On Wiener type filters in SPECT

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Abstract

For 2D data with Poisson noise we give explicit formulas for the optimal space-invariant Wiener type filter with some a priori geometric restrictions on the window function. We show that, under some natural geometric condition, this restrictedly optimal Wiener type filter admits a very efficient approximation by an approximately optimal filter with unknown object power spectrum. Generalizations to the case of some more general noise model are also given. Proceeding from these results we (a) explain, in particular, an efficiency of some well-known "1D" approximately optimal space-invariant Wiener type filtering scheme with unknown object power spectrum in single-photon emission computed tomography (SPECT) and positron emission tomography (PET) imaging based on the classical FBP algorithm or its iterative use and (b) propose also an efficient 2D approximately optimal space-invariant Wiener type filter with unknown object power spectrum for SPECT imaging based on the generalized FBP algorithm (implementing the explicit formula for the nonuniform attenuation correction) and/or the classical FBP algorithm (used iteratively). An efficient space-variant version of the latter 2D filter is also announced. Numerical examples illustrating the aforementioned results in the framework of simulated SPECT imaging are given.

1. Introduction

In the single-photon emission computed tomography (SPECT) one considers a body containing radioactive isotopes emitting photons. The emission data in SPECT consist in the radiation measured outside the body by a family of detectors during some fixed time. The basic problem of SPECT consists in finding the distribution of these isotopes in the body from the emission data and some a priori information concerning the body. Usually this a priori information consists in the photon attenuation coefficient in the points of body, where this coefficient is found in advance by the methods of the transmission computed tomography. Under some conditions, this attenuation coefficient can be also approximately found directly from the emission data in the frameworks of the "identification" problem. In 2D SPECT, that is when the problem is restricted to a fixed two-dimensional plane Ξ intersecting the body and identified with $\mathbb{R}^2$, the emission data are modeled, in some approximation, as 2D attenuated ray transform with Poisson noise (or, more precisely, as a function p of formula (1.4) given below). Let us remind now related mathematical definitions.

The 2D attenuated ray transformation P_a is defined by the formula

$$P_af(\gamma) = \int\limits_{\mathbb{R}} \exp\left(-\mathcal{D}a(s\theta^\perp + t\theta, \theta)\right) f(s\theta^\perp + t\theta)dt, \tag{1.1a}$$
$$\gamma = (s,\theta) \in \mathbb{R}\times\mathbb{S}^1, \quad \theta^\perp = (-\theta_2, \theta_1) \ \text{ for } \ \theta = (\theta_1, \theta_2) \in \mathbb{S}^1,$$

$$\mathcal{D}a(x,\theta) = \int\limits_0^{+\infty} a(x + t\theta)dt, \quad (x,\theta) \in \mathbb{R}^2 \times \mathbb{S}^1, \tag{1.1b}$$

where a and f are real-valued, sufficiently regular functions on $\mathbb{R}^2$ with sufficient decay at infinity, a is a parameter (the attenuation coefficient), $\mathcal{D}a$ is the divergent beam transform of a, f is a test function. In (1.1a) we interpret $\mathbb{R}\times\mathbb{S}^1$ as the set of all oriented straight lines in $\mathbb{R}^2$. If $\gamma = (s,\theta) \in \mathbb{R}\times\mathbb{S}^1$, then $\gamma = \{x \in \mathbb{R}^2 : \ x = s\theta^\perp + t\theta, \ t \in \mathbb{R}\}$ (modulo orientation) and θ gives the orientation of γ.

In SPECT, $f \geq 0$ is the density of radioactive isotopes, $a \geq 0$ is the linear photon attenuation coefficient of the medium, and, in some approximation, CP_af is the expected emission data (the expected sinogram), where C is a positive constant depending on detection parameters.

More precisely, saying about the emission data in 2D SPECT, we assume that

$$a(x) \geq 0, \ f(x) \geq 0, \ \text{ for } \ x \in \mathbb{R}^2, \ a(x) \equiv 0, \ \ f(x) \equiv 0 \ \text{ for } \ |x| \geq R \tag{1.2}$$

and consider in $\mathbb{R}\times\mathbb{S}^1$ a discrete subset of the form

$$\begin{aligned}&\Gamma = \{\gamma_{i,j} = (s_i, \theta(\varphi_j)) : \ s_i = -R + (i-1)\Delta s, \ \varphi_j = (j-1)\Delta\varphi,\\ &\Delta s = 2R/(n_s - 1), \ \Delta\varphi = 2\pi/n_\varphi, \ \ i = 1,\dots,n_s, \ j = 1,\dots,n_\varphi\},\end{aligned} \tag{1.3}$$

where $\theta(\varphi) = (\cos\varphi, \sin\varphi)$, R is the radius of image support of (1.2), n_s, n_φ are sufficiently large natural numbers, and n_φ is even. We say that Γ is a detector set. Note that $\Gamma \subset \{(s,\theta) \in \mathbb{R}\times\mathbb{S}^1 : \ |s| \leq R\}$, where R is the number of (1.2).

In 2D SPECT, in some approximation, the emission data consist of a function p on Γ, where

$$\begin{aligned}&p(\gamma) \ \text{ is a realization of a Poisson variate } \ \mathbf{p}(\gamma)\\ &\text{with the mean } \ M\mathbf{p}(\gamma) = g(\gamma) = CP_af(\gamma) \ \text{ for any } \ \gamma \in \Gamma\\ &\text{and all } \ \mathbf{p}(\gamma), \ \ \gamma \in \Gamma, \ \text{ are independent.}\end{aligned} \tag{1.4}$$

In addition, it is assumed that $C = C_1 t$, where t is the detection time per projection and C_1 is independent of t. We say that p of (1.4) is the 2D attenuated ray transform (CP_af on Γ) with Poisson noise.

For more information concerning the aforementioned basic points of SPECT, see, for example, [NW], [LM], [Br] and references therein.

In the present work we consider the following two problems:

Problem 1.1. Find (as well as possible) g from p, where g and p are the function of (1.4).

Problem 1.2. Find (as well as possible) Cf from the p and a, where f, a and p are the function of (1.2), (1.4) and C is the constant of (1.4).

More precisely, in the present work we develop space-invariant Wiener type filtering approach (of [KDS]) for solving Problem 1.1 and apply this approach to solving Problem 1.2 in the framework of the scheme

$$Cf \approx P_a^{-1}\mathcal{W}p, \tag{1.5}$$

where $\mathcal{W}$ is a filter for solving Problem 1.1 and P_a^{-1} is an inversion method for P_a for the noiseless case.

The main theoretical results of the present work can be summarized as follows:

I. For the noise model (3.4) including (1.4), as a particular case, we give explicit formulas (3.7) for the optimal space-invariant Wiener type filter with a priori geometric restrictions (3.6) on the window function. For the Poisson case, these formulas are completed also by (3.21). These results are given as Theorem 3.1 and Corollary 3.1 of Section 3. We say that the filter of Theorem 3.1 is restrictedly optimal in the Wiener sense and denote it as $\mathcal{W}^{r.o.}$. It is assumed that the object power spectrum $|\hat{g}|^2$ and the variance parameter $V = (n_s n_\varphi)^{-1/2}\widehat{D\mathbf{p}}(0)$ are known in this filter.

II. For the case when $|\hat{g}|^2$ and V are not known, we approximate $\mathcal{W}^{r.o.}$ as $\mathcal{A}$ by formulas (4.7)-(4.9), (4.11). For the Poisson case, where V is not an additional parameter to $|\hat{g}|^2$ (see (3.21)), these formulas are completed also by (4.12). We show that, at least for the Poisson case, $\mathcal{A}$ is a very efficient approximation to $\mathcal{W}^{r.o.}$ if geometric condition (4.13) is fulfilled for each $j \in \hat{I}$. Moreover, for the Poisson case with unknown $|\hat{g}|^2$, we consider the approximately optimal filter $\mathcal{A}$ with adequate level sets S_α of (4.9) as a reasonable approximation to "fully" optimal filter $\mathcal{W}^{opt}$ of (3.3). See Section 4 for details.

III. We show that in an important particular case, under the Poisson assumptions, our filter $\mathcal{A}$ is reduced to the well-known "one-dimensional" filter $\mathcal{A}^{1d}$ going back to [KDS]. This permits to explain a relative efficiency of the "1D" filtering scheme of [KDS] in SPECT and PET imaging based on the classical FBP algorithm (or its iterative use). Besides, by the symmetric choice (4.15), (4.16) of the level sets S_α, we reduce $\mathcal{A}$ to $\mathcal{A}^{sym}$. We consider $\mathcal{A}^{sym}$ as an efficient "2D" approximation to $\mathcal{W}^{opt}$ of (3.3) for the Poisson model (3.1) with sufficiently regular g for the case when $|\hat{g}|^2$ is unknown. (We do not know whether the filter $\mathcal{A}^{sym}$ in its precise form of Section 4 was mentioned in the literature.) See Section 4 for details.

IV. An efficient space-variant version $\mathcal{A}^{sym}_{l_1,l_2}$ of the space-invariant filter $\mathcal{A}^{sym}$ is announced in Section 5 (in Subsection 5.3).

The aforementioned theoretical results were developed in the framework of applications to Problems 1.1 and 1.2. However, these results contribute to the general theory of filters of the Wiener type and, therefore, are not limited by particular tomographical applications considered in the present work.

Actually, in the present work, as P_a^{-1} of (1.5) we use the explicit formula of [No] and the iterative method of [MNOY]. Related results are reminded in Section 6.

As characteristics of filter efficiency we consider, in particular, the numbers describing image error and image bias. In the framework of the reconstruction (1.5) these numbers depend also on P_a^{-1}. Related definitions are reminded in Section 7.

Numerical examples illustrating the aforementioned results on Problems 1.1 and 1.2 are given in Section 8. In these examples we consider a version of the well-known elliptical chest phantom used for numerical simulations of cardiac SPECT imaging. One can see, in particular, that in these examples the symmetric 2D approximately optimal space-invariant filter $\mathcal{A}^{sym}$ of the present work (see Subsection 4.4) is more efficient than the space-invariant filters $\mathcal{A}^{simp}$, $\mathcal{A}^{1d}$, Φ_1, where $\mathcal{A}^{simp}$ is the simplest approximation to $\mathcal{W}^{opt}$ (see Subsection 4.1) , $\mathcal{A}^{1d}$ is the filter of [KDS] (see Subsection 4.3) and Φ_1 is the filter of [GN1] (see Subsection 5.1).

Finally, it should be mentioned also that the noise level in the emission data p of (1.4) is not space-invariant and in this respect all space-invariant filtering schemes are not optimal for Problems 1.1 and 1.2. Space-variant versions of the space-invariant data dependent filtering of [GN1] are constructed in [GN2] (see, in particular, Subsections 5.1, 5.2 of the present paper) . Space-variant versions of the space-invariant Wiener type filters considered in the present work are constructed and analyzed in [GN3]. In addition, our simplest space-variant version $\mathcal{A}^{sym}_{l_1,l_2}$ of the space-invariant Wiener type filter $\mathcal{A}^{sym}$ is already mentioned and illustrated numerically in Subsections 5.3 and 8.4. In particular, our best (iterative) reconstruction Cf_3 (of (8.11)) is obtained using namely $\mathcal{A}^{sym}_{l_1,l_2}$ (for $l_1 = l_2 = 8$). To our knowledge no complete generalization to the space-variant case of the filtering approach of [KDS] was presented in the literature before the present work.

2. Frequency domain form of space invariant filters

Consider the functions p and g of (1.4). Suppose that

$$g(s_i, \theta(\varphi_j)) \equiv 0, \quad \text{if} \quad ||s_i| - R| < L, \tag{2.1}$$

where $\Delta s << L$, where s_i, φ_j, R and Δs are the numbers of (1.3). This condition can always be satisfied by zero-padding the data. Then p and g of (1.4) can be considered as functions on a discrete torus identified with Γ. Note that Γ of (1.3) can be identified with

$$I = \{(i_1, i_2) \in \mathbb{Z}^2 : \ 0 \le i_1 \le n_s - 1, \ \ 0 \le i_2 \le n_\varphi - 1\}. \tag{2.2}$$

Let us suppose that n_φ and n_s of (1.3), (2.2) are even. Let

$$\hat{I} = \{(j_1, j_2) \in \mathbb{Z}^2 : \ -\frac{n_s}{2} \le j_1 \le \frac{n_s}{2} - 1, \ \ -\frac{n_\varphi}{2} \le j_2 \le \frac{n_\varphi}{2} - 1\}. \tag{2.3}$$

Let

$$\|q\|_{L^\alpha(\Gamma')} = (\Delta s \Delta \varphi \sum_{\gamma \in \Gamma'} |q(\gamma)|^\alpha)^{1/\alpha}, \tag{2.4}$$

$$\begin{aligned} \|u\|_{L^\alpha(I')} &= (\sum_{(i_1,i_2)\in I'} |u(i_1, i_2)|^\alpha)^{1/\alpha}, \\ \|\hat{u}\|_{L^\alpha(\hat{I}')} &= (\sum_{(j_1,j_2)\in \hat{I}'} |\hat{u}(j_1, j_2)|^\alpha)^{1/\alpha}, \end{aligned} \tag{2.5}$$

where q, u, $\hat{u}$ are test functions on $\Gamma' \subseteq \Gamma$, $I' \subseteq I$, $\hat{I}' \subseteq \hat{I}$, respectively, $\alpha \in \mathbb{N}$.

Let F denote the 2D discrete Fourier transformation

$$\begin{aligned}
&F:\ L^2(I)\to L^2(\hat I),\ (Fu)(j_1,j_2)=\\
&\frac{1}{\sqrt{n_s n_\varphi}}\sum_{(i_1,i_2)\in I} u(i_1,i_2)\times\\
&\exp\left(-2\pi i\left(\frac{j_1 i_1}{n_s}+\frac{j_2 i_2}{n_\varphi}\right)\right),\quad (j_1,j_2)\in\hat I,\quad i=\sqrt{-1},
\end{aligned}\tag{2.6}$$

where u is a test function on I.

To use $F:\ L^2(I)\to L^2(\hat I)$ and $F^{-1}:\ L^2(\hat I)\to L^2(I)$ for filtering p of (1.4) we use also, in particular, the identification operators

$$\Lambda:\ L^2(\Gamma)\to L^2(I),\quad (\Lambda q)(i_1,i_2)=q(\gamma_{i_1,i_2}),\quad (i_1,i_2)\in I,\tag{2.7}$$
$$\Lambda^{-1}:\ L^2(I)\to L^2(\Gamma),\quad (\Lambda u)(\gamma_{i_1,i_2})=u(i_1,i_2),\quad (i_1,i_2)\in I,\tag{2.8}$$

where $\gamma_{i,j}$ is defined in (1.3), q and u are test functions on Γ and I, respectively.

