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ABSTRACT. We show that the image of the monodromy representation of the Hitchin connection on the sheaf of generalized SL(2)-theta functions over a family of complex smooth projective curves of genus $g \geq 3$ contains an element of infinite order.

1. INTRODUCTION

Let $\pi : \mathcal{C} \to \mathcal{B}$ be a family of smooth connected complex projective curves of genus[1] $g \geq 3$. For any positive integer l, we denote $\mathcal{Z}_l$ the vector bundle over $\mathcal{B}$ having fibers $H^0(\mathrm{M}_{\mathcal{C}_b}(\mathrm{SL}(2)), \mathcal{L}^{\otimes l})$, where $\mathrm{M}_{\mathcal{C}_b}(\mathrm{SL}(2))$ is the moduli space of semistable rank-2 vector bundles with trivial determinant over the curve $\mathcal{C}_b = \pi^{-1}(b)$ for $b \in \mathcal{B}$ and $\mathcal{L}$ is the ample generator of the Picard group. Following Hitchin [H], the bundle $\mathcal{Z}_l$ is equipped with a projectively flat connection called the Hitchin connection.

Theorem. *Assume that the level $l \neq 1, 2, 4, 8$ and that the genus $g \geq 3$. Then there exists a family $\pi : \mathcal{C} \to \mathcal{B}$ of smooth connected projective curves of genus g such that the monodromy representation of the Hitchin connection*

$$\rho_l : \pi_1(\mathcal{B}, b) \longrightarrow \mathbf{P}\mathrm{GL}(\mathcal{Z}_{l,b})$$

has an element of infinite order in its image.

Remark 1.1. An analagous statement is true for more general simple groups, at least for $\mathrm{SL}(n), n \geq 2$ (see remark 4.3 below).We will discuss this question in a future paper.

In the context of Topological Quantum Field Theory as defined by Blanchet-Habegger-Masbaum-Vogel [BHMV], the analogue of the above theorem is well known due to work of Funar [F] for the infiniteness of the image and of Masbaum [Ma] for the exhibition of an explicit element in the mapping class group with image of infinite order.

It is enough to show the above theorem in the context of Conformal Field Theory as defined by Tsuchiya-Ueno-Yamada [TUY]: following a result of the first author [L], the monodromy representation associated to Hitchin's connection coincides with the monodromy representation of the WZW connection. As both, the above Conformal Field Theory and the above Topological Quantum Field Theory are predicted to be equivalent by work in progress of Andersen and Ueno ([AU1], [AU2] and [AU3]), the above theorem should follow from the work of Funar and Masbaum.

In this short note, we give a direct algebraic proof, avoiding the above identification: we first recall Masbaum's initial argument applied to Tsuchiya-Kanie's description of the monodromy

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[1] The genus assumption is just a simplicity assumption and can be relaxed.

representation for the WZW connection in the case of the projective line with 4 points labeled with the standard 2-dimensional representation (see also [AMU]), then we observe that the sewing procedure induces a projectively flat map, enabling us to make an induction on the genus.

A couple of words about the exceptional levels $l = 1, 2, 4, 8$ are in order. For $l = 1$ the monodromy representation ρ_1 is finite for any g. This follows from the fact that Beauville's [B1] strange duality isomorphism $\mathbf{P}H^0(\mathrm{M}_{\mathcal{C}_b}(\mathrm{SL}(2)), \mathcal{L})^* \xrightarrow{\sim} \mathbf{P}H^0(\mathrm{Pic}^{g-1}(\mathcal{C}_b), 2\Theta)$ is projectively flat over $\mathcal{B}$ for any family $\pi : \mathcal{C} \to \mathcal{B}$ (see e.g. [Bel1]) and that ρ_1 thus identifies with the monodromy representation on a space of abelian theta functions, which is known to have finite image (see e.g. [W]). For $l = 2$ there is a canonical morphism $H^0(\mathrm{M}_{\mathcal{C}_b}(\mathrm{SL}(2)), \mathcal{L}^{\otimes 2}) \to H^0(\mathrm{Pic}^{g-1}(\mathcal{C}_b), 4\Theta)_+$, which is an isomorphism if and only if $\mathcal{C}_b$ has no vanishing theta-null [B2]. But this map is not projectively flat having not constant rank. So the question about finiteness of ρ_2 remains open — see also [Bel2]. For $l = 4$ there is a canonical isomorphism [OP] between the dual $H^0(\mathrm{M}_{\mathcal{C}_b}(\mathrm{SL}(2)), \mathcal{L}^{\otimes 4})^*$ and a space of abelian theta functions of order 3. We expect this isomorphism to be projectively flat. For $l = 8$ no isomorphism with spaces of abelian theta functions seems to be known.

Our motivation to study the monodromy representation of the Hitchin connection comes from the Grothendieck-Katz conjectures on the p-curvatures of a local system [K]. In a forthcoming paper we will discuss the consequences of the above theorem in this set-up.

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2. Review of mapping class groups, moduli spaces of pointed curves and braid groups

2.1. Mapping class groups.

In this section we recall the basic definitions and properties of the mapping class groups. We refer the reader e.g. to [I] or [HL].

2.1.1. *Definitions.* Let S be a compact oriented surface of genus g without boundary and with n marked points $x_1, \ldots, x_n \in S$. Associated to the n-pointed surface S are the mapping class groups Γ_g^n and $\Gamma_{g,n}$ defined as the groups of isotopy classes of orientation-preserving diffeomorphisms $\phi : S \to S$ such that $\phi(x_i) = x_i$ for each i, respectively such that $\phi(x_i) = x_i$ and the differential $d\phi_{x_i} : T_{x_i}S \to T_{x_i}S$ at the point x_i is the identity map for each i.

An alternative definition of the mapping class groups Γ_g^n and $\Gamma_{g,n}$ can be given in terms of surfaces with boundary. We consider the surface R obtained from S by removing a small disc around each marked point x_i. The boundary ∂R consists of n circles. Equivalently, the groups Γ_g^n and $\Gamma_{g,n}$ coincide with the groups of isotopy classes of orientation-preserving diffeomorphisms $\phi : R \to R$ such that ϕ preserves each boundary component of R, respectively such that ϕ is the identity on ∂R.

The mapping class group Γ_g is defined to be $\Gamma_g^0 = \Gamma_{g,0}$.

2.1.2. *Dehn twists.* Given an (unparametrized) oriented, embedded circle γ in $R \subset S$ we can associate to it a diffeomorphism T_γ up to isotopy, i.e., an element T_γ in the mapping class groups Γ_g^n and $\Gamma_{g,n}$, the so-called Dehn twist along the curve γ. It is known that the mapping class groups Γ_g^n and $\Gamma_{g,n}$ are generated by a finite number of Dehn twists. We recall the following exact sequence

$$1 \longrightarrow \mathbf{Z}^n \longrightarrow \Gamma_{g,n} \longrightarrow \Gamma_g^n \longrightarrow 1.$$

The n generators of the abelian kernel $\mathbf{Z}^n$ are given by the Dehn twists T_{γ_i}, where γ_i is a loop going around the boundary circle associated to x_i for each i.

