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Abstract. We consider a non-relativistic electron interacting with a classical magnetic field pointing along the x_3-axis and with a quantized electromagnetic field. The system is translation invariant in the x_3-direction and we consider the reduced Hamiltonian $H(P_3)$ associated with the total momentum P_3 along the x_3-axis. For a fixed momentum P_3 sufficiently small, we prove that $H(P_3)$ has a ground state in the Fock representation if and only if $E'(P_3) = 0$, where $P_3 \mapsto E'(P_3)$ is the derivative of the map $P_3 \mapsto E(P_3) = \inf \sigma(H(P_3))$. If $E'(P_3) \neq 0$, we obtain the existence of a ground state in a non-Fock representation. This result holds for sufficiently small values of the coupling constant.
MSC: 81V10; 81Q10; 81Q15

1. Introduction

In this paper we pursue the analysis of a model considered in [**AGG1**], describing a non-relativistic particle (an electron) interacting both with the quantized electromagnetic field and a classical magnetic field pointing along the x_3-axis. An ultraviolet cutoff is imposed in order to suppress the interaction between the electron and the photons of energies bigger than a fixed, arbitrary large parameter Λ. The total system being invariant by translations in the x_3-direction, it can be seen (see [**AGG1**]) that the corresponding Hamiltonian admits a decomposition of the form $H \simeq \int_{\mathbb{R}}^{\oplus} H(P_3)dP_3$ with respect to the spectrum of the total momentum along the x_3-axis that we denote by P_3. For any given P_3 sufficiently close to 0, the existence of a ground state for $H(P_3)$ is proven in [**AGG1**] provided an infrared regularization is introduced (besides a smallness assumption on the coupling parameter). Our aim is to address the question of the existence of a ground state without requiring any infrared regularization.

The model considered here is closely related to similar non-relativistic QED models of freely moving electrons, atoms or ions, that have been studied recently (see [**BCFS, FGS1, Hi, CF, Ch, HH, CFP, FP**] for the case of one single electron, and [**AGG2, LMS, FGS2, HH, LMS2**] for atoms or ions). In each of these papers, the physical systems are translation invariant, in the sense that the associated Hamiltonian H commutes with the operator of total momentum P. As a consequence, $H \simeq \int_{\mathbb{R}^3} H(P)dP$, and one is led to study the spectrum of the fiber Hamiltonian $H(P)$ for fixed P's.

For the one-electron case, an aspect of the so-called *infrared catastrophe* lies in the fact that, for $P \neq 0$, $H(P)$ does not have a ground state in the Fock space (see [**CF, Ch, HH, CFP**]). More precisely, if an infrared cutoff of parameter σ is introduced in the model in order to remove the interaction between the electron and the photons of energies less than σ, the associated Hamiltonian $H_\sigma(P)$ does have a ground state $\Phi_\sigma(P)$ in the Fock space. Nevertheless as $\sigma \to 0$, it is shown that $\Phi_\sigma(P)$ "leaves" the Fock space. Physically this can be interpreted by saying that a free moving electron in its ground state is surrounded by a cloud of infinitely many "soft" photons.

For negative ions, the absence of a ground state for $H(P)$ is established in [**HH**] under the assumption $\nabla E(P) \neq 0$, where $E(P) = \inf \sigma(H(P))$.

In [**CF**], with the help of operator-algebra methods, a representation of a *dressed 1-electron state* non-unitarily equivalent to the usual Fock representation of the canonical commutation relations is given. We shall obtain in this paper a related result, following a different approach, under the further assumption that the electron interact with a classical magnetic field and an electrostatic potential.

We shall first provide a necessary and sufficient condition for the existence of a ground state for $H(P_3)$. Namely we shall prove that the bottom of the spectrum, $E(P_3) = \inf \sigma(H(P_3))$, is an eigenvalue of $H(P_3)$ if and only if $E'(P_3) = 0$ where $E'(P_3)$ denotes the derivative of the map $P_3 \mapsto E(P_3)$. In the case $E'(P_3) \neq 0$, thanks to a (non-unitary) Bogoliubov transformation, in the same way as in [**Ar, DG2**], we shall define a "renormalized" Hamiltonian $H^{\mathrm{ren}}(P_3)$ which can be seen as an expression of the physical Hamiltonian in a non-Fock representation. Then we shall prove that $H^{\mathrm{ren}}(P_3)$ has a ground state. These results have been announced in [**AFGG**].

The regularity of the map $P_3 \mapsto E(P_3)$ plays a crucial role in our proof. Adapting [**Pi, CFP**] we shall see that $P_3 \mapsto E(P_3)$ is of class $\mathrm{C}^{1+\gamma}$ for some strictly positive γ. Let us also mention that our method can be adapted to the case of free moving hydrogenoid ions without spin, the condition $E'(P_3) = 0$ being replaced by $\nabla E(P) = 0$ (see Subsection 1.2 for a further discussion on this point).

The remainder of the introduction is organized as follows: In Subsection 1.1, a precise definition of the model considered in this paper is given, next, in Subsection 1.2, we state our results and compare them to the literature.

1.1. The model. We consider a non-relativistic electron of charge e and mass m interacting with a classical magnetic field pointing along the x_3-axis, an electrostatic potential, and the quantized electromagnetic field in the Coulomb gauge. The Hilbert space for the electron and the photon field is written as

$$\mathcal{H} = \mathcal{H}_{\mathrm{el}} \otimes \mathcal{H}_{\mathrm{ph}}, \tag{1.1}$$

where $\mathcal{H}_{\mathrm{el}} = \mathrm{L}^2(\mathbb{R}^3; \mathbb{C}^2)$ is the Hilbert space for the electron, and $\mathcal{H}_{\mathrm{ph}}$ is the symmetric Fock space over $\mathrm{L}^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ for the photons,

$$\mathcal{H}_{\mathrm{ph}} = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} S_n \left[\mathrm{L}^2(\mathbb{R}^3 \times \mathbb{Z}_2)^{\otimes^n} \right]. \tag{1.2}$$

Here S_n denotes the orthogonal projection onto the subspace of symmetric functions in $\mathrm{L}^2(\mathbb{R}^3 \times \mathbb{Z}_2)^{\otimes^n}$ in accordance with Bose-Einstein statistics. We shall use the notation $\mathbf{k} = (k, \lambda)$ for any $(k, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$, and

$$\int_{\mathbb{R}^3 \times \mathbb{Z}_2} d\mathbf{k} = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk. \tag{1.3}$$

Likewise, the scalar product in $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ is defined by

$$(1.4) \qquad (h_1, h_2) = \int_{\mathbb{R}^3 \times \mathbb{Z}_2} \bar{h}_1(\mathbf{k}) h_2(\mathbf{k}) d\mathbf{k} = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \bar{h}_1(k,\lambda) h_2(k,\lambda) dk.$$

The position and the momentum of the electron are denoted respectively by $x = (x_1, x_2, x_3)$ and $p = (p_1, p_2, p_3) = -i\nabla_x$. The classical magnetic field is of the form $(0, 0, b(x'))$, where $x' = (x_1, x_2)$ and $b(x') = (\partial a_2/\partial x_1)(x') - (\partial a_1/\partial x_2)(x')$. Here $a_j(x')$, $j = 1, 2$, are real functions in $C^1(\mathbb{R}^2)$. The electrostatic potential is denoted by $V(x')$. The quantized electromagnetic field in the Coulomb gauge is defined by

$$(1.5) \qquad \begin{aligned} A(x) &= \frac{1}{\sqrt{2\pi}} \int \frac{\epsilon^\lambda(k)}{|k|^{1/2}} \rho^\Lambda(k) \Big[e^{-ik\cdot x} a^*(\mathbf{k}) + e^{ik\cdot x} a(\mathbf{k}) \Big] d\mathbf{k}, \\ B(x) &= -\frac{i}{\sqrt{2\pi}} \int |k|^{1/2} \left(\frac{k}{|k|} \wedge \epsilon^\lambda(k) \right) \rho^\Lambda(k) \Big[e^{-ik\cdot x} a^*(\mathbf{k}) - e^{ik\cdot x} a(\mathbf{k}) \Big] d\mathbf{k}, \end{aligned}$$

where $\rho^\Lambda(k)$ denotes the characteristic function $\rho^\Lambda(k) = \mathbb{1}_{|k|\le\Lambda}(k)$ and Λ is an arbitrary large positive real number. Note that this explicit choice of the ultraviolet cutoff function ρ^Λ is made mostly for convenience. Our results would hold without change for any ρ^Λ satisfying $\int_{|k|\le 1} |k|^{-2} |\rho^\Lambda(k)|^2 d^3k + \int_{|k|\ge 1} |k| |\rho^\Lambda(k)|^2 d^3k < \infty$. The vectors $\epsilon^1(k)$ and $\epsilon^2(k)$ in (1.5) are real polarization vectors orthogonal to each other and to k. Besides $a^*(\mathbf{k})$ and $a(\mathbf{k})$ are the usual creation and annihilation operators obeying the canonical commutation relations

$$(1.6) \qquad \big[a^\#(\mathbf{k}), a^\#(\mathbf{k}')\big] = 0 \quad , \quad [a(\mathbf{k}), a^*(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}') = \delta_{\lambda\lambda'} \delta(k - k').$$

The Pauli Hamiltonian H_g associated with the system we consider is formally given by

$$(1.7) \qquad \begin{aligned} H_g = &\frac{1}{2m} \Big(p - ea(x') - gA(x) \Big)^2 - \frac{e}{2m} \sigma_3 b(x') \\ &- \frac{g}{2m} \sigma \cdot B(x) + V(x') + H_{\mathrm{ph}}, \end{aligned}$$

where the charge of the electron is replaced by a coupling parameter g in the terms containing the quantized electromagnetic field. The Hamiltonian for the photons in the Coulomb gauge is given by

$$(1.8) \qquad H_{\mathrm{ph}} = \mathrm{d}\Gamma(|k|) = \int |k| a^*(\mathbf{k}) a(\mathbf{k}) d\mathbf{k}.$$

Finally $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the 3-component vector of the Pauli matrices.

Noting that H_g formally commutes with the operator of total momentum in the direction x_3, $P_3 = p_3 + \mathrm{d}\Gamma(k_3)$, one can consider the reduced Hamiltonian associated with $P_3 \in \mathbb{R}$ that we denote by $H_g(P_3)$. For any fixed P_3, $H_g(P_3)$ acts on $L^2(\mathbb{R}^2; \mathbb{C}^2) \otimes \mathcal{H}_{\mathrm{ph}}$ and is formally given by

$$(1.9) \qquad \begin{aligned} H_g(P_3) = &\frac{1}{2m} \sum_{j=1,2} \Big(p_j - ea_j(x') - gA_j(x', 0) \Big)^2 - \frac{e}{2m} \sigma_3 b(x') + V(x') \\ &+ \frac{1}{2m} \Big(P_3 - \mathrm{d}\Gamma(k_3) - gA_3(x', 0) \Big)^2 - \frac{g}{2m} \sigma \cdot B(x', 0) + H_{\mathrm{ph}}. \end{aligned}$$

We define the infrared cutoff Hamiltonian $H_g^\sigma(P_3)$ by replacing $A(x)$ in (1.5) with

$$A_\sigma(x) = \frac{1}{\sqrt{2\pi}} \int \frac{\epsilon^\lambda(k)}{|k|^{1/2}} \rho_\sigma^\Lambda(k) \Big[e^{-ik\cdot x} a^*(\mathbf{k}) + e^{ik\cdot x} a(\mathbf{k}) \Big] d\mathbf{k}, \tag{1.10}$$

where $\rho_\sigma^\Lambda = \mathbb{1}_{\sigma \le |k| \le \Lambda}$, and similarly for $B_\sigma(x)$. We set $E_g(P_3) = \inf \sigma(H_g(P_3))$ and $E_{g\sigma}(P_3) = \inf \sigma(H_g^\sigma(P_3))$.

The electronic Hamiltonian $h(b, V)$ on $\mathrm{L}^2(\mathbb{R}^2; \mathbb{C}^2)$ is defined by

$$h(b, V) = \sum_{j=1,2} \frac{1}{2m} (p_j - e a_j(x'))^2 - \frac{e}{2m} \sigma_3 b(x') + V(x'). \tag{1.11}$$

Let $e_0 = \inf \sigma(h(b, V))$. We make the following hypothesis:

($\mathbf{H_0}$) $h(b, V)$ is essentially self-adjoint on $\mathrm{C}_0^\infty(\mathbb{R}^2; \mathbb{C}^2)$ and e_0 is an isolated eigenvalue of multiplicity 1.

