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Summability of solutions of the heat equation with inhomogeneous thermal conductivity in two variables

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Abstract

We investigate Gevrey order and 1-summability properties of the formal solution of a general heat equation in two variables. In particular, we give necessary and sufficient conditions for the 1-summability of the solution in a given direction. When restricted to the case of constants coefficients, these conditions coincide with those given by D.A. Lutz, M. Miyake, R. Schäfke in a 1999 article ([LMS99]), and we thus provide a new proof of their result.

Keywords: Heat equation, Gevrey series, 1-summability.

AMS classification: 35C10, 35C20, 35K05,40-99, 40B05.

Contents

1 The problem

A formal solution of the classical heat initial conditions problem

$$(1) \qquad \begin{cases} \partial_t\, u - \partial_z^2\, u = 0 \\[2mm] u(0, z) = \varphi(z) \end{cases}$$

in one dimensional spatial variable z reads in the form

$$\begin{aligned} \widehat{u}(t, z) \ &= \ \exp\left(t\,\partial_z^2 t\right)\varphi(z) \\ &= \ \sum_{j \geq 0} \frac{t^j}{j!}\varphi^{(2j)}(z) \end{aligned}$$

provided that all derivatives $\varphi^{(2j)}$ exist[1]. When $\varphi \in \mathcal{O}(D_p)$ is holomorphic in a disc D_ρ with center 0 and radius ρ and hence satisfies, for any $r < \rho$, estimates of the form

$$\left|\varphi^{(2j)}(z)\right| \leq C\, K^{2j}\, \Gamma(1 + 2j)!,$$

for all $j \geq 0$ and positive constants C and K, on D_r then, $\widehat{u}(t, z) \in \mathcal{O}(D_\rho)[[t]]$ is a series of Gevrey type of order 1 in t for all $z \in D_\rho$ (in short, a 1-Gevrey series). The Gevrey estimates are locally uniform with respect to z in D_ρ. These conditions are optimal as shown by the following example: Let consider $\varphi(z) = \dfrac{1}{1 - z} = \sum_{n \geq 0} z^n$ so that $\varphi^{(2j)}(0) = (2j)!$. The corresponding solution $\widehat{u}(t, z)$ is of exact Gevrey order 1 and, in particular, is divergent. It turns out that it is actually 1-summable in all direction but $\mathbb{R}^+$ in the sense of Definition 3.1 below, that is, 1-summable in t uniformly with respect to z near 0.

    In 1999, D. Lutz, M. Miyake and R. Schäfke ([LMS99]) gave necessary and sufficient conditions on φ for $\widehat{u}$ to be 1-summable in a given direction $\arg t = \theta$. Various works have been done towards the summability of divergent solutions of partial differential equations with constant coefficients ([Bal99], [Miy99], [BM99], [Bal04],...) or variable coefficients ([H99], [Ou02], [PZ97], [Mk08], [Mk09],...)

---

[1]We denote $\widehat{u}$, with a hat, to emphasize the possible divergence of the series $\widehat{u}$.

in two variables. In [Mk05], S. Malek has investigated the case of linear partial differential equations with constant coefficients in more variables.

In this article we are interested in the very general heat initial conditions problem with inhomogeneous thermal conductivity and internal heat generation

(2)
$$\begin{cases} \partial_t u - a(z)\,\partial_z^2 u = q(t,z) & a(z) \in \mathcal{O}(D_\rho) \\ u(0,z) = \varphi(z) \in \mathcal{O}(D_\rho). \end{cases}$$

The heat equation describes heat propagation under thermodynamics and Fourier laws. The coefficient $a(z)$, named thermal diffusivity, is related to the thermal conductivity κ by the formula $a = \dfrac{\kappa}{c\rho}$ where c is the capacity and ρ the density of the medium. We assume that $a(z)$ and $\varphi(z)$ are analytic on a neighborhood of $z = 0$. The internal heat input q may be smooth or not. An important case is the case with no internal heat generation corresponding to a homogeneous heat equation:

(3)
$$\begin{cases} \partial_t u - a(z)\,\partial_z^2 u = 0 & a(z) \in \mathcal{O}(D_\rho) \\ u(0,z) = \varphi(z) \in \mathcal{O}(D_\rho). \end{cases}$$

In case of an isotropic and homogeneous medium, κ, c, ρ and hence a are constants. An adequate choice of units allows then to assume $a = 1$ and the equation reduces to the reference heat equation $\partial_t u - \partial_z^2 u = 0$.

Actually, for notational convenience, we consider the problem in the form

(4)
$$\boxed{\left(1 - a(z)\,\partial_t^{-1}\partial_z^2\right)\widehat{u} = \widehat{f}(t,z)} \; , \; a(z) \in \mathcal{O}(D_\rho) \text{ and } \widehat{f}(t,z) \in \mathcal{O}(D_\rho)[[t]]$$

where $\partial_t^{-1}\,\widehat{u}$ stands for the anti-derivative $\int_0^t \widehat{u}(s,z)ds$ of $\widehat{u}$ with respect to t which vanishes at $t = 0$.

Problem (4) is equivalent to

$$\begin{cases} \partial_t \widehat{u} - a(z)\,\partial_z^2\widehat{u} = \partial_t \widehat{f}(t,z) \\ \widehat{u}(0,z) = \widehat{f}(0,z). \end{cases}$$

and hence to Problem (2) by choosing $q(t,z) = \partial_t\widehat{f}(t,z)$ and $\varphi(z) = \widehat{f}(0,z)$. Moreover, Problem (4) reduces to the homogeneous case (3) if and only if the inhomogenuity $\widehat{f}$ does not depend on t.

From now, we denote $D = 1 - a(z)\,\partial_t^{-1}\partial_z^2$ and, given a series $\widehat{u} \in \mathcal{O}(D_\rho)[[t]]$, we denote

$$\widehat{u}(t, z) = \sum_{j \geq 0} \frac{t^j}{j!} u_{j,*}(z) = \sum_{n \geq 0} \widehat{u}_{*,n}(t)\frac{z^n}{n!} = \sum_{j,n \geq 0} \widehat{u}_{j,n}\frac{t^j}{j!}\frac{z^n}{n!}.$$

Since $\big(\mathcal{O}(D_\rho)[[t]], \partial_t, \partial_z\big)$ is a differential algebra and $a(z) \in \mathcal{O}(D_\rho)$ the operator D acts inside $\mathcal{O}(D_\rho)[[t]]$. More precisely, we can state:

Proposition 1.1 *The map*

$$D :\ \mathcal{O}(D_\rho)[[t]] \longrightarrow \mathcal{O}(D_\rho)[[t]]$$

is a linear isomorphism.

