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Abstract

We consider a magnetic Laplacian $-\Delta_A = (id + A)^\star (id + A)$ on a hyperbolic surface $\mathbf{M}$, when the magnetic field dA is infinite at the boundary at infinity. We prove that the counting function of the eigenvalues has a particular asymptotic behavior when $\mathbf{M}$ has an infinite area. [1]

1 Introduction

We consider a smooth, connected, complete and oriented Riemannian surface $(\mathbf{M}, g)$ and a smooth, real one-form A on $\mathbf{M}$. We define the magnetic Laplacian

$$\begin{gathered} -\Delta_A \ = \ (i\ d + A)^\star (i\ d + A)\ , \\ (\ (i\ d + A)u = i\ du + uA\ ,\ \forall\ u\ \in\ C_0^\infty(\mathbf{M};\mathbb{C})\ )\ . \end{gathered} \tag{1.1}$$

The magnetic field is the exact two-form $\rho_B \ = \ dA$.

If dm is the Riemannian measure on $\mathbf{M}$, then

$$\rho_B \ = \ \widetilde{\mathbf{b}}\, dm\ , \quad \text{with} \quad \widetilde{\mathbf{b}}\ \in\ C^\infty(\mathbf{M};\mathbb{R})\ . \tag{1.2}$$

The magnetic intensity is $\mathbf{b} \ = \ |\widetilde{\mathbf{b}}|$.

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It is well known, (see [Shu]), that $-\Delta_A$ has a unique self-adjoint extension on $L^2(\mathbf{M})$, containing in its domain $C_0^\infty(\mathbf{M};\mathbb{C})$, the space of smooth and compactly supported functions.

When $\mathbf{b}$ is infinite at the infinity, (with some additional assumption), the spectrum of $-\Delta_A$ is discrete, and we denote by $(\lambda_j)_j$ the increasing sequence of eigenvalues of $-\Delta_A$, (each eigenvalue is repeated according to its multiplicity). Let

$$N(\lambda) \;=\; \sum_{\lambda_j<\lambda} 1\;. \tag{1.3}$$

We are interested by the hyperbolic surfaces $\mathbf{M}$, when the curvature of $\mathbf{M}$ is constant and negative.

In this case, when $\mathbf{M}$ has finite area, the asymptotic behavior of $N(\lambda)$ seems to be the Weyl formula : $N(\lambda)\;\sim_{+\infty}\;\dfrac{\lambda}{4\pi}|\mathbf{M}|$.
S. Golénia and S. Moroianu in [Go-Mo] have such examples.

In the case of the Poincaré half-plane, $\mathbf{M}=\mathbb{H}$, we prove in [Mo-Tr] that the Weyl formula is not valid : $\lim_{\lambda\to+\infty}\lambda^{-1}N(\lambda)\;=\;+\infty$.
For example when $\mathbf{b}(z)=a_0^2(x/y)^{2m_0}+a_1^2y^{m_1}+a_2^2/y^{m_2}$, $a_j>0$ and $m_j\in\mathbb{N}^\star$, then

$$N(\lambda)\;\sim_{+\infty}\;\lambda^{1+1/(2m_0)}\ln(\lambda)\alpha(m_0,m_1,m_2)\;.$$

In this paper, we are interested by the hyperbolic surfaces with infinite area. When $\mathbf{M}$ is a geometrically finite hyperbolic surface of infinite area and when the above example is arranged for this new situation, (m_0 is absent, m_1 appears in the cusps and m_2 in the funnels), we get

$$N(\lambda)\;\sim_{+\infty}\;\lambda^{1+1/m_2}\alpha(m_2)\;:$$

the cusps do not contribute to the leading part of $N(\lambda)$.

2 Main result

We assume that $(\mathbf{M},\;g)$ is a smooth connected Riemannian manifold of dimension two, which is a geometrically finite hyperbolic surface of infinite area; (see [Per] or [Bor] for the definition and the related references). More precisely

$$\mathbf{M}\;=\;\left(\bigcup_{j=0}^{J_1}M_j\right)\bigcup\left(\bigcup_{k=1}^{J_2}F_k\right)\;; \tag{2.1}$$

where the M_j and the F_k are open sets of $\mathbf{M}$, such that the closure of M_0 is compact, and if $J_1 > 0$, the other M_j are cuspidal ends of $\mathbf{M}$, and the F_k are funnel ends of $\mathbf{M}$.

