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Finite time stability conditions for non autonomous continuous systems

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Finite time stability is defined for continuous non autonomous systems. Starting with a result from Haimo Haimo (1986) we then extend this result to $n-$dimensional non autonomous systems through the use of smooth and nonsmooth Lyapunov functions as in Perruquetti and Drakunov (2000). One obtains two different sufficient conditions and a necessary one for finite time stability of continuous non autonomous systems.

1   Introduction

Since the end of the 19th and the beginning of the 20th century, various concepts dedicated to the qualitative behavior of the solutions for dynamical systems have been introduced (for example the seminal definitions and results from A.M. Lyapunov Lyapunov (1992)). But rapidly, one had to face some more precise time specifications of the behavior of the state variables (or outputs) for real process. For example, some finite time stabilizing control design relies upon sliding mode theory (see Utkin (1992)) and another one upon optimal control theory (see Ryan (1979)). In both cases the controller leads to some state discontinuous feedback control laws. Over the years much work has been dedicated to this concept, getting some sufficient conditions and the application to the construction of finite-time stabilization control (see for example Bhat and Bernstein (1998), Hong (2002), Hong *et al.* (2001, 2002), Moulay and Perruquetti (2006a)). The main idea lies in assigning infinite eigenvalue to the closed loop system at the origin. Let us mention the following illustrative example

$$\dot{x} = - |x|^a \, sgn(x), x \in \mathbb{R}, a \in \,]0, 1[, \tag{1}$$

for which the solutions starting at $\tau = 0$ from x are:

$$\phi^x(\tau) = \begin{cases} sgn(x) \left(|x|^{1-a} - \tau(1-a) \right)^{\frac{1}{1-a}}) \text{ if } 0 \leq \tau \leq \frac{|x|^{1-a}}{1-a} \\ 0 \qquad\qquad\qquad\qquad\qquad \text{ if } \tau > \frac{|x|^{1-a}}{1-a} \end{cases}, \tag{2}$$

and they reach the origin in finite time. In fact, there exists a function called *settling time* that increases the time for a solution to reach the equilibrium. Usually, this function depends on the initial condition of a solution. But, for non autonomous systems, it may also depend on the initial time. Lastly, notice from this example, that in order to obtain finite time stability, the right hand side of the ordinary differential equation *cannot be locally Lipschitz at the origin.*

The paper is organized as follows. After defining the notion of finite time stability for continuous non autonomous systems in section 2 , we recall the necessary and sufficient conditions for finite time stability of autonomous scalar systems (section 3) (result which appears in Haimo (1986), Moulay and Perruquetti (2006b) and whose proof is given here. Then in section 4, we give a generalization for continuous non

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autonomous systems of any dimension. For this we introduce a Lyapunov function whose derivative satisfies an increase to obtain sufficient conditions for finite time stability. In the subsection 4.1, one uses a smooth Lyapunov function and in the subsection 4.2, one uses a nonsmooth one. Nevertheless, as we will see the use of a smooth Lyapunov function is not a weaker result, only a different one. Then, a Lyapunov function whose derivative verifies a decrease is used in order to obtain necessary conditions for finite time stability (see subsection 4.3).

Through the paper, the following notations will be used:

- for $a > 0$, the following function will be used $\varphi_a(x) = |x|^a \, sgn(x)$,
- $\mathcal{V}$ denotes a neighborhood of the origin in $\mathbb{R}^n$,
- $\mathcal{B}_\epsilon$ is the open ball centered at the origin of radius $\epsilon > 0$.

The *upper Dini derivative* of a function $f : \mathbb{R} \to \mathbb{R}$ is the function $D^+ f : \mathbb{R} \to \overline{\mathbb{R}}$ defined by:

$$D^+ f(x) = \limsup_{h \to 0^+} \frac{f\left(x + h\right) - f\left(x\right)}{h}.$$

2   What is finite time stability?

Consider the system

$$\dot{x} = f(t, x), \quad t \in \mathbb{R}_{\geq 0}, x \in \mathbb{R}^n, \tag{3}$$

where $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. Then $\phi_t^x(\tau)$ denotes a solution of the system (3) starting from $(t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ and $\mathcal{S}(t, x)$ represents the set of all solutions ϕ_t^x. The existence of solutions is given by the well known Cauchy-Peano Theorem given for example in Hale (1980). A continuous function $\alpha : [0, a] \to [0, +\infty[$ belongs to *class* $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to *class* $\mathcal{K}_\infty$ if $a = +\infty$ and $\alpha(r) \to +\infty$ as $r \to +\infty$. Moreover, a continuous function $V : \mathbb{R}_{\geq 0} \times \mathcal{V} \to \mathbb{R}_{\geq 0}$ such that

