Combinatorial rigidity of multicritical maps

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Abstract

In this paper, we combine the KSS nest constructed in [KSS] and the analytic method in [AKLS] to prove the combinatorial rigidity of multicritical maps.

1 Introduction

Rigidity is one of the fundamental and remarkable phenomena in holomorphic dynamics. The general rigidity problem can be posed as

Rigidity problem [L]. Any two combinatorially equivalent rational maps are quasi-conformally equivalent. Except for the Lattès examples, the quasi-conformal deformations come from the dynamics of the Fatou set.

In the quadratic case, the rigidity problem is equivalent to the famous hyperbolic conjecture. The MLC conjecture asserting that the Mandelbrot set is locally connected is stronger than the hyperbolic conjecture (cf. [DH]). In 1990, Yoccoz [Hu] proved MLC for all parameter values which are at most finitely renormalizable. Lyubich [L] proved MLC for infinitely renormalizable quadratic polynomials of bounded type. In [KSS], Kozlovski, Shen and van Strien gave the proof of the rigidity for real polynomials with all critical points real. In [AKLS], Avila, Kahn, Lyubich and Shen proved that any unicritical polynomial $f_c : z \mapsto z^d + c$ which is at most finitely renormalizable and has only repelling periodic points is combinatorially rigid, which implies that the connectedness locus (the Multibrot set) is locally connected at the corresponding parameter values. The rigidity problem for the rational maps with Cantor Julia sets is totally solved (cf. [YZ], [Z]). In [Z], Zhai took advantage of a length-area method introduced by Kozlovski, Shen and van Strien (cf. [KSS]) to prove the quasi-conformal rigidity for rational maps with Cantor Julia sets. Kozlovski and van Strien proved that topologically conjugate non-renormalizable polynomials are quasi-conformally conjugate (cf. [KS]).

In the following, we list some other cases in which the rigidity problem is researched (see also [Z]).
(i) Robust infinitely renormalizable quadratic polynomials [Mc1].
(ii) Summable rational maps with small exponents [GS].
(iii) Holomorphic Collet-Eckmann repellers [PR].
(iv) Uniformly weakly hyperbolic rational maps [Ha].

In [PT], we have discussed the combinatorial rigidity of unicritical maps. In this paper, we will give a proof of the combinatorial rigidity of multicritical maps (see the definition in section 2).

In the proof, we will exploit the powerful combinatorial tool called "puzzle" and a sophisticated choice of puzzle pieces called the KSS nest constructed in [KSS] (see Theorem 7.11). To get the quasi-conformal conjugation, we adapt the analytic method in [AKLS] (see Lemma 3.2).

The paper is organized as follows. In section 2, we introduce the definition of the multicritical maps which we study in this paper and present the results of this paper,

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Theorems 2.1, 2.2, 2.3. In section 3, we apply the Spreading Principle appeared in [KSS] to prove Theorem 2.1. In section 4, we resort to the quasiconformal surgery to prove Theorem 2.2. Proof of Theorem 2.3 (a) is given in section 5. We reduce Theorem 2.3 (b) to Main Proposition in section 6. The proof of Main Proposition is presented in section 7. In subsection 7.1, we reduce Main Proposition to Proposition 7.3. The proof of Proposition 7.3 is given in subsection 7.2.

2 Statement

The Set up. $\mathbf{V} = \sqcup_{i \in I} V_i$ is the disjoint union of finitely many Jordan domains in the complex plane $\mathbb{C}$ with disjoint and quasi-circle boundaries, $\mathbf{U}$ is compactly contained in $\mathbf{V}$, and is the union of finitely many Jordan domains with disjoint closures; $f : \mathbf{U} \to \mathbf{V}$ is a proper holomorphic map with all critical points contained in $\mathbf{K}_f := \{z \in \mathbf{U} \mid f^n(z) \in \mathbf{U}\ \forall n\}$, with each $\mathbf{V}$-component containing at most one connected component of $\mathbf{K}_f$ containing critical points.

Denote by $\mathrm{Crit}(f)$ the set of critical points of f and by $\mathbf{P} := \bigcup_{n \geq 1} \bigcup_{c \in \mathrm{Crit}(f)} \{f^n(c)\}$ the postcritical set of f.

Let $\mathrm{int}\mathbf{K}_f$ denote the interior of $\mathbf{K}_f$. For $x \in \mathbf{K}_f$, denote by $\mathbf{K}_f(x)$ the component of $\mathbf{K}_f$ containing the point x. We call a component of $\mathbf{K}_f$ a critical component if it contains a critical point. The map f maps each component of $\mathbf{K}_f$ onto a component of $\mathbf{K}_f$. A component K of $\mathbf{K}_f$ is called *periodic* if $f^p(K) = K$ for some $p \geq 1$, *wandering* if $f^i(K) \cap f^j(K) = \emptyset$ for all $i \neq j \geq 0$.

Two maps in the set-up $(f : \mathbf{U} \to \mathbf{V}), (\tilde{f} : \tilde{\mathbf{U}} \to \tilde{\mathbf{V}})$ are said to be *c-equivalent* (combinatorially equivalent), if there is a pair of orientation preserving homeomorphisms $h_0, h_1 : \mathbf{V} \to \tilde{\mathbf{V}}$ such that

$$\begin{cases} h_1(\mathbf{U}) = \tilde{\mathbf{U}} \text{ and } h_1(\mathbf{P}) = \tilde{\mathbf{P}} \\ h_1 \text{ is isotopic to } h_0 \text{ rel } \partial\mathbf{V} \cup \mathbf{P} \\ h_0 \circ f \circ h_1^{-1}|_{\tilde{\mathbf{U}}} = \tilde{f} \\ h_1|_{\overline{\mathbf{V}} \setminus \mathbf{U}} \text{ is } C_0\text{-qc (an abbreviation of quasi-conformal) for some } C_0 \geq 1, \end{cases} \tag{1}$$

$$\text{in particular} \quad \begin{array}{ccc} \mathbf{V} \supset \mathbf{U} & \xrightarrow{h_1} & \tilde{\mathbf{U}} \subset \tilde{\mathbf{V}} \\ f \downarrow & & \downarrow \tilde{f} \\ \mathbf{V} & \xrightarrow[h_0]{} & \tilde{\mathbf{V}} \end{array} \quad \text{commutes.}$$

This definition is to be compared with the notion of combinatorial equivalence introduced by McMullen in [Mc2]. Notice that this definition is slightly different from the definitions of combinatorial equivalence in [AKLS] and [KS], since we define it without using the external rays and angles.

We say that f and $\tilde{f}$ are *qc-conjugate off* $\mathbf{K}_f$ if there is a qc map $H : \mathbf{V} \setminus \mathbf{K}_f \to \tilde{\mathbf{V}} \setminus \mathbf{K}_{\tilde{f}}$ so that $H \circ f = \tilde{f} \circ H$ on $\mathbf{U} \setminus \mathbf{K}_f$,

$$\text{i.e.} \quad \begin{array}{ccc} \mathbf{U} \setminus \mathbf{K}_f & \xrightarrow{H} & \tilde{\mathbf{U}} \setminus \mathbf{K}_{\tilde{f}} \\ f \downarrow & & \downarrow \tilde{f} \\ \mathbf{V} \setminus \mathbf{K}_f & \xrightarrow[H]{} & \tilde{\mathbf{V}} \setminus \mathbf{K}_{\tilde{f}} \end{array} \quad \text{commutes.}$$

We say that f and $\tilde f$ are *qc-conjugate off* $\mathrm{int}\mathbf{K}_f$ if there is a qc map $\tilde H : \mathbf{V} \to \tilde{\mathbf{V}}$ so that $\tilde H \circ f = \tilde f \circ \tilde H$ on $\mathbf{U} \backslash \mathrm{int}\mathbf{K}_f$,

$$\text{i.e.}\quad \begin{array}{ccc} \mathbf{U}\backslash\mathrm{int}\mathbf{K}_f & \xrightarrow{\tilde H} & \tilde{\mathbf{U}}\backslash\mathrm{int}\mathbf{K}_{\tilde f} \\ f\downarrow & & \downarrow \tilde f \\ \mathbf{V}\backslash\mathrm{int}\mathbf{K}_f & \xrightarrow[\tilde H]{} & \tilde{\mathbf{V}}\backslash\mathrm{int}\mathbf{K}_{\tilde f} \end{array} \quad \text{commutes.}$$

We say that f and $\tilde f$ are *qc-conjugate* if there is a qc map $H' : \mathbf{V} \to \tilde{\mathbf{V}}$ so that $H' \circ f = \tilde f \circ H'$ on $\mathbf{U}$,

$$\text{i.e.}\quad \begin{array}{ccc} \mathbf{V} \supset \mathbf{U} & \xrightarrow{H'} & \tilde{\mathbf{U}} \subset \tilde{\mathbf{V}} \\ f\downarrow & & \downarrow \tilde f \\ \mathbf{V} & \xrightarrow[H']{} & \tilde{\mathbf{V}} \end{array} \quad \text{commutes.}$$

Theorem 2.1. *Let $f, \tilde f$ be two maps in the set-up. Suppose f and $\tilde f$ are qc-conjugate off $\mathbf{K}_f$ by a qc map H. Assume that f satisfies the following property $(*)$:*

For every critical component $\mathbf{K}_f(c)$ of $\mathbf{K}_f$, $c \in \mathrm{Crit}(f)$, and every integer $n \geq 1$, there exists a puzzle piece $Q_n(c)$ containing c such that:

(i) For every critical component $\mathbf{K}_f(c)$, the pieces $\{Q_n(c)\}_{n\geq 1}$ form a nested sequence with $\bigcap_n Q_n(c) = \mathbf{K}_f(c)$ (the depth of $Q_n(c)$ may not equal to n).

(ii) For each $n \geq 1$, the union $\bigcup_{c\in\mathrm{Crit}(f)} Q_n(c)$ is a nice set.

(iii) There is a constant $\tilde C$, such that for each pair $(n, \mathbf{K}_f(c))$ with $n \geq 1$, a critical component $\mathbf{K}_f(c)$, the map $H|_{\partial Q_n(c)}$ admits a $\tilde C$-qc extension inside $Q_n(c)$.

Then the map H extends to a qc map from $\mathbf{V}$ onto $\tilde{\mathbf{V}}$ which is a conjugacy off $\mathrm{int}\mathbf{K}_f$.

See Definition 1 (1) and (2) in the next section for the definitions of a puzzle piece, the depth of it and a nice set.

