Global finite-time observers for non linear systems

Tomas Menard, Emmanuel Moulay, Wilfrid Perruquetti

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Tomas Ménard,[*] Emmanuel Moulay[†] and Wilfrid Perruquetti[‡]

Abstract

A global finite-time observer is designed for nonlinear systems which are uniformly observable and globally Lipschitz. This result is based on a high-gain approach.

1. Introduction

The problem of observation has been studied by a number of researchers these last years. The linear case has been solved by Kalman and Luenberger, but the nonlinear case is still an active domain of research. Several ways of investigation have been borrowed. Linearization of nonlinear system with algebraic methods have been investigated in [1] and [2]. The high-gain observer approach which is closely related to triangular structure has been developped by Gauthier et al. (see [3] and [4]) and is derived from the uniform observability of nonlinear systems. Other methods have been developped

- Kazantzis and Kravaris observer which uses the Lyapunov auxiliary theorem and a direct coordinate transformation in [5],

- backstepping design in [6],

- adaptive observer in [7],

and many other ones. Almost all of these approaches give us asymptotic convergence. In some applicative fields, finite time convergence is needed for example like in secure communication, synchronisation or robot walks (see for instance [8, 9, 10]).

The problem of finite time observers has been less considered because it requires nonsmooth techniques. Nevertheless new conditions of stability and stabilization have been developed in the continuous finite-time domain, involving the settling time function, associated with the Lyapunov theory and homogeneous systems by Bhat and Bernstein in [11, 12], and by Moulay and Perruquetti in [13, 14].

There exists several methods achieving finite-time convergence, e.g. sliding mode observers (see [15, 16]), moving horizon observer (see [17]), but some of these are not continuous like sliding mode observers.

We consider here continuous finite-time observer. Such one has been designed for linear systems in [18] extended to linear time-varying system in [19]. A global finite time observer for a linearizable system via input output injection is constructed in [9] and extended to uniformly observable systems in [20] in a semi-global way. Let us note that such output injection structure : a linear part plus a homogeneous part can be derived from the results obtained by Andrieu-Praly-Astolfi ([21, 22] using the concept of homogeneity in the bi-limit : finite time convergence being obtained if the homogeneity degree in the 0-limit is negative. However, the construction and proof are recursive one.

This paper aim at designing a global observer for systems which are uniformly observable. For this class of systems, there exists a change of coordinates which transforms the system into an observability normal form.

This paper is organized as follows. The class of considered systems, the definitions and the properties of finite time stable systems are given in section 2. Section 3 presents a global finite-time observer followed by the proof of convergence and an illustrative example.

2. Preliminaries

Notations

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- We denote by $t \mapsto x(t, x_{t_0})$ a solution of the system:

$$\begin{cases} \dot{x} = f(x), & x \in \mathbb{R}^n \\ x(t_0) = x_{t_0} \end{cases} \tag{1}$$

  where f is a continuous vector field,

- $\lceil x \rceil^\alpha = \text{sign}(x).|x|^\alpha$, with $\alpha > 0$ and $x \in \mathbb{R}$,

- $\|.\|_{i,k}$ denotes the i-norm on $\mathbb{R}^k$,

- if $x \in \mathbb{R}^n$, $\bar{x}_i$ denotes the truncated vector in $\mathbb{R}^i$ with the ith first components of x ($1 \leq i \leq n$),

- $\mathscr{B}_{\|.\|}(\varepsilon)$ is the ball centered at the origin and of radius ε, w.r.t. (with respect to) the norm $\|.\|$,

- $L_f h(x) = \frac{\partial}{\partial x} h(x).f(x)$ denotes the Lie derivative of h along the vector field f, and $L_f^n h = L_f(L_f^{n-1} h)$ is the n-th Lie derivative of h along the vector field f.

Context

Let us consider a multi-input single-output nonlinear system on $\mathbb{R}^n$

$$\begin{cases} \dot{x} = F(x) + \sum_{i=1}^m G_i(x) u_i, \\ y = h(x), \end{cases} \tag{2}$$

where F and G_i are analytic functions. If system (2) is uniformly observable (see [3]). Then a coordinate change can be found to transform system (2) into the form

$$\begin{cases} \dot{x}_1 = x_2 + \sum_{j=1}^m g_{1,j}(x_1) u_j \\ \dot{x}_2 = x_3 + \sum_{j=1}^m g_{2,j}(x_1, x_2) u_j \\ \vdots \\ \dot{x}_{n-1} = x_n + \sum_{j=1}^m g_{n-1,j}(x) u_j \\ \dot{x}_n = \varphi(x) + \sum_{j=1}^m g_{n,j}(x) u_j \\ y = x_1 = Cx \end{cases} \tag{3}$$

where $C = (1 \quad 0 \dots 0)$.