2.1.3. *The mapping class groups Γ_0^4 and $\Gamma_{0,4}$.* Because of their importance in this paper we recall the presentation of the mapping class groups Γ_0^4 and $\Gamma_{0,4}$ by generators and relations. Keeping the notation of the previous section, we denote by R the 4-holed sphere and by $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ the circles in R around the four boundary circles. We denote by γ_{ij} the circle dividing R into two parts containing two holes each and such that the two circles γ_i and γ_j are in the same part. It is known (see e.g. [I] section 4) that $\Gamma_{0,4}$ is generated by the Dehn twists T_{γ_i} for $1 \leq i \leq 4$ and $T_{\gamma_{ij}}$ for $1 \leq i, j \leq 3$ and that, given a suitable orientation of the circles γ_i and γ_{ij}, there is a relation (the *lantern* relation)

$$T_{\gamma_1} T_{\gamma_2} T_{\gamma_3} T_{\gamma_4} = T_{\gamma_{12}} T_{\gamma_{13}} T_{\gamma_{23}}.$$

Note that the images of the Dehn twists T_{γ_i} under the natural homomorphism

$$\Gamma_{0,4} \longrightarrow \Gamma_0^4, \qquad T_\gamma \mapsto \overline{T}_\gamma,$$

are trivial. Thus the group Γ_0^4 is generated by the three Dehn twists $\overline{T}_{ij}$ for $1 \leq i, j \leq 3$ with the relation $\overline{T}_{\gamma_{12}} \overline{T}_{\gamma_{13}} \overline{T}_{\gamma_{23}} = 1$.

For each 4-holed sphere being contained in a closed genus g surface without boundary one can consider the Dehn twists T_{ij} as elements in the mapping class group Γ_g.

2.2. **Moduli spaces of curves.** Let $\mathfrak{M}_{g,n}$ denote the moduli space parameterizing n-pointed smooth projective curves of genus g. The moduli space $\mathfrak{M}_{g,n}$ is a (possibly singular) algebraic variety. It can also be thought as an orbifold (or Deligne-Mumford stack) and one has an isomorphism

$$j : \pi_1(\mathfrak{M}_{g,n}, x) \xrightarrow{\sim} \Gamma_g^n, \tag{1}$$

where $\pi_1(\mathfrak{M}_{g,n}, x)$ stands for the orbifold fundamental group of $\mathfrak{M}_{g,n}$. In case the space $\mathfrak{M}_{g,n}$ is a smooth algebraic variety, the orbifold fundamental group coincides with the usual fundamental group.

2.3. **The isomorphism between $\pi_1(\mathfrak{M}_{0,4}, x)$ and Γ_0^4.** The moduli space $\mathfrak{M}_{0,4}$ parametrizes ordered sets of 4 points on the complex projective line $\mathbf{P}^1_{\mathbf{C}}$ up to the diagonal action of $\mathbf{P}\mathrm{GL}(2, \mathbf{C})$. The double ratio induces an isomorphism with the projective line $\mathbf{P}^1_{\mathbf{C}}$ with 3 punctures at $0, 1$ and ∞

$$\mathfrak{M}_{0,4} \xrightarrow{\sim} \mathbf{P}^1_{\mathbf{C}} \setminus \{0, 1, \infty\}.$$

We deduce that the fundamental group of $\mathfrak{M}_{0,4}$ is the group with three generators

$$\pi_1(\mathfrak{M}_{0,4}, x) = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_3 \sigma_2 \sigma_1 = 1 \rangle,$$

where σ_1, σ_2 and σ_3 are the loops starting at $x \in \mathbf{P}^1_{\mathbf{C}} \setminus \{0, 1, \infty\}$ and going once around the points $0, 1$ and ∞ with the same orientation. We choose the orientation such that the generators

σ_i satisfy the relation $\sigma_3\sigma_2\sigma_1 = 1$. Clearly $\pi_1(\mathfrak{M}_{0,4}, x)$ coincides with the fundamental group $\pi_1(Q, x)$ of the 3-holed sphere Q.

In this particular case the isomorphism $j : \pi_1(\mathfrak{M}_{0,4}, x) \xrightarrow{\sim} \Gamma_0^4$ can be explicitly described as follows (see e.g. [I] Theorem 2.8.C): we may view the 3-holed sphere Q as the union of the 4-holed sphere R with a disc D glued on the boundary corresponding to the point x_4. Given a loop $\sigma \in \pi_1(Q, x)$ we may find an isotopy $\{f_t : Q \to Q\}_{0\leq t\leq 1}$ such that the map $t \mapsto f_t(x)$ coincides with the loop σ, $f_0 = \mathrm{id}_Q$ and $f_1(D) = D$. Then the isotopy class of f_1 resticted to $R \subset Q$ determines an element $j(\sigma) = [f_1] \in \Gamma_0^4$. Moreover, with the previous notation, we have the equalities (see e.g. [I] Lemma 4.1.I)

$$j(\sigma_1) = \overline{T}_{\gamma_{23}}, \qquad j(\sigma_2) = \overline{T}_{\gamma_{13}}, \qquad j(\sigma_3) = \overline{T}_{\gamma_{12}}.$$

Remark 2.1. At this stage we observe that under the isomorphism j the two elements $\sigma_1^{-1}\sigma_2 \in \pi_1(\mathfrak{M}_{0,4}, x)$ and $\overline{T}_{\gamma_{23}}^{-1}\overline{T}_{\gamma_{13}} \in \Gamma_0^4$ coincide. It was shown by G. Masbaum in [Ma] that the latter element has infinite order in the TQFT-representation of the mapping class group Γ_g — note that $T_{\gamma_{23}}^{-1}T_{\gamma_{13}}$ also makes sense in Γ_g. We will show in Proposition 5.1 that the loop $\sigma_1^{-1}\sigma_2$ has infinite order in the monodromy representation of the WZW connection.

2.4. **Braid groups and configuration spaces.** We recall some basic results about braid groups and configuration spaces. We refer the reader e.g. to [KT] Chapter 1. For our purposes it will be sufficient to deal with braid groups on 3 braids.

2.4.1. *Definitions.* The braid group B_3 is the group generated by two generators g_1 and g_2 and one relation

$$g_1g_2g_1 = g_2g_1g_2.$$

The pure braid group is the kernel $P_3 = \ker(B_3 \to \Sigma_3)$ of the group homomorphism which associates to the generator g_i the transposition $(i, i+1)$ in the symmetric group Σ_3. The braid groups B_3 and P_3 can be identified with the fundamental groups

$$P_3 = \pi_1(X_3, p_3), \qquad B_3 = \pi_1(\overline{X}_3, \overline{p}_3),$$

where X_3 and $\overline{X}_3$ are the complex manifolds parametrizing ordered respectively unordered triples of distinct points in the complex plane

$$X_3 = \{(z_1, z_2, z_3) \in \mathbf{C}^3 \mid z_i \neq z_j\} \qquad \text{and} \qquad \overline{X}_3 = X_3/\Sigma_3.$$

The points p_3 and $\overline{p}_3$ are base points in X_3 and $\overline{X}_3$.

