\mathbf{A})_{2,3,l}(\theta^j,y)dy,-\int_{\Pi_{\theta^j}}\tilde{w_4}(\mathbf{A})_{1,3,l}\,(\theta^j,y)dy,\right.\right.$$
$$\left.\left.\int_{\Pi_{\theta^j}}\tilde{w_4}(\mathbf{A})_{1,2,l}(\theta^j,y)dy\right)\right]\theta^j,$$

for any orthonormal basis $(\theta^1,\theta^2,\theta^3)$ *and where* $\Pi_{p'}$ *is the vector plane* $\{y\in\mathbb{R}^3|y\star p'=0\}$ *for* $p'\in\mathbb{R}^3\backslash\{0\}$, *and* $\mathcal{F}$ *denotes the classical Fourier transform on* $L^1(\mathbb{R}^3)$;

$$(-\mathcal{F}B_{2,3,l}(p),\mathcal{F}B_{1,3,l}(p),-\mathcal{F}B_{1,2,l}(p)) = \tag{5.17b}$$

$$(2\pi)^{-3/2}\sum_{j=1}^{2}\left[\theta_p^j\star\left(\int_{\Pi_{\theta_p^j}}e^{-iy\star p}\tilde{w_4}(\mathbf{A})_{2,3,l}(\theta_p^j,y)dy,\ -\int_{\Pi_{\theta_p^j}}e^{-iy\star p}\tilde{w_4}(\mathbf{A})_{1,3,l}(\theta_p^j,y)dy,\right.\right.$$
$$\left.\left.\int_{\Pi_{\theta_p^j}}e^{-iy\star p}\tilde{w_4}(\mathbf{A})_{1,2,l}(\theta_p^j,y)dy\right)\right]\theta_p^j,$$

for $p\in\mathbb{R}^3\backslash\{0\}$ *and any orthonormal family* $\{\theta_p^1,\theta_p^2\}$ *of the plane* Π_p *(and where* $i=\sqrt{-1}$*);*

if $d\geq 4$ *then*

$$PB_{j,k}(\theta,x)=\frac{\partial}{\partial x_k}\tilde{w_4}(\mathbf{A})_j(\theta,x)-\frac{\partial}{\partial x_j}\tilde{w_4}(\mathbf{A})_k(\theta,x) \tag{5.17c}$$

for $(\theta,x)\in\tilde{\mathcal{V}}_{j,k}$, *where* $\tilde{\mathcal{V}}_{j,k}$ *is the* $(2d-4)$*-dimensional manifold* $\{(\theta,x)\in T\mathbb{S}^{d-1}|\theta=(\theta_1,...,\theta_d),\theta_j=\theta_k=0\}$.

Proof of Proposition 5.3. We first prove the item (i). Let $\xi\in C^1(\mathbb{R}^+,\mathbb{R})$ be such that

$$\mathbf{A}(x)=(-x_2\xi(|x|^2),x_1\xi(|x|^2)),\ x\in\mathbb{R}^2,$$

satisfies (5.1) and

$$B(x)=\begin{pmatrix}0 & 2|x|^2\xi'(|x|^2)\\ -2|x|^2\xi'(|x|^2) & 0\end{pmatrix}\not\equiv 0$$

(e.g. $\xi(t)=\frac{1}{(1+t)^\sigma},t\in\mathbb{R}^+,\ \sigma>1$ or $\xi(t)=e^{-t},\ t\in\mathbb{R}^+$). We define $w_5(\mathbf{A},\theta,x)=\int_{-\infty}^0\int_{-\infty}^\tau 2|\sigma\theta+x|^2\xi'(|\sigma\theta+x|^2)d\sigma d\tau-\int_0^{+\infty}\int_\tau^{+\infty}2|\sigma\theta+x|^2\xi'(|\sigma\theta+x|^2)d\sigma d\tau$, for $(\theta,x)\in T\mathbb{S}^{d-1}$. Let $(\theta,x)\in T\mathbb{S}^{d-1}$ be fixed. Using $|\sigma\theta+x|^2=\sigma^2+|x|^2$ we obtain $w_5(\mathbf{A},\theta,x)=0$. From this equality and (5.3b) it follows that $w_4(\mathbf{A},\theta,x)=w_5(\mathbf{A},\theta,x)(\theta_2,-\theta_1)=0$ $(\theta=(\theta_1,\theta_2))$. The item (i) is proved.