A general linear space invariant filter in $L^2(\Gamma)$, where Γ is considered as a discrete torus, can be written in the form

$$\mathcal{W}:\ L^2(\Gamma)\to L^2(\Gamma),\quad \mathcal{W}=\Lambda^{-1}W\Lambda,\tag{2.9}$$

where

$$W:\ L^2(I)\to L^2(I),\quad W=F^{-1}\hat W F,\tag{2.10}$$
$$\hat W:\ L^2(\hat I)\to L^2(\hat I),\quad (\hat W\hat u)(j)=\hat W(j)\hat u(j),\quad j=(j_1,j_2)\in\hat I,\tag{2.11}$$

where F, Λ, Λ^{-1} are defined in (2.6)-(2.8),

$$\hat W(j)\ \text{ is a real bounded function of }\ j\in\hat I,\tag{2.12a}$$

$$\begin{aligned}
&\hat W(-j)=\hat W(j)\ \text{ for }\ \hat W\ \text{ considered as a periodic}\\
&\text{function on }\ \mathbb{Z}^2\ \text{ with the fundamental domain }\ \hat I,
\end{aligned}\tag{2.12b}$$

and $\hat u$ is a test function. Here the multiplication operator $\hat W$ of (2.11) is the frequency domain form of the space invariant filters $\mathcal{W}$ and W of (2.9), (2.10). In addition, $\hat W(j)$ is the related window function.

Note also that in the simplest space-invariant data independent schemes for filtering p of (1.4) the window function $\hat W$ of (2.12) is given by

$$\hat W(j)=\hat w_1\Big(\frac{2j_1}{\omega_1 n_s}\Big)\hat w_2\Big(\frac{2j_2}{\omega_2 n_\varphi}\Big),\quad j=(j_1,j_2)\in\hat I,\quad \omega_1>0,\ \omega_2>0,\tag{2.13}$$

where $\hat w_1(k)$, $\hat w_2(k)$ are real-valued functions of k such that

$$\begin{aligned}
&\hat w_i(k)=\hat w_i(-k),\quad k\in\mathbb{R},\\
&\lim_{k\to 0}\hat w_i(k)=\hat w_i(0)=1,\quad \hat w_i(k)\equiv 0\ \text{ for }\ |k|>1,\\
&\hat w_i(k_1)\ge\hat w_i(k_2)\ \text{ for }\ |k_1|\le|k_2|,
\end{aligned}\tag{2.14}$$

where $i \in \{1,2\}$. Here $\vec{\omega} = (\omega_1, \omega_2)$ is a filter parameter (and it is usually assumed that $0 < \omega_i \le 1$, $i \in \{1,2\}$).

3. Optimal Wiener filter and its restrictedly optimal analogs

Suppose that:

$$g \text{ is some nonnegative function on } \Gamma \text{ (and } g \not\equiv 0), \tag{3.1a}$$

$$\begin{aligned}&\mathbf{p}(\gamma) \text{ is a Poisson variate with the mean } M\mathbf{p}(\gamma) = g(\gamma),\ \gamma \in \Gamma,\\ &\text{and all } \mathbf{p}(\gamma),\ \gamma \in \Gamma, \text{ are independent,}\end{aligned} \tag{3.1b}$$

$$p \text{ is a realization of } \mathbf{p} \text{ on } \Gamma. \tag{3.1c}$$

Let $\mathcal{W}$ denote a filter of the form (2.9)-(2.12). Then it is well-known (see [GB], [KDS]) that the mean

$$\mu(\mathcal{W}, g) = M\|\mathcal{W}\mathbf{p} - g\|^2_{L^2(\Gamma)} \tag{3.2}$$

is minimal with respect to $\mathcal{W}$ if and only if the window function $\hat{W}(j)$ of (2.11), (2.12) is given by

$$\hat{W}(j) = \hat{W}^{opt}(j) \stackrel{\text{def}}{=} \frac{|\hat{g}(j)|^2}{|\hat{g}(j)|^2 + (n_s n_\varphi)^{-1/2}\hat{g}(0)},\quad j = (j_1, j_2) \in \hat{I}, \tag{3.3}$$

where $\hat{g} = F\Lambda g$ (with F and Λ defined by (2.6), (2.7)). Note that results of such a type go back to [W] and, therefore, the filter $\mathcal{W}$ for p of (3.1), where the window function $\hat{W}$ of (2.11), (2.12) is given by (3.3), is usually referred (see, for example, [KDS], [C]) as an optimal Wiener filter. This filter is denoted as $\mathcal{W}^{opt}$ in the present paper.

Note that an obvious obstacle for a direct use of the optimal Wiener filter $\mathcal{W}^{opt}$ for solving Problem 1.1 consists in the fact that the window $\hat{W}^{opt}$ of (3.3) is given in terms of g which is an unknown of Problem 1.1.

Below in this section, we generalize the "optimal" formula (3.3) to the case of some a priori geometric restrictions on the window function. In some cases such restrictions are rather natural and satisfactory and (that is the key point) result in "regularized" optimal filters which are much more appropriate for the case with unknown $|\hat{g}|$ (than the initial optimal filter with $\hat{W}$ given by (3.3)). In our results on restrictedly optimal Wiener type filters we consider also some more general noise model than (3.1). Note that applications of restrictedly optimal Wiener type filters (of this section) to Problem 1.1 involve also approximations considered in Section 4.

Suppose that:

$$g \text{ is a real function on } \Gamma \text{ (and } g \not\equiv 0), \tag{3.4a}$$

$$\begin{aligned}&\mathbf{p}(\gamma) \text{ is a real variate with the mean } M\mathbf{p}(\gamma) = g(\gamma),\ \gamma \in \Gamma,\\ &\text{and all } \mathbf{p}(\gamma),\ \gamma \in \Gamma, \text{ are independent (and } D\mathbf{p} = M(\mathbf{p} - M\mathbf{p})^2 \not\equiv 0),\end{aligned} \tag{3.4b}$$

$$p \text{ is a realization of } \mathbf{p} \text{ on } \Gamma. \tag{3.4c}$$

One can see that the noise model (3.4) is more general than (3.1).

Let $S_1,\ldots,S_{n^*}$ be subsets of $\hat{I}$ such that

$$\hat{I}=\cup_{\alpha=1}^{n^*}S_\alpha,\quad \text{each}\ \ S_\alpha\neq\emptyset,\ S_\alpha\cap S_\beta=\emptyset\ \text{ if }\ \alpha\neq\beta, \tag{3.5a}$$

$$-S_\alpha=S_{\beta[\alpha]}\ \ (\text{in }\ \mathbb{Z}^2\ \text{ factorized to } \hat{I})\ \text{ for each } S_\alpha, \tag{3.5b}$$

where $\beta[\alpha]$ denotes β depending on α, $\hat{I}$ is considered as a discrete torus and the factorization of $\mathbb{Z}^2$ to $\hat{I}$ is used because of the case when $j\in\hat{I}$ but $-j\notin\hat{I}$.

Now for the noise model (3.4) we consider the problem of finding $\mathcal{W}$ of the form (2.9)-(2.12) such that $\mu(\mathcal{W},g)$ of (3.2) is minimal for fixed g of (3.4) under the restrictions that

$$\hat{W}\ \text{ is constant on each fixed }\ S_\alpha,\quad \alpha=1,\ldots,n^*, \tag{3.6}$$

where $\hat{W}$ is the window function of $\mathcal{W}$. This problem is solved in the next Theorem:

Theorem 3.1. *Let g and $\mathbf{p}$ be defined as in* (3.4a), (3.4b). *Let $\mathcal{W}$ denote a filter of the form* (2.9)-(2.12) *with a priori restrictions* (3.6) *on its window function $\hat{W}$, where $S_1,\ldots,S_{n^*}$ satisfy* (3.5). *Then $\mu(\mathcal{W},g)$ of* (3.2) *is minimal with respect to $\mathcal{W}$ if and only if*

$$\hat{W}(j)=\hat{W}^{r.o.}(j)\stackrel{\text{def}}{=}\frac{\Sigma_{g,\alpha(j)}}{\Sigma_{g,\alpha(j)}+V},\quad j\in\hat{I}, \tag{3.7a}$$

$$\Sigma_{g,\alpha}\stackrel{\text{def}}{=}\frac{1}{|S_\alpha|}\sum_{i\in S_\alpha}|\hat{g}(i)|^2,\quad \alpha=1,\ldots,n^*,\quad V=(n_s n_\varphi)^{-1/2}\widehat{D\mathbf{p}}(0), \tag{3.7b}$$

where $\hat{g}=F\Lambda g$, $\widehat{D\mathbf{p}}=F\Lambda(\widehat{D\mathbf{p}})$ (with F and Λ defined by (2.6), (2.7)), *$|S_\alpha|$ denotes the number of elements in S_α* and *$\alpha(j)$ denotes α such that $j\in S_\alpha$.*

The filter of the form (2.9)-(2.12) with the window given by (3.7) is denoted as $\mathcal{W}^{r.o.}$ in the present paper. We say that $\mathcal{W}^{r.o.}$ is a restrictedly optimal Wiener type filter for the noise model (3.4).

Proof of Theorem 3.1. Due to (3.2),(2.9)-(2.12) and the property

$$\|\Lambda^{-1}F^{-1}\hat{u}\|^2_{L^2(\Gamma)}=\Delta s\Delta\varphi\|\hat{u}\|^2_{L^2(\hat{I})},\quad \hat{u}\in L^2(\hat{I}), \tag{3.8}$$

we have that

$$\mu(\mathcal{W},g)=\Delta s\Delta\varphi M\|\hat{W}\hat{\mathbf{p}}-\hat{g}\|^2_{L^2(\hat{I})}, \tag{3.9}$$

where $\hat{\mathbf{p}}=F\Lambda\mathbf{p}$, $\hat{g}=F\Lambda g$. Further,

$$\begin{aligned} M\|\hat{W}\hat{\mathbf{p}}-\hat{g}\|^2_{L^2(\hat{I})} &\stackrel{(2.5),(3.5a)}{=} M\sum_{\alpha=1}^{n^*}\|\hat{W}\hat{\mathbf{p}}-\hat{g}\|^2_{L^2(S_\alpha)}= \\ \sum_{\alpha=1}^{n^*}M\|\hat{W}\hat{\mathbf{p}}-\hat{g}\|^2_{L^2(S_\alpha)} &\stackrel{(3.6)}{=}\sum_{\alpha=1}^{n^*}M\|\hat{w}_\alpha\hat{\mathbf{p}}-\hat{g}\|^2_{L^2(S_\alpha)}, \end{aligned} \tag{3.10}$$

where $\hat{w}_\alpha$ are real constants such that

$$\hat{W}\equiv\hat{w}_\alpha\ \text{ on each fixed }\ S_\alpha,\quad \alpha=1,\ldots,n^*. \tag{3.11}$$

Due to (3.9)-(3.11), $\mathcal{W}$ minimizes $\mu(\mathcal{W}, g)$ (for fixed g) if and only if for each α (and fixed g) w_α minimizes

$$\hat{\mu}_\alpha(\hat{w}_\alpha, \hat{g}) \stackrel{\text{def}}{=} M\|\hat{w}_\alpha\hat{\mathbf{p}} - \hat{g}\|^2_{L^2(S_\alpha)}, \quad \alpha = 1, \ldots, n^*. \tag{3.12}$$

We have that

$$\begin{aligned}&\hat{\mu}_\alpha(\hat{w}_\alpha, \hat{g}) = M\sum_{j\in S_\alpha}(\hat{w}_\alpha\hat{\mathbf{p}}(j) - \hat{g}(j))\overline{(\hat{w}_\alpha\hat{\mathbf{p}}(j) - \hat{g}(j))} = \\ &\sum_{j\in S_\alpha} M((\hat{w}_\alpha)^2|\hat{\mathbf{p}}(j)|^2 - \hat{w}_\alpha(\hat{\mathbf{p}}(j)\overline{\hat{g}(j)} + \hat{g}(j)\overline{\hat{\mathbf{p}}(j)}) + |\hat{g}(j)|^2) = \\ &C_{g,\alpha,2}\hat{w}_\alpha^2 + C_{g,\alpha,1}\hat{w}_\alpha + C_{g,\alpha,0},\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}C_{g,\alpha,2} &= \sum_{j\in S_\alpha} M|\hat{\mathbf{p}}(j)|^2, \\ C_{g,\alpha,1} &= -\sum_{j\in S_\alpha}(\overline{\hat{g}(j)}M\hat{\mathbf{p}}(j) + \hat{g}(j)M\overline{\hat{\mathbf{p}}(j)}), \\ C_{g,\alpha,0} &= \sum_{j\in S_\alpha} M|\hat{g}(j)|^2.\end{aligned} \tag{3.14}$$

In addition, $C_{g,\alpha,2} \neq 0$ due to the assumption that $D\mathbf{p} \not\equiv 0$ and formula (3.17). Therefore, $\hat{\mu}_\alpha(\hat{w}_\alpha, \hat{g})$ is minimal with respect to $\hat{w}_\alpha$ (for fixed g and α) if and only if

$$\hat{w}_\alpha = -\frac{C_{g,\alpha,1}}{2C_{g,\alpha,2}}. \tag{3.15}$$

Formulas (3.7) follow from (3.11), (3.15), (3.14) and the formulas

$$M\hat{\mathbf{p}}(j) = \hat{g}(j), \quad j \in \hat{I}, \tag{3.16}$$

$$M|\hat{\mathbf{p}}(j)|^2 = |\hat{g}(j)|^2 + V, \quad j \in \hat{I}, \quad V = (n_s n_\varphi)^{-1/2}\widehat{D\mathbf{p}}(0). \tag{3.17}$$

Formula (3.16) is rather obvious. Formula (3.17) follows from the definition $\hat{\mathbf{p}} = F\Lambda\mathbf{p}$ (with F and Λ defined by (2.6), (2.7)) and the formulas

$$M|\xi|^2 = D\xi + |M\xi|^2, \tag{3.18}$$

$$D(c_1\xi_1 + c_2\xi_2) = |c_1|^2 D\xi_1 + |c_2|^2 D\xi_2, \tag{3.19}$$

where ξ is a complex-valued variate, $D\xi = M|\xi - M\xi|^2$, c_1 and c_2 are complex constants, ξ_1 and ξ_2 are independent complex-valued variates.

Property (2.12b) for $\hat{W}$ of (3.7a) follows from (3.5b), (3.7) and the property $|\hat{g}(-j)| = |\hat{g}(j)|$.

Theorem 3.1 is proved.

Note that for the Poisson case (3.1)

$$D\mathbf{p} = M\mathbf{p} = g \tag{3.20}$$

and, consequently,

$$V = (n_s n_\varphi)^{-1/2}\hat{g}(0), \tag{3.21}$$

where V is the number of (3.17). For the Poisson case (3.1) formula (3.17) is, actually, a formula of [GB] (see also [KDS]).