This means that, for any j, $1 \leq j \leq J_1$, there exist strictly positive constants a_j and L_j such that M_j is isometric to $\mathbb{S}\times]a_j^2, +\infty[$, equipped with the metric

$$ds_j^2 \ = \ y^{-2}(\ L_j^2 \ d\theta^2 \ + \ dy^2 \ ) \ ; \tag{2.2}$$

($\mathbb{S} \ = \ \mathbb{S}^1$ is the unit circle.)

In the same way, for any k, $1 \leq k \leq J_2$,there exist strictly positive constants α_k and τ_k such that F_k is isometric to $\mathbb{S}\times]\alpha_k^2, +\infty[$, equipped with the metric

$$ds_k^2 \ = \ \tau_k^2 \cosh^2(t) d\theta^2 \ + \ dt^2 \ ; \tag{2.3}$$

moreover, for any two integers $j, \ k \ > \ 0$, we have $M_j \cap F_k \ = \ \emptyset$ and $M_j \cap M_k \ = \ F_j \cap F_k \ = \ \emptyset$ if $j \neq k$.

Let us choose some $z_0 \ \in \ M_0$ and let us define

$$d \ : \ \mathbf{M} \ \to \ \mathbb{R}_+ \ ; \quad d(z) \ = \ d_g(z, z_0) \ ; \tag{2.4}$$

$d_g(\ . \ , \ . \ )$ denotes the distance with respect to the metric g.

We assume the smooth one-form A to be given such that the magnetic field $\widetilde{\mathbf{b}}$ satisfies

$$\lim_{d(z)\to\infty} \mathbf{b}(z) \ = \ +\infty \ . \tag{2.5}$$

If $J_1 \ > \ 0$, there exists a constant $C_1 \ > \ 0$ such

$$|X\widetilde{\mathbf{b}}(z)| \ \leq \ C_1(\mathbf{b}(z) \ + \ 1)e^{d(z)}|X|_g \ ; \tag{2.6}$$

$$\forall \ z \ \in \ M_j \ , \ \forall \ X \ \in \ T_z\mathbf{M} \ \text{ and } \forall \ j = 1, \dots J_1 \ .$$

There exists a constant $C_2 \ > \ 0$ such

$$|X\widetilde{\mathbf{b}}(z)| \ \leq \ C_2(\mathbf{b}(z) \ + \ 1)|X|_g \ ; \tag{2.7}$$

$$\forall \ z \ \in \ F_k \ , \ \forall \ X \ \in \ T_z\mathbf{M} \ \text{ and } \forall \ k = 1, \dots J_2 \ .$$

For any self-adjoint operator P , and for any real λ , we will denote by $E_\lambda(P)$ its spectral projection, and when its trace is finite we will denote it by

$$N(\lambda; P) \ = \ Tr(E_\lambda(P)) \ .$$

$N(\lambda; P)$ is the number of eigenvalues of P , (counted with their multiplicity), which are in $]-\infty, \lambda[$.

Theorem 2.1 *Under the above assumptions,* $-\Delta_A$ *has a compact resolvent and for any* $\delta \in]\frac{1}{3}, \frac{2}{5}[$ *, there exists a constant* $C > 0$ *such that*

$$\frac{1}{2\pi}\int_{\mathbf{M}}(1 - \frac{C}{(\mathbf{b}(m)+1)^{(2-5\delta)/2}})\ \mathcal{N}(\lambda(1 - C\lambda^{-3\delta+1}) - \frac{1}{4}, \mathbf{b}(m))\ dm$$

$$\leq\ N(\lambda, -\Delta_A)\ \leq \tag{2.8}$$

$$\frac{1}{2\pi}\int_{\mathbf{M}}(1 + \frac{C}{(\mathbf{b}(m)+1)^{(2-5\delta)/2}})\ \mathcal{N}(\lambda(1 + C\lambda^{-3\delta+1}) - \frac{1}{4}, \mathbf{b}(m))\ dm$$

where

$$\mathcal{N}(\mu, \mathbf{b}(m))\ =\ \mathbf{b}(m)\sum_{k=0}^{+\infty}[\mu - (2k+1)\mathbf{b}(m)]_+^0 \quad \textit{if}\ \ \mathbf{b}(m)\ >\ 0\ ,$$

and

$$\mathcal{N}(\mu, \mathbf{b}(m))\ =\ \mu/2 \quad \textit{if}\ \ \mathbf{b}(m)\ =\ 0\ .$$

$[\rho]_+^0$ *is the Heaviside function:*

$$[\rho]_+^0\ =\ \begin{cases} 1\ , & \text{if} \quad \rho > 0 \\ 0\ , & \text{if} \quad \rho \leq 0\ . \end{cases}$$

The Theorem remains true if we replace $\int_{\mathbf{M}}$ by $\sum_{k=1}^{J_2}\int_{F_k}$, due to the fact that the other parts are bounded by $C\lambda$.

