On problem of polarization tomography,I

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Abstract

The polarization tomography problem consists of recovering a matrix function f from the fundamental matrix of the equation $D\eta/dt = \pi_{\dot\gamma} f\eta$ known for every geodesic γ of a given Riemannian metric. Here $\pi_{\dot\gamma}$ is the orthogonal projection onto the hyperplane $\dot\gamma^\perp$. The problem arises in optical tomography of slightly anisotropic media. The local uniqueness theorem is proved: a C^1-small function f can be recovered from the data uniquely up to a natural obstruction. A partial global result is obtained in the case of the Euclidean metric on $\mathbb{R}^3$.

1 Introduction

First of all we shortly recall the physical motivation of the problem. See Section 5.1 of [5] for a detailed discussion.

We consider propagation of time-harmonic electromagnetic waves of frequency ω in a medium with the zero conductivity, unit magnetic permeability, and the dielectric permeability tensor of the form

$$\varepsilon_{ij} = n^2\delta_{ij} + \frac{1}{k}\chi_{ij}, \tag{1.1}$$

where $k = \omega/c$ is the wave number, c being the light velocity. Here $n > 0$ is a function of a point $x \in \mathbb{R}^3$, and the tensor $\chi_{ij} = \chi_{ij}(x)$ determines a small anisotropy of the medium. The smallness is emphasized by the factor $1/k$. Equation (1.1) was suggested by Yu. Kravtsov [2]. By some physical arguments [3], the tensor χ must be Hermitian, $\chi_{ij} = \bar\chi_{ji}$.

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In the scope of the zero approximation of geometric optics, propagation of electromagnetic waves in such media is described as follows. Exactly as in the background isotropic medium, light rays are geodesics of the Riemannian metric

$$dt^2 = n^2(x)|dx|^2; \tag{1.2}$$

the electric vector $E(x)$ and magnetic vector $H(x)$ are orthogonal to each other as well as to the ray; and the amplitude $A^2 = |E|^2 = |H|^2$ satisfies $A = C/\sqrt{nJ}$ along a ray, where J is the geometric divergence and the constant C depends on the ray. The only difference between a slightly anisotropic medium and the background isotropic one consists of the wave polarization. The polarization vector $\eta = n^{-1}A^{-1}E$ satisfies the equation (generalized Rytov's law)

$$\frac{D\eta}{dt} = \frac{i}{2n^2}\pi_{\dot\gamma}\chi\eta \tag{1.3}$$

along a geodesic ray $\gamma(t)$. Here t is the arc length of γ in metric (1.2), $\dot\gamma = d\gamma/dt$ is the speed vector of γ, $\pi_{\dot\gamma}$ is the orthogonal projection onto the plane $\dot\gamma^\perp$, and $D/dt = \dot\gamma^k\nabla_k$ is the covariant derivative along γ in metric (1.2). The right-hand side of (1.3) is understood as follows: $\pi_{\dot\gamma}$ and χ are considered as linear operators on $T^{\mathbb{C}}_{\gamma(t)} = \mathbb{C}^3$, and $\pi_{\dot\gamma}\chi\eta$ is the result of action of the operator $\pi_{\dot\gamma}\chi$ on the complex vector $\eta \in \dot\gamma^\perp$. Here $\dot\gamma^\perp$ is the two-dimensional complex vector space consisting of complex vectors orthogonal to the real vector $\dot\gamma \in \mathbb{R}^3 = T_{\gamma(t)} \subset T^{\mathbb{C}}_{\gamma(t)} = \mathbb{C}^3$. Introducing the notation

$$f = \frac{i}{2n^2}\chi, \tag{1.4}$$

we rewrite (1.3) in the form

$$\frac{D\eta}{dt} = \pi_{\dot\gamma} f\eta. \tag{1.5}$$

Observe that f is a skew-Hermitian operator, $f_{ij} = -\bar f_{ji}$.

Let us now consider the inverse problem. Assume a medium under investigation to be contained in a bounded domain $D \subset \mathbb{R}^3$ with a smooth boundary. The background isotropic medium is assumed to be known, i.e., metric (1.2) is given. The domain D is assumed to be convex with respect to the metric, i.e., for any two boundary points $x_0, x_1 \in \partial D$, there exists a unique geodesic $\gamma : [0,1] \to D$ such that $\gamma(0) = x_0,\ \gamma(1) = x_1$. We consider the inverse problem of determining the anisotropic part χ_{ij} of the dielectric permeability tensor or, equivalently, of determining the tensor f on (1.5). To this end we can fulfill tomographic measurements of the following type. For any unit speed geodesic $\gamma : [0,l] \to D$ between boundary points, we can choose an initial value $\eta_0 = \eta(0) \in \dot\gamma^\perp(0)$ of the polarization vector and measure the final value $\eta_1 = \eta(l) \in \dot\gamma^\perp(l)$ of the solution to equation (1.5). In other words, we assume the linear operator $\dot\gamma^\perp(0) \to \dot\gamma^\perp(l),\ \eta_0 \mapsto \eta_1$ to be known for every unit speed geodesic $\gamma : [0,l] \to D$ between boundary points. Instead of (1.5), we will consider the corresponding operator equation

$$\frac{D\tilde U(t)}{dt} = f_{\dot\gamma(t)}\tilde U(t), \tag{1.6}$$

where $f_{\dot\gamma(t)} : \dot\gamma^\perp(t) \to \dot\gamma^\perp(t)$ is the restriction of the operator $\pi_{\dot\gamma(t)}f(\gamma(t))$ to the plane $\dot\gamma^\perp(t)$, and the solution is considered as a linear operator $\tilde U(t) : \dot\gamma^\perp(t) \to \dot\gamma^\perp(t)$. Equation (1.6) has a unique solution satisfying the initial condition

$$\tilde U(0) = E, \tag{1.7}$$

where E is the identity operator. Since $f_{\dot\gamma(t)}$ is a skew-Hermitian operator, the solution $\tilde U(t)$ is a unitary operator. The final value of the solution

$$\tilde\Phi[f](\gamma) = \tilde U(l) \in GL(\dot\gamma^\perp(l))$$

is the data for the inverse problem. Given the function $\tilde\Phi[f]$ on the set of unit speed geodesics between boundary points, we have to determine the tensor field $f = (f_{ij}(x))$ on the domain D.

We consider the inverse problem in a more general setting. Instead of a domain $D \subset \mathbb{R}^3$ with metric (1.2), we will consider a compact Riemannian manifold (M, g) of an arbitrary dimension $n \geq 3$, and an arbitrary complex tensor field $f = (f_{ij})$ on M. In such a setting, equation (1.6) makes sense along a geodesic γ. We will subordinate the manifold (M, g) to some conditions that guarantee smoothness of the data $\tilde\Phi[f]$ in the case of a smooth f.

The two-dimensional case of $n = 2$ is not interesting since $f_{\dot\gamma}$ and $\tilde U$ become scalar functions and the solution to the scalar equation (1.6) is given by an explicit formula in this case. Therefore the inverse problem is reduced to the inversion of the ray transform I on second rank tensor fields, see the remark before Theorem 5.2.1 of [5].

Equation (1.6) can be slightly simplified. For a point $x \in M$, let $T^{\mathbb{C}}_x M$ be the complexification of the tangent space $T_x M$. Instead of considering the operator $\tilde U(t)$ on $\dot\gamma^\perp(t)$, we define the linear operator

$$U(t) : T^{\mathbb{C}}_{\gamma(t)} M \to T^{\mathbb{C}}_{\gamma(t)} M$$

by

$$U(t)|_{\dot\gamma^\perp(t)} = \tilde U(t), \quad U(t)\dot\gamma(t) = \dot\gamma(t).$$

If $\tilde U(t)$ satisfies (1.6)–(1.7), then $U(t)$ solves the initial value problem

$$\frac{DU}{dt} = (\pi_{\dot\gamma} f \pi_{\dot\gamma})U, \quad U(0) = E. \tag{1.8}$$

Equation (1.8) is more handy than (1.6) since all operators participating in (1.8) are defined on the whole of $T^{\mathbb{C}}_{\gamma(t)} M$. The inverse problem consists of recovering the tensor field f from the data $\Phi[f](\gamma) = U(l)$ known for every unit speed geodesic $\gamma : [0, l] \to M$ between boundary points. Let us emphasize that the inverse problem is strongly nonlinear, i.e., the data $\Phi[f]$ depends on f in a nonlinear manner.

The three-dimensional case, $n = \dim M = 3$, is of the most importance for applications as we have shown above. On the other hand, the three-dimensional case is mathematically the exceptional one because, for a skew-symmetric f, the solution to the inverse problem is not unique. The non-uniqueness is discussed in Section 4.

The main result of the present article is the local uniqueness theorem: the solution to the inverse problem is unique (up to a natural obstruction in the three-dimensional case) if the tensor field f is C^1-small. See Theorem 5.1 below for the precise statement.

Our method of investigating the inverse problem is a combination of approaches used in [9] and in Chapter 5 of [5]. First of all, following [9], we reduce our nonlinear problem to a linear one as follows. Let f_i $(i = 1, 2)$ be two tensor fields and $U_i(t)$ be the corresponding solutions to (1.8) with $f = f_i$. Then $u = U_1^{-1}U_2 - E$ satisfies

$$\frac{Du}{dt} = p\pi_{\dot\gamma}(f_2 - f_1)\pi_{\dot\gamma} q, \quad u(0) = 0, \tag{1.9}$$

where $p = U_1^{-1}$ and $q = U_2$. We consider p and q as operator-valued weights which are close to the unit operator if f_i are C^1-small. Assuming the weights p and q to be fixed, the solution $u(t)$ to the initial value problem (1.9) depends linearly on $f = f_2 - f_1$. We study the linear inverse problem of recovering the tensor field $f = f_2 - f_1$ from the data $F[f](\gamma) = u(l)$ given for all geodesics $\gamma : [0, l] \to M$ between boundary points. In the case of a symmetric tensor field f and of unit weights, this linear problem was considered in Chapter 5 of [5]. We will demonstrate that the same approach works in the case of an arbitrary f and of weights close to the unit one.

There is one more opportunity to extract a linear inverse problem from equation (1.8). Indeed, if $W(t) = \det U(t)$ is the Wronskian, then the function $\varphi(t) = \ln W(t)$ satisfies

$$\frac{d\varphi(t)}{dt} = \mathrm{tr}\,(\pi_{\dot\gamma} f \pi_{\dot\gamma}).$$

Therefore, for every unit speed geodesic $\gamma : [0, l] \to M$ between boundary points, the integral

$$S[f](\gamma) = \int\limits_0^l \mathrm{tr}\,(\pi_{\dot\gamma(t)} f(\gamma(t)) \pi_{\dot\gamma(t)}) dt \tag{1.10}$$

is expressed through the data $\Phi[f]$ by the formula

$$S[f](\gamma) = \ln\det \Phi[f](\gamma).$$

The data $S[f]$ depends linearly on f. Of course, some information is lost while the data $\Phi[f]$ is replaced with $S[f]$. In particular, $S[f]$ is independent of the skew-symmetric part of f.

We finally note that main results of the article are new and nontrivial in the case of $M \subset \mathbb{R}^n$, $n \ge 3$, with the standard Euclidean metric. If a reader is not familiar with the tensor analysis machinery on the tangent bundle of a Riemannian manifold, he/she can first read the article for the latter simplest case.

The article is organized as follows. Section 2 contains some preliminaries concerning Riemannian geometry and tensor analysis. In particular, we define some class of Riemannian manifolds for which the problem can be posed in the most natural way. Instead of considering the ordinary differential equation (1.8) along individual geodesics, we introduce a partial differential equation on the unit tangent bundle and pose an equivalent version of the problem in terms of the latter equation. In Section 3, we consider the corresponding linear problem and prove the uniqueness for weights sufficiently close to the unit in the case of $n \ge 4$. Section 4 discusses the three-dimensional case. In Section 5, we check that the weights p and q are sufficiently close to the unit for a C^1-small f and prove our main result, Theorem 5.1, on the local uniqueness in the nonlinear problem. In the final Section 6, we investigate the question: to which extent is a symmetric tensor field f determined by data (1.10). We give a complete answer to the question in the case of $M = \mathbb{R}^3$ with the Euclidean metric.

2 Posing the problem and introducing some notations

A smooth compact Riemannian manifold (M, g) with boundary is said to be a *convex non-trapping manifold* (CNTM briefly) if it satisfies two conditions: (1) the boundary ∂M is strictly convex, i.e., the second fundamental form

$$\mathrm{II}(\xi, \xi) = \langle \nabla_\xi \nu, \xi \rangle \quad \text{for} \quad \xi \in T_x(\partial M)$$

is positive definite for every boundary point $x \in \partial M$, where ν is the outward unit normal vector to the boundary and ∇_ξ is the covariant derivative in the direction ξ; and (2) for every $x \in M$ and $0 \neq \xi \in T_x M$, the maximal geodesic $\gamma_{x,\xi}(t)$ determined by the initial conditions $\gamma_{x,\xi}(0) = x$ and $\dot\gamma_{x,\xi}(0) = \xi$ is defined on a finite segment $[\tau_-(x, \xi), \tau_+(x, \xi)]$. In what follows, we use the notations $\gamma_{x,\xi}$ and $\tau_\pm(x, \xi)$ many times. They are always understood in the sense of this definition.

Remark. In [5], the term CDRM (compact dissipative Riemannian manifold) is used instead of CNTM. In the case of $M \subset \mathbb{R}^n$ with the standard Euclidean metric, this definition means that M is strictly convex.

By $TM = \{(x, \xi) \mid x \in M, \xi \in T_x M\}$ we denote the tangent bundle and by

$$\Omega M = \{(x, \xi) \in TM \mid |\xi|^2 = \langle \xi, \xi \rangle = g_{ij}(x)\xi^i\xi^j = 1\},$$

the unit sphere bundle. Its boundary can be represented as the union $\partial\Omega M = \partial_+\Omega M \cup \partial_-\Omega M$, where

$$\partial_\pm\Omega M = \{(x, \xi) \in \Omega M \mid x \in \partial M,\ \pm\langle \xi, \nu(x)\rangle \geq 0\}$$

is the manifold of outward (inward) unit vectors. If $\gamma : [0, l] \to M$ is a unit speed geodesic between boundary points, then $(\gamma(0), \dot\gamma(0)) \in \partial_-\Omega M$ and $(\gamma(l), \dot\gamma(l)) \in \partial_+\Omega M$.