PROOF. The operator D is linear. A series $\widehat{u}(t, z) = \displaystyle\sum_{j \geq 0} \frac{t^j}{j!}\widehat{u}_{j,*}(z)$ is a solution of Problem (4) if and only if

(5) $\widehat{u}_{j,*}(z) = \widehat{f}_{j,*}(z) + a(z)\,\widehat{u}''_{j-1,*}(z)$ for all $j \geq 0$ starting from $\widehat{u}_{-1,*}(z) \equiv 0$.

Consequently, to any $\widehat{f}(t, z) \in \mathcal{O}(D_\rho)[[t]]$ there is a unique solution $\widehat{u}(t, z) \in \mathcal{O}(D_\rho)[[t]]$, which proves that D is bijective. $\square$

In Section 2 we show that the inhomogenuity $\widehat{f}(t, z)$ and the unique solution $\widehat{u}(t, z)$ are together 1-Gevrey.

In Section 3 we prove necessary and sufficient conditions for $\widehat{u}$ to be 1-summable in a given direction $\arg t = \theta$. The conditions are valid in the case when either $a(0) \neq 0$ or $a'(0) \neq 0$. When $a(z) = O(z^2)$ an easy counter-example shows that even the rationality of $\widehat{f}(t, z)$ is insufficient.

In Section 4 we discuss the accessibility of our necessary and sufficient conditions. Indeed, the conditions are given not only in terms of the data $\widehat{f}$ but also in terms of the first two terms $\widehat{u}_{*,0}$ and $\widehat{u}_{*,1}$ of the solution $\widehat{u}$ itself.

In the particular case $a = 1$ our conditions coincide with those of [LMS99]. We thus provide a new proof of the result of [LMS99].

2   Gevrey properties

In this article, we consider t as the variable and z as a parameter. The classical notion of a series of Gevrey type of order 1 is extended to z-families as follows.

Definition 2.1 (1-Gevrey series) *A series* $\widehat{u}(t,z) = \sum_{j \geq 0} \dfrac{t^j}{j!} \widehat{u}_{j,*}(z) \in \mathcal{O}(D_\rho)[[t]]$ *is of Gevrey type of order 1 if there exist* $0 < r \leq \rho$, $C > 0$, $K > 0$ *such that for all* $j \geq 0$ *and* $|z| \leq r$ *we have*

$$|\widehat{u}_{j,*}(z)| \leq C\,K^j\,\Gamma(1+2j).$$

In other words, $\widehat{u}(t,z)$ is 1-Gevrey in t, uniformly in z on a neighbourhood of $z = 0$.

We denote $\mathcal{O}(D_\rho)[[t]]_1$ the subset of $\mathcal{O}(D_\rho)[[t]]$ made of the series which are of Gevrey type of order 1.

Proposition 2.2 $\left(\mathcal{O}(D_\rho)[[t]]_1, \partial_t, \partial_z\right)$ *is a differential algebra stable under* ∂_t^{-1} *and* ∂_z^{-1}.

 PROOF. The proof is similar to the one without parameter. Stability under ∂_z is proved using the Cauchy Integral Formula and is guaranted by the condition "there exist $r \leq \rho$ ..." in Definition 2.1. □

It results from this Proposition that the operator $D = 1 - a(z)\partial_t^{-1}\partial_z^2$ acts inside the space $\mathcal{O}(D_\rho)[[t]]_1$.

Because the main result of this section (Theorem 2.5) is set up using Nagumo norms on $\mathcal{O}(D_\rho)$ we begin with a recall of their definition and main properties and we refer to [N42] or to [CRSS00] for more details.

Definition 2.3 (Nagumo norms)
Let $f \in \mathcal{O}(D_\rho)$, $p \geq 0$, $0 < r \leq \rho$ *and let* $d_r(z) = |z| - r$ *denote the euclidian distance of* z *to the boundary of the disc* D_r.

The Nagumo norm $\|f\|_{p,r}$ *of* f *is defined by*

$$\boxed{\|f\|_{p,r} = \sup_{|z|<r} \left| f(z)d_r(z)^p \right|}\ .$$

Proposition 2.4 (Properties of Nagumo norms)

(i) $\|.\|_{p,r}$ *is a norm on* $\mathcal{O}(D_\rho)$;

(ii) For all $z \in D_r$, $|f(z)| \leq \|f\|_{p,r}d(z)^{-p}$;

(iii) $\|f\|_{0,r} = sup_{z \in D_r} |f(z)|$ *is the usual sup-norm on D_r;*

(iv) $\|fg\|_{p+q,r} \leq \|f\|_{p,r} \|g\|_{q,r}$;

(v) (most important) $\|f'\|_{p+1,r} \leq e(p+1) \|f\|_{p,r}$.

Note that the same index r occurs on both sides of the inequality *(v)*. One gets thus an estimate of the derivative f' in terms of f without having to shrink the domain D_r.

Theorem 2.5 *The map*

$$D : \begin{cases} \mathcal{O}(D_\rho)[[t]]_1 & \longrightarrow & \mathcal{O}(D_\rho)[[t]]_1 \\ \widehat{u}(t,z) & \mapsto & \widehat{f}(t,z) = D\widehat{u}(t,z) \end{cases}$$

is a linear isomorphism.

Proof. It results from Proposition 2.2 that $D\big(\mathcal{O}(D_\rho)[[t]]_1\big) \subset \mathcal{O}(D_\rho)[[t]]_1$ and from Proposition 1.1 that D is linear and injective. We are left to prove that D is also surjective.

Let $\widehat{f}(t,z) = \sum_{j \geq 0} \dfrac{t^j}{j!} \widehat{f}_{j,*}(z) \in \mathcal{O}(D_\rho)[[t]]_1$. The coefficients $\widehat{f}_{j,*}(z)$ satisfy

$$\begin{cases} \bullet \widehat{f}_{j,*}(z) \in \mathcal{O}(D_\rho) \text{ for all } j \geq 0. \\ \bullet \text{ There exist } 0 < r \leq \rho,\ C > 0,\ K > 0 \text{ such that for all } j \geq 0 \text{ and } |z| \leq r \\ \quad |\widehat{f}_{j,*}(z)| \leq CK^j \Gamma(1+2j)! \end{cases}$$

and we look forward to similar conditions on the coefficients $\widehat{u}_{j,*}(z)$ of $\widehat{u}(t,z) = \sum_{j \geq 0} \dfrac{t^j}{j!} \widehat{u}_{j,*}(z)$.

