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Abstract: Using an industrial process from the car sector, we show how dioid algebra may be used for the performance evaluation, sizing, and control of this discrete-event dynamic system. Based on a Petri net model as an event graph, max-plus algebra and min-plus algebra permit to write linear equations of the behavior. From this formalism, the cycle time is determined and an optimal sizing is characterized for a required cyclic behavior. Finally, a strict temporal constraint the system is subject to is reformulated in terms of inequalities that the (min, +) system should satisfy, and a control law is designed so that the controlled system satisfies the constraint.

Key words: Dioid algebra, Petri net, sizing, cycle time, control.

1.   INTRODUCTION

For efficient design and operation techniques of manufacturing systems, it is well known that methods and tools are needed to model complex material and control flows, to analyze behavior and interactions of manufacturing resources and to predict the performance measures such as productivity, cycle times and work-in-process. Thanks to significant advances over the last decade, discrete-event system techniques have gained maturity and provide a wide spectrum of tools for solving problems encountered in design, supervisory control and performance evaluation of manufacturing systems. Petri nets, queuing networks, dioid algebra, perturbation analysis, formal language theory, state-charts have, among many others, proven techniques for modeling, specification and analysis of complex discrete-event dynamic systems (DEDS).

The problem tackled here is the performance evaluation, the optimal sizing and the control of a manufacturing plant from the sector of the car industry. The studied process can be embedded in the discrete-event dynamic system class. Among various techniques of analysis and control [2], a theory of linear algebra for DEDS also known as dioid theory has been recently developed [1]. More precisely, as long as the DEDS has no conflict, it can be modeled by an event graph or marked graph, i.e. by a Petri net in which each place has exactly one input and one output transitions. Such a graph is called a Timed Event Graph (TEG) when a sojourn time is associated to each place, or by duality, when a delay is associated to each transition. Hence, the timed behavior of such a TEG is represented by the dates at which the transitions of a TEG are fired for the kth time or equivalently, by the number of firings of transitions at time t. The former description, as daters in the events domain, leads to linear equations in the so-called max-plus algebra, while in the latter, as counters in the time domain, equations are linear in the min-plus algebra. In both cases, there exists a spectral theory that leads to one of the main results in that frameworks concerning performance evaluation of DEDS. The cycle time, or the throughput, is the solution of an eigenvalue problem in the considered algebra. Beside these performance evaluation interests, control problems have been addressed in the context of DEDS using this algebra. In this paper, the first step concerns the cycle time of the industrial plant using the max-plus algebra. Next, for a required cycle time, the sizing and the control of the plant are determined using the min-plus algebra.

The article is organized as follows. Backgrounds concerning the dioid algebra are recalled in Section 2. A brief description of the industrial plant is given in Section 3. A Timed Event Graph model of the plant is provided in Section 4, and linear equations associated to the system in the max-plus dioid algebra are derived in Section 5. On the basis of that model, the cycle time is determined in the next section for a given configuration, that is a given number of available pallets. The sizing problem is tackled in Section 7, where the number of free pallets is optimized for a required cycle time. Finally a control problem is addressed and solved in Section 8.

2.   DIOID ALGEBRA

Definition 1 *(Dioid algebra).*
(i) An abelian monoid is a set D endowed with an internal law $\oplus$ that is associative, commutative, and admits a neutral element denoted ε.
(ii) An abelian semiring is a monoid endowed with a second operation $\otimes$, also associative, distributive with respect to the first law $\oplus$, admitting a

neutral element e, and so that the neutral element of the first law is absorbing for the second law $\otimes$ (i.e. $\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon, \forall a \in D$).

(iii) A dioid is a semiring with an idempotent first law (i.e. $a \oplus a = a, \forall a \in D$).

One says that the dioid is commutative provided that the law $\otimes$ is commutative.