By $T_x^{\mathbb{C}}M$ we denote the complexification of the tangent space $T_x M$. The metric g determines the Hermitian scalar product on $T_x^{\mathbb{C}}M$

$$\langle \eta, \zeta \rangle = g_{ij}\eta^i\bar\zeta^j. \tag{2.1}$$

For $0 \neq \xi \in T_x M$, by $T_{x,\xi}^\perp M = \{\eta \in T_x^{\mathbb{C}}M \mid \langle \eta, \xi \rangle = 0\}$ we denote the orthogonal complement of ξ and by $\pi_\xi : T_x^{\mathbb{C}}M \to T_x^{\mathbb{C}}M$, the orthogonal projection onto $T_{x,\xi}^\perp M$.

Let $\tau_s^r M$ be the bundle of complex tensors that are r times contravariant an s times covariant. Elements of the section space $C^\infty(\tau_s^r M)$ are smooth tensor fields of rank (r, s) on M. In the domain of a local coordinate system, such a field $u \in C^\infty(\tau_s^r M)$ can be represented by the family of smooth functions, $u = (u^{i_1\dots i_r}_{j_1\dots j_s}(x))$, the coordinates of u, where each index takes values from 1 to $n = \dim M$. The metric g determines canonical isomorphisms $\tau_s^r M \cong \tau_0^{r+s} M \cong \tau_{r+s}^0 M$. We will consider the isomorphisms as identifications. So, we do not distinct contra- and covariant tensors but use contra- and covariant coordinates of the same tensor. For example, for $u \in C^\infty(\tau_2^0 M) = C^\infty(\tau_1^1 M) = C^\infty(\tau_0^2 M)$,

$$u_{ij} = g_{ik}u^{k\cdot}_{\cdot j} = g_{jk}u^{\cdot k}_{i\cdot} = g_{ik}g_{jl}u^{kl}.$$

In particular, such a tensor field determines the linear operator

$$u(x) : T_x^{\mathbb{C}}M \to T_x^{\mathbb{C}}M, \quad (u\eta)^i = u^{i\cdot}_{\cdot j}\eta^j$$

at any point $x \in M$. The product of two such operators is written in coordinates as $(uv)_{ij} = u_{ik}v^{k\cdot}_{\cdot j}$. The dual operator has the coordinates $u^*_{ij} = \bar{u}_{ji}$. The operator is Hermitian (symmetric) if and only if $u_{ij} = \bar{u}_{ji}$ ($u_{ij} = u_{ji}$). The scalar product (2.1) is extended to tensors by the formula $\langle u, v\rangle = u^{i_1\dots i_m}\bar{v}_{i_1\dots i_r}$ and determines the norm $|u|^2 = \langle u, u\rangle$. For $u, v \in C^\infty(\tau^1_1 M)$, the norm of the product satisfies $|uv| \le \sqrt{n}|u||v|$, where $n = \dim M$.

We will also widely use semibasic tensor fields introduced in [4], see either Section 3.4 of [5] or Section 2.5 of [7] for a detailed presentation. Let $\beta^r_s M$ be the bundle of complex semibasic tensor fields of rank (r, s). It is a subbundle of $\tau^r_s(TM)$ isomorphic to the induced bundle $\pi^*(\tau^r_s M)$, where $\pi : TM \to M$ is the projection of the tangent bundle. A tensor $u \in T^r_{s,(x,\xi)}(TM)$ at $(x, \xi) \in TM$ is semibasic if it is "pure contravariant in the ξ-variable and pure covariant in x", i.e.,

$$u = u^{i_1\dots i_r}_{j_1\dots j_s}\frac{\partial}{\partial\xi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial\xi^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}.$$

For $U \subset TM$, by $C^\infty(\beta^r_s M; U)$ we denote the space of smooth sections over U. The notation $C^\infty(\beta^r_s M; TM)$ is abbreviated to $C^\infty(\beta^r_s M)$. In the domain of a local coordinate system, such a field $u \in C^\infty(\beta^r_s M)$ can be represented by the family of its coordinates, $u = (u^{i_1\dots i_r}_{j_1\dots j_s}(x, \xi))$, which are smooth functions of $2n$ variables $(x, \xi) = (x^1, \dots, x^n, \xi^1, \dots, \xi^n)$. All the content of the previous paragraph is extended to semibasic tensor fields, where g remains the metric on M in the identification of contra- and covariant tensors. In particular, $u \in C^\infty(\beta^1_1 M)$ determines the linear operator $u(x, \xi) : T^{\mathbb{C}}_x M \to T^{\mathbb{C}}_x M$ for every $(x, \xi) \in TM$. There are two important first order differential operators

$$\overset{v}{\nabla}, \overset{h}{\nabla} : C^\infty(\beta^r_s M) \to C^\infty(\beta^r_{s+1} M)$$

which are called the vertical and horizontal covariant derivatives. The operators are defined in local coordinates by the formulas

$$\overset{v}{\nabla}_k u^{i_1\dots i_r}_{j_1\dots j_s} = \frac{\partial}{\partial\xi^k}u^{i_1\dots i_r}_{j_1\dots j_s},$$

$$\overset{h}{\nabla}_k u^{i_1\dots i_r}_{j_1\dots j_s} = \frac{\partial}{\partial x^k}u^{i_1\dots i_r}_{j_1\dots j_s} - \Gamma^p_{kq}\xi^q\frac{\partial}{\partial\xi^p}u^{i_1\dots i_r}_{j_1\dots j_s} +$$
$$+ \sum_{a=1}^{r}\Gamma^{i_a}_{kp}u^{i_1\dots i_{a-1}pi_{a+1}\dots i_r}_{j_1\dots j_s} - \sum_{a=1}^{s}\Gamma^p_{kj_a}u^{i_1\dots i_r}_{j_1\dots j_{a-1}pj_{a+1}\dots j_s},$$

where Γ^i_{jk} are Christoffel symbols. See Sections 3.4–3.6 of [5] for properties of these operators. Note that $\overset{v}{\nabla} = \partial/\partial\xi$ and $\overset{h}{\nabla} = \partial/\partial x$ in the case of $M \subset \mathbb{R}^n$ with the standard Euclidean metric and of Cartesian coordinates.

The operator

$$H = \xi^i\overset{h}{\nabla}_i : C^\infty(\beta^r_s M) \to C^\infty(\beta^r_s M)$$

is of the most importance in the present article. It is called the differentiation with respect to the geodesic flow.

Given a tensor field $f \in C^\infty(\tau_1^1 M)$ on a CNTM (M, g), let us consider the boundary value problem

$$HU(x,\xi) = \pi_\xi f(x)\pi_\xi U(x,\xi) \quad \text{on} \quad \Omega M, \qquad U|_{\partial_-\Omega M} = E, \tag{2.2}$$

where E is the identity operator. A solution $U = U(x,\xi)$ is assumed to be a section of the bundle $\beta_1^1 M$ over ΩM, i.e., $U \in C(\beta_1^1 M; \Omega M)$. In the case of $M \subset \mathbb{R}^n$ with the standard Euclidean metric, f and U can be considered as $n \times n$-matrix valued functions of $x \in M$ and of $(x,\xi) \in M \times \mathbb{S}^{n-1}$ respectively. Problem (2.2) has a unique solution. Indeed, if we restrict (2.2) to an orbit of the geodesic flow, i.e., if we set $x = \gamma(t)$ and $\xi = \dot\gamma(t)$ for a unit speed geodesic $\gamma : [0,l] \to M$ with $\gamma(0) \in \partial M$, then we immediately arrive to the initial value problem (1.8). The boundary value problem (2.2) is thus equivalent to the family of initial value problems (1.8) considered for all unit speed geodesics simultaneously. The inverse problem is now formulated as follows: one has to recover the tensor field f given the trace

$$\Phi[f] = U|_{\partial_+\Omega M} \tag{2.3}$$

of the solution to (2.2).

In order to abbreviate further formulas, let us introduce the operator P_ξ on tensors which maps $f(x)$ to $\pi_\xi f(x)\pi_\xi$ for $(x,\xi) \in \Omega M$, and write (2.2) in the shorter form

$$HU = (P_\xi f)U, \qquad U|_{\partial_-\Omega M} = E. \tag{2.4}$$

Because of the factor P_ξ and of the boundary condition on $\partial_-\Omega M$, the solution U to (2.4) satisfies

$$U(x,\xi)\xi = U^*(x,\xi)\xi = \xi. \tag{2.5}$$

Therefore the non-trivial part of the data (2.3) consists of the restrictions $\Phi[f](x,\xi)|_{T^\perp_{x,\xi}M}$ for $(x,\xi) \in \partial_+\Omega M$. This agrees with the above discussion of the relationship between (1.6) and (1.8). The solution U is continuous on ΩM and C^∞-smooth on $\Omega M \setminus \Omega(\partial M)$ as one can easily prove using the strict convexity of the boundary.

Concluding the section, let us mention one more inverse problem that is not considered in the present article. Let $gl(T_x^{\mathbb{C}} M)$ be the space of all linear operators on $T_x^{\mathbb{C}} M$. The operator P_ξ participating in (2.4) is the orthogonal projection of the space $gl(T_x^{\mathbb{C}} M)$ onto the subspace

$$\{f \in gl(T_x^{\mathbb{C}} M) \mid f\xi = f^*\xi = 0\}.$$

Let us introduce the smaller subspace

$$\{f \in gl(T_x^{\mathbb{C}} M) \mid f\xi = f^*\xi = 0,\ \operatorname{tr} f = f_i^i = 0\}$$

and denote by Q_ξ the orthogonal projection onto the latter subspace. The corresponding inverse problem for the equation

$$HU = (Q_\xi f)U, \qquad U|_{\partial_-\Omega M} = E \tag{2.6}$$

is also of a great applied interest. To explain the physical meaning of (2.6), let us return to equation (1.5) considered in the three-dimensional case for a skew-Hermitian tensor f. The polarization vector η on (1.5) is a complex two-dimensional vector subordinate to one real condition $|\eta| = 1$. Therefore η can be described by three real parameters. Two of these parameters can be chosen to determine the shape and position of the polarization ellipse on the plane $\dot\gamma^\perp$, while the last parameter is the phase of the electromagnetic wave. See section 6.1 of [5] for a detailed discussion of the subject. Only the first two of these parameters are measured in practice. Deleting the wave phase from the data is mathematically equivalent to replacing the operator P_ξ with Q_ξ. The authors intend to consider the corresponding inverse problem for (2.6) in a subsequent paper.

3 Linear problem

Let (M,g) be a CNTM. Choose two semibasic tensor fields $p,q\in C^\infty(\beta^1_1M;\Omega M)$ satisfying

$$p^*(x,\xi)\xi=\xi,\quad q(x,\xi)\xi=\xi. \tag{3.1}$$

For a tensor field $f\in C^\infty(\tau^1_1M)$, consider the boundary value problem on ΩM

$$Hu=p(P_\xi f)q,\qquad u|_{\partial_-\Omega M}=0. \tag{3.2}$$

The problem has a unique solution $u\in C(\beta^1_1M;\Omega M)$ and, in virtue of (3.1), the solution satisfies

$$u(x,\xi)\xi=u^*(x,\xi)\xi=0. \tag{3.3}$$

In this section, we consider the inverse problem of recovering the tensor field f from the data

$$F[f]=u|_{\partial_+\Omega M}. \tag{3.4}$$

The factors p and q on (3.2) are considered as weights. We will assume the weights to be close to the unit weight E in the following sense: the inequalities

$$|p-E|<\varepsilon,\quad |q-E|<\varepsilon,\quad |\overset{v}{\nabla}p|<\varepsilon,\quad |\overset{v}{\nabla}q|<\varepsilon \tag{3.5}$$

hold uniformly on ΩM with the norm $|\cdot|$ defined in Section 2. The value of ε will be specified later.

Equation (3.2) is initially considered on ΩM. To get some freedom in treating the equation, we extend it to the manifold $T^0M=\{(x,\xi)\in TM\mid \xi\neq 0\}$ of nonzero vectors. The weights are assumed to be positively homogeneous of zero degree in ξ

$$p(x,t\xi)=p(x,\xi),\quad q(x,t\xi)=q(x,\xi)\quad\text{for}\quad t>0.$$

Then the right-hand side of (3.2) is positively homogeneous in ξ of zero degree because f is independent of ξ. The solution u must be extended to T^0M as a homogeneous function of degree -1

$$u(x,t\xi)=t^{-1}U(x,\xi)\quad\text{for}\quad t>0$$

because the operator H increase the degree of homogeneity by 1.

Let us discuss smoothness properties of the solution u. It can be expresses by the explicit formula

$$u(x,\xi)=\int\limits_{\tau_-(x,\xi)}^{0}\Upsilon^{t,0}_\gamma\Big[p(\gamma(t),\dot\gamma(t))P_{\dot\gamma(t)}f(\gamma(t))q(\gamma(t),\dot\gamma(t))\Big]\,dt,$$

where $\gamma = \gamma_{x,\xi}$ and $\Upsilon_\gamma^{t,0}$ is the parallel transport of tensors along the geodesic γ from the point $\gamma(t)$ to $\gamma(0) = x$. The integrand is a smooth function. Therefore smoothness properties of u are determined by that of the integration limit $\tau_-(x, \xi)$. The latter function is C^∞-smooth on $T^0M \setminus T(\partial M)$ but has singularities on $T^0(\partial M)$. Therefore some of integrals considered below are improper and we have to verify their convergence. The verification is performed in the same way as in Section 4.6 of [5]. So, in order to simplify the presentation, we will pay no attention to these singularities.

Besides (3.5), we will impose some smallness condition on the curvature of (M, g). For $(x, \xi) \in \Omega M$, let $K(x, \xi)$ be the supremum of the absolute values of sectional curvatures at the point x over all two-dimensional subspaces of T_xM containing ξ. Define

$$k(M,g) = \sup_{(x,\xi)\in\partial_-\Omega M} \int\limits_0^{\tau_+(x,\xi)} tK(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))dt. \tag{3.6}$$

Theorem 3.1 *For any $n \ge 4$, there exist positive numbers $\delta = \delta(n)$ and $\varepsilon = \varepsilon(n)$ such that, for any n-dimensional CNTM (M, g) satisfying*

$$k(M, g) < \delta \tag{3.7}$$

and for any weights $p, q \in C^\infty(\beta_1^1 M; \Omega M)$ satisfying (3.1) and (3.5), every tensor field $f \in C^\infty(\tau_1^1 M)$ can be uniquely recovered from the trace (3.4) of the solution to the boundary value problem (3.2) and the stability estimate

$$\|f\|_{L^2} \le C\|F[f]\|_{H^1} \tag{3.8}$$

holds with a constant C independent of f. In the case of $n = 3$, the same statement is true for a symmetric tensor field f.

