# A mathematical model for the Fermi weak interactions

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# A MATHEMATICAL MODEL FOR THE FERMI WEAK INTERACTIONS

*by*

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***Abstract.*** — We consider a mathematical model of the Fermi theory of weak interactions as patterned according to the well-known current-current coupling of quantum electrodynamics. We focuss on the example of the decay of the muons into electrons, positrons and neutrinos but other examples are considered in the same way. We prove that the Hamiltonian describing this model has a ground state in the fermionic Fock space for a sufficiently small coupling constant. Furthermore we determine the absolutely continuous spectrum of the Hamiltonian and by commutator estimates we prove that the spectrum is absolutely continuous away from a small neighborhood of the thresholds of the free Hamiltonian. For all these results we do not use any infrared cutoff or infrared regularization even if fermions with zero mass are involved.

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## 1. Introduction

In this note we consider a mathematical model of the Fermi theory of weak interactions as patterned according to the well-known current-current coupling of quantum electrodynamics (see [**GM89, Wei96**]). The weak interaction processes are well described at low energy by the current-current coupling. We choose the example of the decay of the muons into electrons, positrons and neutrinos. The beta decay of the neutron could be considered too.

The mathematical framework involves a fermionic Fock space for the particles and the antiparticles and the interaction is described in terms of annihilation and creation operators together with an $L^2$-kernel with respect to the momenta. The total Hamiltonian, which is the sum of the free energy of the particles and the antiparticles and of the interaction, is a self-adjoint operator in the Fock space. We prove that this Hamiltonian has a ground state in the Fock space for a sufficiently small coupling constant. Furthermore we determine the absolutely continuous spectrum of the Hamiltonian and by commutator estimates we prove that the spectrum is absolutely continuous away from a small neighborhood of the thresholds of the free Hamiltonian.
From the mathematical point of view, the interaction is no more invariant by translation and the singularity of the kernel at the origin is not too strong. In fact the physical formal kernel is locally bounded at the origin. This means that there is no infrared problem even if fermions with zero mass are involved in the model in contrast to the case of QED. Detailed proofs are only given for the Hamiltonian associated with the decay of muons.
We also describe the mathematical model for the beta dacay of quarks $u$ and $d$ for which the results will be the same. We also consider the decay of the massive bosons $W^+$ and $W^-$.

For the proofs we essentially follow the methods developed in [**BFS98**] [**BDG04**] and in [**AGG06**] for the existence of the ground state and those developed by [**BFS98**] and [**Ski98**] for the study of the continuous singular spectrum.

Let us finally mention that the same results should hold in Fock spaces associated to the Dirac equation in Schwarschild, Reisner-Nordstrøm and Kerr black holes as soon as a generalized eigenfunction expansion for the Dirac equation in that context is known.

## 2. The model

The decay of the muons involves four species of particles and antiparticles, the muons $\mu^-$ and $\mu^+$, the electron $e^-$ and the positron $e^+$, the neutrino $\nu_e$ and the antineutrino $\bar{\nu}_e$ associated to the electron and the neutrino $\nu_\mu$ and the antineutrino $\bar{\nu}_\mu$ associated to the muon.

In this article we consider the neutrinos $\nu_e$ and $\nu_\mu$ together with the antineutrinos $\bar{\nu}_e$ and $\bar{\nu}_\mu$ as neutrinos and antineutrinos with different quantum leptonic numbers (see [**GM89**], [**PD95**]). Thus, according to the convention described in section 4.1 of [**Wei95**] and from the mathematical point of view, in what follows the corresponding creation and annihilation operators for $\nu_e$ and $\bar{\nu}_e$ will anticommute with those for $\nu_\mu$ and $\bar{\nu}_\mu$. Our proof does not work if the neutrinos $\nu_e$ and $\nu_\mu$ are considered as particles of different species i.e., if the corresponding creation and annihilation operators for $\nu_e$ and $\bar{\nu}_e$ commute with those for $\nu_\mu$ and $\bar{\nu}_\mu$.

Concerning our notations from now on the particles and antiparticles 1 will be the electrons $e^-$ and the positrons $e^+$, the particles and antiparticles 2 will be the neutrinos $\nu_e$, $\bar{\nu}_e$, the particles and antiparticles 3 will be the neutrinos $\nu_\mu$, $\bar{\nu}_\mu$ and, finally, the particles and antiparticles 4 will be the muons $\mu^-$ and $\mu^+$.

Let $\xi = (p, s)$ be the quantum variables of a particle of spin 1/2. Here $p \in \mathbb{R}^3$ is the momentum, $s \in \{-1/2, 1/2\}$ is the spin polarization of particles and antiparticles 1 and 4 and $s \in \{-1, 1\}$ is the helicity of particles and antiparticles 2 and 3. We set $\Sigma_1 = \mathbb{R}^3 \times \{-1/2, 1/2\}$ for the particles and antiparticles 1 and 4 and $\Sigma_2 = \mathbb{R}^3 \times \{-1, 1\}$ for particles and antiparticles 2 and 3. We will denote by $\bar{\xi}$ the quantum variables of an antiparticle.

Let us define the Fock space. Set

$$Q = (q, \bar{q}, r, \bar{r}, s, \bar{s}, t, \bar{t}) \in \mathbb{N}^8$$

where $q$ (resp. $r,s,t$) is the number of particles 1 (resp. 2,3,4) and $\bar{q}$ (resp. $\bar{r},\bar{s},\bar{t}$) is the number of antiparticles 1 (resp. 2,3,4). For $i=q,r,s,t$ and $\bar{i}=\bar{q},\bar{r},\bar{s},\bar{t}$ we introduce the following sets of variables:

$$\Xi_i=(\xi_1,\xi_2,\ldots,\xi_i) \quad \Xi_{\bar{i}}=(\bar{\xi}_1,\bar{\xi}_2,\ldots,\bar{\xi}_i).$$

Notice that for the neutrinos and antineutrinos we could use another sets of variables by adding leptonic quantum numbers to the $\xi$'s in order to get an equivalent framework.

Let us denote by $\Psi^{(Q)}(\cdot)$ a measurable function of the set of variables $\Xi_q,\Xi_{\bar{q}},\ldots,\Xi_t,\Xi_{\bar{t}}$ which is antisymmetric with respect to each set of variables $\Xi_i$ and $\Xi_{\bar{i}}$ separately and which is square integrable:

$$\|\Psi^{(Q)}\|^2=\int\left|\Psi^{(Q)}(\Xi_q,\Xi_{\bar{q}},\Xi_r,\Xi_{\bar{r}},\Xi_s,\Xi_{\bar{s}},\Xi_t,\Xi_{\bar{t}})\right|^2\prod_{i=(q,r,s,t)}d\Xi_i\prod_{\bar{i}=(\bar{q},\bar{r},\bar{s},\bar{t})}d\Xi_{\bar{i}}<\infty$$

where $d\Xi_i=\prod_{k=1}^{i}d\xi_k$, $d\xi=\sum_s\int d^3p$ and $d\Xi_{\bar{i}}=\prod_{k=1}^{i}d\bar{\xi}_k$, $d\bar{\xi}=\sum_s\int d^3\bar{p}$. When $i=0$ or $\bar{i}=0$, the corresponding variables do not appear in $\Psi^{(Q)}$.

The space $\mathcal{F}^{(Q)}=\{\Psi^{(Q)}\mid\|\Psi^{(Q)}\|<\infty\}$ is an Hilbert space and the Fock space is defined by

$$\mathcal{F}=\oplus_{Q\in\mathbb{N}^8}\mathcal{F}^{(Q)}$$

where $\mathcal{F}^{(0)}=\mathbb{C}$. The vacuum $\Omega$ is the state $(\Psi^{(Q)})_Q$ with $\Psi^{(Q)}=0$ for $Q\neq 0$ and $\Psi^{(0)}=1$. $\mathcal{F}$ is an Hilbert space and if $\Psi=(\Psi^{(Q)})_Q\in\mathcal{F}$ we have

$$\|\Psi\|^2=\sum_{Q\in\mathbb{N}^8}\|\Psi^{(Q)}\|^2.$$

We can now define the formal annihilation and creation operators $b_{j,\epsilon}(\xi)$ and $b^{\star}_{j,\epsilon}(\xi)$ for each type of particles and antiparticles. We have

$$\begin{aligned}(b_{1,+}(\xi)\Psi)^{(Q)}&(\xi_1,\ldots,\xi_q;\Xi_{\bar{q}};\Xi_r;\Xi_{\bar{r}};\Xi_s;\Xi_{\bar{s}};\Xi_t;\Xi_{\bar{t}})=\\&\sqrt{q+1}\,\Psi^{(q+1,\bar{q},\ldots,\bar{t})}(\xi,\xi_1,\ldots,\xi_q;\Xi_{\bar{q}};\Xi_r;\Xi_{\bar{r}};\Xi_s;\Xi_{\bar{s}};\Xi_t;\Xi_{\bar{t}})\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}(b_{1,-}(\xi)\Psi)^{(Q)}&(\Xi_q;\bar{\xi}_1,\ldots,\bar{\xi}_{\bar{q}};\Xi_r;\ldots;\Xi_{\bar{t}})=\\&\sqrt{\bar{q}+1}(-1)^q\Psi^{(q,\bar{q}+1,\ldots,\bar{t})}(\Xi_q;\xi,\bar{\xi}_1,\ldots,\bar{\xi}_{\bar{q}};\Xi_r;\ldots;\Xi_{\bar{t}}).\end{aligned}\tag{2.2}$$

The operators $b_{2,\pm}(\xi)$ (resp. $b_{4,\pm}(\xi)$) are defined similarly by substituting $r$ and $\bar{r}$ (resp. $t$ and $\bar{t}$) for $q$ and $\bar{q}$ in an obvious way.