Example 1 *(Max-plus algebra)*. $\overline{\mathfrak{R}}_{max} = \{\mathfrak{R} \cup \{-\infty\} \cup \{+\infty\}, \max, +\}$ is a commutative dioid, with zero element ε equal to $-\infty$, and the unit element e equal to 0. We adopt the usual notation, so that the symbol $\oplus$ stands for the max operation, and $\otimes$ stands for the addition. Notice that $\varepsilon \otimes +\infty = (-\infty) + (+\infty) = \varepsilon = -\infty$ in $\overline{\mathfrak{R}}_{max}$.

Example 2 *(Min-plus algebra)*. $\overline{\mathfrak{R}}_{min} = \{\mathfrak{R} \cup \{-\infty\} \cup \{+\infty\}, \min, +\}$ is also a commutative dioid, for which ε equals to $+\infty$ and e equals to 0. We shall denote $\oplus'$ the min operation in the sequel, and the symbol $\otimes'$ will stand for the addition. Notice that $\varepsilon \otimes -\infty = (+\infty) + (-\infty) = \varepsilon = +\infty$ in $\overline{\mathfrak{R}}_{min}$.

Example 3 *(Matrix dioid)*. Let $(D, \oplus, \otimes)$ be a given dioid, and denote $D^{n \times n}$ the set of square $n \times n$ matrices with entries over D. The sum and the product over D extend as usually over $D^{n \times n}$ as follows:

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij} \text{ and } (A \otimes B)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \oplus B_{kj}.$$

One can see that $(D^{n \times n}, \oplus, \otimes)$ is a dioid. The neutral elements for the law $\oplus$ is the matrix the entries of which equal ε, the neutral element for the law $\otimes$ is the matrix the entries of which equal e on the diagonal and ε outside. Notice that the products of matrices in $\overline{\mathfrak{R}}_{max}$ and in $\overline{\mathfrak{R}}_{min}$ are not equal, (and do not equal the usual sum of matrices.)

Definition 2 *(Trace)*. The trace of a square matrix $A \in D^{n \times n}$ is the sum of its diagonal entries, denoted:

$$tr(A) = \bigoplus_{i=1}^{n} A_{ii}.$$

In the max-plus algebra, we shall also use the notation:

$$\left(tr(A^k)\right)^{\frac{1}{k}} = \frac{1}{k}\left(\bigoplus_{i=1}^{n}\left(A^k\right)_{ii}\right), \text{ where } A^k = \underbrace{A \otimes \cdots \otimes A}_{k \text{ times}}.$$

These concepts are useful to model Timed Event Graphs with sojourn delays associated to places. A TEG is an ordinary Petri net where each place has a single input transition and a single output transition. We denote p_{ij} the place from transition t_j to transition t_i. We assume that n is the number of transitions which are upstream and downstream transitions of places. The number of sink transitions, i.e. without output places, is equal to s. The number of transitions called source, having no downstream place, is m.

Delays or holding times are associated to places of the TEG. Thus the graph is timed on places. The max-plus approach [1] allows us to model with a system of linear inequations the dynamic behavior of such a TEG.

$$\begin{cases} x(k+1) \geq A \otimes x(k) \oplus B \otimes u(k+1) \\ y(k) \geq C \otimes x(k) \end{cases} \tag{1}$$

where the components of the vector $x(k)$ are the firing times of the n transitions t_i for the kth occurrence, the components of $u(k)$ and of $y(k)$ are respectively the firing dates of the source and of the sink transitions. A, B and C are respectively matrices of size $n \times n$, $n \times m$ and $s \times n$.

3.   PLANT DESCRIPTION

The process we study here (see reference [6] for more details) is composed of three conveyors belt connected by loops (Figure 1). The parts are made on an extruding machine on loop 3. Loops 1 and 2 are both similar one to each other; they are dedicated to a thermal processing of the parts. Loop 3 processes parts that are conveyed on pallets to one of the other loops. On loop 1, respectively on loop 2, pallets are devoted for a certain type of parts coming from loop 3. Hence, synchronization is needed between these three transfer elements. The machine on loop 3 is very flexible and therefore can provide any type of part needed on loop 1 or loop 2. The main problem is to achieve the thermal treatment on loop 1 or loop 2 without major failures.