In the case of a real symmetric f and unit weights, this theorem is a partial case of Theorem 5.2.2 of [5]. We will show that the same proof works with some modifications for Theorem 3.1.

Proof of Theorem 3.1. We rewrite equation (3.2) in the form

$$Hu = P_\xi f + r, \tag{3.9}$$

where

$$r = (p - E)P_\xi f + pP_\xi f(q - E). \tag{3.10}$$

The remainder r is small by (3.5). Because of (3.3), the function $u = (u_{ij}(x, \xi))$ is orthogonal to ξ in both indices

$$\xi^i u_{ij} = \xi^j u_{ij} = 0. \tag{3.11}$$

We write down the Pestov identity for the semibasic tensor field u (see Lemma 4.4.1 of [5] for the case of a real u and Lemma 5.1 of [6] for the general case)

$$2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla} Hu\rangle = |\overset{h}{\nabla} u|^2 + \overset{h}{\nabla}_i v^i + \overset{v}{\nabla}_i w^i - \mathcal{R}_1[u], \tag{3.12}$$

where $\langle\cdot,\cdot\rangle$ and $|\cdot|$ are the scalar product and norm on semibasic tensors defined in Section 2,

$$v^i = \mathrm{Re}\Big(\xi^i \overset{h}{\nabla}{}^j u^{i_1 i_2}\cdot \overset{v}{\nabla}_j \bar u_{i_1 i_2} - \xi^j \overset{v}{\nabla}{}^i u^{i_1 i_2}\cdot \overset{h}{\nabla}_j \bar u_{i_1 i_2}\Big), \tag{3.13}$$

$$w^i = \mathrm{Re}\Big(\xi^j \overset{h}{\nabla}{}^i u^{i_1 i_2} \cdot \overset{h}{\nabla}_j \bar{u}_{i_1 i_2}\Big), \tag{3.14}$$

$$\mathcal{R}_1[u] = R_{kplq}\xi^p\xi^q \overset{v}{\nabla}{}^k u^{i_1 i_2} \cdot \overset{v}{\nabla}{}^l \bar{u}_{i_1 i_2} + \mathrm{Re}\Big((R^{i_1}_{pqj}u^{pi_2} + R^{i_2}_{pqj}u^{i_1 p})\xi^q \overset{v}{\nabla}{}^j \bar{u}_{i_1 i_2}\Big), \tag{3.15}$$

and (R_{ijkl}) is the curvature tensor. Identity (3.12) holds on any open $D \subset TM$ for every $u \in C^2(\beta^1_1 M; D)$. In our case, $D = T^0M \setminus T(\partial M)$ and $u \in C^\infty(\beta^1_1 M; D)$ as is shown above.

The most part of the proof deals with the left-hand side of (3.12). We will first transform it by distinguishing some divergent terms and then will estimate it.

From (3.9)

$$\langle \overset{h}{\nabla} u, \overset{v}{\nabla} Hu\rangle = \langle \overset{h}{\nabla} u, \overset{v}{\nabla}(P_\xi f)\rangle + \langle \overset{h}{\nabla} u, \overset{v}{\nabla} r\rangle. \tag{3.16}$$

We will first investigate the first term on the right-hand side of (3.16). To this end we represent f as

$$f_{ij}(x) = \tilde{f}_{ij}(x,\xi) + \xi_j a_i(x,\xi) + \xi_i \bar{b}_j(x,\xi) + \xi_i\xi_j c(x,\xi), \tag{3.17}$$

where $(\tilde{f}_{ij})$ is a semibasic tensor field orthogonal to ξ in both indices

$$\tilde{f}_{ij}\xi^i = \tilde{f}_{ij}\xi^j = 0, \tag{3.18}$$

semibasic covector fields a and b are orthogonal to ξ

$$a_i\xi^i = b_i\xi^i = 0, \tag{3.19}$$

and $c(x,\xi)$ is a scalar function. One can easily check the existence and uniqueness of the representation. The (vector versions of the) fields a and b are expressed through f by the formulas

$$a = \frac{1}{|\xi|^2}\pi_\xi f\xi, \quad b = \frac{1}{|\xi|^2}\pi_\xi f^*\xi. \tag{3.20}$$

As follows from (3.17)–(3.19),

$$P_\xi f = \tilde{f}$$

or in coordinates

$$(P_\xi f)_{ij} = \tilde{f}_{ij} = f_{ij} - a_i\xi_j - \bar{b}_j\xi_i - c\xi_i\xi_j. \tag{3.21}$$

Differentiating the last equality with respect to ξ and using the fact that f is independent of ξ, we obtain

$$\overset{v}{\nabla}_k(P_\xi f)_{ij} = -\xi_j\overset{v}{\nabla}_k a_i - \xi_i\overset{v}{\nabla}_k\bar{b}_j - \xi_i\xi_j\overset{v}{\nabla}_k c - g_{jk}a_i - g_{ik}\bar{b}_j - (g_{ik}\xi_j + g_{jk}\xi_i)c.$$

Therefore

$$\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(P_\xi f)\rangle = \overset{h}{\nabla}{}^k u^{ij} \cdot \overset{v}{\nabla}_k (P_\xi \bar{f})_{ij} =$$

$$= \overset{h}{\nabla}{}^k u^{ij}\Big(-\xi_j\overset{v}{\nabla}_k\bar{a}_i - \xi_i\overset{v}{\nabla}_k b_j - \xi_i\xi_j\overset{v}{\nabla}_k\bar{c} - g_{jk}\bar{a}_i - g_{ik}b_j - (g_{ik}\xi_j + g_{jk}\xi_i)\bar{c}\Big).$$

The tensor $\overset{h}{\nabla}{}^k u^{ij}$ is orthogonal to ξ in the indices i and j as follows from (3.11). Therefore the last formula is simplified to the following one:

$$\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(P_\xi f)\rangle = -\overset{h}{\nabla}{}^p u_{ip} \cdot \bar{a}^i - \overset{h}{\nabla}{}^p u_{pi} \cdot b^i.$$

Introducing the semibasic covector fields $\overset{h}{\delta}_1 u$ and $\overset{h}{\delta}_2 u$ by the equalities

$$(\overset{h}{\delta}_1 u)_i = \overset{h}{\nabla}{}^p u_{ip}, \quad (\overset{h}{\delta}_2 u)_i = \overset{h}{\nabla}{}^p u_{pi}, \tag{3.22}$$

we write the result in the form

$$\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(P_\xi f)\rangle = -\langle \overset{h}{\delta}_1 u, a\rangle - \langle \overset{h}{\delta}_2 u, \bar b\rangle. \tag{3.23}$$

This implies the estimate

$$2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(P_\xi f)\rangle \le \frac{\beta}{2}(|\overset{h}{\delta}_1 u|^2 + |\overset{h}{\delta}_2 u|^2) + \frac{2}{\beta}(|a|^2 + |b|^2), \tag{3.24}$$

where β is an arbitrary positive number.

Next, we transform the expression $|\overset{h}{\delta}_1 u|^2$ by distinguishing a divergent term

$$|\overset{h}{\delta}_1 u|^2 = (\overset{h}{\delta}_1 u)^i (\overset{h}{\delta}_1 \bar u)_i = \overset{h}{\nabla}_p u^{ip} \cdot \overset{h}{\nabla}{}^q \bar u_{iq} =$$

$$= \overset{h}{\nabla}_p (u^{ip} \overset{h}{\nabla}{}^q \bar u_{iq}) - u^{ip} \overset{h}{\nabla}_p \overset{h}{\nabla}{}^q \bar u_{iq} = \overset{h}{\nabla}_p (u^{ip} \overset{h}{\nabla}{}^q \bar u_{iq}) - u_{i\cdot}^{\cdot p} \overset{h}{\nabla}_p \overset{h}{\nabla}_q \bar u^{iq}. \tag{3.25}$$

By the commutator formula for horizontal derivatives (see Theorem 3.5.2 of [5]),

$$\overset{h}{\nabla}_p \overset{h}{\nabla}_q \bar u^{iq} = \overset{h}{\nabla}_q \overset{h}{\nabla}_p \bar u^{iq} - R^k_{jpq} \xi^j \overset{v}{\nabla}_k \bar u^{iq} + R^i_{jpq} \bar u^{jq} + R^q_{jpq} \bar u^{ij}.$$

Substituting this value into the previous formula, we obtain

$$|\overset{h}{\delta}_1 u|^2 = \mathrm{Re}\Big(- u_{i\cdot}^{\cdot p} \overset{h}{\nabla}_q \overset{h}{\nabla}_p \bar u^{iq} + \overset{h}{\nabla}_p (u^{ip} \overset{h}{\nabla}{}^q \bar u_{iq})\Big) + \mathcal{R}_2[u], \tag{3.26}$$

where

$$\mathcal{R}_2[u] = \mathrm{Re}\Big(u_{i\cdot}^{\cdot p} (R^k_{jpq} \xi^j \overset{v}{\nabla}_k \bar u^{iq} - R^i_{jpq} \bar u^{jq} - R^q_{jpq} \bar u^{ij})\Big).$$

We now transform the first summand on the right-hand side of (3.26) in the order reverse to that used in (3.25)

$$|\overset{h}{\delta}_1 u|^2 = \mathrm{Re}\Big(- \overset{h}{\nabla}_q (u_{i\cdot}^{\cdot p} \overset{h}{\nabla}_p \bar u^{iq}) + \overset{h}{\nabla}_q u_{i\cdot}^{\cdot p} \cdot \overset{h}{\nabla}_p \bar u^{iq} + \overset{h}{\nabla}_p (u^{ip} \overset{h}{\nabla}{}^q \bar u_{iq})\Big) + \mathcal{R}_2[u].$$

Introducing the semibasic vector field $\tilde v_1$ by the formula

$$(\tilde v_1)^i = \mathrm{Re}\Big(u^{ji} \overset{h}{\nabla}{}^k \bar u_{jk} - u_{jk} \overset{h}{\nabla}{}^k \bar u^{ji}\Big), \tag{3.27}$$

we write the result in the form

$$|\overset{h}{\delta}_1 u|^2 = \mathrm{Re}\Big(\overset{h}{\nabla}{}^i u^{jk} \cdot \overset{h}{\nabla}_k \bar u_{ji}\Big) + \overset{h}{\nabla}_i (\tilde v_1)^i + \mathcal{R}_2[u]. \tag{3.28}$$

In the same way, we obtain

$$|\overset{h}{\delta}_2 u|^2 = \mathrm{Re}\Big(\overset{h}{\nabla}{}^i u^{jk} \cdot \overset{h}{\nabla}_j \bar u_{ik}\Big) + \overset{h}{\nabla}_i (\tilde v_2)^i + \mathcal{R}_3[u] \tag{3.29}$$

with

$$(\tilde{v}_2)^i = \mathrm{Re}\left(u^{ij}\overset{h}{\nabla}{}^k\bar{u}_{kj} - u_{kj}\overset{h}{\nabla}{}^k\bar{u}^{ij}\right) \tag{3.30}$$

and

$$\mathcal{R}_3[u] = \mathrm{Re}\left(u^{p\cdot}_{\cdot i}(R^k_{jpq}\xi^j\overset{v}{\nabla}_k\bar{u}^{qi} - R^q_{jpq}\bar{u}^{ji} - R^i_{jpq}\bar{u}^{qi})\right).$$

Taking the sum of (3.28) and (3.29), we have

$$|\overset{h}{\delta}_1 u|^2 + |\overset{h}{\delta}_2 u|^2 = \mathrm{Re}\left(\overset{h}{\nabla}{}^i u^{jk}\cdot\overset{h}{\nabla}_j\bar{u}_{ik} + \overset{h}{\nabla}{}^i u^{jk}\cdot\overset{h}{\nabla}_k\bar{u}_{ji}\right) + \overset{h}{\nabla}_i\tilde{v}^i + \mathcal{R}_4[u], \tag{3.31}$$

where

$$\tilde{v} = \tilde{v}_1 + \tilde{v}_2, \quad \mathcal{R}_4[u] = \mathcal{R}_2[u] + \mathcal{R}_3[u].$$

Introduce the semibasic tensor field $z = (z_{ijk})$ by the formula

$$\overset{h}{\nabla}_i u_{jk} = \frac{\xi_i}{|\xi|^2}(Hu)_{jk} + z_{ijk}. \tag{3.32}$$

The idea of this new notation is that the tensor z is orthogonal to ξ in all its indices

$$\xi^i z_{ijk} = \xi^j z_{ijk} = \xi^k z_{ijk} = 0, \tag{3.33}$$

while the tensor $\overset{h}{\nabla}u = (\overset{h}{\nabla}_i u_{jk})$ has the mentioned property only in the last two indices. The summands on the right-hand side of (3.32) are orthogonal to each other, so

$$|\overset{h}{\nabla}u|^2 = \frac{1}{|\xi|^2}|Hu|^2 + |z|^2. \tag{3.34}$$

The first two terms on the right-hand side of (3.31) can be expressed through z. Indeed, one easily see with the help of (3.32) and (3.33) that

$$\mathrm{Re}\left(\overset{h}{\nabla}{}^i u^{jk}\cdot\overset{h}{\nabla}_j\bar{u}_{ik} + \overset{h}{\nabla}{}^i u^{jk}\cdot\overset{h}{\nabla}_k\bar{u}_{ji}\right) = \mathrm{Re}\,(z^{ijk}\bar{z}_{jik} + z^{ijk}\bar{z}_{kji}) \le 2|z|^2.$$

With the help of the last inequality, (3.31) implies the estimate

$$|\overset{h}{\delta}_1 u|^2 + |\overset{h}{\delta}_2 u|^2 \le 2|z|^2 + \overset{h}{\nabla}_i\tilde{v}^i + \mathcal{R}_4[u] \tag{3.35}$$

which, together with (3.24), gives

$$2\,\mathrm{Re}\langle\overset{h}{\nabla}u, \overset{v}{\nabla}(P_\xi f)\rangle \le \beta|z|^2 + \frac{2}{\beta}(|a|^2 + |b|^2) + \frac{\beta}{2}\overset{h}{\nabla}_i\tilde{v}^i + \frac{\beta}{2}\mathcal{R}_4[u].$$