We refer to [**AHS, So, IT, Ra**] for possible choices of b, V satisfying Hypothesis **($\mathbf{H_0}$)**. The following proposition is established in [**AGG1**, Theorem 2.3]:

Proposition 1.1. *Suppose Hypothesis* ($\mathbf{H_0}$). *For sufficiently small values of $|g|$, H_g is self-adjoint with domain $D(H_g) = D(H_0)$, and for any $\sigma \ge 0$ and $P_3 \in \mathbb{R}$, $H_g^\sigma(P_3)$ identifies with a self-adjoint operator with domain $D(H_g^\sigma(P_3)) = D(H_0(P_3))$. Moreover H_g admits the decomposition*

$$H_g = \int_{\mathbb{R}}^{\oplus} H_g(P_3) dP_3. \tag{1.12}$$

1.2. Results and comments. The key ingredient that we shall need in order to prove our main theorem (see Theorem 1.3 below) lies in the regularity of the map $P_3 \mapsto E'_{g\sigma}(P_3)$ uniformly in $\sigma \ge 0$.

Theorem 1.2. *Assume that* ($\mathbf{H_0}$) *holds. There exists $g_0 > 0$, $\sigma_0 > 0$ and $P_0 > 0$ such that for all $|g| \le g_0$, for all $0 \le \sigma \le \sigma_0$, for all P_3, k_3 such that $|P_3| \le P_0$, $|P_3 + k_3| \le P_0$, for all $\delta > 0$,*

$$|E'_{g\sigma}(P_3 + k_3) - E'_{g\sigma}(P_3)| \le \mathrm{C}_\delta |k_3|^{\frac{1}{4} - \delta}, \tag{1.13}$$

where C_δ is a positive constant depending only on δ.

Similar results for a free electron (that is for $b = V = 0$) interacting with the quantized electromagnetic field have been obtained recently (see [**Ch, CFP, FP**]). The model studied in the latter papers is technically simpler than the one considered here in that the fiber Hamiltonian $H(P)$ associated with a free electron does not contain the electronic part $h(b, V)$ and its (minimal) coupling to the quantized electromagnetic field. In particular the operator $H(P)$ in [**Ch, CFP, FP**] acts only on the Fock space, whereas in our case $H_{g\sigma}(P_3)$ still contains interactions between the electromagnetic field and the electronic degrees of freedom. We shall use the exponential decay of the ground states $\Phi_g^\sigma(P_3)$ in x' in order to overcome this difficulty.

It is proved in [**Ch**] (for a free electron) that $P \mapsto E(P) = \inf \sigma(H(P))$ is of class C^2 in a neighborhood of 0 thanks to a renormalization group analysis (see also [**BCFS**]). The author also shows that, still in a neighborhood of $P = 0$, the derivative $\nabla E(P)$ vanishes only at $P = 0$. In [**CFP**], with the help of what the authors call "iterative analytic perturbation theory", following a multiscale analysis developed in [**Pi**], it is proved, among other results, that $P \mapsto E(P)$ is of class $C^{5/4-\delta}$ for arbitrary small $\delta > 0$. The method has later been improved in [**FP**] leading to the C^2 property of $P \mapsto E(P)$.

In order to establish our main theorem, Theorem 1.3, the "degree of regularity" we need is reached as soon as $P_3 \mapsto E_{g\sigma}(P_3)$ is at least of order $C^{1+\gamma}$, uniformly in σ, for some $\gamma > 0$. Theorefore, although one can conjecture that $P_3 \mapsto E_{g\sigma}(P_3)$ is of class C^2 uniformly in σ, Theorem 1.2 is sufficient for our purpose. In order to prove it we shall adapt [**Pi, CFP**]: First, we shall give a short proof of the existence of a spectral gap for $H_g^\sigma(P_3)$ (restricted to the space of photons of energies bigger than σ) above the non-degenerate eigenvalue $E_{g\sigma}(P_3)$. Next we shall apply "iterative analytic perturbation theory".

We postpone the proof of Theorem 1.2 to the appendix. Since several parts are taken from [**Pi, CFP**], we shall not give all the details, rather we shall emphasize the differences with [**Pi, CFP**].

For $h \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, let us define the field operator $\Phi(h)$ by

$$\Phi(h) = \frac{1}{\sqrt{2}}(a^*(h) + a(h)), \tag{1.14}$$

where the creation operator $a^*(h)$ and the annihilation operator $a(h)$ are defined respectively by

$$a^*(h) = \int_{\mathbb{R}^3\times\mathbb{Z}_2} h(\mathbf{k}) a^*(\mathbf{k}) d\mathbf{k}, \quad a(h) = \int_{\mathbb{R}^3\times\mathbb{Z}_2} \bar{h}(\mathbf{k}) a(\mathbf{k}) d\mathbf{k}. \tag{1.15}$$

Hence, letting $h_{j,\sigma}(x')$ and $\tilde{h}_{j,\sigma}(x')$ for $j = 1, 2, 3$ be defined respectively by

$$\begin{aligned} h_{j,\sigma}(x', \mathbf{k}) &= \pi^{-1/2} \frac{\epsilon_j^\lambda(k)}{|k|^{1/2}} \rho_\sigma^\Lambda(k) e^{ik'\cdot x'}, \\ \tilde{h}_{j,\sigma}(x', \mathbf{k}) &= -i\pi^{-1/2} |k|^{1/2} \left(\frac{k}{|k|} \wedge \epsilon^\lambda(k) \right)_j \rho_\sigma^\Lambda(k) e^{ik'\cdot x'}, \end{aligned} \tag{1.16}$$

where $k' = (k_1, k_2)$, we have $A_{j,\sigma}(x', 0) = \Phi(h_{j,\sigma}(x'))$ and $B_{j,\sigma}(x', 0) = \Phi(\tilde{h}_{j,\sigma}(x'))$. The Weyl operator associated with $h \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ is denoted by $W(h) = e^{i\Phi(h)}$. Let $f_\sigma : \mathbb{R}^3 \times \mathbb{Z}_2 \to \mathbb{C}$ be defined by

$$f_\sigma(\mathbf{k}) = -\frac{g}{\sqrt{4\pi}} \frac{\rho_\sigma^\Lambda(k)\epsilon_\lambda^3(k)}{k_3|k|^{1/2}} \frac{E_{g\sigma}(P_3 - k_3) - E_{g\sigma}(P_3)}{E_{g\sigma}(P_3 - k_3) - E_{g\sigma}(P_3) + |k|}. \tag{1.17}$$

If $\sigma = 0$ we remove the subindex σ in the preceding notations. We recall from [**AGG1**, Lemma 4.3] that for g, σ, P_3 and $|k|$ sufficiently small,

$$E_{g\sigma}(P_3 - k_3) - E_{g\sigma}(P_3) \geq -\frac{3}{4}|k|. \tag{1.18}$$

Hence in particular for $\sigma > 0$, we have $f_\sigma \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, whereas if $\sigma = 0$ and $P_3 \mapsto E_g(P_3)$ is of class $C^{1+\gamma}$ with $\gamma > 0$, then

$$f \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \iff E_g'(P_3) = 0. \tag{1.19}$$

Similarly as in [**Ar**] (see also [**DG2, Pa**]), we define the "renormalized" (Bogoliubov transformed) Hamiltonian $H_{g\sigma}^{\text{ren}}(P_3)$ by the expression

$$H_{g\sigma}^{\text{ren}}(P_3) = W(if_\sigma) H_g^\sigma(P_3) W(if_\sigma)^*. \tag{1.20}$$

Notice that the identity (1.20) might only be formal for $\sigma = 0$ since in this case, by (1.19), f might not be in L^2. Nevertheless using usual commutation relations (see for instance [**DG1**]), we define for any $\sigma \geq 0$:

$$H^{\mathrm{ren}}_{g\sigma}(P_3) = \frac{1}{2m}\sum_{j=1,2}\Big(p_j - ea_j(x') - gA_{j,\sigma}(x',0) + g\mathrm{Re}(h_{j,\sigma}(x'), f_\sigma)\Big)^2$$
$$+ \frac{1}{2m}\Big(P_3 - \mathrm{d}\Gamma(k_3) - \Phi(k_3 f_\sigma) - \frac{1}{2}(k_3 f_\sigma, f_\sigma) - gA_{3,\sigma}(x',0) + g\mathrm{Re}(h_{3,\sigma}(x'), f_\sigma)\Big)^2$$
$$- \frac{e}{2m}\sigma_3 b(x') - \frac{g}{2m}\sigma\cdot\Big(B_\sigma(x',0) - \mathrm{Re}(\tilde{h}_\sigma(x'), f_\sigma)\Big) + V(x')$$
$$+ H_f + \Phi(|k| f_\sigma) + \frac{1}{2}(|k| f_\sigma, f_\sigma).$$

In the same way as for $H^\sigma_g(P_3)$ (see Proposition 1.1), one can verify that $H^{\mathrm{ren}}_{g\sigma}(P_3)$ is self-adjoint with domain $D(H^{\mathrm{ren}}_{g\sigma}(P_3)) = D(H_0(P_3))$ for any $\sigma \geq 0$. Besides for $\sigma > 0$, we have that $H^{\mathrm{ren}}_{g\sigma}(P_3)$ is unitarily equivalent to $H^\sigma_g(P_3)$, whereas for $\sigma = 0$, one can verify that $H^{\mathrm{ren}}_g(P_3)$ is unitarily equivalent to $H_g(P_3)$ if and only if $f \in \mathrm{L}^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. Our main result is:

THEOREM 1.3. *Suppose Hypothesis* ($\mathbf{H_0}$). *There exist* $g_0 > 0$ *and* $P_0 > 0$ *such that for all* $0 \leq |g| \leq g_0$ *and* $0 \leq |P_3| \leq P_0$,

(i) $H_g(P_3)$ *has a ground state if and only if* $E_g'(P_3) = 0$,
(ii) $H^{\mathrm{ren}}_g(P_3)$ *has a ground state.*

The proof of Theorem 1.3 can be adapted to the case of free moving hydrogenoid ions without spins[1], the condition $E_g'(P_3) = 0$ being replaced by $\nabla E_g(P) = 0$, where $E_g(P)$ denotes the bottom of the spectrum of the fiber Hamiltonian $H_g(P)$. The existence of ground states for atoms has been obtained in [**AGG2**] thanks to a Power-Zienau-Wooley transformation and the crucial property $Q = 0$ (here Q denotes the total charge of the atomic system). Indeed, in [**HH**], it is proved that for negative ions ($Q < 0$) $H_g(P)$ does not have a ground state if $\nabla E_g(P) \neq 0$. Let us also mention [**LMS**] where the existence of ground states for atoms is proven for any value of the coupling constant g, by adapting [**GLL**], under the further assumption $E_g(P) \geq E_g(0)$ which has not been proven yet. Thus in addition to these results, our method provides the existence of ground states for spinless hydrogenoid ions, both for $H_g(P)$ in the case $\nabla E_g(P) = 0$ and for $H^{\mathrm{ren}}_g(P)$.

The two statements "$H_g(P_3)$ has a ground state if $E_g'(P_3) = 0$" and "$H^{\mathrm{ren}}_g(P_3)$ has a ground state" shall be established following the same standard procedure: An infrared cutoff σ is introduced into the model so that the Hamiltonian $H^\sigma_g(P_3)$ (respectively $H^{\mathrm{ren}}_{g\sigma}(P_3)$) has a ground state $\Phi^\sigma_g(P_3)$ (respectively $\Phi^{\mathrm{ren}}_{g\sigma}(P_3)$). We then need to prove that $\Phi^\sigma_g(P_3)$ and $\Phi^{\mathrm{ren}}_{g\sigma}(P_3)$ converge strongly as $\sigma \to 0$. To this end we control the number of photons in the states $\Phi^\sigma_g(P_3)$ and $\Phi^{\mathrm{ren}}_{g\sigma}(P_3)$ thanks to a pull-through formula and (1.13).

We emphasize that, in the case $E_g'(P_3) \neq 0$, $H^{\mathrm{ren}}_g(P_3)$ can be seen as an expression of the physical Hamiltonian in a representation of the canonical commutation relations non-unitarily equivalent to the Fock representation. Besides, regarding [**Ch**] for the case of a single freely moving electron, one can conjecture that for sufficiently small values of $|P_3|$, $E_g'(P_3) = 0$ if and only if $P_3 = 0$.

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[1] The hypothesis of simplicity for the electronic ground state ($\mathbf{H_0}$) imposes this restriction to hydrogenoid atoms or ions.

Our proof of the absence of a ground state for $H_g(P_3)$ in the case $E_g'(P_3) \neq 0$ is based on a contradiction argument and [**DG2**, Lemma 2.6] (see also Lemma 2.2). Again the result is achieved by deriving a suitable expression of $a(\mathbf{k})\Phi_g(P_3)$ thanks to a pull-through formula (assuming here that $H_g(P_3)$ has a ground state $\Phi_g(P_3)$). Note that the regularity property (1.13) appears again as a key property (although here only (1.13) for $\sigma = 0$ is required).

