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# Time series aggregation, disaggregation and long memory \*

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**Abstract** The paper studies the aggregation/disaggregation problem of random parameter AR(1) processes and its relation to the long memory phenomenon. We give a characterization of a subclass of aggregated processes which can be obtained from simpler, "elementary", cases. In particular cases of the mixture densities, the structure (moving average representation) of the aggregated process is investigated.

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*Keywords:* random coefficient AR(1), long memory, aggregation, disaggregation

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# 1 Introduction

In this paper we consider the aggregation scheme introduced in the paper of Granger (1980), where it was shown that aggregation of random parameter AR(1) processes can produce long memory. Since this work, a large number of papers were devoted to the question how do micro level processes imply the long memory at macro level and applications (mainly in economics) (see Haubrich and Lo (2001), Oppenheim and Viano (2004), Zaffaroni (2004), Chong (2006), etc.).

Recently, Dacunha-Castelle and Oppenheim (2001) (see also Dacunha-Castelle and Fermin (2006)) stated the following problem: which class of long memory processes can be obtained by the aggregation of short memory models with random coefficients?

Let us give a precise formulation of the problem when the underlying short memory models are described by AR(1) dynamics. Let  $\varepsilon = \{\varepsilon_t, t \in \mathbf{Z}\}$  be a sequence of independent identically distributed (i.i.d.) random variables (r.v.) with  $E\varepsilon_t = 0$ ,  $E\varepsilon_t^2 = \sigma_\varepsilon^2$ , and let  $a$  be a random variable supported by  $(-1, 1)$  and satisfying

$$E\left[\frac{1}{1-a^2}\right] < \infty. \quad (1.1)$$

Consider a sequence of i.i.d. processes  $Y^{(j)} = \{Y_t^{(j)}, t \in \mathbf{Z}\}$ ,  $j \geq 1$  defined by the random AR(1) dynamics

$$Y_t^{(j)} = a^{(j)}Y_{t-1}^{(j)} + \varepsilon_t^{(j)}, \quad (1.2)$$

where  $\varepsilon^{(j)} = \{\varepsilon_t^{(j)}, t \in \mathbf{Z}\}$ ,  $j \geq 1$ , are independent copies of  $\varepsilon = \{\varepsilon_t, t \in \mathbf{Z}\}$  and  $a^{(j)}$ 's are independent copies of  $a$ . Here, the sequences  $a, a^{(j)}$ ,  $j \geq 1$  and  $\varepsilon, \varepsilon^{(j)}$ ,  $j \geq 1$ , are independent. Under these conditions, (1.2) admits a covariance-stationary solution  $Y_t^{(j)}$  and the finite dimensional distributions of the process  $X_t^{(N)} = N^{-1/2} \sum_{j=1}^N Y_t^{(j)}$ ,  $t \in \mathbf{Z}$ , weakly converge, as  $N \rightarrow \infty$ , to those of a zero mean Gaussian process  $X_t$ , called the *aggregated process* (see Oppenheim and Viano (2004)).

Assume that the distribution of r.v.  $a$  admits a *mixture density*  $\varphi$ , which by (1.1) satisfies

$$\int_{-1}^1 \frac{\varphi(x)}{1-x^2} dx < \infty. \quad (1.3)$$

The covariance function and spectral density of aggregated process  $X_t$  are given, respectively, by

$$\gamma(h) := \text{Cov}(X_h, X_0) = \sigma_\varepsilon^2 \int_{-1}^1 \frac{x^{|h|}}{1-x^2} \varphi(x) dx \quad (1.4)$$

and

$$f(\lambda) = \frac{\sigma_\varepsilon^2}{2\pi} \int_{-1}^1 \frac{\varphi(x)}{|1 - xe^{i\lambda}|^2} dx. \quad (1.5)$$

Note that an aggregated process  $X_t$  possess the long memory property (i.e.  $\sum_{h=-\infty}^{\infty} |\gamma(h)| = \infty$ ) if and only if

$$\int_{-1}^1 \frac{\varphi(x)}{(1-x^2)^2} dx = \infty. \quad (1.6)$$

If the mixture density  $\varphi$  is *a priori given* and our aim is to characterize the properties of the induced aggregated process (moving average representation, spectral density, covariance function, etc.), we call this problem an *aggregation problem*. And vice versa, if we observe the aggregated process  $X_t$  with spectral density  $f$  and we need to find the individual processes (if they exist) of form (1.2) with some mixture density  $\varphi$ , which produce the aggregated process, then we call this problem a *disaggregation problem*. The second problem, which is much harder than the first one, is equivalent to the finding of  $\varphi$  such that (1.5) (or (1.4)) and (1.3) hold. In the latter case we say that *the mixture density  $\varphi$  is associated with the spectral density  $f$* .

Sections 2 and 3 are devoted to the disaggregation problem. Equality (1.4) shows that the covariance function  $\gamma(h)$  can be interpreted as a  $h$ -moment of the density function  $\varphi(x)(1-x^2)^{-1}$  supported by  $(-1, 1)$ , and thus finding of the mixture density is related to the moments' problem (see Feller (1976)). In Section 2, we prove that, under some mild conditions, the mixture density associated with the product spectral density can be obtained from the "elementary" mixture densities associated with the multipliers. In Section 3, we apply the obtained result to the spectral density which is a product of to FARIMA-type spectral densities. Using the form of mixture density for the classical FARIMA model we show that one can solve the disaggregation problem for more complex long memory stationary sequences.

For the aggregation problem, it is important to characterize, for a given class of mixture densities, the behavior of the coefficients in its linear representation and the behavior of the spectral density. We address this problem to Section 4. In the Appendix, we provide a proof of the form of the mixture density in FARIMA case.

## 2 Mixture density for the product of aggregated spectral densities

Let  $X_{1,t}$  and  $X_{2,t}$  are two aggregated processes obtained from the independent copies of AR(1) sequences  $Y_{1,t} = a_1 Y_{1,t-1} + \varepsilon_{1,t}$  and  $Y_{2,t} = a_2 Y_{2,t-1} + \varepsilon_{2,t}$ , respectively, where  $a_1, a_2$  satisfy (1.1),  $\varepsilon_i = \{\varepsilon_{i,t}, t \in \mathbf{Z}\}$  and  $a_i$  are independent,  $i = 1, 2$ . Denote  $\sigma_{i,\varepsilon}^2 := E\varepsilon_{i,t}^2$ ,  $i = 1, 2$ .

Assume that  $\varphi_1$  and  $\varphi_2$  are the mixture densities associated with spectral densities  $f_1$  and  $f_2$ , respectively, i.e.

$$f_i(\lambda) = \frac{\sigma_{i,\varepsilon}^2}{2\pi} \int_{-1}^1 \frac{\varphi_i(x)}{|1 - xe^{i\lambda}|^2} dx, \quad i = 1, 2. \quad (2.1)$$

The following proposition shows that, if  $\varphi_1$  and  $\varphi_2$  in (2.1) are supported by  $[0, 1]$  and  $[-1, 0]$  respectively, then the stationary Gaussian process with spectral density  $f(\lambda) = f_1(\lambda)f_2(\lambda)$  can also be obtained by aggregation of the i.i.d. AR(1) processes with some mixture density  $\varphi$  and noise sequence  $\varepsilon$ .