From the recurrence relation (5) the relation

$$\frac{\widehat{u}_{j,*}(z)}{\Gamma(1+2j)} = \frac{\widehat{f}_{j,*}(z)}{\Gamma(1+2j)} + a(z) \frac{\widehat{u}''_{j-1,*}(z)}{\Gamma(1+2j)}$$

starting from $\widehat{u}_{-1,*}(z) \equiv 0$ holds for all $j \geq 0$. Applying the Nagumo norms of indices $(2j, r)$ and properties *(iv)* and *(v)* of Proposition 2.4 we get

$$\begin{aligned} \frac{\|\widehat{u}_{j,*}(z)\|_{2j,r}}{\Gamma(1+2j)} &\leq \frac{\|\widehat{f}_{j,*}(z)\|_{2j,r}}{\Gamma(1+2j)} + \|a(z)\|_{0,r} \frac{\|\widehat{u}''_{j-1,*}(z)\|_{2j,r}}{\Gamma(1+2j)} \\[2mm] &\leq \quad '' \quad + \|a(z)\|_{0,r}\, e^2 \frac{\|\widehat{u}_{j-1,*}(z)\|_{2j-2,r}}{\Gamma\big(1+(2j-2)\big)} \end{aligned}$$

Denote $g_j = \dfrac{\|\widehat{f}_{j,*}(z)\|_{2j,r}}{\Gamma(1+2j)}$ and $\alpha = \|a(z)\|_{0,r}\, e^2$ and consider the numerical sequence

$$
\begin{cases}
v_{-1} = 0 \\[4pt]
v_j = g_j + \alpha\, v_{j-1} \text{ for all } j \geq 0.
\end{cases}
$$

By construction, $\dfrac{\|\widehat{u}_{j,*}(z)\|_{2j,r}}{\Gamma(1+2j)} \leq v_j$ for all $j \geq 0$.

Let us bound v_j as follows. By assumption, $0 \leq g_j \leq \dfrac{CK^j\Gamma(1+2j)}{\Gamma(1+2j)}r^{2j} = C(Kr^2)^j$ for all j and the series $g(X) = \sum_{j\geq 0} g_j X^j$ is convergent. Due to the recurrence relation defining the v_j's the series $v(X) = \sum_{j\geq 0} v_j X^j$ satisfy $(1-\alpha X)v(X) = g(X)$. It is then convergent and there exist constants $C' > 0, K' > 0$ such that $v_j \leq C'\, K'^j$ for all j. Hence,

$$
\|\widehat{u}_{j,*}(z)\|_{2j,r} \leq C'\, K'^j \Gamma(1+2j) \quad \text{for all } j \geq 0.
$$

We deduce a similar estimate on the sup-norm by shrinking the domain D_r. Indeed, let $0 < r' < r$. For all $j \geq 0$ and $z \in D_{r'}$,

$$
\begin{aligned}
|\widehat{u}_{j,*}(z)| &= \left| \widehat{u}_{j,*}(z) d_r(z)^{2j}\frac{1}{d_r(z)^{2j}} \right| \\[6pt]
&\leq \frac{1}{(r-r')^{2j}}\left| \widehat{u}_{j,*}(z) d_r(z)^{2j} \right|
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sup_{z \in D_{r'}} |\widehat{u}_{j,*}(z)| &\leq \frac{1}{(r-r')^{2j}}\|\widehat{u}_{j,*}\|_{2j,r} \\[6pt]
&\leq C'\left(\frac{K'}{(r-r')^2} \right)^j \Gamma(1+2j)
\end{aligned}
$$

$\square$

3   1-summability

Still considering t as the variable and z as a parameter, one extends the classical notions of summability to families parameterized by z in requiring similar conditions, the estimates being however uniform with respect to the parameter z. For a general study of series with coefficients in a Banach space we refer to [Bal00]. Among the many equivalent definitions of 1-summability in a given direction $\arg t = \theta$ at $t = 0$

we choose here a generalization of Ramis definition which states that a series $\widehat{f}$ is 1-summable in the direction θ if there exists a holomorphic function f which is 1-Gevrey asymptotic to $\widehat{f}$ on an open sector $\Sigma_{\theta,>\pi}$ bisected by θ with opening larger than π (*cf.* [R80] Déf 3.1). There are various equivalent ways of expressing the 1-Gevrey asymptoticity. We choose to extend the one which sets conditions on the successive derivatives of f (see [Mal95] p. 171 or [R80] Thm 2.4, for instance).

Definition 3.1 (1-summability) *A series $\widehat{u}(t,z) \in \mathcal{O}(D_\rho)[[t]]$ is* 1-summable in the direction $\arg t = \theta$ *if there exist a sector $\Sigma_{\theta,>\pi}$, a radius $0 < r \leq \rho$ and a function $u(t,z)$ called* 1-sum *of $\widehat{u}(t,z)$ in the direction θ such that*

1. *u is defined and holomorphic on $\Sigma_{\theta,>\pi} \times D_r$;*

2. *For any $z \in D_r$ the map $t \mapsto u(t,z)$ has $\widehat{u}(t,z) = \sum_{j \geq 0} \dfrac{t^j}{j!} \widehat{u}_{j,*}(z)$ as Taylor series at 0 on $\Sigma_{\theta,>\pi}$;*

3. *For any proper[2] subsector $\Sigma \subset\subset \Sigma_{\theta,>\pi}$ there exist constants $C > 0, K > 0$ such that for all $\ell \geq 0$, all $t \in \Sigma$ and $z \in D_r$*

$$\boxed{\left| \partial_t^\ell u(t,z) \right| \leq CK^\ell \Gamma(1 + 2\ell)} \quad .$$

We denote $\mathcal{O}(D_\rho)\{\{t\}\}_{1,\theta}$ the subset of $\mathcal{O}(D_\rho)[[t]]$ made of all 1-summable series in the direction $\arg t = \theta$. Actually, $\mathcal{O}(D_\rho)\{\{t\}\}_{1,\theta}$ is included in $\mathcal{O}(D_\rho)[[t]]_1$.