Corollary 2.2 *Under the assumptions of Theorem 2.1 and if the function*

$$\omega(\mu)\ =\ \int_{\mathbf{M}}[\mu - \mathbf{b}(m)]_+^0 dm$$

satisfies, $\exists\, C_1 > 0$ s.t. $\forall\, \mu > C_1$, $\forall\, \tau \in]0, 1[$,

$$\omega\ ((1+\tau)\ \mu) - \omega(\mu) \leq C_1\ \tau\ \omega(\mu)\ , \tag{2.9}$$

then

$$N(\lambda; -\Delta_A)\ \sim\ \frac{1}{2\pi}\int_{\mathbf{M}}\mathcal{N}(\lambda - \frac{1}{4}, \mathbf{b}(m))\ dm\ . \tag{2.10}$$

For example this allows us to consider magnetic fields of the following type:

$$\text{on } F_k \; , \quad \mathbf{b}(\theta,t) \; = \; p_k(\; 1/\cosh(t)\;) \; ,$$
$$\text{and on } M_j \; , \; j>0 \; , \; \mathbf{b}(\theta,y) \; = \; q_j(y) \; ,$$

where the $p_k(s)$ and the $q_j(s)$ are, for large s, polynomial functions of order ≥ 1 . In this case, if d is the largest order of the $p_k(s)$, then

$$N(\lambda; -\Delta_A) \; \sim \; \alpha \lambda^{1+1/d} \; ,$$

for some constant $\alpha > 0$, depending only on the funnels F_k where the order of $p_k(s)$ is d .

3 Estimate for Dirichlet operators

3.1 The main propositions

In this section, we consider some particular open set Ω of $\mathbf{M}$ with smooth boundary. To Ω and $-\Delta_A$, we associate the Dirichlet operator $-\Delta_A^\Omega$, and we estimate $N(\lambda; -\Delta_A^\Omega)$.

Proposition 3.1 *Let Ω an open set of M_0 with smooth boundary. Then there exists a constant $C_\Omega > 0$ s.t.*

$$\left| N(\lambda; -\Delta_A^\Omega) \; - \; \frac{|\Omega|}{4\pi}\lambda \right| \; \leq \; C_\Omega \sqrt{\lambda} \; ; \quad \forall \; \lambda > 1 \; .$$

As $\overline{\Omega}$ is compact, the above estimate is well known. See for example Theorem 29.3.3 in [Hor].

Proposition 3.2 *Let $j > 0$ and Ω an open set of the cusp M_j, isometric to $\mathbb{S}\times]a^2, +\infty[$, equipped with the metric*

$$ds^2 \; = \; y^{-2}(\; L^2 \; d\theta^2 \; + \; dy^2 \;) \; ; \quad (a \text{ and } L \text{ are strictly positive constants}) \; .$$

Then $-\Delta_A^\Omega$ has a compact resolvent and

$$N(\lambda; -\Delta_A^\Omega) \; \sim \; \frac{|\Omega|}{4\pi}\lambda \; ; \quad \text{as} \quad \lambda \to +\infty \; .$$

We will prove it in the next subsection.

Proposition 3.3 *Let Ω an open set of a funnel F_k, isometric to* $\mathbb{S}\times]a^2,+\infty[$, *equipped with the metric*

$$ds^2 \;=\; L^2\cosh^2(t)\;d\theta^2 \;+\; dt^2\;;\quad (a \text{ and } L \text{ are strictly positive constants})\,.$$

Then $-\Delta_A^\Omega$ *has a compact resolvent and for any* $\delta\;\in\,]\frac{1}{3},\frac{2}{5}[$, *there exists a constant* $C\;>\;0$ *such that*

$$\frac{1}{2\pi}\int_\Omega(1-\frac{C}{(\mathbf{b}(m)+1)^{(2-5\delta)/2}})\;\mathcal{N}(\lambda(1-C\lambda^{-3\delta+1})-\frac{1}{4},\mathbf{b}(m))\;dm$$

$$\leq\;N(\lambda,-\Delta_A^\Omega)\;\leq$$

$$\frac{1}{2\pi}\int_\Omega(1+\frac{C}{(\mathbf{b}(m)+1)^{(2-5\delta)/2}})\;\mathcal{N}(\lambda(1+C\lambda^{-3\delta+1})-\frac{1}{4},\mathbf{b}(m))\;dm$$

The proof comes easily following the ones in the Poincaré half-plane of [Mo-Tr], using the method of [Col], in the neighbourhood of the boundary at infinity. It corresponds to a context where the partitions of unity were fine, so they can be performed on $\mathbb{S}\times]a^2,+\infty[$, (instead of $\mathbb{R}\times]-\infty,0[$) .

