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Abstract

We consider a magnetic Laplacian $-\Delta_A = (id+A)^\star(id+A)$ on a noncompact hyperbolic surface $\mathbf{M}$ with finite area. A is a real one-form and the magnetic field dA is constant in each cusp. When the harmonic component of A satifies some quantified condition, the spectrum of $-\Delta_A$ is discrete. In this case we prove that the counting function of the eigenvalues of $-\Delta_A$ satisfies the classical Weyl formula, even when $dA = 0$. [1]

1 Introduction

We consider a smooth, connected, complete and oriented Riemannian surface $(\mathbf{M}, g)$ and a smooth, real one-form A on $\mathbf{M}$. We define the magnetic Laplacian, the Bochner Laplacian

$$-\Delta_A = (i\ d + A)^\star (i\ d + A)\ , \tag{1.1}$$

$$(\ (i\ d + A)u = i\ du + uA\ ,\ \forall\ u\ \in\ C_0^\infty(\mathbf{M};\mathbb{C})\ .$$

The magnetic field is the exact two-form $\rho_B = dA$.
If dm is the Riemannian measure on $\mathbf{M}$, then

$$\rho_B = \widetilde{\mathbf{b}}\, dm\ , \quad \text{with} \quad \widetilde{\mathbf{b}} \in C^\infty(\mathbf{M};\mathbb{R})\ . \tag{1.2}$$

[1] *Keywords*: spectral asymptotics, magnetic field, Aharanov-Bohm, hyperbolic surface.

The magnetic intensity is $\mathbf{b} = |\widetilde{\mathbf{b}}|$.

It is well known, (see [Shu]), that $-\Delta_A$ has a unique self-adjoint extension on $L^2(\mathbf{M})$, containing in its domain $C_0^\infty(\mathbf{M};\mathbb{C})$, the space of smooth and compactly supported functions. The spectrum of $-\Delta_A$ is gauge invariant : for any $f \in C^1(\mathbf{M};\mathbb{R})$, $-\Delta_A$ and $-\Delta_{A+df}$ are unitarily equivalent, hence they have the same spectrum.

We are interested in constant magnetic fields on $\mathbf{M}$ in the case when $(\mathbf{M},\ g)$ is a non-compact geometrically finite hyperbolic surface of finite area; (see [Per] or [Bor] for the definition and the related references). More precisely

$$\mathbf{M} = \bigcup_{j=0}^{J} M_j \tag{1.3}$$

where the M_j are open sets of $\mathbf{M}$, such that the closure of M_0 is compact, and (when $J \geq 1$) the other M_j are cuspidal ends of $\mathbf{M}$.

This means that, for any j, $1 \leq j \leq J$, there exist strictly positive constants a_j and L_j such that M_j is isometric to $\mathbb{S}\times]a_j^2, +\infty[$, equipped with the metric

$$ds_j^2 = y^{-2}(\ L_j^2\ d\theta^2 + dy^2\ )\ ; \tag{1.4}$$

($\mathbb{S} = \mathbb{S}^1$ is the unit circle and $M_j \cap M_k = \emptyset$ if $j \neq k$) .
Let us choose some $z_0 \in M_0$ and let us define

$$d\ :\ \mathbf{M} \to \mathbb{R}_+\ ; \quad d(z) = d_g(z, z_0)\ ; \tag{1.5}$$

$d_g(\ .\ ,\ .\ )$ denotes the distance with respect to the metric g.