**Proposition 2.1** *Let  $\varphi_1$  and  $\varphi_2$  be the mixture densities associated with spectral densities  $f_1$  and  $f_2$ , respectively. Assume that  $\text{supp}(\varphi_1) \subset [0, 1]$ ,  $\text{supp}(\varphi_2) \subset [-1, 0]$  and*

$$f(\lambda) = f_1(\lambda)f_2(\lambda). \quad (2.2)$$

*Then the mixture density  $\varphi(x)$ ,  $x \in [-1, 1]$  associated with  $f$  is given by equality*

$$\varphi(x) = \frac{1}{C_*} \left( \varphi_1(x) \int_{-1}^0 \frac{\varphi_2(y)}{(1-xy)(1-y/x)} dy + \varphi_2(x) \int_0^1 \frac{\varphi_1(y)}{(1-xy)(1-y/x)} dy \right), \quad (2.3)$$

*where  $C_* := \int_0^1 \left( \int_{-1}^0 \varphi_1(x)\varphi_2(y)(1-xy)^{-1} dy \right) dx$ . The variance of the noise is*

$$\sigma_\varepsilon^2 = \frac{\sigma_{1,\varepsilon}^2 \sigma_{2,\varepsilon}^2 C_*}{2\pi}. \quad (2.4)$$

PROOF. Obviously, the covariance function  $\gamma(h) = \text{Cov}(X_h, X_0)$  has a form

$$\gamma(h) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma_1(j+|h|)\gamma_2(j), \quad (2.5)$$

where  $\gamma_1$  and  $\gamma_2$  are the covariance functions of the aggregated processes obtained from the mixture densities  $\varphi_1$  and  $\varphi_2$ , respectively:

$$\gamma_1(j) = \sigma_{1,\varepsilon}^2 \int_0^1 \varphi_1(x) \frac{x^{|j|}}{1-x^2} dx, \quad \gamma_2(j) = \sigma_{2,\varepsilon}^2 \int_{-1}^0 \varphi_2(x) \frac{x^{|j|}}{1-x^2} dx.$$

Clearly,  $\gamma_1(j) > 0$  and  $\gamma_2(j) = (-1)^j |\gamma_2(j)|$ . Let  $h \geq 0$ . Then

$$\begin{aligned}
2\pi\sigma_{1,\varepsilon}^{-2}\sigma_{2,\varepsilon}^{-2}\gamma(h) &= \sigma_{1,\varepsilon}^{-2}\sigma_{2,\varepsilon}^{-2} \sum_{j=-\infty}^{\infty} \gamma_1(j+h)\gamma_2(j) \\
&= \sigma_{1,\varepsilon}^{-2}\sigma_{2,\varepsilon}^{-2} \left( \sum_{j=0}^{\infty} \gamma_1(j+h)\gamma_2(j) + \sum_{j=0}^{\infty} \gamma_1(j)\gamma_2(j+h) + \sum_{j=1}^{h-1} \gamma_1(-j+h)\gamma_2(j) \right) \\
&=: s_1 + s_2 + s_3.
\end{aligned} \tag{2.6}$$

We have  $s_1 = \lim_{N \rightarrow \infty} s_1^{(N)}$ , where

$$\begin{aligned}
s_1^{(N)} &= \sum_{j=0}^N \int_0^1 \varphi_1(x) \frac{x^{j+h}}{1-x^2} dx \int_{-1}^0 \varphi_2(y) \frac{y^j}{1-y^2} dy \\
&= \int_0^1 \int_0^1 \varphi_1(x)\varphi_2(-y) \frac{1}{1-x^2} \frac{1}{1-y^2} \sum_{j=0}^N (-1)^j x^{j+h} y^j dx dy \\
&= \int_0^1 \int_0^1 \varphi_1(x)\varphi_2(-y) \frac{x^h}{1-x^2} \frac{1}{1-y^2} \frac{1-(-xy)^{N+1}}{1+xy} dx dy.
\end{aligned}$$

Note that

$$\left| \varphi_1(x)\varphi_2(-y) \frac{x^h}{1-x^2} \frac{1}{1-y^2} \frac{1-(-xy)^{N+1}}{1+xy} \right| \leq 2\varphi_1(x)\varphi_2(-y) \frac{1}{1-x^2} \frac{1}{1-y^2}$$

and

$$\int_0^1 \int_0^1 \varphi_1(x)\varphi_2(-y) \frac{1}{1-x^2} \frac{1}{1-y^2} dx dy = \gamma_1(0)\gamma_2(0) < \infty.$$

Therefore, by the dominated convergence theorem, we obtain

$$\begin{aligned}
s_1 &= \int_0^1 \int_0^1 \varphi_1(x)\varphi_2(-y) \frac{x^h}{1-x^2} \frac{1}{1-y^2} \lim_{N \rightarrow \infty} \frac{1-(-xy)^{N+1}}{1+xy} dx dy \\
&= \int_0^1 \int_0^1 \varphi_1(x)\varphi_2(-y) \frac{x^h}{1-x^2} \frac{1}{1-y^2} \frac{1}{1+xy} dx dy \\
&= \int_0^1 \frac{x^h}{1-x^2} \left\{ \varphi_1(x) \int_{-1}^0 \varphi_2(y) \frac{1}{1-y^2} \frac{1}{1-xy} dy \right\} dx.
\end{aligned} \tag{2.7}$$

Analogously, we have

$$s_2 = \int_{-1}^0 \frac{y^h}{1-y^2} \left\{ \varphi_2(y) \int_0^1 \varphi_1(x) \frac{1}{1-x^2} \frac{1}{1-xy} dx \right\} dy. \tag{2.8}$$

For the last term in the decomposition (2.6) we have

$$\begin{aligned}
s_3 &= \sum_{j=1}^{h-1} \int_0^1 \varphi_1(x) \frac{x^{-j+h}}{1-x^2} dx \int_{-1}^0 \varphi_2(x) \frac{x^j}{1-x^2} dx \\
&= \int_0^1 \int_0^1 \varphi_1(x) \frac{x^h}{1-x^2} \varphi_2(-y) \frac{1}{1-y^2} (-y/x) \frac{1-(-y/x)^{h-1}}{1+y/x} dx dy \\
&= \int_0^1 \frac{x^h}{1-x^2} \left\{ \varphi_1(x) \int_{-1}^0 \varphi_2(y) \frac{1}{1-y^2} \frac{y/x}{1-y/x} dy \right\} dx \\
&\quad - \int_{-1}^0 \frac{y^h}{1-y^2} \left\{ \varphi_2(y) \int_0^1 \varphi_1(x) \frac{1}{1-x^2} \frac{1}{1-y/x} dx \right\} dy. \tag{2.9}
\end{aligned}$$