3.2 Proof of Proposition 3.2

For simplicity we change the unit circle $\mathbb{S}=\mathbb{S}_1$ into the circle $\mathbb{S}_L$, of radius L , so

$$\Omega=\mathbb{S}_L\times]a^2,+\infty[\,,\quad ds^2\;=\;y^{-2}(dx^2\;+\;dy^2)\,,\quad\text{and}\qquad(3.1)$$
$$-\Delta_A^\Omega u(z)=y^2[(D_x-A_1)^2u(z)+(D_y-A_2)^2u(z)]\;;$$

moreover $d(z,z')=\arg\cosh\dfrac{y^2+y'^2+d^2_{\mathbb{S}_L}(x,x')}{2yy'}$.

We begin by proving the compactness of the resolvent of $-\Delta_A^\Omega$.

Lemma 3.4 *There exists* $C_0>1$ *such that*

$$\int_\Omega(\mathbf{b}(z)-C_0)|u(z)|^2dm\;\leq\;\int_\Omega -\Delta_A^\Omega u(z)\overline{u(z)}dm\;;\quad\forall\,u\;\in\;C_0^\infty(\Omega)\,.$$

Proof. Let us denote the quadratic form

$$q_A^\Omega(u) \;=\; \int_\Omega -\Delta_A^\Omega u(z)\overline{u(z)}dm \quad \forall\, u \,\in\, C_0^\infty(\Omega)\,. \tag{3.2}$$

Then $q_A^\Omega(u) \;=\; \int_\Omega \left[|(D_x - A_1)u|^2 \;+\; (D_y - A_2)u|^2\right] dxdy$,

and $\left|\int_\Omega \widetilde{\mathbf{b}}(z)|u(z)|^2 dm\right|$

$$= \left|\int_\Omega [(D_x - A_1)u(z)\overline{(D_y - A_2)u(z)} \;-\; (D_y - A_2)u(z)\overline{(D_x - A_1)u(z)}]dxdy\right| .$$

Therefore we get that $\left|\int_\Omega \widetilde{\mathbf{b}}(z)|u(z)|^2 dm\right| \;\leq\; q_A^\Omega(u)$.
As $\mathbf{b}(z) = |\widetilde{\mathbf{b}}(z)| \;\to\; +\infty$ at the infinity, the Lemma comes easily.

The Lemma 3.4 and the assumption (2.5) prove that $-\Delta_A^\Omega$ has compact resolvent.

Later on, we will need that the assumptions (2.5) and (2.6) ensure that there exists $C > 1$ such that $\forall\, z = (x, y)\,,\; z' = (x', y') \;\in\; \Omega$,

$$\mathbf{b}(z)/C \;\leq\; \mathbf{b}(z') \;\leq\; C\mathbf{b}(z)\,, \quad \text{if } |y - y'| \leq 1 \text{ and } y > C\,. \tag{3.3}$$

This comes from the fact that $d(z)$ is equivalent to $\ln(y)$ for $y(> 1)$ large enough, so the assumption (2.6) ensures that $|\partial_x b(z)| + |\partial_y b(z)| \leq C(|b(z)| + 1)$.

Lemma 3.5 *There exists a constant $C_0 > 1$ such that, for any $\lambda > 1$ and for any $K \subset \Omega$ isometric to $I_1 \times I_2$, endowed with the metric in (3.1), with*

$$I_1 =]x_0 - \epsilon_1, x_0 + \epsilon_1[\;,\;\; I_2 =]y_0 - \epsilon_2, y_0 + \epsilon_2[\;,$$

$$\epsilon_1 \;\in]C_0^{-1}, 1[\;,\;\; \epsilon_2 = \sqrt{y_0}/\sqrt{\mathbf{b}(z_0)}\;,\;\; (y_0 > C_0)\;;$$

the following estimates hold:

$$[\lambda(1 - \frac{1}{\sqrt{y_0}}) - C_0]\frac{|K|_g}{4\pi} \;\leq\; N(\lambda; -\Delta_A^K) \;\leq\; [\lambda(1 + \frac{1}{\sqrt{y_0}}) + C_0]\frac{|K|_g}{4\pi}\,. \tag{3.4}$$

Proof. If $\mathbf{b}(z_0) > C\lambda$, then, according to the estimate of Lemma 3.4 with K instead of Ω , $\quad N(\lambda; -\Delta_A^K) \;=\; 0$.
So we can assume that $\mathbf{b}(z_0) \leq C\lambda$.