Furthermore, taking into account the anticommutation between $b_{3,\pm}$ and $b_{2,\pm}$, we have

(2.3)

$$(b_{3,+}(\xi)\Psi)^{(Q)}(\Xi_q;\Xi_{\bar{q}};\Xi_r;\Xi_{\bar{r}};\xi_1,\dots,\xi_s;\Xi_{\bar{s}};\Xi_t;\Xi_{\bar{t}}) = \sqrt{s+1}(-1)^{r+\bar{r}}\Psi^{(q,\bar{q},r,\bar{r},s+1,\bar{s},t,\bar{t})}(\Xi_q;\Xi_{\bar{q}};\Xi_r;\Xi_{\bar{r}};\xi,\xi_1,\dots,\xi_s;\Xi_{\bar{s}};\Xi_t;\Xi_{\bar{t}})$$

and

(2.4)

$$(b_{3,-}(\xi)\Psi)^{(Q)}(\Xi_q;\Xi_{\bar{q}};\Xi_r;\Xi_{\bar{r}};\Xi_s;\bar{\xi}_1,\dots,\bar{\xi}_{\bar{s}};\Xi_t;\Xi_{\bar{t}}) = \sqrt{\bar{s}+1}(-1)^{r+\bar{r}+s}\Psi^{(q,\bar{q},r,\bar{r},s,\bar{s}+1,t,\bar{t})}(\Xi_q;\Xi_{\bar{q}};\Xi_r;\Xi_{\bar{r}};\Xi_s;\xi,\bar{\xi}_1,\dots,\bar{\xi}_{\bar{s}};\Xi_t;\Xi_{\bar{t}}).$$

As usual $b^\star_{j,\epsilon}(\xi)$ is the formal adjoint of $b_{j,\epsilon}(\xi)$, for example

(2.5)

$$(b^\star_{1,+}(\xi)\Psi)^{(q+1,\bar{q},r,\bar{r},s,\bar{s},t,\bar{t})}(\xi_1,\dots,\xi_{q+1};\Xi_{\bar{q}};\Xi_r;\Xi_{\bar{r}};\Xi_s;\Xi_{\bar{s}};\Xi_t;\Xi_{\bar{t}}) = \frac{1}{\sqrt{q+1}}\sum_{i=1}^{q+1}(-1)^{i+1}\delta(\xi-\xi_i)\Psi^{(q,\bar{q},r,\bar{r},s,\bar{s},t,\bar{t})}(\xi_1,\dots,\hat{\xi_i},\dots,\xi_{q+1};\Xi_{\bar{q}};\Xi_r;\Xi_{\bar{r}};\Xi_s;\Xi_{\bar{s}};\Xi_t;\Xi_{\bar{t}})$$

where $\hat{\cdot}$ denotes that the ith variable has to be omitted.
The following canonical anticommutation relations hold

$$\{b_{j,\epsilon}(\xi), b^\star_{j,\epsilon'}(\xi')\} = \delta_{\epsilon,\epsilon'}\delta(\xi-\xi'), \quad j=1,2,3,4,\ \epsilon,\epsilon'=\pm$$

where $\delta(\xi-\xi') = \delta_{s,s'}\delta(p-p')$,

$$\{b_{j,\epsilon}(\xi), b_{j,\epsilon'}(\xi')\} = \{b^\star_{j,\epsilon}(\xi), b^\star_{j,\epsilon'}(\xi')\} = 0, \quad j=1,2,3,4,\ \epsilon,\epsilon'=\pm$$

$$\{b_{2,\epsilon}(\xi), b^\sharp_{3,\epsilon'}(\xi')\} = \{b^\star_{2,\epsilon}(\xi), b^\sharp_{3,\epsilon'}(\xi')\} = 0$$

where $b^\sharp$ is $b$ or $b^\star$.
Note that

$$[b_{i,\epsilon}(\xi), b^\sharp_{j,\epsilon'}(\xi')] = [b^\star_{i,\epsilon}(\xi), b^\sharp_{j,\epsilon'}(\xi')] = 0, \quad j=1,2,3,4,\ i=1,4 \text{ and } j\neq i.$$

Let $\mathcal{F}_0$ be the subspace of functions $\Psi = (\Psi^{(Q)})_Q$ such that $\Psi^{(Q)}$ is a function in the Schwartz space and $\Psi^{(Q)} = 0$ for all but finitely many $Q$. The $b_{j,\epsilon}(\xi)$'s are well defined operators on $\mathcal{F}_0$ but they are not closable. It is better to introduce the following operators:

$$b_{j,\epsilon}(\phi) = \int b_{j,\epsilon}(\xi)\overline{\phi(\xi)}d\xi,$$

$$b^\star_{j,\epsilon}(\phi) = \int b^\star_{j,\epsilon}(\xi)\phi(\xi)d\xi$$

where $\phi \in L^2(\Sigma)$ and $\Sigma = \Sigma_1$ when $j = 1, 4$ and $\Sigma = \Sigma_2$ when $j = 2, 3$. Both $b_{j,\epsilon}(\phi)$ and $b^\star_{j,\epsilon}(\phi)$ are bounded operators on $\mathcal{F}$ and

$$\|b^\star_{j,\epsilon}(\phi)\| = \|b_{j,\epsilon}(\phi)\| = \|\phi\|.$$

The $b_{j,\epsilon}(\phi)$'s and the $b^\star_{j,\epsilon}(\phi)$'s satisfy similar anticommutation relations (see [**Tha92**]).
The free Hamiltonian $H_0$ is given by

$$H_0 = \sum_{j=1}^{4} \sum_{\epsilon=+,-} \int d\xi \omega_j(\xi) b^\star_{j,\epsilon}(\xi) b_{j,\epsilon}(\xi) \tag{2.6}$$

where

$$\begin{aligned} \omega_1(\xi) &= \omega_1(p) = \sqrt{|p|^2 + m_1^2} \\ \omega_4(\xi) &= \omega_4(p) = \sqrt{|p|^2 + m_4^2} \\ \omega_j(\xi) &= \omega_j(p) = |p|,\ j = 2, 3 \end{aligned} \tag{2.7}$$

and the mass $m_1$ and $m_4$ are strictly positive. We know that $m_1 < m_4$.
$H_0$ is essentially self-adjoint on $\mathcal{F}_0$, we still denote $H_0$ its self-adjoint extension.
The interaction, denoted by $H_I$ is given by

$$\begin{aligned} H_I = &\sum_{\epsilon\neq\epsilon'} \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 G_{\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3, \xi_4) \\ &\quad b^\star_{1,\epsilon}(\xi_1) b^\star_{2,\epsilon'}(\xi_2) b^\star_{3,\epsilon}(\xi_3) b_{4,\epsilon}(\xi_4) \\ &+ \sum_{\epsilon\neq\epsilon'} \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 \overline{G_{\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3, \xi_4)} \\ &\quad b^\star_{4,\epsilon}(\xi_4) b_{3,\epsilon}(\xi_3) b_{2,\epsilon'}(\xi_2) b_{1,\epsilon}(\xi_1) \end{aligned} \tag{2.8}$$

where $G_{\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3, \xi_4)$ is a kernel.
In particular this interaction describes the decay of the muon $\mu$ into an electron and two neutrinos $\bar{\nu}_e$ and $\nu_\mu$.
The total Hamiltonian is then

$$H = H_0 + gH_I \tag{2.9}$$

where $g \in \mathbb{R}$ is the coupling constant.
We first show that a self-adjoint operator in $\mathcal{F}$ is associated with the total Hamiltonian $H$ if the kernels $G_{\epsilon,\epsilon'}$ are in $L^2$.