Substitute values (3.20) for a and b into the last formula

$$2\,\mathrm{Re}\langle\overset{h}{\nabla}u, \overset{v}{\nabla}(P_\xi f)\rangle \le \beta|z|^2 + \frac{2}{\beta|\xi|^4}(|\pi_\xi f\xi|^2 + |\pi_\xi f^*\xi|^2) + \frac{\beta}{2}\overset{h}{\nabla}_i\tilde{v}^i + \frac{\beta}{2}\mathcal{R}_4[u]. \tag{3.36}$$

Next, we estimate the second term on the right-hand side of (3.16). We differentiate equality (3.10) with respect to ξ taking the independence f of ξ into account

$$\overset{v}{\nabla}r = \overset{v}{\nabla}\left((p-E)P_\xi\right)f + \overset{v}{\nabla}(pP_\xi)f(q-E) + pP_\xi f\overset{v}{\nabla}q.$$

In what follows in the proof, we denote different constants depending only on $n = \dim M$ by the same letter C. On using (3.5), we obtain from the last formula

$$|\overset{v}{\nabla} r| \le \frac{C\varepsilon}{|\xi|}|f|.$$

From this

$$2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla} r\rangle \le C\varepsilon(|\overset{h}{\nabla} u|^2 + \frac{1}{|\xi|^2}|f|^2). \tag{3.37}$$

Combining (3.36) and (3.37), we obtain from (3.16)

$$2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla} Hu\rangle \le \beta|z|^2 + \frac{2}{\beta|\xi|^4}(|\pi_\xi f\xi|^2 + |\pi_\xi f^*\xi|^2) + \frac{\beta}{2}\overset{h}{\nabla}_i \tilde{v}^i + C\varepsilon(|\overset{h}{\nabla} u|^2 + \frac{1}{|\xi|^2}|f|^2) + \frac{\beta}{2}\mathcal{R}_4[u]. \tag{3.38}$$

Estimating the left-hand side of the Pestov identity (3.12) by (3.38), we obtain for $|\xi| = 1$

$$|\overset{h}{\nabla} u|^2 + \overset{v}{\nabla}_i w^i - \beta|z|^2 - \frac{2}{\beta}(|\pi_\xi f\xi|^2 + |\pi_\xi f^*\xi|^2) - C\varepsilon(|\overset{h}{\nabla} u|^2 + |f|^2) \le \overset{h}{\nabla}_i(\frac{\beta}{2}\tilde{v}^i - v^i) + \mathcal{R}[u], \tag{3.39}$$

where

$$\mathcal{R}[u] = \mathcal{R}_1[u] + \frac{\beta}{2}\mathcal{R}_4[u].$$

We multiply inequality (3.39) by the volume form $d\Sigma = |d_x\omega(\xi) \wedge dV^n(x)|$, integrate over ΩM, and transform the integrals of divergent terms by Gauss-Ostrogradskii formulas (see Theorem 3.6.3 of [5])

$$\int\limits_{\Omega M} \Big[|\overset{h}{\nabla} u|^2 + (n-2)|Hu|^2 - \beta|z|^2 - \frac{2}{\beta}(|\pi_\xi f\xi|^2 + |\pi_\xi f^*\xi|^2) - C\varepsilon(|\overset{h}{\nabla} u|^2 + |f|^2)\Big] d\Sigma \le$$

$$\le \int\limits_{\partial\Omega M} \langle \frac{\beta}{2}\tilde{v} - v, \nu\rangle d\Sigma^{2n-2} + \int\limits_{\Omega M} \mathcal{R}[u] d\Sigma. \tag{3.40}$$

The second term on the left-hand side has appeared because w is positively homogeneous of degree -1 in ξ and satisfies $\langle \xi, w\rangle = |Hu|^2$ as is seen from (3.14). Substituting the value $|\overset{h}{\nabla} u|^2 = |z|^2 + |Hu|^2$ from (3.34), we write the result in the form

$$\int\limits_{\Omega M} \Big[(1-\beta-C\varepsilon)|z|^2 + (n-1-C\varepsilon)|Hu|^2 - \frac{2}{\beta}(|\pi_\xi f\xi|^2 + |\pi_\xi f^*\xi|^2) - C\varepsilon|f|^2\Big] d\Sigma \le$$

$$\le \int\limits_{\partial\Omega M} \langle \frac{\beta}{2}\tilde{v} - v, \nu\rangle d\Sigma^{2n-2} + \int\limits_{\Omega M} \mathcal{R}[u] d\Sigma. \tag{3.41}$$

Assuming $\beta \le 1$, integrals on the right-hand side of (3.41) can be estimated exactly as in Section 5.5 of [5]

$$\left|\int\limits_{\Omega M} \mathcal{R}[u] d\Sigma\right| \le Ck(M,g) \int\limits_{\Omega M} |\overset{h}{\nabla} u|^2 d\Sigma \le C\delta \int\limits_{\Omega M} |\overset{h}{\nabla} u|^2 d\Sigma, \tag{3.42}$$

$$\left|\int\limits_{\partial\Omega M}\langle\frac{\beta}{2}\tilde{v}-v,\nu\rangle d\Sigma^{2n-2}\right|\le D\|u|_{\partial_+\Omega M}\|^2_{H^1}, \tag{3.43}$$

where $k(M,g)$ is defined by (3.6). The second inequality on (3.42) is valid because of (3.7). The constant D on (3.43) depends on (M,g), unlike the constant C on (3.42) which depends only on n.

Combining (3.41)–(3.43) and using again the equality $|\overset{h}{\nabla}u|^2=|z|^2+|Hu|^2$, we obtain

$$\int\limits_{\Omega M}\Big[(1-\beta-C\varepsilon-C\delta)|z|^2+(n-1-C\varepsilon-C\delta)|Hu|^2-\frac{2}{\beta}(|\pi_\xi f\xi|^2+|\pi_\xi f^*\xi|^2)-C\varepsilon|f|^2\Big]d\Sigma\le$$

$$\le D\|u|_{\partial_+\Omega M}\|^2_{H^1} \tag{3.44}$$

with some new constant C depending only on n.

Let us compare $|Hu|$ and $|P_\xi f|$. The estimate $|r|<C\varepsilon|f|$ follows from (3.5) and (3.10). The latter, together with (3.9), implies

$$|Hu|^2\ge|P_\xi f|^2-C\varepsilon|f|^2. \tag{3.45}$$

Using the last inequality, we transform (3.44) to the final form

$$\int\limits_M\int\limits_{\Omega_xM}\Big[(1-\beta-C\varepsilon-C\delta)|z|^2+(n-1-C\varepsilon-C\delta)|P_\xi f|^2-$$

$$-\frac{2}{\beta}(|\pi_\xi f\xi|^2+|\pi_\xi f^*\xi|^2)-C\varepsilon|f|^2\Big]d\omega_x(\xi)dV^n(x)\le D\|u|_{\partial_+\Omega M}\|^2_{H^1}. \tag{3.46}$$

Let us remind that β is an arbitrary number satisfying $0<\beta\le 1$.

Lemma 3.2 *For every Riemannian manifold (M,g) of dimension $n\ge 4$ and every point $x\in M$, the Hermitian form*

$$B(f,f)=\int\limits_{\Omega_xM}\Big[(n-1)|P_\xi f|^2-2(|\pi_\xi f\xi|^2+|\pi_\xi f^*\xi|^2)\Big]d\omega_x(\xi)$$

is positive definite on the space of second rank tensors at x. Moreover, the estimate

$$B(f,f)\ge c|f|^2$$

holds with a positive constant c depending only on n. In the case of $n=3$, the same statement is valid for symmetric tensors.

The proof of the lemma will be given later, and now we finish the proof of Theorem 3.1 by making use of the lemma.

By the lemma, the inequality

$$c\|f\|^2_{L^2}=c\int\limits_M|f|^2\,dV^n(x)\le\int\limits_M\int\limits_{\Omega_xM}\Big[(1-\beta-C\varepsilon-C\delta)|z|^2+$$

$$+(n-1-C\varepsilon-C\delta)|P_\xi f|^2-\frac{2}{\beta}(|\pi_\xi f\xi|^2+|\pi_\xi f^*\xi|^2)-C\varepsilon|f|^2\Big]d\omega_x(\xi)dV^n(x) \qquad (3.47)$$

holds for $\beta=1,\ \varepsilon=\delta=0$. By continuity and by estimates $|P_\xi f|\le|f|,\ |\pi_\xi f\xi|\le|f|,\ |\pi_\xi f^*\xi|\le|f|$ for $|\xi|=1$, the same inequality holds for some positive c,ε,δ, and β independent of f and satisfying $1-\beta-C\varepsilon-C\delta\ge0$. Combining (3.46) and (3.47), we obtain

$$\|f\|^2_{L^2}\le C\|u|_{\partial_+\Omega M}\|^2_{H^1}=C\|F[f]\|^2_{H^1}$$

with $C=D/c$. This finishes the proof of Theorem 3.1.

Proof of Lemma 3.2. One easily checks the equality $B(f,f)=B(u,u)+B(v,v)$ for a complex tensor f represented as $f=u+iv$ with real u and v. Therefore it suffices to prove the statement for a real tensor f.

The corresponding symmetric bilinear form is

$$B(f,h)=\int\limits_{\Omega_xM}\Big[(n-1)\langle P_\xi f,h\rangle-2(\langle\pi_\xi f\xi,h\xi\rangle+\langle\pi_\xi f^*\xi,h^*\xi\rangle)\Big]d\omega_x(\xi).$$

Obviously, $B(f,h)=0$ for a symmetric f and skew-symmetric h. Therefore, it suffices to prove the positiveness of B on the spaces of real symmetric and skew-symmetric tensors separately.

The positiveness of B on the space of real symmetric tensors is proved in Lemma 5.6.1 of [5], where $\pi_\xi f\xi$ is denoted by $P_\xi j_\xi f$.

Thus, we have to consider the quadratic form B on the space of real skew-symmetric tensors. On making use of an orthonormal basis, we identify T_xM with $\mathbb{R}^n$ endowed with the standard scalar product and identify Ω_xM with the unit sphere $\Omega\subset\mathbb{R}^n$. So

$$\frac{1}{\omega}B(f,f)=\frac{1}{\omega}\int\limits_\Omega\Big[(n-1)|P_\xi f|^2-4|\pi_\xi f\xi|^2\Big]d\omega(\xi), \qquad (3.48)$$

where ω is the volume of Ω and $d\omega$ is the standard volume form on Ω.

For a skew-symmetric f

$$\pi_\xi f\xi=f\xi \qquad (3.49)$$

since $f\xi$ is orthogonal to ξ. Therefore formulas (3.20) and (3.21) are simplified to the following ones:

$$a=-b=f\xi,\quad (P_\xi f)_{ij}=f_{ij}-(f\xi)_i\xi_j+\xi_i(f\xi)_j$$

for $|\xi|=1$. Using the last equality and $\langle f\xi,\xi\rangle=0$, we easily calculate

$$|P_\xi f|^2=|f|^2-2|f\xi|^2\quad\text{for}\quad|\xi|=1. \qquad (3.50)$$

Substitute (3.49) and (3.50) into (3.48)

$$\frac{1}{\omega}B(f,f)=(n-1)|f|^2-\frac{2(n+1)}{\omega}\int\limits_\Omega|f\xi|^2\,d\omega. \qquad (3.51)$$

On using the obvious relation

$$\frac{1}{\omega}\int\limits_\Omega\xi_i\xi_j\,d\omega=\begin{cases}0 & \text{for}\quad i\ne j,\\ 1/n & \text{for}\quad i=j,\end{cases}$$

we find

$$\frac{1}{\omega}\int\limits_{\Omega}|f\xi|^2\,d\omega=\frac{1}{n}|f|^2.$$

Inserting this value into (3.51), we see that

$$\frac{1}{\omega}B(f,f)=\frac{n^2-3n-2}{n}|f|^2.$$

This implies the positiveness of B on skew-symmetric tensors for $n\geq 4$.

4 Three-dimensional case

We will first show that both our problems, linear and nonlinear, possess some non-uniqueness in the three-dimensional case.

Let (M,g) be a three-dimensional CNTM which is assumed to be oriented. Every tangent space T_xM is a three-dimensional oriented Euclidean space. So, the vector product

$$T_xM\times T_xM\to T_xM,\quad (v,w)\mapsto v\times w$$

is well defined. It is extended to the $\mathbb{C}$-bilinear operation

$$T_x^{\mathbb{C}}M\times T_x^{\mathbb{C}}M\to T_x^{\mathbb{C}}M,\quad (v,w)\mapsto v\times w$$

on complex vectors. For a complex vector field $v\in C^\infty(\tau_0^1M)$, we denote by $L_v\in C^\infty(\tau_1^1M)$ the operator of vector multiplication by v,

$$L_v(x)\eta=v(x)\times\eta\quad\text{for}\quad \eta\in T_x^{\mathbb{C}}M.$$

Note that L_v is a skew-symmetric tensor field. Quite similarly, for a semibasic vector field $v\in C^\infty(\beta_0^1M)$, the operator $L_v\in C^\infty(\beta_1^1M)$ is defined. Let us prove the formula

$$P_\xi L_v=\pi_\xi L_v\pi_\xi=\frac{\langle v,\xi\rangle}{|\xi|^2}L_\xi. \tag{4.1}$$

We remind that $\langle\cdot,\cdot\rangle$ and $|\cdot|$ are defined in Section 2. This is a pure algebraic local formula. So, we can use a positive orthonormal basis $(e_1,e_2,e_3=\xi/|\xi|)$ in T_xM. In such a basis

$$\pi_\xi=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix},\ L_v=\begin{pmatrix}0&-v_3&v_2\\v_3&0&-v_1\\-v_2&v_1&0\end{pmatrix},\ L_\xi=\begin{pmatrix}0&-|\xi|&0\\|\xi|&0&0\\0&0&0\end{pmatrix},\ \langle v,\xi\rangle=v_3|\xi|,$$

and the formula follows immediately.