Theorem 2.2. *Let f be a map in the set-up. Then for any component D of* $\mathrm{int}\mathbf{K}_f$*, $f^i(D) \cap f^j(D) = \emptyset$ for all $i \neq j \geq 0$.*

Theorem 2.3. *Let $f, \tilde f$ be two maps in the set-up. Then the following statements hold.*

(a) If f and $\tilde f$ are c-equivalent, then they are qc-conjugate off $\mathbf{K}_f$.

(b) Suppose $H : \mathbf{V}\backslash\mathbf{K}_f \to \tilde{\mathbf{V}}\backslash\mathbf{K}_{\tilde f}$ is a qc conjugacy off $\mathbf{K}_f$. Assume that for every critical component $\mathbf{K}_f(c)$, $c \in \mathrm{Crit}(f)$, satisfying that $f^l(\mathbf{K}_f(c))$ is a critical periodic component of $\mathbf{K}_f$ for some $l \geq 1$, there are a constant M_c and an integer $N_c \geq 0$ such that for each $n \geq N_c$, the map $H|_{\partial P_n(c)}$ admits an M_c-qc extension inside $P_n(c)$, where $P_n(c)$ is a puzzle piece of depth n containing c. Then the map H extends to a qc conjugacy off $\mathrm{int}\mathbf{K}_f$*. Furthermore, if for every component K of $\mathbf{K}_f$ with non-empty interior, $H|_{\partial K}$ extends to a qc conjugacy inside K, then f and $\tilde f$ are qc-conjugate by an extension of H.*

3 Proof of Theorem 2.1

Suppose that f and $\tilde f$ are qc-conjugate off $\mathbf{K}_f$ by a C_0-qc map H. Starting from the property $(*)$, we will prove that H admits a qc extension across $\mathbf{K}_f$ which is a conjugacy off $\mathrm{int}\mathbf{K}_f$.

Definition 1. *(1) For every $n \geq 0$, we call each component of $f^{-n}(\mathbf{V})$ a* ***puzzle piece*** *of* ***depth*** *n for f. Similarly, we call each component of $\tilde{f}^{-n}(\tilde{\mathbf{V}})$ a puzzle piece of depth n for $\tilde{f}$. Denote by $\operatorname{depth}(P)$ the depth of a puzzle piece P.*

We list below three basic properties of the puzzle pieces.

(a) Every puzzle piece is a quasi-disk and there are finitely many puzzle pieces of the same depth.

(b) Given two puzzle pieces P and Q with $\operatorname{depth}(P) > \operatorname{depth}(Q)$, either $P \subset\subset Q$ or $\overline{P} \cap \overline{Q} = \emptyset$.

(c) For $x \in \mathbf{K}_f$, for every $n \geq 0$, there is a unique puzzle piece of depth n containing x. Denote the piece by $P_n(x)$. Then $P_{n+1}(x) \subset\subset P_n(x)$ and $\cap_{n\geq 0} P_n(x)$ is exactly the component of $\mathbf{K}_f$ containing x.

(2) Suppose $X \subset \mathbf{V}$ is a finite union of puzzle pieces (not necessarily of the same depth). We say X is ***nice*** *if for any $z \in \partial X$ and any $n \geq 1$, $f^n(z) \notin X$ as long as $f^n(z)$ is defined, that is, for any component P of X, for any $n \geq 1$, $f^n(P)$ is not strictly contained in X. For example, if X has a unique component, obviously it is a nice set.*

(3) Let A be an open set and $z \in A$. Denote the component of A containing z by $\operatorname{Comp}_z(A)$.

Given an open set X consisting of finitely many puzzle pieces, let

$$D(X) = \{z \in \mathbf{V} \mid \exists k \geq 0, f^k(z) \in X\} = \cup_{k\geq 0} f^{-k}(X).$$

For $z \in D(X)\backslash X$, let $k(z)$ be the minimal positive integer such that $f^{k(z)}(z) \in X$. Set

$$\mathcal{L}_z(X) := \operatorname{Comp}_z(f^{-k(z)}(\operatorname{Comp}_{f^{k(z)}(z)}(X))).$$

Obviously, $f^{k(z)}(\mathcal{L}_z(X)) = \operatorname{Comp}_{f^{k(z)}(z)}(X)$.

Lemma 3.1. *Suppose X is a finite union of puzzle pieces. The following statements hold.*

(1) For any $z \in D(X)\backslash X$, $\mathcal{L}_z(X), f(\mathcal{L}_z(X)), \cdots, f^{k(z)-1}(\mathcal{L}_z(X))$ are pairwise disjoint.

(2) Suppose X is nice and $z \in D(X)\backslash X$. Then for all $0 \leq i < k(z)$, $f^i(\mathcal{L}_z(X)) \cap X = \emptyset$. In particular, if $X \supset \operatorname{Crit}(f)$, then $\mathcal{L}_z(X)$ is conformally mapped onto a component of X by the iterate of $f^{k(z)}$.

Proof. (1) Assume there exist $0 \leq i < j < k(z)$ with $f^i(\mathcal{L}_z(X)) \cap f^j(\mathcal{L}_z(X)) \neq \emptyset$. Then $f^i(\mathcal{L}_z(X)) \subset\subset f^j(\mathcal{L}_z(X))$ and

$$f^{k(z)-j}(f^i(\mathcal{L}_z(X))) \subset\subset f^{k(z)-j}(f^j(\mathcal{L}_z(X))) = f^{k(z)}(\mathcal{L}_z(X)) = \operatorname{Comp}_{f^{k(z)}(z)}(X).$$

So $f^{k(z)-j+i}(z) \in X$. But $0 < k(z) - j + i < k(z)$. This is a contradiction with the minimality of $k(z)$.

(2) Assume there is some $0 \leq i_0 < k(z)$ with $f^{i_0}(\mathcal{L}_z(X)) \cap X \neq \emptyset$. We can show that $f^{i_0}(\mathcal{L}_z(X)) \cap X \subset\subset f^{i_0}(\mathcal{L}_z(X))$. In fact, when $i_0 \neq 0$, this is due to the minimality of $k(z)$; when $i_0 = 0$, it is because $z \notin X$. Let P be a component of X with $P \subset\subset f^{i_0}(\mathcal{L}_z(X))$. So $f^{k(z)-i_0}(P) \subset\subset f^{k(z)-i_0}(f^{i_0}(\mathcal{L}_z(X))) = \operatorname{Comp}_{f^{k(z)}(z)}(X)$. It contradicts the condition that X is nice. $\square$

The corollary below follows directly from the above lemma.

Corollary 3.2. *Suppose X is a finite union of puzzle pieces. The following statements hold.*

(i) For any $z \in D(X)\backslash X$, $\{\mathcal{L}_z(X), f(\mathcal{L}_z(X)), \cdots, f^{k(z)-1}(\mathcal{L}_z(X))\}$ meets every critical point at most once and

$$\deg(f^{k(z)} : \mathcal{L}_z(X) \to \mathrm{Comp}_{f^{k(z)}(z)}(X)) \leq (\max_{c\in\mathrm{Crit}(f)} \deg_c(f))^{\#\mathrm{Crit}(f)}$$

(ii) Suppose X is nice and $z \in D(X)\backslash X$. Then $\mathcal{L}_w(X) = \mathcal{L}_z(X)$ for all $w \in \mathcal{L}_z(X)$ and $\mathcal{L}_{w'}(X) \cap \mathcal{L}_z(X) = \emptyset$ for all $w' \notin \mathcal{L}_z(X)$.

(iii) Suppose X is nice and $z \in D(X)\backslash X$. Then for all $0 < i < k(z)$, $f^i(\mathcal{L}_z(X)) = \mathcal{L}_{f^i(z)}(X)$.

Let K be a critical component of $\mathbf{K}_f$ and $c_1, c_2, \cdots, c_l$ be all the critical points on K. Then $P_n(c_1) = P_n(c_2) = \cdots = P_n(c_l)$ and

$$\deg(f|_{P_n(c_1)}) = (\deg_{c_1}(f) - 1) + \cdots + (\deg_{c_l}(f) - 1) + 1$$

for all $n \geq 0$. We can view K as a component containing one critical point of degree $\deg(f|_{P_n(c_1)})$. Hence in the following, we assume that each $\mathbf{V}$-component contains at most one critical point.

Now we will combine the property $(*)$ and the Spreading Principle appeared in [KSS] to prove Theorem 2.1.

Proof of Theorem 2.1. First fix $n \geq 1$. We shall repeat the proof of the Spreading Principle in [KSS] to get a qc map H_n from $\mathbf{V}$ onto $\tilde{\mathbf{V}}$.

Set $W_n := \bigcup_{c\in\mathrm{Crit}(f)} Q_n(c)$. Then by Lemma 3.1 (2), each component of $D(W_n)$ is mapped conformally onto a component of W_n by some iterate of f.

For every puzzle piece P, we can choose an arbitrary qc map $\phi_P : P \to \tilde{P}$ with $\phi_P|_{\partial P} = H|_{\partial P}$ since H is a qc map from a neighborhood of ∂P to a neighborhood of $\partial\tilde{P}$ and $\partial P, \partial\tilde{P}$ are quasi-circles (see e.g. [CT], Lemma C.1). Note that by the definition of W_n, there are finitely many critical puzzle pieces not contained in W_n. So we can take C_n' to be an upper bound for the maximal dilatation of all the qc maps ϕ_P, where P runs over all puzzle pieces of depth 0 and all critical puzzle pieces not contained in W_n.

Given a puzzle piece P, let $0 \leq k \leq \mathrm{depth}(P)$ be the minimal nonnegative integer such that $f^k(P)$ is a critical puzzle piece or has depth 0. Set $\tau(P) = f^k(P)$. Then $f^k : P \to \tau(P)$ is a conformal map and so is $\tilde{f}^k : \tilde{P} \to \tau(\tilde{P})$, where $\tilde{P}$ is the puzzle piece bounded by $H(\partial P)$ for $\tilde{f}$ and $\tau(\tilde{P}) = \tilde{f}^k(\tilde{P})$. Given a qc map $q : \tau(P) \to \tau(\tilde{P})$, we can lift it through the maps f^k and $\tilde{f}^k$, that is, there is a qc map $p : P \to \tilde{P}$ such that $\tilde{f}^k \circ p = q \circ f^k$. Notice that the maps p and q have the same maximal dilatation, and if $q|_{\partial\tau(P)} = H|_{\partial\tau(P)}$, then $p|_{\partial P} = H|_{\partial P}$.