Equalities (2.7)–(2.9), together with (2.6), imply

$$\begin{aligned}
2\pi\sigma_{1,\varepsilon}^{-2}\sigma_{2,\varepsilon}^{-2}\gamma(h) &= \int_0^1 \frac{x^h}{1-x^2} \varphi_1(x) \left\{ \int_{-1}^0 \varphi_2(y) \frac{1}{1-y^2} \frac{y/x}{1-y/x} dy \right. \\
&\quad \left. + \int_{-1}^0 \varphi_2(y) \frac{1}{1-y^2} \frac{1}{1-xy} dy \right\} dx \\
&\quad + \int_{-1}^0 \frac{y^h}{1-y^2} \varphi_2(y) \left\{ \int_0^1 \varphi_1(x) \frac{1}{1-x^2} \frac{1}{1-xy} dx \right. \\
&\quad \left. - \int_0^1 \varphi_1(x) \frac{1}{1-x^2} \frac{1}{1-y/x} dx \right\} dy \\
&= \int_0^1 \frac{x^h}{1-x^2} \varphi_1(x) \left\{ \int_{-1}^0 \varphi_2(y) \frac{1}{(1-y/x)(1-xy)} dy \right\} dx \\
&\quad + \int_{-1}^0 \frac{y^h}{1-y^2} \varphi_2(y) \left\{ \int_0^1 \varphi_1(x) \frac{1}{(1-x/y)(1-xy)} dx \right\} dy.
\end{aligned}$$

This and (1.4) imply (2.3), taking into account that  $\int_{-1}^1 \varphi(x) dx = 1$ .  $\square$

### 3 Seasonal long memory case

In this section, we apply the obtained result to the spectral densities  $f_1$  and  $f_2$  having the forms:

$$f_1(\lambda; d_1) = \frac{1}{2\pi} |1 - e^{i\lambda}|^{-2d_1} \tag{3.1}$$

$$f_2(\lambda; d_2) = \frac{1}{2\pi} |1 + e^{i\lambda}|^{-2d_2}, \quad 0 < d_1, d_2 < 1/2. \tag{3.2}$$

We call these spectral densities (and corresponding processes) *fractionally integrated*, FI( $d_1$ ), and *seasonal fractionally integrated*, SFI( $d_2$ ), spectral densities. The

mixture density associated with the  $\text{FI}(d_1)$  spectral density (3.1) is given by the following expression (for the sketch of proof see Dacunha-Castelle and Oppenheim (2001)):

$$\varphi_1(x; d_1) = C(d_1)x^{d_1-1}(1-x)^{1-2d_1}(1+x)\mathbf{1}_{[0,1]}(x) \quad (3.3)$$

with

$$C(d) = \frac{\Gamma(3-d)}{2\Gamma(d)\Gamma(2-2d)} = 2^{2d-2} \frac{\sin(\pi d)}{\sqrt{\pi}} \frac{\Gamma(3-d)}{\Gamma((3/2)-d)} \quad (3.4)$$

and the variance of the noise

$$\sigma_{1,\varepsilon}^2 = \frac{\sin(\pi d_1)}{C(d_1)\pi}. \quad (3.5)$$

For convenience, we provide the rigorous proof of this result in Proposition 5.1 of Appendix.

Similarly, the mixture density associated with the spectral density (3.2) is given by

$$\varphi_2(x; d_2) = \varphi_1(-x; d_2) = C(d_2)|x|^{d_2-1}(1+x)^{1-2d_2}(1-x)\mathbf{1}_{[-1,0]}(x) \quad (3.6)$$

and

$$\sigma_{2,\varepsilon}^2 = \frac{\sin(\pi d_2)}{C(d_2)\pi}, \quad (3.7)$$

since

$$f_2(\lambda; d_2) = f_1(\pi - \lambda; d_2) = \frac{\sigma_{2,\varepsilon}^2}{2\pi} \int_0^1 \frac{\varphi_1(x; d_2)}{|1 + xe^{i\lambda}|^2} dx = \frac{\sigma_{2,\varepsilon}^2}{2\pi} \int_{-1}^0 \frac{\varphi_1(-x; d_2)}{|1 - xe^{i\lambda}|^2} dx.$$

Clearly, mixture densities  $\varphi_1(x; d_1)$  and  $\varphi_2(x; d_2)$ , given in (3.3), (3.6), satisfy (1.3) and, hence, assumptions of Proposition 2.1, whenever  $0 < d_1 < 1/2$  and  $0 < d_2 < 1/2$ . Moreover, since  $d_1 > 0$  and  $d_2 > 0$ , both  $\varphi_1$  and  $\varphi_2$  satisfy (1.6). The mixture density associated with the spectral density

$$f(\lambda; d_1, d_2) = f(\lambda; d_1)f(\lambda; d_2) = \frac{1}{(2\pi)^2} |1 - e^{i\lambda}|^{-2d_1}|1 + e^{i\lambda}|^{-2d_2}, \quad 0 < d_1, d_2 < 1/2 \quad (3.8)$$

can be derived from representation (2.3).

Denote  $F(a, b, c; x)$  a hypergeometric function

$$F(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a} dt,$$

where  $c > b > 0$  if  $x < 1$  and, in addition,  $c - a - b > 0$  if  $x = 1$ .



**Proposition 3.1** *The mixture density associated with  $f(\cdot; d_1, d_2)$  (3.8) is given by equality*

$$\begin{aligned}\varphi(x; d_1, d_2) &= C(d_1, d_2)x^{d_1-1}(1-x)^{1-2d_1}G(-x; d_2)\mathbf{1}_{[0,1]}(x) \\ &\quad + C(d_2, d_1)|x|^{d_2-1}(1+x)^{1-2d_2}G(x; d_1)\mathbf{1}_{[-1,0]}(x),\end{aligned}\quad (3.9)$$

where

$$G(x; d) := F\left(1, d, 2-d; \frac{1}{x}\right) - xF(1, d, 2-d; x)$$

and

$$\begin{aligned}C(d_1, d_2) &= (C^*)^{-1} \frac{\Gamma(d_2)\Gamma(2-2d_2)}{\Gamma(2-d_2)}, \\ C^* &= \int_0^1 x^{d_1-1}(1-x)^{1-2d_1}(1+x) \left\{ \int_0^1 \frac{y^{d_2-1}(1-y)^{1-2d_2}(1+y)}{1+xy} dy \right\} dx.\end{aligned}$$

The variance of the noise is

$$\sigma_\varepsilon^2 = (2\pi)^{-1} C^* C(d_1) C(d_2) \sigma_{1,\varepsilon}^2 \sigma_{2,\varepsilon}^2 = \frac{\sin(\pi d_1) \sin(\pi d_2) C^*}{2\pi^3}. \quad (3.10)$$