We use that the spectrum of $-\Delta_A^K$ is gauge-invariant, so we can suppose that in K

$$A_2 = 0 \quad \text{and} \quad A_1(x,y) = -\int_{y_0}^{y} \frac{\widetilde{\mathbf{b}}(x,\rho)}{\rho^2} d\rho \ .$$

Then $|A_1(x,y)| \leq C\epsilon_2 \dfrac{\mathbf{b}(z_0)}{y_0^2}$.

From this estimate, we get that for any $\epsilon \in]0,1[$,

$$-(1-\epsilon)\Delta_0^K \ - \ C\epsilon_2^2 \frac{\mathbf{b}^2(z_0)}{\epsilon y_0^2} \ \leq \ -\Delta_A^K \ \leq \ -(1+\epsilon)\Delta_0^K \ + \ C\epsilon_2^2 \frac{\mathbf{b}^2(z_0)}{\epsilon y_0^2} \ .$$

We take $\epsilon = 1/\sqrt{y_0}$, to get

$$-(1-\frac{1}{\sqrt{y_0}})\Delta_0^K \ - \ C\frac{\mathbf{b}(z_0)}{\sqrt{y_0}} \ \leq \ -\Delta_A^K \ \leq \ -(1+\frac{1}{\sqrt{y_0}})\Delta_0^K \ + \ C\frac{\mathbf{b}(z_0)}{\sqrt{y_0}} \ .$$

As $\mathbf{b}(z_0) \leq C\lambda$, the Lemma follows easily from the min-max principle and the well-known estimate for $N(\lambda; -\Delta_0^K)$.

Proof of Proposition 3.2.

It follows easily from Lemma 3.5, (for large y), using the same tricks as in [Mo-Tr].

4 Proof of the main Theorem 2.1

The proof comes easily from the three propositions 3.1 - - 3.3, following the method developped in [Mo-Tr].

5 Remark on the case of constant magnetic field

It is not always possible to have a constant magnetic field on $\mathbf{M}$, (for topological reason), but for any $(b,\beta) \in \mathbb{R}^{J_1} \times \mathbb{R}^{J_2}$, there exists a one-form A , such that the corresponding magnetic field dA satisfies

$$dA \ = \ \widetilde{\mathbf{b}}(z)dm \qquad \begin{cases} \widetilde{\mathbf{b}}(z) \ = \ b_j \ \forall \ z \ \in \ M_j \\ \widetilde{\mathbf{b}}(z) \ = \ \beta_k \ \forall \ z \ \in \ F_k \end{cases} \tag{5.1}$$

Theorem 5.1 *Assume (2.1) and (5.1).*

If $J_1 = 0$ *and* $J_2 > 0$ *, then the essential spectrum of* $-\Delta_A$ *is*

$$\mathrm{sp}_{ess}(-\Delta_A) \; = \; [\frac{1}{4} + \inf_k \beta_k^2 \; , \; +\infty[\bigcup \left(\bigcup_{k=1}^{J2} S(\beta_k) \right) \tag{5.2}$$

with $S(\beta_k) = \emptyset$ *when* $|\beta_k| \leq 1/2$ *and when* $|\beta_k| > 1/2$
$S(\beta_k) = \{(2j+1)|\beta_k| - j(j+1) \; ; \; j \in \mathbb{N}, \; j < |\beta_k| - 1/2\}$.