Theorem 3.1 and formula (3.21) imply the following corollary:

Corollary 3.1. *For the Poisson model* (3.1) *the mean* $\mu(\mathcal{W}, g)$ of (3.2) *is minimal with respect to* $\mathcal{W}$ *of the form* (2.9)-(2.12) *with a priori restrictions* (3.6) *if and only if* $\hat{W}$ *is given by* (3.7) *with* V *given by* (3.21).

Note that if

$$S_{\alpha(j)} = \{j\} \ \text{ for any } \ j \in \hat{I}, \tag{3.22}$$

then the window $\hat{W} = \hat{W}^{r.o.}$ of Corollary 3.1 is reduced to $\hat{W}^{opt}$ of (3.3).

Note, finally, that Theorem 3.1 and Corollary 3.1 admit straightforward generalizations to the case of any dimension (and, in particular, to the 3D case).

4. Approximations to the Wiener optimal filter and to its restrictedly optimal analogs

4.1. Simplest approximation $\mathcal{A}^{simp}$. To apply the Wiener optimal filter $\mathcal{W}^{opt}$ to Problem 1.1 one needs to express approximately the window $\hat{W}^{opt}$ of (3.2) in terms of the data p of (1.4), (3.1c). To construct such approximations one can proceed from formulas (3.16), (3.17), (3.21). In view of (3.3), (3.16), (3.17), (3.21), the simplest approximation to $\hat{W}^{opt}$ is given by (see, for example, [KDS], [C]):

$$\hat{W}^{opt}(j) \approx \hat{A}^{simp}(j), \quad j \in \hat{I}, \tag{4.1}$$

where

$$\begin{aligned} &\hat{A}^{simp}(j) = \frac{|\hat{p}(j)|^2 - (n_s n_\varphi)^{-1/2}\hat{p}(0)}{|\hat{p}(j)|^2} \ \text{ if } \ |\hat{p}(j)|^2 - (n_s n_\varphi)^{-1/2}\hat{p}(0) > 0, \\ &\hat{A}^{simp}(j) = 0 \ \text{ if } \ |\hat{p}(j)|^2 - (n_s n_\varphi)^{-1/2}\hat{p}(0) \le 0, \end{aligned} \tag{4.2}$$

where $\hat{p} = F\Lambda p$ (with F, Λ defined by (2.6), (2.7)).

The filter of the form (2.9)-(2.11) with the window given by (4.2) is denoted as $\mathcal{A}^{simp}$ in the present paper. We consider $\mathcal{A}^{simp}$ as the simplest approximation to $\mathcal{W}^{opt}$ for the case of unknown $|\hat{g}|^2$.

Note that $\hat{p}(j)$ is a good approximation to $\hat{g}(j)$, $|\hat{p}(j)|^2$ is a good approximation to $|\hat{g}(j)|^2 + (n_s n_\varphi)^{-1/2}\hat{g}(0)$ and $\hat{A}^{simp}(j)$ is a good approximation to $\hat{W}^{opt}(j)$ if

$$|\hat{g}(j)| \gg ((n_s n_\varphi)^{-1/2}\hat{g}(0))^{1/2} \ \text{ for fixed } \ j \in \hat{I}. \tag{4.3}$$

This statement follows from formulas (3.16), (3.17), (3.21) and their corollary that

$$D\hat{\mathbf{p}}(j) = (n_s n_\varphi)^{-1/2}\hat{g}(0), \quad j \in \hat{I}, \tag{4.4}$$

the Chebyshev inequality written in the form

$$Prob\,\{|\xi - M\xi| \le \varepsilon|M\xi|\} \ge 1 - \frac{D\xi}{\varepsilon^2|M\xi|^2}, \tag{4.5}$$

where $D\xi = M|\xi - M\xi|^2$, and the formulas

$$\begin{aligned} &|\xi|^2 - M|\xi|^2 = |\xi|^2 - |M\xi|^2 - D\xi = (|\xi| - |M\xi|)(|\xi| + |M\xi|) - D\xi, \\ &||\xi|^2 - M|\xi|^2| \le |\xi - M\xi|(2|M\xi| + |\xi - M\xi|) + D\xi \quad (\text{for } \ \xi = \hat{\mathbf{p}}(j)). \end{aligned} \tag{4.6}$$

As a rule, condition (4.3) is satisfied if j is sufficiently close to 0 but is not satisfied otherwise. Therefore, approximation (4.1), (4.2) to the Wiener optimal filter is not very efficient in the framework of applications to Problem 1.1 and 1.2 (numerical examples are given in Section 8) . Actually, more satisfactory approximations to the Wiener optimal filter can be given proceeding from Theorem 3.1 and Corollary 3.1 with appropriate subsets S_α; see Subsections 4.2-4.5.

4.2. General approximation $\mathcal{A}$. Consider the noise model (3.4). In a similar way with (4.1), (4.2), in view of formulas (3.7) (for the optimal window $\hat{W}^{r.o.}$ with a priori restrictions (3.6)) and formulas (3.16), (3.17) (for $M\mathbf{p}$ and $M|\hat{\mathbf{p}}|^2$) we have that

$$\hat{W}^{r.o.}(j) \approx \hat{A}(j), \quad j \in \hat{I}, \tag{4.7}$$

where

$$\begin{aligned} &\hat{A}(j) = \frac{\Sigma_{p,\alpha(j)} - V_p}{\Sigma_{p,\alpha(j)}} \ \text{ if } \ \Sigma_{p,\alpha(j)} - V_p > 0, \\ &\hat{A}(j) = 0 \ \text{ if } \ \Sigma_{p,\alpha(j)} - V_p \le 0, \end{aligned} \tag{4.8}$$

where

$$\Sigma_{p,\alpha(j)} = \frac{1}{|S_{\alpha(j)}|}\sum_{i\in S_{\alpha(j)}} |\hat{p}(i)|^2, \quad j \in \hat{I}, \quad V_p \approx V, \tag{4.9}$$

where S_α, $|S_\alpha|$, $\alpha(j)$ are the same that in (3.7), $\hat{p} = F\Lambda p$ (with F, Λ defined by (2.6), (2.7)), V is the number of (3.7). In addition, as regards precise formulas for V_p, see (4.11), (4.12). The filter of the form (2.9)-(2.12) with the window given by (4.8), (4.9) is denoted as $\mathcal{A}$ in the present paper.

If n_s and n_φ are sufficiently great and g is a sufficiently regular function on Γ considered as a discrete torus, then

$$|\hat{g}(j)|^2 \approx 0 \ \text{ if } \ j \in \hat{I} \ \text{ is sufficiently close to } \ \partial\hat{I}, \tag{4.10}$$

where $\partial\hat{I} \subset \hat{I}$ is the boundary of $\hat{I}$ in $\mathbb{Z}^2$. In this case, due to (3.17), (4.10), the approximation V_p can be defined by the formula

$$V_p = \frac{1}{|\Omega|}\sum_{j\in\Omega}|\hat{p}(j)|^2, \tag{4.11}$$

where Ω is a subset of $\hat{I}$, each point of Ω is sufficiently close to $\partial\hat{I}$, $|\Omega|$ is the number of points in Ω and $|\Omega|$ is sufficiently great.

Besides, for the Poisson case (3.1) the approximation V_p can be defined as

$$V_p = (n_s n_\varphi)^{-1/2}\hat{p}(0). \tag{4.12}$$

Note that V_p of (4.12) does not necessarily coincide completely with V_p of (4.11) for the Poisson case.

All further considerations of this section are given for simplicity for the Poisson case (3.1) (if other indications are not given explicitly).

Note that if S_α are given by (3.18), then the filter $\mathcal{A}$ with V_p given by (4.12) is reduced to $\mathcal{A}^{simp}$ of Subsection 4.1.

The principal advantage of the approximation (4.7) (for the Poisson case) in comparison with (4.1) consists in the fact that if

$$|S_{\alpha(j)}| \ \text{ is great enough in comparison with } \ |j| \ \text{ for fixed } \ j\in\hat{I}, \tag{4.13}$$

where $|j|$ is the distance from j to the origin 0 of $\hat{I}$ in an appropriate norm, then (because of averaging in $\Sigma_{g,\alpha}$, $\Sigma_{p,\alpha}$ of (3.7), (4.9)) $\Sigma_{p,\alpha(j)}$ is a much better approximation to $\Sigma_{g,\alpha(j)} + (n_s n_\varphi)^{-1/2}\hat{g}(0)$ than $|\hat{p}(j)|^2$ to $|\hat{g}(j)|^2 + (n_s n_\varphi)^{-1/2}\hat{g}(0)$ and, as a corollary, $\hat{A}(j)$ is a much better approximation to $\hat{W}^{r.o.}(j)$ than $\hat{A}^{simp}(j)$ to $\hat{W}^{opt}(j)$. Moreover, for appropriate subsets S_α it turns out that $\hat{A}$ (of (4.8)) is, actually a considerably better approximation to $\hat{W}^{opt}$ (of (3.3)) in the framework of applications to Problems 1.1 and 1.2 than A^{simp} (of (4.2)).

4.3. One-dimensional approximation $\mathcal{A}^{1d}$. Let the subsets S_α of (3.5) be defined as

$$S_{\alpha(j)} = \{z = (z_1, z_2)\in\hat{I}:\ z_1 = j_1\}\ \ \forall\ j = (j_1, j_2)\in\hat{I}, \tag{4.14}$$

where $\alpha(j)$ denotes α such that $j\in S_\alpha$. Then the filter $\mathcal{A}$ of Subsection 4.2 (for the Poisson case) is reduced to the "one-dimensional" approximately optimal Wiener type filter $\mathcal{A}^{1d}$ going back to [KDS]. Filters as $\mathcal{A}^{1d}$ are, actually, considered in the literature as rather satisfactory approximations to optimal filters as $\mathcal{W}^{opt}$ in the framework of SPECT and PET imaging based on the classical FBP algorithm or its iterative use (see, for example, [KDS], [SKC], [BCB], [C]).

Note that $\mathcal{A}^{1d}$ is not very interesting as an approximation to $\mathcal{W}^{opt}$ in the framework of pure applications to Problem 1.1. The reason is that the subsets S_α of (4.14) are not symmetric with respect to the indices z_1 and z_2 on $\hat{I}$ and, therefore, $\mathcal{A}^{1d}$ is not symmetric with respect to s and φ variables on Γ. More precisely, due to (4.14) the window function

$\hat{A}^{1d}(j)$, $j = (j_1, j_2)$, is independent of j_2 and, therefore, $\mathcal{A}^{1d}$ does not filtrate at all with respect to the angle variable φ on Γ. However, the classical FBP algorithm is not very sensitive to no filtering in the angle-direction of projections in the framework of the noise model (1.4). This together with Theorem 3.1, Corollary 3.1 and property (4.13) for S_α of (4.14) is our explanation of the fact that the filter $\mathcal{W} = \mathcal{A}^{1d}$ is rather efficient (in the class of space-invariant filters) in the framework of applications to Problem 1.2 via (1.5) with P_a^{-1} based on iterations of the classical FBP algorithm (see Section 8 for numerical illustration).

4.4. Symmetric two-dimensional approximation $\mathcal{A}^{sym}$. As symmetric two-dimensional approximately optimal Wiener type filter we consider $\mathcal{A}^{sym}$ defined as $\mathcal{A}$ of Subsection 4.2 with S_α defined as

$$S_\alpha = \{z = (z_1, z_2) \in \hat{I} : \ \tau_{\alpha-1} \le \max\big(|z_1|, \big|\frac{n_s}{n_\varphi} z_2\big|\big) < \tau_\alpha\}, \ \alpha = 1, \ldots, n^*, \tag{4.15}$$

where $\tau_0, \ldots, \tau_{n^*}$ are some appropriate fixed real numbers such that $\tau_0 = 0$, $\tau_{\alpha-1} < \tau_\alpha$ (and $S_\alpha \neq \emptyset$), $\alpha = 1, \ldots, n^*$, $\tau_{n^*} = (n_s + 1)/2$ and where we assume that $n_\varphi \le n_s$. Actually, in the numerical examples of present work we assume that $n_\varphi = n_s$ and

$$\tau_0 = 0, \ \tau_\alpha = 1/2 + \alpha, \ \text{ for } \ \alpha = 1, \ldots, n^*, \ \ n^* = n_s/2 = n_\varphi/2. \tag{4.16}$$

One can see that the subsets S_α of (4.15), (4.16) are rather symmetric with respect to the indices z_1 and z_2 on $\hat{I}$ in contrast with the subsets S_α of (4.14).

Symmetric S_α of (4.15), (4.16) are much more natural than asymmetric S_α of (4.14) as level sets of filtering window in the framework of the noise model (3.1) as soon as the regularity of g of (3.1) is more or less similar with respect to each of the variables s and φ on Γ. As a result $\mathcal{A}^{sym}$ is of interest as an approximation to $\mathcal{W}^{opt}$ even in the framework of pure applications to Problem 1.1 in contrast with $\mathcal{A}^{1d}$.

In addition, in the framework of further applications to Problem 1.2 via (1.5) (even with P_a^{-1} consisting in the classical FBP algorithm used iteratively) $\mathcal{A}^{sym}$ gives also considerably better results than $\mathcal{A}^{1d}$. This advantage of $\mathcal{A}^{sym}$ in comparison with $\mathcal{A}^{1d}$ is especially strong if P_a^{-1} of (1.5) is the explicit formula of [No]. Numerical examples illustrating $\mathcal{A}^{sym}$ in the framework of applications to Problems 1.1 and 1.2 are given in Section 8. An efficiency of $\mathcal{A}^{sym}$ (in the class of space-invariant filters) in the framework of these applications is explained by Theorem 3.1, Corollary 3.1, property (4.13) and adequate geometry of the subsets S_α of (4.15), (4.16).

4.5. Possibility of the "bowtie shape" geometry for S_α. Finally, note that subsets S_α (arising in (3.7), (4.8), (4.9)) with geometry even more appropriate for applications to Problems 1.1 and 1.2 than in (4.15), (4.16) can be constructed proceeding from the result (see [RL], [MN], [GouNol], [GN2] and figure 2(b) of the present paper) that the Fourier transform $\hat{g} = F\Lambda g$, where g is the function of (1.4), is supported mainly in some rather specific domain (of bowtie shape) dependent on f and a. However, we will not develop this issue in the present work.

5. Some filtering schemes of [GN1], [GN2], [GN3]

All considerations of this section are given for simplicity for the Poisson model (3.1).