Figure 1: The plant.

Assume loop 2 is under study (identical process for loop 1) (see Figure 2). Parts arrive (from loop 3) at point A and an operator fixes them to a pallet. The pallets are then released on a conveyor belt that leads them to point I. Here they enter inside the furnace. This element is a channel divided into two sections. Inside the former section parts are heated and they are next cooled down inside the latter. We can remark that there is no buffer between these two sections. Once pallets come outside the furnace (point O), they are transferred to a second operator who removes parts from the pallets. Thus, parts are taken away at point E according to the external resources. Finally, the free pallets are released and transfer to point A.

Figure 2: Loop 2.

The constraint for the thermal processing is that no overrun on the heating time is permitted. If so, all parts are rejected and the whole process has to be restarted. The problem is that from time to time, parts are not taken away immediately and those perturbations can affect the production achievement if the sizing, in number of pallets involved, is not correctly designed.

4. TIMED EVENT GRAPH OF THE LOOP

Let us consider loop 2. The capacity of conveyor E-A is equal to 7 pallets, while 5 pallets are free on this conveyor. This physical process is modeled thanks to a TEG with sojourn delays associated to places. The durations of operations (noted d in Figure 2) are assumed to be identical for all the parts and are reported beside places on the Petri net model (see Figure 3). For instance, the transfer time from point E to point A is considered to be equal to four time units and is represented by place p_{17} (i.e. place from transition t_7 to transition t_1). Tokens represent the resources of the plant: the pallets, the operators or the conveyor capacities (noted l in Figure 2). Transition t_1 models the beginning of the fixing operation and transition t_2 the end of that operation. Only one token is available for this firing transition, meaning that the operator can treat one part at a time. The delay spent for this task is equal to one time unit. The beginning of the thermal process, point I, is modeled by the firing of transition t_3.
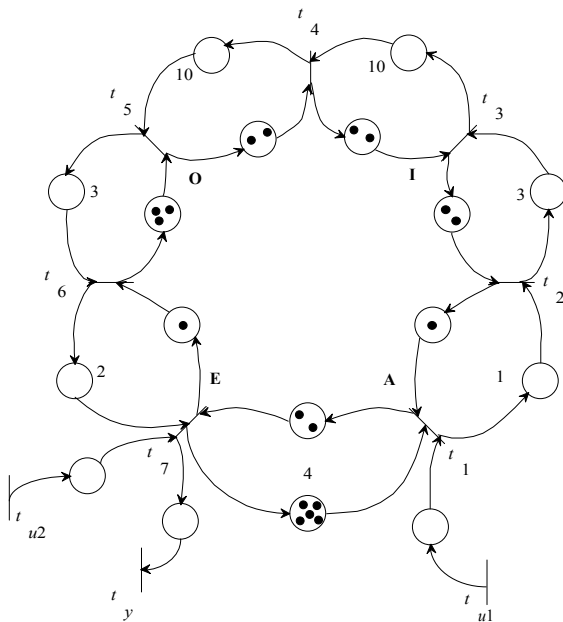
Figure 3: Model of the plant – loop 2.

Transition t_y represents the departure of an achieved part, t_{u2} models the necessity of a resource to carry the terminated part and t_{u1} models parts arrivals from loop 3. Figure 3 shows the process in an initial state where the five pallets involving through the system are available. The maximal capacity of the conveyor from point E to point A is supposed to be equal to seven pallets.