PROOF. (2.3) implies that

$$\varphi(x; d_1, d_2) = \frac{1}{C_*} (C(d_2)\varphi_1(x)F(-x; d_2) + C(d_1)\varphi_2(x)F(x; d_1)), \quad (3.11)$$

where

$$\begin{aligned}F(x; d) &:= \int_0^1 \frac{y^{d-1}(1-y)^{1-2d}(1+y)}{(1-xy)(1-y/x)} dy, \\ C_* &= \int_0^1 \varphi_1(x) \left( \int_{-1}^0 \frac{\varphi_2(y)}{1-xy} dy \right) dx = C(d_1)C(d_2)C^*.\end{aligned}$$

Using equality

$$\frac{1+y}{(1-xy)(1-y/x)} = \frac{1}{(1-x)(1-y/x)} - \frac{x}{(1-x)(1-xy)},$$

we have

$$\begin{aligned}F(x; d) &= \frac{1}{1-x} \int_0^1 \frac{y^{d-1}(1-y)^{1-2d}}{1-y/x} dy - \frac{x}{1-x} \int_0^1 \frac{y^{d-1}(1-y)^{1-2d}}{1-xy} dy \\ &= \frac{\Gamma(d)\Gamma(2-2d)}{\Gamma(2-d)} \frac{1}{1-x} (F(1, d, 2-d; 1/x) - xF(1, d, 2-d; x)) \\ &= \frac{\Gamma(d)\Gamma(2-2d)}{\Gamma(2-d)} \frac{G(x; d)}{1-x}.\end{aligned}\quad (3.12)$$

Now, (3.9) follows from (3.11) and (3.12), whereas (3.10) follows from (2.4), (3.5), (3.7).

To finish the proof note that all the hypergeometric functions appearing in the form of the mixture density are correctly defined.  $\square$

In the next proposition we present the asymptotics of  $\varphi(x; d_1, d_2)$  in the neighborhoods of 0 and  $\pm 1$ .

**Proposition 3.2** *Let the mixture density  $\varphi$  be given in (3.9). Then*

$$\varphi(x; d_1, d_2) \sim \begin{cases} \frac{\pi}{C^* \sin(\pi d_2)} x^{d_1+d_2-1}, & x \rightarrow 0+, \\ \frac{\pi}{C^* \sin(\pi d_1)} |x|^{d_1+d_2-1}, & x \rightarrow 0-, \end{cases} \quad (3.13)$$

$$\varphi(x; d_1, d_2) \sim \begin{cases} \frac{2^{1-2d_2} \pi}{C^* \sin(\pi d_2)} (1-x)^{1-2d_1}, & x \rightarrow 1-, \\ \frac{2^{1-2d_1} \pi}{C^* \sin(\pi d_1)} (1-x)^{1-2d_2}, & x \rightarrow -1+. \end{cases} \quad (3.14)$$

PROOF. Applying identities (see Abramowitz and Stegun (1972))

$$F(a, b; c; 1/x) = \left(\frac{x}{x-1}\right)^b F(b, c-a; c; \frac{1}{1-x}), \quad F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

we have that for  $x \rightarrow 0+$

$$\begin{aligned} G(-x, d_2) &= F\left(1, d_2, 2-d_2; -\frac{1}{x}\right) + xF(1, d_2, 2-d_2; -x) \\ &= \left(\frac{x}{1+x}\right)^{d_2} F(d_2, 1-d_2, 2-d_2; 1/(1+x)) + xF(1, d_2, 2-d_2; -x) \\ &\sim x^{d_2} F(d_2, 1-d_2, 2-d_2; 1) \\ &= \frac{\Gamma(2-d_2)\Gamma(1-d_2)}{\Gamma(2-2d_2)} x^{d_2} \\ &= \frac{\sqrt{\pi} \Gamma(2-d_2)}{2^{1-2d_2}\Gamma((3/2)-d_2)} x^{d_2}, \end{aligned}$$

and similarly for  $x \rightarrow 0-$

$$G(x, d_1) \sim \frac{\sqrt{\pi} \Gamma(2-d_1)}{2^{1-2d_1}\Gamma((3/2)-d_1)} |x|^{d_1}.$$

This and equality (3.9) imply

$$\begin{aligned} \varphi(x; d_1, d_2) &\sim C(d_1, d_2) \frac{\sqrt{\pi} \Gamma(2-d_2)}{2^{1-2d_2}\Gamma((3/2)-d_2)} x^{d_1+d_2-1} \\ &= \frac{C(d_1)C(d_2)}{C_*} \Gamma(d_2)\Gamma(1-d_2)x^{d_1+d_2-1} \\ &= (C^*)^{-1} \frac{\pi}{\sin(\pi d_2)} x^{d_1+d_2-1}, \quad x \rightarrow 0+, \\ \varphi(x; d_1, d_2) &\sim (C^*)^{-1} \frac{\pi}{\sin(\pi d_1)} |x|^{d_1+d_2-1}, \quad x \rightarrow 0-. \end{aligned}$$

For  $x \rightarrow 1-$  we obtain

$$\begin{aligned}\varphi(x; d_1, d_2) &\sim 2C(d_1, d_2)F(1, d_2, 2 - d_2; -1)(1 - x)^{1-2d_1} \\ &= C(d_1, d_2) \frac{\sqrt{\pi}\Gamma(2 - d_2)}{\Gamma((3/2) - d_2)} (1 - x)^{1-2d_1} \\ &= (C^*)^{-1} \frac{2^{1-2d_2}\pi}{\sin(\pi d_2)} (1 - x)^{1-2d_1}\end{aligned}$$

and similarly for  $x \rightarrow -1+$

$$\begin{aligned}\varphi(x; d_1, d_2) &\sim 2C(d_2, d_1)F(1, d_1, 2 - d_1; -1)(1 + x)^{1-2d_2} \\ &= (C^*)^{-1} \frac{2^{1-2d_1}\pi}{\sin(\pi d_1)} (1 + x)^{1-2d_2}.\end{aligned}$$

REMARK 3.1 Clearly,

$$\begin{aligned}\varphi_1(x; d_1) &\sim \begin{cases} C(d_1)x^{d_1-1}, & x \rightarrow 0+, \\ 2C(d_1)(1 - x)^{1-2d_1}, & x \rightarrow 1-, \end{cases} \\ \varphi_2(x; d_2) &\sim \begin{cases} C(d_2)|x|^{d_2-1}, & x \rightarrow 0-, \\ 2C(d_2)(1 + x)^{1-2d_2}, & x \rightarrow -1+. \end{cases}\end{aligned}$$

Hence, by Proposition 3.2, the mixture density  $\varphi$  associated with the product spectral density (3.8) behaves as  $\varphi_1$  when  $x$  approaches 1, and behaves as  $\varphi_2$  when  $x$  approaches  $-1$ . However, at zero,  $\varphi$  behaves as  $|x|^{d_1+d_2-1}$ , i.e. both densities  $\varphi_1$  and  $\varphi_2$  count.