The paper is organized as follows: In Section 2, we prove Theorem 1.3. Next in the appendix we prove Theorem 1.2.

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2. Proof of Theorem 1.3

The following proposition is proven in Subsection A.1 of the appendix.

Proposition 2.1. *Assume that* $(\mathbf{H_0})$ *holds. There exists* $g_0 > 0$, $\sigma_0 > 0$ *and* $P_0 > 0$ *such that for all* $|g| \leq g_0$, *for all* $0 < \sigma \leq \sigma_0$, *for all* $|P_3| \leq P_0$, $H_{g\sigma}(P_3)$ *has a unique normalized ground state* $\Phi_g^\sigma(P_3)$, *i.e.*

$$H_g^\sigma(P_3)\Phi_g^\sigma(P_3) = E_{g\sigma}(P_3)\Phi_g^\sigma(P_3), \quad \|\Phi_g^\sigma(P_3)\| = 1. \tag{2.1}$$

Notice that Proposition 2.1 is also established in [**AGG1**] under the weaker assumption that e_0 is an isolated eigenvalue of $h(b, V)$ of finite multiplicity. Let us recall a lemma, due to [**DG2**], on which is based our proof of the absence of a ground state for $H_g(P_3)$ in the case $E_g'(P_3) \neq 0$.

Lemma 2.2. *Let* $\Psi \in \mathrm{L}^2(\mathbb{R}^2; \mathbb{C}^2) \otimes \mathcal{H}_{\mathrm{ph}}$. *Assume that*

$$\int_{\mathbb{R}^3\times\mathbb{Z}_2} \|(a(\mathbf{k}) - h(\mathbf{k}))\Psi\|^2 d\mathbf{k} < \infty, \tag{2.2}$$

where h *is a measurable function from* $\mathbb{R}^3 \times \mathbb{Z}_2$ *to* $\mathbb{C}$ *such that* $h \notin \mathrm{L}^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. *Then* $\Psi = 0$.

Proof. See [**DG2**, Lemma 2.6]. □

Theorem 1.3 shall follow from a suitable decomposition of $a(\mathbf{k})\Phi_g^\sigma(P_3)$ based on a pull-through formula. The latter is the purpose of the following lemma, where the equalities should be understood as identities between measurable functions from $\mathbb{R}^3 \times \mathbb{Z}_2$ to $\mathrm{L}^2(\mathbb{R}^2; \mathbb{C}^2) \otimes \mathcal{H}_{\mathrm{ph}}$. For a rigorous justification of the commutations used in the next proof, we refer for instance to [**Ge, HH**].

In order to shorten the notations, we shall write

$$\begin{aligned} H &= H_g^\sigma(P_3), \quad E = E_{g\sigma}(P_3), \quad \Phi = \Phi_g^\sigma(P_3), \\ \tilde{H} &= H_g^\sigma(P_3 - k_3), \quad \tilde{E} = E_{g\sigma}(P_3 - k_3). \end{aligned} \tag{2.3}$$

Lemma 2.3. *Let* $\sigma \geq 0$ *and let* $\Phi = \Phi_g^\sigma(P_3)$ *be a normalized ground state of* $H = H_g^\sigma(P_3)$ *(assuming it exists for* $\sigma = 0$*). We have:*

$$a(\mathbf{k})\Phi = L_\sigma(\mathbf{k})\Phi + R_\sigma(\mathbf{k})\Phi + \frac{1}{\sqrt{2}} f_\sigma(\mathbf{k})\Phi, \tag{2.4}$$

where L_σ *and* R_σ *are operator-valued functions such that,*

$$\int_{\mathbb{R}^3\times\mathbb{Z}_2} \|L_\sigma(\mathbf{k})\Phi\|^2 d\mathbf{k} \leq \mathrm{C}g^2, \tag{2.5}$$

and

$$R_\sigma(\mathbf{k}) = -\frac{g}{2\sqrt{2\pi}} \frac{\rho_\sigma^\Lambda(k)\epsilon_3^\lambda(k)|k|^{1/2}}{k_3(\tilde{E} - E + |k|)} \frac{\tilde{H} - \tilde{E}}{\tilde{H} - E + |k|}. \tag{2.6}$$

Proof. It follows from the canonical commutation relations (1.6) that

$$\begin{aligned} a(\mathbf{k})H &= (\tilde{H} + |k|)a(\mathbf{k}) \\ &- \frac{g}{2^{\frac{3}{2}}m} \sum_{j=1,2} \Big(h_{j,\sigma}(x', \mathbf{k})\big(p_j - ea_j(x') - gA_{j,\sigma}(x', 0)\big) + \sigma_j \tilde{h}_{j,\sigma}(x', \mathbf{k}) \Big) \\ &- \frac{g}{2^{\frac{3}{2}}m} \Big(h_{3,\sigma}(x', \mathbf{k})\big(P_3 - \mathrm{d}\Gamma(k_3) - gA_{3,\sigma}(x', 0)\big) + \sigma_3 \tilde{h}_{3,\sigma}(x', \mathbf{k}) \Big). \end{aligned} \tag{2.7}$$

In order to control the term containing $(p_j - ea_j(x') - gA_{j,\sigma}(x', 0))$ in the right-hand-side of the previous equality, we use that (formally)

$$\frac{1}{2m}\big(p_j - ea_j(x') - gA_{j,\sigma}(x', 0)\big) = i[H, x'_j], \tag{2.8}$$

for $j = 1, 2$. Notice that an alternative would be to consider the Hamiltonian obtained through a unitary Power-Zienau-Wooley transformation (see for instance [**GLL**]). For a rigorous justification of (2.8), we refer to [**BFP**, Theorem II.10] which can easily be adapted to our case. In particular it follows that $x'_j\Phi \in D(H)$. Applying (2.7) to Φ then yields

$$\begin{aligned} a(\mathbf{k})\Phi &= \frac{ig}{2^{\frac{1}{2}}} \sum_{j=1,2} h_{j,\sigma}(x', \mathbf{k})[\tilde{H} - E + |k|]^{-1}(H - E)x'_j\Phi \\ &+ \frac{g}{2^{\frac{3}{2}}m}[\tilde{H} - E + |k|]^{-1}\sigma \cdot \tilde{h}_\sigma(x', \mathbf{k})\Phi \\ &+ \frac{g}{2^{\frac{3}{2}}m} h_{3,\sigma}(x', \mathbf{k})[\tilde{H} - E + |k|]^{-1}\big(P_3 - \mathrm{d}\Gamma(k_3) - gA_{3,\sigma}(x', 0)\big)\Phi. \end{aligned} \tag{2.9}$$

Note that the expressions of H and $\tilde{H}$ imply

$$\tilde{H} - H = -\frac{k_3}{m}\big(P_3 - \mathrm{d}\Gamma(k_3) - gA_{3,\sigma}(x', 0)\big) + \frac{k_3^2}{2m}. \tag{2.10}$$

From (1.18), we get

$$\|[\tilde{H} - E + |k|]^{-1}\| \leq \mathrm{C}|k|^{-1}. \tag{2.11}$$

Moreover it is not difficult to show that

$$\Big\| \big(P_3 - \mathrm{d}\Gamma(k_3) - gA_{3,\sigma}(x', 0)\big)[\tilde{H} - E + |k|]^{-1} \Big\| \leq \mathrm{C}|k|^{-1}, \tag{2.12}$$

and consequently, by (2.10),

$$\Big\| (H - E)[\tilde{H} - E + |k|]^{-1} \Big\| \leq \mathrm{C}. \tag{2.13}$$

Introducing (2.11)–(2.13) into (2.9) and recalling the definitions (1.16) of h_j and $\tilde{h}_j$, we thus obtain

$$\begin{aligned} a(\mathbf{k})\Phi &= L_1(\mathbf{k})\Phi \\ &+ \frac{g}{2^{\frac{3}{2}}m} h_{3,\sigma}(0, \mathbf{k})[\tilde{H} - E + |k|]^{-1}\big(P_3 - \mathrm{d}\Gamma(k_3) - gA_{3,\sigma}(x', 0)\big)\Phi, \end{aligned} \tag{2.14}$$

where

$$\|L_1(\mathbf{k})\Phi\| \leq \mathrm{C}|g||k|^{-1/2}\big(\|\Phi\| + \|x'_1\Phi\| + \|x'_2\Phi\|\big). \tag{2.15}$$

In passing from (2.9) to (2.14) we used that

$$|h_{3,\sigma}(x',\mathbf{k}) - h_{3,\sigma}(0,\mathbf{k})| \leq \mathrm{C}|k||x'|. \tag{2.16}$$

Let us now note the following obvious identity:

$$\frac{\tilde{H}-E}{\tilde{H}-E+|k|} = \frac{\tilde{E}-E}{\tilde{E}-E+|k|} + \frac{|k|}{\tilde{E}-E+|k|}\Big(\frac{\tilde{H}-\tilde{E}}{\tilde{H}-E+|k|}\Big). \tag{2.17}$$

Hence, introducing (2.10) and (2.17) into (2.14) leads to

$$\begin{aligned} a(\mathbf{k})\Phi &= L_1(\mathbf{k})\Phi - \frac{gk_3}{2^{\frac{5}{2}}} h_{3,\sigma}(0,\mathbf{k})[\tilde{H}-E+|k|]^{-1}\Phi \\ &- \frac{g}{2^{\frac{3}{2}}k_3}\frac{\tilde{E}-E}{\tilde{E}-E+|k|} h_{3,\sigma}(0,\mathbf{k})\Phi \\ &- \frac{g}{2^{\frac{3}{2}}k_3}\frac{|k|}{\tilde{E}-E+|k|} h_{3,\sigma}(0,\mathbf{k})\frac{\tilde{H}-\tilde{E}}{\tilde{H}-E+|k|}\Phi. \end{aligned} \tag{2.18}$$

We conclude the proof using again that $\|x'_j\Phi\| < \infty$. □

The following lemma shows in particular that if the map $P_3 \mapsto E(P_3)$ is sufficiently regular, then $\mathbf{k} \mapsto \|R(\mathbf{k})\Phi\|$ is in $\mathrm{L}^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, where $R(\mathbf{k})$ denotes the operator defined in (2.6) for $\sigma = 0$.

Lemma 2.4. *Let the parameters g, σ, P_3 be fixed. Assume that there exist $\gamma > 0$, $P_0 > 0$ and a positive constant* C *independent of $\sigma \geq 0$ such that for all $|k_3| \leq P_0$,*

$$\big|E'_{g\sigma}(P_3+k_3) - E'_{g\sigma}(P_3)\big| \leq \mathrm{C}|k_3|^{\gamma}. \tag{2.19}$$

Then there exists a positive constant C'*, independent of σ, such that*

$$\Big\|\big(H_g^{\sigma}(P_3-k_3) - E_{g\sigma}(P_3-k_3)\big)^{1/2}\Phi\Big\| \leq \mathrm{C}'|k_3|^{\frac{1+\gamma}{2}}. \tag{2.20}$$

Proof. We use again the notations (2.3) and let in addition $E' = E'_{g\sigma}(P_3)$. By (2.10), we have

$$\tilde{E}-E \leq (\Phi,(\tilde{H}-H)\Phi) = -\frac{k_3}{m}\big(\Phi,\big(P_3 - \mathrm{d}\Gamma(k_3) - A_{3,\sigma}(x',0)\big)\Phi\big) + \frac{k_3^2}{2m}. \tag{2.21}$$

Dividing by $-k_3$ and letting $k_3 \to 0$ (distinguishing the cases $k_3 > 0$ and $k_3 < 0$), we obtain the Feynman-Hellman formula:

$$E' = \frac{1}{m}\big(\Phi,\big(P_3 - \mathrm{d}\Gamma(k_3) - A_{3,\sigma}(x',0)\big)\Phi\big). \tag{2.22}$$

Hence, by (2.10),

$$\begin{aligned} \Big|(\Phi,(\tilde{H}-\tilde{E})\Phi)\Big| &= \Big|(\Phi,(\tilde{H}-H)-(\tilde{E}-E)\Phi)\Big| \\ &\leq \Big|-k_3E' - (\tilde{E}-E)\Big| + \frac{k_3^2}{2m}. \end{aligned} \tag{2.23}$$

The lemma then follows from (2.19) and the mean value theorem. □

We are now ready to prove Theorem 1.3:

PROOF OF THEOREM 1.3. Let us begin with estimating the term $\|R_\sigma(\mathbf{k})\Phi_g^\sigma(P_3)\|$ appearing in Lemma 2.3. Recalling the notations (2.3), we write

$$\|R_\sigma(\mathbf{k})\Phi\| \leq \frac{\mathrm{C}|g|}{|k_3||k|^{\frac{1}{2}}}\mathbb{1}_{\sigma\leq|k|\leq\Lambda}(k)\left\|\frac{(\tilde{H}-\tilde{E})^{1/2}}{\tilde{H}-E+|k|}\right\|\left\|(\tilde{H}-\tilde{E})^{1/2}\Phi\right\|. \tag{2.24}$$