Finite-time stability

In this paper, we are interested in observer, but especially in finite-time observer. So we recall here the main definitions and properties for finite-time stability.

In system (1), f is a continuous but not Lipschitzian function, so it may happen that any solution of the system converges to zero in finite time.

Example 1. *For $x \in \mathbb{R}$, the solutions of $\dot{x} = -\text{sign}(x)|x|^{\frac{1}{3}}$, are given by:*

$$x(t, x_0) = \text{sign}(x_0)\left(|x_0|^{\frac{1}{3}} - \frac{t}{3}\right)^3, \text{if } 0 < t < 3|x_0|^{\frac{1}{3}}$$

$$x(t, x_0) = 0, \text{if } t \geq 3|x_0|^{\frac{1}{3}},$$

and thus reaches zero in finite time.

It is aimed here to exploit this property of such dynamical nonlinear systems to design a *finite time observer* (FTO). Due to the non Lipschitz condition on the right hand side of (1) backward uniqueness may be lost, and thus we only consider forward uniqueness which is recalled here.

Definition 1. *The system (1) is said to have unique solutions in forward time on a neighbourhood $\mathscr{U} \subset \mathbb{R}^n$ if for any $x_0 \in \mathscr{U}$ and two right maximally defined solutions of (1), $x(t, x_0) : [0, T_x[\rightarrow \mathbb{R}^n$ and $y(t, x_0) : [0, T_y[\rightarrow \mathbb{R}^n$, there exists $0 < T_{x_0} \leq \min\{T_x, T_y\}$ such that $x(t, x_0) = y(t, x_0)$ for all $t \in [0, T_{x_0}[$.*

It can be assumed that for each $x_0 \in \mathscr{U}$, T_{x_0} is chosen to be the largest in $\mathbb{R}_+ \cup \{+\infty\}$. Various sufficient conditions for forward uniqueness can be found in [23]. Let us consider the system (1) where f is continuous on $\mathbb{R}^n$ and where f has unique solutions in forward time. We recall the definition of finite-time stability.

Definition 2. *The origin of the system (1) is said* finite time stable *if:*

1. *there exists a function $T : \mathscr{V} \setminus \{0\} \rightarrow \mathbb{R}_+$ defined on a neighbourhood $\mathscr{V}$ of the origin, such that for all $x_0 \in \mathscr{V} \setminus \{0\}$, $x(t, x_0)$ is defined (and unique) on $[0, T(x_0))$, $x(t, x_0) \in \mathscr{V} \setminus \{0\}$ for all $t \in [0, T(x_0))$ and $\lim_{t \to T(x_0)} x(t, x_0) = 0$.*

   T is called the settling-time function *of the system (1).*

2. *for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for every $x_0 \in \left(\mathscr{B}_{\|.\|_{2,n}}(\delta(\varepsilon)) \setminus \{0\}\right) \cap \mathscr{V}$, $x(t, x_0) \in \mathscr{B}_{\|.\|_{2,n}}(\varepsilon)$ for all $t \in [0, T(x_0))$.*

The following result gives a sufficient condition for system (1) to be finite time stable (see [24, 25] for ordinary differential equations, and [26] for Differential inclusions):

Theorem 1. *[24, 25] Let the origin be an equilibrium point for the system (1), and let f be continuous on an open neighborhood $\mathscr{V}$ of the origin. If there exists a Lyapunov function[1] $V : \mathscr{V} \rightarrow \mathbb{R}_+$ and a function $r : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that*

$$\frac{d}{dt} V(x(t)) \leq -r(V(x)), \tag{4}$$

along the solutions of (1) and $\varepsilon > 0$ such that

$$\int_0^\varepsilon \frac{dz}{r(z)} < +\infty, \tag{5}$$

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[1]V is a continuously differentiable function defined on $\mathscr{V}$ such that V is positive definite and $\frac{dV}{dt}$ is negative definite.

then the origin is finite time stable.