L1) V is positive definite,

L2) $\dot{V}(t, x) = D^+ \left(V \circ \tilde{\phi}_t^x \right)(0)$ is negative definite with $\tilde{\phi}_t^x(\tau) = (\tau, \phi_t^x(\tau))$,

is a *Lyapunov function* for (3). If V is smooth then

$$\dot{V}(t, x) = \frac{\partial V}{\partial t}(t, x) + \sum_{i=1}^n \frac{\partial^i V}{\partial x_i}(t, x) f_i(t, x).$$

A continuous function $v : \mathbb{R}_{\geq 0} \times \mathcal{V} \to \mathbb{R}$ is *decrescent* if there exists a $\mathcal{K}-$ function ψ such that

$$|v(t, x)| \leq \psi\left(||x||\right) \quad \forall(t, x) \in \mathbb{R}_{\geq 0} \times \mathcal{V}.$$

A continuous function $v : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$ is *radially unbounded* if there exists a $\mathcal{K}_\infty-$function φ such that

$$v(t, y) \geq \varphi(||y||), \quad \forall t \in \mathbb{R}_{\geq 0} \quad \forall y \in \mathbb{R}^n.$$

The definition of asymptotic stability is well known (see Hahn (1963)). In this case, the solutions of the system (3) tend to the origin (but without information about the time transient). This information comes from the notion of "settling time" which, when finite and combined with the stability concept, leads to the following definition for finite time stability

Definition 2.1 The origin is *weakly finite time stable* for the system (3) if:

A1) the origin is Lyapunov stable for the system (3),

A2) for all $t \in I$, there exists $\delta(t) > 0$, such that if $x \in \mathcal{B}_{\delta(t)}$ then for all $\Phi_t^x \in \mathcal{S}(t, x)$:

    a) $\phi_t^x(\tau)$ is defined for $\tau \geq t$,

    b) there exists $0 \leq T(\phi_t^x) < +\infty$ such that $\phi_t^x(\tau) = 0$ for all $\tau \geq t + T(\phi_t^x)$.

$$T_0(\phi_t^x) = \inf\{T(\phi_t^x) \geq 0 : \phi_t^x(\tau) = 0 \ \forall \tau \geq t + T(\phi_t^x)\}$$

    is called the *settling time of the solution* ϕ_t^x.

A3) Moreover, if $T_0(t, x) = \sup\limits_{\phi_t^x \in \mathcal{S}(t,x)} T_0(\phi_t^x) < +\infty$, then the origin is *finite time stable* for the system (3).

    $T_0(t, x)$ is called the *settling time with respect to the initial conditions of the system* (3).

Remark 1 When the system is asymptotically stable, the settling time of a solution may be infinite.
If the system (3) is continuous on $I \times \mathcal{V}$ and locally Lipschitz on $I \times \mathcal{V} \setminus \{0\}$, because of solution uniqueness, the settling time of a solution and the settling time with respect to the initial conditions of the system are the same: $T_0(t, x) = T_0(\phi_t^x)$.

Definition 2.2 Let the origin be an equilibrium point of the system (3). The origin is *uniformly finite time stable* for the system (3) if the origin is

B1) uniformly asymptotically stable for the system (3),

B2) finite time stable for the system (3),

B3) there exists a positive definite continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that the settling time with respect initial condition of the system (3) satisfies:

$$T_0(t, x) \leq \alpha(\|x\|).$$

3   Autonomous scalar systems

Let us recall the result for finite time stability of autonomous scalar systems of the form

$$\dot{x} = f(x), \quad x \in \mathbb{R}, \tag{4}$$

where $f : \mathbb{R} \to \mathbb{R}$ given in Haimo (1986). In this particular case there exists a necessary and sufficient condition for finite time stability:

LEMMA 3.1 *Let the origin be an equilibrium point of the system (4) where f is continuous. The origin is finite time stable for the system (4) if and only if there exists a neighborhood of the origin $\mathcal{V}$ such that for all $x \in \mathcal{V} \setminus \{0\}$*

$$xf(x) < 0, \tag{5}$$

$$\int_x^0 \frac{dz}{f(z)} < +\infty. \tag{6}$$

Proof ($\Leftarrow$) As $xf(x) < 0$, $V(x) = x^2$ is a Lyapunov function for the system. So, the origin of the system (4) is asymptotically stable.