For any $b \in \mathbb{R}^J$, there exists a one-form A , such that the corresponding magnetic field dA satisfies

$$dA = \widetilde{\mathbf{b}}(z)dm \quad \text{with} \quad \widetilde{\mathbf{b}}(z) = b_j\ \forall\ z \in M_j\ . \tag{1.6}$$

The following statement on the essential spectrum is proven in [Mo-Tr1] :

Theorem 1.1 *Assume (1.3) and (1.6). Then for any* j , $1 \leq j \leq J$ *and for any* $z \in M_j$ *there exists a unique closed curve through* z , $\mathcal{C}_{j,z}$ *in* $(M_j,\ g)$, *not contractible and with zero* $g-$*curvature. (*$\mathcal{C}_{j,z}$ *is called an horocycle of* M_j *). The following limit exists and is finite:*

$$[A]_{M_j} = \lim_{d(z)\to+\infty} \int_{\mathcal{C}_{j,z}} A\ . \tag{1.7}$$

If $J^A \ = \ \{j \in \mathbb{N}\ ,\ 1 \leq j \leq J \ s.t.\ [A]_{M_j} \in 2\pi\mathbb{Z}\ \} \ \neq \ \emptyset$ *, then*

$$\mathrm{sp}_{ess}(-\Delta_A) \ = \ [\frac{1}{4} + \min_{j \in J^A} b_j^2 \ , \ +\infty[\ . \tag{1.8}$$

If $J^A \ = \ \emptyset$ *, then* $\mathrm{sp}_{ess}(-\Delta_A) \ = \ \emptyset$ *:*
$-\Delta_A$ *has purely discrete spectrum, (its resolvent is compact).*

When the magnetic Laplacian $-\Delta_A$ has purely discrete spectrum, it is called a magnetic bottle, (see [Col2]).

If $A = df + A^H + A^\delta$ is the Hodge decomposition of A with A^H harmonic, ($dA^H = 0$ and $d^\star A^H = 0$) , then $\forall\ j$, $[A]_{M_j} = [A^H]_{M_j}$, so the discreteness of the spectrum of $-\Delta_A$ depends only on the harmonic component of A . So one can see the case $J^A = \emptyset$ as an Aharonov-Bohm phenomenon [Ah-Bo], a situation where the magnetic field dA is not sufficicient to describe $-\Delta_A$ and the use of the magnetic potential A is essential : we can have magnetic bottle with null intensity.

2 The Weyl formula in the case of finite area with a non-integer class one-form

Here we are interested in the pure point part of the spectrum. We assume that $J^A \ = \ \emptyset$, then the spectrum of $-\Delta_A$ is discrete. In this case, we denote by $(\lambda_j)_j$ the increasing sequence of eigenvalues of $-\Delta_A$, (each eigenvalue is repeated according to its multiplicity). Let

$$N(\lambda, -\Delta_A) \ = \ \sum_{\lambda_j < \lambda} 1\ . \tag{2.1}$$

We will show that the asymptotic behavior of $N(\lambda)$ is given by the Weyl formula :

Theorem 2.1 *Consider a geometrically finite hyperbolic surface* $(\mathbf{M},\ g)$ *of finite area, and assume (1.6) with* $J^A \ = \ \emptyset$ *, (see (1.7 for the definition).*

Then

$$N(\lambda, -\Delta_A) \ = \ \lambda\frac{|\mathbf{M}|}{4\pi} \ + \ \mathbf{O}(\sqrt{\lambda}\ln\lambda)\ . \tag{2.2}$$

Remark 2.2 *As J^A depends only on the harmonic component of A , J^A is not empty when* $\mathbf{M}$ *is simply connected. In [Go-Mo] there are some results close to Theorem 2.1, but for simply connected manifolds.*

The cases where the magnetic field prevails were studied in [Mo-Tr1] and in [Mo-Tr2].

Proof of Theorem 2.1. Any constant depending only on the b_j and on $\min_{1\leq j\leq J}\inf_{k\in\mathbb{Z}} |[A]_{M_j} - 2k\pi|$ will be denoted invariably C .

Consider a cusp $M = M_j = \mathbb{S}\times]\alpha^2, +\infty[$ equipped with the metric $ds^2 = L^2e^{-2t}d\theta^2 + dt^2$ for some $\alpha > 0$ and $L > 0$.