For any fixed $z \in D_r$, the 1-summabilty of the series $\widehat{u}(t,z)$ is the classical 1-summability and Watson Lemma implies the unicity of its 1-sum, if any.

Proposition 3.2 $\left(\mathcal{O}(D_\rho)\{\{t\}\}_{1,\theta}, \partial_t, \partial_z \right)$ *is a differential $\mathbb{C}$-algebra stable under ∂_t^{-1} and ∂_z^{-1}.*

PROOF. Let $\widehat{u}(t,z)$ and $\widehat{v}(t,z)$ be two 1-summable series in direction θ. In Definition 3.1 we can choose the same constants r, C, K both for $\widehat{u}$ and $\widehat{v}$. The product $w(t,z) =$

---

[2]In this context a subsector Σ of a sector Σ' is said a proper subsector and one denotes $\Sigma \subset\subset \Sigma'$ if its closure in $\mathbb{C}$ is contained in $\Sigma' \cup \{0\}$.

$u(t, z)v(t, z)$ satisfies conditions *1* and *2* of Definition 3.1. Moreover,

$$
\begin{aligned}
\left|\partial_t^\ell w(t, z)\right| &= \left|\sum_{p=0}^{\ell} \binom{\ell}{p} \partial_t^p u(t, z)\partial_t^{\ell-p} v(t, z)\right| \\
&\leq C^2 K^\ell \Gamma(1+2\ell) \left|\sum_{p=0}^{\ell} \frac{\Gamma(1+\ell)}{\Gamma(1+2\ell)} \frac{\Gamma(1+2p)}{\Gamma(1+p)} \frac{\Gamma\big(1+2(\ell-p)\big)}{\Gamma\big(1+(\ell-p)\big)}\right| \\
&\leq C^2 K^\ell (\ell+1) \Gamma(1+2\ell) \\
&\leq C' K'^\ell \Gamma(1+2\ell) \quad \text{for adequate } C', K' > 0.
\end{aligned}
$$

This proves condition *3* of Definition 3.1 for $w(t, z)$, that is, stability of $\mathcal{O}(D_\rho)\{\{t\}\}_{1,\theta}$ under multiplication.

Stability under ∂_t, ∂_t^{-1} or ∂_z^{-1} is straightforward. Stability under ∂_z is obtained using the Integral Cauchy Formula on a disc $D_{r'}$ with $r' < r$. $\qquad\square$

We may notice that the 1-sum $u(t, z)$ of a 1-summable series $\widehat{u}(t, z) \in \mathcal{O}(D_\rho)\{\{t\}\}_{1,\theta}$ may be analytic with respect to z on a disc D_r smaller than the common disc D_ρ of analyticity of the coefficients $\widehat{u}_{j,*}(z)$ of $\widehat{u}(t, z) = \sum_{j\geq 0} \dfrac{t^j}{j!}\widehat{u}_{j,*}(z)$. With respect to t, the 1-sum $u(t, z)$ is analytic on a sector supposedly open and containing a closed sector $\overline{\Sigma}_{\theta,\pi}$ bisected by θ with opening π; there is no control on the angular opening except that it must be larger than π and no control on the radius of this sector except that it must be positive. Thus, the 1-sum $u(t, z)$ is well defined as a section of the sheaf of analytic functions in (t, z) on a germ of closed sector of opening π (*i.e.*, a closed interval $\overline{I}_{\theta,\pi}$ of length π on the circle S^1 of directions issuing from 0, *cf.* [MalR92] 1.1 or [L-R94] I.2) times $\{0\} \subset \mathbb{C}_z$. We denote $\mathcal{O}_{\overline{I}_{\theta,\pi}\times\{0\}}$ the space of such sections.

Corollary 3.3 *The operator of 1-summation*

$$
\mathcal{S} : \left\{ \begin{aligned}
\mathcal{O}(D_\rho)\{\{t\}\}_{1,\theta} &\longrightarrow \mathcal{O}_{\overline{I}_{\theta,\pi}\times\{0\}} \\
\widehat{u}(t, z) &\mapsto u(t, z)
\end{aligned}\right.
$$

is a homomorphism of differential $\mathbb{C}$-algebras for the derivations ∂_t and ∂_z and it commutes with ∂_t^{-1} and ∂_z^{-1}.

Theorem 3.4

Let a direction $\arg t = \theta$ issuing from 0 and a series $\widehat{f}(t,z) \in \mathcal{O}(D_\rho)[[t]]$ be given. Recall $D = 1 - a(z)\partial_t^{-1}\partial_z^2$ and assume that either $a(0) \neq 0$ or $a(0) = 0$ and $a'(0) \neq 0$.

Then, the unique solution $\widehat{u}(t,z)$ of $D\widehat{u} = \widehat{f}$ in $\mathcal{O}(D_\rho)[[t]]$ is 1-summable in the direction θ if and only if $\widehat{u}_{*,0}(t), \widehat{u}_{*,1}(t)$ and $\widehat{f}(t,z)$ are 1-summable in the direction θ.

Moreover, the 1-sum $u(t,z)$, if any, satisfies equation (4) in which $\widehat{f}(t,z)$ is replaced by the 1-sum $f(t,z)$ of $\widehat{f}(t,z)$ in direction θ.

PROOF. We first place ourselves in the case $a(0) \neq 0$.

Denote $a(z) = \displaystyle\sum_{n \geq 0} a_n z^n$.

As a preliminary remark we notice that, by identification of equal powers of z in Equation

$$(4) \qquad \left(1 - a(z)\,\partial_t^{-1}\partial_z^2\right) \sum_{n \geq 0} \widehat{u}_{*,n}(t)\,\frac{z^n}{n!} = \sum_{n \geq 0} \widehat{f}_{*,n}(t)\,\frac{z^n}{n!},$$

we get

$$\begin{cases} \widehat{u}_{*,0}(t) - a_0\,\partial_t^{-1}\widehat{u}_{*,2}(t) = \widehat{f}_{*,0}(t) \\ \widehat{u}_{*,1}(t) - a_1\,\partial_t^{-1}\widehat{u}_{*,2}(t) - a_0\,\partial_t^{-1}\widehat{u}_{*,3}(t) = \widehat{f}_{*,1}(t) \\ \text{and so on}\ldots \end{cases}$$

so that each $\widehat{u}_{*,n}(t)$ is uniquely and linearly determined from $\widehat{u}_{*,0}(t)$, $\widehat{u}_{*,1}(t)$ and $\widehat{f}(t,z)$.