5.   DIOID MODEL OF THE LOOP

The construction of matrix *A*, see (1), can be done in several different ways. Olsder et al. have proposed in [7] a method to built max-plus models for large scale systems. For relatively small scale systems, like in the case of this study, one can rather apply the method proposed by Mairesse [5]. That method consists in an addition of some places and transitions in order to obtain a TEG with places containing either zero or one token. A null delay is associated to those additional places. One can see in Figure 4 the resulting TEG, applying that method to the model of Figure 3.
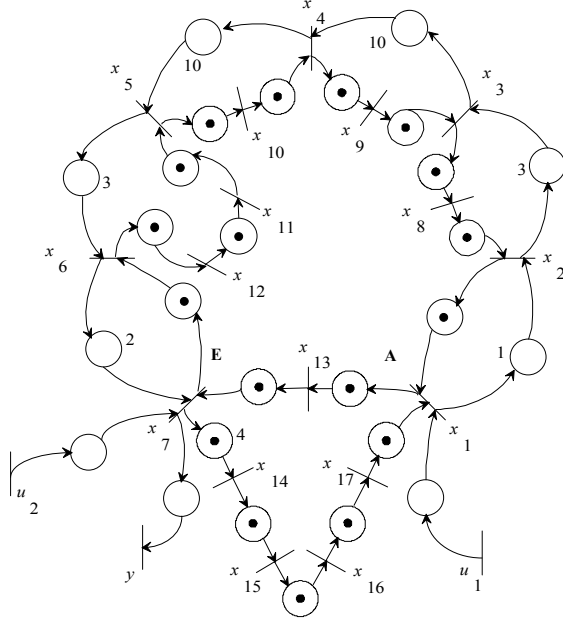
Figure 4: Resulting TEG for loop 2.

The TEG dynamic behavior can therefore be modeled by the following inequations (symbol $\otimes$ is omitted for the seek of readability):

$$\begin{cases} x(k+1) \geq A_0 x(k+1) \oplus A_1 x(k) \oplus B_0 u(k+1) \\ y(k) \geq Cx(k) \end{cases} \tag{2}$$

Then, matrices A_0, A_1 and B_0 are built. A_{0ij} is the delay associated to the empty place that links transition t_j to transition t_i. A_{1ij} is the delay associated to the place containing one token that links transition t_j to transition t_i. B_{0ij} is the delay associated to the empty place that links transition t_{uj} to transition t_i. If there is no place linking transition t_j to transition t_i (resp. transition t_{uj} to

transition t_i) the entry is ε. If the delay is zero, it is noted e. Matrices A_0, A_1, B_0 and C associated to the TEG of Figure3 are given bellow, where ε is replaced by a dot for the seek of readability.

$$
A_0 = \begin{pmatrix}
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & 3 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & 10 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & 10 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & 3 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & 2 & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{pmatrix}
$$

$$
A_1 = \begin{pmatrix}
. & e & . & . & . & . & . & . & . & . & e & . & . & . & . & . & . & e \\
. & . & . & . & . & . & . & e & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & e & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & e & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & e & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & e & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & e & . & . & . & . & . & . & . \\
. & . & e. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & e & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & e & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & e & . & . & . & . & . & . & . & . & . \\
. & . & . & . & e & . & . & . & . & . & . & . & . & . & . & . & . & . \\
e & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 4 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & e & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & e & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & e & .
\end{pmatrix}
$$

$$
B_0 = \begin{pmatrix}
e & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & e & . & . & . & . & . & . & . & . & . & .
\end{pmatrix}^T
\qquad
C = \begin{pmatrix}
. & . & . & . & . & . & e & . & . & . & . & . & . & . & . & .
\end{pmatrix}
$$

For TEG under maximal speed assumption, system of inequations (2) turns into a system of equations. Furthermore one can derive the system of explicit equations (3):

$$
\begin{cases}
x(k+1) = Ax(k) \oplus Bu(k+1) \\
y(k) = Cx(k)
\end{cases}
\tag{3}
$$

with matrices $A = A_0^* A_1$ and $B = A_0^* B_0$, where operator $*$ stands for the Kleene star operator: $a^* = e \oplus a \oplus a^2 \oplus \dots$.