It follows from the Spectral Theorem and (1.18) that

$$\left\|\frac{(\tilde{H}-\tilde{E})^{1/2}}{\tilde{H}-E+|k|}\right\| = \sup_{r\geq 0}\left|\frac{r^{\frac{1}{2}}}{r+\tilde{E}-E+|k|}\right| \leq \sup_{r\geq 0}\left|\frac{r^{\frac{1}{2}}}{r+|k|/4}\right| \leq \frac{\mathrm{C}}{|k|^{\frac{1}{2}}}. \tag{2.25}$$

Thus, Theorem 1.2 together with Lemma 2.4 yield

$$\|R_\sigma(\mathbf{k})\Phi\| \leq \frac{\mathrm{C}|g|}{|k_3|^{\frac{1}{2}-\frac{\gamma}{2}}|k|}\mathbb{1}_{\sigma\leq|k|\leq\Lambda}(k), \tag{2.26}$$

where $\gamma = 1/4-\delta$, and where δ in Theorem 1.2 is chosen such that $0<\delta<1/4$. Hence

$$\int_{\mathbb{R}^3\times\mathbb{Z}_2}\|R_\sigma(\mathbf{k})\Phi\|^2 d\mathbf{k} \leq \mathrm{C}g^2. \tag{2.27}$$

Let us now prove (i). First assume that $E_g'(P_3)=0$. In order to get the existence of a ground state for $H_g(P_3)$ our aim is to prove that $\Phi_g^\sigma(P_3)$ converges strongly as $\sigma\to 0$. Using Lemma A.7 (see also Remark A.8), we obtain from (1.17) that

$$|f_\sigma(\mathbf{k})| \leq \mathrm{C}\left(\frac{g^2\sigma}{|k_3||k|^{\frac{3}{2}}}+\frac{|g|(E_g(P_3-k_3)-E_g(P_3))}{|k_3||k|^{\frac{3}{2}}}\right)\mathbb{1}_{\sigma\leq|k|\leq\Lambda}(k). \tag{2.28}$$

Hence, since $E_g'(P_3)=0$ by assumption, (1.13) implies

$$|f_\sigma(\mathbf{k})| \leq \mathrm{C}\left(\frac{g^2\sigma}{|k_3||k|^{3/2}}+\frac{|g|k_3^{\frac{1}{4}-\delta}}{|k|^{\frac{3}{2}}}\right)\mathbb{1}_{\sigma\leq|k|\leq\Lambda}(k). \tag{2.29}$$

Therefore

$$\|f_\sigma\|_{\mathrm{L}^2(\mathbb{R}^3\times\mathbb{Z}_2)} \leq \mathrm{C}|g|. \tag{2.30}$$

Combining Lemma 2.3 with (2.30) and (2.27), we obtain

$$(\Phi_g^\sigma(P_3),\mathcal{N}\Phi_g^\sigma(P_3)) = \int_{\mathbb{R}^3\times\mathbb{Z}^2}\|a(\mathbf{k})\Phi_g^\sigma(P_3)\|^2 d\mathbf{k} \leq \mathrm{C}g^2, \tag{2.31}$$

where $\mathcal{N}=\mathrm{d}\Gamma(I)$ denotes the number operator. For a sufficiently small fixed $|g|$, the strong convergence of $\Phi_g^\sigma(P_3)$ as $\sigma\to 0$ is then obtained by following for instance [**BFS**], showing that $|(\Phi_g^\sigma(P_3),\Phi_{\mathrm{el}}\otimes\Omega)|\geq \mathrm{C}>0$ uniformly in $\sigma\geq 0$. Here Φ_{el} denotes a normalized ground state of $h(b,V)$.

Assume next that $E_g'(P_3)\neq 0$ and let us prove that $H_g(P_3)$ does not have a ground state. By Lemmata 2.2, 2.3 and Estimate (2.27), it suffices to prove that $f\notin \mathrm{L}^2(\mathbb{R}^3\times\mathbb{Z}_2)$. The latter follows from the fact that

$$\left|\frac{E_g(P_3-k_3)-E_g(P_3)}{k_3}\right| \geq \mathrm{C}>0 \tag{2.32}$$

uniformly for small k_3 since $E_g'(P_3)\neq 0$. Hence Theorem 1.3(i) is proven.

Let us finally prove (ii). For $\sigma>0$, we set

$$\Phi^{\mathrm{ren}} = W(if_\sigma)\Phi_g^\sigma(P_3). \tag{2.33}$$

Obviously Φ^{ren} is a normalized ground state of $H^{\mathrm{ren}}_{g\sigma}(P_3)$. By Lemma 2.3 we have

$$\begin{aligned} a(\mathbf{k})\Phi^{\mathrm{ren}} &= W(if_\sigma)a(\mathbf{k})\Phi + [a(\mathbf{k}), W(if_\sigma)]\Phi \\ &= W(if_\sigma)L_\sigma(\mathbf{k})\Phi + W(if_\sigma)R_\sigma(\mathbf{k})\Phi + \frac{1}{\sqrt{2}} f_\sigma(\mathbf{k})\Phi^{\mathrm{ren}} + [a(\mathbf{k}), W(if_\sigma)]\Phi. \end{aligned}$$

One can compute the commutator $[a(\mathbf{k}), W(if_\sigma)] = -2^{-1/2} f_\sigma(\mathbf{k})$, so that

$$a(\mathbf{k})\Phi^{\mathrm{ren}} = W(if_\sigma)L_\sigma(\mathbf{k})\Phi + W(if_\sigma)R_\sigma(\mathbf{k})\Phi. \tag{2.34}$$

Therefore, since $W(if_\sigma)$ is unitary, $\|a(\mathbf{k})\Phi^{\mathrm{ren}}\|$ can be estimated in the same way as $\|a(\mathbf{k})\Phi\|$ (in the case $E'_g(P_3) = 0$), using (2.5) and (2.27). This leads to the existence of a ground state for $H^{\mathrm{ren}}_g(P_3)$ and concludes the proof of Theorem 1.3. □

Appendix A. Uniform regularity of the map $P_3 \mapsto E_{g\sigma}(P_3)$

In this appendix we shall prove Theorem 1.2. The structure follows [**Pi**] and [**CFP**]: First, we give a simple proof of the existence of a spectral gap for the infrared cutoff Hamiltonian $H^\sigma_g(P_3)$, considered as an operator on the space of photons of energies $\geq \sigma$. Our proof is based on the min-max principle. Then we establish (1.13) by adapting [**Pi, CFP**] (see also [**BFP**]). In comparison to [**CFP**], the main technical difference comes from the terms in $H_g(P_3)$ containing the interaction between the electronic variables x'_j and the quantized electromagnetic field. This shall be handled in Lemma A.11 below thanks to the exponential decay of $\Phi^\sigma_g(P_3)$ in x'_j.

In some parts of our presentation, we shall only sketch the proof, emphasizing the differences that we have to include, and referring otherwise to [**Pi**], [**BFP**], or [**CFP**].

Let us begin with some definitions and notations. Henceforth we remove the subindex g to simplify the notations, and for $\sigma \geq 0$, we replace $H^\sigma(P_3)$ by its Wick-ordered version $H^\sigma(P_3) - \frac{g^2}{2m}(\Lambda^2 - \sigma^2)$ (which we still denote by $H^\sigma(P_3)$). Note that this shall not affect our discussion below on the regularity of the ground state energy since the two operators only differ by a constant. We decompose

$$H^\sigma(P_3) = h_0(P_3) + H^\sigma_I(P_3), \tag{A.1}$$

where

$$h_0(P_3) = h(b, V) \otimes \mathbb{1} + \mathbb{1} \otimes \left[\frac{1}{2m}\Big(P_3 - d\Gamma(k_3)\Big)^2 + H_f \right], \tag{A.2}$$

and

$$\begin{aligned} H^\sigma_I(P_3) = &- \frac{g}{m} \sum_{j=1,2} \left(A_{j,\sigma}(x', 0)\Big(p_j - e a_j(x')\Big) + \frac{g^2}{2m} A_{j,\sigma}(x', 0)^2 \right) \\ &- \frac{g}{2m} A_{3,\sigma}(x', 0)\Big(P_3 - d\Gamma(k_3)\Big) - \frac{g}{2m}\Big(P_3 - d\Gamma(k_3)\Big) A_{3,\sigma}(x', 0) \\ &+ \frac{g^2}{2m} A_{3,\sigma}(x', 0)^2 - \frac{g}{2m}\sigma \cdot B_\sigma(x', 0) - \frac{g^2}{2m}(\Lambda^2 - \sigma^2). \end{aligned} \tag{A.3}$$

Let Φ_{el} denote a normalized ground state of $h(b, V)$. For any $|P_3| < m$, one can easily check that $\Phi_{\mathrm{el}} \otimes \Omega$ is a ground state of $h_0(P_3)$, with ground state energy

$e_0(P_3) = e_0 + P_3^2/2m$. Note that for $\tau \le \sigma$, we have

$$
\begin{aligned}
&H^\tau(P_3) - H^\sigma(P_3) \\
&= -\frac{g}{m}\sum_{j=1,2} A^\sigma_{j,\tau}(x',0)\Big(p_j - ea_j(x') - gA_{j,\sigma}(x',0)\Big) - \frac{g^2}{2m}(\sigma^2-\tau^2) \\
&+ \frac{g^2}{2m}A^\sigma_\tau(x',0)^2 - \frac{g}{2m}A^\sigma_{3,\tau}(x',0)\Big(P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x',0)\Big) \\
&- \frac{g}{2m}\Big(P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x',0)\Big)A^\sigma_{3,\tau}(x',0) - \frac{g}{2m}\sigma\cdot B^\sigma_\tau(x',0),
\end{aligned}
\tag{A.4}
$$

where

$$A^\sigma_\tau(x',0) = \frac{1}{\sqrt{2\pi}}\int \frac{\epsilon^\lambda(k)}{|k|^{1/2}}\rho^\sigma_\tau(k)\left[e^{-ik'\cdot x'}a^*_\lambda(k) + e^{ik'\cdot x'}a_\lambda(k)\right]d\mathbf{k}, \tag{A.5}$$

and likewise for $B^\sigma_\tau(x',0)$. Let $\mathcal{H}_\sigma = \mathrm{L}^2(\mathbb{R}^2;\mathbb{C}^2)\otimes\mathcal{F}_\sigma$, where $\mathcal{F}_\sigma$ denotes the symmetric Fock space over $\mathrm{L}^2(\{\mathbf{k}\in\mathbb{R}^3\times\mathbb{Z}_2, |k|\ge\sigma\})$. The restriction of $H^\sigma(P_3)$ to $\mathcal{H}_\sigma$ is denoted by $H_\sigma(P_3)$:

$$H_\sigma(P_3) = H^\sigma(P_3)|_{\mathcal{H}_\sigma}, \tag{A.6}$$

and, similarly,

$$h_{0,\sigma}(P_3) = h_0(P_3)|_{\mathcal{H}_\sigma} \quad , \quad H_{I,\sigma}(P_3) = H^\sigma_I(P_3)|_{\mathcal{H}_\sigma}. \tag{A.7}$$

Let Ω_σ be the vacuum in $\mathcal{F}_\sigma$. Then for $|P_3| < m$, $\Phi_{\mathrm{el}}\otimes\Omega_\sigma$ is a ground state of $h_{0,\sigma}(P_3)$ with ground state energy $e_0(P_3)$, and

$$\mathrm{Gap}(h_{0,\sigma}(P_3)) \ge (1 - \frac{|P_3|}{m})\sigma, \tag{A.8}$$

where $\mathrm{Gap}(H) = \inf(\sigma(H)\setminus\{E(H)\}) - \inf(\sigma(H))$ for any self-adjoint and semi-bounded operator H with ground state energy $E(H)$. We also define

$$H^\sigma_\tau(P_3) = \left(H^\tau(P_3) - H^\sigma(P_3)\right)|_{\mathcal{H}_\tau}. \tag{A.9}$$

The symmetric Fock space over $\mathrm{L}^2(\{\mathbf{k}\in\mathbb{R}^3\times\mathbb{Z}_2, \tau\le|k|\le\sigma\})$ is denoted by $\mathcal{F}^\sigma_\tau$. Note that there exists a unitary operator $\mathcal{V}:\mathcal{H}_\tau\to\mathcal{H}_\sigma\otimes\mathcal{F}^\sigma_\tau$. We shall identify $\mathcal{H}_\tau$ and $\mathcal{H}_\sigma\otimes\mathcal{F}^\sigma_\tau$ in the sequel in order to simplify the notations. We let Ω^σ_τ be the vacuum in $\mathcal{F}^\sigma_\tau$.