We prove the item (ii). We shall distinguish the case $d=3$ from the case $d\geq 4$.

First let $d\geq 4$. Let $j,k=1..d$ be fixed, $j\neq k$. Formula (5.12a) implies (5.17c). Let $x'\in\mathbb{R}^d$. As $d\geq 4$, the dimension of the vector space $H_{j,k}=\{(x_1,\ldots,x_d)\in\mathbb{R}^d|x_j=x_k=0\}$ is greater than or equal to 2. Let $\{e_1,e_2\}$ be an orthonormal family of $H_{j,k}$. Let Y be the affine plane of $\mathbb{R}^d$ which passes through x' and whose tangent vector space is the vector space spanned by $\{e_1,e_2\}$. From (5.17c), it follows that the integral of $B_{j,k}$ over any straight line of Y is known from

$\frac{\partial}{\partial x_k}\tilde{w_4}(\mathbf{A})_j(\theta,x) - \frac{\partial}{\partial x_j}\tilde{w_4}(\mathbf{A})_k(\theta,x)$ given for all $(\theta,x)\in\tilde{\mathcal{V}}_{j,k}$. Thus using results on inversion of the X-ray transform P (see [GGG], [Na] and [No1]), we obtain that $B_{j,k|Y}$ can be reconstructed from $\frac{\partial}{\partial x_k}\tilde{w_4}(\mathbf{A})_j(\theta,x) - \frac{\partial}{\partial x_j}\tilde{w_4}(\mathbf{A})_k(\theta,x)$ given for all $(\theta,x)\in\tilde{\mathcal{V}}_{j,k}$ (where $B_{j,k|Y}$ denotes the restriction of $B_{j,k}$ to Y). Hence $B_{j,k}(x')$ can be reconstructed from $\frac{\partial}{\partial x_k}\tilde{w_4}(\mathbf{A})_j(\theta,x) - \frac{\partial}{\partial x_j}\tilde{w_4}(\mathbf{A})_k(\theta,x)$ given for all $(\theta,x)\in\tilde{\mathcal{V}}_{j,k}$.

Then let $d=3$ and let $l=1..3$ be fixed. Under conditions (5.1), $B_{j,k,l}\in L^1(\mathbb{R}^3)$, for $j,k=1..3$. Let $p\in\mathbb{R}^3$ be fixed. From (5.12b) and (5.13) we obtain

$$\begin{aligned}&\theta\star(-\mathcal{F}B_{2,3,l}(p),\mathcal{F}B_{1,3,l}(p),-\mathcal{F}B_{1,2,l}(p))= \qquad (5.18)\\ &(2\pi)^{-3/2}\theta\star\bigg(\int_{\Pi_\theta}e^{-iy\star p}\tilde{w_4}(\mathbf{A})_{2,3,l}(\theta,y)dy,\ -\int_{\Pi_\theta}e^{-iy\star p}\tilde{w_4}(\mathbf{A})_{1,3,l}(\theta,y)dy,\\ &\qquad\int_{\Pi_\theta}e^{-iy\star p}\tilde{w_4}(\mathbf{A})_{1,2,l}(\theta,y)dy\bigg)\end{aligned}$$

for any $\theta\in\mathbb{S}^2,\ \theta\star p=0$. The formula (5.18) implies (5.17a). To prove that (5.18) also implies (5.17b), we shall use the following

Lemma 5.1. *Under the conditions* (5.1), $(-\mathcal{F}B_{2,3,l}(p),\mathcal{F}B_{1,3,l}(p),-\mathcal{F}B_{1,2,l}(p))\star p=0$, *for* $p\in\mathbb{R}^3$.

Lemma 5.1 and (5.18) imply (5.17b).