$$A_0^* = \begin{pmatrix}
e & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
1 & e & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
4 & 3 & e & . & . & . & . & . & . & . & . & . & . & . & . & . \\
14 & 13 & 10 & e & . & . & . & . & . & . & . & . & . & . & . & . \\
24 & 23 & 20 & 10 & e & . & . & . & . & . & . & . & . & . & . & . \\
27 & 26 & 23 & 13 & 3 & e & . & . & . & . & . & . & . & . & . & . \\
29 & 28 & 25 & 15 & 5 & 2 & e & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & e & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & e & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & e & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & e & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & e & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & e & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & e & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & e & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & e
\end{pmatrix}$$

$$A = \begin{pmatrix}
. & e & . & . & . & . & . & . & . & . & . & . & . & . & . & e. \\
. & 1 & . & . & . & . & . & e & . & . & . & . & . & . & . & 1 \\
. & 4 & . & . & . & . & . & 3 & e & . & . & . & . & . & . & 4 \\
. & 14 & . & . & . & . & . & 13 & 10 & e. & . & . & . & . & . & 14 \\
. & 24 & . & . & . & . & . & 23 & 20. & 10 & e & . & . & . & . & 24 \\
. & 27 & . & . & . & . & e & 26 & 23 & 13 & 3 & . & . & . & . & 27 \\
. & 29 & . & . & . & . & 2 & 28 & 25 & 15 & 5 & . & e & . & . & 29 \\
. & . & e. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & e & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & e & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & e & . & . & . & . & . \\
. & . & . & . & e & . & . & . & . & . & . & . & . & . & . & . \\
e & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 4 & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & e & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & e & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & e &
\end{pmatrix}$$

$$B = \begin{pmatrix}
e & 1 & 4 & 14 & 24 & 27 & 29 & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & e & . & . & . & . & . & . & . & .
\end{pmatrix}^T$$

6. CYCLE TIME OF THE AUTONOMOUS LOOP

The throughput is maximal if inputs do not constrain the evolution, i.e. if parts are always available at point A and transportation utilities always present at point E. This is modeled by $u_i(k) = \varepsilon$. Then, the system is said to

be autonomous, without source transitions, and can be modeled by equation (4).

$$x(k+1) = Ax(k) \qquad (4)$$

As long as its graph is strongly connected, or equivalently, matrix A is irreducible, after a finite transient behavior, the system reaches a periodic regime. Therefore, one can write:

$$\exists K, \forall k \geq K, x(k+c) = \lambda^c x(k) \qquad (5)$$

where λ refers to equation (6):

$$Ax = \lambda x \qquad (6)$$

and c is the cyclicity of matrix A. In other words, c parts are achieved every $c \times \lambda$ time units. The eigenvalue λ is unique for a strongly connected event graph and the throughput is given directly as $1/\lambda$. This eigenvalue is computed according to the algorithm of Cochet-Terrasson et al. [3].

$$\lambda = \bigoplus_{k=1}^{n} \left(tr A^k \right)^{\frac{1}{k}} \qquad (7)$$

Applying this algorithm to the model of the plant, λ = 33/5 = 6.6 time units.

7. SIZING FOR A REQUIRED CYCLE TIME

Now, the capacity of the conveyor from point E to point A is not fixed (i.e. place p_{71} is empty in Figure 3). The sizing problem is to determine the minimal number of pallets needed to reach the cycle time of 5 units. This required cycle time corresponds to the bottleneck element, i.e. the furnace, which has a production cycle at 5 time units. In other terms, the minimal initial marking of place p_{17} has to be determined. A method to solve this problem is presented by Gaubert in [4]. After having chosen a desired periodic throughput $\overline{\varphi}$, one can calculate in the min-plus algebra, the initial