Let $Y_0 = \mathbf{V}$ denote the union of all the puzzle pieces of depth 0. Set $X_0 = \emptyset$. For $j \geq 0$, we inductively define X_{j+1} to be the union of puzzle pieces of depth $j+1$ such that each of these pieces is contained in Y_j and is a component of $D(W_n)$; set $Y_{j+1} := (Y_j \cap f^{-(j+1)}(\mathbf{V}))\backslash X_{j+1}$. We have the following relations: for any $j \geq 0$,

$$Y_j = (Y_j\backslash f^{-(j+1)}(\mathbf{V})) \sqcup X_{j+1} \sqcup Y_{j+1}, \quad Y_{j+1} \subset\subset Y_j, \quad X_{j'} \cap X_j = \emptyset \text{ for any } j' \neq j\ .$$

Given any component Q of Y_{j+1}, we claim that $\tau(Q)$ is either one of the finitely many critical puzzle pieces not contained in W_n, or one of the finitely many puzzle pieces of depth 0. In fact, for such Q, either $Q \cap D(W_n) = \emptyset$ or $Q \cap D(W_n) \neq \emptyset$. In the former case, since $\mathrm{Crit}(f) \subset W_n \subset D(W_n)$, Q is mapped conformally onto a puzzle piece of depth 0 by the iterate of $f^{\mathrm{depth}(Q)}$. So $\tau(Q)$ is a puzzle piece of depth 0. In the latter case, if $Q \cap D(W_n) \subset\subset D(W_n)$, then Q is compactly contained in a component of $D(W_n)$, denoted by Q', and $Q' \subset\subset X_{j'}$ for some $j' < j+1$. But $Q \subset\subset Y_j \subset\subset Y_{j-1} \subset\subset \cdots \subset\subset Y_0$ and $X_j \cap Y_j = \emptyset, X_{j-1} \cap Y_{j-1} = \emptyset, \cdots, X_0 \cap Y_0 = \emptyset$. This is a contradiction. Hence $Q \cap D(W_n) \subset\subset Q$. If there is a critical point $c \in Q \cap D(W_n)$, then the component of W_n containing c is compactly contained in Q and $\tau(Q) = Q$. Otherwise, $\tau(Q)$ must be a critical puzzle piece not contained in W_n.

Define $H^{(0)} = \phi_P$ on each component P of Y_0. For each $j \geq 0$, assuming that $H^{(j)}$ is defined, we define $H^{(j+1)}$ as follows:

$$H^{(j+1)} = \begin{cases} H^{(j)} & \text{on } \mathbf{V} \backslash Y_j \\ H & \text{on } Y_j \backslash f^{-(j+1)}(\mathbf{V}) \\ \text{the univalent pullback of } \phi & \text{on each component of } X_{j+1} \\ \text{the univalent pullback of } \phi_{\tau(Q)} & \text{on each component } Q \text{ of } Y_{j+1}, \end{cases}$$

where the map ϕ is the qc-extension obtained by the assumption $(*)$.

Set $C_n = \max\{C_0, C_n', \tilde{C}\}$. The $\{H^{(j)}\}_{j \geq 0}$ is a sequence of C_n-qc maps. Hence it is precompact in the uniform topology.

By definition, $H^{(j)} = H^{(j+1)}$ outside Y_j. Thus, the sequence $\{H^{(j)}\}$ converges pointwise outside

$$\bigcap_j Y_j = \{x \in \mathbf{K}_f \mid f^k(x) \notin W_n,\ k \geq 0\}.$$

This set is a hyperbolic subset, on which f is uniformly expanding, and hence has zero Lebesgue measure, in particular no interior. So any two limit maps of the sequence $\{H^{(j)}\}_{j \geq 0}$ coincide on a dense open set of $\mathbf{V}$, therefore coincides on $\mathbf{V}$ to a unique limit map. Denote this map by H_n. It is C_n-qc.

By construction, H_n coincides with H on $\mathbf{V} \backslash ((\bigsqcup_j X_j) \cup (\bigcap_j Y_j))$ is therefore C_0-qc there; and is $\tilde{C}$-qc on $\bigsqcup_j X_j$. It follows that the maximal dilatation of H_n is bounded by $\max\{C_0, \tilde{C}\}$ except possibly on the set $\bigcap_j Y_j$. But this set has zero Lebesgue measure. It follows that the maximal dilatation of H_n is $\max\{C_0, \tilde{C}\}$, which is independent of n.

The sequence $H_n : \mathbf{V} \to \tilde{\mathbf{V}}$ has a subsequence converging uniformly to a limit qc map $H' : \mathbf{V} \to \tilde{\mathbf{V}}$, with $H'|_{\mathbf{V} \backslash \mathbf{K}_f} = H$. Therefore H' is a qc extension of H. On the other hand, $H \circ f = \tilde{f} \circ H$ on $\mathbf{U} \backslash \mathbf{K}_f$. So $H' \circ f = \tilde{f} \circ H'$ holds on $\mathbf{U} \backslash \mathrm{int}\mathbf{K}_f$ by continuity. Therefore H' is a qc-conjugacy off $\mathrm{int}\mathbf{K}_f$. This ends the proof of Theorem 2.1. □

4 Proof of Theorem 2.2

In this section let $f : \mathbf{U} \to \mathbf{V}$ be a map in the set-up. Let $q \geq 1$ denote the number of components of $\mathbf{V}$. Enumerate the $\mathbf{V}$-components by $V_1, V_2, \cdots, V_q$.

Lemma 4.1. *Let $\mathbf{W}$ be an open round disk centered at 0 with radius > 1 containing $\overline{\mathbf{V}}$. The map $f : \mathbf{U} \to \mathbf{V}$ extends to a map F on $\mathbf{V}$ so that*
–on each component V_i of $\mathbf{V}$, the restriction $F|_{V_i} : V_i \to \mathbf{W}$ is a quasi-regular branched covering;
–every component of $\mathbf{U}$ is a component of $F^{-1}(\mathbf{V})$;
–the restriction F on $F^{-1}(\mathbf{V})$ is holomorphic.

Proof. **Part I**. Fix any component V_i of $\mathbf{V}$ such that $V_i \cap \mathbf{U} \neq \emptyset$. We will extend $f|_{V_i \cap \mathbf{U}}$ to a map F on V_i with the required properties. It will be done in three steps. Refer to Figure 1 for the construction of F on V_i.

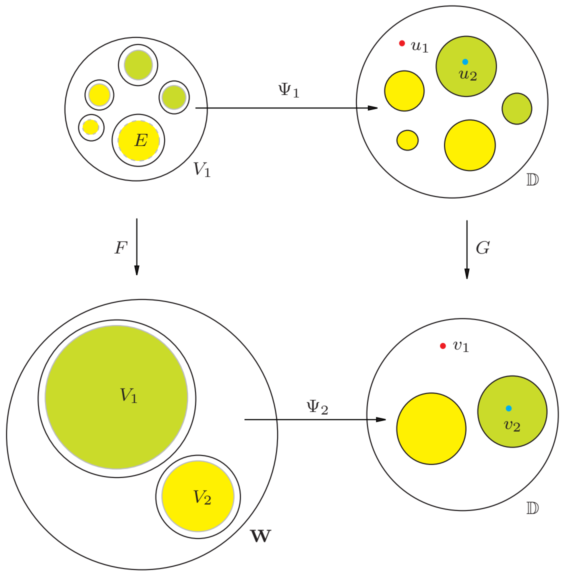

Figure 1 The construction of F on V_1. In this figure, $q = 2, i = 1$.

Step 1. The first step is to construct a Blaschke product $G : \mathbb{D} \to \mathbb{D}$ of degree d_i, where $\mathbb{D}$ denotes the unit disk and d_i is determined below.

For every component V_j of $\mathbf{V}$, we define

$$q_{ij} = \#\{U \text{ a component of } \mathbf{U} \mid U \subset V_i,\ f(U) = V_j\}$$

Set $q_i = \max_j q_{ij}$. Then $q_i \geq 1$ and

$$\#\{\text{components of } \mathbf{U} \cap V_i\} = \sum_j q_{ij} \leq q \cdot q_i \ .$$

We construct a Blaschke product $G : \mathbb{D} \to \mathbb{D}$, as well as a set $\mathcal{D}$ which is the union of q Jordan domains in $\mathbb{D}$ with pairwise disjoint closures, as follows:

• If V_i does not contain critical points of f, then set $G(z) = z^{q_i}$, and choose $\mathcal{D}$ to be a collection of q Jordan domains compactly contained in $\mathbb{D} \backslash \{0\}$ with pairwise disjoint closures. Set $d_i = q_i$. Note that each component of $\mathcal{D}$ has exactly q_i preimages. So

$$\begin{aligned} \#\{\text{components of } G^{-1}(\mathcal{D})\} &= q \cdot q_i \\ &\geq \#\{\text{components of } \mathbf{U} \cap V_i\} \ . \end{aligned}$$

• Otherwise, by assumption in the set-up, the set $\mathrm{Crit}(f)$ intersects exactly one component U of $\mathbf{U} \cap V_i$. Set $d_i = q_i + \deg f|_U - 1$. Choose G so that it has degree d_i, and has two distinct critical points u_1 and u_2 such that $\deg_{u_1}(G) = q_i$, $\deg_{u_2}(G) = \deg(f|_U)$ and $G(u_1) \neq G(u_2)$.[2] Set $v_i = G(u_i)$, $i = 1, 2$. Now choose $\mathcal{D}$

[2]One way to construct such a map G is as follows: Consider the map $z \mapsto z^{q_i}$ together with a preimage $x \in]0, 1[$ of $1/2$. Cut $\mathbb{D}$ along $[x, 1[$, glue in $\deg(f|_U)$ consecutive sectors to define a new space $\tilde{D}$. Define

to be a collection of q Jordan domains compactly contained in $\mathbb{D}\backslash\{v_1\}$ with pairwise disjoint closures and with $v_2 \in \mathcal{D}$. Note that the preimage of any $\mathcal{D}$-component not containing v_2 has d_i components, whereas the preimage of the $\mathcal{D}$-component containing v_2 has $d_i - \deg(f|_U) + 1 = q_i$ components. So

$$\begin{aligned}\#\{\text{components of } G^{-1}(\mathcal{D})\} &= (q-1)d_i + q_i \\ &= (q-1)(q_i + \deg f|_U - 1) + q_i \\ &= q \cdot q_i + (q-1)(\deg f|_U - 1) \\ &> q \cdot q_i \\ &\geq \#\{\text{components of } \mathbf{U} \cap V_i\} \ .\end{aligned}$$

In both cases $G : \overline{\mathbb{D}}\backslash G^{-1}(\mathcal{D}) \to \overline{\mathbb{D}}\backslash\mathcal{D}$ is a proper map with a unique critical point.