In particular, assuming forward uniqueness of the solution and the continuity of the settling time function, Bhat and Bernstein (see [12, Definition 2.2]) showed that finite time stability of the origin is equivalent to the existence of a Lyapunov function satisfying (4) where $r(x) = cx^a$, with $a \in]0,1[, c > 0$. In order to circumvent the classical Lyapunov function art of design, one can use homogeneity conditions recalled hereafter.

Homogeneity

Definition 3. *A function $V : \mathbb{R}^n \to \mathbb{R}$ is* homogeneous *of degree d w.r.t. the weights $(r_1,\ldots,r_n) \in \mathbb{R}^n_{>0}$ if*

$$V(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) = \lambda^d V(x_1, \ldots, x_n), \forall \lambda > 0. \quad (6)$$

Definition 4. *A vector field f is* homogeneous *of degree d w.r.t. the weights $(r_1,\ldots,r_n) \in \mathbb{R}^n_{>0}$ if for all $1 \le i \le n$, the i-th component f_i is a homogeneous function of degree $r_i + d$. The system (1) is homogeneous of degree d if the vector field f is homogeneous of degree d.*

Theorem 2. *[27, Theorem 5.8 and Corollary 5.4] Let g be defined on $\mathbb{R}^n$ and be a continuous vector field homogeneous of degree $d < 0$ with respect to dilation Λ_r. If the origin of (1) is locally asymptotically stable, then it is globally finite time stable.*

Previous observers

Perruquetti et al. first construct a finite time observer for a canonical observable form in [9], i.e. for a linear system in the state. The proof of finite-time stability is based on homogeneity property (specifically on Theorem 2). Then, a semi-global observer was constructed by Shen and Xia in [20] based on the Perruquetti et al. observer. This result is recall here:

Theorem 3. *System (3) admits a semi-global observer of the form:*

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + k_1 \lceil y - \hat{x}_1 \rfloor^{\alpha_1} + \sum_{j=1}^{m} g_{1,j}(\hat{x}_1) u_j \\ \dot{\hat{x}}_2 = \hat{x}_2 + k_3 \lceil y - \hat{x}_1 \rfloor^{\alpha_2} + \sum_{j=1}^{m} g_{2,j}(\hat{x}_1, \hat{x}_2) u_j \\ \vdots \\ \dot{\hat{x}}_n = \varphi(\hat{x}) + k_n \lceil y - \hat{x}_1 \rfloor^{\alpha_n} + \sum_{j=1}^{m} g_{n,j}(\hat{x}) u_j \end{cases} \quad (7)$$

where the powers α_i are defined by

$$\alpha_i = i\alpha + (i-1), \ \alpha \in \left] 1 - \frac{1}{n}, 1 \right[\quad (8)$$

and

$$[k_1, \ldots, k_n]^T = S_\infty^{-1}(\theta) C^T, \quad (9)$$

where $S_\infty(\theta)$ is the unique solution of the Riccati equation:

$$\begin{cases} \theta S_\infty(\theta) + A^T S_\infty(\theta) + S_\infty(\theta) A - C^T C = 0 \\ S_\infty(\theta) = S_\infty^T(\theta) \end{cases} \quad (10)$$

where $(A)_{ij} = \delta_{i,j-1}$ and $C = (1 \quad 0 \ldots 0)$.

3. Global Observers

In this section, we propose a global finite-time observer for system (3) based on the semi-global finite-time observer (7). In order to prove our result, we need the following lemma (issued from [28], remark 1):

Lemma 1. *Assume system (1) is globally asymptotically stable and finite-time attractive on a neighborhood of the origin. Then system (1) is globally finite-time stable.*

The proof of the following technical lemma is not given here, but an explicit computation (easy but long) gives the first equality and the second easily follows from the first.

Lemma 2. *The matrix $S_\infty(\theta)$ and $S_\infty^{-1}(\theta)$ verify the following properties:*

$$S_\infty(\theta)_{i,j} = S_\infty(1)_{i,j} \frac{1}{\theta^{i+j-1}} \quad (11)$$

$$S_\infty^{-1}(\theta)_{i,j} = S_\infty^{-1}(1)_{i,j} \theta^{i+j-1} \quad (12)$$

for any $\theta > 0$ and $1 \le i, j \le n$.