minimal marking q. It is shown that the throughput constraint $\varphi(q) \geq \overline{\varphi}$ is equivalent to the existence of a finite subeigenvector of $\overline{\varphi}$. We denote by T_{ij} the holding time of place p_{ij} and by N_{ij} its initial marking. If place p_{ij} does

not exist, by convention, $T_{ij} = -\infty$ and $N_{ij} = +\infty$. For the TEG of Figure 3, with an initial marking $q_{71} = 0$ and q_{17} unknown, these matrix are defined as follow, where the dot replaces the infinite values:

$$
N(q) = \begin{bmatrix}
. & 1 & . & . & . & . & q_{17} \\
e & . & 2 & . & . & . & . \\
. & e & . & 2 & . & . & . \\
. & . & e & . & 2 & . & . \\
. & . & . & e & . & 3 & . \\
. & . & . & . & e & . & 1 \\
e & . & . & . & . & e & .
\end{bmatrix}
\quad \text{and} \quad
T = \begin{bmatrix}
. & e & . & . & . & . & 4 \\
1 & . & e & . & . & . & . \\
. & 3 & . & e & . & . & . \\
. & . & 10 & . & e & . & . \\
. & . & . & 10 & . & e & . \\
. & . & . & . & 3 & . & e \\
e & . & . & . & . & 2 & .
\end{bmatrix}
$$

From the method of Gaubert [4], the following assertions are equivalent:

i) $\varphi(q) \geq \bar{\varphi}$

ii) $\rho(C(q)) \geq 0$, (with $\rho(C(q)) = \bigoplus_{k=1}^{n}{}' \left(trC(q)^k \right)^{\frac{1}{k}}$)

where matrix $C(q)$ is defined by: $C_{ij}(q) = N_{ij}(q) - \bar{\varphi} T_{ij}$.

iii) There exists a vector $\alpha \in \mathfrak{R}^n$ such that $\forall i, \ \min_j (C_{ij}(q) + \alpha_j) \geq \alpha_i$.

We can notice that $C_{ij}(q) = +\infty$ if either $N_{ij}(q) = +\infty$ or $T_{ij}(q) = -\infty$. This ensures the coherence of the notation 'dot'.

For $\bar{\varphi} = 1/5$, $C(q) = N(q) - \dfrac{1}{5}T$, and thus:

$$
C(q) = \begin{bmatrix}
. & 1 & . & . & . & . & (q_{17} - \frac{4}{5}) \\
-\frac{1}{5} & . & 2 & . & . & . & . \\
. & -\frac{3}{5} & . & 2 & . & . & . \\
. & . & -\frac{10}{5} & . & 2 & . & . \\
. & . & . & -\frac{10}{5} & . & 3 & . \\
. & . & . & . & -\frac{3}{5} & . & 1 \\
e & . & . & . & . & -\frac{2}{5} & .
\end{bmatrix}
$$

Applying assertion ii) with $\overline{\varphi} = 1/5$, a minimal initial marking of place p_{17} is equal to 7, as $q_{17} \geq 33/5$. Finally, a minimum of 7 free pallets guarantees to the process a cycle time of 5 units.

8. CONTROL UNDER STRICT TIME CONSTRAINT

The production unit includes a critical section, which is the warm part of the furnace. The parts should not stay more than 10 time units in this section. Hence the production system is submitted to a strict timed constraint, which is the required sojourn time in the oven. The non respect of the constraint can occur, for instance, because of the lack of transportation resource (point E of Figure 2) during a certain period of time which leads to a blocking of pieces in the furnace. For the normal behavior of the process, the finite product is taken away and the pallets are recycled at the level of point E. In case of a problem at this point, the parts pile up downstream the working station, which may lead to the blocking of parts in the furnace and the cessation of the production.