5.1. Space-invariant data dependent filter Φ_ε *of [GN1].* The window function of the space-invariant data dependent filter $\Phi_\varepsilon = \Phi_{\varepsilon,\omega}$ of [GN1] is given by (2.13), where

$$\hat{w}_1(s) = \hat{w}_2(s) = \left(\frac{\sin(s)}{s}\right)^2, \quad \omega_1 = \omega_2 = \omega \tag{5.1}$$

and $\omega = \omega(p, \varepsilon)$ is data dependent and is determined from the equation

$$\frac{\|p - \Phi_{\varepsilon,\omega} p\|_{L^2(\Gamma)}}{\|\Phi_{\varepsilon,\omega} p\|_{L^2(\Gamma)}} \approx \varepsilon \left(\frac{\|p\|_{L^1(\Gamma)}}{\|p\|^2_{L^2(\Gamma)} - \|p\|_{L^1(\Gamma)}}\right)^{1/2} \tag{5.2}$$

for any fixed realization p of $\mathbf{p}$ of (3.1). Here ε is a filter parameter and the "optimal" value for ε is 1. Actually, there is some similarity in geometric structure of the windows of Φ_1 and $\mathcal{A}^{sym}$.

5.2. Space-variant data dependent filter $\Phi_{l_1,l_2,\varepsilon}$ *of [GN2].* Let

$$\Gamma_\infty = \{\gamma_{i,j} : \quad i \in \mathbb{Z}, \quad j = 1, \ldots, n_\varphi\}, \tag{5.3}$$

where $\gamma_{i,j}$ are defined as in (1.3). One can see that $\Gamma = \Gamma_\infty \cap \{(s,\theta) \in \mathbb{R} \times \mathbb{S}^1 : \ |s| \le R\}$ and that Γ_∞ is an extension of Γ. Let

$$\begin{aligned} &\mathcal{D}_{\gamma,l_1,l_2} = \{\gamma' = (s', \theta(\varphi')) \in \Gamma_\infty : \ -[(l_1-1)/2]\Delta s \le s' - s \le [l_1/2]\Delta s, \\ &\ -[(l_2-1)/2]\Delta\varphi \le \varphi' - \varphi \le [l_2/2]\Delta\varphi\}, \\ &\ \gamma = (s, \theta(\varphi)) \in \Gamma, \quad l_1, l_2 \in \mathbb{N}, \quad l_1 \le n_s, \ l_2 \le n_\varphi, \end{aligned} \tag{5.4}$$

where $[\lambda]$ is the integer part of real nonnegative λ. One can see that $\mathcal{D}_{\gamma,l_1,l_2}$ is $l_1 \times l_2$ neighborhood of $\gamma \in \Gamma$ in Γ_∞.

The space-variant data dependent filter $\Phi_{l_1,l_2,\varepsilon}$ of [GN2] is defined by the formula

$$(\Phi_{l_1,l_2,\varepsilon} p)(\gamma) = (\Phi_\varepsilon(p\big|_{\mathcal{D}_{\gamma,l_1,l_2}}))(\gamma), \ \gamma \in \Gamma, \tag{5.5}$$

for any fixed p of (3.1), where $\mathcal{D}_{\gamma,l_1,l_2}$ is defined by (5.4), $p\big|_{\mathcal{D}_{\gamma,l_1,l_2}}$ is defined using zero-padding if $\Gamma \backslash \mathcal{D}_{\gamma,l_1,l_2} \ne \emptyset$, Φ_ε is the filter of Subsection 5.1 with Γ replaced by $\mathcal{D}_{\gamma,l_1,l_2}$. In addition, ε, l_1, l_2 are filter parameters and the basic value for ε is 1. One can see that $\Phi_{l_1,l_2,\varepsilon}$ is a space-variant version of Φ_ε.

5.3. Space-variant approximately optimal Wiener type filter $\mathcal{A}^{sym}_{l_1,l_2}$ *of [GN3].* The space-variant approximately optimal Wiener type filter $\mathcal{A}^{sym}_{l_1,l_2}$ of [GN3] is defined by the formula

$$(\mathcal{A}^{sym}_{l_1,l_2} p)(\gamma) = (\mathcal{A}^{sym}(p\big|_{\mathcal{D}_{\gamma,l_1,l_2}}))(\gamma), \ \gamma \in \Gamma, \tag{5.6}$$

for any fixed p of (3.1), where $p\big|_{\mathcal{D}_{\gamma,l_1,l_2}}$ is the same that in (5.5), $\mathcal{A}^{sym}$ is the filter of Subsection 4.4 with V_p defined by (4.12) and Γ replaced by $\mathcal{D}_{\gamma,l_1,l_2}$. In addition, l_1, l_2 are filter parameters. One can see that $\mathcal{A}^{sym}_{l_1,l_2}$ is a space-variant version of $\mathcal{A}^{sym}$. Numerical examples illustrating $\mathcal{A}^{sym}_{l_1,l_2}$ in the framework of applications to Problems 1.1 and 1.2 are given in Section 8.

6. Reconstruction of Cf from CP_af and a

First, we consider the following explicit inversion formula

$$Cf = \mathcal{N}_a g, \tag{6.1}$$

where $g = CP_af$,

$$\mathcal{N}_a q(x) = \frac{1}{4\pi}\Big(-\frac{\partial}{\partial x_1}\int\limits_{\mathbb{S}^1} K(x,\theta)\theta_2 d\theta + \frac{\partial}{\partial x_2}\int\limits_{\mathbb{S}^1} K(x,\theta)\theta_1 d\theta\Big), \tag{6.2a}$$

$$K(x,\theta) = \exp\left[-\mathcal{D}a(x,-\theta)\right]\tilde{q}_\theta(x\theta^\perp), \tag{6.2b}$$

$$\begin{aligned}\tilde{q}_\theta(s) = {} & \exp\big(A_\theta(s)\big)\cos\left(B_\theta(s)\right) H\big(\exp\left(A_\theta\right)\cos\left(B_\theta\right) q_\theta\big)(s) + \\ & \exp\big(A_\theta(s)\big)\sin\left(B_\theta(s)\right) H\big(\exp\left(A_\theta\right)\sin\left(B_\theta\right) q_\theta\big)(s),\end{aligned} \tag{6.2c}$$

$$A_\theta(s) = \frac{1}{2}Pa(s,\theta), \quad B_\theta(s) = H\,A_\theta(s), \quad q_\theta(s) = q(s,\theta), \tag{6.2d}$$

where q is a test function, $P = P_0$ is the classical two-dimensional ray transformation (i.e. P_0 is defined by (1.1a) with $a \equiv 0$), H is the Hilbert transformation defined by the formula

$$H\,u(s) = \frac{1}{\pi}\,p.v.\int\limits_{\mathbb{R}} \frac{u(t)}{s-t}dt, \tag{6.3}$$

where u is a test function, $x = (x_1, x_2) \in \mathbb{R}^2$, $\theta = (\theta_1, \theta_2) \in \mathbb{S}^1$, $\theta^\perp = (-\theta_2, \theta_1)$, $s \in \mathbb{R}$, $d\theta$ is arc-length measure on the circle $\mathbb{S}^1$.

In a slightly different form (using complex notations) formula (6.1) was obtained in [No]. Some new proofs of this formula were given in [Na] and [BS]. Formula (6.1) was successfully implemented numerically in [Ku] and [Na] via a direct generalization of the (classical) filtered back-projection (FBP) algorithm. However, this generalized FBP algorithm turned out to be less stable, in general, than its classical analogue. Some possibilities for improving the stability of SPECT imaging based on (6.1), (6.2) with respect to the Poisson noise in the emission data g were proposed, in particular, in [Ku] (preprint version), [GJKNT] and [GN]. Some fast numerical implementation of formula (6.1) was proposed in [BM].

Second, assuming (1.2), we consider the iterative reconstruction method with the following step. If Cf_n is an approximation with the number n to Cf (as an approximation Cf_n may have some negative values) and $g = CP_af$, then we

(1) compute

$$h_n(s,\theta) = (g(s,\theta) + \mu_n)\frac{PCf_n(s,\theta) + \mu_n}{P_aCf_n(s,\theta) + \mu_n} - \mu_n, \tag{6.4}$$

where μ_n is some sufficiently small positive constant depending on P_aCf_n such that $P_aCf_n(s,\theta)+\mu_n>0$ for $(s,\theta)\in\mathbb{R}\times\mathbb{S}^1$, $P=P_0$ is defined by (1.1a) with $a\equiv 0$,

(2) enforce the conditions

$$0\le g(s,\theta)\le h_n(s,\theta)\le \exp\left(Pa(s,\theta)\right)g(s,\theta),\ (s,\theta)\in\mathbb{R}\times\mathbb{S}^1, \tag{6.5}$$

and (3) compute

$$Cf_{n+1}=P^{-1}h_n \tag{6.6}$$

using (6.1) with $a\equiv 0$ (i.e. using a variant of the classical FBP algorithm). This step (i.e. the passage from Cf_n to Cf_{n+1} via (6.4)-(6.6)) is a variation of the step of the iterative SPECT reconstruction algorithm of [MNOY] (see also [MIMIKIH] and [GJKNT]). This algorithm (with the step (6.4)-(6.6)) is rather stable or, more precisely, if a is not too strong, then its stability properties with respect to the Poisson noise in the emission data g are comparable with the stability properties of (6.1) for $a\equiv 0$ (i.e. with the stability properties of the classical FBP algorithm).

In the present work we improve the stability of SPECT reconstruction based on (6.1), (6.2) or/and on (6.4)-(6.6) with respect to the Poisson noise in the emission data g by means of approximately optimal space-invariant Wiener type filters (with unknown object power spectrum) of Section 4 and one of their space-variant versions (of [GN3]) mentioned in Section 5 (in Subsection 5.3).

Actually, in the present work we consider, mainly, the reconstructions Cf_1 and Cf_3, where Cf_1 is reconstructed via (6.1), (6.2) and Cf_2, Cf_3 are obtained proceeding from Cf_1 via (6.4)-(6.6). Actually, the iterations Cf_2 and Cf_3 are rather close to each other, but nevertheless Cf_3 is still somewhat more stable and more properly illustrates stability properties of the classical FBP algorithm used iteratively. This can be considered as a stabilization of (6.1) or as an acceleration of the iterative reconstruction based on (6.4)-(6.6).

Note also that in the numerical studies of the present work the attenuation map a and the emitter activity f (and all reconstructions of f) are actually considered on

$$\begin{aligned} &X=\{x_{i,j}:\ x_{i,j}=(-R+(i-1)\Delta s,-R+(j-1)\Delta s),\\ &\Delta s=2R/(n_s-1),\quad i=1,\ldots,n_s,\ j=1,\ldots,n_s\}, \end{aligned} \tag{6.7}$$

where R, Δs, n_s are the same that in (1.3). See Sections 7 and 8 for further presentation of our numerical studies.

7. Characteristics of filter efficiency

We consider, in particular,

$$\begin{aligned} M_k\mathcal{W}\mathbf{p}&=\frac{1}{k}\sum_{i=1}^{k}\mathcal{W}p_i\quad\text{on}\quad\Gamma,\\ D_k\mathcal{W}\mathbf{p}&=\frac{1}{k}\sum_{i=1}^{k}(\mathcal{W}p_i-M_k\mathcal{W}\mathbf{p})^2\quad\text{on}\quad\Gamma, \end{aligned} \tag{7.1}$$

$$M_k P_a^{-1}\mathcal{W}\mathbf{p} = \frac{1}{k}\sum_{i=1}^{k} P_a^{-1}\mathcal{W}p_i \quad \text{on} \quad X,$$
$$D_k P_a^{-1}\mathcal{W}\mathbf{p} = \frac{1}{k}\sum_{i=1}^{k}(P_a^{-1}\mathcal{W}p_i - M_k P_a^{-1}\mathcal{W}\mathbf{p})^2 \quad \text{on} \quad X, \tag{7.2}$$

where $\mathcal{W}$ is a fixed filtering method for solving Problem 1.1, $\mathbf{p}$ is the Poisson field of (1.4), $p_1, \ldots, p_k$ are some k independent realizations of $\mathbf{p}$, P_a^{-1} is a fixed inversion method for P_a of (1.1), (1.4) for the noiseless case. In addition, k is rather great so that $M_k \approx M = M_\infty$, $D_k \approx D = D_\infty$.

The functions $M_k\mathcal{W}\mathbf{p}$ and $M_k P_a^{-1}\mathcal{W}\mathbf{p}$ are used for evaluating the bias (or nonrandom errors) of $\mathcal{W}$ in comparison with g and $P_a^{-1}g$, where g is the noiseless data of (1.4). For example, a typical bias effect of filtering consists in too strong smoothing some important image details. In turn, $D_k\mathcal{W}\mathbf{p}$ and $D_k P_a^{-1}\mathcal{W}\mathbf{p}$ describe the variance of $\mathcal{W}\mathbf{p}$ and $P_a^{-1}\mathcal{W}\mathbf{p}$ with respect to the mean results $M_k\mathcal{W}\mathbf{p}$ and $M_k P_a^{-1}\mathcal{W}\mathbf{p}$.

We emphasize that definitions (7.1), (7.2), as well as (3.2) and (7.6), (7.7), do not mean that k independent realizations of $\mathbf{p}$ are available in practice. Images like the images of (7.2) are actually standard in tomographical studies, see, for example, [C].

We use also the following notations

$$\zeta(q_2, q_1, \Gamma') = \frac{\|q_2 - q_1\|_{L^2(\Gamma')}}{\|q_1\|_{L^2(\Gamma')}}, \tag{7.3}$$

where q_1, q_2 are test functions on $\Gamma' \subseteq \Gamma$ and $\|\cdot\|_{L^2(\Gamma')}$ is defined by (2.4), and

$$\eta(u_1, u_2, X') = \frac{\|u_2 - u_1\|_{L^2(X')}}{\|u_1\|_{L^2(X')}}, \tag{7.4}$$

$$\|u\|_{L^n(X')} = \big((\Delta s)^2 \sum_{x\in X'} |u(x)|^n\big)^{1/n}, \quad n \in \mathbb{N}, \tag{7.5}$$

where u, u_1, u_2 are test functions on $X' \subseteq X$. Note that for p and g of (1.4) the quantity $\zeta(p, g, \Gamma)$ is the noise level (in the L^2- sense) of p on Γ.