A.1. Existence of a spectral gap.

LEMMA A.1. *There exist $g_0 > 0$, $\sigma_0 > 0$ and $P_0 > 0$ such that the following holds: Let $|g| \le g_0$, $0 \le \sigma \le \sigma_0$ and $|P_3| \le P_0$ be such that $H_\sigma(P_3)$ has a normalized ground state $\Phi_\sigma(P_3)$ and $\mathrm{Gap}(H_\sigma(P_3)) \ge \gamma\sigma$ for some $\gamma > 0$. Then for all $0 \le \tau \le \sigma$, $\Phi_\sigma(P_3)\otimes\Omega^\sigma_\tau$ is a normalized ground state of $H^\sigma(P_3)|_{\mathcal{H}_\tau}$, and*

$$\mathrm{Gap}(H^\sigma(P_3)|_{\mathcal{H}_\tau}) \ge \min(\gamma\sigma, \tau/4). \tag{A.10}$$

PROOF. To simplify the notations, let us remove the dependence on P_3 throughout the proof. First, one can readily check that $\Phi_\sigma\otimes\Omega^\sigma_\tau$ is an eigenstate of $H^\sigma|_{\mathcal{H}_\tau}$

associated with the eigenvalue E_σ. For any v we let $[v]$ and $[v]^\perp$ denote respectively the subspace spanned by v and its orthogonal complement. We write

$$\inf_{\Phi\in[\Phi_\sigma\otimes\Omega_\tau^\sigma]^\perp,\|\Phi\|=1}(\Phi, H^\sigma|_{\mathcal{H}_\tau}\Phi)$$
$$\geq \min\Big(\inf_{\Phi\in[\Phi_\sigma]^\perp\otimes[\Omega_\tau^\sigma],\|\Phi\|=1}(\Phi, H^\sigma|_{\mathcal{H}_\tau}\Phi), \inf_{\Phi\in\mathcal{H}_\sigma\otimes[\Omega_\tau^\sigma]^\perp,\|\Phi\|=1}(\Phi, H^\sigma|_{\mathcal{H}_\tau}\Phi)\Big).$$

The assumption $\mathrm{Gap}(H_\sigma)\geq\gamma\sigma$ implies

$$\inf_{\Phi\in[\Phi_\sigma]^\perp\otimes[\Omega_\tau^\sigma],\|\Phi\|=1}(\Phi, H^\sigma|_{\mathcal{H}_\tau}\Phi)\geq E_\sigma+\gamma\sigma.$$

On the other hand, using that the number operator $\int_{\tau\leq|k|\leq\sigma}a^*(\mathbf{k})a(\mathbf{k})d\mathbf{k}$ commutes with $H^\sigma|_{\mathcal{H}_\tau}$, one can prove as in [**Pi**] that

$$\inf_{\Phi\in\mathcal{H}_\sigma\otimes[\Omega_\tau^\sigma]^\perp,\|\Phi\|=1}(\Phi, H^\sigma|_{\mathcal{H}_\tau}\Phi)\geq\inf_{\tau\leq|k|\leq\sigma}(E_\sigma(P_3-k_3)-E_\sigma(P_3)+|k|).$$

We conclude the proof thanks to (1.18) □

Corollary A.2. *Under the conditions of Lemma A.1, for all* $0\leq\tau\leq\sigma$,

$$E_\tau(P_3)\leq E_\sigma(P_3)\leq e_0(P_3). \tag{A.11}$$

Proof. It follows from Lemma A.1 that

$$\begin{aligned}E_\tau(P_3)&\leq(\Phi_\sigma(P_3)\otimes\Omega_\tau^\sigma, H_\tau(P_3)\Phi_\sigma(P_3)\otimes\Omega_\tau^\sigma)\\&=(\Phi_\sigma(P_3)\otimes\Omega_\tau^\sigma, H^\sigma(P_3)|_{\mathcal{H}_\tau}\Phi_\sigma(P_3)\otimes\Omega_\tau^\sigma)=E_\sigma(P_3).\end{aligned}\tag{A.12}$$

Hence the first inequality in (A.11) is proven. To prove the second one, it suffices to write similarly

$$\begin{aligned}E_\sigma(P_3)&\leq(\Phi_{\mathrm{el}}\otimes\Omega_\sigma, H_\sigma(P_3)\Phi_{\mathrm{el}}\otimes\Omega_\sigma)\\&=(\Phi_{\mathrm{el}}\otimes\Omega_\sigma, h_{0,\sigma}(P_3)\Phi_{\mathrm{el}}\otimes\Omega_\sigma)=e_0(P_3).\end{aligned}\tag{A.13}$$

□

We shall establish the existence of a spectral gap of order $O(\sigma)$ above the bottom of the spectrum of $H_\sigma(P_3)$ by induction. More precisely, let $\mathbf{Gap}(\sigma)$ denote the assertion

$$\mathbf{Gap}(\sigma)\begin{cases}\text{(i)} & E_\sigma(P_3)\text{ is a simple eigenvalue of }H_\sigma(P_3),\\ \text{(ii)} & \mathrm{Gap}(H_\sigma(P_3))\geq\sigma/8.\end{cases}$$

We shall prove

Proposition A.3. *There exists $g_0>0$, $\sigma_0>0$ and $P_0>0$ such that, for all $|g|\leq g_0$, $0<\sigma\leq\sigma_0$ and $|P_3|\leq P_0$, the assertion $\mathbf{Gap}(\sigma)$ above holds.*

Let us begin with two preliminary useful estimates:

Lemma A.4. *Fix the parameters g, σ and P_3 such that $0\leq|g|\leq g_0$, $0\leq\sigma\leq\sigma_0$ and $0\leq|P_3|\leq P_0$, for some sufficiently small small g_0, σ_0 and P_0. For any $0<\rho<1$,*

$$\begin{aligned}&\Big\|[h_{0,\sigma}(P_3)-e_0(P_3)+\rho]^{-1/2}H_{I,\sigma}(P_3)[h_{0,\sigma}(P_3)-e_0(P_3)+\rho]^{-1/2}\Big\|\\&\leq\mathrm{C}|g|\rho^{-1/2},\end{aligned}\tag{A.14}$$

where C *is a positive constant (depending only on* Λ*). Likewise,*

$$\Big\| \left[H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho\right]^{-1/2} H^\sigma_\tau(P_3) \left[H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho\right]^{-1/2} \Big\| \leq \mathrm{C}|g|\sigma^{1/2}\rho^{-1/2}. \tag{A.15}$$

Proof. Let us prove (A.15), Estimate (A.14) would follow similarly. We introduce the expression of $H^\sigma_\tau(P_3)$ given by (A.4) and (A.9) and estimate each term separately. Consider for instance

$$|g| \Big\| \Big[H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho\Big]^{-1/2} \int_{\tau \leq |k| \leq \sigma} \frac{\epsilon^{(3)}_\lambda(k)}{|k|^{1/2}} e^{ik'\cdot x'} a^*(\mathbf{k}) d\mathbf{k} \Big(P_3 - d\Gamma(k_3) + gA_{3,\sigma}(x',0)\Big) \Big[H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho\Big]^{-1/2} \Big\|. \tag{A.16}$$

Using that

$$\Big\| \Big(P_3 - d\Gamma(k_3) + gA_{3,\sigma}(x',0)\Big) \Big[H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho\Big]^{-1/2} \Big\| \leq \mathrm{C}\rho^{-1/2}, \tag{A.17}$$

we get

$$\text{(A.16)} \leq \mathrm{C}|g|\rho^{-1/2} \Big\| \Big[H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho\Big]^{-1/2} \int_{\tau \leq |k| \leq \sigma} \frac{\epsilon^{(3)}_\lambda(k)}{|k|^{1/2}} e^{ik'\cdot x'} a^*(\mathbf{k}) d\mathbf{k} \Big\|.$$

Moreover, for any $\Phi \in D(H^\sigma(P_3)|_{\mathcal{H}_\tau})$,

$$\Big\| \Big[H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho\Big]^{-1/2} \int_{\tau \leq |k| \leq \sigma} \frac{\epsilon^{(3)}_\lambda(k)}{|k|^{1/2}} e^{ik'\cdot x'} a^*(\mathbf{k}) d\mathbf{k}\Phi \Big\|^2$$
$$\leq \int_{\tau \leq |k|, |\tilde{k}| \leq \sigma} \frac{\mathrm{C}}{|k|^{1/2}|\tilde{k}|^{1/2}} \left| \Big(\Phi, a(\mathbf{k}) \Big[H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho\Big]^{-1} a^*(\tilde{\mathbf{k}})\Phi\Big) \right| d\mathbf{k} d\tilde{\mathbf{k}}.$$

Now, for any $\mathbf{k}$ such that $\tau \leq |k| \leq \sigma$, we have the pull-through formula

$$a(\mathbf{k}) H^\sigma(P_3)|_{\mathcal{H}_\tau} = \Big[H^\sigma(P_3 - k_3)|_{\mathcal{H}_\tau} + |k|\Big] a(\mathbf{k}), \tag{A.18}$$

since $a(\mathbf{k})$ commutes with $A_\sigma(x',0)$. Hence

$$\Big(\Phi, a(\mathbf{k}) \Big[H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho\Big]^{-1} a^*(\tilde{\mathbf{k}})\Phi\Big)$$
$$= \delta(\mathbf{k} - \tilde{\mathbf{k}}) \Big(\Phi, \Big[H^\sigma(P_3 - k_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + |k| + \rho\Big]^{-1} \Phi\Big)$$
$$+ \Big(a(\tilde{\mathbf{k}})\Phi, \Big[H^\sigma(P_3 - k_3 - \tilde{k}_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + |k| + |\tilde{k}| + \rho\Big]^{-1} a(\mathbf{k})\Phi\Big).$$

Using that $H^\sigma(P_3 - k_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + |k| \geq |k|/4$ for any k sufficiently small (see (1.18)), we get

$$\Big\| \Big[H^\sigma(P_3 - k_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + |k| + \rho\Big]^{-1} \Big\| \leq \frac{\mathrm{C}}{|k|}.$$

Let $H^\sigma_{f,\tau} = \int_{\tau \leq |k| \leq \sigma} |k| a^*(\mathbf{k}) a(\mathbf{k}) d\mathbf{k}$. As in [**Pi**, Lemma 1.1], it follows from the proof of Lemma A.1 that $H^\sigma_{f,\tau} \leq \mathrm{C}(H^\sigma(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3))$ for any P_3 sufficiently small. This yields

$$\left\| \left[H^{\sigma}_{f,\tau} + |k| + |\tilde{k}|\right] \left[H^{\sigma}(P_3 - k_3 - \tilde{k}_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + |k| + |\tilde{k}| + \rho\right]^{-1} \right\| \leq \mathrm{C}.$$

Thus, combining the previous estimates we obtain

$$\begin{aligned} &\left\| \left[H^{\sigma}(P_3)|_{\mathcal{H}_\tau} - E_\sigma(P_3) + \rho\right]^{-1/2} \int_{\tau \leq |k| \leq \sigma} \frac{\epsilon^{(3)}_\lambda(k)}{|k|^{1/2}} e^{ik'\cdot x'} a^*(\mathbf{k}) d\mathbf{k} \Phi \right\|^2 \\ &\leq \mathrm{C} \int_{\tau \leq |k| \leq \sigma} \frac{d\mathbf{k}}{|k|^2} + \mathrm{C} \left[\int_{\tau \leq |k| \leq \sigma} \frac{d\mathbf{k}}{|k|^{\frac{1}{2}}} \left\| [H^{\sigma}_{f,\tau} + |k|]^{-1/2} a(\mathbf{k}) \Phi \right\| \right]^2 \leq \mathrm{C}\sigma. \end{aligned}$$

Since $D(H^{\sigma}(P_3)|_{\mathcal{H}_\tau})$ is dense in $\mathcal{H}_\tau$, the result is proven as for the term we have chosen to consider, that is (A.16) $\leq \mathrm{C}|g|\sigma^{1/2}\rho^{-1/2}$. Since the other terms in the expression of H^{σ}_{τ} given by (A.4) can be treated in the same way, the lemma is established. □

The next lemma corresponds to the root in the induction procedure leading to the proof of Proposition A.3.