8.1 Min-plus model: explicit equation

To each transition t_i we associate the time function $\theta_i(t)$, which is the number of firings of transition t_i at time t. For the TEG of Figure 3, $U_1(t)$, which is the number of firings of transition t_{u1} at time t, can be seen as a control, which permits to retain the parts arriving from the central loop 3, and postpone their arrival in the production process, if necessary. Similarly, $U_2(t)$ can be seen as a disturbance, i.e. the number of firings of transition t_{u2} at time t. In the sequel, $U(t)$ denotes the vector with components $U_1(t)$ and $U_2(t)$. The min-plus equations of the production unit are deduced from the event graph of Figure 3. They read like follows:

$$
\begin{aligned}
\theta(t) = \ & A_0^{'*} A_1' \theta(t-1) \oplus' A_0^{'*} A_2' \theta(t-2) \oplus' A_0^{'*} A_3' \theta(t-3) \oplus' A_0^{'*} A_4' \theta(t-4) \\
& \oplus' A_0^{'*} A_{10}' \theta(t-10) \oplus' A_0^{'*} B' U(t)
\end{aligned}
\tag{8}
$$

where:

$$A_1' = \begin{bmatrix} . & . & . & . & . & . & . \\ e & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix} \quad A_2' = \begin{bmatrix} . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & e & . \end{bmatrix} \quad A_3' = \begin{bmatrix} . & . & . & . & . & . & . \\ . & . & e & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & e & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

$$A_4' = \begin{bmatrix} . & . & . & . & . & . & 7 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix} \quad A_{10}' = \begin{bmatrix} . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & e & . & . & . & . \\ . & . & . & e & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix} \quad B' = \begin{bmatrix} e & . \\ . & . \\ . & . \\ . & . \\ . & . \\ . & . \\ . & e \end{bmatrix}$$

$$A_0' = \begin{bmatrix} . & 1 & . & . & . & . & . \\ . & . & 2 & . & . & . & . \\ . & . & . & 2 & . & . & . \\ . & . & . & . & 2 & . & . \\ . & . & . & . & . & 3 & . \\ . & . & . & . & . & . & 1 \\ e & . & . & . & . & . & . \end{bmatrix} \quad \text{so that} \quad A_0'^* = \begin{bmatrix} e & 1 & 3 & 5 & 7 & 10 & 11 \\ 10 & e & 2 & 4 & 6 & 9 & 10 \\ 8 & 9 & e & 2 & 4 & 7 & 8 \\ 6 & 7 & 9 & e & 2 & 5 & 6 \\ 4 & 5 & 7 & 9 & e & 3 & 4 \\ 1 & 2 & 4 & 6 & 8 & e & 1 \\ e & 1 & 3 & 5 & 7 & 10 & e \end{bmatrix}$$

8.2   Time constraint

The strict timed constraint holds at the level of the place p_{43} (linking transition t_3 to transition t_4) which represents the warm part of the furnace. The duration of the thermal treatment is fixed and equals to 10 time units. This constraint can be expressed in terms of the variables $\theta_i(t)$, as follows:

$$\theta_3(t) - \theta_4(t+10) = 0, \forall t \geq 10 \tag{9}$$

The control problem we face consists in synthesizing a control $U_1(t)$, knowing $\theta_i(t)$ and $U_2(t)$, which guarantees the respect of the constraint (9).

Replacing the matrices in the explicit equation (8), one obtains:

$$\begin{aligned} \theta_4(t) = {} & 7\theta_1(t-1) \oplus' 6\theta_6(t-2) \oplus' 9\theta_2(t-3) \oplus' 5\theta_5(t-3) \\ & \oplus' 13\theta_7(t-4) \oplus' \theta_3(t-10) \oplus' 2\theta_4(t-10) \oplus' 6U_1(t) \oplus' 6U_2(t) \end{aligned} \tag{10}$$

from which the following equivalence is deduced:

$$\theta_4(t+10) = \theta_3(t) \qquad \Leftrightarrow \qquad \begin{cases} 2\,\theta_4(t) \geq \theta_3(t) & (a) \\ 5\theta_5(t+7) \geq \theta_3(t) & (b) \\ 6\theta_6(t+8) \geq \theta_3(t) & (c) \\ 13\theta_7(t+6) \geq \theta_3(t) & (e) \\ 9\theta_2(t+7) \geq \theta_3(t) & (f) \\ 7\theta_1(t+9) \geq \theta_3(t) & (g) \\ 6U_1(t+10) \geq \theta_3(t) & (h) \\ 6U_2(t+10) \geq \theta_3(t) & (i) \end{cases} \qquad (11)$$