Let $m,n=1,2,3$ $m\neq n$. Using the injectivity of the Fourier transform and (5.17a) and (5.17b), we obtain that $B_{m,n,l}$ is uniquely determined by $w_4(\mathbf{A},\theta,x)$ given for all $(\theta,x)\in T\mathbb{S}^{d-1}$. Since $B_{m,n}$ vanishes at infinity, we deduce that $B_{m,n}$ is uniquely determined by $w_4(\mathbf{A},\theta,x)$ given for all $(\theta,x)\in T\mathbb{S}^{d-1}$.

Proposition 5.3 is proved. □

Proof of Lemma 5.1. We define $\lambda:\mathbb{R}^3\to\mathbb{R}$ by

$$\lambda(p)=(-\mathcal{F}B_{2,3,l}(p),\mathcal{F}B_{1,3,l}(p),-\mathcal{F}B_{1,2,l}(p))\star p,\ p=(p_1,p_2,p_3)\in\mathbb{R}^3. \qquad (5.19)$$

Now we shall use the tempered distributions and $\mathcal{F}$ shall denotes the Fourier transform of tempered distributions. Under conditions (5.1), $\mathbf{A}_i$ defines a tempered distribution of $\mathcal{S}'(\mathbb{R}^3)$.

From (5.2) and (5.19) it follows that

$$\begin{aligned}<\lambda(p),\phi>=&<p_1p_lp_2\mathcal{F}\mathbf{A}_3-p_1p_lp_3\mathcal{F}\mathbf{A}_2-p_2p_lp_1\mathcal{F}\mathbf{A}_3\\ &+p_2p_lp_3\mathcal{F}\mathbf{A}_1+p_3p_lp_1\mathcal{F}\mathbf{A}_2-p_3p_lp_2\mathcal{F}\mathbf{A}_1,\phi>\\ =&0 \qquad (5.20)\end{aligned}$$

for $\phi\in\mathcal{S}(\mathbb{R}^3)$. Since $B_{m,n,l}\in L^1(\mathbb{R}^3)$, $\mathcal{F}B_{m,n,l}$ is a continuous function on $\mathbb{R}^3$ for any $m,n=1,2,3$. Thus λ is continuous on $\mathbb{R}^3$. From the continuity of λ and (5.20), it follows that $\lambda\equiv 0$.

Lemma 5.1 is proved. □

Now we are ready to prove Proposition 1.2. We note that

$$w_4(\mathbf{A},\theta,x)=\frac{1}{2}(w_2(V,\mathbf{A},\theta,x)-w_2(V,\mathbf{A},-\theta,x)), \qquad (5.21a)$$

$$\begin{aligned} PV(\theta,x)\theta+&\int_0^{+\infty}\int_\tau^{+\infty}\nabla V(s\theta+x)dsd\tau-\int_{-\infty}^0\int_{-\infty}^\tau\nabla V(s\theta+x)dsd\tau \qquad (5.21b)\\ &=\frac{1}{2}(w_2(V,\mathbf{A},\theta,x)+w_2(V,\mathbf{A},-\theta,x)), \end{aligned}$$

for $(\theta,x)\in T\mathbb{S}^{d-1}$. The formulas (5.21), Proposition 1.1 of [Jo] and Proposition 5.3 imply Proposition 1.2. □

Remark 5.2. If we replace the conditions (5.1) and the formula (5.2) by the conditions (4.11b), (4.11c) and (4.11d), then Propositions 5.1, 5.2, 5.3 and Lemma 5.1 still hold. To prove Propositions 5.1, 5.2 and 5.3 under the conditions (4.11b), (4.11c) and (4.11d), we use the transversal gauge (given by (4.12)) and we follow Proof of Propositions 5.1, 5.2 and 5.3 under the conditions (5.1). To prove Lemma 5.1 under the conditions (4.11b), (4.11c) and (4.11d), we use the transversal gauge $\mathbf{A}$ (given by (4.12)) and we note that (4.13a) implies that $\mathbf{A}_i$ defines a tempered distribution on $\mathcal{S}(\mathbb{R}^d)$ for $i=1..d$, and we follow Proof of Lemma 5.1 under the conditions (5.1).