2.4.2. *Relation between P_3 and $\pi_1(\mathfrak{M}_{0,4}, x)$.* The natural map

$$\mathfrak{M}_{0,4} = \mathbf{P}^1_{\mathbf{C}} \setminus \{0, 1, \infty\} \longrightarrow X_3, \qquad z \mapsto (0, 1, z)$$

induces a group homomorphism at the level of fundamental groups

$$\Psi : \pi_1(\mathfrak{M}_{0,4}, x) = \langle \sigma_1, \sigma_2 \rangle \longrightarrow P_3 = \pi_1(X_3, p_3),$$

with $p_3 = (0, 1, x)$. Then Ψ is a monomorphism by [KT] Theorem 1.16. Moreover, the image of Ψ coincides with the kernel of the natural group homomorphism

$$\mathrm{im}\ \Psi = \ker\left(P_3 = \pi_1(X_3, p_3) \longrightarrow P_2 = \pi_1(X_2, p_2)\right)$$

induced by the projection onto the first two factors $X_3 \to X_2$, $(z_1, z_2, z_3) \mapsto (z_1, z_2)$ and $p_2 = (0, 1)$. One computes explicitly (see [KT] section 1.4.2) that

$$\Psi(\sigma_1) = g_2g_1^2g_2^{-1}, \qquad \text{and} \qquad \Psi(\sigma_2) = g_2^2.$$

For later use we introduce the element

$$\sigma = \sigma_1^{-1}\sigma_2 \in \pi_1(\mathfrak{M}_{0,4}, x). \tag{2}$$

3. Conformal blocks and the projective WZW connection

3.1. **General set-up.** We consider the simple Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$. The set of irreducible $\mathfrak{sl}_2$-modules, i.e. the set of dominant weights of $\mathfrak{sl}_2$, is in bijection with the set of positive integers $\mathbf{N}$. We fix an integer $l \geq 1$, called the level, and introduce the set $P_l = \{\lambda \in \mathbf{N} \mid \lambda \leq l\}$. Given an integer $n \geq 1$, a collection $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (P_l)^n$ of dominants weights of $\mathfrak{sl}_2$ and a family

$$\mathcal{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_n; \xi_1, \ldots, \xi_n)$$

of n-pointed stable curves of arithmetic genus g parameterized by a base variety $\mathcal{B}$ with sections $s_i : \mathcal{B} \to \mathcal{C}$ and formal coordinates ξ_i at the divisor $s_i(\mathcal{B}) \subset \mathcal{C}$, one constructs (see [TUY] section 4.1) a locally free sheaf

$$\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F})$$

over the base variety $\mathcal{B}$, called the *sheaf of conformal blocks* or the *sheaf of vacua*. We recall that $\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F})$ is a subsheaf of $\mathcal{O}_{\mathcal{B}} \otimes \mathcal{H}^\dagger_{\vec{\lambda}}$, where $\mathcal{H}^\dagger_{\vec{\lambda}}$ denotes the dual of the tensor product $\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n}$ of the integrable highest weight representations $\mathcal{H}_{\lambda_i}$ of level l and weight λ_i of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. The formation of the sheaf of conformal blocks commutes with base change. In particular, we have for any point $b \in \mathcal{B}$

$$\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_b \cong \mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_b),$$

where $\mathcal{F}_b$ denotes the data $(\mathcal{C}_b = \pi^{-1}(b); s_1(b), \ldots, s_n(b); \xi_{1|\mathcal{C}_b}, \ldots, \xi_{n|\mathcal{C}_b})$ consisting of a stable curve $\mathcal{C}_b$ with n-marked points $s_1(b), \ldots, s_n(b)$ and formal coordinates $\xi_{i|\mathcal{C}_b}$ at the points $s_i(b)$.

We recall that the sheaf of conformal blocks $\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F})$ does not depend (up to a canonical isomorphism) on the formal coordinates ξ_i (see e.g. [U] Theorem 4.1.7). We therefore omit the formal coordinates in the notation.

We will denote

$$\vec{1}_n = (1, 1, \ldots, 1) \in (P_l)^n$$

the collection having all dominants weights equal to 1, i.e., corresponding to the standard 2-dimensional representation of $\mathfrak{sl}_2$.

3.2. **The projective WZW connection.** We now outline the definition of the projective WZW connection on the sheaf $\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F})$ over the smooth locus $\mathcal{B}^s \subset \mathcal{B}$ parameterizing smooth curves and refer to [TUY] or [U] for a detailed account. Let $\mathcal{D} \subset \mathcal{B}$ be the discriminant locus and let $\mathcal{S} = \coprod_{i=1}^n s_i(\mathcal{B})$ be the union of the images of the n sections. We recall the exact sequence

$$0 \longrightarrow \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*\mathcal{S}) \longrightarrow \pi_*\Theta'_{\mathcal{C}}(*\mathcal{S})_\pi \xrightarrow{\theta} \Theta_{\mathcal{B}}(-\log \mathcal{D}) \longrightarrow 0,$$

where $\Theta_{\mathcal{C}/\mathcal{B}}(*\mathcal{S})$ denotes the sheaf of vertical rational vector fields on $\mathcal{C}$ with poles only along the divisor $\mathcal{S}$, and $\Theta'_{\mathcal{C}}(*\mathcal{S})_\pi$ the sheaf of rational vector fields on $\mathcal{C}$ with poles only along the divisor $\mathcal{S}$ and with constant horizontal components along the fibers of π. There is an $\mathcal{O}_{\mathcal{B}}$-linear map

$$p : \pi_*\Theta'_{\mathcal{C}}(*\mathcal{S})_\pi \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathcal{B}}((\xi_i))\frac{d}{d\xi_i},$$

which associates to a vector field $\vec{l}$ in $\Theta'_{\mathcal{C}}(*\mathcal{S})_\pi$ the n Laurent expansions $l_i \frac{d}{d\xi_i}$ around the divisor $s_i(\mathcal{B})$. Abusing notation we also write $\vec{l}$ for its image under p

$$\vec{l} = (l_1 \frac{d}{d\xi_1}, \cdots, l_n \frac{d}{d\xi_n}) \in \bigoplus_{i=1}^{n} \mathcal{O}_{\mathcal{B}}((\xi_i)) \frac{d}{d\xi_i}.$$

We then define for any vector field $\vec{l}$ in $\Theta'_{\mathcal{C}}(*\mathcal{S})_\pi$ the endomorphism $D(\vec{l})$ of $\mathcal{O}_{\mathcal{B}} \otimes \mathcal{H}^\dagger_{\vec{\lambda}}$ by

$$D(\vec{l})(f \otimes u) = \theta(\vec{l}).f \otimes u + \sum_{i=1}^{n} f \otimes (T[l_i].u)$$

for f a local section of $\mathcal{O}_{\mathcal{B}}$ and $u \in \mathcal{H}^\dagger_{\vec{\lambda}}$. Here $T[l_i]$ denotes the action of the energy-momentum tensor on the i-th component $\mathcal{H}^\dagger_{\lambda_i}$. It is shown in [TUY] that $D(\vec{l})$ preserves $\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F})$ and that $D(\vec{l})$ only depends on the image $\theta(\vec{l})$ up to homothety. One therefore obtains a projective connection ∇ on the sheaf $\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F})$ given by

$$\nabla_{\theta(\vec{l})} = \theta(\vec{l}) + T[\vec{l}].$$

Remark 3.1. For a family of smooth n-pointed curves of genus 0 the projective WZW connection is actually a connection.