Step 2. Make $\mathbf{U}, \mathbf{V}$ 'thick'.

In $\mathbf{W}$, take q Jordan domains with smooth boundaries $\widehat{V}_j$, $j = 1, \cdots, q$, such that each $\widehat{V}_j$ is compactly contained in $\mathbf{W}$, $V_j \subset \widehat{V}_j$ for each $j = 1, \cdots, q$, and all of the $\widehat{V}_j$ have pairwise disjoint closures. Denote $\widehat{\mathbf{V}} = \cup_{j=1}^q \widehat{V}_j$.

In V_i, take $\widehat{\mathbf{U}}$ to be a union of $\#\{\text{components of } G^{-1}(\mathcal{D})\}$ (which is greater than the number of $\mathbf{U}$-components in V_i) Jordan domains with smooth boundaries with the following properties:

– $\widehat{\mathbf{U}}$ is compactly contained in V_i;
– $(\mathbf{U} \cap V_i)$ is compactly contained in $\widehat{\mathbf{U}}$;
– each component of $\widehat{\mathbf{U}}$ contains at most one component of $(\mathbf{U} \cap V_i)$;
– the components of $\widehat{\mathbf{U}}$ have pairwise disjoint closures.

There exists a qc map $\Psi_2 : \overline{\mathbf{W}} \to \overline{\mathbb{D}}$ such that $\Psi_2(\widehat{\mathbf{V}}) = \mathcal{D}$.

Let now U be any component of $\mathbf{U}$. There is a unique component $\widehat{U}$ of $\widehat{\mathbf{U}}$ containing U. Also $f(U) = V_j \subset \widehat{V}_j$ for some j, and $\Psi_2(\widehat{V}_j)$ is a component, denoted by $D(U)$, of $\mathcal{D}$. See the following diagram:

$$\begin{array}{lcc} U \subset \widehat{U} & & G^{-1}(D(U)) \\ \downarrow f & & \downarrow G \\ V_j \subset \widehat{V}_j & \xrightarrow{\Psi_2} & D(U) \end{array}$$

There is a qc map $\Psi_1 : \overline{V}_i \to \overline{\mathbb{D}}$ so that $\Psi_1(\widehat{\mathbf{U}}) = G^{-1}(\mathcal{D})$ and, for any component U of $V_i \cap \mathbf{U}$, the set $\Psi_1(\widehat{U})$ is a component of $G^{-1}(D(U))$. Then we can define a quasiregular branched covering $F : \overline{V}_i\backslash\widehat{\mathbf{U}} \to \overline{\mathbf{W}}\backslash\widehat{\mathbf{V}}$ of degree d_i to be

$$\Psi_2^{-1} \circ G|_{\overline{\mathbb{D}}\backslash G^{-1}(\mathcal{D})} \circ \Psi_1 \ .$$

Step 3. Glue.

Define at first $F = f$ on $V_i \cap \mathbf{U}$. For each component $\widehat{E}$ of $\widehat{\mathbf{U}}$ not containing a component of $\mathbf{U}$, take a Jordan domain E with smooth boundary compactly

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a new map that maps each sector homeomorphically onto $\mathbb{D}\backslash[\frac{1}{2}, 1[$, and agrees with $z \mapsto z^{q_i}$ elsewhere. This gives a branched covering $\hat{G}$ from $\tilde{D}$ onto $\mathbb{D}$ with two critical points and two critical values. Use $\hat{G}$ to pull back the standard complex structure of $\mathbb{D}$ turn $\tilde{D}$in a Riemann surface. Uniformize $\tilde{D}$ by a map $\phi : \mathbb{D} \to \tilde{D}$. Then $G = \hat{G} \circ \phi$ suites what we need.

contained in $\widehat{E}$. Then F maps $\partial\widehat{E}$ homeomorphically onto $\partial\widehat{V}_j$ for some j. Define F to be a conformal map from E onto V_j by Riemann Mapping Theorem and F extends homeomorphically from $\overline{E}$ onto $\overline{V}_j$.

Notice that the map F is defined everywhere except on a disjoint union of annular domains, one in each component of $\widehat{\mathbf{U}}$. Furthermore F maps the two boundary components of each such annular domain onto the boundary of $\widehat{V}_j \backslash V_j$ for some j, and is a covering of the same degree on each boundary component.

This shows that F admits an extension as a covering of these annular domains. As all boundary curves are smooth and F is quasi-regular outside the annular domains, the extension can be made quasi-regular as well.

Part II. We may now extend F to every $\mathbf{V}$ component intersecting $\mathbf{U}$ following the same procedure as shown in Part I. Assume that V_i is a $\mathbf{V}$-component disjoint from $\mathbf{U}$. We define $F : V_i \to \mathbf{W}$ to be a conformal homeomorphism and we set $d_i = 1$. We obtain a quasi-regular map $F : \mathbf{V} \to \mathbf{W}$ as an extension of $f : \mathbf{U} \to \mathbf{V}$. By construction, F is holomorphic on $F^{-1}(\mathbf{V})$. □

Lemma 4.2. *There is an integer d so that for the map $g : z \mapsto z^d$, the map F has an extension on $\mathbf{W}\backslash\mathbf{V}$ so that $F : \mathbf{W}\backslash\mathbf{V} \to g(\mathbf{W})\backslash\mathbf{W}$ is a quasi-regular branched covering, coincides with g on $\partial\mathbf{W}$ and is continuous on $\overline{\mathbf{W}}$. In particular $F^{-1}(\mathbf{W}) = \mathbf{V}$ and F is holomorphic on $F^{-2}(\mathbf{W}) = F^{-1}(\mathbf{V})$.*

Proof. Set $d = \sum_{i=1}^{q} d_i$, where the d_i's are defined in the proof of Lemma 4.1. See Figure 2 for the proof of this lemma.

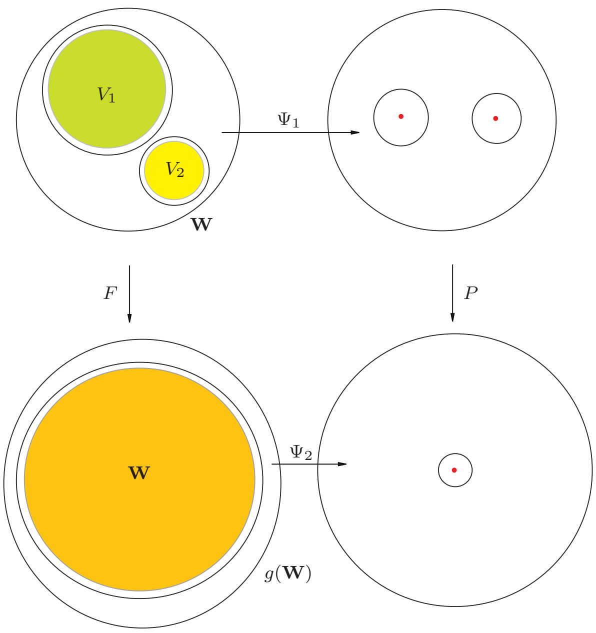


Figure 2

The domain $\widehat{\mathbf{V}}$ is defined as in the proof of the previous lemma. Now take a Jordan domain $\widehat{\mathbf{W}}$ with smooth boundary such that $\mathbf{W} \subset\subset \widehat{\mathbf{W}} \subset\subset g(\mathbf{W})$.

Let

$$P(z) = (z - a_1)^{d_1}(z - a_2)^{d_2} \cdots (z - a_q)^{d_q},$$

where $a_1, a_2, \cdots, a_q \in \mathbb{C}$ are distinct points.

Note that for each $1 \leq i \leq q$, we have $P(a_i) = 0$, and a_i is a critical point of P whenever $d_i > 1$.

Take $r > 0$ small enough and $R > 0$ large enough such that $\{0 < |z| \leq r\} \cup \{R \leq z| < \infty\}$ contains no critical value of P. Obviously $P : P^{-1}(\{r \leq |z| \leq R\}) \to \{r \leq |z| \leq R\}$ is a holomorphic proper map of degree d.

Note that $P^{-1}(\{|z| \leq R\}$ is a closed Jordan domain, and the set $P^{-1}(\{|z| \leq r\})$ consists of q disjoint closed Jordan domains, each containing exactly one of the a_i's in the interior.

There exist qc maps $\Phi_1 : \overline{\mathbf{W}} \to P^{-1}(\{|z| \leq R\})$, $\Phi_2 : g(\overline{\mathbf{W}}) \to \{|z| \leq R\}$ such that for $i = 1, \cdots, q$, the set $\Phi_1(\widehat{V_i})$ is equal to the component of $P^{-1}(\{|z| \leq r\})$ containing a_i, and $\Phi_2(\widehat{\mathbf{W}}) = \{|z| \leq r\}$, and that

$$P(\Phi_1(z)) = \Phi_2(g(z)),\ z \in \partial\mathbf{W}.$$

Set $F = \Phi_2^{-1} \circ P \circ \Phi_1$ on $\overline{\mathbf{W}} \backslash \widehat{\mathbf{V}}$.

Fix any $i = 1, \cdots, q$. Both maps $F : \partial\widehat{V_i} \to \partial\widehat{\mathbf{W}}$ and $F : \partial V_i \to \partial\mathbf{W}$ are coverings of degree d_i. We may thus extend as before F to a qusiregular covering map from $\widehat{V_i} \backslash V_i$ onto $\widehat{\mathbf{W}} \backslash \mathbf{W}$.

This ends the construction of F. □

Proof of Theorem 2.2. Extend the map F in Lemma 4.2 to $\overline{\mathbb{C}}$ by setting $F = g$ on $\overline{\mathbb{C}} \backslash \mathbf{W}$.

This F is quasi-regular, and is holomorphic on $(\overline{\mathbb{C}} \backslash \mathbf{W}) \cup F^{-2}(\mathbf{W})$. So every orbit passes at most twice the region $\mathbf{W} \backslash F^{-2}(\mathbf{W})$. By Surgery Principle (see Page 130 Lemma 15 in [Ah]), the map F is qc-conjugate to a polynomial h. The set $\mathbf{K}_F$ can be defined as for h and the two dynamical systems $F|_{\mathbf{K}_F}$ and $h|_{\mathbf{K}_h}$ are topologically conjugate.