Theorem 4. *Let us consider system (3) with a bounded input $u = (u_1, \ldots, u_m)$ and assume functions $(g_{i,j})_{1 \le i,j \le n}$, and φ are globally Lipschitz. Then there exists $\theta^* > 0$ and $\varepsilon > 0$ such that for all $\theta > \theta^*$ and $\alpha \in]1 - \varepsilon, 1[$, system (3) admits the following* **global** *finite-time high-gain observer:*

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + k_1 (\lceil e_1 \rfloor^{\alpha_1} + \rho e_1) + \sum_{j=1}^{m} g_{1,j}(\hat{x}_1) u_j \\ \dot{\hat{x}}_2 = \hat{x}_3 + k_2 (\lceil e_1 \rfloor^{\alpha_2} + \rho e_1) + \sum_{j=1}^{m} g_{2,j}(\hat{x}_1, \hat{x}_2) u_j \\ \vdots \\ \dot{\hat{x}}_n = \varphi(\hat{x}) + k_n (\lceil e_1 \rfloor^{\alpha_n} + \rho e_1) + \sum_{j=1}^{m} g_{n,j}(\hat{x}) u_j \end{cases} \quad (13)$$

where the powers α_i are defined by (8), the gains k_i are defined by (9), and $\rho = \left(\frac{n^2 \theta^{\frac{2}{3}} S_1 + 1}{2} \right)$, where $S_1 = \max_{1 \le i, j \le n} |S_\infty(1)_{i,j}| . |S_\infty^{-1}(1)_{j,1}|.$

Proof. We denote $e = x - \hat{x}$. By using

$$D(x, \hat{x}, u) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \varphi(x) - \varphi(\hat{x}) \end{pmatrix} + \sum_{j=1}^{m} (g_j(x) - g_j(\hat{x})) u_j(t), \quad (14)$$

where $g_j = (g_{1,j}, \ldots, g_{n,j})$, and

$$F(K,e) = \begin{pmatrix} k_1 \lceil e_1 \rfloor^{\alpha_1} \\ \vdots \\ k_n \lceil e_1 \rfloor^{\alpha_n} \end{pmatrix}, \qquad (15)$$

the error system is given by:

$$\dot{e} = Ae - F(K,e) - \rho S_\infty^{-1}(\theta) C^T C e + D(x,\hat{x},u). \quad (16)$$

In order to prove the global convergence of the observer, we prove first the existence of a "Lyapunov" function V for the error system (16) which is positive definite on $\mathbb{R}^n$ and such that $\frac{d}{dt}V$ is negative definite on $\mathbb{R}^n - \mathscr{B}_{S_\infty(\theta)}(r)$. Then we will prove that the error system is finite time stable at the origin on $\mathscr{B}_{S_\infty(\theta)}(2r)$. The negativeness of $\frac{d}{dt}V$ on $\mathbb{R}^n - \mathscr{B}_{S_\infty(\theta)}(r)$ and the finite time stability on $\mathscr{B}_{S_\infty(\theta)}(2r)$ prove that the error equation is globally asymptotic stable and finite-time stable at the origin. We apply then the Lemma 1 which gives the result.

Let us consider:

$$V(e) = e^T S_\infty(\theta) e. \qquad (17)$$

By using (10) and (16), the derivative of V is given by:

$$\begin{aligned} \frac{d}{dt}(e^T S_\infty(\theta) e) &= -\theta e^T S_\infty(\theta) e - (2\rho - 1)(Ce)^2 \\ &\quad -2e^T S_\infty(\theta) F(K,e) \\ &\quad +2e^T S_\infty(\theta) D(x,\hat{x},u). \end{aligned} \qquad (18)$$

It leads to:

$$\begin{aligned} \frac{d}{dt}(e^T S_\infty(\theta) e) &\leq -\theta \|e\|^2_{S_\infty(\theta)} - (2\rho - 1)(Ce)^2 \\ &\quad -2e^T S_\infty(\theta) F(K,e) \\ &\quad +2\|e\|_{S_\infty(\theta)} \|D(x,\hat{x},u)\|_{S_\infty(\theta)}. \end{aligned} \qquad (19)$$