- The condition is necessary by Proposition 3.2. Indeed, if $\widehat{u}$ is 1-summable then so are $\widehat{u}_{*,0}(t) = \widehat{u}(t,0)$, $\widehat{u}_{*,1}(t) = \dfrac{1}{z}\left(\widehat{u}(t,z) - \widehat{u}_{*,0}(t)\right)\Big|_{z=0}$ and $\widehat{f} = D\widehat{u}$.

- Prove that the condition is sufficient. Assume that $\widehat{u}_{*,0}(t)$, $\widehat{u}_{*,1}(t)$ and $\widehat{f}(t,z)$ are 1-summable in direction θ.

  Set $\widehat{u}(t,z) = \widehat{u}_{*,0}(t) + z\,\widehat{u}_{*,1}(t) + \partial_z^{-2}\widehat{v}(t,z)$ and $\widehat{w} = \partial_t^{-1}\widehat{v}$.

  With these notations Equation (4) becomes

$$(6) \quad \left(1 - \frac{1}{a(z)}\partial_t\partial_z^{-2}\right)\widehat{w}(t,z) = \widehat{g}(t,z) \quad \text{where} \quad \widehat{g} = \frac{1}{a(z)}(\widehat{u}_{*,0} + z\widehat{u}_{*,1} - \widehat{f})$$

  and it suffices to prove that $\widehat{w}$ is 1-summable in direction θ when $\widehat{g}$ is. To this end, we proceed through a fixed point method as follows.

Setting $\widehat{w}(t,z) = \sum\limits_{p \geq 0} \widehat{w}_p(t,z)$ Equation (6) reads

$$\widehat{w}_0 - \frac{1}{a(z)}\partial_t\partial_z^{-2}\,\widehat{w}_0 \quad = \widehat{g}$$

$$+\widehat{w}_1 - \frac{1}{a(z)}\partial_t\partial_z^{-2}\,\widehat{w}_1$$

$$+\cdots$$

$$+\widehat{w}_p - \frac{1}{a(z)}\partial_t\partial_z^{-2}\,\widehat{w}_p$$

$$+\cdots$$

and we choose the solution given by the system

(7)
$$\begin{cases} \widehat{w}_0 &= \widehat{g} \\[2mm] \widehat{w}_1 &= \dfrac{1}{a(z)}\partial_t\partial_z^{-2}\,\widehat{w}_0 \\[2mm] \cdots \\[2mm] \widehat{w}_p &= \dfrac{1}{a(z)}\partial_t\partial_z^{-2}\,\widehat{w}_{p-1} \\[2mm] \cdots \end{cases}$$

We can check that, for all $p \geq 0$, the formal series $\widehat{w}_p(t,z)$ are of order $O(z^{2p})$ in z and consequently, the series $\widehat{w}(t,z) = \sum_{p \geq 0} \widehat{w}_p(t,z)$ itself makes sense as a formal series in t and z.

Let $w_0(t,z)$ denote the 1-sum of $\widehat{w}_0 = \widehat{g}$ in direction θ and for all $p > 0$, let $w_p(t,z)$ be determined as the solution of System (7) in which all $\widehat{w}_p$ are replaced by w_p. All w_p are defined on a common domain $\Sigma_{\theta,>\pi} \times D_{\rho'}$.

We are willing to prove that the series $\sum\limits_{p \geq 0} w_p(t,z)$ is convergent with sum $w(t,z)$, the 1-sum of $\widehat{w}(t,z)$ in direction θ.

The 1-summability of $\widehat{w}_0$ implies that there exists $0 < r' < \rho'$ and, for any subsector $\Sigma \subset\subset \Sigma_{\theta,>\pi}$, there exist constants $C' > 0, \quad K' > 0$ such that for all $\ell \geq 0$ and $(t,z) \in \Sigma \times D_{r'}$,

$$\left|\partial_t^\ell w_0(t,z)\right| \leq C'K'^\ell\,\Gamma(1 + 2\ell).$$

Denote $B = \max\limits_{z \in D_r}\left|\dfrac{1}{a(z)}\right|$

From $w_1 = \dfrac{1}{a(z)}\,\partial_t\partial_z^{-2}w_0$ we deduce that

$$
\begin{aligned}
\left|\partial_t^\ell w_1\right| &= \left|\frac{1}{a(z)}\,\partial_t^{\ell+1}\partial_z^{-2}\,w_0\right| \\
&\leq B \max_{z\in D_r}\left|\partial_t^{\ell+1}w_0\right|\frac{|z|2}{2!} \\
&\leq C'\,K'^{\ell+1}\,\Gamma\big(1+2(\ell+1)\big)\frac{B|z|^2}{2!}
\end{aligned}
$$

and, by recursion, that

$$
(8)\qquad \left|\partial_t^\ell w_p(t,z)\right| \leq C'K'^{\ell+p}\,\Gamma\big(1+2(\ell+p)\big)\frac{(B|z|^2)^p}{(2p)!}\quad\text{ for all } p\geq 0.
$$

This implies

$$
\begin{aligned}
\sum_{p\geq 0}\left|\partial_t^\ell w_p(t,z)\right| &\leq C'\,K'^{\ell}\,\Gamma(1+2\ell)\sum_{p\geq 0}\binom{2\ell+2p}{2p}\left(K'\,B\,|z|^2\right)^p \\
&\leq C'\,(4K')^\ell\,\Gamma(1+2\ell)\sum_{p\geq 0}\left(4K'B|z|^2\right)^p \\
&\text{since }\binom{2\ell+2p}{2p}\leq\sum_{k=0}^{2\ell+2p}\binom{2\ell+2p}{k}=2^{2\ell+2p}.
\end{aligned}
$$

Denote $L = 4K'Br^2$ and choose r so small that $L < 1$.

Denote $C = C'\sum_{p\geq 0}L^p < \infty$ and $K = 4K'$.

Then,

$$
(9)\qquad \sum_{p\geq 0}\left|\partial_t^\ell w_p(t,z)\right| \leq CK^\ell\Gamma(1+2\ell)\quad\text{ on }\Sigma\times D_r.
$$

In particular, for $\ell = 0$, the series $\sum w_p(t,z)$ is normally convergent on $\Sigma\times D_r$. Consequently, its sum $w(t,z)$ exists and is analytic on $\Sigma\times D_r$. This proves point *1* of Definition 3.1 if we choose as sector $\Sigma\subset\Sigma_{\theta,>\pi}$ a sector bisected by θ with opening larger than π .