Let $\phi^x(\tau)$ be a solution of the system which tend to the origin with time $T(\phi^x)$. We have to show that $T(\phi^x) < +\infty$. With the asymptotic stability, if x is small enough, then $\tau \mapsto \phi^x(\tau)$ is strictly monotone for $\tau \geq 0$.

Moreover,

$$T(\phi^x) = \int_0^{T(\phi^x)} d\tau.$$

As $xf(x) < 0$ for all $x \in \mathcal{V} \backslash \{0\}$, $\frac{1}{f}$ is defined on $\mathcal{V} \backslash \{0\}$. The following change of variables, $[0, T(\phi^x)[\; \rightarrow \;]0, x]$, $\tau \mapsto \phi^x(\tau)$ leads to

$$\int_x^0 \frac{dz}{f(z)} = \int_0^{T(\phi^x)} \frac{\dot{\phi}^x(\tau)}{f(\phi^x(\tau))} d\tau = T(\phi^x) < +\infty.$$

$T(x) = \sup\limits_{\phi^x \in \mathcal{S}(x)} T(\phi^x)$ does not depend on ϕ^x. So, we conclude that the origin of the system (4) is finite time stable with the following settling time $T_0(x) = T(\phi^x)$ satisfying $T_0(x) = \int_x^0 \frac{dz}{f(z)}$.

($\Rightarrow$) Suppose that the origin of the system (4) is finite time stable. Let $\delta > 0$ given by definition 2.1. Suppose that there exists $x \in]-\delta, \delta[\, \backslash \, \{0\}$ such that $xf(x) \geq 0$.

- If $xf(x) = 0$, then $f(x) = 0$ et $\phi^x(\tau) \equiv x$ is a solution of the system (4) which does not tend to the origin.
- If $xf(x) > 0$, with no loss of generality, we can suppose that $x > 0$ et $f(x) > 0$. The continuity of f and the fact that $f(x) > 0$ lead to $f(z) > 0$ for z in a neighborhood of x. Then, the function $\tau \mapsto \phi^x(\tau)$ increases in a neighborhood of the origin. With its continuity, this solution can not tend to the origin.

Let $x \in]-\delta, \delta[\, \backslash \, \{0\}$ and consider the solution $\phi^x(\tau)$. By assumption, there exists $0 \leq T_0(\phi^x) < +\infty$ such that $\phi^x(\tau) = 0$ for all $\tau \geq t + T_0(\phi^x)$. With the asymptotic stability, x can be chosen small enough such that $\tau \mapsto \phi^x(\tau)$ decreases for $\tau \geq t$. The following change of variables $[0, T_0(\phi^x)[\; \rightarrow \;]0, x]$, $\tau \mapsto \phi^x(\tau)$ leads to

$$\int_x^0 \frac{dz}{f(z)} = \int_0^{T_0(\phi^x)} d\tau = T_0(\phi^x) < +\infty.$$

$\square$

For x in a neighborhood of the origin, the settling time of a solution of (4) and the settling time with respect to the initial conditions of the system (4) are equal and

$$T_0(x) = \int_x^0 \frac{dz}{f(z)}.$$

If the system (4) is globally defined and if conditions (5) and (6) hold globally, then the origin is globally finite time stable. Moreover, it is obvious that for autonomous systems, uniform finite time stability is finite time stability.

Example 3.2 Let $a \in]0, 1[$ and consider system (1). Obviously $-x\varphi_a(x) < 0$ for $x \neq 0$, and let $x \in \mathbb{R}$ then

$$\int_x^0 \frac{dz}{-|z|^a \, sgn(z)} = \frac{|x|^{1-a}}{1-a} < +\infty.$$

The assumptions of Lemma 3.1 are satisfied. Thus the origin is uniformly finite time stable and the solutions $\phi^x(\tau)$ tend to the origin with the settling time $T_0(x) = \frac{|x|^{1-a}}{1-a}$. These conclusions were directly obtained in the introduction by explicit computation of the solutions (2).

4   General case

For the more general systems described by Equation (3), a natural extension will invoke the use of Lyapunov functions to give sufficient or necessary conditions for finite time stability.