Theorem 2.2 holds for the pair $(h, \mathbf{K}_h)$ (in place of $(f, \mathbf{K}_f)$) by Sullivan's no-wandering-domain theorem. It follows that the result also holds for $(F, \mathbf{K}_F)$. But $\mathbf{K}_f$ is an F-invariant subset of $\mathbf{K}_F$ with every component of $\mathbf{K}_f$ being a component of $\mathbf{K}_F$, and with $F|_{\mathbf{K}_f} = f|_{\mathbf{K}_f}$. So the theorem holds for the pair $(f, \mathbf{K}_f)$. □

5 Proof of Theorem 2.3 (a)

We just repeat the standard argument (see for example Appendix in [Mc2]).

Assume that $f, \tilde{f}$ are c-equivalent. Set $\mathbf{U} = \mathbf{U}_1$, and $\mathbf{U}_n = f^{-n}(\mathbf{V})$. The same objects gain a tilde for $\tilde{f}$. For $t \in [0, 1]$, let $h_t : \overline{\mathbf{V}} \to \overline{\tilde{\mathbf{V}}}$ be an isotopy path linking h_0 to h_1.

Then there is a unique continuous extension $(t, z) \mapsto h(t, z)$, $[0, \infty[\times \overline{\mathbf{V}} \to \overline{\tilde{\mathbf{V}}}$ such that

0) each $h_t : z \to h(t, z)$ is a homeomorphism,

1) $h_t|_{\partial\mathbf{V} \cup \mathbf{P}} = h_0|_{\partial\mathbf{V} \cup \mathbf{P}}$, $\forall t \in [0, +\infty[$,

2) for $n \geq 1$, $t > n$ and $x \in \mathbf{V} \backslash \mathbf{U}_n$ we have $h_t(x) = h_n(x)$,

3) for $t \in [0, 1]$ the following diagram commutes:

$$\begin{array}{ccc} \vdots & & \vdots \\ \mathbf{U}_2 & \xrightarrow{h_{t+2}} & \tilde{\mathbf{U}}_2 \\ \downarrow f & & \downarrow \tilde{f} \\ \mathbf{U}_1 & \xrightarrow{h_{t+1}} & \tilde{\mathbf{U}}_1 \\ \downarrow f & & \downarrow \tilde{f} \\ \mathbf{V} & \xrightarrow{h_t} & \tilde{\mathbf{V}}. \end{array}$$

Set then $\Omega = \bigcup_{n\geq 1} \mathbf{V}\backslash\mathbf{U}_n = \mathbf{V}\backslash\mathbf{K}_f$, and $\tilde{\Omega} = \tilde{\mathbf{V}}\backslash\mathbf{K}_{\tilde{f}}$. Then there is a qc map $H : \Omega \to \tilde{\Omega}$ such that $H(x) = h_n(x)$ for $n \geq 1$ and $x \in \mathbf{V}\backslash\mathbf{U}_n$ and that $H \circ f|_{\Omega\cap\mathbf{U}} = \tilde{f} \circ H|_{\tilde{\Omega}\cap\tilde{\mathbf{U}}}$, i.e. H realizes a qc-conjugacy from f to $\tilde{f}$ off $\mathbf{K}_f$. The qc constant of H is equal to C_0, the qc constant of h_1 on $\mathbf{V}\backslash\mathbf{U}$.

6 Proof of Theorem 2.3 (b)

Main Proposition. *Let f be a map as in the set-up. Assume that for every critical component $\mathbf{K}_f(c)$, $c \in \mathrm{Crit}(f)$, satisfying that $f^l(\mathbf{K}_f(c))$ is a critical periodic component of $\mathbf{K}_f$ for some $l \geq 1$, there are a constant M_c and an integer $N_c \geq 0$ such that for each $n \geq N_c$, the map $H|_{\partial P_n(c)}$ admits an M_c-qc extension inside $P_n(c)$, where $P_n(c)$ is a puzzle piece of depth n containing c. Then f satisfies the property $(*)$ stated in Theorem 2.1.*

We will postpone the proof of Main Proposition in the next section. Here we combine this proposition and Theorem 2.1 to give a proof of Theorem 2.3 (b).

Proof of Theorem 2.3 (b). Combining Main Proposition and Theorem 2.1, we have a qc conjugacy off $\mathbf{K}_f$ admits a qc extension across $\mathbf{K}_f$ which is a conjugacy off $\mathrm{int}\mathbf{K}_f$. $\square$

7 Proof of Main Proposition

In this section, we always assume f is a map in the set-up with the assumption that each $\mathbf{V}$-component contains at most one critical point.

7.1 Reduction of Main Proposition

Definition 2. *(1) For $x, y \in \mathbf{K}_f$, we say that the forward orbit of x **combinatorially accumulates** to y, written as $x \to y$, if for any $n \geq 0$, there is $j \geq 1$ such that $f^j(x) \in P_n(y)$.*

Clearly, if $x \to y$ and $y \to z$, then $x \to z$.

Let $\mathrm{Forw}(x) = \{y \in \mathbf{K}_f \mid x \to y\}$ for $x \in \mathbf{K}_f$.

(2) Define an equivalence relation in $\mathrm{Crit}(f)$ as follows:

$$\text{for } c_1, c_2 \in \mathrm{Crit}(f), c_1 \sim c_2 \Longleftrightarrow c_1 = c_2, \text{ or } c_1 \to c_2 \text{ and } c_2 \to c_1.$$

Let $[c]$ denote the equivalence class containing c for $c \in \mathrm{Crit}(f)$.

It is clear that $[c] = \{c\}$ if $c \not\to c$.

(3) We say that $[c_1]$ accumulates to $[c_2]$, written as $[c_1] \to [c_2]$, if

$$\exists\, c_1' \in [c_1], \exists\, c_2' \in [c_2] \text{ such that } c_1' \to c_2'.$$

It is easy to check that if $[c_1] \to [c_2]$, then

$$\forall\, c_1'' \in [c_1], \exists\, c_2'' \in [c_2] \text{ such that } c_1'' \to c_2''.$$

It follows from this property that if $[c_1] \to [c_2], [c_2] \to [c_3]$, then $[c_1] \to [c_3]$.
(4) Define $\mathcal{D}(f) := \mathrm{Crit}(f)/\sim$. Define a partial order $\leq$ in $\mathcal{D}(f)$:

$$[c_1] \leq [c_2] \Longleftrightarrow [c_1] = [c_2] \text{ or } [c_2] \to [c_1].$$

We can decompose the quotient $\mathcal{D}(f)$ as follows. Let $\mathcal{D}_0(f)$ be the set of elements in $\mathcal{D}(f)$ which are minimal in the partial order $\leq$, that is, $[c] \in \mathcal{D}_0(f)$ if and only if $[c]$ doesn't accumulate to any element in $\mathcal{D}(f)\backslash\{[c]\}$. For every $k \geq 0$, assume $\mathcal{D}_k(f)$ is defined, then $\mathcal{D}_{k+1}(f)$ is defined to be the set of elements in $\mathcal{D}(f)$ which are minimal in the set $\mathcal{D}(f) \setminus (\mathcal{D}_k(f) \cup \mathcal{D}_{k-1}(f) \cup \cdots \cup \mathcal{D}_0(f))$ in the partial order $\leq$.

For the construction above, we can prove the properties below.

Lemma 7.1. *(P1) There is an integer $M \geq 0$ such that $\mathcal{D}(f) = \bigsqcup_{k=0}^{M} \mathcal{D}_k(f)$.*

(P2) For every $k \geq 0$, given $[c_1], [c_2] \in \mathcal{D}_k(f), [c_1] \neq [c_2]$, we have $[c_1] \not\to [c_2]$ and $[c_2] \not\to [c_1]$.

(P3) Let $[c_1] \in \mathcal{D}_s(f), [c_2] \in \mathcal{D}_t(f)$ with $s < t$. Then $[c_1] \not\to [c_2]$.

(P4) For every $k \geq 1$, every $[c]$ in $\mathcal{D}_k(f)$ accumulates to some element in $\mathcal{D}_{k-1}(f)$.

Proof. (P1) holds because $\mathcal{D}(f)$ is a finite set and $\mathcal{D}_i(f) \cap \mathcal{D}_j(f) = \emptyset$ for $i \neq j$.

(P2) and (P3) follow directly from the minimal property of the elements in $\mathcal{D}_k(f)$ for every $0 \leq k \leq M$.

(P4) Let $k = 1$. If there is some element $[c_1] \in \mathcal{D}_1(f)$ such that it doesn't accumulate to any element in $\mathcal{D}_0(f)$, then combining with (P2) and (P3), we have $[c_1]$ doesn't accumulate to any element in $\mathcal{D}(f)\backslash\{[c_1]\}$. Consequently, $[c_1] \in \mathcal{D}_0(f)$. But $[c_1] \in \mathcal{D}_1(f)$ and by (P1), $\mathcal{D}_0(f) \cap \mathcal{D}_1(f) = \emptyset$. We get a contradiction. So any element in $\mathcal{D}_1(f)$ will accumulate to some element in $\mathcal{D}_0(f)$.

Now we suppose $k \geq 2$ and (P4) holds for $\mathcal{D}_1(f), \mathcal{D}_2(f), \cdots, \mathcal{D}_{k-1}(f)$. Assume (P4) is not true for $\mathcal{D}_k(f)$, that is, there is some $[c_k] \in \mathcal{D}_k(f)$ such that $[c_k]$ doesn't accumulate to any element in $\mathcal{D}_{k-1}(f)$.

If $[c_k]$ doesn't accumulate to any element in $\cup_{j=0}^{k-2}\mathcal{D}_j(f)$, then by (P2) and (P3), we conclude that $[c_k] \in \mathcal{D}_0(f)$ which contradicts the condition that $[c_k] \in \mathcal{D}_k(f)$ and the fact that $\mathcal{D}_k(f) \cap \mathcal{D}_0(f) = \emptyset$ by (P1).

Let $0 \leq i \leq k-2$ be an integer satisfying that $[c_k]$ won't accumulate to any element in $\cup_{j=i+1}^{k-1}\mathcal{D}_j(f)$ and $[c_k]$ accumulates to some element in $\mathcal{D}_i(f)$. Then $[c_k]$ won't accumulate to any element in $\cup_{j=i+1}^{M} D_j(f)\backslash\{[c_k]\}$ and hence $[c_k] \in \mathcal{D}_{i+1}(f)$. But notice that $i+1 \leq k-1$ and $[c_k] \in \mathcal{D}_k(f)$. A contradiction. □

Combining with the transitive property stated in Definition 2 (3), we can consecutively apply (P3) above and prove the following.