One can observe that all these inequalities, except (11-*i*), readily follow from system (8) of explicit equations. The conclusion is that, provided that (8) holds, the time constraint (9) is satisfied if and only if (11-*i*) is satisfied.

8.3   Control synthesis

Remark that, on the one hand, $U_2(t+10) \geq \theta_7(t+10)$, which leads to $U_2(t+10) \geq \theta_7(t)$, since the function $\theta_7(t)$ is nondecreasing, and that, on the other hand, $U_1(t) \geq \theta_1(t) \geq \theta_2(t) \geq \theta_3(t)$ as system (8) is satisfied. It readily follows that the control law

$$U_1(t) = 6\,\theta_7(t) \qquad (12)$$

implies that the condition (11-*i*) is satisfied, and consequently that the strict timed constraint is fulfilled.

The control law (12) can be interpreted as a feedback, determining at each instant t the control $U_1(t)$ in terms of the state $\theta_7(t)$ of the system. This control law is actually a Kanban, which lets a maximum of 6 pallets enter the production loop between point A and point E at a time, to avoid any congestion that may be caused by a failure in the working place E. Repeating for the new configuration the computation of the time cycle as in Section 6, one finds that the cycle time value is unchanged and equal to 5 time units.

9.   CONCLUSION

We have applied to an industrial process in the car industry, a method for dealing with performance evaluation and sizing based on dioid algebra. First, the system has been specified and modeled as a timed discrete event graph, i.e. a Petri net structure without conflict situation. Next, such model has been dealt with max-plus algebra that provides, through semi-linear analysis type methods, an easy way to evaluate the cycle time. Using min-plus algebra, the minimal number of resources required have been determined for an imposed cycle time, and finally a control law has been designed for ensuring the respect of a strict time constraint which exists at the level of a furnace. The control law is optimal since it does not change the requiered cycle time and ensures the robustness of the production.

REFERENCES

[1] Baccelli F., G. Cohen, G. Olsder, et J. Quadrat. 1992. *Synchronization and Linearity: An algebra for Discrete Event Systems*. Willey.

[2] Cassandras C.G. and S. Lafortune. 1999. *Introduction to discrete event systems*. Kluwer Academic.

[3] Cochet-Terrasson J., G.Cohen, S.Gaubert, M.Mc Gettrick, J.P.Quadrat. 1998. 'Numerical computation of spectral elements in max-plus algebra'. *IFAC conference on system structure and control*, France, pp. 699-706.

[4] Gaubert S. 1995. 'Resource optimization and (min,+) spectral theory'. *IEEE trans. on automatic control*, 40(11), pp. 1931-1934.

[5] Mairesse J. 1998. 'Petri nets, (max,+) algebra and scheduling'. *Actes de la 26ième école de printemps d'informatique théorique: algebre max-plus et applications en informatique et automatique,* INRIA, LIAFA, IRCyN, France, pp. 329-357.

[6] Martinez C. and P. Castagna. 2003. 'Sizing of an industrial plant using tight time constraints using complementary approaches: (max, +) algebra and computer simulation'. *Simulation Modelling Practice and Theory,* Elsevier, vol. 11, pp. 75-88.

[7] Olsder G.J., Subiono, M.Mc Guettrick. 1998. 'On large scale max-plus algebra model in railway systems'. *Actes de la 26ième école de printemps d'informatique théorique: algebre max-plus et applications en informatique et automatique,* INRIA, LIAFA, IRCyN, France, pp. 177-192.