4. Monodromy of the WZW connection for a family of 4-pointed rational curves

In this section we review the results by Tsuchiya and Kanie [TK] on the monodromy of the WZW connection for a family of rational curves with 4 marked points. We take $\mathcal{B} = X_3 = \{(z_1, z_2, z_3) \in \mathbf{C}^3 \mid z_i \neq z_j\}$ (see section 2.4.1) and consider the universal family

$$\mathcal{F} = (\pi : \mathcal{C} = \mathcal{B} \times \mathbf{P}^1 \to \mathcal{B}; s_1, s_2, s_3, s_\infty),$$

where the section s_i is given by the natural projection $X_3 \to \mathbf{C}$ on the i-th component followed by the inclusion $\mathbf{C} \subset \mathbf{P}^1_{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ and s_∞ is the constant section corresponding to $\infty \in \mathbf{P}^1_{\mathbf{C}}$. We will denote

$$\mathcal{F}_4^{univ} = (\pi : \mathcal{C} = \mathfrak{M}_{0,4} \times \mathbf{P}^1 \to \mathfrak{M}_{0,4}; t_0, t_1, t, t_\infty) \tag{3}$$

the pull-back of the family $\mathcal{F}$ under the natural embedding $\mathfrak{M}_{0,4} \to X_3$ (see section 2.4.2). We consider for $l \geq 1$ and for $\vec{\lambda} = \vec{1}_4 \in (P_l)^4$ the sheaf of conformal blocks $\mathcal{V}^\dagger_{l,\vec{1}_4}(\mathcal{F})$. The rank of this locally free sheaf equals 2 for any $l \geq 1$(see e.g. [TK] Theorem 3.3). Moreover $\mathcal{V}^\dagger_{l,\vec{1}_4}(\mathcal{F})$ is equipped with a flat actual connection ∇ (not only projective) (see section 3.2).

Remark 4.1. It is known [TK] that the differential equations satisfied by the flat sections of $(\mathcal{V}^\dagger_{l,\vec{1}_4}(\mathcal{F}), \nabla)$ coincide with the Knizhnik-Zamolodchikov equations (see e.g. [EFK]). Moreover, we will show in a forthcoming paper that the local system $(\mathcal{V}^\dagger_{l,\vec{1}_4}(\mathcal{F}), \nabla)$ also coincides with a certain Gauss-Manin local system.

We observe that the symmetric group Σ_3 acts naturally on the base variety X_3. The local system $(\mathcal{V}^\dagger_{l,\vec{1}_4}(\mathcal{F}), \nabla)$ is invariant under this Σ_3-action and admits a natural Σ_3-linearization. Thus by descent we obtain a local system $(\overline{\mathcal{V}^\dagger_{l,\vec{1}_4}(\mathcal{F})}, \overline{\nabla})$ over $\overline{X}_3$. Therefore, we obtain a monodromy representation

$$\widetilde{\rho}_l : B_3 = \pi_1(\overline{X}_3, \overline{p}_3) \longrightarrow \mathrm{GL}(\overline{\mathcal{V}^\dagger_{l,\vec{1}_4}(\mathcal{F})}_{\overline{p}_3}) = \mathrm{GL}(2, \mathbf{C})$$

Proposition 4.2 ([TK] Theorem 5.2)**.** *We put $q = \exp(\frac{2i\pi}{l+2})$. There exists a basis B of the vector space $\overline{\mathcal{V}^\dagger_{l,\vec{1}_4}(\mathcal{F})}_{\overline{p}_3} = \mathcal{V}^\dagger_{l,\vec{1}_4}(\mathcal{F})_{p_3}$ such that*

$$\mathrm{Mat}_B(\widetilde{\rho}_l(g_1)) = q^{-\frac{3}{4}} \begin{pmatrix} q & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathrm{Mat}_B(\widetilde{\rho}_l(g_2)) = \frac{q^{-\frac{3}{4}}}{q+1} \begin{pmatrix} -1 & t \\ t & q^2 \end{pmatrix},$$

with $t = \sqrt{q(1+q+q^2)}$. Note that both matrices have eigenvalues $q^{\frac{1}{4}}$ and $-q^{-\frac{3}{4}}$.

Remark 4.3. We would like to mention that the above theorem has been generalized to the Lie algebra $\mathfrak{sl}_n$ in [Ka].

5. Infinite monodromy over $\mathfrak{M}_{0,4}$

We denote by ρ_l the restriction of the monodromy representation $\widetilde{\rho}_l$ to the subgroup $\pi_1(\mathfrak{M}_{0,4}, x)$

$$\rho_l : \pi_1(\mathfrak{M}_{0,4}, x) \to \mathrm{GL}(2, \mathbf{C}).$$

Proposition 5.1. *Let $\sigma \in \pi_1(\mathfrak{M}_{0,4}, x)$ be the element introduced in* (2)*. If the level $l \neq 1, 2, 4$ and* 8*, then the element $\rho_l(\sigma)$ has infinite order in both* $\mathbf{P}\mathrm{GL}(2, \mathbf{C})$ *and* $\mathrm{GL}(2, \mathbf{C})$

Proof. Using the explicit form of the monodromy representation ρ_l given in Proposition 4.2 we compute the matrix associated to $\Psi(\sigma) = \Psi(\sigma_1^{-1}\sigma_2) = g_2 g_1^{-2} g_2$

$$\mathrm{Mat}_B(\widetilde{\rho}_l(\Psi(\sigma))) = \frac{1}{(q+1)^2} \begin{pmatrix} q^{-2} + t^2 & t(q^2 - q^{-2}) \\ t(q^2 - q^{-2}) & t^2 q^{-2} + q^4 \end{pmatrix}.$$

This matrix has determinant 1 and trace $2 - q - q^{-1} + q^2 + q^{-2}$. Hence the matrix has finite order if and only if there exists a primitive root of unity λ such that

$$\lambda + \lambda^{-1} = 2 - q - q^{-1} + q^2 + q^{-2}.$$

In [Ma] it is shown that this can only happen if $l = 1, 2, 4$ or 8: using the transitive action of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ on primitive roots of unity, one gets that for *any* primitive $(l+2)$-th root $\tilde{q}$ there exists a primitive root $\tilde{\lambda}$ such that

$$\tilde{\lambda} + \tilde{\lambda}^{-1} = 2 - \tilde{q} - \tilde{q}^{-1} + \tilde{q}^2 + \tilde{q}^{-2}.$$

In particular, we have the inequality $|1 - \mathbf{R}\mathrm{e}(\tilde{q}) + \mathbf{R}\mathrm{e}(\tilde{q}^2)| \leq 1$ for any primitive $(l+2)$-th root $\tilde{q}$. But for $l \neq 1, 2, 4$ and 8, one can always find a primitive $(l+2)$-th root $\tilde{q}$ such that $\mathbf{R}\mathrm{e}(\tilde{q}^2) > \mathbf{R}\mathrm{e}(\tilde{q})$ — for the explicit root $\tilde{q}$ see [Ma].