Next, we prove the formula

$$\overset{h}{\nabla}L_\xi=0. \tag{4.2}$$

The formula is quite expectable since $\overset{h}{\nabla}\xi=0$. Nevertheless, it needs a proof. Here, we have to use a general coordinate system since $\overset{h}{\nabla}$ is a differential operator. The vector product is expressed by the formula

$$(v\times w)^i=\frac{1}{\sqrt{g}}(v_{i+1}w_{i+2}-v_{i+2}w_{i+1})$$

in general coordinates, where $g = \det(g_{ij})$ and indices are reduced modulo 3. Therefore

$$(L_v)_{i,i+1} = -\sqrt{g}v^{i+2} \tag{4.3}$$

and (4.2) is proved by straightforward calculations with making use of $\nabla_i g_{jk} = 0$ and $\overset{h}{\nabla}_i \xi^j = 0$.

We derive from (4.1)–(4.2) and the formula $H(1/|\xi|^2) = 0$ that

$$H(\frac{\lambda}{|\xi|^2}L_\xi) = \frac{H\lambda}{|\xi|^2}L_\xi = P_\xi L_{\nabla\lambda}$$

for a complex function $\lambda \in C^\infty(M)$. Thus

$$H(\frac{\lambda}{|\xi|^2}L_\xi) = P_\xi L_{\nabla\lambda}. \tag{4.4}$$

This gives us the following non-uniqueness in the linear problem. If the function λ vanishes on the boundary, $\lambda|_{\partial M} = 0$, then $u(x,\xi) = \lambda L_\xi/|\xi|^2$ solves the boundary value problem

$$Hu = P_\xi f, \quad u_{\partial_-\Omega M} = 0 \tag{4.5}$$

with $f = L_{\nabla\lambda}$ and satisfies

$$F[f] = u|_{\partial_+\Omega M} = 0. \tag{4.6}$$

The boundary value problem (4.5) coincides with (3.2) for the unit weights $p = q = E$, and (4.6) means that f cannot be recovered from $F[f]$.

The same arguments give us some non-uniqueness in the nonlinear problem. Fix a function $\lambda \in C^\infty(M)$ vanishing on the boundary, $\lambda|_{\partial M} = 0$, and, for $(x,\xi) \in \Omega M$, define the linear operator $U(x,\xi)$ on $T_x^{\mathbb{C}}M$ whose matrix in a positive orthonormal basis $(e_1, e_2, e_3 = \xi)$ is

$$\begin{pmatrix} \cos\lambda(x) & -\sin\lambda(x) & 0 \\ \sin\lambda(x) & \cos\lambda(x) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For a real function λ, $U(x,\xi)$ is the rotation of T_xM around the axis ξ by the angle $\lambda(x)$. In the case of a complex λ, the operator is also well defined although its geometric sense is more complicated. The semibasic tensor field $U \in C^\infty(\beta_1^1 M; \Omega M)$ satisfies the equation

$$HU = (P_\xi L_{\nabla\lambda})U \tag{4.7}$$

and boundary condition

$$U|_{\partial\Omega M} = E. \tag{4.8}$$

Indeed, (4.8) is obvious. Let us prove (4.7). In virtue of (4.1), equation (4.7) is equivalent to the following one:

$$HU(x,\xi) = \langle\nabla\lambda(x), \xi\rangle L_\xi U(x,\xi). \tag{4.9}$$

Let γ be a unit speed geodesic. Setting $x = \gamma(t)$, $\xi = \dot\gamma(t)$ in (4.9), we arrive to the equation

$$\frac{DU(t)}{dt} = \frac{d\lambda(t)}{dt}L_{\dot\gamma(t)}U(t), \tag{4.10}$$

where $\lambda(t) = \lambda(\gamma(t))$ and $U(t) = U(\gamma(t), \dot\gamma(t))$. Conversely, if (4.10) holds for any unit speed geodesic γ, then (4.9) is true. To prove (4.10), we choose an orthonormal basis $(e_1(t), e_2(t), e_3(t) = \dot\gamma(t))$ of $T_{\gamma(t)}M$ which is parallel along γ. In such a basis, (4.10) is equivalent to the matrix equation

$$\frac{d}{dt}\begin{pmatrix} \cos\lambda(t) & -\sin\lambda(t) & 0 \\ \sin\lambda(t) & \cos\lambda(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{d\lambda}{dt}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\lambda(t) & -\sin\lambda(t) & 0 \\ \sin\lambda(t) & \cos\lambda(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is obviously true.

Comparing (4.7)–(4.8) with (2.3)–(2.4), we see that a field $f = L_{\nabla\lambda}$ with $\lambda|_{\partial M} = 0$ cannot be recovered from data (2.3).

Let us introduce the definition: $f \in C^\infty(\tau_1^1 M)$ is said to be a *potential field* if it can be represented as $f = L_{\nabla\lambda}$ for some function $\lambda \in C^\infty(M)$ vanishing on the boundary, $\lambda|_{\partial M} = 0$. Potential fields constitute the natural obstruction for the uniqueness in the both, linear and nonlinear, problems. We are going to prove that a solution to the linear problem is unique up to the obstruction. The corresponding local result for the nonlinear problem will be obtained in the next section. First of all we prove

Lemma 4.1 *(on decomposition). Let (M, g) be a compact oriented three-dimensional Riemannian manifold with boundary. Every tensor field $f \in C^\infty(\tau_1^1 M)$ can be uniquely represented as*

$$f = L_{\nabla\lambda} + \tilde f \tag{4.11}$$

with some $\lambda \in C^\infty(M)$ satisfying

$$\lambda|_{\partial M} = 0 \tag{4.12}$$

and some tensor field $\tilde f \in C^\infty(\tau_1^1 M)$ satisfying the condition: the 2-form $\tilde f_{ij} dx^i \wedge dx^j$ is closed,

$$d(\tilde f_{ij} dx^i \wedge dx^j) = 0. \tag{4.13}$$

The summands of decomposition (4.11) are called the *potential* and *closed* parts of f respectively. Note that (4.13) involves only the skew-symmetric part of $\tilde f$, i.e., a symmetric tensor field is closed. Lemma 4.1 can be derived from the Hodge-Morrey decomposition [8] but we give a shorter independent proof.

Proof of Lemma 4.1. We first prove the uniqueness statement. Assume (4.11)–(4.13) to be valid. Applying the exterior derivative d to the form $f_{ij} dx^i \wedge dx^j$ and using (4.11) and (4.13), we obtain

$$d\Big((L_{\nabla\lambda})_{ij} dx^i \wedge dx^j\Big) = d(f_{ij} dx^i \wedge dx^j).$$

On using (4.3), one can check by a straightforward calculation in coordinates that

$$d\Big((L_{\nabla\lambda})_{ij} dx^i \wedge dx^j\Big) = -2\Delta\lambda\sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3,$$

where Δ is the Laplace-Beltrami operator. Thus, the function λ solves the Dirichlet problem

$$\Delta\lambda = -\frac{1}{\sqrt{g}}\Big(\frac{\partial f_{\overline{23}}}{\partial x^1} + \frac{\partial f_{\overline{31}}}{\partial x^2} + \frac{\partial f_{\overline{12}}}{\partial x^3}\Big), \quad \lambda|_{\partial M} = 0, \tag{4.14}$$

where f^- is the skew-symmetric part of f, i.e., $f^-_{ij} = \frac{1}{2}(f_{ij} - f_{ji})$. The solution to the Dirichlet problem is unique.

The existence statement is proved by reverse arguments. Given f, let λ be the solution to the Dirichlet problem (4.14) and $\tilde{f} = f - L_{\nabla\lambda}$. Then (4.11)–(4.13) holds. The lemma is proved.

We restrict ourselves to considering the inverse problem for closed fields only.

Theorem 4.2 *There exist such positive numbers δ and ε that, for any oriented 3-dimensional CNTM (M, g) satisfying (3.7) and for any weights $p, q \in C^\infty(\beta^1_1 M; \Omega M)$ satisfying (3.1) and (3.5), every closed tensor field $f \in C^\infty(\tau^1_1 M)$ can be uniquely recovered from the trace (3.4) of the solution to the boundary value problem (3.2) and the stability estimate*

$$\|f\|^2_{L^2} \le C\left(\|F[f]\|^2_{H^1} + \|f|_{\partial M}\|_{L^2} \cdot \|F[f]\|_{L^2}\right) \tag{4.15}$$

holds with a constant C independent of f.

Proof follows the same line as the proof of Theorem 3.1 with the following difference: the left-hand side of the Pestov identity (3.12) will be estimated in a different way by making use of the closeness of f.

We represent f as the sum of symmetric and skew-symmetric fields

$$f = f^+ + f^-, \quad f^+_{ij} = f^+_{ji}, \quad f^-_{ij} = -f^-_{ji}.$$

Taking the symmetry of Christoffel symbols into account, the closeness condition for f can be written as

$$\nabla_i f^-_{jk} + \nabla_j f^-_{ki} + \nabla_k f^-_{ij} = 0. \tag{4.16}$$

The vector $f^-\xi$ is orthogonal to ξ and therefore $\pi_\xi f^-\xi = f^-\xi$. Formulas (3.20) take now the form

$$a = \frac{1}{|\xi|^2}\pi_\xi f^+\xi + \frac{1}{|\xi|^2} f^-\xi, \quad b = \frac{1}{|\xi|^2}\pi_\xi \bar{f}^+\xi - \frac{1}{|\xi|^2}\bar{f}^-\xi. \tag{4.17}$$

We write equation (3.2) in the form (3.9) with the remainder r defined by (3.10). Then we write the Pestov identity (3.12) for u with terms defined by (3.13)–(3.15). The left hand-side of the identity can be written as in (3.16).

The main problem is estimating the first term $\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(P_\xi f)\rangle$ on the right-hand side of (3.16). To this end we represent f in form (3.17) with a and b defined by (4.17). Then (3.23) holds. In view of (4.17), equation (3.23) can be written as

$$\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(P_\xi f)\rangle = -\frac{1}{|\xi|^2}\langle \overset{h}{\delta}_1 u + \overset{h}{\delta}_2 u, \pi_\xi f^+\xi\rangle - \frac{1}{|\xi|^2}\langle \overset{h}{\delta}_1 u - \overset{h}{\delta}_2 u, f^-\xi\rangle. \tag{4.18}$$

We transform the second term on the right-hand side of (4.18) by distinguishing divergent terms. Using definition (3.22) of $\overset{h}{\delta}_1 u$ and $\overset{h}{\delta}_2 u$, we write

$$\langle \overset{h}{\delta}_1 u - \overset{h}{\delta}_2 u, f^-\xi\rangle = (\overset{h}{\nabla}_p u^{ip} - \overset{h}{\nabla}_p u^{pi})\bar{f}^-_{ik}\xi^k =$$

$$= \overset{h}{\nabla}_p(\xi^k u^{ip}\bar{f}^-_{ik} + \xi^k u^{pi}\bar{f}^-_{ki}) - \xi^k u^{ip}(\overset{h}{\nabla}_p \bar{f}^-_{ik} + \overset{h}{\nabla}_i \bar{f}^-_{kp}).$$

Now, we use the closeness condition (4.16) to transform this formula to the following one:

$$\langle \overset{h}{\delta}_1 u - \overset{h}{\delta}_2 u, f^-\xi\rangle = \xi^k u^{ip} \overset{h}{\nabla}_k \bar{f}^-_{pi} + \overset{h}{\nabla}_p(\xi^k u^{ip} \bar{f}^-_{ik} + \xi^k u^{pi} \bar{f}^-_{ki}).$$

Finally, we distinguish a divergent term from the first summand on the right-hand side

$$\langle \overset{h}{\delta}_1 u - \overset{h}{\delta}_2 u, f^-\xi\rangle = \overset{h}{\nabla}_k(\xi^k u^{ip} \bar{f}^-_{pi}) - \xi^k \overset{h}{\nabla}_k u^{ip} \cdot \bar{f}^-_{pi} + \overset{h}{\nabla}_p(\xi^k u^{ip} \bar{f}^-_{ik} + \xi^k u^{pi} \bar{f}^-_{ki}).$$

This can be written as

$$\langle \overset{h}{\delta}_1 u - \overset{h}{\delta}_2 u, f^-\xi\rangle = \langle Hu, f^-\rangle + \overset{h}{\nabla}_i(\xi^i u^{jk} \bar{f}^-_{kj} + \xi^k u^{ji} \bar{f}^-_{jk} + \xi^k u^{ij} \bar{f}^-_{kj}). \tag{4.19}$$

Next, we calculate the first term on the right-hand side of (4.19) by making use of (3.9) and (3.21)

$$\langle Hu, f^-\rangle = (Hu)^{ij} \bar{f}^-_{ij} = ((P_\xi f)^{ij} + r^{ij})\bar{f}^-_{ij} = (f^{ij} - a^i\xi^j - \bar{b}^j\xi^i - c\xi^i\xi^j + r^{ij})\bar{f}^-_{ij}. \tag{4.20}$$

Since f^- is a skew-symmetric tensor, $c\xi^i\xi^j\bar{f}^-_{ij} = 0$. This means that the term $c\xi^i\xi^j$ on the right-hand side of (4.20) can be omitted. By the same reason, the term f^{ij} can be replaced with $(f^-)^{ij}$, the vector $a = \frac{1}{|\xi|^2}\pi_\xi f\xi$ can be replaced with $\frac{1}{|\xi|^2} f\xi$, and the vector b can be replaced with $\frac{1}{|\xi|^2} f^*\xi$. In such the way, (4.20) takes the form

$$\langle Hu, f^-\rangle = |f^-|^2 - \frac{1}{|\xi|^2}(f^{ik}\xi_k\xi^j + f^{kj}\xi_k\xi^i)\bar{f}^-_{ij} + \langle r, f^-\rangle.$$

The second term on the right-hand side is independent of f^+, and the formula can be written as

$$\langle Hu, f^-\rangle = |f^-|^2 - \frac{2}{|\xi|^2}|f^-\xi|^2 + \langle r, f^-\rangle. \tag{4.21}$$

Substitute (4.21) into (4.19)

$$\langle \overset{h}{\delta}_1 u - \overset{h}{\delta}_2 u, f^-\xi\rangle = |f^-|^2 - \frac{2}{|\xi|^2}|f^-\xi|^2 + \langle r, f^-\rangle + \overset{h}{\nabla}_i(\xi^i u^{jk} \bar{f}^-_{kj} + \xi^k u^{ji} \bar{f}^-_{jk} + \xi^k u^{ij} \bar{f}^-_{kj}).$$

Inserting this expression into (4.18) and estimating the first term on the right-hand side of (4.18) in a similar way as in deriving (3.24), we arrive to the inequality

$$\begin{aligned} 2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(P_\xi f)\rangle \le \frac{\beta}{2}(|\overset{h}{\delta}_1 u|^2 + |\overset{h}{\delta}_2 u|^2) + \frac{4}{\beta|\xi|^4}|\pi_\xi f^+\xi|^2 - \\ -\frac{2}{|\xi|^2}|\bar{f}^-|^2 + \frac{4}{|\xi|^4}|f^-\xi|^2 + \overset{h}{\nabla}_i(\tilde{v}_3)^i + \frac{2}{|\xi|^2}|r|\cdot|f|, \end{aligned} \tag{4.22}$$

where

$$(\tilde{v}_3)^i = -\frac{2}{|\xi|^2}\,\mathrm{Re}(\xi^i u^{jk} \bar{f}^-_{kj} + \xi^k u^{ji} \bar{f}^-_{jk} + \xi^k u^{ij} \bar{f}^-_{kj}) \tag{4.23}$$

and β is an arbitrary positive number. The last term on the right-hand side of (4.22) can be estimated by $C\varepsilon|f|^2/|\xi|^2$ as follows from (3.5) and (3.10). Estimating the first term on the right-hand side of (4.22) by (3.35), we obtain

$$2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(P_\xi f)\rangle \le \beta|z|^2 + \frac{4}{\beta|\xi|^4}|\pi_\xi f^+\xi|^2 - \frac{2}{|\xi|^2}|\bar{f}^-|^2 + \frac{4}{|\xi|^4}|f^-\xi|^2 + \frac{C\varepsilon}{|\xi|^2}|f|^2 + \overset{h}{\nabla}_i\tilde{v}^i + \frac{\beta}{2}\mathcal{R}_4[u] \tag{4.24}$$

with the same curvature dependent term $\mathcal{R}_4[u]$ as in (3.36) and

$$\tilde{v} = \frac{\beta}{2}(\tilde{v}_1 + \tilde{v}_2) + \tilde{v}_3,$$

where $\tilde{v}_i$ $(i = 1, 2, 3)$ are defined by (3.27), (3.30), and (4.23) respectively.