4.1   *Sufficient condition using smooth Lyapunov function*

In this section, we extend a result coming from Haimo (1986) to non autonomous systems. In the following, one needs the existence of a Lyapunov function $V : \mathbb{R}_{\geq} \times \mathcal{V} \to \mathbb{R}_{\geq}$ and a continuous function $r : \mathbb{R}_{\geq} \to \mathbb{R}_{\geq}$ such that $r(0) = 0$ satisfying the following differential inequality

$$\dot{V}(t, x) \leq -r(V(t, x)) \tag{7}$$

for all $(t, x) \in I \times \mathcal{V}$.

Since the use of a Lyapunov function will lead to some scalar differential inequality, the following proposition will give sufficient condition for finite time stability: the existence of a Lyapunov function satisfying the condition (7).

PROPOSITION 4.1 *Let the origin be an equilibrium point for the system (3) where f is continuous.*

  i) *If there exists a continuously differentiable Lyapunov function satisfying condition (7) with a positive definite continuous function $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for some $\epsilon > 0$*

$$\int_0^\epsilon \frac{dz}{r(z)} < +\infty \tag{8}$$

   then the origin (3) is finite time stable for the system (3).
 ii) *If in addition to i), V is decrescent, then the origin of the system (3) is uniformly finite time stable for the system (3).*
iii) *If in addition to i), the system (3) is globally defined and V is radially unbounded, then the origin of the system (3) is globally finite time stable for the system (3).*

Proof i) Since $V : I \times \mathcal{V} \to \mathbb{R}_{\geq 0}$ is a Lyapunov function (thus satisfies L1 and L2), then the Lyapunov Theorem tells us that the origin is asymptotically stable. Let $\phi_t^x(\tau)$ be a solution of (3) which tends to the origin with the settling time $T_0(\phi_t^x(\tau))$ $(0 \leq T_0(\phi_t^x(\tau)) \leq +\infty$: from the attractivity of the origin). We have to prove that $T_0(t, x) < +\infty$. By using the asymptotic stability definition, x can be chosen small enough to ensure that $\phi_t^x(\tau) \in \mathcal{V}$ for $\tau \geq t$ and $\tau \mapsto V(\tau, \phi_t^x(\tau))$ strictly decreases for $\tau \geq t$. By using the change of variables: $[t, t + T_0(\phi_t^x)] \to [0, V(t, x)]$ given by $z = V(\tau, \phi_t^x(\tau))$, one obtains

$$\int_{V(t,x)}^0 \frac{dz}{-r(z)} = \int_t^{t+T_0(\phi_t^x)} \frac{\dot{V}(\tau, \phi_t^x(\tau))}{-r(V(\tau, \phi_t^x(\tau)))} d\tau.$$

By assumption, $\dot{V}(\tau, \phi_t^x(\tau)) \leq -r(V(\tau, \phi_t^x(\tau))) \leq 0$ for all $\tau \geq t$. This shows that

$$T_0(\phi_t^x) = \int_t^{t+T_0(\phi_t^x)} d\tau \leq \int_0^{V(t,x)} \frac{dz}{r(z)}.$$

This implies that $T_0(\phi_t^x) < +\infty$. As $\int_0^{V(t,x)} \frac{dz}{r(z)}$ is independent of ϕ_t^x, $T_0(t, x) < +\infty$. Thus, the origin of the system (3) is finite time stable.

ii) If V is decrescent, then the system is uniformly asymptotically stable. Moreover, there exists a

$\mathcal{K}-$function β such that $V(t,x) \leq \beta\left(\|x\|\right)$. So

$$T_0(t,x) \leq \int_0^{\beta(\|x\|)} \frac{dz}{r(z)} = \alpha\left(\|x\|\right)$$

with α positive definite.

$iii)$ If V is radially unbounded, the system is globally asymptotically stable. Then, for all x in $\mathbb{R}^n$ all the functions $\tau \mapsto V(\tau, \phi_t^x(\tau))$ decrease. Thus the system is globally finite time stable. $\square$

Remark 1 The settling time with respect to initial conditions of the system (3) satisfies the following inequality

$$T_0(t,x) \leq \int_0^{V(t,x)} \frac{dz}{r(z)}.$$

so it is continuous at the origin.