Finally, since $\rho_l(\sigma)$ has trivial determinant, its class in $\mathbf{P}\mathrm{GL}(2, \mathbf{C})$ will also have infinite order. □

Remark 5.2. The same computation shows that the element $\rho_l(\sigma_1\sigma_2^{-1}) \in \mathrm{GL}(2, \mathbf{C})$ also has infinite order if $l \neq 1, 2, 4$ and 8. This implies that the orientation chosen for both loops σ_1 and σ_2 around 0 and 1 is irrelevant. On the other hand, it is immediately seen that the elements $\rho_l(\sigma_1), \rho_l(\sigma_2)$ and $\rho_l(\sigma_1\sigma_2)$ have finite order for any level l.

Proposition 5.3. *In the four cases $l = 1, 2, 4$ and 8, the image $\rho_l(\pi_1(\mathcal{M}_{0,4}, x))$ in the projective linear group* $\mathbf{PGL}(2, \mathbf{C})$ *is finite and isomorphic to the groups given in table*

l	1	2	4	8
$\rho_l(\pi_1(\mathcal{M}_{0,4}, x))$	μ_3	$\mu_2 \times \mu_2$	A_4	A_5

Here A_n denotes the alternating group on n letters.

Proof. We denote by $m_1, m_2 \in \mathbf{PGL}(2, \mathbf{C})$ the elements defined by the matrices $\mathrm{Mat}_B(\rho_l(\sigma_1))$ and $\mathrm{Mat}_B(\rho_l(\sigma_2))$ and denote by $\mathrm{ord}(m_i)$ their order in the group $\mathbf{PGL}(2, \mathbf{C})$. In the first two cases one immediately checks the relations $m_1 = m_2$, $\mathrm{ord}(m_1) = \mathrm{ord}(m_2) = 3$ (for $l = 1$) and $\mathrm{ord}(m_1) = \mathrm{ord}(m_2) = \mathrm{ord}(m_1 m_2) = 2$ (for $l = 2$).

In the case $l = 4$ we recall that the alternating group A_4 has the following presentation by generators and relations
$$A_4 = \langle a, b \mid a^3 = b^2 = (ab)^3 = 1 \rangle.$$
Using the formulae of Proposition 4.2 and 5.1 we check that $\mathrm{ord}(m_1) = \mathrm{ord}(m_2) = 3$ and $\mathrm{ord}(m_1^{-1} m_2) = 2$, so that $a = m_1$ and $b = m_1^{-1} m_2$ generate the group A_4.

In the case $l = 8$ we recall that the alternating group A_5 has the following presentation by generators and relations
$$A_5 = \langle a, b \mid a^2 = b^3 = (ab)^5 = 1 \rangle.$$
Using the formulae of Proposition 4.2 and 5.1 we check that $\mathrm{ord}(m_1) = \mathrm{ord}(m_2) = 5$ and $\mathrm{ord}(m_1^{-1} m_2) = 3$. Moreover a straightforward computation shows that the element $m_1^{-1} m_2 m_1^{-1}$ is (up to a scalar) conjugate to the matrix
$$\mathrm{Mat}_B(\widetilde{\rho}_l(g_1^{-2} g_2^2 g_1^{-2})) = * \begin{pmatrix} q^{-4}(1 + t^2) & t(1 - q^{-2}) \\ t(1 - q^{-2}) & t^2 + q^4 \end{pmatrix},$$
which has trace zero. Note that $t^2 = q + q^2 + q^3$ and $q^{-4} = -q$. Hence $\mathrm{ord}(m_1^{-1} m_2 m_1^{-1}) = \mathrm{ord}(m_1 m_2^{-1} m_1) = 2$. Therefore if we put $a = m_1 m_2^{-1} m_1$ and $b = m_1^{-1} m_2$, we have $ab = m_1$ and $ab^2 = m_2$, so that $\mathrm{ord}(a) = 2$, $\mathrm{ord}(b) = 3$, and $\mathrm{ord}(ab) = 5$, i.e. a, b generate the group A_5. □

Corollary 5.4. *In the four cases $l = 1, 2, 4$ and 8, the image $\widetilde{\rho}_l(B_3)$ in* $\mathrm{GL}(2, \mathbf{C})$ *is finite.*

Proof. First, we observe that the image $\rho_l(\pi_1(\mathcal{M}_{0,4}, x))$ in $\mathrm{GL}(2, \mathbf{C})$ is finite. In fact, by Proposition 5.3 its image in $\mathbf{PGL}(2, \mathbf{C})$ is finite and its intersection $\rho_l(\pi_1(\mathcal{M}_{0,4}, x)) \cap \mathbf{C}^* \mathrm{Id}$ with the center of $\mathrm{GL}(2, \mathbf{C})$ is also finite. The latter follows from the fact that the determinant $\det \mathrm{Mat}_B(\widetilde{\rho}_l(g_i)) = -q^{-\frac{1}{2}}$ has finite order in $\mathbf{C}^*$.

Secondly, we recall that P_3 is generated by the normal subgroup $\pi_1(\mathcal{M}_{0,4}, x)$ and by the element g_1^2. Since $\widetilde{\rho}_l(g_1^2)$ has finite order and since $B_3/P_3 = \Sigma_3$ is finite, we obtain that $\widetilde{\rho}_l(B_3)$ is a finite subgroup. □

6. Infinite monodromy over $\mathfrak{M}_{g,n}$

6.1. **Desingularization of families of nodal curves.** We introduce the notation $R = \mathbf{C}[[\tau]]$, $K = \mathbf{C}((\tau))$ and $\overline{K}$ the algebraic closure of K. For a variety $\mathcal{B}$ defined over $\mathbf{C}$ we denote $\mathcal{B}_R = \mathcal{B} \times \mathrm{Spec}(R)$ and $\mathcal{B}_{\overline{K}} = \mathcal{B} \times \mathrm{Spec}(\overline{K})$.