Corollary 7.2. *For every $k \geq 1$, every $[c]$ in $\mathcal{D}_k(f)$ accumulates to some element in $\mathcal{D}_0(f)$.*

We will deduce Main Proposition from the following result.

Proposition 7.3. *Assume that for every critical component $\mathbf{K}_f(c)$, $c \in \mathrm{Crit}(f)$, satisfying that $f^l(\mathbf{K}_f(c))$ is a periodic critical component of $\mathbf{K}_f$ for some $l \geq 1$, there are a constant M_c and an integer $N_c \geq 0$ such that for each $n \geq N_c$, the map $H|_{\partial P_n(c)}$ admits an M_c-qc extension inside $P_n(c)$, where $P_n(c)$ is a puzzle piece of depth n containing c. Then for every $c \in [c_0]$ and every integer $n \geq 1$, there is a puzzle piece $K_n(c)$ containing c with the following properties.*

(i) For every $c \in [c_0]$, the pieces $\{K_n(c)\}_{n\geq 1}$ is a nested sequence.

(ii) For each $n \geq 1$, $\bigcup_{c\in[c_0]} K_n(c)$ is a nice set.

(iii) There is a constant $\tilde{M} = \tilde{M}([c_0])$, such that for each $n \geq 1$ and each $c \in [c_0]$, $H|_{\partial K_n(c)}$ admits an $\tilde{M}$-qc extension inside $K_n(c)$.

We will postpone the proof of Proposition 7.3 to the next subsection. Here we prove the following lemma and then use it and Proposition 7.3 to prove Main Proposition.

Lemma 7.4. *Let $[c_1]$ and $[c_2]$ be two distinct equivalence classes. Suppose that for each $i = 1, 2$, W_i is a nice set consisting of finitely many puzzle pieces such that each piece contains a point in $[c_i]$.*

(1) If $[c_1] \not\to [c_2]$ and $[c_2] \not\to [c_1]$, then $W_1 \cup W_2$ is a nice set containing $[c_1] \cup [c_2]$.

(2) Suppose $[c_2] \not\to [c_1]$ and

$$\min_{P_2 \text{ a comp. of } W_2} \mathrm{depth}(P_2) \geq \max_{P_1 \text{ a comp. of } W_1} \mathrm{depth}(P_1),$$

i.e., the minimal depth of the components of W_2 is not less than the maximal depth of those of W_1. Then $W_1 \cup W_2$ is nice.

Before proving this lemma, we need to give an assumption for simplicity. Notice that given two critical points c, c', if $c \not\to c'$, then there is some integer $n(c, c')$ depending on c and c' such that for all $j \geq 1$, for all $n \geq n(c, c')$, $f^j(c) \notin P_n(c')$. Since $\#\mathrm{Crit}(f) < \infty$, we can take $n_0 = \max\{n(c, c') \mid c, c' \in \mathrm{Crit}(f)\}$. Without loss of generality, we may assume that $n_0 = 0$, that is to say we assume that

$$(**) \qquad \text{for any two critical points } c, c', \text{ for all } j \geq 1, f^j(c) \notin P_0(c') \text{ if } c \not\to c'.$$

In the following paragraphs until the end of this article, we always assume $(**)$ holds.

Proof of Lemma 7.4. (1) According to Definition 2 (3) and the assumption $(**)$, we know that

$$\begin{aligned}
[c_1] \not\to [c_2] &\iff \forall c_1' \in [c_1], \forall c_2' \in [c_2], c_1' \not\to c_2' \\
&\iff \forall c_1' \in [c_1], \forall c_2' \in [c_2], \forall n \geq 0, \forall j \geq 0, f^j(c_1') \notin P_n(c_2') \\
&\iff \forall \text{ a puzzle piece } P \ni c_1', \forall n \geq 0, \forall j \geq 0, f^j(P) \cap P_n(c_2') = \emptyset.
\end{aligned}$$

In particular, for any $c_1' \in [c_1]$, any $c_2' \in [c_2]$, for the component P_1 of W_1 containing c_1' and the component P_2 of W_2 containing c_2', for any $j \geq 0$, $f^j(P_1) \cap P_2 = \emptyset$. It is equivalent to say that for any component P of W_1, for any $j \geq 0$, $f^j(P) \cap W_2 = \emptyset$.

Similarly, from the condition that $[c_2] \not\to [c_1]$, we can conclude that for any component Q of W_2, for any $j \geq 0$, $f^j(Q) \cap W_1 = \emptyset$. Hence $W_1 \cup W_2$ is a nice set.

(2) On one hand, from the proof of (1), we know that $[c_2] \not\to [c_1]$ implies that for any component Q of W_2, for any $j \geq 0$, $f^j(Q) \cap W_1 = \emptyset$.

On the other hand, for any component P of W_1, for any $j \geq 0$, we have

$$\begin{aligned} \mathrm{depth}(f^j(P)) &= \mathrm{depth}(P) - j \\ &\leq \max_{P_1 \text{ a comp. of } W_1} \mathrm{depth}(P_1) - j \\ &\leq \min_{P_2 \text{ a comp. of } W_2} \mathrm{depth}(P_2) \end{aligned}$$

and then $f^j(P)$ can not be strictly contained in W_2.

Hence $W_1 \cup W_2$ is nice. □

Now we can derive Main Proposition from Proposition 7.3.

Proof of Main Proposition. (i) follows immediately from Proposition 7.3 (i).

(ii) For every $[\tilde{c}] \in \mathcal{D}(f)$ and every $\hat{c} \in [\tilde{c}]$, let $\{K_n(\hat{c})\}_{n\geq 1}$ be the puzzle pieces obtained in Proposition 7.3.

Given $[c_0] \in \mathcal{D}_k(f)$, $0 \leq k < M$, let $A_k([c_0]) = \{[c] \in \mathcal{D}_{k+1}(f) \mid [c] \to [c_0]\}$. Clearly, $\#A_k([c_0]) < \infty$.

Recall that $\mathcal{D}(f) = \sqcup_{i=0}^{M} \mathcal{D}_i(f)$. For every $[c_0] \in \mathcal{D}_M(f)$, set $Q_n(c) = K_n(c)$ for each $c \in [c_0]$.

Now consider $[c_0] \in \mathcal{D}_{M-1}(f)$.

If $A_{M-1}([c_0]) = \emptyset$, then set $Q_n(c) = K_n(c)$ for each $c \in [c_0]$.

Otherwise, there exists a subsequence $\{l_n\}_{n\geq 1}$ of $\{n\}$ such that

$$\min_{c' \in [c_0]} \mathrm{depth}(K_{l_n}(c')) \geq \max_{[c'] \in A_{M-1}([c_0])} \mathrm{depth}(Q_n(c'))$$

because $\min_{c' \in [c_0]} \mathrm{depth}(K_n(c'))$ increasingly tends to the infinity as $n \to \infty$.

We repeat this process consecutively to $\mathcal{D}_{M-2}(f), \cdots, \mathcal{D}_0(f)$ and then all $Q_n(c)$ are defined. Combining the properties (P2), (P3) stated in Lemma 7.1 and Lemma 7.4, we easily conclude that $\cup_{c \in \mathrm{Crit}(f)} Q_n(c)$ is a nice set for every $n \geq 1$.

(iii) Since $\#\mathcal{D}(f) < \infty$, we can take the constant $\tilde{C} = \max\{\tilde{M}([\tilde{c}]) \mid [\tilde{c}] \in \mathcal{D}(f)\}$. □

7.2 Proof of Proposition 7.3

First, we need to introduce a classification of the set $\mathrm{Crit}(f)$ and several preliminary results.

Definition 3. *(i) Suppose $c \to c$. For $c_1, c_2 \in [c]$, we say that $P_{n+k}(c_1)$ is a **child** of $P_n(c_2)$ if $f^k(P_{n+k}(c_1)) = P_n(c_2)$ and $f^{k-1} : P_{n+k-1}(f(c_1)) \to P_n(c_2)$ is conformal.*

*c is called **persistently recurrent** if for every $n \geq 0$, every $c' \in [c]$, $P_n(c')$ has finitely many children. Otherwise, c is said to be **reluctantly recurrent**.*

It is easy to check that if c is persistently recurrent, then so is every $c' \in [c]$ and this is also true for a reluctantly recurrent c.

(ii) Let

$$\begin{aligned} \mathrm{Crit_n}(f) &= \{c \in \mathrm{Crit}(f) \mid c \not\to c' \textit{ for any } c' \in \mathrm{Crit}(f)\}, \\ \mathrm{Crit_e}(f) &= \{c \in \mathrm{Crit}(f) \mid c \not\to c \textit{ and } \exists\, c' \in \mathrm{Crit}(f) \textit{ such that } c \to c'\}, \\ \mathrm{Crit_r}(f) &= \{c \in \mathrm{Crit}(f) \mid c \to c \textit{ and } c \textit{ is reluctantly recurrent}\}, \\ \mathrm{Crit_p}(f) &= \{c \in \mathrm{Crit}(f) \mid c \to c \textit{ and } c \textit{ is persistently recurrent}\}. \end{aligned}$$

Then $\mathrm{Crit}(f) = \mathrm{Crit_n}(f) \sqcup \mathrm{Crit_e}(f) \sqcup \mathrm{Crit_r}(f) \sqcup \mathrm{Crit_p}(f)$ *is a classification of* $\mathrm{Crit}(f)$.

In this section, we will use sometimes the combinatorial tool – the tableau defined by Branner-Hubbard in [BH]. The reader can also refer to [QY] and [PQRTY] for the definition of the tableau.

For $x \in \mathbf{K}_f$, the tableau $\mathcal{T}(x)$ is the graph embedded in $\{(u,v) \mid u \in \mathbb{R}^-, v \in \mathbb{R}\}$ with the axis of u pointing upwards and the axis of v pointing rightwards (this is the standard $\mathbb{R}^2$ with reversed orientation), with vertices indexed by $-\mathbb{N}\times\mathbb{N}$, where $\mathbb{N} = \{0, 1, \cdots\}$, with the vertex at $(-m, 0)$ being $P_m(x)$, the puzzle piece of depth m containing x, and with $f^j(P_m(x))$ occupying the $(-m+j, j)$th entry of $\mathcal{T}(x)$. The vertex at $(-m+j, j)$ is called critical if $f^j(P_m(x))$ contains a critical point. If $f^j(P_m(x))$ contains some $y \in \mathbf{K}_f$, we call the vertex at $(-m+j, j)$ is a y-vertex.