For all $\ell\geq 1$, the series $\sum\partial_t^\ell w_p(t,z)$ is also normally convergent on $\Sigma\times D_r$ so that the series $\sum w_p(t,z)$ can be derivated termwise infinitely many times with respect to t and the estimates (9) imply

$$
(10)\qquad \left|\partial_t^\ell w(t,z)\right| \leq CK^\ell\Gamma(1+2\ell)\quad\text{ on }\Sigma\times D_r
$$

which proves the condition *3* of Definition 3.1.

Moreover, summing the Equations (7) for w_p and the 1-sum $g(t, z)$ instead of $\widehat{w}_p$ and $\widehat{g}(t, z)$ we get $w(t, z) = g(t, z) + \dfrac{1}{a(z)} \sum_{p \geq 0} \partial_t \partial_z^{-2} w_p(t, z) = g(t, z) + \dfrac{1}{a(z)} \partial_t \partial_z^{-2} w(t, z)$. Hence, $w(t, z)$ satisfies Equation (6) with right hand side $g(t, z)$ in place of $\widehat{g}(t, z)$.

Finally, the fact that all derivatives of $w(t, z)$ with respect to t are bounded on Σ implies the existence of $\lim\limits_{\substack{t \to 0 \\ t \in \Sigma}} \partial_t^\ell w(t, z)$ for all $z \in D_r$ and hence the existence of the Taylor series of w at 0 on Σ for all $z \in D_r$. Since $w(t, z)$ satisfies Equation (6), so does its Taylor series. Since Equation (6) has a unique formal solution $\widehat{w}(t, z)$, we can conclude that the Taylor expansion of $w(t, z)$ is $\widehat{w}(t, z)$, which proves part 2 of Definition 3.1.

This achieves the proof of the 1-summability of $\widehat{u}(t, z)$ in direction θ in the case when $a(0) \neq 0$.

- The fact that the 1-sum $u(t, z)$ of $\widehat{u}(t, z)$ in direction θ satisfies Equation (4) with right hand side the 1-sum $f(t, z)$ of $\widehat{f}(t, z)$ instead of $\widehat{f}(t, z)$ is equivalent to the fact that $w(t, z)$ satisfies Equation (6) with right hand side $g(t, z)$ instead of $\widehat{g}(t, z)$, which we proved above. It is also a consequence of Corollary 3.3.

In the case when $a(0) = 0$ and $a'(0) \neq 0$ the necessary condition again results from Proposition 3.2. The fact that $u(t, z)$ satisfies Equation (4) results from Corollary 3.3. We sketch the proof of the sufficient condition.

Denote $a(z) = zA(z)$ with $A(0) \neq 0$.
In this case, identification of equal powers of z shows that $\widehat{u}_{*,0} = \widehat{f}_{*,0}$ and that all $\widehat{u}_{*,n}$ for $n \geq 1$ are uniquely determined by $\widehat{u}_{*,1}$ and $\widehat{f}$.
We set again $\widehat{u}(t, z) = \widehat{u}_{*,0} + z\widehat{u}_{*,1} + \partial_t \partial_z^{-2} \widehat{w}$ so that $\widehat{w}$ satisfies the equation

$$(11) \quad \left(1 - \frac{1}{zA(z)} \partial_t \partial_z^{-2}\right) \widehat{w}(t, z) = \widehat{g}(t, z) \quad \text{where } \widehat{g} = \frac{1}{A(z)}\left(\widehat{u}_{*,1} + \frac{\widehat{u}_{*,0} - \widehat{f}}{z}\right).$$

Still, $\widehat{g}$ is a formal series, assumed to be 1-summable in direction θ and we look for $\widehat{w}$ in the form $\widehat{w} = \sum\limits_{p \geq 0} \widehat{w}_p$ as previously. The operator $\dfrac{1}{z} \partial_z^{-2}$ implies that $\widehat{w}^p = O(z^p)$ instead of $O(z^{2p})$. If we denote $B = \max\limits_{z \in D_r} \dfrac{1}{|A(z)|}$, then, for all p and ℓ,

$$\left|\partial_t^\ell w_p\right| \leq C' K'^{\ell+p} \Gamma\left(1 + 2(\ell + p)\right) \frac{(B|z|)^p}{p!}$$

and it follows that, for a convenient choice of $r > 0$,

$$\left| \partial_t^\ell w(t, z) \right| \leq C \, K^\ell \, \Gamma(1 + 2\ell)$$

with $C = C' \sum_{p \geq 0} (4KBr)^p < \infty$ and $K = 4K'$. $\hfill \square$

The case of a thermal diffusivity $a(z) = O(z^2)$ gives rise to the conditions $\widehat{u}_{*,0}(t) = \widehat{f}_{*,0}(t)$ and $\widehat{u}_{*,1}(t) = \widehat{f}_{*,1}(t)$ and we could hope of similar necessary and sufficient conditions which apply to the inhomogenuity $\widehat{f}(t, z)$ only. This is not the case since the previous proof cannot be extended to that situation. Indeed, the appearance of $\dfrac{\partial_z^{-2}}{z^2}$ instead of ∂_z^{-2} or $\dfrac{\partial_z^{-2}}{z}$ implies that no power of z remains in the estimates (8) and we cannot guaranty the convergence of the estimate for $\partial_t^\ell w$.

The counter-example below shows that even with $\widehat{f}(t, z)$ independent of t and rational the 1-summability of $\widehat{u}(t, z)$ may fail.

Counter-example 3.5

Consider the heat initial conditions problem (4) with $\widehat{f}(t, z) = \displaystyle\sum_{n \geq 0} z^n = \dfrac{1}{1-z}$ and $a(z) \equiv 1$. The series $\widehat{f}(t, z)$ is independent of t and is convergent in z near 0 with rational sum. The problem is equivalent to the heat initial conditions problem without internal heat generation

$$(12) \qquad \begin{cases} \partial_t \widehat{u} - z^2 \, \partial_z^2 \widehat{u} = 0 \\[2mm] \widehat{u}(0, z) = \displaystyle\sum_{n \geq 0} z^n \end{cases}$$

In this case, $\widehat{u}_{*,0}(t) = \widehat{f}_{*,0}(t) \equiv 1$, $\widehat{u}_{*,1}(t) = \widehat{f}_{*,1}(t) \equiv 1$ and for all $n \geq 2$, $\widehat{u}_{*,n}(t)$ satisfies

$$\widehat{u}'_{*,n}(t) - n(n-1)\widehat{u}_{*,n}(t) = 0 \quad \text{and} \quad \widehat{u}_{*,n}(0) = n!.$$

Consequently, $\widehat{u}_{*,n}(t) = n! \, e^{n(n-1)t}$.