If J_1 *and* J_2 *are* > 0 *, then for any* j *,* $1 \leq j \leq J_1$ *and for any* $z \in M_j$ *there exists a unique closed curve through* z *,* $\mathcal{C}_{j,z}$ *in* $(M_j, \; g)$ *, not contractible and with zero* $g-$*curvature. The following limit exists and is finite:*

$$[A]_{M_j} \; = \; \lim_{d(z) \to +\infty} \int_{\mathcal{C}_{j,z}} A \; . \tag{5.3}$$

If $J_1^A = \{j \in \mathbb{N} \; , \; 1 \leq j \leq J_1 \; s.t. \; [A]_{M_j} \in 2\pi\mathbb{Z} \; \}$ *, then*

$$\mathrm{sp}_{ess}(-\Delta_A) \; = \; [\frac{1}{4} + \min\{ \inf_{j \in J_1^A} b_j^2 \; , \; \inf_{1 \leq k \leq J_2} \beta_k^2 \; \} \; , \; +\infty[\bigcup \left(\bigcup_{k=1}^{J2} S(\beta_k) \right) \; . \tag{5.4}$$

If $J_2 = 0$ *and* $J_1^A = \emptyset$ *, then* $\mathrm{sp}_{ess}(-\Delta_A) = \emptyset$ *:*
$-\Delta_A$ *has purely discrete spectrum, (its resolvent is compact).*

Remark 5.2 *In Theorem 5.1, one can change* $\mathcal{C}_{j,z}$ *into* $\mathcal{S}_{j,z}$ *, the unique closed curve through* z *, not contractible and with minimal* $g-$*length.*
$\mathcal{S}_{j,z}$ *is not smooth at* z*,* $\mathcal{S}_{j,z}$ *is part of two geodesics through* z *, so there is an out-going tangent and an incoming tangent at* z *. It is easy to see that* $\mathcal{C}_{j,z} \cap \mathcal{S}_{j,z} = \{z\}$ *, so by Stokes formula*

$$\int_{\mathcal{S}_{j,z}} (A - A^0) \; = \; \int_{\mathcal{C}_{j,z}} (A - A^0) \; ,$$

where A^0 *is a one-form on* M *, such that*

$$dA \; = \; dA^0 \quad \text{on} \quad M_j \quad \text{and} \quad [A^0]_{M_j} \; = \; 0 \; ; \; \forall \, j \; .$$

The orientation in both cases $\mathcal{C}_{j,z}$ *and* $\mathcal{S}_{j,z}$ *, is chosen such that, if* $u_z, \; v_z \; \in T_z M_j$ *,* $g_z(u_z, v_z) = 0$ *,* $dm(u_z, v_z) > 0$ *, and* u_z *is tangent to the curve (in the positive direction), then* v_z *points to boundary at infinity; (for* $\mathcal{S}_{j,z}$ *, one can take as* u_z *the out-going tangent, or the incoming tangent).*

Proof of Theorem 5.1. It is clear that

$$\mathrm{sp}_{ess}(-\Delta_A) \;=\; \left(\bigcup_{j=1}^{J_1} \mathrm{sp}_{ess}(-\Delta_A^{M_j})\right) \bigcup \left(\bigcup_{k=1}^{J_2} \mathrm{sp}_{ess}(-\Delta_A^{F_k})\right) \;; \qquad (5.5)$$

so the proof will result on the two lemmas below.

Lemma 5.3

$$\mathrm{sp}_{ess}(-\Delta_A^{F_k}) \;=\; [\frac{1}{4} + \beta_k^2 \;,\; +\infty[\cup S(\beta_k) \;\; .$$

Proof. We have $-\Delta_A^{F_k} \;=\; \tau_k^{-2} \cosh^{-2}(t)(D_\theta - A_1)^2 + \cosh^{-1}(t)(D_t - A_2) \left[\cosh(t)(D_t - A_2)\right]$.
Since $\widetilde{\mathbf{b}} = \beta_k = \tau_k^{-1} \cosh^{-1}(t)(\partial_\theta A_2 - \partial_t A_1)$, there exists a function φ such that $A - \widetilde{A} \;=\; d\varphi$ if $\widetilde{A} = (\xi - \beta_k \tau_k \sinh(t))d\theta$, (for some constant ξ) .
So we can assume that $A = \widetilde{A}$.

We change the density $dm = \tau_k \cosh(t) d\theta dt$ for $d\theta dt$, using the unitary operator $Uf = (\tau_k \cosh(t))^{1/2} f$, so

$$P = -U\Delta_A^{F_k} U^\star = \tau_k^{-2} \cosh^{-2}(t)(D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}(1 + \cosh^{-2}(t)) \; .$$

We remind that $\lambda \;\in\; \mathrm{sp}_{ess}(-\Delta_A^{F_k})$ iff there exists a sequence $(u_j)_j \;\in\; Dom(-\Delta_A^{F_k})$ converging weekly in $L^2(F_k)$ to zero, $\|u_j\|_{L^2(F_k)} = 1$ and such that the sequence $(-\Delta_A^{F_k} u_k - \lambda u_k)_k$ converges strongly to zero.