Lemma A.5. *There exist $g_0 > 0$, $\sigma_0 > 0$, $P_0 > 0$ and a positive constant C_0 such that for all $|g| \leq g_0$ and $|P_3| \leq P_0$, for all σ such that $\mathrm{C}_0 g^2 \leq \sigma \leq \sigma_0$, the assertion* $\mathbf{Gap}(\sigma)$ *holds.*

Proof. To simplify the notations, we write H_σ for $H_\sigma(P_3)$, E_σ for $E_\sigma(P_3)$, and similarly for other quantities depending on P_3. Let μ_σ denote the first point above E_σ in the spectrum of H_σ. By the min-max principle,

$$\mu_\sigma \geq \inf_{\psi \in [\Phi_{\mathrm{el}} \otimes \Omega_\sigma]^\perp, \|\psi\| = 1} (\psi, H_\sigma \psi), \tag{A.19}$$

where $[v]^\perp$ denotes the orthogonal complement of the vector space spanned by v. It follows from (A.14) that for any $\psi \in [\Phi_{\mathrm{el}} \otimes \Omega_\sigma]^\perp$, $\|\psi\| = 1$, and any $\rho > 0$,

$$\begin{aligned} (\psi, H_\sigma \psi) &\geq (\psi, H_{0,\sigma}\psi) - \mathrm{C}|g|\rho^{-1/2}(\psi, [h_{0,\sigma} - e_0(P_3) + \rho]\psi) \\ &\geq \left(1 - \mathrm{C}|g|\rho^{-1/2}\right)(\psi, H_{0,\sigma}\psi) + \mathrm{C}|g|\rho^{-1/2} e_0(P_3) - \mathrm{C}|g|\rho^{1/2}. \end{aligned} \tag{A.20}$$

By (A.8), for any $\psi \in [\Phi_{\mathrm{el}} \otimes \Omega_\sigma]^\perp$, $(\psi, h_{0,\sigma}\psi) \geq e_0(P_3) + (1 - |P_3|/m)\sigma$ provided that σ_0 is chosen sufficiently small. Hence for any ρ such that $\rho^{1/2} > \mathrm{C}|g|$,

$$(\psi, H_\sigma \psi) \geq e_0(P_3) + \left(1 - \mathrm{C}|g|\rho^{-1/2}\right)\left(1 - \frac{|P_3|}{m}\right)\sigma - \mathrm{C}|g|\rho^{1/2}. \tag{A.21}$$

Choosing $\rho^{1/2} = 4\mathrm{C}|g|$ and P_0 sufficiently small, by Corollary A.2, we obtain

$$\begin{aligned} (\psi, H_\sigma \psi) &\geq E_\sigma + \frac{3}{4}\left(1 - \frac{|P_3|}{m}\right)\sigma - 4\mathrm{C}^2 g^2 \\ &\geq E_\sigma + \frac{1}{2}\sigma - 4\mathrm{C}^2 g^2. \end{aligned} \tag{A.22}$$

Together with (A.19), this leads to the statement of the lemma provided that the constant C_0 is chosen such that $\mathrm{C}_0 > 32\mathrm{C}^2/3$. □

The following lemma corresponds to the induction step of the induction process in the proof of Proposition A.3.

Lemma A.6. *There exists $g_0 > 0$, $\sigma_0 > 0$ and $P_0 > 0$ such that for all $|g| \le g_0$ and $|P_3| \le P_0$, for all σ such that $0 < \sigma \le \sigma_0$,*

$$\mathbf{Gap}(\sigma) \Rightarrow \mathbf{Gap}(\sigma/2).$$

Proof. Again, throughout the proof, we drop the dependence on P_3 in all the considered quantities. Let $\mathbf{Gap}(\sigma)$ be satisfied for some $0 < \sigma$, let Φ_σ be a ground state of H_σ, and let $\tau = \sigma/2$. As in the proof of Lemma A.5, let μ_τ denote the first point above E_τ in the spectrum of H_τ. By the min-max principle,

$$\mu_\tau \ge \inf_{\psi \in [\Phi_\sigma \otimes \Omega_\tau^\sigma]^\perp, \|\psi\|=1} (\psi, H_\tau \psi), \tag{A.23}$$

where Ω_τ^σ is the vacuum in $\mathcal{F}_\tau^\sigma$ and where $[\Phi_\sigma \otimes \Omega_\tau^\sigma]^\perp$ denotes the orthogonal complement of the vector space spanned by $\Phi_\sigma \otimes \Omega_\tau^\sigma$ in $\mathcal{H}_\sigma \otimes \mathcal{F}_\tau^\sigma$. It follows from (A.15) that for any $\rho > 0$,

$$\begin{aligned}(\psi, H_\tau \psi) &\ge (\psi, H^\sigma|_{\mathcal{H}_\tau} \psi) + (\psi, H_\tau^\sigma \psi)\\ &\ge \left[1 - \mathrm{C}|g|\sigma^{1/2}\rho^{-1/2}\right](\psi, H^\sigma|_{\mathcal{H}_\tau}\psi) + \mathrm{C}|g|\sigma^{1/2}\rho^{-1/2}E_\sigma - \mathrm{C}|g|\sigma^{1/2}\rho^{1/2}.\end{aligned}$$

Next, from $\mathbf{Gap}(\sigma)$ and Property (A.10), since $\tau = \sigma/2$, we obtain that for any ψ in $[\Phi_\sigma \otimes \Omega_\tau^\sigma]^\perp$, $\|\psi\| = 1$,

$$(\psi, H^\sigma|_{\mathcal{H}_\tau}\psi) \ge E_\sigma + \min\left(\frac{\sigma}{8}, \frac{\tau}{4}\right) \ge E_\sigma + \sigma/8, \tag{A.24}$$

provided that $|g|$ is sufficiently small. Hence for any $\rho > 0$ such that $\rho^{1/2} > \mathrm{C}|g|\sigma^{1/2}$,

$$(\psi, H_\tau\psi) \ge E_\sigma + \left[1 - \mathrm{C}|g|\sigma^{1/2}\rho^{-1/2}\right]\frac{\sigma}{8} - \mathrm{C}|g|\sigma^{1/2}\rho^{1/2}. \tag{A.25}$$

Choosing $\rho^{1/2} = 4\mathrm{C}|g|\sigma^{1/2}$, by Corollary A.2, we get

$$(\psi, H_\tau\psi) \ge E_\sigma + \frac{3}{32}\sigma - 4\mathrm{C}^2 g^2 \sigma \ge E_\tau + \frac{3}{16}\tau - 8\mathrm{C}^2 g^2 \tau. \tag{A.26}$$

Hence $\mu_\tau \ge E_\tau + \tau/8$ provided that $|g| \le (8\mathrm{C})^{-1}$, which proves the lemma. □

Proof of Proposition A.3 As mentioned above, Proposition A.3 easily follows from Lemmata A.5 and A.6, and an induction argument. □

Let us conclude this Subsection with a bound on the difference $|E_\tau - E_\sigma|$.

Lemma A.7. *Under the conditions of Proposition A.3, there exists a positive constant* C *such that for all $0 \le \tau \le \sigma \le \sigma_0$,*

$$|E_\tau(P_3) - E_\sigma(P_3)| \le \mathrm{C}|g|\sigma. \tag{A.27}$$

Proof. By Corollary A.2, we already have $E_\tau(P_3) \le E_\sigma(P_3)$. The inequality $E_\sigma(P_3) \le E_\tau(P_3) + \mathrm{C}|g|\sigma$ follows similarly, using (A.15) and a variational argument. □

Remark A.8. Lemma A.7 remains true if the operators under consideration are not Wick-ordered. More precisely in this case we have

$$E_\tau(P_3) \le E_\sigma(P_3) + \mathrm{C}g^2\sigma \le E_\tau(P_3) + \mathrm{C}|g|\sigma. \tag{A.28}$$

A.2. Proof of Theorem 1.2. The key property used in the proof of Theorem 1.2 lies in the estimate of $|E'_\tau(P_3) - E'_\sigma(P_3)|$ for $\tau \le \sigma$.

Proposition A.9. *There exits $g_0 > 0$, $\sigma_0 > 0$ and $P_0 > 0$ such that for all $0 < |g| \le g_0$ and $|P_3| \le P_0$, for all $\sigma, \tau > 0$ such that $\tau \le \sigma \le \sigma_0$, for all $\delta > 0$,*

$$|E'_\tau(P_3) - E'_\sigma(P_3)| \le \mathrm{C}_\delta \sigma^{1/2-\delta},$$

where C_δ is a positive constant depending only on δ.

We shall divide the main part of the proof of Proposition A.9 into two lemmata. Let us begin with some definitions and notations. For $\sigma > 0$ and $\rho \ge 0$, we define the function $g_{\sigma,\rho} \in \mathrm{L}^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ by

$$g_{\sigma,\rho}(\mathbf{k}) = g \mathbb{1}_{\sigma \le |k| \le \Lambda}(k) \frac{\epsilon^3_\lambda(k)}{\sqrt{2\pi}|k|^{1/2}} \frac{\rho}{|k| - k_3\rho}.$$

Depending on the context, the Weyl operator $W(ig_{\sigma,\rho})$ will represent an operator on $\mathcal{H}_\sigma$, $\mathcal{H}_\tau$ (for $\tau \le \sigma$), or $\mathcal{H}$.

From now on, to simplify the notations, we drop the dependence on P_3 everywhere unless a confusion may arise. For g, σ and P_3 as in Proposition A.3, let Φ_σ denote a normalized ground state of H_σ. Define

$$H^{\mathrm{ren}}_{\sigma,\rho} = W(ig_{\sigma,\rho}) H_\sigma W(ig_{\sigma,\rho})^*, \quad \Phi^{\mathrm{ren}}_{\sigma,\rho} = W(ig_{\sigma,\rho})\Phi_\sigma,$$

and let $P^{\mathrm{ren}}_{\sigma,\rho}$ be the orthogonal projection onto the vector space spanned by $\Phi^{\mathrm{ren}}_{\sigma,\rho}$. Note that $\Phi^{\mathrm{ren}}_{\sigma,\rho}$ is a normalized, non-degenerate ground state of $H^{\mathrm{ren}}_{\sigma,\rho}$, associated with the ground state energy E_σ. Recall that, by Lemma A.1, $[\Phi_\sigma \otimes \Omega^\sigma_\tau]$ is a ground state of $H^\sigma|_{\mathcal{H}_\tau}$. We set

$$H^{\mathrm{ren}}_{\sigma,\rho,\tau} = W(ig_{\sigma,\rho}) H^\sigma|_{\mathcal{H}_\tau} W(ig_{\sigma,\rho})^*, \quad \Phi^{\mathrm{ren}}_{\sigma,\rho,\tau} = W(ig_{\sigma,\rho})[\Phi_\sigma \otimes \Omega^\sigma_\tau],$$

and the projection onto the vector space spanned by $\Phi^{\mathrm{ren}}_{\sigma,\rho,\tau}$ is denoted by $P^{\mathrm{ren}}_{\sigma,\rho,\tau}$. Since $W(ig_{\sigma,\rho}) = e^{i\Phi(ig_{\sigma,\rho})\otimes\mathbb{1}}$, it can be seen that $\Phi^{\mathrm{ren}}_{\sigma,\rho,\tau} = [W(ig_{\sigma,\rho})\Phi_\sigma] \otimes \Omega^\sigma_\tau = \Phi^{\mathrm{ren}}_{\sigma,\rho} \otimes \Omega^\sigma_\tau$.

Lemma A.10. *There exists $g_0 > 0$, $\sigma_0 > 0$ and $P_0 > 0$ such that for all $0 < |g| \le g_0$ and $|P_3| \le P_0$, for all $\sigma, \tau > 0$ such that $\tau \le \sigma \le \sigma_0$,*

$$|E'_\sigma - E'_\tau| \le \mathrm{C}\left[\left\| P^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} - P^{\mathrm{ren}}_{\tau,E'_\sigma} \right\| + g^2\sigma \right], \tag{A.29}$$

where C is a positive constant.