Proposition 2.1 allows us construct the mixture density also in the case when the spectral density  $f$  of aggregated process has the form

$$f(\lambda) = \frac{1}{2\pi} \left(2 \sin \frac{|\lambda|}{2}\right)^{-2d} g(\lambda), \quad 0 < d < 1/2,$$

where  $g(\lambda)$  is analytic spectral density on  $[-\pi, \pi]$ . In general, the existence of the mixture density associated with any analytic spectral density is not clear. For example, AR(1) is aggregated process only if the mixture density is the Dirac delta function, what is difficult to apply in practice. Similar inference concerns also the ARMA processes, i.e. rational spectral densities. Another class of spectral densities obtained by aggregating "non-degenerated" mixture densities is characterized in the following proposition.

**Proposition 3.3** *A mixture density  $\varphi_g$  is associated with some analytic spectral density  $f$  if and only if there exists  $0 < a_* < 1$  such that  $\text{supp}(\varphi) \subset [-a_*, a_*]$ .*

PROOF. For sufficiency, assume that there exists  $0 < a_* < 1$  such that  $\text{supp}(\varphi) \subset [-a_*, a_*]$ . The covariance function of the corresponding process satisfies

$$\begin{aligned} |\gamma(h)| &\leq \sigma_\varepsilon^2 \int_{-1}^1 \frac{|x|^{|h|}}{1-x^2} \varphi(x) \, dx \\ &= \sigma_\varepsilon^2 \int_{-a_*}^{a_*} \frac{|x|^{|h|}}{1-x^2} \varphi(x) \, dx \\ &\leq \sigma_\varepsilon^2 a_*^{|h|} \int_{-a_*}^{a_*} \frac{\varphi(x)}{1-x^2} \, dx \\ &= C a_*^{|h|}, \end{aligned}$$

i.e. the covariance function decays exponentially to zero. This implies that the spectral density  $f(\lambda) = (2\pi)^{-1} \sum_{h=-\infty}^{\infty} \gamma(h) e^{ih\lambda}$  is analytic function on  $[-\pi, \pi]$  (see, e.g., Bary (1964, p. 80–82)).

To prove the necessity, assume that  $f$  is an analytic function on  $[-\pi, \pi]$  or, equivalently, the corresponding covariance function decays exponentially to zero, i.e. there exists  $\theta \in (0, 1)$  and a constant  $C > 0$  such that  $|\gamma(h)| \leq C\theta^{|h|}$ . Assume to the contrary that  $\text{supp}(\varphi) = [-1, 1]$ .

Let  $h \geq 0$  be an even integer. Then

$$|\gamma(h)| = \sigma_\varepsilon^2 \int_{-1}^1 \frac{|x|^h}{1-x^2} \varphi(x) \, dx \leq C\theta^h$$

implies that

$$\int_{-1}^1 \left(\frac{|x|}{\theta}\right)^h \frac{\varphi(x)}{1-x^2} \, dx \leq C. \quad (3.15)$$

Rewrite the last integral as

$$\begin{aligned} \int_{-1/\theta}^{1/\theta} |x|^h \frac{\varphi(\theta x)}{1-(\theta x)^2} \, dx &= \int_{-1}^1 |x|^h \frac{\varphi(\theta x)}{1-(\theta x)^2} \, dx + \int_{-1/\theta}^{-1} |x|^h \frac{\varphi(\theta x)}{1-(\theta x)^2} \, dx \\ &\quad + \int_1^{1/\theta} |x|^h \frac{\varphi(\theta x)}{1-(\theta x)^2} \, dx =: I_1(h) + I_2(h) + I_3(h). \end{aligned}$$

For every  $x \in [-1, 1]$  we have

$$|x|^h \frac{\varphi(\theta x)}{1-(\theta x)^2} \leq \frac{\varphi(\theta x)}{1-(\theta x)^2}.$$

Hence, by the dominated convergence theorem and (1.3),  $I_1(h) \rightarrow 0$  as  $h \rightarrow \infty$ . The Fatou lemma, however, implies that both integrals  $I_2(h)$  and  $I_3(h)$  tend to infinity as  $h \rightarrow \infty$ :

$$\liminf_{h \rightarrow \infty} I_2(h) \geq \int_1^{1/\theta} \frac{\varphi(\theta x)}{1-(\theta x)^2} \liminf_{h \rightarrow \infty} x^h \, dx = \infty$$

since  $1/\theta > 1$ . This contradicts (3.15).  $\square$

**EXAMPLE 3.1** Assume that the mixture density  $\varphi_g$  has a uniform distribution on  $[a, b]$ , where  $-1 < a < b < 1$ . By Proposition 3.3, the associated spectral density is analytic function on  $[-\pi, \pi]$  and can be easily calculated:

$$\begin{aligned} f_g(\lambda) &= \frac{\sigma_\varepsilon^2}{2\pi(b-a)} \int_a^b \frac{dx}{1-2x\cos\lambda+x^2} \\ &= \frac{\sigma_\varepsilon^2}{2\pi(b-a)\sin|\lambda|} \left( \arctan\left(\frac{b-\cos\lambda}{\sin|\lambda|}\right) - \arctan\left(\frac{a-\cos\lambda}{\sin|\lambda|}\right) \right), \quad \lambda \neq 0, \pm\pi. \end{aligned}$$

$$f_g(0) = \sigma_\varepsilon^2(2\pi)^{-1}(1-a)^{-1}(1-b)^{-1}, \quad f_g(\pm\pi) = \sigma_\varepsilon^2(2\pi)^{-1}(1+a)^{-1}(1+b)^{-1}.$$

We obtain the following corollary.

**Corollary 3.1** *Let  $\varphi_1(x; d)$  (3.3) and  $\varphi_g(x)$  be the mixture densities associated with spectral densities  $f_1(\lambda; d)$  (3.1) and analytic spectral density  $g(\lambda)$ , respectively. Assume that  $\text{supp}(\varphi_g) \subset [-a_*, 0]$ ,  $0 < a_* < 1$ , and*

$$f(\lambda) = \frac{1}{2\pi} \left( 2 \sin \frac{|\lambda|}{2} \right)^{-2d} g(\lambda). \quad (3.16)$$

*Then the mixture density  $\varphi(x)$ ,  $x \in [-a_*, 1]$  associated with  $f$  is given by equality*

$$\varphi(x) = C_*^{-1} \left( \varphi_1(x; d) \int_{-a_*}^0 \frac{\varphi_g(y)}{(1-xy)(1-y/x)} dy + \varphi_g(x) \int_0^1 \frac{\varphi_1(y; d)}{(1-xy)(1-y/x)} dy \right),$$

*where*

$$C_* := \int_0^1 \left( \int_{-a_*}^0 \frac{\varphi_1(x; d)\varphi_g(y)}{1-xy} dy \right) dx.$$

## 4 The structure of aggregated process

In order to make further inference about the aggregated process  $X_t$ , e.g., estimation of the mixture density, limit theorems, forecasting, etc., it is necessary to investigate more precise structure of  $X_t$ . In particular, it is important to obtain the linear (moving average) representation of the aggregated process.