Let $\{e_{+,i}, e_{-,\bar{i}},\ i, \bar{i} = 1, 2, \ldots\}$ (resp. $\{f_{+,i}, f_{-,\bar{i}},\ i, \bar{i} = 1, 2, \ldots\}$, $\{g_{+,i}, g_{-,\bar{i}},\ i, \bar{i} = 1, 2, \ldots\}$, $\{h_{+,i}, h_{-,\bar{i}},\ i, \bar{i} = 1, 2, \ldots\}$) be two basis of $L^2(\Sigma_1)$ (resp. $L^2(\Sigma_2)$, $L^2(\Sigma_2)$, $L^2(\Sigma_1)$). We assume that the $e$'s, $f$'s, $g$'s and $h$'s are smooth functions

in the Schwartz space with respect to $p$.
For every $Q = (q, \bar{q}, r, \bar{r}, s, \bar{s}, T, \bar{t}) \in \mathbb{N}^8$ we now consider vectors in $\mathcal{F}$ of the following form:

$$
\begin{aligned}
\Psi^{(Q)} =& b^\star_{1,+}(e_{+i_1}) \dots b^\star_{1,+}(e_{+i_q}) b^\star_{1,-}(e_{-\bar{i}_1}) \dots b^\star_{1,-}(e_{-\bar{i}_{\bar{q}}}) \\
& b^\star_{2,+}(f_{+j_1}) \dots b^\star_{2,+}(f_{+j_r}) b^\star_{2,-}(f_{-\bar{j}_1}) \dots b^\star_{2,-}(f_{-\bar{j}_{\bar{r}}}) \\
& b^\star_{3,+}(g_{+k_1}) \dots b^\star_{3,+}(g_{+k_s}) b^\star_{3,-}(g_{-\bar{k}_1}) \dots b^\star_{3,-}(g_{-\bar{k}_{\bar{r}}}) \\
& b^\star_{4,+}(h_{+l_1}) \dots b^\star_{4,+}(h_{+l_t}) b^\star_{4,-}(h_{-\bar{l}_1}) \dots b^\star_{4,-}(h_{-\bar{l}_{\bar{t}}})\ \Omega.
\end{aligned}
\tag{2.10}
$$

The indexes are ordered such that $i_1 < \dots < i_q$, $\bar{i}_1 < \dots < \bar{i}_{\bar{q}}$ and similarly for the indexes $j, k, l$. The set $\{\Psi^{(Q)} \mid Q \in \mathbb{N}^8\}$ is an orthonormal basis of $\mathcal{F}$ (see [**Tha92**]) and the set

$$\mathcal{F}_{\text{f}in} = \{ \text{ finite linear combination of the basis vectors of the form (2.10) } \}$$

is dense in $\mathcal{F}$.
As the formal expression of $H$ shows, we have to deal with operators in $\mathcal{F}$ built from the product of creation and annihilation operators.
For $H_{\epsilon,\epsilon'}(\cdot,\cdot,\cdot) \in L^2(\Sigma_1 \times \Sigma_2 \times \Sigma_2)$ the formal operator

$$\int_{\Sigma_1\times\Sigma_2\times\Sigma_2} d\xi_1 d\xi_2 d\xi_3 \overline{H_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)} b_{3,\epsilon}(\xi_3) b_{2,\epsilon'}(\xi_2) b_{1,\epsilon}(\xi_1)$$

is defined as a quadratic form on $\mathcal{F}_{\text{f}in} \times \mathcal{F}_{\text{f}in}$:

$$\int_{\Sigma_1\times\Sigma_2\times\Sigma_2} d\xi_1 d\xi_2 d\xi_3 < \Psi\ ,\ \overline{H_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)} b_{3,\epsilon}(\xi_3) b_{2,\epsilon'}(\xi_2) b_{1,\epsilon}(\xi_1) \Phi > .$$

By mimicking the proof of Theorem X.44 in [**RS75**], we get an operator, denoted by $A_{\epsilon,\epsilon'}$, associated with the form such that $A_{\epsilon,\epsilon'}$ is the unique operator in $\mathcal{F}$ such that $\mathcal{F}_{\text{f}in} \subset D(A_{\epsilon,\epsilon'})$ is a core for $A_{\epsilon,\epsilon'}$ and

$$A_{\epsilon,\epsilon'} = \int_{\Sigma_1\times\Sigma_2\times\Sigma_2} d\xi_1 d\xi_2 d\xi_3 \overline{H_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)} b_{3,\epsilon}(\xi_3) b_{2,\epsilon'}(\xi_2) b_{1,\epsilon}(\xi_1)$$

as a quadratic forms on $\mathcal{F}_{\text{f}in} \times \mathcal{F}_{\text{f}in}$. Note that the formal operator

$$\int_{\Sigma_1\times\Sigma_2\times\Sigma_2} d\xi_1 d\xi_2 d\xi_3 H_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3) b^\star_{1,\epsilon}(\xi_1) b^\star_{2,\epsilon'}(\xi_2) b^\star_{3,\epsilon}(\xi_3)$$

is similarly associated with $A^\star_{\epsilon,\epsilon'}$ and we have

$$A^\star_{\epsilon,\epsilon'} = \int_{\Sigma_1\times\Sigma_2\times\Sigma_2} d\xi_1 d\xi_2 d\xi_3 H_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3) b^\star_{1,\epsilon}(\xi_1) b^\star_{2,\epsilon'}(\xi_2) b^\star_{3,\epsilon}(\xi_3)$$

as a quadratic forms on $\mathcal{F}_{\text{f}in} \times \mathcal{F}_{\text{f}in}$.
The proofs of the following propositions are similar to those in [**BDG04**]. For

sake of completeness we give here complete proofs.
We have

***Proposition 2.1.*** — *Suppose that $H_{\epsilon,\epsilon'}(\cdot,\cdot,\cdot) \in L^2(\Sigma_1\times\Sigma_2\times\Sigma_2)$. Then $A_{\epsilon,\epsilon')}$ and $A^\star_{\epsilon,\epsilon'}$ are bounded operators in $\mathcal{F}$ with*

$$\|A_{\epsilon,\epsilon'}\| = \|A^\star_{\epsilon,\epsilon'}\| \leq \|H_{\epsilon,\epsilon'}\|_{L^2(\Sigma_1\times\Sigma_2\times\Sigma_2)}.$$

*Proof.* — Let $\Psi^{(Q)}$ be a vector of the form (2.10). For simplicity we assume that $\{i_1,\dots,i_q\} = \{1,\dots,q\}$, $\{\bar{i}_1,\dots,\bar{i}_{\bar{q}}\} = \{1,\dots,\bar{q}\}$, etc...Let us consider $A_{+,-}$, the other choices of $\epsilon$ and $\epsilon'$ are treated similarly. A straightforward computation shows that

(2.11)

$$\begin{aligned} A_{+,-}\Psi^{(Q)} = &\sum_{\alpha=1}^{q}\sum_{\beta=1}^{\bar{r}}\sum_{\gamma=1}^{s}(-1)^{\alpha+\beta+\gamma+1}(H_{+,-}, e_{+\alpha}\otimes f_{-\beta}\otimes g_{+\gamma})_{L^2(\Sigma_1\times\Sigma_2\times\Sigma_2)} \\ &\prod_{i=1\ i\neq\alpha}^{q} b^\star_{1+}(e_{+i})\prod_{\bar{i}=1}^{\bar{q}} b^\star_{1-}(e_{-\bar{i}})\prod_{j=1}^{r} b^\star_{2+}(f_{+j})\prod_{\bar{j}=1\ \bar{k}\neq\beta}^{\bar{r}} b^\star_{2-}(f_{-\bar{j}}) \\ &\prod_{k=1\ k\neq\gamma}^{s} b^\star_{3+}(g_{+k})\prod_{\bar{k}=1}^{\bar{s}} b^\star_{3-}(g_{-\bar{k}})\prod_{l=1}^{t} b^\star_{4+}(h_{+l})\prod_{\bar{k}=1}^{\bar{t}} b^\star_{4-}(h_{-\bar{l}})\ \Omega. \end{aligned}$$

As the right hand side of (2.11) is a linear combination of orthogonal vectors, we get

$$\begin{aligned} \|A_{+,-}\Psi^{(Q)}\|^2 = &\sum_{\alpha=1}^{q}\sum_{\beta=1}^{\bar{r}}\sum_{\gamma=1}^{s}|(H_{+,-}, e_{+\alpha}\otimes f_{-\beta}\otimes g_{+\gamma})|^2 \\ \leq &\|H_{+,-}\|^2\|\Psi^{(Q)}\|^2. \end{aligned} \tag{2.12}$$

Therefore, in order to prove proposition 2.1, it is enough to show that (2.12) holds for any finite linear combination of the $\Psi^{(Q)}$'s. This can be done as in the proposition 3.4 of [**BDG04**]. We omit the details. □

We now investigate operators in $\mathcal{F}$ associated with the interaction $H_I$. Let us introduce the operators number of each particle:

$$N_i = \sum_{\epsilon}\int d\xi b^\star_{i\epsilon}(\xi)b_{i\epsilon}(\xi) \quad i = 1,2,3,4. \tag{2.13}$$