The second term on the right-hand side of (3.16) is estimated by (3.37) as before. Combining (3.37) and (4.24), we obtain from (3.16)

$$2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla} Hu\rangle \le \beta|z|^2 + \frac{4}{\beta}|\pi_\xi f^+\xi|^2 - 2|\bar{f}^-|^2 + 4|f^-\xi|^2 + C\varepsilon(|f|^2 + |\overset{h}{\nabla} u|^2) + \overset{h}{\nabla}_i \tilde{v}^i + \frac{\beta}{2}\mathcal{R}_4[u] \tag{4.25}$$

for $|\xi| = 1$.

We estimate the left-hand side of the Pestov identity (3.12) by (4.25) and write the result in the form

$$(1 - C\varepsilon)|\overset{h}{\nabla} u|^2 + \overset{v}{\nabla}_i w^i - \beta|z|^2 - \frac{4}{\beta}|\pi_\xi f^+\xi|^2 + 2|\bar{f}^-|^2 - 4|f^-\xi|^2 - C\varepsilon|f|^2 \le \overset{h}{\nabla}_i(\tilde{v}^i - v^i) + \frac{\beta}{2}\mathcal{R}_4[u].$$

Integrating this inequality and transforming the integrals of divergent terms in the same way as in (3.40), we obtain

$$\int\limits_{\Omega M} \Big[(1 - C\varepsilon)|\overset{h}{\nabla} u|^2 + |Hu|^2 - \beta|z|^2 - \frac{4}{\beta}|\pi_\xi f^+\xi|^2 + 2|\bar{f}^-|^2 - 4|f^-\xi|^2 - C\varepsilon|f|^2\Big] d\Sigma \le$$

$$\le \int\limits_{\partial\Omega M} \langle \tilde{v} - v, \nu\rangle d\Sigma^{2n-2} + \int\limits_{\Omega M} \mathcal{R}_4[u] d\Sigma.$$

Substituting the value $|\overset{h}{\nabla} u|^2 = |z|^2 + |Hu|^2$ from (3.34), we write the result in the form

$$\int\limits_{\Omega M} \Big[(1 - \beta - C\varepsilon)|z|^2 + (2 - C\varepsilon)|Hu|^2 - \frac{4}{\beta}|\pi_\xi f^+\xi|^2 + 2|\bar{f}^-|^2 - 4|f^-\xi|^2 - C\varepsilon|f|^2\Big] d\Sigma \le$$

$$\le \int\limits_{\partial\Omega M} \langle \tilde{v} - v, \nu\rangle d\Sigma^{2n-2} + \int\limits_{\Omega M} \mathcal{R}_4[u] d\Sigma. \tag{4.26}$$

The curvature dependent integral on (4.26) is estimated as before in (3.42)

$$\Bigg|\int\limits_{\Omega M} \mathcal{R}_1[u] d\Sigma\Bigg| \le C\delta \int\limits_{\Omega M} |\overset{h}{\nabla} u|^2 d\Sigma, \tag{4.27}$$

while the boundary integral is estimated in a little bit different way. Namely, instead of (3.43), we have the estimate

$$\Bigg|\int\limits_{\partial\Omega M} \langle \tilde{v} - v, \nu\rangle d\Sigma^{2n-2}\Bigg| \le D\Big(\|u|_{\partial_+\Omega M}\|^2_{H^1} + \|f|_{\partial M}\|_{L^2} \cdot \|u|_{\partial_+\Omega M}\|_{L^2}\Big). \tag{4.28}$$

The second term on the right-hand side of (4.28) appears because of the dependence of $\tilde{v}$ on f as is seen from (4.23). Inequality (4.28) is proved in the same way as estimate (4.7.2) of [5].

Combining (4.26)–(4.28) and using again the equality $|\overset{h}{\nabla}u|^2 = |z|^2 + |Hu|^2$, we obtain

$$\int\limits_{\Omega M} \Big[(1-\beta-C\delta-C\varepsilon)|z|^2+(2-C\delta-C\varepsilon)|Hu|^2-\frac{4}{\beta}|\pi_\xi f^+\xi|^2+2|\bar{f}^-|^2-4|f^-\xi|^2-C\varepsilon|f|^2\Big]d\Sigma \le$$

$$\le D\Big(\|u|_{\partial_+\Omega M}\|^2_{H^1} + \|f|_{\partial M}\|_{L^2}\cdot\|u|_{\partial_+\Omega M}\|_{L^2}\Big). \tag{4.29}$$

The tensors f^+ and f^- are orthogonal to each other as well as $P_\xi f^+$ and $P_\xi f^-$ are orthogonal to each other. Therefore

$$|f|^2 = |f^+|^2 + |f^-|^2, \quad |P_\xi f|^2 = |P_\xi f^+|^2 + |P_\xi f^-|^2.$$

With the help of these equalities, (3.45) gives

$$|Hu|^2 \ge |P_\xi f^+|^2 - C\varepsilon|f^+|^2 - C\varepsilon|f^-|^2. \tag{4.30}$$

We use (4.30) to transform the estimate (4.29) to the final form

$$(1-\beta-C\delta-C\varepsilon)\int\limits_{\Omega M}|z|^2 d\Sigma + \int\limits_{\Omega M}\Big[(2-C\delta-C\varepsilon)|P_\xi f^+|^2 - \frac{4}{\beta}|\pi_\xi f^+\xi|^2 - C\varepsilon|f^+|^2\Big]d\Sigma +$$

$$+2\int\limits_{\Omega M}(|\bar{f}^-|^2 - 2|f^-\xi|^2 - C\varepsilon|f^-|^2)d\Sigma \le D\Big(\|u|_{\partial_+\Omega M}\|^2_{H^1} + \|f|_{\partial M}\|_{L^2}\cdot\|u|_{\partial_+\Omega M}\|_{L^2}\Big) \tag{4.31}$$

with some new constant C.

For $\beta = 1$ and $\varepsilon = \delta = 0$, the left-hand side of (4.31) is the integral over M of the Hermitian form

$$A(f,f) = B(f^+,f^+) + 2Q(f^-,f^-),$$

where B is the same as in Lemma 3.2 and

$$Q(f,f) = \int\limits_{\Omega_x M}(|f|^2 - 2|f\xi|^2)d\omega_x(\xi). \tag{4.32}$$

By Lemma 3.2, the form B is positively definite on symmetric tensors in the 3-dimensional case. The form Q is positively definite on skew-symmetric tensors. Indeed, repeating the arguments of the proof of Lemma 3.2, we derive in the 3-dimensional case

$$\frac{1}{\omega}\int\limits_{\Omega_x M}(|f|^2 - 2|f\xi|^2)d\omega_x(\xi) = \frac{1}{3}|f|^2$$

for a skew-symmetric tensor f. Now, the proof is finished in the same way as in Theorem 3.1.

5 Nonlinear problem

We return to considering the inverse problem of recovering a tensor field $f \in C^\infty(\tau_1^1 M)$ on a CNTM (M, g) from the data

$$\Phi[f] = U|_{\partial_+\Omega M}, \tag{5.1}$$

where $U \in C(\beta_1^1 M; \Omega M)$ is the solution to the boundary value problem on ΩM

$$HU = (P_\xi f)U, \qquad U|_{\partial_-\Omega M} = E. \tag{5.2}$$

We will prove the uniqueness under the following smallness assumptions on f:

$$|f(x)| < \varepsilon \quad \text{for} \quad x \in \partial M, \tag{5.3}$$

$$\int\limits_{\tau_-(x,\xi)}^{0} |f(\gamma_{x,\xi}(t))|dt < \varepsilon, \qquad \int\limits_{\tau_-(x,\xi)}^{0} |\nabla f(\gamma_{x,\xi}(t))|dt < \varepsilon \quad \text{for} \quad (x,\xi) \in \partial_+\Omega M. \tag{5.4}$$

We remind that we use notations introduced in Section 2. In particular, ∇ is the covariant derivative. Note that these smallness conditions are quite similar to that of Theorem 2 of [9].

Theorem 5.1 *It is possible to choose a positive number $\delta = \delta(n)$ for $n \geq 4$ such that, for an n-dimensional CNTM (M, g) satisfying the curvature condition (3.7), there exists a positive number $\varepsilon = \varepsilon(M, g)$ such that the following statement is true. Let two tensor fields $f_i \in C^\infty(\tau_1^1 M)$ $(i = 1, 2)$ satisfy (5.3)–(5.4) and $\Phi_i = \Phi[f_i]$ be the corresponding data. Then the estimate*

$$\|f_2 - f_1\|_{L^2} \leq C\|\Phi_1^{-1}\Phi_2 - E\|_{H^1} \tag{5.5}$$

holds with a constant C independent of f_i. In particular, $f_1 = f_2$ if $\Phi_1 = \Phi_2$. In the case of $n = 3$, the same statement is true under the additional assumption that $f_2 - f_1$ is a closed tensor field.

Proof. Let $U_i \in C(\beta_1^1 M; \Omega M)$ $(i = 1, 2)$ be the solution to the boundary value problem

$$HU_i = (P_\xi f_i)U_i, \qquad U_i|_{\partial_-\Omega M} = E. \tag{5.6}$$

According to (5.2), the solution satisfies

$$U_i\xi = U_i^*\xi = \xi. \tag{5.7}$$

Set $u = U_1^{-1}U_2 - E$. Using the equalities $HE = 0$, $H(U_1^{-1}U_2) = HU_1^{-1} \cdot U_2 + U_1^{-1}HU_2$, and $HU_1^{-1} = -U_1^{-1}HU_1 \cdot U_1^{-1}$, one easily derive from (5.6) that u solves the boundary value problem

$$Hu = U_1^{-1}\big(P_\xi(f_2 - f_1)\big)U_2, \qquad u|_{\partial_-\Omega M} = 0 \tag{5.8}$$

and

$$u|_{\partial_+\Omega M} = \Phi_1^{-1}\Phi_2 - E. \tag{5.9}$$

Setting $f=f_2-f_1,\ p=U_1^{-1},\ q=U_2$, we write (5.8)–(5.9) in the form

$$Hu=p(P_\xi f)q,\qquad u|_{\partial_-\Omega M}=0, \tag{5.10}$$

$$F[f]=u|_{\partial_+\Omega M}=\Phi_1^{-1}\Phi_2-E. \tag{5.11}$$

We have arrived to the linear problem considered in Sections 3 and 4. If we will prove that the weights $p=U_1^{-1}$ and $q=U_2$ satisfy conditions (3.1) and (3.5), we would be able to apply Theorems 3.1 and 4.2 to obtain the statement of Theorem 5.1. Condition (3.1) is satisfied by (5.7). The weight $p=U_1^{-1}$ solves the boundary value problem

$$Hp=-p(P_\xi f_1),\qquad p|_{\partial_-\Omega M}=E$$

which is very similar to (5.2). Theorem 5.1 is thus reduced to the following

Lemma 5.2 *Let (M,g) be a CNTM and a tensor field $f\in C^\infty(\tau_1^1M)$ satisfy (5.3)–(5.4). Then the solution U to the boundary value problem (5.2) satisfies the estimates*

$$|U-E|<C\varepsilon,\quad |\overset{v}{\nabla}U|<C\varepsilon\quad on\quad \Omega M \tag{5.12}$$

with some constant C depending on (M,g) but not on f.

To prove Lemma 5.2, we need the following estimate for solutions to linear ordinary differential equations (see Lemma 4.1 of Chapter IV of [1]):

Lemma 5.3 *Let $y=(y_1(t),\dots,y_N(t))$ be the solution to the initial value problem*

$$\frac{dy}{dt}=A(t)y+f(t),\quad y(0)=y_0,$$

where $f(t)$ is an N-dimensional vector and $A(t)$ is an $N\times N$-matrix. Then

$$|y(t)|\le\Big(|y_0|+\int\limits_0^t|f(\tau)|d\tau\Big)\exp\int\limits_0^t|A(\tau)|d\tau,$$

where $|A(\tau)|$ is the operator norm of the matrix $A(\tau)$ defined with the help of the standard norm $|\cdot|$ on $\mathbb{C}^N$.

Let us adjust this statement to our geometric setting.