Example 4.2 Scalar non autonomous systemConsider the following system:

$$\dot{x} = -(1+t)\varphi_a(x), \quad t \geq 0, x \in \mathbb{R}. \tag{9}$$

The function $V(x) = x^2$ is a decrescent Lyapunov function for (9) and

$$\dot{V}(t,x) = -2(1+t)x\varphi_a(x) \leq -2\varphi_{\frac{a+1}{2}}(x^2) = -r(V(x)).$$

Since $\int_0^\epsilon \frac{dz}{r(z)} < +\infty$, the origin is uniformly finite time stable with the settling time with respect to initial conditions of the system

$$T_0(t,x) \leq \frac{4\left|x\right|^{1-a}}{1-a}.$$

Example 4.3 two dimensional systemConsider $a \in]0,1[$ and the system:

$$\begin{cases} \dot{x}_1 = -\varphi_a(x_1) - x_1^3 + x_2 \\ \dot{x}_2 = -\varphi_a(x_2) - x_2^3 - x_1. \end{cases}$$

Taking $V(x) = \frac{\|x\|^2}{2}$, we obtain

$$\dot{V}(x_1,x_2) = -\sum_{i=1}^2 (x_i^4 + |x_i|^{a+1}) \leq 0.$$

V is a Lyapunov function for the system. Moreover, if $r(z) = \varphi_{\frac{a+1}{2}}(z)$, then

$$\dot{V}(x_1,x_2) \leq -r\left(V\left(x_1,x_2\right)\right).$$

In fact

$$\sum_{i=1}^2 (x_i^4 + |x_i|^{a+1}) \geq \left(x_1^2 + x_2^2\right)^{\frac{a+1}{2}} = \|x\|^{a+1}.$$

Thus the origin is uniformly finite time stable with

$$T_0(t,x) \leq 2^{\frac{1+a}{2}} \frac{\|x\|^{1-a}}{1-a}.$$

In order to test condition (8) and to conclude to the finite time stability, first one must have the existence of a pair (V, r) satisfying condition (7). For this, it is enough that there exists a decrescent Lyapunov function for the system (3). Indeed, if V is decrescent, then there exists a class $\mathcal{K}$-function α such that

$$V(t,x) \leq \alpha(\|x\|)$$

for all $(t,x) \in \mathbb{R}_{\geq 0} \times \mathcal{V}$. As $\dot{V}$ is negative definite, there exists a class $\mathcal{K}$-function β such that

$$-\dot{V}(t,x) \geq \beta(\|x\|)$$

for all $(t,x) \in \mathbb{R}_{\geq 0} \times \mathcal{V}$. Combining both results, one obtains that

$$\dot{V}(t,x) \leq -\beta\left(\alpha^{-1}(V(t,x))\right)$$

for all $(t,x) \in \mathbb{R}_{\geq 0} \times \mathcal{V}$. Note that this condition does not imply that the constructed pair (V, r) will be a good candidate for condition (8).

4.2   *Sufficient condition using nonsmooth Lyapunov function*

The question is to know if it is possible to use a continuous only Lyapunov function to show finite time stability. This is possible by adding the condition that r is locally Lipschitz. We shall use the comparison lemma which can be found in (Khalil 1996, Lemma 3.4):

LEMMA 4.4 Comparison lemma*If the scalar differential equation*

$$\dot{x} = f(x), \quad x \in \mathbb{R},$$

has a global semi-flow $\Phi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$, *where f is continuous, and if $g : [a,b] \to \mathbb{R}$ (b could be infinity) is a continuous function such that for all $t \in [a,b)$,*

$$D^+ g(t) \leq f(g(t))$$

then $g(t) \leq \Phi(t, g(a))$ for all $t \in [a,b)$.

Now, one may give a proposition which generalizes a result given in Bhat and Bernstein (2000) to continuous non autonomous systems.

PROPOSITION 4.5 *Let the origin be an equilibrium point for the system (3) where f is continuous. If there exists a continuous Lyapunov function for the system (3) satisfying condition (7) with a positive definite continuous function $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ locally Lipschitz outside the origin such that $\int_0^\epsilon \frac{dz}{r(z)} < +\infty$ with $\epsilon > 0$, then the origin is finite time stable. Moreover, the settling time with respect to initial conditions of the system (3) satisfies*

$$T_0(t,x) \leq \int_0^{V(t,x)} \frac{dz}{r(z)}.$$

Proof Since $V : \mathbb{R}_{\geq 0} \times \mathcal{V} \to \mathbb{R}_{\geq 0}$ is Lyapunov function of the system (3) satisfying condition (7), then from the Lyapunov's theorem, the origin is asymptotically stable. Let $x_0 \in \mathcal{V}$ and $\phi_t^x(\tau)$ be a solution