Let us denote by $-\Delta_A^M$ the Dirichlet operator on M , associated to $-\Delta_A$. The first step will be to prove that

$$N(\lambda, -\Delta_A^M) = \lambda\frac{|M|}{4\pi} + \mathbf{O}(\sqrt{\lambda}\ln\lambda) . \tag{2.3}$$

Since $-\Delta_A^M$ and $-\Delta_{A+d\varphi+kd\theta}^M$ are gauge equivalent for any $\varphi \in C^\infty(\overline{\mathbf{M}}; \mathbb{R})$ and any $k \in \mathbb{Z}$, we can assume that

$$-\Delta_A^M = L^{-2}e^{2t}(D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}, \quad \text{with} \quad A_1 = -\xi \pm bLe^{-t} , \ \xi \in]0, 1[,$$

$(b = b_j , \ 2\pi\xi - [A]_M \in 2\pi\mathbb{Z})$. Then we get that

$$\mathrm{sp}(-\Delta_A^M) = \bigcup_{\ell\in\mathbb{Z}} \mathrm{sp}(P_\ell) \ ; \ P_\ell = D_t^2 + \frac{1}{4} + \left(e^t\frac{(\ell+\xi)}{L} \pm b\right)^2 ,$$

for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha^2, +\infty[$. This implies that

$$N(\lambda, -\Delta_A^M) = \sum_{\ell\in\mathbb{Z}} N(\lambda, P_\ell) = \sum_{\ell\in X_\lambda} N(\lambda, P_\ell) \tag{2.4}$$

with $X_\lambda = \{\ell \ / \ e^{\alpha^2}\frac{|\ell+\xi|}{L} < \sqrt{\lambda - 1/4} - b \}$.

Denoting by Q_ℓ the Dirichlet operator on I associated to

$$Q_\ell = D_t^2 + \frac{1}{4} + \frac{(\ell+\xi)^2}{L^2}e^{2t} ,$$

we easily get that

$$Q_\ell - C\sqrt{Q_\ell} \leq P_\ell \leq Q_\ell + C\sqrt{Q_\ell} . \tag{2.5}$$

Therefore one can find a constant $C(b)$, depending only on b , such that, for any $\lambda >> 1 + C(b)$,

$$N(\lambda - \sqrt{\lambda}C(b), Q_\ell) \ \leq \ N(\lambda, P_\ell) \ \leq \ N(\lambda + \sqrt{\lambda}C(b), Q_\ell) \ . \qquad (2.6)$$

Following Titchmarsh's method ([Tit], Theorem 7.4) we establish the following bounds

Lemma 2.3 *There exists $C > 1$ so that for any $\mu >> 1$ and any $\ell \in X_\mu$,*

$$w_\ell(\mu) - \pi \ \leq \ \pi N(\mu - \frac{1}{4}, Q_\ell) \ \leq \ w_\ell(\mu) + \frac{1}{12}\ln\mu + C \ , \qquad (2.7)$$

with

$$\begin{aligned} w_\ell(\mu) \ &= \ \int_{\alpha^2}^{+\infty} \left[\mu - \frac{(\ell+\xi)^2}{L^2} e^{2t}\right]_+^{1/2} dt \qquad (2.8) \\ &= \ \int_{\alpha^2}^{T_{\mu,L}} \left[\mu - \frac{(\ell+\xi)^2}{L^2} e^{2t}\right]_+^{1/2} dt \ ; \end{aligned}$$

$(e^{T_{\mu,L}} = L\sqrt{\mu}/(\inf_{k\in\mathbb{Z}} |\xi - k|) \ ) \ .$

Proof of Lemma 2.3

The lower bound is easily obtained (see [Tit], Formula 7.1.2 p 143) so we focus on the upper bound.