Lemma 5.4 *Let (M,g) be a CNTM and $f\in C^\infty(\beta_m^0M;\Omega M)$, $A\in C^\infty(\beta_m^mM;\Omega M)$ be two semibasic tensor fields. Let $y\in C(\beta_m^0M;\Omega M)$ be the solution to the boundary value problem*

$$Hy=Ay+f,\qquad y|_{\partial_-\Omega M}=y_0$$

with some $y_0\in C(\beta_m^0M;\partial_-\Omega M)$. Then

$$|y(x,\xi)|\le C\Big(|y_0(\gamma(\tau_-(x,\xi)),\dot\gamma(\tau_-(x,\xi)))|+\int\limits_{\tau_-(x,\xi)}^0|f(\gamma(t),\dot\gamma(t))|dt\Big)\times$$

$$\times\exp\Big[C\int\limits_{\tau_-(x,\xi)}^0|A(\gamma(t),\dot\gamma(t))|dt\Big]$$

for $(x,\xi)\in\Omega M$, where $\gamma=\gamma_{x,\xi}$. Here the norm $|\cdot|$ is defined in Section 2, and the constant C depends on m and (M,g).

The constant C appears in Lemma 5.4 since different norms are used in this Lemma and Lemma 5.3. In what follows in this section, we denote different constants depending on (M, g) by the same letter C.

Proof of Lemma 5.2. Let U be the solution to (5.2). Then $U - E$ solves the boundary value problem

$$H(U-E) = (P_\xi f)(U-E) + P_\xi f, \quad (U-E)|_{\partial_-\Omega M} = 0.$$

Applying Lemma 5.4, we obtain the estimate

$$|(U-E)(x,\xi)| \le C \int\limits_{\tau_-(x,\xi)}^{0} |P_{\dot\gamma(t)} f(\gamma(t))| dt \ \exp\Big[C \int\limits_{\tau_-(x,\xi)}^{0} |P_{\dot\gamma(t)} f(\gamma(t))| dt\Big] \le$$

$$\le C \int\limits_{\tau_-(x,\xi)}^{0} |f(\gamma(t))| dt \exp\Big[C \int\limits_{\tau_-(x,\xi)}^{0} |f(\gamma(t))| dt\Big],$$

Together with (5.4), this implies the first of inequalities (5.12).

The proof of the second of estimates (5.12) is more troublesome because $\overset{v}{\nabla} U$ and $\overset{h}{\nabla} U$ must be estimated together but $\overset{h}{\nabla} U$ is unbounded near $\Omega(\partial M)$.

We start with estimating $\overset{h}{\nabla} U$ on $\partial_-\Omega M$. To this end we consider equation (5.2) at a boundary point $(x,\xi) \in \partial_-\Omega M$. Because of the boundary condition $U|_{\partial_-\Omega M} = E$, equation (5.2) gives

$$(HU)|_{\partial_-\Omega M} = P_\xi f. \tag{5.13}$$

Let us choose boundary normal coordinates $(x^1, \dots, x^n)$ in a neighborhood V of the boundary point such that the boundary is determined by the equation $x^n = 0$, $x^n \ge 0$ in V, and $g_{in} = \delta_{in}$. Because of the boundary condition $U^i_j|_{x^n=0} = \delta^i_j$, the equalities

$$\overset{h}{\nabla}_\alpha U^i_j|_{x^n=0} = 0 \quad (0 \le \alpha \le n-1)$$

hold, and equation (5.13) becomes

$$(\xi_n \overset{h}{\nabla}_n U^i_j)|_{x^n=0} = (P_\xi f)^i_j.$$

Since $\xi_n = -\langle \xi, \nu \rangle$, where $\nu = \nu(x)$ is the unit outward normal to the boundary, this gives with the help of (5.3)

$$|\overset{h}{\nabla} U(x,\xi)| \le \frac{C\varepsilon}{|\langle \xi, \nu \rangle|} \quad \text{for} \quad (x,\xi) \in \partial_-\Omega M, \ \langle \nu, \xi \rangle \ne 0. \tag{5.14}$$

$\overset{v}{\nabla} U$ vanishes on $\partial_-\Omega M$ as follows from the condition $U|_{\partial_-\Omega M} = E$,

$$\overset{v}{\nabla} U|_{\partial_-\Omega M} = 0. \tag{5.15}$$

Applying the operators $\overset{v}{\nabla}$ and $\overset{h}{\nabla}$ to equation (5.2), we obtain

$$\overset{v}{\nabla} HU = (P_\xi f)\overset{v}{\nabla} U + (\overset{v}{\nabla} P_\xi) f U, \quad \overset{h}{\nabla} HU = (P_\xi f)\overset{h}{\nabla} U + (P_\xi \nabla f) U. \tag{5.16}$$

We have used that $\overset{v}{\nabla} f = 0$ and $\overset{h}{\nabla} f = \nabla f$ since f is independent of ξ.

Operators $\overset{v}{\nabla}$ and H satisfy the commutator formula

$$\overset{v}{\nabla} H = H\overset{v}{\nabla} + \overset{h}{\nabla} \tag{5.17}$$

as follows immediately from the definition $H = \xi^i \overset{h}{\nabla}_i$ and from the fact that $\overset{v}{\nabla}$ and $\overset{h}{\nabla}$ commute. The commutator formula for $\overset{h}{\nabla}$ and H is a little bit more complicated. Indeed, using the commutator formula for horizontal derivatives (Theorem 3.5.2 of [5]), we see that

$$\overset{h}{\nabla}_i (HU)^j_k = \overset{h}{\nabla}_i (\xi^p \overset{h}{\nabla}_p U^j_k) = \xi^p \overset{h}{\nabla}_i \overset{h}{\nabla}_p U^j_k =$$
$$= \xi^p \overset{h}{\nabla}_p \overset{h}{\nabla}_i U^j_k + \xi^p \Big(- R^l_{qip} \xi^q \overset{v}{\nabla}_l U^j_k + R^j_{lip} U^l_k - R^l_{kip} U^j_l \Big).$$

This can be written as

$$\overset{h}{\nabla} HU = H\overset{h}{\nabla} U + \mathcal{R}_1[\overset{v}{\nabla} U] + \mathcal{R}_2[U] \tag{5.18}$$

with some algebraic operators $\mathcal{R}_1$ and $\mathcal{R}_2$ on semibasic tensors which are determined by the curvature tensor. The operator $\mathcal{R}_2$ satisfies $\mathcal{R}_2[E] = 0$. Therefore the first of estimates (5.12), which is already proved, implies the inequality

$$|\mathcal{R}_2[U]| = |\mathcal{R}_2[U - E]| \le C\varepsilon \tag{5.19}$$

with some constant C depending on the curvature bound.

Using commutator formulas (5.17) and (5.18), we write (5.16) as

$$H(\overset{v}{\nabla} U) = (P_\xi f)\overset{v}{\nabla} U - \overset{h}{\nabla} U + F, \tag{5.20}$$

$$H(\overset{h}{\nabla} U) = -\mathcal{R}_1[\overset{v}{\nabla} U] + (P_\xi f)\overset{h}{\nabla} U + G, \tag{5.21}$$

where

$$F = (\overset{v}{\nabla} P_\xi) f U, \quad G = (P_\xi \nabla f)U - \mathcal{R}_2[U]. \tag{5.22}$$

We first consider (5.20)–(5.21) as a linear system of ordinary differential equations in coordinates of $\overset{v}{\nabla} U$ and $\overset{h}{\nabla} U$ with free terms F and G. Applying Lemma 5.4 to the system, we obtain the estimate

$$|\overset{h}{\nabla} U(x,\xi)| \le C\Big\{ |\overset{v}{\nabla} U(\gamma(\tau_-(x,\xi)), \dot\gamma(\tau_-(x,\xi)))| + |\overset{h}{\nabla} U(\gamma(\tau_-(x,\xi)), \dot\gamma(\tau_-(x,\xi)))| +$$
$$+ \int\limits_{\tau_-(x,\xi)}^{0} (|F(\gamma(t),\dot\gamma(t))| + |G(\gamma(t),\dot\gamma(t))|)dt \Big\} \exp\Big[C \int\limits_{\tau_-(x,\xi)}^{0} (|\mathcal{R}_1((\gamma(t),\dot\gamma(t))| + |P_{\dot\gamma(t)} f(\gamma(t))| + |E|)dt \Big]. \tag{5.23}$$

The first summand of the expression in braces is equal to zero by (5.15). In virtue of (5.14), the second summand of this expression is estimated as

$$|\overset{h}{\nabla} U(\gamma(\tau_-(x,\xi)), \dot\gamma(\tau_-(x,\xi)))| \le \frac{C\varepsilon}{|\langle \dot\gamma(\tau_-(x,\xi)), \nu(\gamma(\tau_-(x,\xi)))\rangle|}.$$

By Lemma 4.1.2 of [5], the estimate

$$\frac{|\tau_-(x,\xi)|}{|\langle\dot\gamma(\tau_-(x,\xi)),\nu(\gamma(\tau_-(x,\xi)))\rangle|}\le C$$

holds. Combining two last estimates, we obtain

$$|\overset{h}{\nabla}U(\gamma(\tau_-(x,\xi)),\dot\gamma(\tau_-(x,\xi)))|\le\frac{C\varepsilon}{|\tau_-(x,\xi)|}.$$

As is seen from (5.22), (5.19), and (5.4), the integral inside the braces in (5.23) can be estimated by $C\varepsilon$. Finally, the integral under the exponent on (5.23) is estimated by some constant. Therefore (5.23) implies the estimate

$$|\overset{h}{\nabla}U(x,\xi)|\le\frac{C\varepsilon}{|\tau_-(x,\xi)|}\quad\text{for}\quad(x,\xi)\in\Omega M. \tag{5.24}$$

Now, we consider (5.20) as a linear system of ordinary differential equations in coordinates of $\overset{v}{\nabla}U$ with the free term $-\overset{h}{\nabla}U+F$. Applying Lemma 5.4 to the system and using the homogeneous initial condition (5.15), we obtain the estimate

$$|\overset{v}{\nabla}U(x,\xi)|\le C\int\limits_{\tau_-(x,\xi)}^{0}\Big(|\overset{h}{\nabla}U(\gamma(t),\dot\gamma(t))|+|F(\gamma(t),\dot\gamma(t))|\Big)dt\exp\Big[C\int\limits_{\tau_-(x,\xi)}^{0}|P_{\dot\gamma(t)}f(\gamma(t))|dt\Big]. \tag{5.25}$$

In virtue of (5.24), the first integral on the right-hand side of (5.25) can be estimated by $C\varepsilon$. Estimating then the second integral by (5.4), we obtain the second of inequalities (5.12). The lemma is proved.

6 Kernel of the operator S

In this section, we restrict ourselves to considering symmetric matrix functions on the whole of $\mathbb{R}^3$ endowed with the standard scalar product $\langle\cdot,\cdot\rangle$.

Let $M(3)$ be the space of complex-valued symmetric 3×3-matrices. Such a matrix $f\in M(3)$ is considered as the linear operator $f:\mathbb{C}^3\to\mathbb{C}^3$. By $\mathbb{S}^2$ we denote the unit sphere in $\mathbb{R}^3$ and by $T\mathbb{S}^2=\{(x,\xi)\mid\xi\in\mathbb{S}^2,x\in\mathbb{R}^3,\langle x,\xi\rangle=0\}$, the tangent bundle of the sphere. Given $\xi\in\mathbb{S}^2$, let $\xi^\perp=\{\eta\in\mathbb{C}^3\mid\langle\eta,\xi\rangle=0\}$ be the complex two-dimensional space of vectors orthogonal to ξ and $\pi_\xi:\mathbb{C}^3\to\mathbb{C}^3$, the orthogonal projection onto $\xi^\perp$.

Let $\mathcal{S}(\mathbb{R}^3;M(3))$ be the Schwartz space of $M(3)$-valued functions on $\mathbb{R}^3$. The linear operator

$$S:\mathcal{S}(\mathbb{R}^3;M(3))\to C^\infty(T\mathbb{S}^2)$$

is defined by

$$S[f](x,\xi)=\int\limits_{-\infty}^{\infty}\mathrm{tr}\,(\pi_\xi f(x+t\xi)\pi_\xi)dt\quad\text{for}\quad(x,\xi)\in T\mathbb{S}^2, \tag{6.1}$$

compare with (1.10). We are going to answer the question: to which extent is a symmetric matrix function $f\in\mathcal{S}(\mathbb{R}^3;M(3))$ determined by the data $S[f]$? Since $S[f]$ depends linearly on f, the question is equivalent to studying the kernel of the operator S.

Let us recall the definition of the ray transform

$$I : \mathcal{S}(\mathbb{R}^3; M(3)) \to C^\infty(T\mathbb{S}^2),$$

$$I[f](x,\xi) = \int\limits_{-\infty}^{\infty} \langle f(x+t\xi)\xi, \xi\rangle dt = \int\limits_{-\infty}^{\infty} f_{ij}(x+t\xi)\xi^i\xi^j\, dt \quad \text{for} \quad (x,\xi) \in T\mathbb{S}^2. \tag{6.2}$$

See Chapter 2 of [5] for the theory of the ray transform on the Euclidean space. Our approach to studying the kernel of S is based on the following observation: $S[f] = -I[f]$ for a symmetric matrix function f with zero trace, see equation (6.7) below.

Theorem 6.1 *A symmetric matrix function $f = (f_{ij}(x)) \in \mathcal{S}(\mathbb{R}^3; M(3))$ belongs to the kernel of the operator S if and only if it satisfies the system of partial differential equations*

$$\left.\begin{aligned} R_1[f] &:= f_{11;11} + 2f_{12;12} + f_{22;22} + f_{33;11} + f_{33;22} = 0 \\ R_2[f] &:= f_{11;11} + 2f_{13;13} + f_{22;11} + f_{22;33} + f_{33;33} = 0 \\ R_3[f] &:= f_{11;22} + f_{11;33} + 2f_{23;23} + f_{22;22} + f_{33;33} = 0 \end{aligned}\right\} \tag{6.3}$$

where the tensor notations for partial derivatives $f_{ij;kl} = \partial^2 f_{ij}/\partial x_k \partial x_l$ are used for brevity.

The theorem gives the full system $\{R_i[f] \mid i = 1, 2, 3\}$ of local linear functionals that can be recovered from the data $S[f]$.