Proposition 6.1. *Let $\widetilde{\mathcal{F}} = (\widetilde{\pi} : \widetilde{\mathcal{C}} \to \mathcal{B}; s_1, \ldots, s_{n+2})$ be a family of smooth $n + 2$ pointed (not necessarily connected) curves parameterized by a base variety $\mathcal{B}$ and let $\mathcal{F}_0 = (\pi_0 : \mathcal{C}_0 \to$*

$\mathcal{B}; s_1, \ldots, s_n)$ be the n-pointed family of nodal curves obtained from $\widetilde{\mathcal{F}}$ by identifying the two points $s_{n+1}(b)$ and $s_{n+2}(b)$ of $\widetilde{\mathcal{C}}_b = \widetilde{\pi}^{-1}(b)$ for each point $b \in \mathcal{B}$. Then there exists a flat family

$$\mathcal{F}_R = (\pi_R : \mathcal{C}_R \to \mathcal{B}_R; s_{1,R}, \ldots, s_{n,R}),$$

such that

(1) *the restriction of $\mathcal{F}_R$ to the special fiber $(\mathcal{F}_R)_0$ is isomorphic to $\mathcal{F}_0$,*
(2) *the generic fiber $\mathcal{F}_{\overline{K}}$ is a family of smooth n-pointed curves over $\mathcal{B}_{\overline{K}}$.*

Proof. Let $b \in \mathcal{B}$. We denote by $A = \widehat{\mathcal{O}}_{\mathcal{B},b}$ the completion of the local ring $\mathcal{O}_{\mathcal{B},b}$ and by $\pi_0 : \mathcal{C}_0 \to \mathrm{Spec}(A)$ the pull-back of the family $\mathcal{F}_0$ of nodal curves of genus g to $\mathrm{Spec}(A)$. We introduce the formal deformation space $\Gamma \to \mathcal{M}$ of the stable n-pointed nodal curve $(\mathcal{C}_0)_b = C$ with one node $z \in C$ (see [DM] section 1). Then we have the cartesian diagram

$$\begin{array}{ccc} \mathcal{C}_0 & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ A & \xrightarrow{s} & \mathcal{M}, \end{array}$$

as well as n sections $\sigma_i : \mathcal{M} \to \Gamma$. By [DM] page 82, we have

$$\mathcal{M} = \mathrm{Spec}\ \mathbf{C}[[t_1, \ldots, t_{3g-3+n}]], \qquad \text{and} \qquad \widehat{\mathcal{O}}_{\Gamma,z} \cong \mathbf{C}[[t_1, \ldots, t_{3g-3+n}, u, v]]/(uv - t_1),$$

where $t_1 = 0$ is the equation of the locus of singular curves in $\mathcal{M}$. The classifying map $s : \mathrm{Spec}\ A \to \mathrm{Spec}\ \mathbf{C}[[t_1, \ldots, t_{3g-3+n}]] = \mathcal{M}$ sends t_1 to $0 \in A$ (since Spec A parameterizes singular curves) and t_i to an element $f_i \in A$ for $i \geq 2$. We extend s to $\widehat{s} : \mathrm{Spec}\ A[[\tau]] \longrightarrow \mathrm{Spec}\ \mathbf{C}[[t_1, \ldots, t_{3g-3+n}]]$ by mapping t_1 to τ and t_i to f_i for $i \geq 2$. The base change by $\widehat{s}$ then defines an n-pointed family $\mathcal{F}_R$ over Spec $A[[\tau]]$ such that $\mathcal{F}_{|\mathrm{Spec}\ A((\tau))}$ is smooth.

Hence, we have constructed the n-pointed family $\mathcal{F}_R$ over Spec $A[[\tau]]$ for any complete local ring $A = \widehat{\mathcal{O}}_{\mathcal{B},b}$, which proves the theorem. □

6.2. **The sewing procedure.** We will briefly sketch the construction of the sewing homomorphism and give some of its properties (for the details see [TUY] or [U]).

We consider two versal families $\widetilde{\mathcal{F}}$ and $\mathcal{F}_R$ parameterized by the base varieties $\mathcal{B}$ and $\mathcal{B}_R$ as in Proposition 6.1. For any dominant weight μ the Virasoro operator L_0 induces a decomposition of the representation space $\mathcal{H}_\mu$ into a direct sum of eigenspaces $\mathcal{H}_\mu(d)$ for the eigenvalue $d + \Delta_\mu$ of L_0, where $\Delta_\mu \in \mathbf{Q}$ is the trace anomaly and $d \in \mathbf{N}$. We recall that there exists a unique (up to a scalar) bilinear pairing $(.|.) : \mathcal{H}_\mu \times \mathcal{H}_{\mu^\dagger} \to \mathbf{C}$ such that $(X(n)u|v) + (u|X(-n)v) = 0$ for any $X \in \mathfrak{sl}_2$, $n \in \mathbf{Z}$, $u \in \mathcal{H}_\mu$, $v \in \mathcal{H}_{\mu^\dagger}$ and $(.|.)$ is zero on $\mathcal{H}_\mu(d) \times \mathcal{H}_{\mu^\dagger}(d')$ if $d \neq d'$. We choose a basis $\{v_1(d), \ldots, v_{m_d}(d)\}$ of $\mathcal{H}_\mu(d)$ and let $\{v^1(d), \ldots, v^{m_d}(d)\}$ be its dual basis of $\mathcal{H}_{\mu^\dagger}(d)$ with respect to the above bilinear form. Then the element

$$\gamma_d = \sum_{i=1}^{m_d} v_i(d) \otimes v^i(d) \in \mathcal{H}_\mu(d) \otimes \mathcal{H}_{\mu^\dagger}(d) \subset \mathcal{H}_\mu \otimes \mathcal{H}_{\mu^\dagger}$$

does not depend on the basis. Given $\psi \in \mathcal{V}^\dagger_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}})$ we define an element $\widetilde{\psi} \in \mathcal{H}^\dagger_{\vec{\lambda}} \otimes \mathcal{O}_{\mathcal{B}_R} = \mathcal{H}^\dagger_{\vec{\lambda}}[[\tau]] \otimes \mathcal{O}_{\mathcal{B}}$ by the formula

$$\langle \widetilde{\psi} | \phi \rangle = \sum_{d=0}^{\infty} \langle \psi | \phi \otimes \gamma_d \rangle \tau^d, \qquad \text{for any } \phi \in \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{O}_{\mathcal{B}}.$$

Here $\langle .|. \rangle$ denotes the standard pairing between $\mathcal{H}_{\vec{\lambda},\mu,\mu^\dagger}$ and its dual $\mathcal{H}^\dagger_{\vec{\lambda},\mu,\mu^\dagger}$. It is shown in [TUY] that $\widetilde{\psi} \in \mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_R)$, hence we obtain for any $\mu \in P_l$ and any $\vec{\lambda} \in (P_l)^n$ an $\mathcal{O}_{\mathcal{B}_R}$-linear map

$$s_\mu : \mathcal{V}^\dagger_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}}) \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{O}_{\mathcal{B}_R} \longrightarrow \mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_R), \qquad \psi \mapsto \widetilde{\psi}.$$

Proposition 6.2. *For any $\mu \in P_l$ and any $\vec{\lambda} \in (P_l)^n$ the sewing map over the generic fiber $\mathcal{B}_{\overline{K}}$*

$$s_\mu : \mathcal{V}^\dagger_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}}) \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{O}_{\mathcal{B}_{\overline{K}}} \longrightarrow \mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_{\overline{K}})$$

is projectively flat for the WZW connections on both sheaves of conformal blocks.