In our studies we consider, in particular, the following numbers

$$e_{1,k}(\mathcal{W}, g) = (M_k(\zeta(\mathcal{W}\mathbf{p}, g, \Gamma))^2)^{1/2} = \big(\frac{1}{k}\sum_{i=1}^{k}(\zeta(\mathcal{W}p_i, g, \Gamma))^2)^{1/2},$$
$$b_{1,k}(\mathcal{W}, g) = \zeta(M_k\mathcal{W}\mathbf{p}, g, \Gamma), \tag{7.6}$$
$$d_{1,k}(\mathcal{W}, g) = \frac{\big(\|D_k\mathcal{W}\mathbf{p}\|_{L^1(\Gamma)}\big)^{1/2}}{\|g\|_{L^2(\Gamma)}},$$

$$\begin{aligned}
&e_{2,k}(P_a^{-1},\mathcal{W},g) = (M_k(\eta(P_a^{-1}\mathcal{W}\mathbf{p},P_a^{-1}g,X))^2)^{1/2} = \\
&(\frac{1}{k}\sum_{i=1}^{k}(\eta(P_a^{-1}\mathcal{W}p_i,P_a^{-1}g,X))^2)^{1/2}, \\
&b_{2,k}(P_a^{-1},\mathcal{W},g) = \eta(M_kP_a^{-1}\mathcal{W}\mathbf{p},P_a^{-1}g,X), \\
&d_{2,k}(P_a^{-1},\mathcal{W},g) = \frac{\left(\|D_kP_a^{-1}\mathcal{W}\mathbf{p}\|_{L^1(X)}\right)^{1/2}}{\|P_a^{-1}g\|_{L^2(X)}},
\end{aligned} \tag{7.7}$$

where $\mathcal{W}$, P_a^{-1}, $\mathbf{p}$, $p_1,\ldots,p_k$, $M_k\mathcal{W}\mathbf{p}$, $D_k\mathcal{W}\mathbf{p}$, $M_kP_a^{-1}\mathcal{W}\mathbf{p}$, $D_kP_a^{-1}\mathcal{W}\mathbf{p}$ are the same that in (7.1), (7.2), g is the function of (1.4) and ζ, η, $\|\cdot\|_{L^n(\Gamma)}$, $\|\cdot\|_{L^n(X)}$ are defined in (7.3), (7.4), (2.4), (7.5). One can see that the numbers $e_{1,k}$, $b_{1,k}$, $d_{1,k}$, $e_{2,k}$, $b_{2,k}$, $d_{2,k}$ of (7.6), (7.7) have the following sense:

(1) $e_{1,k}$ is a relative mean error, $b_{1,k}$ is a relative mean bias and $d_{1,k}$ is a relative mean deviation from the mean result (of k tests) for $\mathcal{W}\mathbf{p}$ with fixed g and

(2) $e_{2,k}$ is a relative mean error, $b_{2,k}$ is a relative mean bias and $d_{2,k}$ is a relative mean deviation from the mean result (of k tests) for $P_a^{-1}\mathcal{W}\mathbf{p}$ with fixed g. In addition,

$$(e_{1,k})^2 \approx \mu \tag{7.8}$$

for $e_{1,k}$ of (7.6) and μ of (3.2) (where $\mathcal{W}$ and g are the same that in (7.6)) and for sufficiently great k.

Note also that

$$(e_{i,k})^2 \approx (b_{i,k})^2 + (d_{i,k})^2, \quad i = 1,2, \tag{7.9}$$

for $e_{i,k}$, $b_{i,k}$, $d_{i,k}$, $i = 1,2$, of (7.6), (7.7) with sufficiently great k.

To compare different filters we consider also the numbers

$$c_{1,k}(\mathcal{W},g) = \frac{(e_{1,k}(\mathcal{W},g)b_{1,k}(\mathcal{W},g))^{1/2}}{|e_{1,k}(Id,g) - e_{1,k}(\mathcal{W},g)|}, \tag{7.10}$$

$$c_{2,k}(P_a^{-1},\mathcal{W},g) = \frac{(e_{2,k}(P_a^{-1},\mathcal{W},g)b_{2,k}(P_a^{-1},\mathcal{W},g))^{1/2}}{|e_{2,k}(P_a^{-1},Id,g) - e_{2,k}(P_a^{-1},\mathcal{W},g)|}, \tag{7.11}$$

where we use the same notations that in (7.1), (7.2), (7.6), (7.7) and, in addition, Id denotes the identity filter that is $Id(\mathbf{p}) = \mathbf{p}$. We consider $c_{i,k}$ as an error-bias trade-off coefficient between $e_{i,k}$ and $b_{i,k}$, where we take also into account the initial error $e_{i,k}^{initial}$, where $i = 1,2$, $e_{1,k}^{initial} = e_{1,k}(Id,g)$, $e_{2,k}^{initial} = e_{2,k}(P_a^{-1},Id,g)$. This trade-off is better if $c_{i,k}$ is smaller.

In addition to the numbers $e_{i,k}$, $b_{i,k}$, $d_{i,k}$, $c_{i,k}$, $i = 1,2$, of (7.6), (7.7), (7.10), (7.11), one can consider also similar numbers for $\Gamma' \subset \Gamma$ in place of Γ in (7.6), (7.10) and $X' \subset X$ in place of X in (7.7), (7.11). However, in the present paper we consider the global numbers of (7.6), (7.7), (7.10), (7.11) only. Additional local information on $\mathcal{W}\mathbf{p}$ and $P_a^{-1}\mathcal{W}\mathbf{p}$ is available from related images.

8. Numerical examples

8.1. Preliminary remarks. We assume that $n_s = 128$, $n_\varphi = 128$ in (1.3), (2.2), (2.3), (6.7).

Given f and a on X, we assume that $P_a f$ is defined on Γ and is the numerical realization of (1.1) as in [Ku]. Given a on X and q on Γ, we assume that $\mathcal{N}_a q$ is defined on X and denote the numerical realization of (6.2) as in [Ku], [Na] without any regularization. Given Cf_1 and a on X and g on Γ, we assume that $Cf_m(Cf_1, a, g)$ is defined on X and is obtained numerically proceeding from Cf_1 via (6.4)-(6.6) by $m-1$ steps without any regularization in (6.6) (here we do not assume that $g = CP_a f$).

In addition, the 2D discrete Fourier transform $F\Lambda q$ is considered on $\hat{I}$ defined by (2.3) for any q on Γ.

Notice that all two-dimensional images of the present work, except the spectrum of projections, are drawn using a linear grayscale, in such a way that the dark gray color represents zero (or negative values, if any) and white corresponds to the maximum value of the imaged function. For the spectrum of projections, a non-linear grayscale was used, because of too great values of the spectrum for small frequencies.

8.2. Elliptical chest phantom. We consider a version of the elliptical chest phantom (used for numerical simulations of cardiac SPECT imaging; see [HL], [Br], [GN1]). This version is, actually, the same that in [GN1], [GN2] and its description consists in the following:

(1) The major axis of the ellipse representing the body is 30 cm.

(2) The attenuation map is shown in figure 1(a); the attenuation coefficient a is 0.04 cm^{-1} in the lung regions (modeled as two interior ellipses), 0.15 cm^{-1} elsewhere within the body ellipse, and zero outside the body.

(3) The emitter activity f is shown in figure 1(b); f is in the ratio 8:0:1:0 in myocardium (represented as a ring), lungs, elsewhere within the body, and outside the body.

(4) The attenuated ray transform $g = CP_a f$ and noisy emission data p of (1.4) are shown in figures 2(a), 2(c). In addition, the constant C was specified by the equation

$$\|g\|_{L^1(\Gamma)}/\|g\|^2_{L^2(\Gamma)} = 0.30 \tag{8.1}$$

in order to have that the noise level $\zeta(p, g, \Gamma) \approx 0.30$ (where ζ is defined by (7.3)). Actually, we have that

$$\zeta(p, g, \Gamma) = 0.298, \quad \sum_{\gamma\in\Gamma} p(\gamma) = 125450 \tag{8.2}$$

for p shown in figure 2(c).

Figures 2(b), 2(d) show the spectrum $|F\Lambda g|$ and $|F\Lambda p|$.

Figures 3(a)-(d) show the reconstructions

$$Cf_1^0 = \mathcal{N}_a g, \quad Cf_3^0 = Cf(Cf_1^0, a, g) \tag{8.3}$$

(from the noiseless emission data g) and their profiles for $j = 64$.

Figures 4(a)-(d), 5(a)-(d) show the reconstructions

$$Cf_1 = \mathcal{N}_a p, \quad Cf_3 = Cf_3(\mathcal{N}_a p, a, p) \tag{8.4}$$

for p shown in figure 2(c), their profiles for $j = 64$ and the images

$$\begin{aligned} &M_{200}Cf_1 = M_{200}\mathcal{N}_a\mathbf{p}, \quad D_{200}Cf_1 = D_{200}\mathcal{N}_a\mathbf{p}, \\ &M_{200}Cf_3 = M_{200}Cf_3(\mathcal{N}_a\mathbf{p}, a, \mathbf{p}), \quad D_{200}Cf_3 = D_{200}Cf_3(\mathcal{N}_a\mathbf{p}, a, \mathbf{p}), \end{aligned} \tag{8.5}$$

where $\mathbf{p}$ is the Poisson field of (1.4) for the case of our phantom.

In addition:

$$\eta(Cf_1, Cf_1^0, X) = 1.58, \quad \eta(Cf_3, Cf_3^0, X) = 0.74 \tag{8.6}$$

for Cf_1^0, Cf_3^0, Cf_1, Cf_3 of (8.3), (8.4);

$$e_{2,200} = 1.55, \quad b_{2,200} = 0.11, \quad d_{2,200} = 1.55 \tag{8.7a}$$

for $\mathcal{W} = Id$ (that is $\mathcal{W}(\mathbf{p}) = \mathbf{p}$) and $P_a^{-1} = \mathcal{N}_a$;

$$e_{2,200} = 0.75, \quad b_{2,200} = 0.06, \quad d_{2,200} = 0.75 \tag{8.7b}$$

for $\mathcal{W} = Id$ and

$$P_a^{-1}q = Cf_3(\mathcal{N}_a q, a, q), \tag{8.8}$$

where q is a test function on Γ.

We remind that we use the notations of Subsection 8.1 and Section 7. In particular, we use definitions of Section 7 with $k = 200$, where we consider that $k = 200$ is already sufficiently great for our numerical examples. In addition, we use i, j of (6.7) as coordinates on X in the profile indications.

8.3. Illustrations of space-invariant Wiener type filters of Sections 3 and 4.

Figures 6-9 show the filtration result $\mathcal{W}p$ and its spectrum $|F\Lambda\mathcal{W}p|$ (for p shown in Figure 2(c)) and also $M_{200}\mathcal{W}\mathbf{p}$ and $D_{200}\mathcal{W}\mathbf{p}$ (where $\mathbf{p}$ is the Poisson field for our phantom) for $\mathcal{W} = \mathcal{W}^{opt}$, $\mathcal{W}^{sym}$, $\mathcal{A}^{simp}$, $\mathcal{A}^{sym}$, where $\mathcal{W}^{opt}$ is the optimal space-invariant Wiener filter (of Section 3) with the window function given by (3.3), $\mathcal{W}^{sym}$ is the restrictly optimal space-invariant Wiener filter $\mathcal{W}^{r.o.}$ (of Section 3) with the symmetric window function given by (3.7), (3.21), (4.15), (4.16), $\mathcal{A}^{simp}$ is the space-invariant data dependent filter (of Section 4) with the window function defined by (4.2), $\mathcal{A}^{sym}$ is the space-invariant data dependent filter (of Section 4) with the window function defined by (4.8), (4.9), (4.12), (4.15), (4.16).

We remind that: (1) in $\mathcal{W}^{opt}$ and $\mathcal{W}^{sym}$ it is assumed that $|\hat{g}|^2$ is known; (2) $\mathcal{A}^{simp}$ is the simplest approximation to $\mathcal{W}^{opt}$ for the case when $|\hat{g}|^2$ is not known; (3) $\mathcal{A}^{sym}$ is a direct approximation to $\mathcal{W}^{sym}$ and is a regularized approximation to $\mathcal{W}^{opt}$ for the case when $|\hat{g}|^2$ is not known.

Table 1 shows the number $\zeta = \zeta(\mathcal{W}p, g, \Gamma)$ and $e_{1,200}$, $b_{1,200}$, $d_{1,200}$, $c_{1,k}$ of (7.6), (7.10) for $\mathcal{W} = \mathcal{W}^{opt}$, $\mathcal{W}^{sym}$, $\mathcal{A}^{simp}$, $\mathcal{A}^{1d}$, $\mathcal{A}^{sym}$ (and for the filters Φ_1, $\Phi_{8,8,1}$, $\mathcal{A}^{sym}_{8,8}$ of [GN1], [GN2], [GN3], see Section 5 of the present paper).

Figures 6-9 and table 1 show that $\mathcal{A}^{simp}$ is not a very efficient approximation to $\mathcal{W}^{opt}$, whereas $\mathcal{A}^{sym}$ is a very efficient approximation to $\mathcal{W}^{sym}$. Moreover, Figures 6, 9 and (related part of) table 1 show that, actually, $\mathcal{A}^{sym}$ is also a rather efficient approximation to $\mathcal{W}^{opt}$ in the framework of solving Problem 1.1. We remind that a theoretical explanation of these numerical results was given in Sections 3 and 4.

We do not show the images $\mathcal{W}p$, $|F\Lambda\mathcal{W}p|$, $M_{200}\mathcal{W}\mathbf{p}$, $D_{200}\mathcal{W}\mathbf{p}$ for $\mathcal{W}=\mathcal{W}^{1d}$ and $\mathcal{W}=\mathcal{A}^{1d}$, where $\mathcal{W}^{1d}$ is the restrictly optimal space-invariant Wiener filter $\mathcal{W}^{r.o.}$ of Section 3 with "1d"- window function given by (3.7), (3.21), (4.14) and $\mathcal{A}^{1d}$ is the data dependent approximation to $\mathcal{W}^{1d}$ with the window function defined by (4.8), (4.9), (4.12), (4.14). The reasons are that: (1) it was explained already in Section 4 that $\mathcal{A}^{1d}$ is not interesting in the framework of pure applications to Problem 1.1 and (2) we try to avoid too many images in our paper. Nevertheless, the numbers ζ, $e_{1,200}$, $b_{1,200}$, $d_{1,200}$, $c_{1,200}$ for $\mathcal{W}=\mathcal{A}^{1d}$ are shown in table 1. One can see that the numbers of table 1 for $\mathcal{W}=\mathcal{A}^{sym}$ and $\mathcal{W}=\mathcal{A}^{1d}$ confirm the aforementioned critical remarks concerning $\mathcal{A}^{1d}$.

One can see that in our numerical examples namely $\mathcal{A}^{sym}$ has the least $c_{1,200}$ (that is the best trade-off between the error and bias numbers $e_{1,200}$ and $b_{1,200}$) among all space-invariant filters $\mathcal{A}^{simp}$, $\mathcal{A}^{1d}$, $\mathcal{A}^{sym}$, Φ_1 mentioned in table 1 for the case when $|\hat{g}|^2$ is not known.