Proof. Represent the matrix f as

$$f = \tilde{f} + \lambda E, \qquad \operatorname{tr} \tilde{f} = 0, \tag{6.4}$$

where E is the unit matrix and $\lambda \in \mathcal{S}(\mathbb{R}^3)$ is a scalar function. Then

$$S[f] = S[\tilde{f}] + S[\lambda E]. \tag{6.5}$$

As follows from definition (6.1) of the operator S,

$$S[\lambda E] = 2I[\lambda E], \tag{6.6}$$

where

$$I[\lambda E](x,\xi) = \int\limits_{-\infty}^{\infty} \lambda(x+t\xi)dt \quad \text{for} \quad (x,\xi) \in T\mathbb{S}^2$$

is the ray transform of the matrix function λE.

If (ξ, η, ζ) is an orthonormal basis of $\mathbb{R}^3$, then

$$0 = \operatorname{tr} \tilde{f} = \langle \tilde{f}(x)\xi, \xi\rangle + \langle \tilde{f}(x)\eta, \eta\rangle + \langle \tilde{f}(x)\zeta, \zeta\rangle$$

and

$$\operatorname{tr}(\pi_\xi \tilde{f}(x)\pi_\xi) = \langle \tilde{f}(x)\eta, \eta\rangle + \langle \tilde{f}(x)\zeta, \zeta\rangle.$$

This implies

$$\operatorname{tr}(\pi_\xi \tilde{f}(x+t\xi)\pi_\xi) = -\langle \tilde{f}(x+t\xi)\xi, \xi\rangle.$$

Integrating the last equality with respect to t and recalling definitions (6.1) and (6.2) of S and I, we see that

$$S[\tilde f] = -I[\tilde f]. \tag{6.7}$$

Substituting (6.6)–(6.7) into (6.5), we obtain

$$S[f] = -I[\tilde f - 2\lambda E].$$

Thus, f is in the kernel of S if and only if

$$I[\tilde f - 2\lambda E] = 0. \tag{6.8}$$

Now, we apply Theorem 2.2.1 of [5] which states that (6.8) is equivalent to the existence of a vector field v such that

$$\tilde f - 2\lambda E = dv, \tag{6.9}$$

where

$$d : C^\infty(\mathbb{R}^3;\mathbb{R}^3) \to C^\infty(\mathbb{R}^3; M(3)), \quad (dv)_{ij} = \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big)$$

is the inner derivative. Theorem 2.2.1 of [5] is formulated and proved for compactly supported tensor fields. Nevertheless, the same proof works for $\tilde f - 2\lambda E \in \mathcal{S}(\mathbb{R}^3; M(3))$ and gives the vector field v belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^3;\mathbb{R}^3)$.

Let us express λ and $\tilde f$ through f. Applying the trace operator to the first of equations (6.4) and taking the second one into account, we see that

$$\lambda = \frac{1}{3}\mathrm{tr}\, f. \tag{6.10}$$

From (6.4) and (6.10),

$$\tilde f_{ij} = f_{ij} - \frac{1}{3}\mathrm{tr}\, f \cdot \delta_{ij}. \tag{6.11}$$

Substitute (6.10)–(6.11) into (6.9) to obtain

$$dv = f - \mathrm{tr}\, f \cdot E. \tag{6.12}$$

Equation (6.12) represents the overdetermined system of six first order partial differential equations in three unknowns (v_1, v_2, v_3). The solvability condition for the system is presented by Theorem 2.2.2 of [5]: equation (6.12) is solvable if and only if the right-hand side of (6.12) belongs to the kernel of the Saint-Venant operator. Here, we prefer to use the version R of the Saint-Venant operator which is defined by the equation before formula (2.4.6) of [5]. So, the solvability condition for (6.12) is

$$Rh = 0, \quad \text{where} \quad h = f - \mathrm{tr}\, f \cdot E. \tag{6.13}$$

The operator R is defined by the formula

$$4(Rh)_{ijkl} = h_{ik;jl} - h_{jk;il} - h_{il;jk} + h_{jl;ik}.$$

It possesses the following symmetries:

$$(Rh)_{ijkl} = -(Rh)_{jikl} = -(Rh)_{ijlk} = (Rh)_{klij}.$$

Because of the symmetries, the tensor Rh has six linearly independent components, and equation (6.13) is equivalent to the system

$$\begin{aligned}
-R_1[f] &:= (Rh)_{1212} = h_{11;22} - 2h_{12;12} + h_{22;11} = 0,\\
-R_2[f] &:= (Rh)_{1313} = h_{11;33} - 2h_{13;13} + h_{33;11} = 0,\\
-R_3[f] &:= (Rh)_{2323} = h_{22;33} - 2h_{23;23} + h_{33;22} = 0,\\
-R_4[f] &:= (Rh)_{1213} = h_{11;23} - h_{12;13} - h_{13;12} + h_{23;11} = 0,\\
-R_5[f] &:= (Rh)_{2123} = h_{22;13} - h_{12;23} - h_{23;12} + h_{13;22} = 0,\\
-R_6[f] &:= (Rh)_{1323} = h_{33;12} - h_{13;23} - h_{23;13} + h_{12;33} = 0.
\end{aligned}$$

Substitute the value $h_{ij} = f_{ij} - (f_{11} + f_{22} + f_{33})\delta_{ij}$ into the last system

$$\left.\begin{aligned}
R_1[f] &:= f_{11;11} + 2f_{12;12} + f_{22;22} + f_{33;11} + f_{33;22} = 0,\\
R_2[f] &:= f_{11;11} + 2f_{13;13} + f_{22;11} + f_{22;33} + f_{33;33} = 0,\\
R_3[f] &:= f_{11;22} + f_{11;33} + 2f_{23;23} + f_{22;22} + f_{33;33} = 0,\\
R_4[f] &:= f_{12;13} + f_{13;12} + f_{22;23} - f_{23;11} + f_{33;23} = 0,\\
R_5[f] &:= f_{11;13} + f_{12;23} - f_{13;22} + f_{23;12} + f_{33;13} = 0,\\
R_6[f] &:= f_{11;12} - f_{12;33} + f_{13;23} + f_{22;12} + f_{23;13} = 0.
\end{aligned}\right\} \tag{6.14}$$

For a symmetric matrix function $f = (f_{ij}(x)) \in \mathcal{S}(\mathbb{R}^3; M(3))$, system (6.14) contains only three independent equations. More precisely: each of the last three equations of (6.14) can be obtained from the first three equations by taking linear combinations, differentiation, and integration. To prove this, let us rewrite system (6.14) in terms of the Fourier transform $g(\xi) = \hat{f}$. Applying the Fourier transform to each equation of (6.14), we arrive to the system

$$\left.\begin{aligned}
\hat{R}_1[g] &:= \xi_1^2 g_{11} + 2\xi_1\xi_2 g_{12} + \xi_2^2 g_{22} + (\xi_1^2 + \xi_2^2) g_{33} = 0,\\
\hat{R}_2[g] &:= \xi_1^2 g_{11} + 2\xi_1\xi_3 g_{13} + (\xi_1^2 + \xi_3^2) g_{22} + \xi_3^2 g_{33} = 0,\\
\hat{R}_3[g] &:= (\xi_2^2 + \xi_3^2) g_{11} + \xi_2^2 g_{22} + 2\xi_2\xi_3 g_{23} + \xi_3^2 g_{33} = 0,\\
\hat{R}_4[g] &:= \xi_1\xi_3 g_{12} + \xi_1\xi_2 g_{13} + \xi_2\xi_3 g_{22} - \xi_1^2 g_{23} + \xi_2\xi_3 g_{33} = 0,\\
\hat{R}_5[g] &:= \xi_1\xi_3 g_{11} + \xi_2\xi_3 g_{12} - \xi_2^2 g_{13} + \xi_1\xi_2 g_{23} + \xi_1\xi_3 g_{33} = 0,\\
\hat{R}_6[g] &:= \xi_1\xi_2 g_{11} - \xi_3^2 g_{12} + \xi_2\xi_3 g_{13} + \xi_1\xi_2 g_{22} + \xi_1\xi_3 g_{23} = 0.
\end{aligned}\right\} \tag{6.15}$$

One can easily see the following three relations between equations of system (6.15):

$$2\xi_2\xi_3 \hat{R}_4[g] = \xi_3^2 \hat{R}_1[g] + \xi_2^2 \hat{R}_2[g] - \xi_1^2 \hat{R}_3[g],$$

$$2\xi_1\xi_3 \hat{R}_5[g] = \xi_3^2 \hat{R}_1[g] - \xi_2^2 \hat{R}_2[g] + \xi_1^2 \hat{R}_3[g],$$

$$2\xi_1\xi_2 \hat{R}_5[g] = -\xi_3^2 \hat{R}_1[g] + \xi_2^2 \hat{R}_2[g] + \xi_1^2 \hat{R}_3[g].$$

Therefore three last equations of (6.15) follow from three first equations at least if $g(\xi)$ depends continuously on ξ.

Deleting three last equations from system (6.15), we obtain the equivalent system

$$\left.\begin{aligned}\hat{R}_1[g] &:= \xi_1^2 g_{11} + 2\xi_1\xi_2 g_{12} + \xi_2^2 g_{22} + (\xi_1^2+\xi_2^2) g_{33} = 0,\\ \hat{R}_2[g] &:= \xi_1^2 g_{11} + 2\xi_1\xi_3 g_{13} + (\xi_1^2+\xi_3^2) g_{22} + \xi_3^2 g_{33} = 0,\\ \hat{R}_3[g] &:= (\xi_2^2+\xi_3^2) g_{11} + \xi_2^2 g_{22} + 2\xi_2\xi_3 g_{23} + \xi_3^2 g_{33} = 0.\end{aligned}\right\} \tag{6.16}$$

The same is true for system (6.14): deleting three last equations from (6.14), we will obtain the equivalent system (6.3). The theorem is proved.

System (6.16) enables us to answer the question: which integral moments of f can be determined from $S[f]$? Indeed, the Tailor series of the function $g(\xi) = \hat{f}$ is

$$g_{jk}(\xi) \sim \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \frac{i^m}{\alpha!} \mu_{jk,\alpha}^{(m)}[f] \xi^\alpha, \tag{6.17}$$

where

$$\mu_{jk,\alpha}^{(m)}[f] = \int\limits_{\mathbb{R}^3} x^\alpha f_{jk}(x) dx \qquad |\alpha| = m$$

are the integral moments of order m. Assuming f to be in the kernel of S, let us insert series (6.17) into system (6.16). Since the coefficients of (6.16) are homogeneous functions of ξ, the system does not mix moments of different orders. This means that we can take $g(\xi)$ in the form

$$g_{jk}(\xi) = \sum_{|\alpha|=m} \frac{i^m}{\alpha!} \mu_{jk,\alpha}^{(m)}[f] \xi^\alpha$$

if we are looking for moments of order m.

Let us start with considering zero order moments. We substitute the expressions $g_{ij} = \mu_{ij}^{(0)}$ into (6.16). Equating coefficients at the same degrees of ξ at the resulting equations, we easily find that $\mu_{ij}^{(0)} = 0$ for every (i,j). This means that the integral $\int_{\mathbb{R}^3} f(x)dx$ can be determined from the data $S[f]$.

Next, we consider first order moments. We substitute the expressions

$$g_{ij}(\xi) = \mu_{ij,1}^{(1)} \xi_1 + \mu_{ij,2}^{(1)} \xi_2 + \mu_{ij,3}^{(1)} \xi_3$$

into system (6.16). Equating coefficients at the same degrees of ξ at the resulting equations, we arrive to the system

$$\mu_{11,1}^{(1)} + \mu_{22,1}^{(1)} = 0, \quad \mu_{11,1}^{(1)} + \mu_{33,1}^{(1)} = 0, \quad \mu_{11,2}^{(1)} + \mu_{22,2}^{(1)} = 0,$$

$$\mu_{22,2}^{(1)} + \mu_{33,2}^{(1)} = 0, \quad \mu_{11,3}^{(1)} + \mu_{33,3}^{(1)} = 0, \quad \mu_{22,3}^{(1)} + \mu_{33,3}^{(1)} = 0,$$

$$\mu_{11,2}^{(1)} + 2\mu_{12,1}^{(1)} + \mu_{33,2}^{(1)} = 0, \quad \mu_{11,3}^{(1)} + 2\mu_{13,1}^{(1)} + \mu_{22,3}^{(1)} = 0, \quad \mu_{11,3}^{(1)} + \mu_{22,3}^{(1)} + 2\mu_{23,2}^{(1)} = 0,$$

$$2\mu_{12,2}^{(1)} + \mu_{22,1}^{(1)} + \mu_{33,1}^{(1)} = 0, \quad 2\mu_{13,3}^{(1)} + \mu_{22,1}^{(1)} + \mu_{33,1}^{(1)} = 0, \quad \mu_{11,2}^{(1)} + 2\mu_{23,3}^{(1)} + \mu_{33,2}^{(1)} = 0.$$

The general solution to the system looks as follows:

$$\mu_{11,1}^{(1)} = a_1, \quad \mu_{12,1}^{(1)} = a_2, \quad \mu_{13,1}^{(1)} = a_3, \quad \mu_{22,1}^{(1)} = -a_1, \quad \mu_{23,1}^{(1)} = 0, \quad \mu_{33,1}^{(1)} = -a_1,$$

$$\mu^{(1)}_{11,2} = -a_2, \quad \mu^{(1)}_{12,2} = a_1, \quad \mu^{(1)}_{13,2} = 0, \quad \mu^{(1)}_{22,2} = a_2, \quad \mu^{(1)}_{23,2} = a_3, \quad \mu^{(1)}_{33,2} = -a_2,$$

$$\mu^{(1)}_{11,3} = -a_3, \quad \mu^{(1)}_{12,3} = 0, \quad \mu^{(1)}_{13,3} = a_1, \quad \mu^{(1)}_{22,3} = -a_3, \quad \mu^{(1)}_{23,3} = a_2, \quad \mu^{(1)}_{33,3} = a_3,$$

where (a_1, a_2, a_3) are arbitrary constants. Eliminating the constants, we obtain the following independent system of 15 linear combinations of first order moments of f which can be recovered from the data $S[f]$:

$$(\mu^{(1)}_{ij,k} + \delta_{ij}\mu^{(1)}_{kk,k} - \delta_{ik}\mu^{(1)}_{jj,j} - \delta_{jk}\mu^{(1)}_{ii,i})[f], \tag{6.18}$$

where δ_{ij} is the Kronecker tensor. System (6.18) is considered for such (i, j, k) that at least two of these indices are different. A similar consideration is possible for integral moments $\mu^{(m)}_{jk,\alpha}[f]$ of an arbitrary order m.

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