Proof. By definition of the sewing map s_μ it is clear that $\nabla_D(\widetilde{\psi}) = 0$ if $\nabla_D(\psi) = 0$ for any local vector field D coming from $\mathcal{B}$, i.e. independent of τ. Therefore the theorem is a corollary of the following result proved in [TUY]. □

Theorem 6.3 ([TUY] Theorem 6.2.2)**.** *For any section $\psi \in \mathcal{V}^\dagger_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}})$ the multi-valued formal power series $\widehat{\psi} = \tau^{\Delta_\mu}\widetilde{\psi}$ satisfies the relation*

$$\nabla_{\tau\frac{d}{d\tau}}(\widehat{\psi}) = 0 \qquad (\mathrm{mod}\ \mathcal{O}_{\mathcal{B}_{\overline{K}}}\widehat{\psi}).$$

Remark 6.4. We note that the statement given in [TUY] Theorem 6.2.2 says that there exists a vector field $\vec{l}$ such that

$$\left(-\tau\frac{d}{d\tau} + T[\vec{l}]\right).\widehat{\psi} = 0 \qquad (\mathrm{mod}\ \mathcal{O}_{\mathcal{B}_{\overline{K}}}\widehat{\psi}),$$

which is equivalent to the above statement using the property $\theta(\vec{l}) = -\tau\frac{d}{d\tau}$. This last equality is actually proved in [TUY] Corollary 6.1.4, but there is a sign error. The correct formula of [TUY] Corollary 6.1.4 is $\theta(\vec{l}) = -\tau\frac{d}{d\tau}$, which is obtained by writing the 1-cocycle $\theta_{12}(u,\tau) = \tilde{l}'_{u,\tau|U_2} - \tilde{l}_{u,\tau|U_1}$.

Remark 6.5. By making the base change $\nu^k = \tau$, where k is the denominator of the trace anomaly Δ_μ, we obtain a section $\widehat{\psi} \in \mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_{R'})$ with $R' = \mathbf{C}[[\nu]]$ satisfying $\nabla_{\nu\frac{d}{d\nu}}(\widehat{\psi}) = 0$ (mod $\mathcal{O}_{\mathcal{B}_{\overline{K}}}\widehat{\psi}$).

Moreover, summing over all dominant weights $\mu \in P_l$ we obtain a $\mathcal{O}_{\mathcal{B}_R}$-linear isomorphism (the factorization rules, see e.g. [TUY] Theorem 6.2.6 or [U] Theorem 4.4.9)

$$\oplus s_\mu : \bigoplus_{\mu\in P_l} \mathcal{V}^\dagger_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}}) \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{O}_{\mathcal{B}_R} \xrightarrow{\sim} \mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_R).$$

Hence, the fiber over a point $b \in \mathcal{B}_{\overline{K}}$ has a direct sum decomposition (as $\overline{K}$-vector spaces)

$$\bigoplus_{\mu\in P_l} \mathcal{V}^\dagger_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}})_b \otimes_\mathbf{C} \overline{K} \xrightarrow{\sim} \mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_{\overline{K}})_b. \tag{4}$$

We denote by D the subgroup of $\mathbf{P}\mathrm{GL}(\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_{\overline{K}})_b)$ consisting of projective $\overline{K}$-linear maps preserving the direct sum decomposition (4) and by $p_\mu : \mathrm{D} \longrightarrow \mathbf{P}\mathrm{GL}(\mathcal{V}^\dagger_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}})_b \otimes_\mathbf{C} \overline{K})$ the projection onto the summand corresponding to $\mu \in P_l$.

The next proposition is an immediate consequence of the fact that the maps s_μ are projectively flat (Proposition 6.2).

Proposition 6.6. *Let $\widetilde{\mathcal{F}}$ and $\mathcal{F}_R$ be two families of curves as in Proposition 6.1. Then for any $\mu \in P_l$ and any $\vec{\lambda} \in (P_l)^n$*

(1) *the monodromy representation of the sheaf of conformal blocks $\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_{\overline{K}})$ over $\mathcal{B}_{\overline{K}}$ takes values in the subgroup* D*, i.e.,*

$$\rho_{l,\vec{\lambda}} : \pi_1(\mathcal{B}_{\overline{K}}, b) \longrightarrow \mathrm{D} \subset \mathbf{P}\mathrm{GL}(\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_{\overline{K}})_b).$$

(2) *we have a commutative diagram*

$$\begin{array}{ccc} \pi_1(\mathcal{B}_{\overline{K}}, b) & \xrightarrow{\rho_{l,\vec{\lambda}}} & \mathrm{D} \\ \downarrow \cong & & \downarrow p_\mu \\ \pi_1(\mathcal{B}, b) & \xrightarrow{\rho_{l,\vec{\lambda},\mu,\mu^\dagger}} & \mathbf{P}\mathrm{GL}(\mathcal{V}^\dagger_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}})_b \otimes_{\mathbf{C}} \overline{K}) \end{array}$$

6.3. **Proof of the Theorem.** We will now prove the theorem stated in the introduction. This theorem will be a corollary of the following more general result (Theorem 6.7) since we know by [L] assuming $g \geq 3$ that there is a projectively flat isomorphism between the two projectivized vector bundles

$$\mathbf{P}\mathcal{Z}_l \xrightarrow{\sim} \mathbf{P}\mathcal{V}^\dagger_{l,\emptyset}$$

equipped with the Hitchin connection and the WZW connection respectively. Here $\mathcal{V}^\dagger_{l,\emptyset}$ stands for the sheaf of conformal blocks $\mathcal{V}^\dagger_{l,0}(\mathcal{F})$ associated to the family $\mathcal{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1)$ of curves with one point labeled with the trivial representation $\lambda_1 = 0$ (propagation of vacua).

Theorem 6.7. *Assume that the level $l \neq 1, 2, 4, 8$. For the following values of g, n and $\vec{\lambda} \in (P_l)^n$ there exists a family $\mathcal{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_n)$ of smooth connected n-pointed projective curves of genus g such that the projective monodromy representation $\rho_{l,\vec{\lambda}}$ of the WZW connection on the sheaf of conformal blocks*

$$\rho_{l,\vec{\lambda}} : \pi_1(\mathcal{B}, b) \longrightarrow \mathbf{P}\mathrm{GL}(\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F})_b)$$

has an element of infinite order in its image:

(1) $g \geq 0$, $n = 2m \geq 2$, $g + m \geq 2$ *and* $\vec{\lambda} = \vec{1}_{2m}$,
(2) $g \geq 2$, $n = 1$ *and* $\vec{\lambda} = 0$.