### 4.1 Behavior of spectral density of the aggregated process

In this subsection we will study the behavior of the spectral densities corresponding to the general class of semiparametric mixture densities of the form (see Viano and Oppenheim (2004), Leipus et al. (2006))

$$\varphi(x) = (1-x)^{1-2d_1}(1+x)^{1-2d_2}\psi(x), \quad 0 < d_1 < 1/2, \quad 0 < d_2 < 1/2, \quad (4.1)$$

where  $\psi(x)$  is continuous and nonvanishes at the points  $x = \pm 1$ . As it is seen from the mixture densities appearing in Section 3, this form is natural, in particular (4.1) covers the mixture density in (3.9). The corresponding spectral density behaves as a long memory spectral density.

**Lemma 4.1** *Let the density  $\varphi(x)$  be given in (4.1),  $\psi(x)$  is nonnegative function on  $[-1, 1]$  and continuous at the points  $x = \pm 1$  with  $\psi(\pm 1) \neq 0$ . Then the following relations for the corresponding spectral density hold:*

$$f(\lambda) \sim \frac{\sigma_\varepsilon^2 \psi(1)}{2^{2d_2+1} \sin(\pi d_1)} |\lambda|^{-2d_1}, \quad |\lambda| \rightarrow 0, \quad (4.2)$$

$$f(\lambda) \sim \frac{\sigma_\varepsilon^2 \psi(-1)}{2^{2d_1+1} \sin(\pi d_2)} |\pi \mp \lambda|^{-2d_2}, \quad \lambda \rightarrow \pm \pi. \quad (4.3)$$

PROOF. Let  $0 < \lambda < \pi$ . (4.1), (1.5) and change of variables  $u = (x - \cos \lambda) / \sin \lambda$  lead to

$$\begin{aligned} f(\lambda) &= \frac{\sigma_\varepsilon^2}{2\pi} \int_{-1}^1 \frac{(1-x)^{1-2d_1} (1+x)^{1-2d_2} \psi(x)}{|1 - xe^{i\lambda}|^2} dx \\ &= \left(2 \sin \frac{\lambda}{2}\right)^{-2d_1} \left(2 \cos \frac{\lambda}{2}\right)^{-2d_2} g^*(\lambda), \end{aligned}$$

where

$$g^*(\lambda) = \frac{\sigma_\varepsilon^2}{\pi} \int_{-\cot \frac{\lambda}{2}}^{\tan \frac{\lambda}{2}} \frac{(\sin \frac{\lambda}{2} - u \cos \frac{\lambda}{2})^{1-2d_1} (\cos \frac{\lambda}{2} + u \sin \frac{\lambda}{2})^{1-2d_2}}{1+u^2} \psi(u \sin \lambda + \cos \lambda) du.$$

By assumption of continuity at the point 1, the function  $\psi(u \sin \lambda + \cos \lambda)$  is bounded in some neighbourhood of zero, i.e.  $\psi(u \sin \lambda + \cos \lambda) \leq C_1(\lambda_0)$  for  $0 < \lambda < \lambda_0$  and  $\lambda_0 > 0$  sufficiently small. Hence,

$$\frac{(\sin \frac{\lambda}{2} - u \cos \frac{\lambda}{2})^{1-2d_1} (\cos \frac{\lambda}{2} + u \sin \frac{\lambda}{2})^{1-2d_2}}{1+u^2} \psi(u \sin \lambda + \cos \lambda) \leq \frac{C_2(\lambda_0)(1+|u|)^{1-2d_1}}{1+u^2}$$

for  $0 < \lambda < \lambda_0$  and, by the dominated convergence theorem, as  $\lambda \rightarrow 0$ ,

$$\begin{aligned} g^*(\lambda) &\rightarrow \frac{\sigma_\varepsilon^2 \psi(1)}{\pi} \int_0^\infty \frac{u^{1-2d_1}}{1+u^2} du \\ &= \frac{\sigma_\varepsilon^2 \psi(1)}{2\pi} \Gamma(d_1) \Gamma(1-d_1) \\ &= \frac{\sigma_\varepsilon^2 \psi(1)}{2 \sin(\pi d_1)} \end{aligned}$$

implying (4.2). The same argument leads to relation (4.3).  $\square$

REMARK 4.1 Note, differently from Viano and Oppenheim (2004), we do not require the boundedness of function  $\psi(x)$  on interval  $[-1, 1]$ . In fact,  $\psi(x)$  can have singularity points within  $(-1, 1)$ , see (3.3), (3.6).

## 4.2 Moving average representation of the aggregated process

Any aggregated process admits an absolutely continuous spectral measure. If, in addition, its spectral density, say,  $f(\lambda)$  satisfies

$$\int_{-\pi}^{\pi} \log f(\lambda) d\lambda > -\infty, \quad (4.4)$$

then the function

$$h(z) = \exp \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\lambda} + z}{e^{i\lambda} - z} \log f(\lambda) d\lambda \right\}, \quad |z| < 1,$$

is an outer function from the Hardy space  $H^2$ , does not vanish for  $|z| < 1$  and  $f(\lambda) = |h(e^{i\lambda})|^2$ . Then, by the Wold decomposition theorem, corresponding process  $X_t$  is purely nondeterministic and has the MA( $\infty$ ) representation (see Anderson (1971, Ch. 7.6.3))

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \quad (4.5)$$

where the coefficients  $\psi_j$  are defined from the expansion of normalized outer function  $h(z)/h(0)$ ,  $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ ,  $\psi_0 = 1$ , and  $Z_t = X_t - \widehat{X}_t$ ,  $t = 0, 1, \dots$  ( $\widehat{X}_t$  is the optimal linear predictor of  $X_t$ ) is the innovation process, which is zero mean, uncorrelated, with variance

$$\sigma^2 = 2\pi \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(\lambda) d\lambda \right\}. \quad (4.6)$$

By construction, the aggregated processes are Gaussian, implying that the innovations  $Z_t$  are i.i.d.  $N(0, \sigma^2)$  random variables.

We obtain the following results.