Figures 12, 13 show the reconstructions

$$Cf_1=\mathcal{N}_a\mathcal{W}p,\quad Cf_3=Cf_3(\mathcal{N}_a\mathcal{W}p,a,\mathcal{W}p) \tag{8.9}$$

(from p shown in Figure 2(c)), their profiles for $j=64$ and

$$\begin{aligned}&M_{200}Cf_1=M_{200}\mathcal{N}_a\mathcal{W}\mathbf{p},\quad D_{200}Cf_1=D_{200}\mathcal{N}_a\mathcal{W}\mathbf{p},\\&M_{200}Cf_3=M_{200}Cf_3(\mathcal{N}_a\mathcal{W}\mathbf{p},a,\mathcal{W}\mathbf{p}),\quad D_{200}Cf_3=D_{200}Cf_3(\mathcal{N}_a\mathcal{W}\mathbf{p},a,\mathcal{W}\mathbf{p})\end{aligned} \tag{8.10}$$

(where $\mathbf{p}$ is the Poisson field for our phantom) for $\mathcal{W}=\mathcal{A}^{sym}$. Besides, figures 10, 11 show the reconstruction Cf_3 of (8.9), its profile for $j=64$ and $M_{200}Cf_3$, $D_{200}Cf_3$ of (8.10) for $\mathcal{W}=\mathcal{A}^{simp}$, $\mathcal{A}^{1d}$.

We do not show Cf_1, Cf_3 of (8.9) and related images for $\mathcal{W}=\mathcal{W}^{opt},\mathcal{W}^{sym}$ (these images are shown in the first version of our paper). The reasons are that: (1) the filters $\mathcal{W}^{opt}$, $\mathcal{W}^{sym}$ are given for the case of known $|\hat{g}|^2$ and, therefore, can not be used directly in real SPECT imaging (modeled by Problem 1.2) and (2) we try to avoid too many images. Nevertheless, for the completeness of presentation the error and bias numbers for Cf_1 and Cf_3 of (8.9) with $\mathcal{W}=\mathcal{W}^{opt},\mathcal{W}^{sym}$ are mentioned in tables 2 and 3 considered below.

Table 2 shows the numbers $\eta^{(1)}=\eta(Cf_1,Cf_1^0,X)$ for Cf_1^0, Cf_1 of (8.3), (8.9) and $e_{2,200}^{(1)}$, $b_{2,200}^{(1)}$, $d_{2,200}^{(1)}$, $c_{2,200}^{(1)}$ of (7.7), (7.11) for $\mathcal{W}=\mathcal{W}^{opt}$, $\mathcal{W}^{sym}$, $\mathcal{A}^{simp}$, $\mathcal{A}^{1d}$, $\mathcal{A}^{sym}$ (and for Φ_1 of [GN1]) and $P_a^{-1}=\mathcal{N}_a$.

Table 3 shows, in particular, the numbers $\eta^{(3)}=\eta(Cf_3,Cf_3^0,X)$ for Cf_3, Cf_3^0 of (8.3), (8.9) and $e_{2,200}^{(3)}$, $b_{2,200}^{(3)}$, $d_{2,200}^{(3)}$, $c_{2,200}^{(3)}$ of (7.7) (7.11) for $\mathcal{W}=\mathcal{W}^{opt}$, $\mathcal{W}^{sym}$, $\mathcal{A}^{simp}$, $\mathcal{A}^{1d}$, $\mathcal{A}^{sym}$ (and for Φ_1 of [GN1]) and P_a^{-1} defined by (8.8).

Figures 10, 11 and the numbers of table 3 for $\mathcal{W}=\mathcal{A}^{1d},\mathcal{A}^{simp},\Phi_1$ confirm the well-known numerical result (see [KDS], [SKC], [BCB], [C]) that filters like $\mathcal{A}^{1d}$ are relatively efficient in the framework of reconstructions like Cf_3 of (8.9) (in particular, in table 3, $\mathcal{A}^{1d}$ has smaller $c_{2,200}^{(3)}$ than $\mathcal{A}^{simp}$ and Φ_1). A theoretical explanation of this numerical result was given in Subsection 4.3. Nevertheless, figures 11, 13 and (related part of) table 3 show that $\mathcal{A}^{1d}$ is less optimal than $\mathcal{A}^{sym}$ in the framework of the reconstruction Cf_3 of (8.9).

Note also that the reconstruction Cf_1 of (8.9) is not interesting for $\mathcal{W}=\mathcal{A}^{simp}$, $\mathcal{A}^{1d}$ as one can see, in particular, from (related part of) table 2. The reason is that Cf_1 of

(8.9) is much more sensitive to residual noise in $\mathcal{W}p$ than Cf_3 and that the residual noise in $\mathcal{A}^{simp}p$ and $\mathcal{A}^{1d}p$ is rather strong. Therefore, to avoid too many images in our paper we do not show Cf_1 of (8.9) and related images for $\mathcal{W} = \mathcal{A}^{simp}$, $\mathcal{A}^{1d}$.

Similarly with the case of table 1, one can see that in our numerical examples namely $\mathcal{A}^{sym}$ has the least $c^{(i)}_{2,200}$ (that is the best trade-off between the error and bias numbers $e^{(i)}_{2,200}$ and $b^{(i)}_{2,200}$), $i = 1, 3$, among all space-invariant filters $\mathcal{A}^{simp}$, $\mathcal{A}^{1d}$, $\mathcal{A}^{sym}$, Φ_1 mentioned in tables 2 and 3 for the case when $|\hat{g}|^2$ is not known.

To avoid too many images in our paper we do not show images obtained using Φ_1 of [GN1]. Actually, these images confirm that $\mathcal{A}^{sym}$ works better than Φ_1. Nevertheless, as it was already mentioned in Subsection 5.1 and will be mentioned in Subsection 8.4 there is also some similarity between Φ_1 and $\mathcal{A}^{sym}$.

8.4. Illustration of space-variant filtrations of [GN2] and [GN3]. An important property of the space-invariant filters Φ_ε and $\mathcal{A}^{sym}$ consists in the fact that they have efficient space-variant analogs $\Phi_{l_1,l_2,\varepsilon}$ and $\mathcal{A}^{sym}_{l_1,l_2}$ constructed in [GN2], [GN3] and mentioned in Subsections 5.2 and 5.3 of the present paper.

Figures 14(a)-(d) show $\mathcal{W}p$, $|F\Lambda\mathcal{W}p|$ (for p shown in figure 2(c)) and $M_{200}\mathcal{W}\mathbf{p}$, $D_{200}\mathcal{W}\mathbf{p}$ (where $\mathbf{p}$ is the Poisson field of (1.4) for our phantom) for $\mathcal{W} = \mathcal{A}^{sym}_{8,8}$.

Table 1 shows the numbers ζ, $e_{1,200}$, $b_{1,200}$, $d_{1,200}$, $c_{1,200}$ of (7.6), (7.10) for $\mathcal{W} = \mathcal{A}^{sym}_{8,8}$ and $\mathcal{W} = \Phi_{8,8,1}$.

Figures 15(a)-(d) show

$$Cf_3 = Cf_3(\mathcal{N}_a\mathcal{W}_1 p, a, \mathcal{W}p), \tag{8.11}$$

its profile for $j = 64$ and related $M_{200}Cf_3$, $D_{200}Cf_3$ (defined as in (8.10) but with $Cf_3(\mathcal{N}_a\mathcal{W}_1\mathbf{p}, a, \mathcal{W}\mathbf{p})$ in place of $Cf_3(\mathcal{N}_a\mathcal{W}\mathbf{p}, a, \mathcal{W}\mathbf{p})$) for $\mathcal{W}_1 = \mathcal{A}^{sym}$, $\mathcal{W} = \mathcal{A}^{sym}_{8,8}$.

Table 3 shows the numbers $\eta^{(3)} = \eta(Cf_3, Cf^0_3, X)$ for Cf^0_3, Cf_3 of (8.3), (8.11) and $e^{(3)}_{2,200}$, $b^{(3)}_{2,200}$, $d^{(3)}_{2,200}$, $c^{(3)}_{2,200}$ defined as in (7.7) (7.11) with $Cf_3(\mathcal{N}_a\mathcal{W}_1\mathbf{p}, a, \mathcal{W}\mathbf{p})$ in place of $P_a^{-1}\mathcal{W}\mathbf{p}$ and $P_a^{-1}g = Cf^0_3$ of (8.3) for $(\mathcal{W}_1, \mathcal{W}) = (\mathcal{A}^{sym}, \mathcal{A}^{sym}_{8,8})$ and $(\mathcal{W}_1, \mathcal{W}) = (\Phi_1, \Phi_{8,8,1})$.

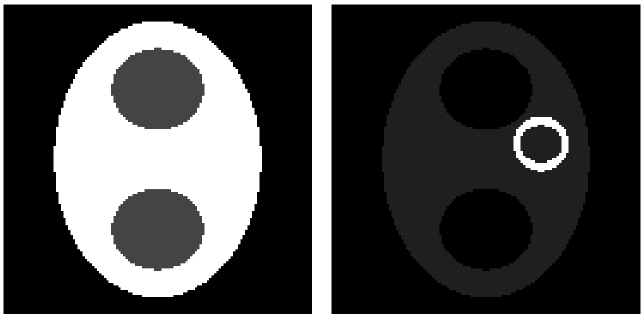
Note that the high-frequency component of the residual noise in $\mathcal{W}p$ for $\mathcal{W} = \mathcal{A}^{sym}_{8,8}$, $\Phi_{8,8,1}$ is less negligible than for $\mathcal{W} = \mathcal{A}^{sym}$, Φ_1 and that (in general and in our case in particular) $\mathcal{N}_a$ is rather sensitive to this noise component or, more precisely, much more sensitive than the classical FBP algorithm. In addition, $\mathcal{W}_1 p$ is used in (8.11) for the first approximation $Cf_1 = \mathcal{N}_a\mathcal{W}_1 p$ only and $\mathcal{W}p$ is used in (8.11) in the framework of iterations of the classical FBP algorithm only. Therefore, to obtain the best Cf_3 we deal with $\mathcal{W}_1 \neq \mathcal{W}$ in (8.11).

We do not show $\mathcal{W}p$ for $\mathcal{W} = \Phi_{8,8,1}$ and Cf_3 of (8.11) and related images for $\mathcal{W} = \Phi_{8,8,1}$, $\mathcal{W}_1 = \Phi_1$. The reasons are that: (1) for our phantom (described in subsection 8.2) the aforementioned images are more or less similar to the corresponding images for $\mathcal{W} = \mathcal{A}^{sym}_{8,8}$ and $\mathcal{W}_1 = \mathcal{A}^{sym}$ (actually, $\mathcal{W}p$ and Cf_3 of (8.11) are somewhat more smooth for $\mathcal{W} = \Phi_{8,8,1}$, $\mathcal{W}_1 = \Phi_1$ than for $\mathcal{W} = \mathcal{A}^{sym}_{8,8}$ and $\mathcal{W}_1 = \mathcal{A}^{sym}$) and (2) we try to avoid too many images in our paper.

One can see that in our numerical examples among all filtering schemes mentioned in the present work (1) namely $\mathcal{A}^{sym}_{8,8}$ and $\Phi_{8,8,1}$ have the best trade-off (the least $c_{1,200}$)

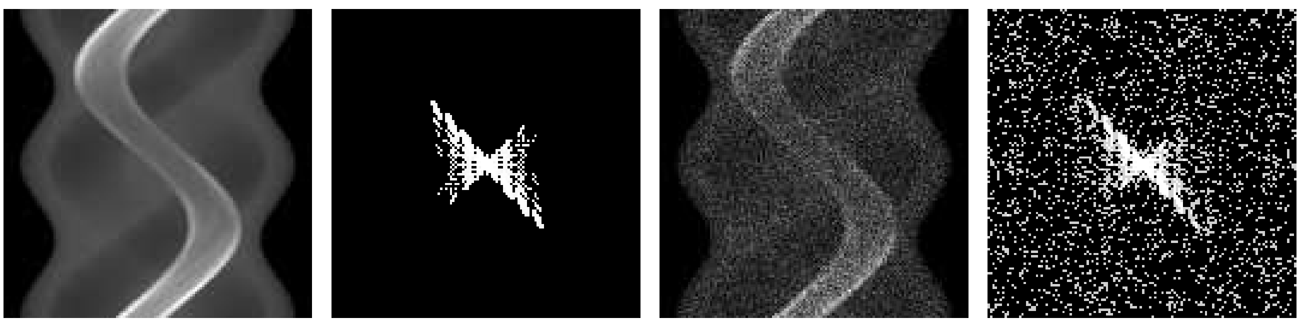
between the error and bias numbers $e_{1,200}$ and $b_{1,200}$ for the case when $|\hat{g}|$ is not known (see table 1) and (2) namely $\mathcal{W} = \mathcal{A}^{sym}_{8,8}$ (with $\mathcal{W}_1 = \mathcal{A}^{sym}$) has the best trade-off (the least $c^{(3)}_{2,200}$) between the error and bias numbers $e^{(3)}_{2,200}$ and $b^{(3)}_{2,200}$ for the case when $|\hat{g}|$ is not known (see table 3).

For more information on $\Phi_{l_1,l_2,\varepsilon}$ see [GN2]. For more information on $\mathcal{A}^{sym}_{l_1,l_2}$ (and on some other space- variant Wiener type filters for solving Problem 1.1 and Problem 1.2 via (1.5)) see [GN3].



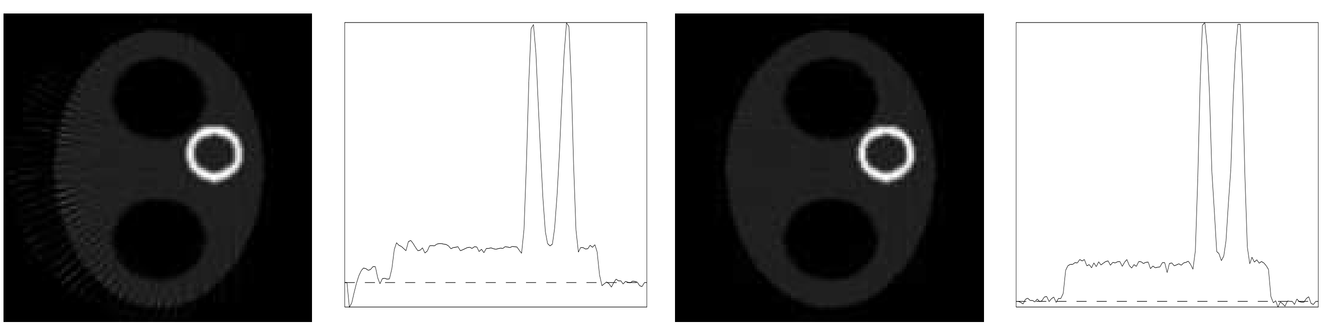
(a) (b)

Figure 1. Attenuation map a (a) and emitter activity f (b).