of (3) which tends to the origin with the settling time $T_0(\phi_t^x)$. It remains to prove that $T_0(t,x) < +\infty$. Because of asymptotic stability, one may suppose with no loss of generality that $\phi_t^x(\tau) \in \mathcal{V}$ for $t \geq 0$ and $V(t,x) \in [0,\epsilon]$. Let us consider the system

$$\dot{z} = -r(z), \quad z \geq 0,$$

with the global semi flow $\Phi(\tau, z)$ for $z \geq 0$. Now, applying the comparison lemma (4.4), one deduces that

$$V(\tau, \phi_t^x(\tau)) \leq \Phi(\tau, V(t,x)), \quad \tau \geq 0, x \in \mathcal{V} \setminus \{0\}.$$

From Lemma 3.1, one knows that

$$\Phi(\tau, z) = 0 \text{ for } \tau \geq \int_0^{V(t,x)} \frac{dz}{r(z)}.$$

With the positive definiteness of V, one concludes that

$$\phi_t^x(\tau) = 0 \text{ for } \tau \geq \int_0^{V(t,x)} \frac{dz}{r(z)}.$$

Moreover, $\int_0^{V(t,x)} \frac{dz}{r(z)}$ is independent of ϕ_t^x, this shows that $T_0(t,x) < +\infty$. Thus, the origin of the system (3) is finite time stable. $\square$

One wants to give an example using a continuous only Lyapunov function.

Example 4.6 Let $0 < a < 1$, if we consider the simplest system

$$\dot{x} = -x^a sgn(x), \quad x \in \mathbb{R},$$

and the continuous Lyapunov function $V(x) = |x|$, we have

$$\dot{V}(x) = -|x|^a = -V(x)^a.$$

So the system is finite time stable.

Let us sum up the two previous results: in order to obtain the finite time stability, we have the choice between a pair (V_1, r_1) with V_1 which is continuously differentiable and r_1 only continuous, or a pair (V_2, r_2) with V_2 which is continuous and r_2 locally Lipschitz. Moreover, the two conditions are not equivalent because there is no converse theorem for general non autonomous continuous systems.

4.3   *Necessary conditions*

For the moment, there is no necessary and sufficient condition for finite time stability of general continuous (even autonomous) systems. The only converse theorem appears in Bhat and Bernstein (2000) for autonomous systems with uniqueness of solutions in forward time and when the settling time is continuous at the origin. So, one proposes a sufficient condition for non autonomous continuous systems. In the following, one needs the existence of a Lyapunov function $V : \mathbb{R}_\geq \times \mathcal{V} \to \mathbb{R}_\geq$ and a continuous function $s : \mathbb{R}_\geq \to \mathbb{R}_\geq$ such that $s(0) = 0$ satisfying the following differential inequality

$$\dot{V}(t,x) \geq -s(V(t,x)) \tag{10}$$

for all $(t,x) \in \mathbb{R}_{\geq 0} \times \mathcal{V}$. Then, the following proposition gives necessary conditions for finite time stability: the existence of such a pair (V,s) such that the differential inequality (10) holds.

PROPOSITION 4.7 *Let the origin be an equilibrium point for the system (3) where f is continuous. If the origin is weakly finite time stable for the system (3) then for all Lyapunov functions for the system (3) satisfying condition (10) with a continuous positive definite function $s : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, there exists $\epsilon > 0$ such that*

$$\int_0^\epsilon \frac{dz}{s(z)} < +\infty. \tag{11}$$

Proof Suppose that $V : I \times \mathcal{V} \to \mathbb{R}_{\geq 0}$ is a Lyapunov function satisfying condition 10. Let $\phi_t^x(\tau)$ be a solution of (3) with the settling time $0 \leq T_0(\phi_t^x) < +\infty$. Because of the asymptotic stability, one may choose x small enough to ensure that $\phi_t^x(\tau) \in \mathcal{V}$ for all $\tau \geq t$ and $\tau \mapsto V(\tau, \phi_t^x(\tau))$ strictly decreases for $\tau \geq t$. By using the change of variables, $[t, t+T_0(\phi_t^x)] \to [0, V(t,x)]$ given by $z = V(\tau, \phi_t^x(\tau))$, one obtains

$$\int_{V(t,x)}^0 \frac{dz}{-s(z)} = \int_t^{t+T_0(\phi_t^x)} \frac{\dot{V}(\tau, \phi_t^x(\tau))}{-s(V(\tau, \phi_t^x(\tau)))} d\tau.$$