Each $N_i$ is self-adjoint in $\mathcal{F}$ and $\mathcal{F}_{fin}$ is a core for it.
For $G_{\epsilon,\epsilon'}(\cdot,\cdot,\cdot,\cdot) \in L^2(\Sigma_1\times\Sigma_2\times\Sigma_2\times\Sigma_1)$ the formal operators

$$\int_{\Sigma_1\times\Sigma_2\times\Sigma_2\times\Sigma_1} d\xi_1 d\xi_2 d\xi_3 d\xi_4 G_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4)b^\star_{1,\epsilon}(\xi_1)b^\star_{2,\epsilon'}(\xi_2)b^\star_{3,\epsilon}(\xi_3)b_{4,\epsilon}(\xi_4)$$

and

$$\int_{\Sigma_1\times\Sigma_2\times\Sigma_2\times\Sigma_1} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \overline{G_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4)} b^\star_{4,\epsilon}(\xi_4) b_{3,\epsilon}(\xi_3) b_{2,\epsilon'}(\xi_2) b_{1,\epsilon}(\xi_1)$$

are defined as a quadratic form on $\mathcal{F}_{\text{f}in}\times\mathcal{F}_{\text{f}in}$. Again by mimicking the proof of Theorem X.44 in [**RS75**], we get an operator, denoted by $B_{\epsilon,\epsilon'}$, associated with the form such that $B_{\epsilon,\epsilon'}$ is the unique operator in $\mathcal{F}$ such that $\mathcal{F}_{\text{f}in} \subset D(A_{\epsilon,\epsilon'})$ is a core for $B_{\epsilon,\epsilon'}$ and

$$B_{\epsilon,\epsilon'} = \int_{\Sigma_1\times\Sigma_2\times\Sigma_2\times\Sigma_1} d\xi_1 d\xi_2 d\xi_3 d\xi_4 G_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4) b^\star_{1,\epsilon}(\xi_1) b^\star_{2,\epsilon'}(\xi_2) b^\star_{3,\epsilon}(\xi_3) b_{4,\epsilon}(\xi_4)$$

and

$$B^\star_{\epsilon,\epsilon'} = \int_{\Sigma_1\times\Sigma_2\times\Sigma_2\times\Sigma_1} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \overline{G_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4)} b^\star_{4,\epsilon}(\xi_4) b_{3,\epsilon}(\xi_3) b_{2,\epsilon'}(\xi_2) b_{1,\epsilon}(\xi_1)$$

as quadratic forms on $\mathcal{F}_{\text{f}in} \times \mathcal{F}_{\text{f}in}$.
We then have

***Proposition 2.2.*** — *Suppose that $G_{\epsilon,\epsilon'}(\cdot,\cdot,\cdot,\cdot) \in L^2(\Sigma_1\times\Sigma_2\times\Sigma_2\times\Sigma_1)$. Then $D(B_{\epsilon,\epsilon'}), D(B^\star_{\epsilon,\epsilon'}) \supset D(N_4^{1/2})$ and*

$$\begin{aligned} \|B_{\epsilon,\epsilon'}\Psi\| &\le \|G_{\epsilon,\epsilon'}\|_{L^2(\Sigma_1\times\Sigma_2\times\Sigma_2\times\Sigma_1)}\|N_4^{1/2}\Psi\|, \\ \|B^\star_{\epsilon,\epsilon'}\Psi\| &\le \|G_{\epsilon,\epsilon'}\|_{L^2(\Sigma_1\times\Sigma_2\times\Sigma_2\times\Sigma_1)}\|N_4^{1/2}\Psi\|. \end{aligned} \tag{2.14}$$

*for $\Psi \in D(N_4^{1/2})$.*

*Proof.* — We only investigate $B_{+,-}$. The proof for the other cases is quite similar. Set $Q = (q,\bar q, r, \bar r, s, \bar s, t, \bar t)$ and $Q' = (q+1,\bar q, r, \bar r+1, s+1, \bar s, t-1, \bar t)$. Let $\Psi^{(Q)}$ and $\Psi^{(Q')}$ be two vectors in $\mathcal{F}_{\text{f}in}\cap\mathcal{F}^{(Q)}$ and $\mathcal{F}_{\text{f}in}\cap\mathcal{F}^{(Q')}$ respectively. We have

$$\begin{aligned} &(\Psi^{(Q')}, B_{+,-}\Psi^{(Q)}) = \int_{\Sigma_1\times\Sigma_2\times\Sigma_2\times\Sigma_1} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\ &\Big(\overline{G_{+,-}(\xi_1,\xi_2,\xi_3,\xi_4)} b_{3,+}(\xi_3) b_{2,-}(\xi_2) b_{1,+}(\xi_1)\Psi^{(Q')}\ ,\ b_{4,+}(\xi_4)\Psi^{(Q)}\Big) \end{aligned} \tag{2.15}$$

and by the Fubini theorem, we get

$$\begin{aligned} &\left|(\Psi^{(Q')}, B_{+,-}\Psi^{(Q)})\right|^2 = \Big| \int_{\Sigma_1} d\xi_4 \Big( b_{4,+}(\xi_4)\Psi^{(Q)}, \\ &\int_{\Sigma_1\times\Sigma_2\times\Sigma_2} \overline{G_{+,-}(\xi_1,\xi_2,\xi_3,\xi_4)} b_{3,+}(\xi_3) b_{2,-}(\xi_2) b_{1,+}(\xi_1)\Psi^{(Q')}\Big)\Big|^2. \end{aligned} \tag{2.16}$$

By the Cauchy-Schwarz inequality and proposition 2.1, we obtain

(2.17)
$$\Big|(\Psi^{(Q')}, B_{+,-}\Psi^{(Q)})\Big|^2 \leq$$
$$\Big(\int_{\Sigma_1} d\xi_4 \|b_{4,+}(\xi_4)\Psi^{(Q)}\| \big(\int_{\Sigma_1\times\Sigma_2\times\Sigma_2} d\xi_1 d\xi_2 d\xi_3 |G_{+,-}(\xi_1,\xi_2,\xi_3,\xi_4)|^2\big)^{1/2}\Big)^2 \|\Psi^{(Q')}\|^2.$$

Applying again the Cauchy-Schwarz inequality and by the definition of $b_{4+}(\xi_4)$ we finally get

$$\Big|(\Psi^{(Q')}, B_{+,-}\Psi^{(Q)})\Big|^2 \leq t\|G_{+,-}\|^2\|\Psi^{(Q)}\|^2\|\Psi^{(Q')}\|^2 = \|G_{+,-}\|^2\|N_4^{1/2}\Psi^{(Q)}\|^2\|\Psi^{(Q')}\|^2.$$

Since $B_{+,-}\Psi^{(Q)} \in \mathcal{F}^{(Q')}$ we deduce

$$\Big|(\Phi, B_{+,-}\Psi^{(Q)})\Big|^2 \leq \|G_{+,-}\|^2\|N_4^{1/2}\Psi^{(Q)}\|^2\|\Phi\|^2$$

for every $\Phi \in \mathcal{F}_{fin}$. Now, since $\Phi \in \mathcal{F}_{fin}$ is dense in $\mathcal{F}$, the last inequality still holds for every $\Phi \in \mathcal{F}$ and every $Q \in \mathbb{N}^8$. Therefore we have

$$\|B_{+,-}\Psi^{(Q)}\|^2 \leq \|G_{+,-}\|^2\|N_4^{1/2}\Psi^{(Q)}\|^2$$

which yields

$$\|B_{+,-}\Psi\|^2 \leq \|G_{+,-}\|^2\|N_4^{1/2}\Psi\|^2 \tag{2.18}$$

for every $\Psi \in \mathcal{F}_{\rm fin}$. Since $\mathcal{F}_{fin}$ is a core for $N_4^{1/2}$ and $B_{+,-}$ is closable (see Theorem X.44 in [**RS75**]) we have $D(N_4^{1/2}) \subset D(B_{+,-})$ and the inequality (2.18) is still true for every $\Psi \in D(N_4^{1/2})$. □

Set

$$\begin{aligned} V_2^{\epsilon\epsilon'} = &\int_{\Sigma_1\times\Sigma_2\times\Sigma_1} d\xi_1 d\xi_3 d\xi_4 G^2_{\epsilon\epsilon'}(\xi_1,\xi_3,\xi_4) \\ &b^\star_{1\epsilon}(\xi_1) b^\star_{3\epsilon'}(\xi_3) b_{4\epsilon}(\xi_4), \\ V_3^{\epsilon\epsilon'} = &\int_{\Sigma_1\times\Sigma_2\times\Sigma_1} d\xi_1 d\xi_2 d\xi_4 G^3_{\epsilon\epsilon'}(\xi_1,\xi_2,\xi_4) \\ &b^\star_{1\epsilon}(\xi_1) b^\star_{2\epsilon'}(\xi_2) b_{4\epsilon}(\xi_4), \end{aligned} \tag{2.19}$$

where $G^j_{\epsilon\epsilon'} \in L^2(\Sigma_1\times\Sigma_2\times\Sigma_1)$, $j = 2,3$. $V_j^{\epsilon\epsilon'}$, $j = 2,3$, are defined as quadratic forms on $\mathcal{F}_{\rm fin} \times \mathcal{F}_{\rm fin}$. As above we then have

***Proposition 2.3***. — *Suppose that $G^j_{\epsilon\epsilon'} \in L^2(\Sigma_1 \times \Sigma_2 \times \Sigma_1)$, $j = 2, 3$. Then $D(V_j^{\epsilon\epsilon'}), D(V_j^{\epsilon\epsilon'\star}) \supset D(N_4^{1/2})$ and*

$$\begin{aligned} \|V_j^{\epsilon\epsilon'}\Psi\| &\le \|G^j_{\epsilon\epsilon'}\|_{L^2(\Sigma_1\times\Sigma_2\times\Sigma_1)}\|N_4^{1/2}\Psi\|, \\ \|V_j^{\epsilon\epsilon'\star}\Psi\| &\le \|G^j_{\epsilon\epsilon'}\|_{L^2(\Sigma_1\times\Sigma_2\times\Sigma_1)}\|N_4^{1/2}\Psi\|. \end{aligned} \tag{2.20}$$

*for $\Psi \in D(N_4^{1/2})$ and $j = 2, 3$.*

The proof of proposition 2.3 is exactly the same as the one of proposition 2.2.