Proof. We will prove part (1) of the theorem by induction on the genus g and the number of points $2m$. The first case $g = 0, m = 2$ is given by Proposition 5.1: we can take $\mathcal{B} = \mathfrak{M}_{0,4} = \mathbf{P}^1_{\mathbf{C}} \setminus \{0, 1, \infty\}$ with the universal family $\mathcal{F}_4^{univ}$ of 4-pointed curves (3). Suppose now that the theorem holds for curves of genus $g = 0$ with $2m$ points. Let $\mathcal{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_{2m})$ be a family of $2m$-pointed smooth connected curves of genus 0 having an element of infinite order in the image of the monodromy representation of $\mathcal{V}^\dagger_{l,\vec{1}_{2m}}(\mathcal{F})$. Consider the family $\widetilde{\mathcal{F}}$ given by the disjoint union $\mathcal{F} \cup \mathcal{F}_4^{univ}$, i.e. the family parameterized by $\mathcal{B} \times \mathfrak{M}_{0,4}$ of $2m + 4$ marked curves equal to the disjoint union $\mathbf{P}^1 \cup \mathbf{P}^1$ with $2m$ marked points $s_1, \ldots, s_{2m}$ on the first $\mathbf{P}^1$ and 4 marked points t_0, t_1, t_∞, t on the second $\mathbf{P}^1$. We then consider the family $\mathcal{F}_0$ of nodal reducible curves obtained from $\widetilde{\mathcal{F}}$ by identifying the two points $s_{2m}(b) \in \mathbf{P}^1$ and $t(b') \in \mathbf{P}^1$ for each $(b, b') \in \mathcal{B} \times \mathfrak{M}_{0,4}$ as well as a family $\mathcal{F}_R$ satisfying the conditions of Proposition 6.1. Then the family $\mathcal{F}_{\overline{K}}$ parameterizes $(2m + 2)$-marked smooth connected curves of genus 0 and by the direct sum decomposition (4) the $\overline{K}$-vector space $\mathcal{V}^\dagger_{l,\vec{1}_{2m+2}}(\mathcal{F}_{\overline{K}})_{(b,b')}$ with $(b, b') \in \mathcal{B} \times \mathfrak{M}_{0,4}$

contains the direct summand $\mathcal{V}^\dagger_{l,\vec{1}_{2m+2},1,1}(\widetilde{\mathcal{F}})_{(b,b')} \otimes_{\mathbf{C}} \overline{K}$ corresponding to $\mu = 1 \in P_l$. Note that $\mu^\dagger = \mu$ for any dominant weight μ of $\mathfrak{sl}(2)$. Moreover, by [U] Proposition 3.1.10 we have a decomposition

$$\mathcal{V}^\dagger_{l,\vec{1}_{2m+2},1,1}(\widetilde{\mathcal{F}})_{(b,b')} \cong \mathcal{V}^\dagger_{l,\vec{1}_{2m}}(\mathcal{F})_b \otimes \mathcal{V}^\dagger_{l,\vec{1}_4}(\mathcal{F}_4^{univ})_{b'}.$$

By the induction hypothesis $\mathcal{V}^\dagger_{l,\vec{1}_{2m}}(\mathcal{F})_b \neq \{0\}$. So the monodromy representation of the conformal block $\mathcal{V}^\dagger_{l,\vec{1}_{2m}}(\mathcal{F}) \otimes \mathcal{V}^\dagger_{l,\vec{1}_4}(\mathcal{F}_4^{univ})$ has an element of infinite order in its image (by Proposition 5.1 or by the induction hypothesis). Hence, from Proposition 6.6(2) for $\mu = 1$ we deduce that the monodromy representation of $\mathcal{V}^\dagger_{l,\vec{1}_{2m+2}}(\mathcal{F}_{\overline{K}})$ also has an element of infinite order in its image. By induction this shows part (1) of the theorem for $g = 0$ and any $m \geq 2$.

In order to complete the proof of part (1) we assume that the theorem holds for curves of genus g with $2k$ marked points for any k such that $g + k \geq 2$. We take $k = m + 1$ and let $\widetilde{\mathcal{F}} = (\widetilde{\pi} : \widetilde{\mathcal{C}} \to \mathcal{B}; s_1, \ldots, s_{2m+2})$ be a versal family of $2m+2$-pointed smooth connected curves of genus g having an element of infinite order in the image of the monodromy representation of $\mathcal{V}^\dagger_{l,\vec{1}_{2m+2}}(\widetilde{\mathcal{F}})$. We now apply Proposition 6.1 to the family $\widetilde{\mathcal{F}}$. As before, the direct sum decomposition (4) the $\overline{K}$-vector space $\mathcal{V}^\dagger_{l,\vec{1}_{2m}}(\mathcal{F}_{\overline{K}})_b$ with $b \in \mathcal{B}$ contains the direct summand $\mathcal{V}^\dagger_{l,\vec{1}_{2m+2}}(\widetilde{\mathcal{F}})_b \otimes_{\mathbf{C}} \overline{K}$ corresponding to $\mu = 1 \in P_l$. By the induction hypothesis and by Proposition 6.6(2) for $\mu = 1$, we obtain that the monodromy representation of $\mathcal{V}^\dagger_{l,\vec{1}_{2m}}(\mathcal{F}_{\overline{K}})$ has an element of infinite order in its image. This proves the statement for $2m$-marked curves of genus $g+1$ with $(g+1) + m \geq 2$.

Finally, in order to prove part (2) we shall use the existence of a family $\widetilde{\mathcal{F}} = (\widetilde{\pi} : \widetilde{\mathcal{C}} \to \mathcal{B}; s_1, s_2, s_3)$ of 3-pointed smooth curves of genus $g \geq 1$ having an element of infinite order in the image of the monodromy representation on the conformal block $\mathcal{V}^\dagger_{l,\vec{\lambda}}(\widetilde{\mathcal{F}})$ with $\vec{\lambda} = (0, 1, 1)$. The existence of such a family is shown by induction exactly as in part (1) starting the induction with the 5-pointed family

$$\mathcal{F}_5^{univ} = (\pi : \mathcal{C} = \mathfrak{M}_{0,4} \times \mathbf{P}^1 \to \mathfrak{M}_{0,4}; -t, t_0, t_1, t, t_\infty)$$

and the sheaf of conformal blocks $\mathcal{V}^\dagger_{l,0,\vec{1}_4}(\mathcal{F}_5^{univ})$. Note that there is a projectively flat isomorphism (propagation of vacua, see e.g. [U] Theorem 3.3.1)

$$\mathcal{V}^\dagger_{l,0,\vec{1}_4}(\mathcal{F}_5^{univ}) \xrightarrow{\sim} \mathcal{V}^\dagger_{l,\vec{1}_4}(\mathcal{F}_4^{univ}).$$

We then apply Proposition 6.1 to the 3-pointed family $\widetilde{\mathcal{F}}$, which produces a 1-pointed family $\mathcal{F}_{\overline{K}}$ over $\mathcal{B}_{\overline{K}}$. By the same argument as in part (1) we show that $\mathcal{F}_{\overline{K}}$ has infinite monodromy. □

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