Proof. By the Feynman-Hellman formula (see (2.22)),

$$E'_\sigma = \frac{1}{m}\left(\Phi_\sigma, [P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x',0)]\,\Phi_\sigma\right)_{\mathcal{H}_\sigma}. \tag{A.30}$$

It follows from (A.30) and commutation relations with $W(ig_{\sigma,E'_\sigma})$ that

$$\begin{aligned} E'_\sigma = &\frac{1}{m}\Bigg(\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma}, \Bigg[P_3 - d\Gamma(k_3) - \Phi(k_3 g_{\sigma,E'_\sigma}) - \frac{1}{2}(k_3 g_{\sigma,E'_\sigma}, g_{\sigma,E'_\sigma}) \\ &\qquad - gA_{3,\sigma}(x',0) + g\mathrm{Re}(h_{3,\sigma}(x'), g_{\sigma,E'_\sigma})\Bigg]\,\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma}\Bigg)_{\mathcal{H}_\sigma}, \end{aligned} \tag{A.31}$$

Consequently, for $\tau \leq \sigma$, we can write

$$\begin{aligned} E'_\sigma = & \frac{1}{m}\Big(\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}, \Big[P_3 - d\Gamma(k_3) - \Phi(k_3 g_{\tau,E'_\sigma}) - \frac{1}{2}(k_3 g_{\sigma,E'_\sigma}, g_{\sigma,E'_\sigma}) \\ & - gA_{3,\tau}(x',0) + g\mathrm{Re}(h_{3,\sigma}(x'), g_{\sigma,E'_\sigma}) \Big] \Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} \Big)_{\mathcal{H}_\tau}, \end{aligned} \tag{A.32}$$

whereas

$$\begin{aligned} E'_\tau = & \frac{1}{m}\Big(\Phi^{\mathrm{ren}}_{\tau,E'_\sigma}, \Big[P_3 - d\Gamma(k_3) - \Phi(k_3 g_{\tau,E'_\sigma}) - \frac{1}{2}(k_3 g_{\tau,E'_\sigma}, g_{\tau,E'_\sigma}) \\ & - gA_{3,\tau}(x',0) + g\mathrm{Re}(h_{3,\tau}(x'), g_{\tau,E'_\sigma}) \Big] \Phi^{\mathrm{ren}}_{\tau,E'_\sigma} \Big)_{\mathcal{H}_\tau}. \end{aligned} \tag{A.33}$$

The expression into brackets being uniformly bounded with respect to $H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}$, one can prove that

$$\begin{aligned} \Big\| \Big[& P_3 - d\Gamma(k_3) - \Phi(k_3 g_{\tau,E'_\sigma}) - \frac{1}{2}(k_3 g_{\sigma,E'_\sigma}, g_{\sigma,E'_\sigma}) \\ & - gA_{3,\tau}(x',0) + \mathrm{Re}(h_{3,\sigma}(x'), g_{\sigma,E'_\sigma}) \Big] \Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} \Big\| \leq \mathrm{C}, \end{aligned} \tag{A.34}$$

and likewise with $\Phi^{\mathrm{ren}}_{\tau,E'_\sigma}$ replacing $\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}$. In addition, we have

$$\big| (k_3 g_{\sigma,E'_\sigma}, g_{\sigma,E'_\sigma}) - (k_3 g_{\tau,E'_\sigma}, g_{\tau,E'_\sigma}) \big| \leq \mathrm{C} g^2 \sigma, \tag{A.35}$$

and, similarly,

$$\Big\| \big[\mathrm{Re}(h_{3,\tau}(x'), g_{\tau,E'_\sigma}) - \mathrm{Re}(h_{3,\sigma}(x'), g_{\sigma,E'_\sigma}) \big] \Phi^{\mathrm{ren}}_{\tau,E'_\sigma} \Big\| \leq \mathrm{C}|g|\sigma. \tag{A.36}$$

Estimating the difference of (A.32) and (A.33) then leads to

$$|E'_\sigma - E'_\tau| \leq \mathrm{C} \Big[\Big\| \Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} - \Phi^{\mathrm{ren}}_{\tau,E'_\sigma} \Big\|_{\mathcal{H}_\tau} + g^2\sigma \Big] \tag{A.37}$$

The statement of the lemma now follows by choosing the non-degenerate ground states $\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}$ and $\Phi^{\mathrm{ren}}_{\tau,E'_\sigma}$ in such a way that

$$\Big\| \Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} - \Phi^{\mathrm{ren}}_{\tau,E'_\sigma} \Big\|_{\mathcal{H}_\tau} \leq \mathrm{C} \Big\| P^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} - P^{\mathrm{ren}}_{\tau,E'_\sigma} \Big\|. \tag{A.38}$$

Note that this choice is indeed possible due to the non-degeneracy of the ground states $\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}$ and $\Phi^{\mathrm{ren}}_{\tau,E'_\sigma}$. □

For g, P_3, σ, ρ as above, let us define the operator $\nabla H^{\mathrm{ren}}_{\tau,\rho}$ by

$$\begin{aligned} \nabla H^{\mathrm{ren}}_{\sigma,\rho} &= \frac{1}{m} W(ig_{\sigma,\rho}) \big[P_3 - d\Gamma(k_3) - gA_{3,\sigma}(x',0) \big] W(ig_{\sigma,\rho})^* \\ &= \frac{1}{m} \Big[P_3 - d\Gamma(k_3) - \Phi(k_3 g_{\sigma,\rho}) - \frac{1}{2}(k_3 g_{\sigma,\rho}, g_{\sigma,\rho}) \\ &\qquad - gA_{3,\sigma}(x',0) + g\mathrm{Re}(h_{3,\sigma}(x'), g_{\sigma,\rho}) \Big]. \end{aligned}$$

LEMMA A.11. *Let $\Gamma_{\sigma,\mu}$ be the curve $\Gamma_{\sigma,\mu} = \{\mu\sigma e^{i\nu}, \nu \in [0, 2\pi[\}$. There exist $g_0 > 0$, $\sigma_0 > 0$, $\mu > 0$ and $P_0 > 0$, such that for all $0 < |g| \le g_0$, $|P_3| \le P_0$, for all $\sigma > 0$ and $\tau > 0$ such that $\sigma/2 \le \tau \le \sigma \le \sigma_0$,*

$$\begin{aligned} \left\| P^{\rm ren}_{\sigma,E'_\sigma,\tau} - P^{\rm ren}_{\tau,E'_\sigma} \right\| \le \mathrm{C}|g|^{1/2}\sigma^{1/2} \sup_{z\in\Gamma_{\sigma,\mu}} \Bigg[1 + \Big| \Big(\left(\nabla H^{\rm ren}_{\sigma,E'_\sigma} - E'_\sigma\right) \Phi^{\rm ren}_{\sigma,E'_\sigma}, \\ \left[H^{\rm ren}_{\sigma,E'_\sigma} - E_\sigma - z\right]^{-1} \left(\nabla H^{\rm ren}_{\sigma,E'_\sigma} - E'_\sigma\right) \Phi^{\rm ren}_{\sigma,E'_\sigma} \Big) \Big|^{\frac12} \Bigg], \end{aligned} \tag{A.39}$$

where C *is a positive constant.*

PROOF. By [**BFP**, Lemma II.11],

$$\left\| P^{\rm ren}_{\sigma,E'_\sigma,\tau} - P^{\rm ren}_{\tau,E'_\sigma} \right\| = \left| \left(\Phi^{\rm ren}_{\sigma,E'_\sigma,\tau}, [P^{\rm ren}_{\sigma,E'_\sigma,\tau} - P^{\rm ren}_{\tau,E'_\sigma}] \Phi^{\rm ren}_{\sigma,E'_\sigma,\tau} \right) \right|^{1/2}. \tag{A.40}$$

It follows from Lemma A.1 and Proposition A.3 that $\mathrm{Gap}(H^{\rm ren}_{\sigma,E'_\sigma,\tau}) \ge \sigma/8$ and $\mathrm{Gap}(H^{\rm ren}_{\tau,E'_\sigma}) \ge \tau/8 \ge \sigma/16$. Therefore, since $|E_\sigma - E_\tau| \le \mathrm{C}|g|\sigma$ by Lemma A.7, we can write

$$P^{\rm ren}_{\sigma,E'_\sigma,\tau} - P^{\rm ren}_{\tau,E'_\sigma} = \frac{i}{2\pi} \oint_{\Gamma_{\sigma,\mu}} \left(\left[H^{\rm ren}_{\sigma,E'_\sigma,\tau} - E_\sigma - z \right]^{-1} - \left[H^{\rm ren}_{\tau,E'_\sigma} - E_\sigma - z \right]^{-1} \right) dz,$$

provided $\mu < 1/16$ and $|g|$ is sufficiently small. Expanding $\left[H^{\rm ren}_{\tau,E'_\sigma} - E_\sigma - z \right]^{-1}$ into a (convergent) Neumann series yields

$$\begin{aligned} P^{\rm ren}_{\sigma,E'_\sigma,\tau} - P^{\rm ren}_{\tau,E'_\sigma} = \frac{i}{2\pi} \sum_{n\ge1} \oint_{\Gamma_{\sigma,\mu}} (-1)^n \left[H^{\rm ren}_{\sigma,E'_\sigma,\tau} - E_\sigma - z \right]^{-1} \\ \left(\left[H^{\rm ren}_{\tau,E'_\sigma} - H^{\rm ren}_{\sigma,E'_\sigma,\tau} \right] \left[H^{\rm ren}_{\sigma,E'_\sigma,\tau} - E_\sigma - z \right]^{-1} \right)^n dz. \end{aligned}$$

Let us compute the difference $H^{\rm ren}_{\tau,E'_\sigma} - H^{\rm ren}_{\sigma,E'_\sigma,\tau}$ explicitly. We have:

$$\begin{aligned} H^{\rm ren}_{\sigma,E'_\sigma,\tau} &= \frac{1}{2m} \sum_{j=1,2} \Big(p_j - e a_j(x') - g A_{j,\sigma}(x', 0) + g \mathrm{Re}(h_{j,\sigma}(x'), g_{\sigma,E'_\sigma}) \Big)^2 \\ &+ \frac{m}{2} (\nabla H^{\rm ren}_{\sigma,E'_\sigma})^2 - \frac{e}{2m}\sigma_3 b(x') - \frac{g}{2m}\sigma \cdot \Big(B_\sigma(x', 0) - \mathrm{Re}(\tilde h_\sigma(x'), g_{\sigma,E'_\sigma}) \Big) \\ &+ V(x') + H_f + \Phi(|k| g_{\sigma,E'_\sigma}) + \frac12 (|k| g_{\sigma,E'_\sigma}, g_{\sigma,E'_\sigma}) - \frac{g^2}{2m}(\Lambda^2 - \sigma^2), \end{aligned}$$

and

$$\begin{aligned} H^{\rm ren}_{\tau,E'_\sigma} &= \frac{1}{2m} \sum_{j=1,2} \Big(p_j - e a_j(x') - g A_{j,\tau}(x', 0) + g \mathrm{Re}(h_{j,\tau}(x'), g_{\tau,E'_\sigma}) \Big)^2 \\ &+ \frac{m}{2} (\nabla H^{\rm ren}_{\tau,E'_\sigma})^2 - \frac{e}{2m}\sigma_3 b(x') - \frac{g}{2m}\sigma \cdot \Big(B_\sigma(x', 0) - \mathrm{Re}(\tilde h_\tau(x'), g_{\tau,E'_\sigma}) \Big) \\ &+ V(x') + H_f + \Phi(|k| g_{\tau,E'_\sigma}) + \frac12 (|k| g_{\tau,E'_\sigma}, g_{\tau,E'_\sigma}) - \frac{g^2}{2m}(\Lambda^2 - \tau^2). \end{aligned}$$

Let us decompose:

$$H^{\rm ren}_{\tau,E'_\sigma} - H^{\rm ren}_{\sigma,E'_\sigma,\tau} = [a] + [b] + [c] + [d] + [e], \tag{A.41}$$

with

$$[a] = \frac{1}{m} \sum_{j=1,2} \Big(- gA^{\sigma}_{j,\tau}(0,0) + g\mathrm{Re}(h_{j,\tau}(0), g^{\sigma}_{\tau,E'_\sigma}) \Big) \times \Big(p_j - ea_j(x') - gA_{j,\sigma}(x',0) + g\mathrm{Re}(h_{j,\sigma}(x'), g_{\sigma,E'_\sigma}) \Big),$$

$$[b] = \frac{1}{2m} \sum_{j=1,2} \Big(- gA^{\sigma}_{j,\tau}(x',0) + g\mathrm{Re}(h_{j,\tau}(x'), g^{\sigma}_{\tau,E'_\sigma}) \Big)^2 - \frac{g^2}{2m}(\sigma^2 - \tau^2) + \frac{1}{2m}\Big(- \Phi(k_3 g^{\sigma}_{\tau,E'_\sigma}) - \frac{1}{2}(k_3 g^{\sigma}_{\tau,E'_\sigma}, g^{\sigma}_{\tau,E'_\sigma}) - gA^{\sigma}_{3,\tau}(x',0) + g\mathrm{Re}(h_{3,\tau}(x'), g^{\sigma}_{\tau,E'_\sigma}) \Big)^2, + \frac{g}{2m}\sigma \cdot \mathrm{Re}\Big(\tilde{h}_\tau(x') - \tilde{h}_\sigma(x'), g_{\tau,E'_\sigma} \Big)$$