The following theorem shows that the formal total Hamiltonian is associated with a self-adjoint operator in $\mathcal{F}$, still denoted by $H$, if the interaction kernels are in $L^2$.

***Theorem 2.4***. — *Suppose that $G_{\epsilon\epsilon'}(\cdot,\cdot,\cdot,\cdot) \in L^2(\Sigma_1 \times \Sigma_2 \times \Sigma_2 \times \Sigma_1)$ for $\epsilon \neq \epsilon'$. Then $H = H_0 + gH_I$ is a self-adjoint operator in $\mathcal{F}$ for every $g \in \mathbb{R}$ with domain $D(H_0)$.*

*Proof.* — Recall that $H_0$ with domain $\mathcal{F}_{\mathrm{fin}}$ is essentially self-adjoint. By proposition 2.2 we have, for every $\Psi \in \mathcal{F}_{\mathrm{fin}}$,

$$\|H_I\Psi\| \le 2\left(\sum_{\epsilon\neq\epsilon'}\|G_{\epsilon\epsilon'}\|_{L^2}\right)\|N_4^{1/2}\Psi\|$$

and we get for every $\epsilon > 0$,

$$\|H_I\Psi\| \le 2\left(\sum_{\epsilon\neq\epsilon'}\|G_{\epsilon\epsilon'}\|_{L^2}\right)\left(\sqrt{\epsilon/2}\|N_4\Psi\| + \frac{1}{\sqrt{2\epsilon}}\|\Psi\|\right).$$

Furthermore, since $\omega_4(p) \ge m_4$, we have

$$\|N_4\Psi\| \le \frac{1}{m_4}\|H_0\Psi\|.$$

Thus

$$\|H_I\Psi\| \le 2\left(\sum_{\epsilon\neq\epsilon'}\|G_{\epsilon\epsilon'}\|_{L^2}\right)\left(\frac{1}{m_4}\sqrt{\epsilon/2}\|H_0\Psi\| + \frac{1}{\sqrt{2\epsilon}}\|\Psi\|\right)$$

which means that $H_I$ is relatively bounded with respect to $H_0$ with zero relative bound and the theorem follows from the Kato-Rellich theorem. $\square$

## 3. The results

Our main result states that $H$ has a ground state for $g$ sufficiently small. We have

***Theorem 3.1.*** — *Suppose that for $\epsilon \neq \epsilon'$, $G_{\epsilon\epsilon'}(\cdot,\cdot,\cdot,\cdot) \in L^2(\Sigma_1,\Sigma_2,\Sigma_2,\Sigma_1)$ and*

$$\sum_{i=2}^{3} \int_{\overline{B(0,1)}} \frac{|G_{\epsilon\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4)|^2}{|p_i|^2} d\xi_1 d\xi_2 d\xi_3 d\xi_4 < \infty \tag{3.1}$$

*where $\xi_j = (p_j, s_j)$, $p_j \in \mathbb{R}^3$, $j = 1,2,3,4$ and where $\overline{B(0,1)} = \{(p_1,p_2,p_3,p_4) \in \mathbb{R}^{12} \mid \sum_{j=1}^4 |p_j|^2 \leq 1\}$.*
*Then there exists $g_0 > 0$ such that $H$ has an unique ground state for $|g| \leq g_0$. Furthermore $\sigma(H) = \sigma_{\mathrm{ac}}(H) = [\inf \sigma(H), +\infty)$.*

Notice that Theorem 3.1 is true for sharp cutoffs, i.e., when $G_{\epsilon\epsilon'} = \chi_\Lambda$, $\Lambda > 0$, with

$$\begin{aligned} \chi_\Lambda(p_1,p_2,p_3,p_4) &= 1 \text{ if } |p_j| \leq \Lambda,\ j = 1,2,3,4 \\ &= 0 \text{ otherwise.} \end{aligned} \tag{3.2}$$

This means that the ground state exists without infrared regularization even if particles with zero mass are involved.
The statement concerning the absolutely continuous spectrum of $H$ follows easily from the existence of asymptotic Fock representations of the ACR. Precisely, for $f \in L^2(\mathbb{R}^3)$ we define the operators

$$b^\flat_{j\epsilon,t}(f) = e^{itH} e^{-itH_0} b^\flat_{j\epsilon}(f) e^{itH_0} e^{-itH}, \quad j = 1,2,3,4\ , \epsilon = \pm.$$

Then for $f \in C_0^\infty(\mathbb{R}^3)$ and $\psi \in \mathcal{F}$ the strong limits of $b^\flat_{j\epsilon,t}(f)$ exist:

$$\lim_{t \to \pm\infty} b^\flat_{j\epsilon,t}(f)\psi = b^\flat_{j\epsilon,\pm}(f)\psi.$$

The $b^\flat_{j\epsilon,\pm}(f)$'s satisfy the ACR and if $\phi$ is the ground state of $H$, we have, for $f \in C_0^\infty(\mathbb{R}^3)$,

$$b^\flat_{j\epsilon,\pm}(f)\phi = 0.$$

The fact that $\sigma(H) = \sigma_{\mathrm{ac}}(H) = [\inf \sigma(H), +\infty)$ follows by mimicking [**Hir05**].
Now the next theorem concerns the absolutely continuous spectrum of $H$. We define $S$ as the set of threshold of $H_0$:

$$S = \{k m_1 + l m_4 \mid k, l \in \mathbb{N}\}. \tag{3.3}$$

***Theorem 3.2.*** — *Suppose that for* $\epsilon \neq \epsilon'$, $G_{\epsilon\epsilon'}(\cdot,\cdot,\cdot,\cdot) \in L^2(\Sigma_1\times\Sigma_2\times\Sigma_2\times\Sigma_1)$ *satisfy* (3.1) *and that for* $i = 1,2,3,4$, $p_i \cdot \nabla_{p_i} G_{\epsilon\epsilon'}$ *and* $p_i^2 \Delta_{p_i} G_{\epsilon\epsilon'}$ *are all in* $L^2(\Sigma_1 \times \Sigma_2 \times \Sigma_2 \times \Sigma_1)$. *Then there exists a constant* $C > 0$ *such that, for* $g$ *sufficiently small, the spectrum of* $H$ *in* $\mathbb{R} \setminus (S + [-C\sqrt{g}, C\sqrt{g}])$ *is absolutely continuous.*

## 4. Proof of theorem 3.1

Let $H_{I,\sigma}$ be the operator obtained from (2.8) by substituting

$$G^{\sigma}_{\epsilon\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4) = 1_{\{(p_1,p_2,p_3,p_4)||p_2 \geq \sigma,\, |p_3| \geq \sigma|\}} G_{\epsilon\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4)$$

for $G_{\epsilon\epsilon'}$ where $\sigma$ is a strictly positive parameter. We then define

$$H_\sigma = H_0 + gH_{I,\sigma}.$$

$H_\sigma$ is a self adjoint operator in $\mathcal{F}$ with domain $D(H_\sigma) = D(H_0)$ for any $g \in \mathbb{R}$ and any $\sigma > 0$.
Set

$$H_0^1 = \sum_\epsilon \int \omega_1(\xi) b^\star_{1\epsilon}(\xi) b_{1\epsilon}(\xi) d\xi + \sum_\epsilon \int \omega_4(\xi) b^\star_{4\epsilon}(\xi) b_{4\epsilon}(\xi) d\xi. \tag{4.1}$$

We consider $H_0^1$ as a self-adjoint operator in the Fock space $\mathcal{F}_1$ associated with the particles and antiparticles 1 and 4. We then have $\sigma(H_0^1) = \{0\} \cup [m_1, +\infty)$ because $m_1 < m_4$.
For $0 < \lambda < m_1$ let $P(\lambda)$ be the spectral projection of $H_0^1$ in $\mathcal{F}_1$ corresponding to $(-\infty, \lambda]$ and let $P_{\Omega_{neut}}$ be the orthogonal projection on the vacuum state of the neutrinos and antineutrinos 2 and 3. We consider $P_{\Omega_{\text{neut}}}$ as a projection in the Fock space $\mathcal{F}_2$ associated with the neutrinos and antineutrinos 2 and 3. Note that $\mathcal{F} \equiv \mathcal{F}_1 \otimes \mathcal{F}_2$. As in [**BDG04**] and [**BFS98**] theorem 3.1 is the consequence of the following theorem:

***Theorem 4.1.*** — *There exists* $g_0 > 0$ *such that for every* $g$ *satisfying* $|g| \leq g_0$ *the following properties hold:*

(i) *For every* $\psi \in D(H_0)$ *we have* $H_\sigma\psi \to H\psi$ *as* $\sigma \to 0$.
(ii) *For every* $\sigma \in (0,1]$, $H_\sigma$ *has a normalized ground state* $\phi_\sigma$.
(iii) *We have for every* $\sigma \in (0,1]$

$$(\phi_\sigma, P(\lambda) \otimes P_{\Omega_{\text{neut}}}\phi_\sigma) \geq 1 - \delta_g(\lambda)$$

*where* $\delta_g(\lambda)$ *tends to zero when* $g$ *tends to zero and* $0 \leq \delta_g(\lambda) < 1$ *for* $|g| \leq g_0$.