It is clear that $\mathrm{sp}(-\Delta_A^{F_k}) \;=\; \mathrm{sp}(\bigoplus_{\ell\in\mathbb{Z}} P_\ell)$,

$$P_\ell = D_t^2 + \tau_k^{-2} \cosh^{-2}(t)(\ell + \beta_k \tau_k \sinh(t) - \xi)^2 + \frac{1}{4}(1 + \cosh^{-2}(t)) \; ,$$

for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha_k^2 \;,\; +\infty[$.
So $\mathrm{sp}(-\Delta_A^{F_k}) = \bigcup_{\ell\in\mathbb{Z}} \mathrm{sp}(P_\ell)$.

Writing that $P_\ell \;=\; D_t^2 + \left(\dfrac{\ell - \xi}{\tau_k \cosh(t)} + \beta_k \tanh(t)\right)^2 + \dfrac{1}{4}(1 + \cosh^{-2}(t))$, we get easily that $\mathrm{sp}_{ess}(P_\ell) = [\frac{1}{4} + \beta_k^2, +\infty[$, and that the number of eigenvalues $< \dfrac{1}{4} + \beta_k^2$ is finite for all $\ell < \xi$ and equal to zero for all $\ell \geq \xi$. Here

we assume $\beta_k > 0$. So $[\frac{1}{4}+\beta_k^2, +\infty[\subset \mathrm{sp}_{ess}(-\Delta_A^{F_k})$ and the other part of $\mathrm{sp}_{ess}(-\Delta_A^{F_k})$ is $S_\infty = \{\lambda \ ; \ \lambda = \lim_{j\to+\infty} \lambda_{\ell(j)} \ , \ \lambda_{\ell(j)} \in \mathrm{sp}_d(P_{\ell(j)})\}$,
where $(\ell(j))_j$ denotes any decreasing sequence of negative integers.
Now we use again the formula
$P_\ell \ = \ D_t^2 + \left(\frac{\ell-\xi}{\tau_k \cosh(t)} + \beta_k \tanh(t)\right)^2 + \frac{1}{4}(1+\cosh^{-2}(t))$.
Assuming $\ell - \xi < 0$, we set $\rho = |\ell-\xi|/\tau_k$ and we introduce the new variable $y = 2\rho e^{-t}$. We get that P_ℓ is unitarily equivalent to $\widetilde{P}_\rho$ defined as a Dirichlet type operator in $L^2(]0, 2\rho e^{-\alpha_k^2}[; dy)$, (zero boundary condition is only required on the right boundary):

$$\widetilde{P}_\rho \ = \ D_y(y^2 D_y) + W_\rho(y) \ , \quad \text{with}$$

$$W_\rho(y) = \left(\beta_k \frac{(1-y^2/(4\rho^2)}{1+y^2/(4\rho^2)} - \frac{y}{1+y^2/(4\rho^2)}\right)^2 + \left(\frac{y/(2\rho)}{1+y^2/(4\rho^2)}\right)^2 .$$

So we have $\lim_{\rho\to+\infty} W_\rho(y) = W_\infty(y) = (\beta_k - y)^2$, and the operator
$\widetilde{P}_\infty = D_y(y^2 D_y) + W_\infty(y)$ on $L^2(]0,+\infty[; dy)$ satisfies, (see [Mo-Tr]),
$\mathrm{sp}(\widetilde{P}_\infty) = \mathrm{sp}_{ess}(\widetilde{P}_\infty) \cup \mathrm{sp}_d(\widetilde{P}_\infty)$ with

$$\mathrm{sp}_{ess}(\widetilde{P}_\infty) \ = \ [\frac{1}{4}+\beta_k^2, +\infty[\ ; \qquad \mathrm{sp}_d(\widetilde{P}_\infty) \ = \ S(\beta_k) \ .$$

We remind that the eigenfunctions associated to the eigenvalues in $S(\beta_k)$ of $\widetilde{P}_\infty$ are exponentially decreasing, so if $\lambda_0(\rho) \le \ldots \le \lambda_j(\rho) \le \lambda_{j+1}(\rho) \ldots$ are the eigenvalues of $\widetilde{P}_\rho$ then for any j ,
$\lim_{\rho\to+\infty} \lambda_j(\rho) = \lambda_j(\infty) = (2j+1)\beta_k - j(j+1)$, if $\beta_k > 1/2$ and $j < \beta_k - 1/2$, otherwise $\lim_{\rho\to+\infty} \lambda_j(\rho) = \frac{1}{4} + \beta_k^2$.
Therefore we get that $S_\infty \ = \ S(\beta_k)$, or $S_\infty \ = \ S(\beta_k) \ \cup \ \{\frac{1}{4}+\beta_k^2\}$: the formula of Lemma 5.3 follows.