Let us define $V_\ell = \frac{(\ell+\xi)^2}{L^2}e^{2t}$ and denote by ϕ_μ^ℓ a solution of $Q_\ell\phi = (\mu - \frac{1}{4})\phi$. Consider x_ℓ and y_ℓ so that $V_\ell(x_\ell) = \mu$ and $V_\ell(y_\ell) = \nu$, for a given $0 < \nu < \mu$ to be determined later. We denote by m the number of zeros of ϕ_μ^ℓ on $]\alpha^2, y_\ell[$. Recall that the number n of zeros of ϕ_μ^ℓ on $]\alpha^2, x_\ell[$ is equal to $N(\mu - \frac{1}{4}, Q_\ell)$. Applying Lemma 7.3 p 146 in [Tit] we deduce that

$$m\pi = \int_{\alpha^2}^{y_\ell} [\mu - V_\ell]^{1/2} \, dt + R_\ell$$

with $R_\ell = \frac{1}{4}\ln(\mu - V_\ell(\alpha^2)) - \frac{1}{4}\ln(\mu - V_\ell(y_\ell)) + \pi$, hence

$$|n\pi - \int_{\alpha^2}^{x_\ell} [\mu - V_\ell]^{1/2} \, dt| \leq (x_\ell - y_\ell)(\mu - \nu)^{1/2} + R_\ell + (n - m)\pi$$

According to the Sturm comparison theorem ([Tit], p 107-108), we have

$$(n - m)\pi \leq (x_\ell - y_\ell)(\mu - \nu)^{1/2}$$

and

$$|n\pi - \int_{\alpha^2}^{x_\ell} [\mu - V_\ell]^{1/2}\, dt| \le \ln(\frac{\mu}{\nu})(\mu-\nu)^{1/2} + \frac{1}{4}\ln\mu - \frac{1}{4}\ln(\mu-\nu) + 2\pi$$

Now taking $\nu = \mu - \mu^{2/3}$ we get the desired estimate.

In view of (2.4) we now compute $\sum_{\ell\in\mathbb{Z}} w_\ell(\mu)$. We first get the following

Lemma 2.4 *There exists $C > 1$ such that, for any $\mu >> 1$ and any $t \in [\alpha^2, T_{\mu,L}]$,*

$$\left| \int_{\mathbb{R}} \left[\mu - \frac{(x+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} dx \ - \sum_{\ell\in\mathbb{Z}} \left[\mu - \frac{(\ell+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} \right| \ \le\ C(\sqrt{\mu} + \frac{e^t}{L}) \ .$$

This leads to

Lemma 2.5 *There exists $C > 1$ such that, for any $\mu >> 1$,*

$$\left| \int_{\alpha^2}^{T_{\mu,L}} \int_{\mathbb{R}} \left[\mu - \frac{(x+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} dxdt \ - \sum_{\ell\in\mathbb{Z}} w_\ell(\mu) \right| \ \le\ C\sqrt{\mu}\ln\mu \ .$$

We now compute the integral in the left-hand side.
Making the change of variables $y^2 = \frac{(x+\xi)^2}{L^2\mu}e^{2t}$ we obtain that it is equal to $\mu L \int_{\alpha^2}^{T_{\mu,L}} e^{-t} dt \int_{\mathbb{R}} [1-x^2]_+^{1/2}\, dx$, so we get

Lemma 2.6 *There exists $C > 1$ such that, for any $\mu >> 1$,*

$$\left| \int_{\alpha^2}^{T_{\mu,L}} \int_{\mathbb{R}} \left[\mu - \frac{(x+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} dxdt \ - \mu L e^{-\alpha^2} \int_{\mathbb{R}} \left[1-x^2\right]_+^{1/2} dx \right| \ \le\ C\sqrt{\mu} \ .$$

Noticing that $|M| = 2\pi L e^{-\alpha^2}$ and using Lemmas 2.5 and 2.6 we have

Lemma 2.7

$$\frac{1}{\pi}\sum_{\ell} \in w_\ell(\mu) \ = \ \frac{|M|}{4\pi}\mu \ + \ \mathbf{O}(\sqrt{\mu}\ln\mu) \ , \quad \text{as} \quad \mu \to +\infty \ .$$

In view of (2.4),(2.6) and (2.7) Lemma 2.7 ends the proof of formula (2.3). Now it remains to consider the whole surface $\mathbf{M}$.