Since $\dot{V}(\tau, \phi_t^x(\tau)) \geq -s(V(\tau, \phi_t^x(\tau)))$ and $-s(V(\tau, \phi_t^x(\tau))) < 0$ for all $\tau \geq t$ one obtains

$$\int_0^{V(t,x)} \frac{dz}{s(z)} \leq \int_t^{t+T_0(\phi_t^x)} d\tau = T_0(\phi_t^x) < +\infty.$$

$\square$

Remark 2 This condition may be used to conclude to the non weakly finite time stability and in particular to the non finite time stability for some systems.

Example 4.8 Consider the following system:

$$\dot{x} = \frac{-|x|}{1 + g(t)}$$

where $t \in \mathbb{R}$ and g is a positive function bounded below by $c > 0$. The function $V(x) = \frac{x^2}{2}$ is a Lyapunov function for the system and

$$-\dot{V}(t,x) \leq \frac{x^2}{1+c} = s\left(\frac{x^2}{2}\right)$$

with $s(z) = \frac{2z}{1+c}$. Since $\int_0^x \frac{dz}{s(z)} = +\infty$ for all $x > 0$, the origin is not finite time stable.

To test condition (11) and to conclude to the non finite time stability, one must have the existence of a Lyapunov function satisfying condition (10). A sufficient condition to obtain condition (10) is that there exists a Lyapunov function for the system (3) such that $-\dot{V}$ is decrescent.

There exists a gap between the sufficient and the necessary conditions. If $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a positive definite function such that $\dot{V} = -r(V)$ for the system (3) then there exists a necessary and sufficient condition for finite time stability:

- if $\int_0^\epsilon \frac{dz}{r(z)} < +\infty$ then the origin is finite time stable;
- if $\int_0^\epsilon \frac{dz}{r(z)} = +\infty$ then the origin is not finite time stable.

5   Application

In general, one uses for $r(z)$ the following function $r(z) = \varphi_a(z)$ with $0 < a < 1$. In the following example, one has forced to change the form for $r(z)$ in order to show the finite time stability. Let the function u be defined on $]0,1]$ by its graph given in figure 1 and such that $u\left(\frac{1}{n}\right) = n^2$ for $n \geq 1$ and $0 < a < 1$.
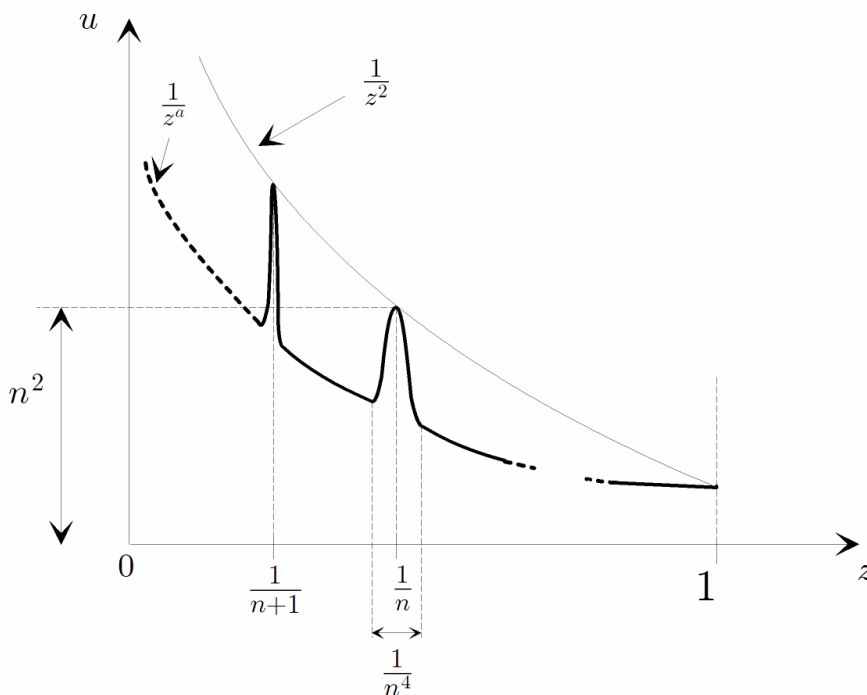