References

[A] S. Arians, Geometric approach to inverse scattering for the Schrödinger equation with magnetic and electric potentials, *J. Math. Phys.* **38**(6), 2761-2773 (1997).

[E] A. Einstein, Über das Relativitätsprinzip und die aus demselben gezogenen Folgerungen, *Jahrbuch der Radioaktivität und Elektronik* **4**, 411-462 (1907).

[ER1] G. Eskin, J. Ralston, Inverse scattering problem for the Schrödinger equation with magnetic potential at a fixed energy, *Comm. Math. Phys.* **173**, 199-224 (1995).

[ER2] G. Eskin, J. Ralston, Inverse scattering problems for the Schrödinger operators with magnetic and electric potentials, *IMA vol. Math. Appl.* **90**, 147-166 (1997).

[EW] V. Enss, R. Weder, Inverse potential scattering: a geometrical approach, in *Mathematical Quantum Theory II: Schrödinger operators* (Feldman, J., Froese, R. and Rosen, L., eds.), CRM Proc. Lecture Notes 8, Amer. Math. Soc., Providence, R.I., 1995.

[F] L.D. Faddeev, Uniqueness of solution of the inverse scattering problem, *Vestnik. Leningrad. Univ.* **11**(7), 126-130 (1956).

[GGG] I.M. Gel'fand, S.G. Gindikin, M.I. Graev, Integral geometry in affine and projective spaces, *Itogi Nauki i Tekhniki, Sovr. Prob. Mat.* **16**, 53-226 (1980) (Russian).

[GN] M.L. Gerver, N. S. Nadirashvili, Inverse problem of mechanics at high energies, *Comput. Seismology* **15**, 118-125 (1983) (Russian).

[Ha] G. Hachem, The Faddeev formula in the inverse scattering for Dirac operators, *Helv. Phys. Acta* **72**(5&6), 301-315 (1999).

[HN] G.M. Henkin, R.G. Novikov, A multidimensional inverse problem in quantum and acoustic scattering, *Inverse Problems* **4**, 103-121 (1988).

[I] H. T. Ito, High-energy behavior of the scattering amplitude for a Dirac operator, *Publ. RIMS, Kyoto Univ.* **31**, 1107-1133 (1995).

[Jo] A. Jollivet, On inverse scattering for the multidimensional relativistic Newton equation at high energies, *J. Math. Phys.* (2006), to appear (2005 preprint, /math-ph/0502040).

[Ju] W. Jung, Geometric approach to inverse scattering for Dirac equation, *J. Math. Phys.* **38**(1), 39-48 (1997).

[LL1] L.D. Landau, E.M. Lifschitz, *Mechanics*, Pergamon Press Oxford, 1960.

[LL2] L.D. Landau, E.M. Lifschitz, *The Classical Theory of Fields*, Pergamon Press New York, 1971.

[Na] F. Natterer, *The Mathematics of Computerized Tomography*, Teubner Stuttgart and Wiley Chichester, 1986.

[Ni] F. Nicoleau, A stationary approach to inverse scattering for Schrödinger operators with first order perturbation, *Commun. partial differ. equ.* **22** (3&4), 527-553 (1997).

[No1] R.G. Novikov, Small angle scattering and X-ray transform in classical mechanics, *Ark. Mat.* **37**, 141-169 (1999).

[No2] R.G. Novikov, The $\overline{\partial}$-approach to approximate inverse scattering at fixed energy in three dimensions, *IMRP Int. Math. Res. Pap.* **6**, 287-349 (2005).

[R] J. Radon, Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten, *Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math.-Nat. K1* **69**, 262-267 (1917).

[S] B. Simon, Wave operators for classical particle scattering, *Comm. Math. Phys.* **23**, 37-48 (1971).

[Y] K. Yajima, Classical scattering for relativistic particles, *J. Fac. Sci., Univ. Tokyo, Sect. I A* **29**, 599-611 (1982).

A. Jollivet
Laboratoire de Mathématiques Jean Leray (UMR 6629)
Université de Nantes
F-44322, Nantes cedex 03, BP 92208, France
e-mail: jollivet@math.univ-nantes.fr