Suppose $\widehat{u}(t, z)$ is 1-summable in a direction θ with sum $u(t, z)$. Then, since $\widehat{u}_{*,n}(t) = \partial_z^n \widehat{u}(t, z)\big|_{z=0}$ all $\widehat{u}_{*,n}(t)$ are 1-summable in direction θ with sum $u_{*,n}(t) = \partial_z^n u(t, z)\big|_{z=0}$. The Integral Cauchy Formula applied to $\partial_z^n u(t, z)$ at $z = 0$ provides estimates of the form

$$|u_{*,n}(t)| = \left| \frac{n!}{2\pi i} \int_{|\zeta| = R < r} \frac{u(t, \zeta)}{\zeta^{n+1}} d\zeta \right| \leq \frac{n!}{2\pi} \frac{C 2\pi R}{R^{n+1}} = C \, k^n \, n!$$

on a sector bisected by θ with opening larger than π. In our case, $\widehat{u}_{*,n}(t) = u_{*,n}(t) = n!\,e^{n(n-1)t}$. The functions $e^{n(n-1)t}$ being unbounded on any sector larger than a half plane such estimates are impossible. Hence, $\widehat{u}(t,z)$ is 1-summable in no direction. $\square$

4   Initial conditions

We end this article with a discussion of how to apply the above result and we develop the cases when $a(z) = a \in \mathbb{C}^*$ or $a(z) = bz$, $b \in \mathbb{C}^*$.

The formal series $\widehat{f}(t,z)$ is a data of the problem and although its 1-summability may be not obvious we assume that it is known. $\widehat{f}(t,z)$ is not itself the initial conditions but is closely connected to (see Section 1).

The series $\widehat{u}_{*,0}(t)$ and $\widehat{u}_{*,1}(t)$ can, at least theoretically, be computed in terms of $\widehat{f}(t,z)$ from the formula

$$\widehat{u}(t,z) = \sum_{k \geq 0} \left(a\partial_t^{-1}\partial_z^2 \right)^k \widehat{f}(t,z)$$

and an explicit computation can be achieved for simple $a(z)$ such as $a(z) = a$ constant, $a(z) = bz$ $(b \in \mathbb{C}^*)$ or $a(z) = a + bz$. However, an explicit computation of $\widehat{u}_{*,0}(t)$ and $\widehat{u}_{*,1}(t)$ looks like hopeless for a general $a(z)$.

4.1   Case $a(z) = a \in \mathbb{C}^*$

When a is a constant then the operators a, ∂_t and ∂_z commute and $\left(a\partial_t^{-1}\partial_z^2 \right)^k = a^k \partial_t^{-k} \partial_z^{2k}$. From the calculation of $\widehat{u}(t,z) = \sum_{k \geq 0} \left(a\partial_t^{-1}\partial_z^2 \right)^k \widehat{f}(t,z)$ we obtain

$$(13) \qquad \begin{cases} \widehat{u}_{*,0}(t) & = & \displaystyle\sum_{k \geq 0} \frac{t^k}{k!} \sum_{j+n=k} a^n \widehat{f}_{j,2n} \\[2em] \widehat{u}_{*,1}(t) & = & \displaystyle\sum_{k \geq 0} \frac{t^k}{k!} \sum_{j+n=k} a^n \widehat{f}_{j,2n+1} \end{cases}$$

Our aim is to characterize the 1-summability of these two series as a property of the inhomogenuity $\widehat{f}$.

- We start with the case where $\widehat{f}(t,z) = \displaystyle\sum_{n \geq 0} \widehat{f}_{0,n} \frac{z^n}{n!}$ is independent of t which corresponds to Problem (3). For simplicity, we denote $\widehat{f}(z)$.

The formulæ (13) become

(14)
$$\begin{cases} \widehat{u}_{*,0}(t) & = & \sum_{k \geq 0} \dfrac{(at)^k}{k!} \widehat{f}_{0,2k} \\[2ex] \widehat{u}_{*,1}(t) & = & \sum_{k \geq 0} \dfrac{(at)^k}{k!} \widehat{f}_{0,2k+1} \end{cases}$$

Define the 2-Laplace transform of $\widehat{f}(z)$ by $\mathcal{L}_z^{[2]} \widehat{f}(\zeta) = \sum_{n \geq 0} \widehat{f}_{0,n} \dfrac{\zeta^n}{n!} \dfrac{n!}{[n/2]!}$ where $[n/2]$ stands for the integer part of $n/2$. Then,

$$\mathcal{L}_z^{[2]} \widehat{f}\big((at)^{1/2}\big) = \widehat{u}_{*,0}(t) + (at)^{1/2} \widehat{u}_{*,1}(t).$$

and we may state

Proposition 4.1 *Suppose $a(z) = a \in \mathbb{C}^*$ and $\widehat{f}(t,z) = \widehat{f}(z)$.*
Then, the following three assertions are equivalent.

(i) $\widehat{u}_{*,0}(t)$ *and* $\widehat{u}_{*,1}(t)$ *are 1-summable in direction θ;*

(ii) $\mathcal{L}_z^{[2]} \widehat{f}(z)$ *is 2-summable in the directions $\frac{1}{2}(\theta + \arg a)$ mod π;*

(iii) $\widehat{f}(z)$ *is analytic near 0 and it can be analytically continued to sectors neighbouring the directions $\frac{1}{2}(\theta + \arg a)$ mod π with exponential growth of order 2 at infinity.*

Assertion *(iii)* with $a = 1$ (hence $\arg a = 0$) is how the conditions are formulated in [LMS99] and proved via direct Borel-Laplace estimations. Our method provides thus a new proof of this result.