**Proposition 4.1** *Let the mixture density  $\varphi$  satisfies the assumptions of Lemma 4.1. Then the aggregated process admits a moving average representation (4.5), where the  $Z_t$  are Gaussian i.i.d. random variables with zero mean and variance (4.6).*

PROOF. We have to verify that (4.4) holds. According to (4.2),  $\log f(\lambda) \sim -C_1 \log |1 - e^{i\lambda}|$ ,  $|\lambda| \rightarrow 0$ , where  $C_1 > 0$ . For any  $\epsilon > 0$  choose  $0 < \lambda_0 \leq \pi/3$ , such that

$$-\frac{\log f(\lambda)}{C_1 \log |1 - e^{i\lambda}|} - 1 \geq -\epsilon, \quad 0 < \lambda \leq \lambda_0.$$

Since  $-\log|1 - e^{i\lambda}| \geq 0$  for  $0 \leq \lambda \leq \pi/3$ , we obtain

$$\int_0^{\lambda_0} \log f(\lambda) d\lambda \geq (C_1 - \epsilon) \int_0^{\lambda_0} \log|1 - e^{i\lambda}| d\lambda > -\infty \quad (4.7)$$

using the well known fact that  $\int_0^\pi \log|1 - e^{i\lambda}| d\lambda = 0$ . Using (4.3) and the same argument, we get

$$\int_{\pi-\lambda_0}^\pi \log f(\lambda) d\lambda > -\infty. \quad (4.8)$$

When  $\lambda \in [\lambda_0, \pi - \lambda_0]$ , there exist  $0 < L_1 < L_2 < \infty$  such that

$$L_1 \leq \frac{1}{2\pi|1 - xe^{i\lambda}|^2} \leq L_2$$

uniformly in  $x \in (-1, 1)$ . Thus, by (1.5),  $L_1 \leq f(\lambda) \leq L_2$  for any  $\lambda \in [\lambda_0, \pi - \lambda_0]$ , and therefore

$$\int_{\lambda_0}^{\pi-\lambda_0} \log f(\lambda) d\lambda > -\infty. \quad (4.9)$$

(4.7)–(4.9) imply inequality (4.4).  $\square$

**Lemma 4.2** *If the spectral density  $g$  of the aggregated process  $X_t$  is analytic function, then  $X_t$  admits representation*

$$X_t = \sum_{j=0}^{\infty} g_j Z_{t-j},$$

where the  $Z_t$  are Gaussian i.i.d. random variables with zero mean and variance

$$\sigma_g^2 = 2\pi \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log g(\lambda) d\lambda \right\} \quad (4.10)$$

and the  $g_j$  satisfy  $|\sum_{j=0}^{\infty} g_j| < \infty$ ,  $g_0 = 1$ .

PROOF. From Proposition 3.3, there exists  $0 < a_* < 1$  such that

$$g(\lambda) = \frac{\sigma_g^2}{2\pi} \int_{-a_*}^{a_*} \frac{\varphi_g(x)}{|1 - xe^{i\lambda}|^2} dx. \quad (4.11)$$

For all  $x \in [-a_*, a_*]$  and  $\lambda \in [0, \pi]$  we have

$$\frac{1}{|1 - xe^{i\lambda}|^2} \geq C > 0.$$

This and (4.11) imply  $\int_0^\pi \log g(\lambda) d\lambda > -\infty$ . Finally,  $|\sum_{j=0}^{\infty} g_j| < \infty$  follows from representation

$$g(\lambda) = \frac{\sigma_g^2}{2\pi} \left| \sum_{j=0}^{\infty} g_j e^{ij\lambda} \right|^2$$

and the assumption of analyticity of  $g$ .  $\square$



**Proposition 4.2** *Let  $X_t$  be an aggregated process with spectral density*

$$f(\lambda) = \frac{1}{2\pi} \left(2 \sin \frac{|\lambda|}{2}\right)^{-2d} g(\lambda), \quad 0 < d < 1/2,$$

*satisfying the assumptions of Corollary 3.1. Then  $X_t$  admits a representation (4.5), where the  $Z_t$  are Gaussian i.i.d. random variables with zero mean and variance*

$$\sigma^2 = 2\pi \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(\lambda) d\lambda \right\} = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log g(\lambda) d\lambda \right\} = \frac{\sigma_g^2}{2\pi}$$

*and the  $\psi_j$  satisfy*

$$\psi_j \sim \frac{\sum_{j=0}^{\infty} g_j}{\Gamma(d)} j^{d-1}, \quad \psi_0 = 1. \quad (4.12)$$

*Here, the  $g_j$  are given in Lemma 4.2.*

PROOF. We have

$$\frac{1}{2\pi} \left(2 \sin \frac{|\lambda|}{2}\right)^{-2d} = \frac{1}{2\pi} \left| \sum_{j=0}^{\infty} h_j e^{ij\lambda} \right|^2 \quad \text{with} \quad h_j = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}$$

and, recall,

$$g(\lambda) = \frac{\sigma_g^2}{2\pi} \left| \sum_{j=0}^{\infty} g_j e^{ij\lambda} \right|^2, \quad \sum_{j=0}^{\infty} g_j^2 < \infty$$

since, by Lemma 4.2,  $\int_{-\pi}^{\pi} \log g(\lambda) d\lambda > -\infty$ . On the other hand,  $\int_{-\pi}^{\pi} \log f(\lambda) d\lambda > -\infty$  implies

$$f(\lambda) = \frac{1}{2\pi} \left| \sum_{j=0}^{\infty} \tilde{\psi}_j e^{ij\lambda} \right|^2, \quad \sum_{j=0}^{\infty} \tilde{\psi}_j^2 < \infty$$

and, by uniqueness of the representation,

$$\tilde{\psi}_k = \frac{\sigma_g}{\sqrt{2\pi}} \sum_{j=0}^k h_{k-j} g_j.$$

It easy to see that,

$$\sum_{j=0}^k h_{k-j} g_j \sim h_k \sum_{j=0}^{\infty} g_j \sim C_2 k^{d-1}, \quad (4.13)$$

where  $C_2 = \Gamma^{-1}(d) \sum_{j=0}^{\infty} g_j$ . Indeed, taking into account that  $h_k \sim \Gamma^{-1}(d) k^{d-1}$ , we can write

$$\sum_{j=0}^k h_{k-j} g_j = \Gamma^{-1}(d) k^{d-1} \sum_{j=0}^{\infty} a_{k,j} g_j,$$

where  $a_{k,j} = h_{k-j}\Gamma(d)k^{1-d}\mathbf{1}_{\{j \leq k\}} \rightarrow 1$  as  $k \rightarrow \infty$  for each  $j$ . On the other hand, we have  $|a_{k,j}| \leq C(1+j)^{1-d}$  uniformly in  $k$  and, since the  $g_j$  decay exponentially fast, the sum  $\sum_{j=0}^{\infty}(1+j)^{1-d}|g_j|$  converges and the dominated convergence theorem applies to obtain (4.13).