*Proof.* — We first estimate $E_\sigma = \inf \sigma(H_\sigma)$, $\sigma \in (0,1]$. One proves that $E_\sigma \leq 0$ as in lemma 4.3 of [**BDG04**].
Recall that there exist a constant $C > 0$ such that for every $\eta > 0$ and for every $\sigma \in (0,1]$

$$\|H_{I,\sigma}\psi\| \leq C(\sqrt{\eta}\|H_0\psi\| + \frac{1}{\sqrt{\eta}}\|\psi\|), \quad \psi \in D(H_0). \tag{4.2}$$

Therefore it follows from the Kato-Rellich theorem that

$$|E_\sigma| \leq \frac{|g|C}{\sqrt{\eta} - |g|\eta C} \tag{4.3}$$

when $|g|\sqrt{\eta}C < 1$.
(i) follows from the following inequality and from the Lebesgue's theorem:

$$\|(H - H_\sigma)\psi\| \leq 2C|g|(\sum_{\epsilon\neq\epsilon'} \|G_{\epsilon\epsilon'} - G^\sigma_{\epsilon\epsilon'}\|_{L^2})(\sqrt{\eta}\|H_0\psi\| + \frac{1}{\sqrt{\eta}}\|\psi\|).$$

(ii) is proved as in [**BFS98**] or in [**BDG04**] (theorem 4.10). We omit the details. Thus we have $H_\sigma\phi_\sigma = E_\sigma\phi_\sigma$ with $\|\phi_\sigma\| = 1$.
Writing $H_0\phi_\sigma = H_\sigma\phi_\sigma - gH_{I,\sigma}\phi_\sigma$ we get using (4.2) and (4.3)

$$\begin{aligned} \|H_0\phi_\sigma\| &\leq (|E_\sigma| + |g|\frac{C}{\sqrt{\eta}})(1 - \sqrt{\eta}|g|C)^{-1} \\ &\leq |g|\frac{C}{\sqrt{\eta}})(1 - \sqrt{\eta}|g|C)^{-2}(2 - |g|\sqrt{\eta}C) \end{aligned} \tag{4.4}$$

for every $\sigma \in (0,1]$ and for $\sqrt{\eta}|g|C < 1$.
It remains to prove (iii). Note that (iii) is equivalent to

$$((P(\lambda)^\perp \otimes P_{\Omega_{\rm neut}} + 1 \otimes P^\perp_{\Omega_{\rm neut}})\phi_\sigma, \phi_\sigma) \leq \delta_g(\lambda) \tag{4.5}$$

for every $\sigma \in (0,1]$.
Note that

$$\begin{aligned} 0 =&(P(\lambda)^\perp \otimes P_{\Omega_{\rm neut}})(H_\sigma - E_\sigma)\phi_\sigma \\ =&P(\lambda)^\perp(H_0^1 \otimes 1 - E_\sigma) \otimes P_{\Omega_{\rm neut}}\phi_\sigma + g(P(\lambda)^\perp \otimes P_{\Omega_{\rm neut}})H_{I,\sigma}\phi_\sigma. \end{aligned} \tag{4.6}$$

Remarking that $P(\lambda)^\perp H_0^1 \geq m_1 P(\lambda)^\perp$ and using $E_\sigma \leq 0$, we get

$$(P(\lambda)^\perp \otimes P_{\Omega_{\rm neut}}\phi_\sigma, \phi_\sigma) \leq -\frac{|g|}{m_1}(P(\lambda)^\perp \otimes P_{\Omega_{\rm neut}}H_{I,\sigma}\phi_\sigma, \phi_\sigma).$$

Furthermore it follows from (4.2) that there exists a constant $C > 0$ such that

$$\Big|(P(\lambda)^\perp \otimes P_{\Omega_{\rm neut}}H_{I,\sigma}\phi_\sigma, \phi_\sigma)\Big| \leq C$$

and thus

$$(P(\lambda)^{\perp} \otimes P_{\Omega_{\rm neut}}\phi_\sigma, \phi_\sigma) \leq C\frac{|g|}{m_1} \tag{4.7}$$

On the other hand one easily verifies that there exists a constant $C > 0$ such that

$$\|P^{\perp}_{\Omega_{\rm neut}}\phi_\sigma\| \leq C(\|N_2^{1/2}\phi_\sigma\| + \|N_3^{1/2}\phi_\sigma\|) \tag{4.8}$$

for every $\sigma \in (0,1]$ where we recall that $N_j = \sum_\epsilon \int b^\star_{j\epsilon}(\xi) b_{j,\epsilon}(\xi) d\xi$.
The proof of (iii) then follows from (4.5), (4.7), (4.8) and the following lemma

***Lemma 4.2.*** — *There exists a constant $C > 0$ such that*

$$\|N_j^{1/2}\phi_\sigma\|^2 \leq g^2 C \left( \sum_{\epsilon\neq\epsilon'} \int \frac{|G_{\epsilon\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4)|^2}{|p_j|^2} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \right) \|H_0\phi_\sigma\|^2 \tag{4.9}$$

*for $j = 2, 3$ and for every $\sigma \in (0,1]$.*

*Proof.* — Recall that,

$$\{b_{2\epsilon}(\xi), b^\flat_{3\epsilon}(\xi')\} = \{b_{2\epsilon}(\xi), b^\flat_{3\epsilon'}(\xi')\} = 0 \tag{4.10}$$

according to our convention. It follows from the CAR and (4.10) that we have the following pull-through formula:

$$0 = (H_\sigma - E_\sigma + \omega_j(\xi)) b_{j,\epsilon}(\xi)\phi_\sigma + g V_j^{\epsilon\epsilon'\sigma}(\xi)\phi_\sigma, \quad j = 2, 3$$

where for $\epsilon \neq \epsilon'$

$$\begin{aligned} V_2^{\epsilon\epsilon'\sigma}(\xi) &= \int d\xi_1 d\xi_3 d\xi_4 G^\sigma_{\epsilon'\epsilon}(\xi_1,\xi,\xi_3,\xi_4) b^\star_{1\epsilon'}(\xi_1) b^\star_{3\epsilon'}(\xi_3) b_{4\epsilon'}(\xi_4) \\ V_3^{\epsilon\epsilon'\sigma}(\xi) &= \int d\xi_1 d\xi_2 d\xi_4 G^\sigma_{\epsilon\epsilon'}(\xi_1,\xi_2,\xi,\xi_4) b^\star_{1\epsilon}(\xi_1) b^\star_{2\epsilon'}(\xi_2) b_{4\epsilon}(\xi_4). \end{aligned} \tag{4.11}$$

We have

$$b_{j,\epsilon}(\xi)\phi_\sigma = -g(H_\sigma - E_\sigma + \omega_j(\xi))^{-1} V_j^{\epsilon\epsilon'\sigma}(\xi)\phi_\sigma.$$

By proposition 2.3 we get

$$\|b_{2\epsilon}(\xi)\phi_\sigma\|^2 \leq \frac{g^2}{m_4^2|p_2|^2} \left( \int |G_{\epsilon'\epsilon}(\xi_1,\xi,\xi_3,\xi_4)|^2 d\xi_1 d\xi_3 d\xi_4 \right) \|H_0\phi_\sigma\|^2 \tag{4.12}$$

and

$$\|b_{3\epsilon}(\xi)\phi_\sigma\|^2 \leq \frac{g^2}{m_4^2|p_3|^2} \left( \int |G_{\epsilon\epsilon'}(\xi_1,\xi_2,\xi,\xi_4)|^2 d\xi_1 d\xi_2 d\xi_4 \right) \|H_0\phi_\sigma\|^2. \tag{4.13}$$

Note that

$$\sum_\epsilon \int \|b_{j,\epsilon}(\xi)\phi_\sigma\|^2 d\xi = \|N_j^{1/2}\phi_\sigma\|^2 \quad j = 2, 3. \tag{4.14}$$

The lemma then follows from (4.12), (4.13) and (4.14) and theorem 3.2 is proved. Note that the uniqueness (up to a phase) of the ground state follows as in [**AGG06**] and [**Hir05**]. Thus theorem 3.1 is proved. □

Let us remark that the proof of lemma 4.2 is rather formal but, by mimicking [**Hir05**], one easily gets a rigorous proof. We omit the details.