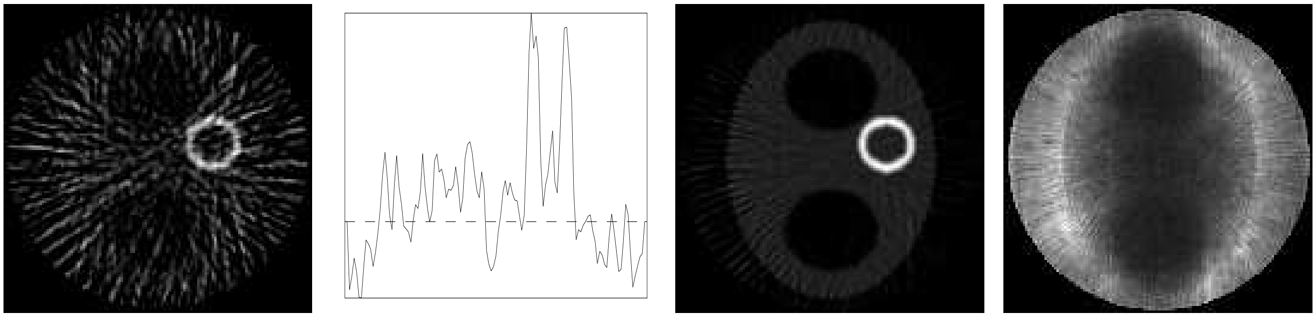
(a) (b) (c) (d)

Figure 2. Noiseless emission data $g = CP_a f$ (a), spectrum $|Fg|$ (b), noisy emission data p (c), spectrum $|Fp|$ (d).



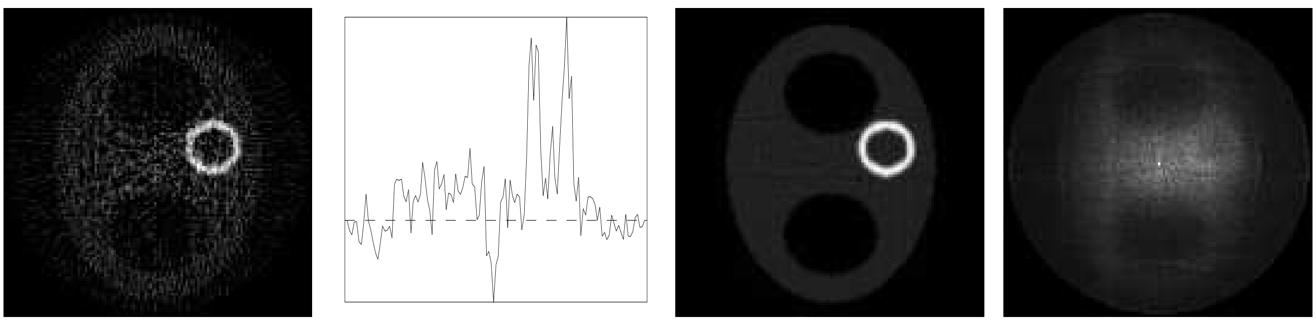
(a) (b) (c) (d)

Figure 3. Reconstructions $Cf_1^0 = \mathcal{N}_a g$ (a) and $Cf_3^0 = Cf_3(Cf_1^0, a, g)$ (c) from the noiseless emission data g, and their profiles for $j = 64$ (b), (d).



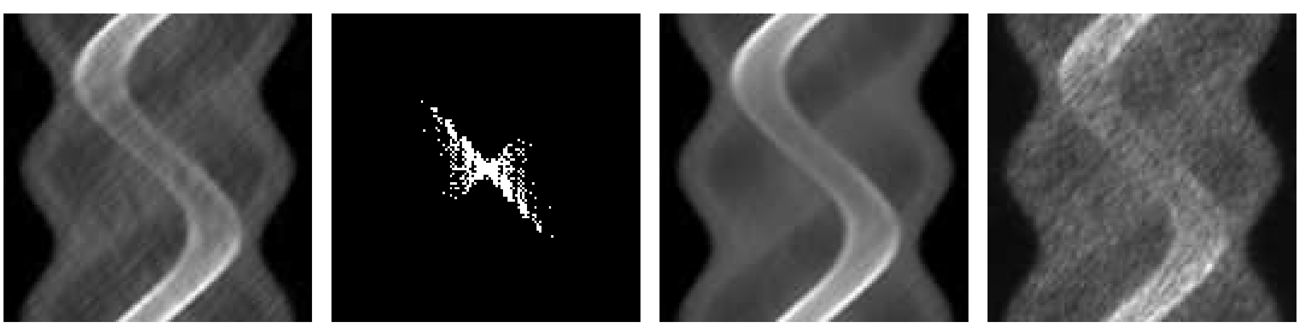
(a) (b) (c) (d)

Figure 4. Reconstruction $Cf_1 = \mathcal{N}_a p$ (a) with its profile for $j = 64$ (b) from the noisy emission data p without any filtration, and related $M_{200}Cf_1$ (c) and $D_{200}Cf_1$ (d).



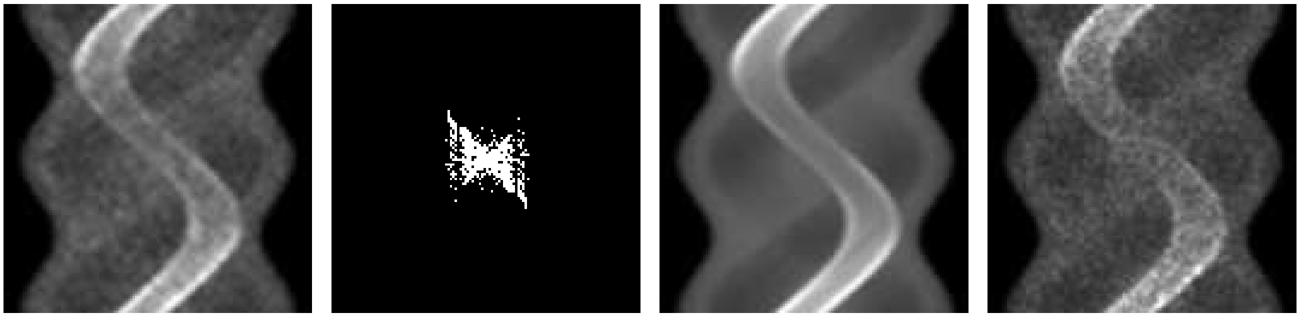
(a) (b) (c) (d)

Figure 5. Reconstruction $Cf_3 = Cf_3(\mathcal{N}_a p, a, p)$ (a) with its profile for $j = 64$ (b) from the noisy emission data p without any filtration, and related $M_{200}Cf_3$ (c) and $D_{200}Cf_3$ (d).



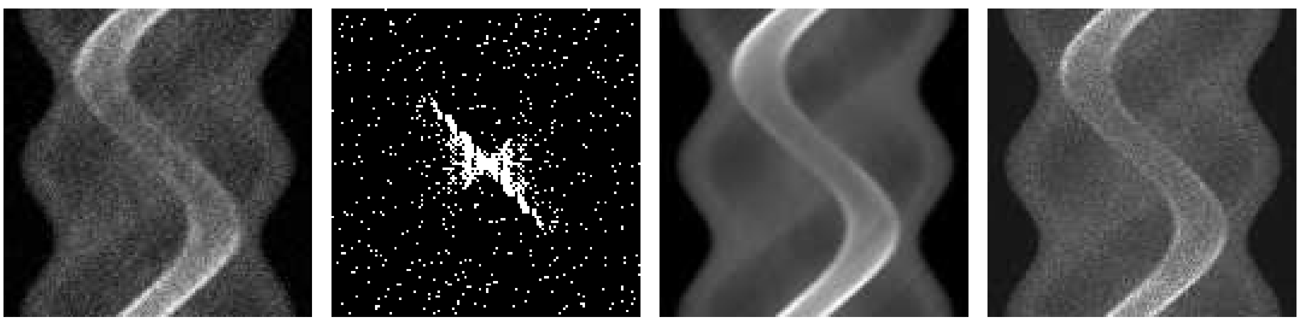
(a) (b) (c) (d)

Figure 6. Filtration result $\mathcal{W}p$ (a), its spectrum $|F\Lambda\mathcal{W}p|$ (b), $M_{200}\mathcal{W}\mathbf{p}$ (c) and $D_{200}\mathcal{W}\mathbf{p}$ (d) for $\mathcal{W} = \mathcal{W}^{opt}$.



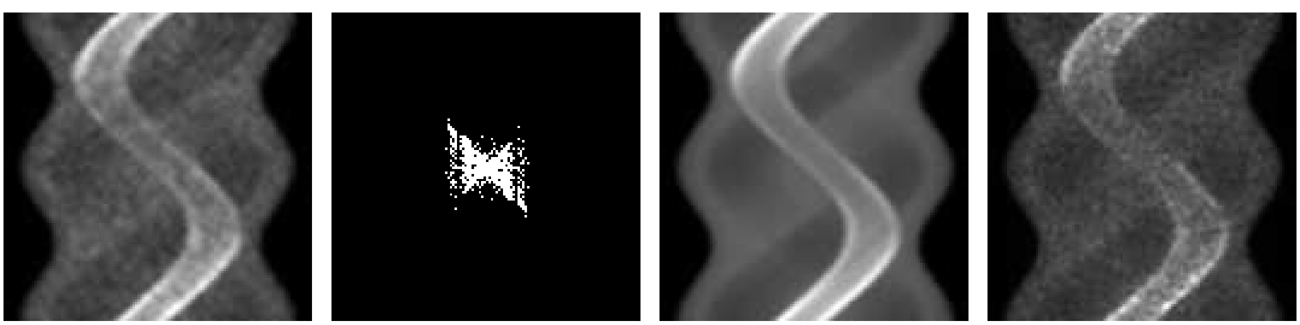
(a) (b) (c) (d)

Figure 7. Filtration result $\mathcal{W}p$ (a), its spectrum $|F\Lambda\mathcal{W}p|$ (b), $M_{200}\mathcal{W}\mathbf{p}$ (c) and $D_{200}\mathcal{W}\mathbf{p}$ (d) for $\mathcal{W} = \mathcal{W}^{sym}$.



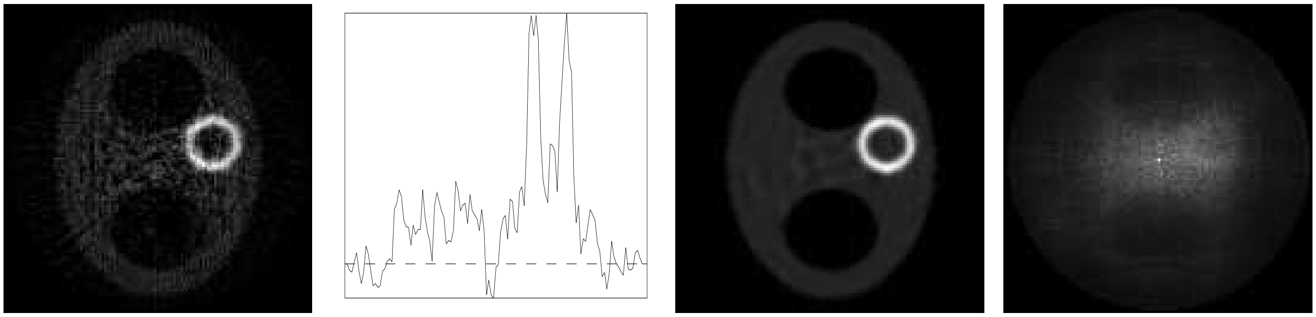
(a) (b) (c) (d)

Figure 8. Filtration result $\mathcal{W}p$ (a), its spectrum $|F\Lambda\mathcal{W}p|$ (b), $M_{200}\mathcal{W}\mathbf{p}$ (c) and $D_{200}\mathcal{W}\mathbf{p}$ (d) for $\mathcal{W} = \mathcal{A}^{simp}$.



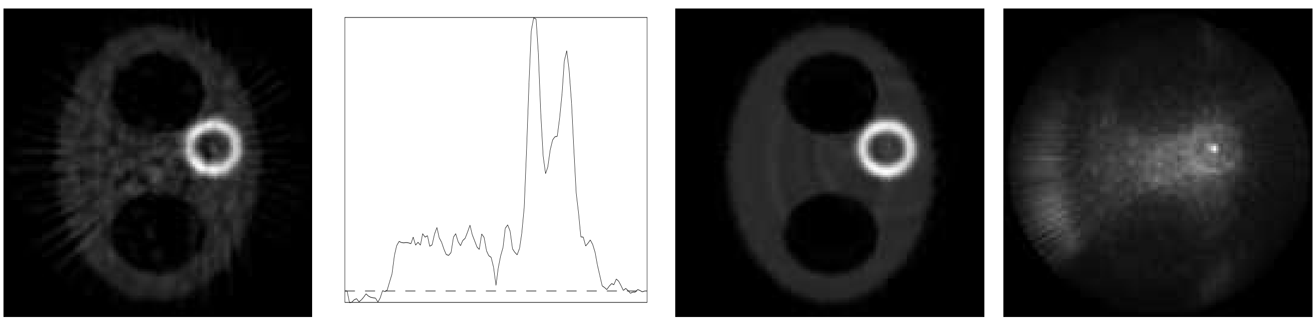
(a) (b) (c) (d)

Figure 9. Filtration result $\mathcal{W}p$ (a), its spectrum $|F\Lambda\mathcal{W}p|$ (b), $M_{200}\mathcal{W}\mathbf{p}$ (c) and $D_{200}\mathcal{W}\mathbf{p}$ (d) for $\mathcal{W} = \mathcal{A}^{sym}$.



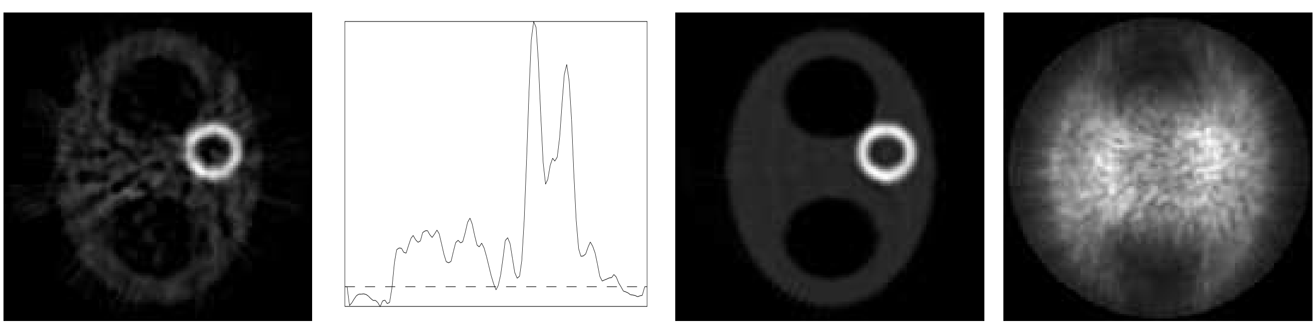
(a) (b) (c) (d)

Figure 10. Reconstruction $Cf_3 = Cf_3(\mathcal{N}_a\mathcal{W}p, a, \mathcal{W}p)$ (a), its profile for $j = 64$ (b) and related $M_{200}Cf_3$ (c) and $D_{200}Cf_3$ (d) for $\mathcal{W} = \mathcal{A}^{simp}$.