Hence, we can write

$$f(\lambda) = \frac{\sigma_g^2}{(2\pi)^2} \left| \sum_{j=0}^{\infty} \psi_j e^{ij\lambda} \right|^2, \quad \psi_0 = 1,$$

where  $\psi_j = \tilde{\psi}_j \sqrt{2\pi}/\sigma_g \sim C_2 j^{d-1}$  and (4.5) follows.  $\square$

## 5 Appendix. Mixture density associated with FI( $d$ ) spectral density

**Proposition 5.1** *Mixture density associated with FI( $d$ ) spectral density*

$$f(\lambda; d) = \frac{1}{2\pi} \left( 2 \sin \frac{|\lambda|}{2} \right)^{-2d}, \quad 0 < d < 1/2,$$

is given by equality

$$\varphi(x) = C(d)x^{d-1}(1-x)^{1-2d}(1+x)\mathbf{1}_{[0,1]}(x), \quad (5.1)$$

where

$$C(d) = \frac{\Gamma(3-d)}{2\Gamma(d)\Gamma(2-2d)} = 2^{2d-2} \frac{\sin(\pi d)}{\sqrt{\pi}} \frac{\Gamma(3-d)}{\Gamma((3/2)-d)}.$$

The variance of the noise is

$$\sigma_\varepsilon^2 = \frac{\sin(\pi d)}{C(d)\pi}.$$

PROOF. Equality

$$f(\lambda; d) = 2^{-d-1}\pi^{-1}(1 - \cos \lambda)^{-d}$$

implies that  $1 - \cos \lambda = (\pi 2^{d+1} f(\lambda; d))^{-1/d}$ . Hence, rewriting

$$|1 - xe^{i\lambda}|^2 = (1-x)^2 \left( 1 + \frac{2x}{(1-x)^2} (1 - \cos \lambda) \right),$$

and assuming  $\text{supp}(\varphi) = [0, 1]$ , we obtain that the spectral density of aggregated process is of the form

$$\begin{aligned} \frac{\sigma_\varepsilon^2}{2\pi} \int_0^1 \frac{\varphi(x)}{|1 - xe^{i\lambda}|^2} dx &= \frac{\sigma_\varepsilon^2}{2\pi} \int_0^1 \frac{\varphi(x)}{(1-x)^2 \left( 1 + \frac{2x}{(1-x)^2} (1 - \cos \lambda) \right)} dx \\ &= \frac{\sigma_\varepsilon^2}{2\pi} \int_0^1 \frac{\varphi(x)}{(1-x)^2 \left( 1 + \frac{2x}{(1-x)^2} (\pi 2^{d+1} f(\lambda; d))^{-1/d} \right)} dx \end{aligned} \quad (5.2)$$

The change of variables  $y^{1/d} = 2x/(1-x)^2$  implies

$$dy = \frac{d2^d x^{d-1}(1+x)}{(1-x)^{2d+1}} dx. \quad (5.3)$$

Consider the density  $\varphi$  defined by

$$dy = \frac{\varphi(x)}{\tilde{C}(d)(1-x)^2} dx, \quad (5.4)$$

where  $\tilde{C}(d)$  is some constant. Then (5.2) becomes

$$\begin{aligned} \frac{\sigma_\varepsilon^2}{2\pi} \int_0^1 \frac{\varphi(x)}{|1 - xe^{i\lambda}|^2} dx &= \frac{\sigma_\varepsilon^2 \tilde{C}(d)}{2\pi} \int_0^\infty \frac{dy}{(1 + y^{1/d}(\pi 2^{d+1} f(\lambda; d))^{-1/d})} \\ &= f(\lambda; d) \sigma_\varepsilon^2 2^d \tilde{C}(d) \int_0^\infty \frac{dz}{1 + z^{1/d}} \end{aligned}$$

after the change of variables  $z = \frac{y}{\pi 2^{d+1} f(\lambda; d)}$ . Therefore, FI( $d$ ) is an aggregated process and, by (5.3)–(5.4), the mixture density has a form

$$\varphi(x) = \tilde{C}(d) d 2^d x^{d-1} (1-x)^{1-2d} (1+x), \quad (5.5)$$

and the variance of the noise is (see formula 6.1.17 in Abramowitz and Stegun (1972))

$$\begin{aligned} \sigma_\varepsilon^2 &= 2^{-d} (\tilde{C}(d))^{-1} \left( \int_0^\infty \frac{dz}{1 + z^{1/d}} \right)^{-1} \\ &= 2^{-d} (d \tilde{C}(d))^{-1} (B(d, 1-d))^{-1} \\ &= 2^{-d} (d \tilde{C}(d))^{-1} \frac{\sin(\pi d)}{\pi}. \end{aligned}$$

Finally, it remains to calculate the constant  $\tilde{C}(d)$  to ensure that the mixture density  $\varphi$  given in (5.5) integrates to one over the interval  $[0, 1]$ . We have

$$\begin{aligned} \int_0^1 \varphi(x) dx &= \tilde{C}(d) d 2^d \left( \int_0^1 x^{d-1} (1-x)^{1-2d} dx + \int_0^1 x^d (1-x)^{1-2d} dx \right) \\ &= \tilde{C}(d) d 2^d \left( B(d, 2-2d) + B(d+1, 2-2d) \right) \\ &= \tilde{C}(d) d 2^{1+d} \frac{\Gamma(d)\Gamma(2-2d)}{(2-d)\Gamma(2-d)} \\ &= \tilde{C}(d) d 2^{2-d} \frac{\sqrt{\pi}}{\sin(\pi d)} \frac{\Gamma((3/2)-d)}{\Gamma(3-d)}. \end{aligned}$$

Hence,

$$\tilde{C}(d) = \frac{1}{d 2^{d+1}} \frac{\Gamma(3-d)}{\Gamma(d)\Gamma(2-2d)} = \frac{2^{d-2}}{d} \frac{\sin(\pi d)}{\sqrt{\pi}} \frac{\Gamma(3-d)}{\Gamma((3/2)-d)}$$

and  $C(d) = \tilde{C}(d) d 2^d$ . □

## References

- Abramowitz, M., Stegun, I.A. (1972) *Handbook of Mathematical Functions*. Dover, New York.
- Anderson, T.W. (1971) *The Statistical Analysis Of Time Series*. John Wiley & Sons, New York.
- Bary, N.K. (1964) *A Treatise On Trigonometric Series, Vol. I*. Pergamon Press, Oxford.
- Chong, T.T. (2006) The polynomial aggregated AR(1) model. *Econometrics Journal* **9**, 98–122.
- Dacunha-Castelle, D. and Oppenheim, G. (2001) Mixtures, aggregations and long memory. Université de Paris-Sud, Mathématiques. Preprint 2001-72.
- Dacunha-Castelle, D. and Fermin, L. (2006) Disaggregation of long memory processes on  $C^\infty$  class. *Electronic Communications in Probability* **11**, 35–44.
- Haubrich, J.G. and Lo, A.W. (2001) The sources and nature of long-term dependence in the business cycle. *Econom. Rev.* (Fed. Reserve Bank Cleveland, Q II) **37**, 15–30.
- Leipus, R., Oppenheim, G., Philippe, A. and Viano, M.-C. (2006) Orthogonal series density estimation in a disaggregation scheme. *Journal of Statistical Planning and Inference* **136**, 2547–2571.
- Oppenheim, G. and Viano, M.-C. (2004) Aggregation of random parameters Ornstein-Uhlenbeck or AR processes: some convergence results. *Journal of Time Series Analysis* **25**, 335–350.
- Szegő, G. (1959) *Orthogonal Polynomials*. American Mathematical Society, New York.
- Tolstov, G.P. (1976) *Fourier Series*. Dover, New York.
- Zaffaroni, P. (2004) Contemporaneous aggregation of linear dynamic models in large economies. *Journal of Econometrics* **120**, 75-102.