## 5. Proof of theorem 3.2

In order to prove the absence of continuous singular spectrum away from the thresholds of $H_0$, we use the Mourre's method originates from [**Mou81**]. Actually this method has been applied successfuly to QED models (see for instance [**BFS98, BFSS99, GGM04a, GGM04b, Amm04**]).
To this end, we estimate from below the commutator of $H$ with an anti-selfadjoint operator $A = -A^\star$. Our choice for $A$ is the sum of the second quantization of dilatation generator on each particle and antiparticle space. Namely, denoting $a_j = \big(p_j \cdot \nabla_{p_j} + \nabla_{p_j} \cdot p_j\big)$, the generator of dilatation in the particle $j$ acting on $L^2(\mathbb{R}^3)$, we set

$$A = \sum_{\epsilon=\pm} \sum_{j=1}^{4} d\Gamma_{j\epsilon}(a_j) \tag{5.1}$$

where giving an operator $a$ on $L^2(\mathbb{R}^3)$, the operator $d\Gamma_{j\epsilon}(a) : \mathcal{F} \to \mathcal{F}$ is defined by

$$d\Gamma_{j\epsilon}(a) = \int d\xi b^\star_{j\epsilon}(\xi)\, a\, b_{j\epsilon}(\xi). \tag{5.2}$$

Note that $iA$ is essentially self-adjoint on $\mathcal{F}_{fin}$. It remains to compute $[A, H]$. We begin with the remark that the second quantization respects commutators, i.e., for given operators $a$, $a'$ on the one particle space $L^2(\mathbb{R}^3)$ and given $f \in L^2(\mathbb{R}^3)$ such that $af$ and $a^\star f$ belong to $L^2(\mathbb{R}^3)$, we have for $j = 1, 2, 3, 4$ and $\epsilon = \pm$:

$$\begin{aligned} [d\Gamma_{j\epsilon}(a), d\Gamma_{j\epsilon}(a')]\psi &= d\Gamma_{j\epsilon}([a, a'])\psi \\ [d\Gamma_{j\epsilon}(a), b^\star_{j\epsilon}(f)]\psi &= b^\star_{j\epsilon}(af)\psi \\ [d\Gamma_{j\epsilon}(a), b_{j\epsilon}(f)]\psi &= -b_{j\epsilon}(a^\star f)\psi, \end{aligned} \tag{5.3}$$

and also for $i, j = 1, 2, 3, 4$ and $\epsilon, \epsilon' = \pm$ with $(j, \epsilon) \neq (i, \epsilon')$:

$$\begin{aligned} [d\Gamma_{j\epsilon}(a), d\Gamma_{i\epsilon'}(a')]\psi &= 0 \\ [d\Gamma_{j\epsilon}(a), b^\star_{i\epsilon'}(f)]\psi &= 0 \\ [d\Gamma_{j\epsilon}(a), b_{i\epsilon'}(f)]\psi &= 0 \end{aligned} \tag{5.4}$$

for every $\psi \in \mathcal{F}_{\mathrm{fin}}$.
Recall that

$$H_0 = \sum_{\epsilon=\pm} \sum_{j=1}^{4} d\Gamma_{j\epsilon}(\omega_j)$$

and a straightforward calculus leads to
(5.5)

$$[A, H_0]\psi = \Big( \sum_{\epsilon=\pm} d\Gamma_{1\epsilon} \left( \frac{p^2}{\sqrt{p^2+m_1^2}} \right) + d\Gamma_{2\epsilon}(|p|) + d\Gamma_{3\epsilon}(|p|) + d\Gamma_{4\epsilon} \left( \frac{p^2}{\sqrt{p^2+m_4^2}} \right) \Big) \psi$$

for $\psi \in \mathcal{F}_{\mathrm{fin}}$.
Let us remark that $[A, H_0]$ is relatively bounded with respect to $H_0$.

***Proposition 5.1***. — *Let $\Delta$ be a closed subset of $\mathbb{R}$ such that $\Delta \cap S = \emptyset$ and set $\beta = dist(\Delta, S) > 0$. Then*

$$E_\Delta(H_0)[A, H_0]E_\Delta(H_0) \geq \beta E_\Delta(H_0)$$

*where $E_\Delta(H_0)$ denotes the spectral projection of $H_0$ for the interval $\Delta$.*

*Proof.* — Using (5.5), we have for a given state $\Psi^{(Q)} \in \mathcal{F}^{(Q)}$ such that $E_\Delta(H_0)\Psi^{(Q)} = \Psi^{(Q)}$,

(5.6)

$$\begin{aligned}
[A, H_0]\Psi^{(Q)}(\Xi_q, \ldots, \Xi_{\bar{t}}) = \Big( & \sum_{j=1}^{q} \frac{p_{1j}^2}{\sqrt{p_{1j}^2 + m_1^2}} + \sum_{j=1}^{\bar{q}} \frac{\bar{p}_{1j}^2}{\sqrt{\bar{p}_{1j}^2 + m_1^2}} \\
& + \sum_{j=1}^{r} |p_{2j}| + \sum_{j=1}^{\bar{r}} |\bar{p}_{2j}| \\
& + \sum_{j=1}^{s} |p_{3j}| + \sum_{j=1}^{\bar{s}} |\bar{p}_{3j}| \\
& + \sum_{j=1}^{t} \frac{p_{4j}^2}{\sqrt{p_{4j}^2 + m_4^2}} + \sum_{j=1}^{\bar{t}} \frac{\bar{p}_{4j}^2}{\sqrt{\bar{p}_{4j}^2 + m_4^2}} \Big) \Psi^{(Q)}(\Xi_q, \ldots, \Xi_{\bar{t}}).
\end{aligned}$$

The free energy of such state $\Psi^{(Q)}$ is given by

$$
\begin{aligned}
H_0\Psi^{(Q)}(\Xi_q,\ldots,\Xi_{\bar{t}}) = \Big(&\sum_{j=1}^{q}\sqrt{p_{1j}^2+m_1^2}+\sum_{j=1}^{\bar{q}}\sqrt{\bar{p}_{1j}^2+m_1^2}\\
&+\sum_{j=1}^{r}|p_{2j}|+\sum_{j=1}^{\bar{r}}|\bar{p}_{2j}|\\
&+\sum_{j=1}^{s}|p_{3j}|+\sum_{j=1}^{\bar{s}}|\bar{p}_{3j}|\\
&+\sum_{j=1}^{t}\sqrt{p_{4j}^2+m_4^2}+\sum_{j=1}^{\bar{t}}\sqrt{\bar{p}_{4j}^2+m_4^2}\Big)\Psi^{(Q)}(\Xi_q,\ldots,\Xi_{\bar{t}})
\end{aligned}
\tag{5.7}
$$

with

$$
\begin{aligned}
&\sum_{j=1}^{q}\sqrt{p_{1j}^2+m_1^2}+\sum_{j=1}^{\bar{q}}\sqrt{\bar{p}_{1j}^2+m_1^2}+\sum_{j=1}^{r}|p_{2j}|+\sum_{j=1}^{\bar{r}}|\bar{p}_{2j}|\\
&+\sum_{j=1}^{s}|p_{3j}|+\sum_{j=1}^{\bar{s}}|\bar{p}_{3j}|+\sum_{j=1}^{t}\sqrt{p_{4j}^2+m_4^2}+\sum_{j=1}^{\bar{t}}\sqrt{\bar{p}_{4j}^2+m_4^2}\ \in\Delta.
\end{aligned}
\tag{5.8}
$$

We decompose $H_0\Psi^{(Q)}$ as follows

$$
\begin{aligned}
H_0\Psi^{(Q)}(\Xi_q,\ldots,\Xi_{\bar{t}}) = \Big[&(q+\bar{q})m_1+(t+\bar{t})m_4\\
&+\sum_{j=1}^{q}(\sqrt{p_{1j}^2+m_1^2}-m_1)+\sum_{j=1}^{\bar{q}}(\sqrt{\bar{p}_{1j}^2+m_1^2}-m_1)\\
&+\sum_{j=1}^{r}|p_{2j}|+\sum_{j=1}^{\bar{r}}|\bar{p}_{2j}|\\
&+\sum_{j=1}^{s}|p_{3j}|+\sum_{j=1}^{\bar{s}}|\bar{p}_{3j}|\\
&+\sum_{j=1}^{t}(\sqrt{p_{4j}^2+m_4^2}-m_4)+\sum_{j=1}^{\bar{t}}(\sqrt{\bar{p}_{4j}^2+m_4^2}-m_4)\Big]\Psi^{(Q)}(\Xi_q,\ldots,\Xi_{\bar{t}}).
\end{aligned}
\tag{5.9}
$$

By (5.9) we get according to the definition of $\beta$

$$\begin{aligned}&\sum_{j=1}^{q}(\sqrt{p_{1j}^2+m_1^2}-m_1)+\sum_{j=1}^{\bar q}(\sqrt{\bar p_{1j}^2+m_1^2}-m_1)\\ &+\sum_{j=1}^{r}|p_{2j}|+\sum_{j=1}^{\bar r}|\bar p_{2j}|+\sum_{j=1}^{s}|p_{3j}|+\sum_{j=1}^{\bar s}|\bar p_{3j}|\\ &+\sum_{j=1}^{t}(\sqrt{p_{4j}^2+m_4^2}-m_4)+\sum_{j=1}^{\bar t}(\sqrt{\bar p_{4j}^2+m_4^2}-m_4)\geq\beta\end{aligned}\tag{5.10}$$

for $(p_1,p_2,p_3,p_4)$ satisfying (5.8).
Therefore using

$$\begin{aligned}\frac{p^2}{\sqrt{p^2+m^2}}&=(\sqrt{p^2+m^2}-m)\frac{\sqrt{p^2+m^2}+m}{\sqrt{p^2+m^2}}\\ &\geq\sqrt{p^2+m^2}-m\end{aligned}\tag{5.11}$$

we conclude the proof. ☐

We now estimate the commutator $[A,H_I]$. By (2.8), (5.3), (5.4) and since $a_j^\star=-a_j$ we have