Since φ and $g_{i,j}$ $(i = 1,\ldots,n, j = 1,\ldots,m)$ are globally Lipschitzian with a constant l and u is bounded by u_0, it follows that:

$$\begin{aligned} &\|D(x,\hat{x},u)\|_{S_\infty(\theta)} \\ &= \left(\sum_{1 \leq i,j \leq n} S_\infty(\theta)_{i,j} D_i(\bar{x}_i,\hat{\bar{x}}_i,u) D_j(\bar{x}_j,\hat{\bar{x}}_j,u) \right)^{\frac{1}{2}} \\ &\leq \left((lu_0)^2 \sum_{1 \leq i,j \leq n} |S_\infty(1)_{i,j}| \cdot \|\bar{e}_i\|_{1,i} \|\bar{e}_j\|_{1,j} \frac{1}{\theta^{i+j-1}} \right)^{\frac{1}{2}} \\ &\leq \left((lu_0)^2 S\theta \sum_{1 \leq i,j \leq n} \left\| \frac{1}{\theta^i} \bar{e}_i \right\|_{1,i} \left\| \frac{1}{\theta^j} \bar{e}_j \right\|_{1,j} \right)^{\frac{1}{2}} \end{aligned} \qquad (20)$$

where $S = \max_{1 \leq i,j \leq n} |S_\infty(1)_{i,j}|$. Set $\xi_i = \frac{e_i}{\theta^i}$, clearly for $\theta \geq 1$:

$$\left\| \frac{1}{\theta^i} \bar{e}_i \right\|_{1,i} \leq \left\| \bar{\xi}_i \right\|_{1,i} \leq \|\xi\|_{1,n}. \qquad (21)$$

By norm equivalence, there exists $C_1 > 0$ such that:

$$\|\xi\|_{1,n} \leq C_1 \|\xi\|_{S_\infty(1)}, \qquad (22)$$

$$\|\xi\|^2_{S_\infty(1)} = \frac{1}{\theta} \|e\|^2_{S_\infty(\theta)}, \qquad (23)$$

it leads to

$$\|D(x,\hat{x},u)\|_{S_\infty(\theta)} \leq n l u_0 C_1 \sqrt{S} \|e\|_{S_\infty(\theta)}. \qquad (24)$$

Finally, we have

$$\begin{aligned} \frac{d}{dt}V(e) &\leq (-\theta + M)V(e) \\ &\quad -(2\rho - 1)(Ce)^2 \\ &\quad -2e^T S_\infty(\theta) F(K,e), \end{aligned} \qquad (25)$$

where $M = 2n l u_0 C_1 \sqrt{S}$.

According to (25) in order to prove the negative definiteness of $\frac{d}{dt}V$ on $\mathbb{R}^n - \mathscr{B}_{S_\infty(\theta)}(r)$, we use an overvaluation of $e^T S_\infty(\theta) F(K,e)$. According to lemma 2, we have the following equalities:

$$\begin{aligned} &e^T S_\infty(\theta) F(K,e) \\ &= \sum_{1 \leq i,j \leq n} e_i \frac{(S_\infty(1))_{i,j}}{\theta^{i+j-1}} \left(S_\infty^{-1}(1) \right)_{j,1} \theta^j \lceil e_1 \rfloor^{\alpha_j}, \\ &= \sum_{j=1}^n \left(S_\infty^{-1}(1) \right)_{j,1} \lceil e_1 \rfloor^{\alpha_j} \sum_{i=1}^n \frac{e_i}{\theta^{i-1}} (S_\infty(1))_{i,j}. \end{aligned} \qquad (26)$$

We overvalue (26) in two times. We cut the subset $P^r = \mathbb{R}^n - \mathscr{B}_{S_\infty(\theta)}(r)$ in two complementary parts:

$$P^r_{<1} = \{e \in \mathbb{R}^n : |e_1| < 1\} \cap P^r, \qquad (27)$$

$$P^r_{\geq 1} = \{e \in \mathbb{R}^n : |e_1| \geq 1\} \cap P^r. \qquad (28)$$

On $P^r_{<1}$, we have:

$$|e_1|^{\alpha_i} < 1, \ i = 1,\ldots,n. \qquad (29)$$

Hence

$$|e^T S_\infty(\theta) F(K,e)| \leq n S_1 \theta \sum_{i=1}^n \left| \frac{e_i}{\theta^i} \right|, \qquad (30)$$

where $S_1 = \max_{1 \leq i,j \leq n} |S_\infty^{-1}(1)_{j,1}| \times |S_\infty(1)_{i,j}|$. Let $\xi_i = \frac{e_i}{\theta^i}$ for $i = 1,\ldots,n$, it follows:

$$|e^T S_\infty(\theta) F(K,e)| \leq n S_1 \theta \|\xi\|_{1,n}. \qquad (31)$$

As

$$\|\xi\|_{1,n} \leq C_1 \|\xi\|_{S_\infty(1)}, \qquad (32)$$

$$\|\xi\|^2_{S_\infty(1)} = \frac{1}{\theta} \|e\|^2_{S_\infty(\theta)}, \qquad (33)$$

we have

$$|e^T S_\infty(\theta) F(K,e)| \leq n S . C_1 \sqrt{\theta} \|e\|_{S_\infty(\theta)}. \qquad (34)$$