All tableau satisfy the following three basic rules (see [BH], [QY], [PQRTY]).

(Rule 1). In $\mathcal{T}(x)$ for $x \in \mathbf{K}_f$, if the vertex at $(-m, n)$ is a y-vertex, then so is the vertex at $(-i, n)$ for every $0 \le i \le m$.

(Rule 2). In $\mathcal{T}(x)$ for $x \in \mathbf{K}_f$, if the vertex at $(-m, n)$ is a y-vertex, then for every $0 \le i \le m$, the vertex at $(-m+i, n+i)$ is a vertex being $P_{-m+i}(f^i(y))$.

(Rule 3) (See Figure 3). Given $x_1, x_2 \in \mathbf{K}_f$. Suppose there exist integers $m_0 \ge 1, n_0 \ge 0, i_0 \ge 1, n_1 \ge 1$ and critical points c_1, c_2 with the following properties.

(i) In $\mathcal{T}(x_1)$, the vertex at $(-(m_0+1), n_0)$ is a c_1-vertex and $(-(m_0+1-i_0), n_0+i_0)$ is a c_2-vertex.

(ii) In $\mathcal{T}(x_2)$, the vertex at $(-m_0, n_1)$ is a c_1-vertex and $(-(m_0+1), n_1)$ is not critical.

If in $\mathcal{T}(x_1)$, for every $0 < i < i_0$, the vertex at $(-(m_0-i), n_0+i)$ is not critical, then in $\mathcal{T}(x_2)$, the vertex at $(-(m_0+1-i_0), n_1+i_0)$ is not critical.

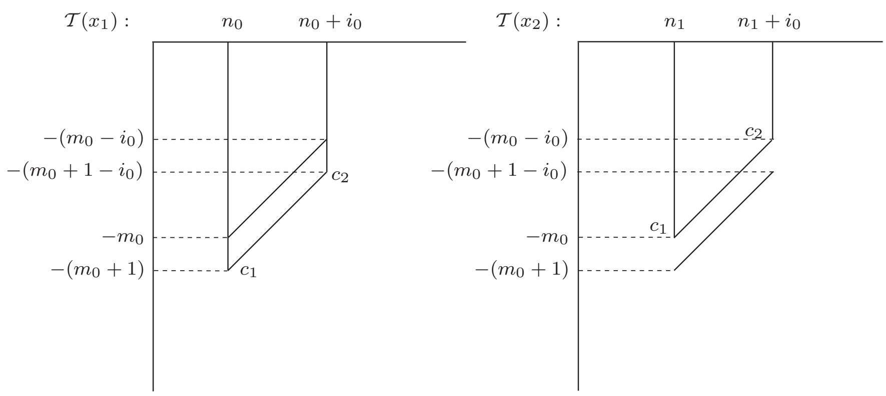


Figure 3

Recall that in subsection 7.1, we make the assumption (**). Here, we translate that assumption in the language of the tableau. It is equivalent to assume that for $c, c' \in \mathrm{Crit}(f)$, c'-vertex appears in $\mathcal{T}(c)$ iff $c' \in \mathrm{Forw}(c)$.

Lemma 7.5. *1. Let $\mathbf{K}_f(c)$ be a periodic component of $\mathbf{K}_f$ with period p. Then the following properties hold.*

(1) $f^i(\mathbf{K}_f(c')) \in \{\mathbf{K}_f(c), f(\mathbf{K}_f(c)), \cdots, f^{p-1}(\mathbf{K}_f(c))\}$, $\forall c' \in \mathrm{Forw}(c)$, $\forall i \ge 0$.

(2) $\mathrm{Forw}(c) = [c]$.

(3) $c \in \mathrm{Crit_p}(f)$.

2. Let $c \in \mathrm{Crit_p}(f)$ with $\mathbf{K}_f(c)$ non-periodic. Then the following properties hold.

(1) $\mathrm{Forw}(c) = [c]$.

(2) For every $c' \in [c]$, $c' \in \mathrm{Crit_p}(f)$ with $\mathbf{K}_f(c')$ non-periodic.

Proof. 1. Notice that $\mathbf{K}_f(c)$ is periodic iff there is a column in $\mathcal{T}(c)\backslash\{0\text{-th column}\}$ such that every vertex on that column is a c-vertex. According to this and using the tableau rules, it is easy to check that the statements in Point 1 are true.

2. (1) This property is the same as Lemma 1 in [QY]. For self-containedness, we introduce the proof here.

Assume there is some $c' \in \mathrm{Crit}(f)$ with $c \to c'$ but $c' \not\to c$. In the following, all the vertices we discuss are in $\mathcal{T}(c)$. One may refer to Figure 4 for the proof.

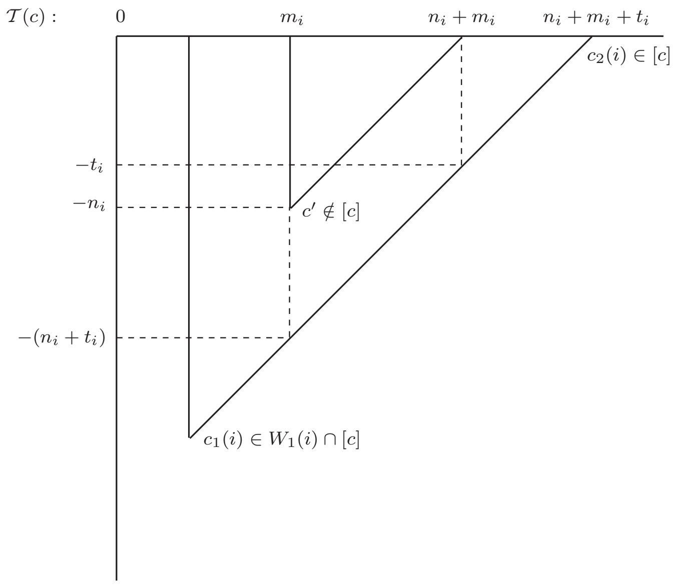


Figure 4

If there exists a column such that every vertex on it is a c'-vertex, then $c' \to c$ because $c \to c$. Hence there are infinitely many c'-vertices $\{(-n_i, m_i)\}_{i\geq 1}$ such that $(-(n_i+1), m_i)$ is not critical and $\lim_{i\to\infty} n_i = \infty$.

By the tableau rule (Rule 2) and the assumption $(**)$, we can see that there are no vertices being critical points in $[c]$ on the diagonal starting from the vertex $(-n_i, m_i)$ and ending at the 0-th row. Since $c \to c$, from the vertex $(0, n_i+m_i)$, one can march horizontally $t_i \geq 1$ steps to the right until the first hit of some $c_2(i)$-vertex in $[c]$. Then by (Rule 1), there are no vertices being critical points in $[c]$ on the diagonal from the vertex $(-t_i, n_i+m_i)$ to the vertex $(0, n_i+m_i+t_i)$. Therefore, there are no vertices being critical points in $[c]$ on the diagonal from the vertex $(-(n_i+t_i), m_i)$ to the vertex $(0, n_i+m_i+t_i)$, denote this diagonal by I.

If there exists a point $\tilde{c} \in \mathrm{Forw}(c)\backslash[c]$ on the diagonal I, then by the assumption $(**)$, every vertex, particularly the end vertex $(0, n_i+m_i+t_i)$ of I, can't be a $\hat{c}$-vertex for any $\hat{c} \in [c]$. This contradicts the choice that the vertex $(0, n_i+m_i+t_i)$ is a $c_2(i)$-vertex for $c_2(i) \in [c]$.

Consequently, there are no critical points in $\mathrm{Forw}(c)$ on the diagonal I. Combining with the assumption $(**)$, we know that there are no critical points on the diagonal I.

Follow the diagonal from the vertex $(-(n_i+t_i), m_i)$ left downwards until we reach a critical vertex $W_1(i)$ (such $W_1(i)$ exists since the 0-th column vertex on that diagonal is critical). Let $c_1(i)$ be the critical point in $W_1(i)$. Then $c_1(i) \in [c]$ follows from the fact that $(0, n_i+m_i+t_i)$ is a $c_2(i)$-vertex for $c_2(i) \in [c]$ and the assumption $(**)$

Therefore, $W_1(i)$ is a child of $P_0(c_2(i))$. Notice that the depth of $W_1(i)$ is greater than n_i. As $c_2(i)$ lives in the finite set $[c]$ and $n_i \to \infty$ when $i \to \infty$, some point in $[c]$ must have infinitely many children. This is a contradiction with the condition that $c \in \mathrm{Crit}_p(f)$.

(2) follows directly from Point 1 (1) and Point 1 (2). $\square$

Set

$$\mathrm{Crit}_{\mathrm{per}}(f) = \{c \in \mathrm{Crit}_{\mathrm{p}}(f) \mid \mathbf{K}_f(c) \text{ is periodic}\}.$$

Lemma 7.6. *(i) If* $c \in \mathrm{Crit}_{\mathrm{n}}(f) \cup \mathrm{Crit}_{\mathrm{p}}(f)$*, then* $[c] \in \mathcal{D}_0(f)$*; if* $c \in \mathrm{Crit}_{\mathrm{e}}(f)$*, then* $[c] \notin \mathcal{D}_0(f)$*.*

(ii) For every $c_0 \in \mathrm{Crit}(f)$*, one of the following cases will occur.*

Case 1. $\mathrm{Forw}(c_0) \cap (\mathrm{Crit}_{\mathrm{n}}(f) \cup \mathrm{Crit}_{\mathrm{r}}(f)) \neq \emptyset$.

Case 2. $\mathrm{Forw}(c_0) \subset \mathrm{Crit}_{\mathrm{p}}(f)$.

Case 3. For any $c \in \mathrm{Forw}(c_0)$*, either* $c \in \mathrm{Crit}_{\mathrm{p}}(f)$ *or,* $c \in \mathrm{Crit}_{\mathrm{e}}(f)$ *and* $\mathrm{Forw}(c)$ *contains a critical point in* $\mathrm{Crit}_{\mathrm{p}}(f)$*, and the latter* c *always exists.*

Proof. (i) By the definitions of $\mathrm{Crit}_{\mathrm{n}}(f)$ and $\mathrm{Crit}_{\mathrm{e}}(f)$, we easily see that if $c \in \mathrm{Crit}_{\mathrm{n}}(f)$, $[c] = \{c\} \in \mathcal{D}_0(f)$ and if $c \in \mathrm{Crit}_{\mathrm{e}}(f)$, $[c] = \{c\} \notin \mathcal{D}_0(f)$. If $c \in \mathrm{Crit}_{\mathrm{p}}(f)$, then by the previous lemma, we know that $\mathrm{Forw}(c) = [c]$ and then $[c] \in \mathcal{D}_0(f)$.