Lemma 5.4 *If* $1 \le j \le J_1$ *and* $j \ \notin \ J_1^A$, *then*

$$\mathrm{sp}_{ess}(-\Delta_A^{M_j}) \ = \ \emptyset \ .$$

If $j \ \in \ J_1^A$, *then*

$$\mathrm{sp}_{ess}(-\Delta_A^{M_j}) \ = \ [\frac{1}{4} + b_j^2 \ , \ +\infty[\ .$$

Proof. Use the coordinate $t = \ln y$ instead of y , so

$$M_j \;=\; \mathbb{S}\times]\alpha_j^2, +\infty[\quad \text{and} \quad ds_j^2 \;=\; L_j^2 e^{-2t} d\theta^2 \;+\; dt^2 \;;\;\; (\alpha_j = e^{a_j}) \;.$$

Then $-\Delta_A^{M_j} \;=\; L_j^{-2} e^{2t}(D_\theta - A_1)^2 + e^t(D_t - A_2)(e^{-t}(D_t - A_2))$,
$\widetilde{\mathbf{b}} = \; L_j^{-1} e^t(\partial_\theta A_2 - \partial_t A_1)$ and $dm = L_j e^{-t} d\theta dt$. As in Lemma 5.3, we have

$$A - \widetilde{A} \;=\; d\varphi \text{ if } \widetilde{A} = (\xi + L_j b_j e^{-t}) d\theta \;,\; (\text{for some constant } \xi) \;.$$

So we can also assume that $A = \widetilde{A}$.

We replace the density dm by $d\theta dt$, using the unitary operator $Uf = \sqrt{L_j} e^{-t/2} f$, so

$$P = -U\Delta_A^{M_j} U^\star = L_j^{-2} e^{2t}(D_\theta - A_1)^2 + D_t^2 + \frac{1}{4} \;.$$

Then we get also that

$$\mathrm{sp}(-\Delta_A^{M_j}) = \mathrm{sp}(P) = \bigcup_{\ell\in\mathbb{Z}} \mathrm{sp}(P_\ell) \;;\; P_\ell = D_t^2 + \frac{1}{4} + \left(e^t \frac{(\ell+\xi)}{L_j} + b_j\right)^2 \;,$$

for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha_j^2, +\infty[$.

When $\ell + \xi \neq 0$, the spectrum of P_ℓ is discrete. More precisely

$$\mathrm{sp}(P_\ell) \;=\; \mathrm{sp}(P^\pm) \;, \quad \text{where} \quad P^\pm = D_t^2 + \frac{1}{4} + (\pm e^t + b_j)^2$$

for the Dirichlet condition on $L^2(I_{j,\ell}; dt)$; $I_{j,\ell} =]\alpha_j^2 + \ln(|\ell+\xi|/L_j), +\infty[$, and $\pm = \dfrac{\ell+\xi}{|\ell+\xi|}$.

So $\lim\limits_{|\ell|\to\infty} \inf \mathrm{sp}(P_\ell) \;=\; +\infty$, and then we get easily that the spectrum of $-\Delta_A^{M_j}$ is discrete, when $\xi = [A]_{M_j}/(2\pi) \;\notin\; \mathbb{Z}$.

If $\ell + \xi = 0$, the spectrum of P_ℓ is absolutely continuous :

$$\mathrm{sp}(P_{-\xi}) \;=\; \mathrm{sp}_{ess}(P_{-\xi}) \;=\; \mathrm{sp}_{ac}(P_{-\xi}) \;\;=\; [\frac{1}{4} + b_j^2 \;,\; +\infty[\;;$$

and then, when $[A]_{M_j} \;\in\; 2\pi\mathbb{Z}$, $\quad \mathrm{sp}_{ess}(-\Delta_A^{M_j}) \;=\; [\frac{1}{4} + b_j^2 \;,\; +\infty[$.
This achieves the proof of Lemma 5.4.

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