Let $C_2 = nSC_1$. Taking $r > 1$, then $\|e\|_{S_\infty(\theta)} \le \|e\|^2_{S_\infty(\theta)}$ for $e \in P^r$, so:

$$|e^T S_\infty(\theta) F(K,e)| \le C_2 \sqrt{\theta} \|e\|^2_{S_\infty(\theta)}. \qquad (35)$$

Thus it leads to:

$$\frac{d}{dt} V(e) \le (-\theta + M + C_2\sqrt{\theta})V(e) \qquad (36)$$
$$-(2\rho - 1)(Ce)^2$$
$$\le (-\theta + M + C_2\sqrt{\theta})V(e). \qquad (37)$$

So there exists $\theta_1 > 0$ such that for all $\theta \ge \theta_1$, $\frac{d}{dt}V(e) < 0$ for all $e \in P^r_{<1}$.

On $P^r_{\ge 1}$, we have $|e_1| \ge 1$ so $|e_1|^{\alpha_i} \le |e_1|$ for $i = 1, \dots, n$.

$$|e^T S_\infty(\theta) F(K,e)|$$
$$\le nS\theta \sum_{i=1}^n \left| \frac{e_i}{\theta^i} \right| \cdot |e_1|,$$
$$\le nS \sum_{i=1}^n \left(\theta^{\frac{2}{3}} \left| \frac{e_i}{\theta^i} \right| \right) \left(\theta^{\frac{1}{3}} |e_1| \right), \qquad (38)$$
$$\le \frac{nS}{2} \left(\theta^{\frac{4}{3}} \left(\sum_{i=1}^n \left| \frac{e_i}{\theta^i} \right|^2 \right) + n\theta^{\frac{2}{3}} |e_1|^2 \right),$$
$$\le \frac{nS\theta^{\frac{4}{3}}}{2} \|\xi\|^2_{2,n} + \frac{n^2\theta^{\frac{2}{3}}S}{2} |e_1|^2.$$

But

$$\|\xi\|^2_{2,n} \le C_3 \|\xi\|^2_{S_\infty(1)}, \qquad (39)$$
$$\|\xi\|^2_{S_\infty(1)} = \frac{1}{\theta} \|e\|^2_{S_\infty(\theta)}, \qquad (40)$$

hence

$$|e^T S_\infty(\theta) F(K,e)| \le C_4 \theta^{\frac{1}{3}} \|e\|^2_{S_\infty(\theta)} + \frac{n^2\theta^{\frac{2}{3}}S}{2}|e_1|^2, \qquad (41)$$

where $C_4 = nSC_3$. Thus we have:

$$\frac{d}{dt}V(e) \le \left(-\theta + M + C_4 \theta^{\frac{1}{3}} \right) V(e), \qquad (42)$$

and there exists $\theta_2 > 0$ such that for all $\theta \ge \theta_2$, $\frac{d}{dt}V(e) < 0$ for all $e \in P^r_{\ge 1}$.
Combining the two previous parts, for all $\theta \ge \max\{\theta_1, \theta_2\}$, we have:

$$\frac{d}{dt}V(e) < 0, \quad e \in P^r, \quad r > 1. \qquad (43)$$

Let us prove the finite-time stability of the error system (16) on $\mathscr{B}_{\|.\|_{S_\infty(\theta)}}(2r)$. We consider the following Lyapunov function:

$$\tilde{V}_\alpha(e) = \tilde{e}^T S_\infty(\theta) \tilde{e}, \qquad (44)$$

where $\tilde{e} = \left(\lceil e_1 \rfloor^{\frac{1}{q}} \ \lceil e_2 \rfloor^{\frac{1}{\alpha_1 q}} \dots \lceil e_n \rfloor^{\frac{1}{\alpha_{n-1}q}} \right)$, $q = \prod_{i=1}^{n-1} [(i-1)\alpha - (i-2)]$ is the product of the weights. We have:

$$\frac{d}{dt}\tilde{V}_\alpha(e) = A + B + C \qquad (45)$$

where

$$A = 2\tilde{e}^T S_\infty(\theta) \begin{pmatrix} \frac{1}{q}|e_1|^{\frac{1}{q}-1}(\frac{1}{2}e_2 - k_1 \lceil e_1 \rfloor^{\alpha_1}) \\ \vdots \\ \frac{1}{\alpha_n q}|e_n|^{\frac{1}{\alpha_n q}-1}(-k_n \lceil e_1 \rfloor^{\alpha_n}) \end{pmatrix}, \qquad (46)$$

and

$$B = 2\tilde{e}^T S_\infty(\theta) \begin{pmatrix} \frac{1}{q}|e_1|^{\frac{1}{q}-1}(\frac{1}{2}e_2 - \rho k_1 e_1) \\ \vdots \\ \frac{1}{\alpha_n q}|e_n|^{\frac{1}{\alpha_n q}-1}(-\rho k_n e_1) \end{pmatrix}, \qquad (47)$$

and

$$C = 2\tilde{e}^T S_\infty(\theta) \begin{pmatrix} \frac{1}{q}|e_1|^{\frac{1}{q}-1}D_1 \\ \vdots \\ \frac{1}{\alpha_{n-1}q}|e_n|^{\frac{1}{\alpha_{n-1}q}-1}D_n \end{pmatrix}. \qquad (48)$$

We prove now that the second term B is negative. We use the same technique as in [9], i.e. we use the tube lemma. As V is proper, $\mathscr{B}_{\|.\|_{S_\infty(\theta)}}(2r)$ is a compact set of $\mathbb{R}^n$. Define the function $\varphi : \mathbb{R}_{>0} \times \mathscr{B}_{\|.\|_{S_\infty(\theta)}}(2r) \to \mathbb{R}$

$$(\alpha, e) \mapsto B. \qquad (49)$$

Using the same technique as in Gauthier et al. [3], we easily prove that $\varphi(1,e) < 0$ for $e \in \mathbb{R}^n$. Since φ is continuous, $\varphi^{-1}(\mathbb{R}_{<0})$ is an open subset of $\mathbb{R}_{>0} \times \mathscr{B}_{\|.\|_{S_\infty(\theta)}}(2r)$ containing the slice $\{1\} \times \mathscr{B}_{\|.\|_{S_\infty(\theta)}}(2r)$. Since $\mathscr{B}_{\|.\|_{S_\infty(\theta)}}(2r)$ is compact, it follows from the tube lemma that $\varphi^{-1}(\mathbb{R}_{<0})$ contains some tube $(1 - \mu_1, 1 + \mu_2) \times \mathscr{B}_{\|.\|_{S_\infty(\theta)}}(2r)$ about $\{1\} \times \mathscr{B}_{\|.\|_{S_\infty(\theta)}}(2r)$. For all $(\alpha, e) \in (1 - \mu_1, 1 + \mu_2) \times \mathscr{B}_{\|.\|_{S_\infty(\theta)}}(2r)$

$$\varphi(\alpha, e) < 0. \qquad (50)$$

Thus there exists $\varepsilon_1 > 0$ such that for $\alpha \in (1 - \varepsilon_1, 1)$:

$$\frac{d}{dt}\tilde{V}_\alpha(e) \le A + C. \qquad (51)$$

According to the proof of the main theorem of [20], it follows that for every compact set $\mathscr{U}$ containing the origin, there exists $\theta_3 > 0$ and $\varepsilon_2 > 0$ such that for $\theta \ge \theta_3$ and $\alpha \in (1 - \varepsilon_2, 1)$, $\mathscr{U} \subset \Omega$, where Ω is the domain of

attraction of the observer. But according to [3], there exists $\delta_0 > 0$ such that:

$$S_\infty(\theta) \geq \delta_0 I, \quad \forall \theta \geq 0. \tag{52}$$

Hence, there exists $\theta_4 > 0$ such that for all $\theta \geq \theta_4$:

$$\mathscr{B}_{S_\infty(\theta)}(2r) \subset \delta_0 \mathscr{B}_{\|.\|_2}(2r) \subset \mathscr{U}. \tag{53}$$

Finally we take $\theta^* = \max\{\theta_1, \ldots, \theta_4\}$ and $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. $\qquad\square$

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