We have : $\mathbf{M} \ = \ \left(\bigcup_{j=0}^{J} M_j \right)$

where the M_j are open sets of $\mathbf{M}$, such that the closure of M_0 is compact, and the other M_j are cuspidal ends of $\mathbf{M}$ and

$M_j \cap M_k = \emptyset$, if $j \neq k$. We denote $M_0^0 = \mathbf{M} \setminus (\bigcup_{j=1}^{J} \overline{M_j})$, then

$$\mathbf{M} \ = \ \overline{M_0^0} \bigcup \left(\bigcup_{j=1}^{J} \overline{M_j} \right) \ . \tag{2.9}$$

Let us denote respectively by $-\Delta_{A,D}^{\Omega}$ and by $-\Delta_{A,N}^{\Omega}$ the Dirichlet operator and the Neumann-like operator on an open set Ω of $\mathbf{M}$ associated to $-\Delta_A$. The minimax principle and (2.9) imply that

$$N(\lambda, -\Delta_{A,D}^{M_0^0}) + \sum_{1\leq j \leq J} N(\lambda, -\Delta_{A,D}^{M_j}) \leq N(\lambda, -\Delta_A) \tag{2.10}$$

$$\leq N(\lambda, -\Delta_{A,N}^{M_0^0}) + \sum_{1\leq j \leq J} N(\lambda, -\Delta_{A,N}^{M_j})$$

The Weyl formula with remainder, (see [Hor] for Dirichlet boundary condition and [Sa-Va] p. 9 for Neumann-like boundary condition), gives that

$$\left\{ \begin{array}{l} N(\lambda, -\Delta_{A,D}^{M_0^0}) = (4\pi)^{-1} |M_0^0| \lambda + \mathbf{O}(\sqrt{\lambda}) \\ N(\lambda, -\Delta_{A,N}^{M_0^0}) = (4\pi)^{-1} |M_0^0| \lambda + \mathbf{O}(\sqrt{\lambda}) \end{array} \right\} . \tag{2.11}$$

The asymptotic formula for $N(\lambda, -\Delta_{A,N}^{M_j})$,

$$N(\lambda, -\Delta_{A,N}^{M_j}) \ = \ \lambda \frac{|M_j|}{4\pi} \ + \ \mathbf{O}(\sqrt{\lambda} \ln \lambda) \ , \tag{2.12}$$

is obtained as for the Dirichlet case (2.3) (with $M = M_j$), by noticing that $N(\lambda, P_{\ell,D}) \leq N(\lambda, P_{\ell,N}) \leq N(\lambda, P_{\ell,D}) + 1$, where $P_{\ell,D}$ and $P_{\ell,N}$ are Dirichlet and Neumann operators on a half-line $I =]\alpha^2, +\infty[$, associated to the same differential Schödinger operator $P_\ell = D_t^2 + \frac{1}{4} + (e^t \frac{(\ell + \xi)}{L} \pm b)^2$.

We get (2.2) from (2.3) with $M = M_j$, (2.12), (for any $j = 1, \ldots, J)$, (2.10) and (2.11).$\square$

Remark 2.8 *Theorem 2.1 still holds if the metric of* $\mathbf{M}$ *is modified in a compact set.*

When $A = 0$, $-\Delta = -\Delta_0$ *has embedded eigenvalues in its essential spectrum,* $(sp_{ess}(-\Delta) = [\frac{1}{4}, +\infty[)$. *If* $N_{ess}(\lambda, -\Delta)$ *denotes the number of these eigenvalues in* $[\frac{1}{4}, \lambda[$, *then it is well known that one has an upper bound* $N_{ess}(\lambda, -\Delta) \leq \lambda \frac{|\mathbf{M}|}{4\pi}$; *see [Col1] and [Hej] for the history and related improvement of the upper bound.*
Recently [Mul] established a sharp asymptotic formula, similar to our case,

$$N_{ess}(\lambda, -\Delta) = \lambda \frac{|\mathbf{M}|}{4\pi} + \mathbf{O}(\sqrt{\lambda} \ln \lambda) ,$$

for some particular $\mathbf{M}$.

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