$$[c] = \frac{1}{m} \sum_{j=1,2} \Big(- g(A^{\sigma}_{j,\tau}(x',0) - A^{\sigma}_{j,\tau}(0)) + g\mathrm{Re}(h_{j,\tau}(x') - h_{j,\tau}(0), g^{\sigma}_{\tau,E'_\sigma}) \Big) \times \Big(p_j - ea_j(x') - gA_{j,\sigma}(x',0) + g\mathrm{Re}(h_{j,\sigma}(x'), g_{\sigma,E'_\sigma}) \Big) - gE'_\sigma[A^{\sigma}_{3,\tau}(x',0) - A^{\sigma}_{3,\tau}(0,0)] + gE'_\sigma \mathrm{Re}(h_{3,\tau}(x') - h_{3,\tau}(0), g^{\sigma}_{\tau,E'_\sigma}),$$

$$[d] = gE'_\sigma(h_{3,\tau}(0), g^{\sigma}_{\tau,E'_\sigma}) - \frac{1}{2}E'_\sigma(k_3 g^{\sigma}_{\tau,E'_\sigma}, g^{\sigma}_{\tau,E'_\sigma}),$$

$$[e] = \frac{1}{2}\Big(- \Phi(k_3 g^{\sigma}_{\tau,E'_\sigma}) - \frac{1}{2}(k_3 g^{\sigma}_{\tau,E'_\sigma}, g^{\sigma}_{\tau,E'_\sigma}) - gA^{\sigma}_{3,\tau}(x',0) + g\mathrm{Re}(h_{3,\tau}(x'), g^{\sigma}_{\tau,E'_\sigma}) \Big) \times \Big(\nabla H^{\mathrm{ren}}_{\sigma,E'_\sigma} - E'_\sigma \Big) + \frac{1}{2}\Big(\nabla H^{\mathrm{ren}}_{\sigma,E'_\sigma} - E'_\sigma \Big) \times \Big(- \Phi(k_3 g^{\sigma}_{\tau,E'_\sigma}) - \frac{1}{2}(k_3 g^{\sigma}_{\tau,E'_\sigma}, g^{\sigma}_{\tau,E'_\sigma}) - gA^{\sigma}_{3,\tau}(x',0) + g\mathrm{Re}(h_{3,\tau}(x'), g^{\sigma}_{\tau,E'_\sigma}) \Big).$$

Note that we have added and subtracted E'_σ, using the identity $(E'_\sigma k_3 - |k|)g_{\sigma,E'_\sigma} = -gE'_\sigma h_{3,\sigma}(0)$ and likewise with g_{τ,E'_σ} replacing g_{σ,E'_σ}. Let us now consider, for some $n \geq 1$,

$$\oint_{\Gamma_{\sigma,\mu}} \Big(\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}, \Big[H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} - E_\sigma - z \Big]^{-1} \Big(\Big[H^{\mathrm{ren}}_{\tau,E'_\sigma} - H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} \Big] \Big[H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} - E_\sigma - z \Big]^{-1} \Big)^n \Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} \Big). \tag{A.42}$$

We insert (A.41) into the right-hand side of (A.42), thus obtaining a sum of terms that we estimate separately. We claim that all the terms where at least one of the operators $[a]$, $[b]$, or $[c]$ appear, are bounded by $\mathrm{C}\sigma(\mathrm{C}'|g|)^n$ where C, C' are two positive constants. The latter can be proven by means of rather standard estimates involving pull-through formulas (see for instance [**BFS, Pi, BFP, CFP**]), so we shall not give all the details. Let us still emphasize that in order to deal with $[a]$ or $[c]$ we need to use the exponential decay of $\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}$ in x' (proven in [**AGG2**, Appendix A]). This is the main difficulty we encounter compared to the proof of [**CFP**]. In order to overcome it, we adapt a method due to [**Si**] (see also [**AFFS**, Section 5]). Let us give an example: Consider

$$\Big(\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}, [e] \Big[H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} - E_\sigma - z \Big]^{-1} [a] \Big[H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} - E_\sigma - z \Big]^{-1} [e] \Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} \Big). \tag{A.43}$$

We shall take advantage of the identity

$$\text{(A.44)} \qquad \Big(p_j - ea_j(x') - gA_{j,\sigma}(x',0) + g\mathrm{Re}(h_{j,\sigma}(x'), g_{\sigma,E'_\sigma})\Big) = 2i\left[H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}, x'_j\right]$$

which holds in the sense of quadratic forms on $D(H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}) \cap D(x'_j)$. The field operator $A^{\tau}_{j,\sigma}(0,0) = \Phi(h^{\tau}_{j,\sigma})$ in $[a]$ decompose into a sum of a creation operator and an annihilation operator that are estimated separately. Take for instance the creation operator. Using a pull-through formula, we have to bound:

$$\text{(A.45)} \qquad \begin{aligned} g\int h^{\sigma}_{j,\tau}(\mathbf{k})\Big(\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}, [e]a^*(\mathbf{k})\left[H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}(P_3-k_3) - E_\sigma + |k| - z\right]^{-1} \\ \left[H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}, x'_j\right]\left[H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} - E_\sigma - z\right]^{-1}[e]\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}\Big)d\mathbf{k}. \end{aligned}$$

Let $\gamma > 0$ be such that $\|e^{\gamma\langle x'\rangle}\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}\| < \infty$. Undoing the commutator $[H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}, x'_j]$ gives two terms. We write the first one under the form

$$\begin{aligned} g\int h^{\sigma}_{j,\tau}(\mathbf{k})\Big(\Big(H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} - E_\sigma\Big)\left[H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}(P_3-k_3) - E_\sigma + |k| - \bar{z}\right]^{-1} a(\mathbf{k})[e]^*\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}, \\ x'_j e^{-\gamma\langle x'\rangle}e^{\gamma\langle x'\rangle}\left[H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} - E_\sigma - z\right]^{-1} e^{-\gamma\langle x'\rangle}[e]e^{\gamma\langle x'\rangle}\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}\Big)d\mathbf{k}. \end{aligned}$$

Now we have the following estimates:

$$\text{(A.46)} \qquad \left\| e^{\gamma\langle x'\rangle}\left[H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} - E_\sigma - z\right]^{-1} e^{-\gamma\langle x'\rangle}[e]e^{\gamma\langle x'\rangle}\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}\right\| \le \mathrm{C}|g|,$$

$$\text{(A.47)} \qquad \left\| x'_j e^{-\gamma\langle x'\rangle}\right\| \le \mathrm{C},$$

$$\text{(A.48)} \qquad \left\| \left[H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}(P_3-k_3) - E_\sigma + |k| - z\right]^{-1}\Big(H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} - E_\sigma\Big)\right\| \le \mathrm{C},$$

$$\text{(A.49)} \qquad \left\| a(\mathbf{k})[e]^*\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}\right\| \le \mathrm{C}|g||k|^{-1/2}.$$

Note that in (A.48) and (A.49), we used that $\tau \le |k| \le \sigma$, and thus in particular that $a(\mathbf{k})\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} = 0$. Since the other term coming from the commutator $[H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}, x'_j]$ can be estimated in the same way, this yields

$$\text{(A.50)} \qquad |(\text{A.45})| \le \mathrm{C}|g|^3\int |h^{\sigma}_{j,\tau}(\mathbf{k})||k|^{-1/2}d\mathbf{k} \le \mathrm{C}|g|^3\sigma^2.$$

Taking into account the factor σ coming from the integration in (A.42) would finally lead to our claim in the case of the example (A.43). The same holds for the terms containing $[c]$ at least once (except that the use of (A.44) is then not required). Besides, since $[d]$ is constant,

$$\oint_{\Gamma_{\sigma,\mu}}\left(\Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}, \left[H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} - E_\sigma - z\right]^{-1}\left([d]\left[H^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau} - E_\sigma - z\right]^{-1}\right)^n \Phi^{\mathrm{ren}}_{\sigma,E'_\sigma,\tau}\right) = 0.$$

Therefore it remains to consider the terms containing only $[d]$ or $[e]$, with $[e]$ appearing at least in one factor. One can prove that this leads to

$$\left\| P^{\rm ren}_{\sigma,E'_\sigma,\tau} - P^{\rm ren}_{\tau,E'_\sigma} \right\| \leq {\rm C}|g|^{1/2}\sigma^{1/2} \sup_{z\in\Gamma_{\sigma,\mu}} \Bigg[1+\sigma^{-1} \Bigg\| \left| H^{\rm ren}_{\sigma,E'_\sigma,\tau} - E_\sigma - z \right|^{-1/2} \left(-\Phi(k_3 g^\sigma_{\tau,E'_\sigma}) - \frac{1}{2}(k_3 g^\sigma_{\tau,E'_\sigma}, g^\sigma_{\tau,E'_\sigma}) - gA^\sigma_{3,\tau}(x',0) + g{\rm Re}(h_{3,\tau}(x'), g^\sigma_{\tau,E'_\sigma}) \right) \left[\nabla H^{\rm ren}_{\sigma,E'_\sigma} - E'_\sigma \right] \Phi^{\rm ren}_{\sigma,E'_\sigma,\tau} \Bigg\| \Bigg].$$

Using again the exponential decay of $\Phi^{\rm ren}_{\sigma,E'_\sigma}$ in x', we may replace ${\rm Re}(f_{3,\tau}(x'), g^\sigma_{\tau,E'_\sigma})$ with ${\rm Re}(f_{3,\tau}(0), g^\sigma_{\tau,E'_\sigma})$ in the previous expression. Proceeding then as in [**CFP**, Lemma A.3], since both

$$(\Phi(k_3 g^\sigma_{\tau,E'_\sigma}) + gA^\sigma_{3,\tau}(x',0))(\nabla H^{\rm ren}_{\sigma,E'_\sigma} - E'_\sigma)\Phi^{\rm ren}_{\sigma,E'_\sigma,\tau} \quad \text{and} \quad (\nabla H^{\rm ren}_{\sigma,E'_\sigma} - E'_\sigma)\Phi^{\rm ren}_{\sigma,E'_\sigma,\tau}$$

are orthogonal to $\Phi^{\rm ren}_{\sigma,E'_\sigma,\tau}$, we obtain Inequality (A.39) (notice in particular that σ_0 and μ must be fixed sufficiently small to pass from the last estimate to (A.39)). □

Proof of Proposition A.9 To conclude the proof of Proposition A.9, in view of Lemmata A.10 and A.11, it suffices to show that

$$\left| \left(\left(\nabla H^{\rm ren}_{\sigma,E'_\sigma} - E'_\sigma \right) \Phi^{\rm ren}_{\sigma,E'_\sigma}, \left[H^{\rm ren}_{\sigma,E'_\sigma} - E_\sigma - z \right]^{-1} \left(\nabla H^{\rm ren}_{\sigma,E'_\sigma} - E'_\sigma \right) \Phi^{\rm ren}_{\sigma,E'_\sigma} \right) \right| \leq \frac{{\rm C}_\delta}{|g|\sigma^{2\delta}},$$

for any $z \in \Gamma_{\sigma,\mu}$ and any $\delta > 0$. This corresponds to the bound (IV.68) in [**CFP**] and can be proven in the same way as in [**CFP**, Subsection IV.5, step (4)], using an induction procedure. We therefore refer the reader to [**CFP**] for a proof. □

Proof of Theorem 1.2 Fix P_3 and k_3 such that $|P_3| \leq P_0$, $|P_3 + k_3| \leq P_0$. One can see that there exist positive constants ${\rm C}_0$ and C such that, for any $0 < \beta < 1$ and $\sigma \geq {\rm C}_0|k_3|^\beta$,

$$|E'_\sigma(P_3 + k_3) - E'_\sigma(P_3)| \leq {\rm C}|k_3|^{\frac{1}{2}(1-\beta)}. \tag{A.51}$$

This can be proven by estimating $|E'_\sigma(P_3 + k_3) - E'_\sigma(P_3)|$ in terms of $\|\Phi_\sigma(P_3 + k_3) - \Phi_\sigma(P_3)\|$, then using the second resolvent equation to estimate $\|[H_\sigma(P_3 + k_3) - z]^{-1} - [H_\sigma(P_3) - z]^{-1}\|$. Now, for $\sigma \leq {\rm C}_0|k_3|^\beta$, we use Proposition A.9, which yields

$$\begin{aligned} &|E'_\sigma(P_3 + k_3) - E'_\sigma(P_3)| \\ &\leq \left| E'_\sigma(P_3 + k_3) - E'_{{\rm C}_0|k_3|^\beta}(P_3 + k_3) \right| + \left| E'_{{\rm C}_0|k_3|^\beta}(P_3 + k_3) - E'_{{\rm C}_0|k_3|^\beta}(P_3) \right| \\ &\quad + \left| E'_\sigma(P_3) - E'_{{\rm C}_0|k_3|^\beta}(P_3) \right| \\ &\leq {\rm C}_\delta \left[|k_3|^{\frac{1}{2}(1-\beta)} + |k_3|^{\frac{1}{2}\beta(1-\delta)} \right]. \end{aligned}$$

The theorem follows by choosing $\beta = [2-\delta]^{-1}$. □

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