• Consider now the case of a general $\widehat{f}(t,z)$.
The interpretation of the 1-summability of $\widehat{u}_{*,0}(t)$ and $\widehat{u}_{*,1}(t)$ becomes more involved and uses Borel and Laplace transforms of $\widehat{f}(t,z)$ in both variables.
We denote $\mathcal{L}_z$ or $\mathcal{B}_z$ and so on. . . the 1-Laplace or 1-Borel transform *w.r.t.* z and so on. . . . These operators are defined here by $\mathcal{L}_z z^n = \zeta^n [n]!$ and $\mathcal{B}_z = \mathcal{L}_z^{-1}$ where $[n]$ denotes the integer part of n.
Consider $\mathcal{L}_t \mathcal{L}_z \widehat{f}\big(\tau, (a\tau)^{1/2}\big) = \sum_{k \geq 0} \tau^k \sum_{j+n=k} \widehat{f}_{j,2n} a^n + (a\tau)^{1/2} \sum_{k \geq 0} \tau^k \sum_{j+n=k} \widehat{f}_{j,2n+1} a^n$

and

$$\mathcal{B}_\tau \mathcal{L}_t \mathcal{L}_z \widehat{f}\big(\tau, (a\tau)^{1/2}\big)(t) = \sum_{k\geq 0} \frac{t^k}{k!} \sum_{j+n=k} \widehat{f}_{j,2n} a^n + (at)^{1/2} \sum_{k\geq 0} \frac{t^k}{k!} \sum_{j+n=k} \widehat{f}_{j,2n+1} a^n \quad \text{(the}$$

terms in τ^k are divided by $k!$ and the terms in $\tau^{k+1/2}$ by $[k + 1/2]! = k!$).
Denote $\widehat{F}(t) = \mathcal{B}_\tau \mathcal{L}_t \mathcal{L}_z \widehat{f}\big(\tau, (a\tau)^{1/2}\big)(t^2)$. Then,

$$\widehat{F}(t^{1/2}) = \widehat{u}_{*,0}(t) + (at)^{1/2} \widehat{u}_{*1}(t)$$

and we may state:

Proposition 4.2 *Suppose $a(z) = a \in \mathbb{C}^*$ and $\widehat{f}(t,z)$ general.*
Then, the series $\widehat{u}_{*,0}(t)$ and $\widehat{u}_{*,1}(t)$ are 1-summable in direction θ if and only if the
series $\widehat{F}$ associated with $\widehat{f}$ as above is 2-summable in the directions $\theta/2 \mod \pi$.

The condition in Proposition 4.1 may be not easy to check but seems reasonnable. In Proposition 4.2, the link between $\widehat{f}$ and $\widehat{F}$ is more complicated and the question remains of how to check the 2-summability of $\widehat{F}$ in practice.

4.2   Case $a(z) = bz, \ b \in \mathbb{C}^*$

In this case, $\big(a(z)\partial_t^{-1}\partial_z^2\big)^k = b^k \partial_t^{-k}(z\partial_z^2)^k$ and

$$(z\partial_z^2)^k \cdot \frac{z^n}{n!} = \begin{cases} \dfrac{z^{n-k}}{(n-k)!} \dfrac{(n-1)!}{(n-k-1)!} & \text{if } 0 \leq k < n \\[2mm] 0 & \text{if } n \leq k. \end{cases}$$

From the calculation of $\widehat{u}(t,z) = \sum_{k\geq 0} \big(bz\partial_t^{-1}\partial_z^2\big)^k \widehat{f}(t,z)$ we obtain

$$(15) \qquad \begin{cases} \widehat{u}_{*,0}(t) &= \displaystyle\sum_{j\geq 0} \frac{t^j}{j!} \widehat{f}_{j,0} = \widehat{f}_{*,0}(t) \\[4mm] \widehat{u}_{*,1}(t) &= \displaystyle\sum_{j,k\geq 0} \widehat{f}_{j,k+1}\, b^k \frac{t^{j+k}}{(j+k)!} k! \end{cases}$$

Since $\widehat{u}_{*,0}(t) = \widehat{f}_{*,0}(t)$ is 1-summable when so is $\widehat{f}(t,z)$, our aim is now to characterize the 1-summability of the series $\widehat{u}_{*,1}(t)$ as a property of $\widehat{f}$.

$\bullet$ Let us first again place ourselves in the situation of Problem (3) where the inhomogenuity $\widehat{f}(t,z) = \sum_{n\geq 0} \widehat{f}_{0,n} \frac{z^n}{n!}$ is independent of t.

Formulæ (15) become

$$
(16) \qquad
\begin{cases}
\widehat{u}_{*,0}(t) & = & = \widehat{f}_{0,0}; \\
\\
\widehat{u}_{*,1}(t) & = & \displaystyle\sum_{k \geq 0} \widehat{f}_{0,k+1}\, b^k\, t^k.
\end{cases}
$$

Thus, $\mathcal{L}_z \widehat{f}(bt) = \widehat{f}_{0,0} + bt\widehat{u}_{*,1}(t)$ and we may state

Proposition 4.3 *Suppose $a(z) = bz$, $b \in \mathbb{C}^*$ and $\widehat{f}(t,z) = \widehat{f}(z)$.*
Then, $\widehat{u}_{*,0}(t)$ is a constant and the following three assertions are equivalent.

(i) $\widehat{u}_{*,1}(t)$ is 1-summable in direction θ;

(ii) $\mathcal{L}_z \widehat{f}(z)$ is 1-summable in the direction $\theta + \arg b$;

(iii) $\widehat{f}(z)$ is analytic near 0 and it can be analytically continued to a sector neighbouring the direction $\theta + \arg b$ with exponential growth of order 1 at infinity.

• Consider the case of a general $\widehat{f}(t,z)$.
The Laplace transform of $\widehat{f}$ w.r.t. z reads $\mathcal{L}_z \widehat{f}(t,z) = \widehat{f}_{*,0}(t) + z\sum_{j,n\geq 0} \frac{t^j}{j!}\widehat{f}_{j,n+1} z^n$.
Consider the series $\widehat{g}(t,z) = \mathcal{L}_t \mathcal{L}_z \left[\frac{1}{z}\left(\mathcal{L}_z \widehat{f}(t,z) - \widehat{f}_{*,0}(t)\right)\right]$, We can check that the Borel transform of the series $\widehat{g}(t,bt)$ is equal to $\widehat{u}_{*,1}(t)$ and we may state:

Proposition 4.4 *Suppose $a(z) = bz$, $b \in \mathbb{C}^*$ and $\widehat{f}(t,z)$ general.*
Then, the series $\widehat{u}_{*,1}(t)$ is 1-summable in direction θ if and only if the Borel transform of $\widehat{g}(t,bt)$ is 1-summable in direction θ.

The comment following Propositions 4.1 and 4.2 keeps valid.

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