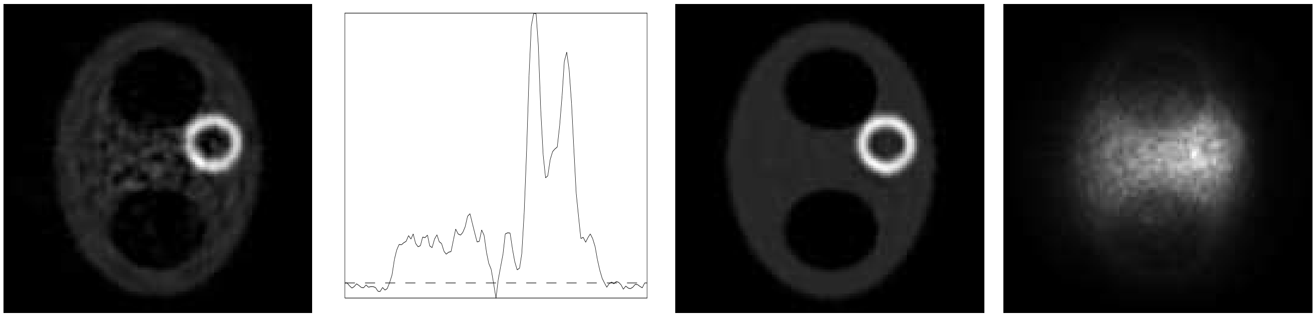
(a) (b) (c) (d)

Figure 11. Reconstruction $Cf_3 = Cf_3(\mathcal{N}_a\mathcal{W}p, a, \mathcal{W}p)$ (a), its profile for $j = 64$ (b) and related $M_{200}Cf_3$ (c) and $D_{200}Cf_3$ (d) for $\mathcal{W} = \mathcal{A}^{1d}$.



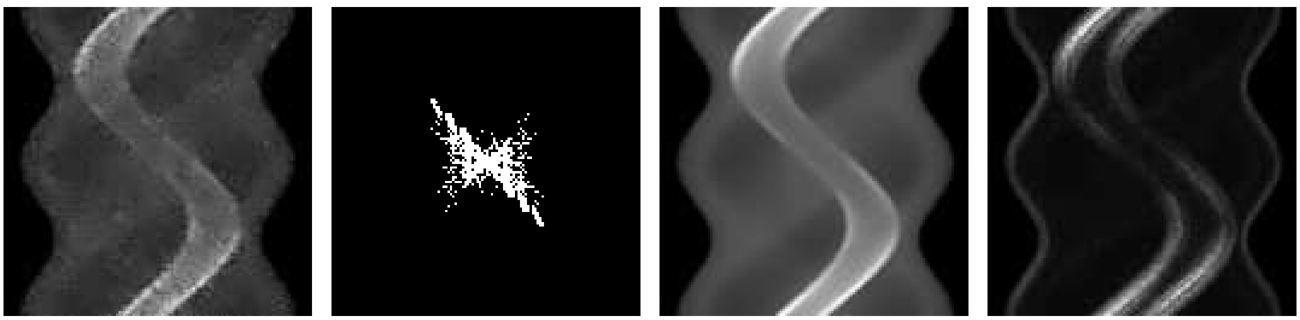
(a) (b) (c) (d)

Figure 12. Reconstruction $Cf_1 = \mathcal{N}_a\mathcal{W}p$ (a), its profile for $j = 64$ (b) and related $M_{200}Cf_1$ (c) and $D_{200}Cf_1$ (d) for $\mathcal{W} = \mathcal{A}^{sym}$.



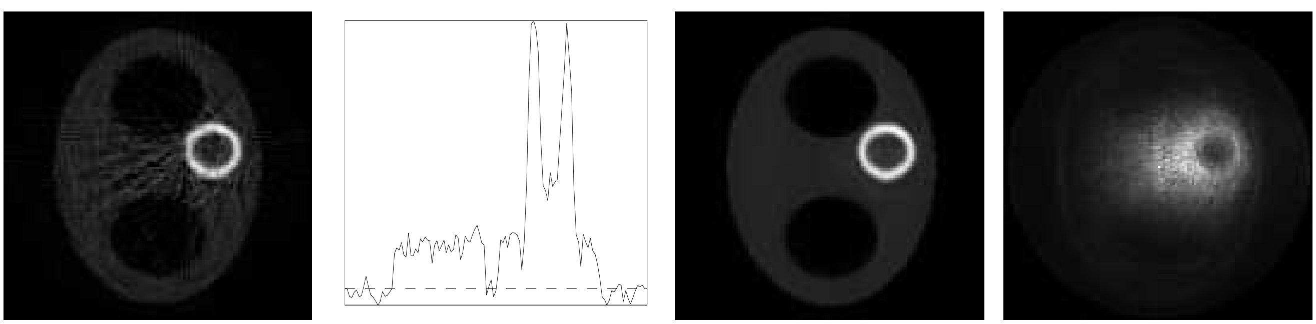
(a) (b) (c) (d)

Figure 13. Reconstruction $Cf_3 = Cf_3(\mathcal{N}_a\mathcal{W}p, a, \mathcal{W}p)$ (a), its profile for $j = 64$ (b) and related $M_{200}Cf_3$ (c) and $D_{200}Cf_3$ (d) for $\mathcal{W} = \mathcal{A}^{sym}$.



(a) (b) (c) (d)

Figure 14. Filtration result $\mathcal{W}p$ (a), its spectrum $|F\Lambda\mathcal{W}p|$ (b), $M_{200}\mathcal{W}\mathbf{p}$ (c) and $D_{200}\mathcal{W}\mathbf{p}$ (d) for $\mathcal{W} = \mathcal{A}^{sym}_{8,8}$.



(a) (b) (c) (d)

Figure 15. Reconstruction $Cf_3 = Cf_3(\mathcal{N}_a\mathcal{W}_1p, a, \mathcal{W}p)$ (a), its profile for $j = 64$ (b) and related $M_{200}Cf_3$ (c) and $D_{200}Cf_3$ (d) for $\mathcal{W}_1 = \mathcal{A}^{sym}$, $\mathcal{W} = \mathcal{A}^{sym}_{8,8}$.

	ζ	$e_{1,200}$	$b_{1,200}$	$d_{1,200}$	$c_{1,200}$
$\mathcal{W}^{opt}$	0.075	0.076	0.050	0.057	0.276
$\mathcal{W}^{sym}$	0.094	0.095	0.064	0.071	0.384
$\mathcal{A}^{simp}$	0.160	0.158	0.044	0.152	0.594
$\mathcal{A}^{1d}$	0.142	0.143	0.085	0.116	0.711
$\mathcal{A}^{sym}$	0.096	0.097	0.064	0.073	0.393
$\mathcal{A}^{sym}_{8,8}$	0.110	0.112	0.032	0.107	0.323
Φ_1	0.105	0.106	0.087	0.061	0.497
$\Phi_{8,8,1}$	0.089	0.091	0.047	0.078	0.315

Table 1. Numbers $\zeta = e_{1,1}$ and $e_{1,200}$, $b_{1,200}$, $d_{1,200}$, $c_{1,200}$ of (7.6), (7.10) for $\mathcal{W} = \mathcal{W}^{opt}$, $\mathcal{W}^{sym}$, $\mathcal{A}^{simp}$, $\mathcal{A}^{1d}$, $\mathcal{A}^{sym}$, $\mathcal{A}^{sym}_{8,8}$, Φ_1, $\Phi_{8,8,1}$.

	$\eta^{(1)}$	$e^{(1)}_{2,200}$	$b^{(1)}_{2,200}$	$d^{(1)}_{2,200}$	$c^{(1)}_{2,200}$
$\mathcal{W}^{opt}$	0.273	0.274	0.215	0.170	0.190
$\mathcal{W}^{sym}$	0.369	0.370	0.240	0.282	0.254
$\mathcal{A}^{simp}$	0.782	0.735	0.180	0.716	0.443
$\mathcal{A}^{1d}$	0.509	0.506	0.277	0.424	0.358
$\mathcal{A}^{sym}$	0.378	0.380	0.240	0.295	0.259
Φ_1	0.376	0.381	0.308	0.224	0.292

Table 2. Numbers $\eta^{(1)} = e^{(1)}_{2,1}$ and $e^{(1)}_{2,200}$, $b^{(1)}_{2,200}$, $d^{(1)}_{2,200}$, $c^{(1)}_{2,200}$ of (7.7), (7.11) for $\mathcal{W} = \mathcal{W}^{opt}$, $\mathcal{W}^{sym}$, $\mathcal{A}^{simp}$, $\mathcal{A}^{1d}$, $\mathcal{A}^{sym}$, Φ_1 and $P_a^{-1} = \mathcal{N}_a$.

	$\eta^{(3)}$	$e^{(3)}_{2,200}$	$b^{(3)}_{2,200}$	$d^{(3)}_{2,200}$	$c^{(3)}_{2,200}$
$\mathcal{W}^{opt}$	0.220	0.221	0.174	0.137	0.374
$\mathcal{W}^{sym}$	0.266	0.266	0.216	0.156	0.495
$\mathcal{A}^{simp}$	0.401	0.411	0.153	0.382	0.741
$\mathcal{A}^{1d}$	0.309	0.306	0.260	0.161	0.637
$\mathcal{A}^{sym}$	0.273	0.270	0.217	0.162	0.506
$\mathcal{A}^{sym}_{8,8}$	0.271	0.274	0.146	0.231	0.418
Φ_1	0.335	0.335	0.311	0.128	0.773
$\Phi_{8,8,1}$	0.252	0.255	0.185	0.175	0.438

Table 3. Numbers $\eta^{(3)} = e^{(3)}_{2,1}$ and $e^{(3)}_{2,200}$, $b^{(3)}_{2,200}$, $d^{(3)}_{2,200}$, $c^{(3)}_{2,200}$ of (7.7), (7.11) for $\mathcal{W} = \mathcal{W}^{opt}$, $\mathcal{W}^{sym}$, $\mathcal{A}^{simp}$, $\mathcal{A}^{1d}$, $\mathcal{A}^{sym}$, $\mathcal{A}^{sym}_{8,8}$ (with $\mathcal{W}_1 = \mathcal{A}^{sym}$), Φ_1, $\Phi_{8,8}$ (with $\mathcal{W}_1 = \Phi_1$) and P_a^{-1} defined by (8.8).

9. Conclusion

For the Poisson model (3.1) and, more generally, for the noise model (3.4) we found explicit formulas for the optimal space-invariant Wiener type filter with a priori geometric restrictions (3.6) on the window function, see Theorem 3.1 and Corollary 3.1 of Section 3. We say that this filter is restrictedly optimal in the Wiener sense and denote it as $\mathcal{W}^{r.o.}$. It is assumed that the object power spectrum $|\hat{g}|^2$ and the variance parameter V are known in $\mathcal{W}^{r.o.}$.

For the case when $|\hat{g}|^2$ and V are not known, we considered the data dependent space-invariant filter $\mathcal{A}$ approximating $\mathcal{W}^{r.o.}$ by formulas of Subsection 4.2. We show that, at least for the Poisson case, this approximation (with $V \approx (n_s n_\varphi)^{-1/2}\hat{p}(0)$) is very efficient if geometric condition (4.13) is fulfilled for each $j \in \hat{I}$. We say that $\mathcal{A}$ is approximately optimal in the Wiener sense.

We showed that in an important particular case, under the Poisson assumptions, our filter $\mathcal{A}$ is reduced to the well-known (see [KDS], [SKC], [BCB], [C]) "one-dimensional" filter $\mathcal{A}^{1d}$ going back to [KDS]. This permits to explain a relative efficiency of the "1D" filtering scheme of [KDS] in SPECT and PET imaging based on the classical FBP algorithm (or its iterative use). See Subsection 4.3.

By the symmetric choice (4.15), (4.16) of the level sets S_α, we reduced $\mathcal{A}$ to $\mathcal{A}^{sym}$. We consider $\mathcal{A}^{sym}$ as a reasonable "2D" approximation to the optimal Wiener type filter $\mathcal{W}^{opt}$ of (3.3) for the Poisson model (3.1) with sufficiently regular g for the case when $|\hat{g}|^2$ is unknown. See Subsection 4.4.

In Subsection 5.3, an efficient space-variant version $\mathcal{A}^{sym}_{l_1,l_2}$ of $\mathcal{A}^{sym}$ is also presented. We do not know whether the space-invariant filter $\mathcal{A}^{sym}$ in its precise form of Section 4 was mentioned in the literature. In any case our principal results concerning $\mathcal{A}^{sym}$ consist in its justification proceeding from Theorem 3.1 and in its completely space-variant version $\mathcal{A}^{sym}_{l_1,l_2}$. To our knowledge no complete generalization to the space-variant case of the filtration approach of [KDS] was mentioned in the literature before the present work.

In Section 8, the optimal, restrictedly optimal and approximately optimal space-invariant Wiener (or Wiener type) filters $\mathcal{W}^{opt}$, $\mathcal{W}^{sym}$, $\mathcal{A}^{simp}$, $\mathcal{A}^{1d}$, $\mathcal{A}^{sym}$ and the space-variant version $\mathcal{A}^{sym}_{l_1,l_2}$ of $\mathcal{A}^{sym}$ are illustrated by numerical examples in the framework of simulated SPECT imaging based on generalized and/or classical FBP algorithms. In addition, a numerical comparison (of the aforementioned filters) with the space-invariant Φ_1 and space-variant $\Phi_{l_1,l_2,1}$ data dependent filters of [GN1] and [GN2] is also given. One can see that in the numerical examples of Section 8 namely $\mathcal{A}^{sym}$ gives the best results on the level of the error-bias trade-off among all filters $\mathcal{A}^{simp}$, $\mathcal{A}^{1d}$, $\mathcal{A}^{sym}$, Φ_1 with unknown $|\hat{g}|^2$ and namely $\mathcal{A}^{sym}_{8,8}$ gives, in particular, the best iterative reconstruction Cf_3 on the level of the error-bias trade-off among all filters $\mathcal{A}^{simp}$, $\mathcal{A}^{1d}$, $\mathcal{A}^{sym}$, Φ_1, $\Phi_{8,8,1}$, $\mathcal{A}^{sym}_{8,8}$ with unknown object power spectrum $|\hat{g}|^2$; see Subsections 8.3 and 8.4.

We emphasize that the present work is not a topical review. In particular, we do not discuss the methods of [SV], [LiH], [HL], [Hs], [CK], [LaR], [F]. Comparative studies of the present work are given in the framework of the most related preceding results only. Further comparisons will be given elsewhere.

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