Figure 1. graph of $u(z)$

Let

$$r(z) = \begin{cases} 0 & \text{if } z = 0 \\ \frac{1}{u(z)} & \text{if } z \neq 0 \end{cases}$$

then r is a continuous positive definite function and

$$\int_0^1 \frac{dz}{r(z)} = \int_0^1 u(z)\, dz \leq \int_0^1 \frac{dz}{z^a} + \sum_{n=2}^{\infty} \frac{1}{n^2} < +\infty.$$

Consider the continuous function

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{r(x^2)}{2x} & \text{if } x \neq 0 \end{cases}$$

As

$$f\left(\frac{1}{n}\right) \leq \frac{-1}{\frac{2}{n} u\left(\frac{1}{n^2}\right)} \leq \frac{1}{2n^3},$$

f is continuous at the origin and thus on $\mathbb{R}$. Consider the system

$$\dot{x} = f(x), \quad x \in \mathbb{R}.$$

$V(x) = x^2$ is a Lyapunov function for the system and

$$\dot{V}(x) \leq -r(V(x)).$$

So, the system is finite time stable. Nevertheless, there is no function φ_b with $0 < b < 1$ such that r is bounded below by φ_b.

6   Conclusion

A necessary and a sufficient condition for finite time stability of non autonomous continuous systems are given. As mentioned, there is still a gap to obtain necessary and sufficient conditions. The main difficulty comes from the non existence of the flow for only continuous systems and the non continuity of the settling time. Thus it is not an easy task to prove the existence of a Lyapunov function satisfying condition 7 under the hypothesis of finite time stability of the origin. However, with the given sufficient conditions, it is possible to investigate the problem of finite time stabilization for general continuous and non autonomous systems.

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References

S.P. Bhat and D.S. Bernstein, "Continuous Finite-Time Stabilization of the Translational and Rotational Double Integrator", *IEEE Trans. Automat. Control*, 43(5), pp. 678–682, 1998.

S.P. Bhat and D.S. Bernstein, "Finite Time Stability of Continuous Autonomous Systems", *SIAM J. Control Optim.*, 38(3), pp. 751–766, 2000.

W. Hahn, *Theory and Application of Liapunov's Direct Method*, n.J., Prentice-Hall inc., 1963.

V.T. Haimo, "Finite Time Controllers", *SIAM J. Control Optim.*, 24(4), pp. 760–770, 1986.

J.K. Hale, *Ordinary Differential Equations, 2nd Edition*, Pure and applied mathematics XXI, Krieger, 1980.

Y. Hong, "Finite-Time Stabilization and Stabilizability of a Class of Controllable Systems", *Systems Control Lett.*, 46, pp. 231–236, 2002.

Y. Hong, J. Huang and Y. Xu, "On an Output Feedback Finite-Time Stabilization Problem", *IEEE Trans. Automat. Control*, 46, pp. 305–309, 2001.

Y. Hong, Y. Xu and J. Huang, "Finite-Time Control for Robot Manipulators", *Systems Control Lett.*, 46, pp. 243–253, 2002.

H.K. Khalil, *Nonlinear Systems*, NJ 07458, Upper Saddle River: Prentice-Hall, 1996.

A.M. Lyapunov, "Stability of Motion: General Problem", *Internat. J. Control*, 55(3), pp. 520–790, lyapunov Centenary issue, 1992.

E. Moulay and W. Perruquetti, "Finite Time Stability of Non Linear Systems", in *IEEE Conference on Decision and Control*, Hawaii, USA, 2003, pp. 3641–3646.

E. Moulay and W. Perruquetti, "Finite Time Stability and Stabilization of a Class of Continuous Systems", *J. Math. Anal. Appl.*, 323(2), pp. 1430–1443, 2006a.

E. Moulay and W. Perruquetti, *Finite-time stability and stabilization: state of the art*, in Advances in Variable Structure and Sliding Mode Control, 334, Lecture Notes in Control and Information Sciences, Springer-Verlag, 2006b.

W. Perruquetti and S. Drakunov, "Finite Time Stability and Stabilisation", in *IEEE Conference on Decision and Control*, Sydney, Australia, 2000.

E. Ryan, "Singular Optimal Controls for Second-Order Saturating Systems", *Internat. J. Control*, 3(4), pp. 549–564, 1979.

V.I. Utkin, *Sliding Modes in Control Optimization*, Berlin: Springer-Verlag, 1992.