$$\begin{aligned}[A,H_I]\psi=&\Big(\sum_{\epsilon\neq\epsilon'}\sum_{j=1}^{4}\int d\xi_1d\xi_2d\xi_3d\xi_4(a_jG_{\epsilon\epsilon'})(\xi_1,\xi_2,\xi_3,\xi_4)\\ &\qquad b_{1,\epsilon}^\star(\xi_1)b_{2,\epsilon'}^\star(\xi_2)b_{3,\epsilon}^\star(\xi_3)b_{4,\epsilon}(\xi_4)\\ &+\sum_{\epsilon\neq\epsilon'}\sum_{j=1}^{4}\int d\xi_1d\xi_2d\xi_3d\xi_4\overline{(a_jG_{\epsilon\epsilon'})(\xi_1,\xi_2,\xi_3,\xi_4)}\\ &\qquad b_{4,\epsilon}^\star(\xi_4)b_{3,\epsilon}(\xi_3)b_{2,\epsilon'}(\xi_2)b_{1,\epsilon}(\xi_1)\Big)\psi\end{aligned}\tag{5.12}$$

for $\psi\in\mathcal{F}_{\rm fin}$. Therefore, if we assume that $a_jG_{\epsilon\epsilon'}\in L^2$ for each $j=1,2,3,4$ and for each $\epsilon\neq\epsilon'$, we deduce as in the proof of theorem 2.4 that $[A,H_I]$ is $H_0$ relatively bounded and in particular there exist $c>0$ such that

$$E_\Delta(H_0)[A,H_I]E_\Delta(H_0)\geq -cE_\Delta(H_0).\tag{5.13}$$

We deduce

***Proposition 5.2.*** — *Assume that $a_jG_{\epsilon\epsilon'}\in L^2$ for each $j=1,2,3,4$ and for each $\epsilon\neq\epsilon'$. There exists $c>0$ such that if $\Delta$ is a closed interval of $\mathbb{R}$ verifying*

*$\Delta \cap S = \emptyset$ then*

$$E_\Delta(H)[A,H]E_\Delta(H) \geq (\frac{\beta}{2} - \frac{cg}{\beta})E_\Delta(H)$$

*where $E_\Delta(H)$ denotes the spectral projection of $H$ for the interval $\Delta$ and $\beta =$ dist$(\Delta, S) > 0$ is suficiently small.*

*Proof.* — Let $\Delta'$ be the closed interval such that $\Delta = \Delta' + [-\beta/2, \beta/2]$ and assume that $0 < \beta < 1$. Using the Helffer-Sjöstrand Functional Calculus (see for instance [**DS99**]), we find that

$$\|E_\Delta(H)(1 - E_{\Delta'}(H_0))\| \leq \frac{c_1 g}{\beta}$$

for some constant $c_1 > 0$ independent of $\Delta$, $g$ and $\beta$.
Therefore, using that $[A,H]$ is $H$ bounded (see the proof of theorem 3.2 just below),

$$E_\Delta(H)[A,H]E_\Delta(H) \geq E_\Delta(H)E_{\Delta'}(H_0)[A,H]E_{\Delta'}(H_0)E_\Delta(H) - c_2\frac{g}{\beta}E_\Delta(H) \tag{5.14}$$

for some constant $c_2 > 0$.
On the other hand, from proposition 5.1 and (5.13), we have

$$E_{\Delta'}(H_0)[A,H]E_{\Delta'}(H_0) \geq (\frac{\beta}{2} - c_3 g)E_{\Delta'}(H_0) \tag{5.15}$$

for some constant $c_3 > 0$.
Inserting (5.15) in (5.14) we get

$$\begin{aligned}
E_\Delta(H)[A,H]E_\Delta(H) &\geq (\frac{\beta}{2} - c_3 g)E_\Delta(H)E_{\Delta'}(H_0)E_\Delta(H) - c_2\frac{g}{\beta}E_\Delta(H) \\
&\geq (\frac{\beta}{2} - c_3 g)(1 - \frac{c_1 g}{\beta})E_\Delta(H) - c_2\frac{g}{\beta}E_\Delta(H) \\
&\geq (\frac{\beta}{2} - \frac{cg}{\beta})E_\Delta(H)
\end{aligned}$$

for some $c > 0$ independent of $\Delta$, $g$ and $\beta$. □

*Proof of theorem 3.2* Theorem 3.2 is a consequence of proposition 5.2 and the Mourre theory. Actually it only remains to verify the applicability of this theory. This means that we have to verify that $[A,H]$ and $[A,[A,H]]$ are $H$ bounded. From (5.5) we deduce that $[A,H_0]$ is $H_0$ bounded. For the second commutator a simple calculus gives

$$[A,[A,H_0]]\psi = \Big[\sum_{\epsilon=\pm} d\Gamma_{1\epsilon}\left(\frac{p^2m_1^2}{(p^2+m_1^2)^{3/2}}\right) + d\Gamma_{2\epsilon}(|p|) + d\Gamma_{3\epsilon}(|p|) + d\Gamma_{4\epsilon}\left(\frac{p^2m_4^2}{(p^2+m_4^2)^{3/2}}\right)\Big]\psi$$

for $\psi \in \mathcal{F}_{\mathrm{fin}}$. Thus $[A,[A,H_0]]$ is $H_0$ bounded.

We have already noted that $[A,H_I]$ is $H_0$ bounded as soon as $a_jG_{\epsilon\epsilon'} \in L^2$ for each $j=1,2,3,4$ and for each $\epsilon \neq \epsilon'$. The computation of the commutator of $A$ with the expression of $[A,H_I]$ given by (5.12) shows that $[A,[A,H_I]]$ is $H_0$ bounded as soon as $a_ja_jG_{\epsilon\epsilon'} \in L^2$ for each $j=1,2,3,4$ and for each $\epsilon \neq \epsilon'$. These conditions on $G_{\epsilon\epsilon'}$ are satisfied when $p_i\cdot\nabla_{p_i}G_{\epsilon\epsilon'}$ and $p_i^2\Delta_{p_i}G_{\epsilon\epsilon'}$ are all in $L^2(\Sigma_1\times\Sigma_2\times\Sigma_2\times\Sigma_1)$ for $i=1,2,3,4$ and $\epsilon\neq\epsilon'$. □

## 6. Other examples

The main other examples of the Fermi-weak interactions are the beta decay of the neutron and of the quarks $u$ and $d$. Let us consider the decay of the quark $d$. This decay involves four species of particles and antiparticles: the quarks $u$ and $d$ and their antiparticles $\bar{u}$ and $\bar{d}$, the electron $e^-$ and the positron $e^+$, the neutrino $\nu_e$ and its antineutrino $\bar{\nu}_e$ (see [**Wei96, GM89**]). The Fock space is the fermionic Fock space associated to these four species of particles and the interaction is given by

(6.1)

$$H_I = \int d\xi_1d\xi_2d\xi_3d\xi_4\ J(\xi_1,\xi_2,\xi_3,\xi_4)\ b_{1,+}^\star(\xi_1)b_{2,-}^\star(\xi_2)b_{3,+}^\star(\xi_3)b_{4,+}(\xi_4) + \int d\xi_1d\xi_2d\xi_3d\xi_4\ \overline{J(\xi_1,\xi_2,\xi_3,\xi_4)}\ b_{4,+}^\star(\xi_4)b_{3,+}(\xi_3)b_{2,-}(\xi_2)b_{1,+}(\xi_1).$$

Here the particles and antiparticles 1 are the electrons and the positrons, the particles and antiparticles 2 are the neutrinos $\nu_e$ and $\bar{\nu}_e$, the particles and antiparticles 3 are the quarks $u$ and $\bar{u}$ and, finally, the particles and antiparticles 4 are the quarks $d$ and $\bar{d}$.

Obviously theorems 3.1 and 3.2 remains valid for the associated Hamiltonian under appropriate conditions on the kernel $J$.

We can also consider the decay of the massive bosons $W^\pm$ into electrons, positrons and neutrinos $\nu_e$ and $\bar{\nu}_e$ (see [**Wei96, GM89**]). The Fock space is the tensor product of the fermionic Fock space associated to the electrons, the positrons and the neutrinos $\nu_e$ and $\bar{\nu}_e$ and of the bosonic Fock space associated to a massive boson of spin 1. The interaction is then given by

$$\begin{aligned} H_I =& \sum_{\epsilon \neq \epsilon'} \int d\xi_1 d\xi_2 d\xi_3 \; K_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3) \; b^\star_{1,\epsilon}(\xi_1) b^\star_{2,\epsilon'}(\xi_2) a_{3,\epsilon}(\xi_3) \\ &+ \sum_{\epsilon \neq \epsilon'} \int d\xi_1 d\xi_2 d\xi_3 \; \overline{K_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)} \; a^\star_{3,\epsilon}(\xi_3) b_{2,\epsilon'}(\xi_2) b_{1,\epsilon}(\xi_1). \end{aligned} \tag{6.2}$$

Here the particles and antiparticles 1 are the electrons and the positrons, the particles and antiparticles 2 are the neutrinos $\nu_e$ and $\bar{\nu}_e$, and $a_+(\xi_3)$ (resp. $a_-(\xi_3)$) is the annihilation operator for the meson $W^-$ (resp. $W^+$).
Once again theorems 3.1 and 3.2 remains valid for the associated Hamiltonian under appropriate conditions on the kernels $K_{\epsilon,\epsilon'}$.
One could also give a mathematical model for the decay of the massive boson $Z^0$.

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