(ii) Suppose that Case 1 and Case 2 do not happen. Let $c \in \mathrm{Forw}(c_0)$ with $c \notin \mathrm{Crit}_{\mathrm{p}}(f)$. Notice that $\mathrm{Crit}(f) = \mathrm{Crit}_{\mathrm{n}}(f) \cup \mathrm{Crit}_{\mathrm{r}}(f) \cup \mathrm{Crit}_{\mathrm{p}}(f) \cup \mathrm{Crit}_{\mathrm{e}}(f)$. So $c \in \mathrm{Crit}_{\mathrm{e}}(f)$ and then by (i), $[c] \in \mathcal{D}_k(f)$ for some $k \geq 1$. It follows from Corollary 7.2 that $[c] = \{c\}$ accumulates to some element $[\tilde{c}] \in \mathcal{D}_0(f)$.

Since Case 1 does not happen, we conclude that for every $[\hat{c}] \in \mathcal{D}_0(f)$ with $[c_0] \to [\hat{c}]$, every point in $[\hat{c}]$ belongs to $\mathrm{Crit}_{\mathrm{p}}(f)$. Note that $[c_0] \to [c] \to [\tilde{c}]$ and $[\tilde{c}] \in \mathcal{D}_0(f)$. Hence every point in $[\tilde{c}]$ belongs to $\mathrm{Crit}_{\mathrm{p}}(f)$, particularly, $\tilde{c} \in \mathrm{Crit}_{\mathrm{p}}(f)$. $\square$

Recall that in section 3, for an open set X consisting of finitely many puzzle pieces, we define the set $D(X)$ and $\mathcal{L}_z(X)$ for $z \in D(X)\backslash X$. The following is a property about $\mathcal{L}_z(X)$ when X consists of a single piece.

Lemma 7.7. *Let* P *be a puzzle piece and the set* $\{x_1, \cdots, x_m\} \subset \mathbf{V}$ *be a finite set of points with each* $x_i \in D(P)\backslash P$ *for* $1 \leq i \leq m$*. Let* $f^{k_i}(\mathcal{L}_{x_i}(P)) = P$ *for some* $k_i \geq 1$*. Then*

(1) for every $1 \leq i \leq m$*, every* $0 \leq j < k_i$*, either*

$$f^j(\mathcal{L}_{x_i}(P)) = \mathcal{L}_{x_s}(P) \textit{ for some } 1 \leq s \leq m,$$

or

$$f^j(\mathcal{L}_{x_i}(P)) \cap \mathcal{L}_{x_t}(P) = \emptyset \textit{ for all } 1 \leq t \leq m;$$

(2) $\cup_{i=1}^m \mathcal{L}_{x_i}(P) \bigcup P$ *is a nice set.*

Proof. (1) (by contradiction). Assume that there are integers $1 \leq i_0 \leq m$, $0 \leq j_0 < k_{i_0}$, and there is some $\mathcal{L}_{x_{i_1}}(P)$ for $1 \leq i_1 \leq m$, such that

$$f^{j_0}(\mathcal{L}_{x_{i_0}}(P)) \neq \mathcal{L}_{x_{i_1}}(P) \text{ and } f^{j_0}(\mathcal{L}_{x_{i_0}}(P)) \cap \mathcal{L}_{x_{i_1}}(P) \neq \emptyset.$$

Then either $f^{j_0}(\mathcal{L}_{x_{i_0}}(P)) \subset\subset \mathcal{L}_{x_{i_1}}(P)$ or $f^{j_0}(\mathcal{L}_{x_{i_0}}(P)) \supset\supset \mathcal{L}_{x_{i_1}}(P)$.

We may assume $f^{j_0}(\mathcal{L}_{x_{i_0}}(P)) \subset\subset \mathcal{L}_{x_{i_1}}(P)$. The proof of the other case is similar to this case.

On one hand , since $f^{k_{i_0}-j_0} : f^{j_0}(\mathcal{L}_{x_{i_0}}(P)) \to P$ and $f^{k_{i_1}} : \mathcal{L}_{x_{i_1}}(P) \to P$, we have $k_{i_0} - j_0 > k_{i_1}$.

On the other hand, we know that $k_{i_0}-j_0$ is the first landing time of the points in $f^{j_0}(\mathcal{L}_{x_{i_0}}(P))$ to P because $f^{j_0}(\mathcal{L}_{x_{i_0}}(P))=\mathcal{L}_{f^{j_0}(x_{i_0})}(P)$ by Corollary 3.2 (3), while from the assumption that $f^{j_0}(\mathcal{L}_{x_{i_0}}(P))\subset\subset\mathcal{L}_{x_{i_1}}(P)$, we have k_{i_1} is also the first landing time of the points in $f^{j_0}(\mathcal{L}_{x_{i_0}}(P))$ to P. So $k_{i_0}-j_0=k_{i_1}$. A contradiction.

(2) For any $q\geq 1$ (as long as $\mathrm{depth}(f^q(P))\geq 0$),

$$\mathrm{depth}(f^q(P))<\mathrm{depth}(P)<\mathrm{depth}(\mathcal{L}_{x_s}(P))$$

for every $1\leq s\leq m$. So $f^q(P)$ can not be strictly contained in $\cup_{i=1}^m\mathcal{L}_{x_i}(P)$ for all $q\geq 1$.

Fix $1\leq i\leq m$. For $1\leq j<k_i$, by (1), we know that $f^j(\mathcal{L}_{x_i}(P))$ is not strictly contained in $\cup_{i=1}^m\mathcal{L}_{x_i}(P)$. Since P is a single puzzle piece and then it is nice, by Lemma 3.1 (2), we have $f^j(\mathcal{L}_{x_i}(P))\cap P=\emptyset$. When $j\geq k_i$, notice that as long as $\mathrm{depth}(f^j(\mathcal{L}_{x_i}(P)))\geq 0$, we have

$$\mathrm{depth}(f^j(\mathcal{L}_{x_i}(P)))\leq\mathrm{depth}(P)<\mathrm{depth}(\mathcal{L}_{x_s}(P))$$

for every $1\leq s\leq m$ which implies that $f^j(\mathcal{L}_{x_i}(P))$ is not strictly contained in $\cup_{i=1}^m\mathcal{L}_{x_i}(P)\bigcup P$. □

Lemma 7.8. *Let Q,Q',P,P' be puzzle pieces with the following properties.*

(a) $Q\subset\subset Q'$, $c_0\in P\subset\subset P'$ for $c_0\in\mathrm{Crit}(f)$.

(b) There is an integer $l\geq 1$ such that $f^l(Q)=P, f^l(Q')=P'$.

(c) $(P'\backslash P)\cap(\cup_{c\in\mathrm{Forw}(c_0)}\cup_{n\geq 0}\{f^n(c)\})=\emptyset$.

Then for all $0\leq i\leq l$, $(f^i(Q')\backslash f^i(Q))\cap\mathrm{Forw}(c_0)=\emptyset$.

Proof. If $f^{l-1}(Q')\backslash f^{l-1}(Q)$ contains some $c\in\mathrm{Forw}(c_0)$, notice that $f(f^{l-1}(Q))=P$ and $\deg(f:f^{l-1}(Q')\to P')=\deg_c(f)$, then $f(c)\in P'\backslash P$. It contradicts the condition (c).

For the case $l=1$, the lemma holds.

Now assume that $l\geq 2$.

We first prove that $(f^{l-2}(Q')\backslash f^{l-2}(Q))\cap\mathrm{Forw}(c_0)=\emptyset$.

If $(f^{l-1}(Q')\backslash f^{l-1}(Q))\cap(\mathrm{Crit}(f)\backslash\mathrm{Forw}(c_0))=\emptyset$, then

$$f:f^{l-1}(Q')\backslash f^{l-1}(Q)\to P'\backslash P.$$

If $f^{l-2}(Q')\backslash f^{l-2}(Q)$ contains some $c'\in\mathrm{Forw}(c_0)$, notice that $f(f^{l-2}(Q))=f^{l-1}(Q)$ and $\deg(f:f^{l-2}(Q')\to f^{l-1}(Q'))=\deg_{c'}(f)$, so $f(c')\in f^{l-1}(Q')\backslash f^{l-1}(Q)$ and then $f^2(c')\in P'\backslash P$ which contradicts the condition (c). Hence under the assumption that $(f^{l-1}(Q')\backslash f^{l-1}(Q))\cap(\mathrm{Crit}(f)\backslash\mathrm{Forw}(c_0))=\emptyset$, we come to the conclusion that $(f^{l-2}(Q')\backslash f^{l-2}(Q))\cap\mathrm{Forw}(c_0)=\emptyset$.

Otherwise, $f^{l-1}(Q')\backslash f^{l-1}(Q)$ contains some $c_1\in\mathrm{Crit}(f)\backslash\mathrm{Forw}(c_0)$. Since $c_1\notin\mathrm{Forw}(c_0)$, $c_1\notin\mathrm{Forw}(c)$ for any $c\in\mathrm{Forw}(c_0)$. By the assumption $(**)$, we conclude that $(f^{l-2}(Q')\backslash f^{l-2}(Q))\cap\mathrm{Forw}(c_0)=\emptyset$.

Continue the similar argument as above, we could prove the lemma for all $0\leq i\leq l-3$. □

The analytic method we will use to prove Proposition 7.3 is the following lemma on covering maps of the unit disk.

Lemma 7.9. *(see [AKLS] Lemma 3.2)*

For every integer $d\geq 2$ and every $0<\rho<r<1$ there exists $L_0=L_0(\rho,r,d)$ with the following property. Let $g,\tilde{g}:(\mathbb{D},0)\to(\mathbb{D},0)$ be holomorphic proper maps *of degree at most d, with critical values contained in $\mathbb{D}_\rho$. Let $\eta, \eta' : \mathbb{T} \to \mathbb{T}$ be two homeomorphisms satisfying $\tilde{g} \circ \eta' = \eta \circ g$, where $\mathbb{T}$ denotes the unit circle. Assume that η admits an L-qc extension $\xi : \mathbb{D} \to \mathbb{D}$ which is the identity on $\mathbb{D}_r$. Then η' admits an L'-qc extension $\xi' : \mathbb{D} \to \mathbb{D}$ which is the identity on $\mathbb{D}_r$, where $L' = \max\{L, L_0\}$.*

In the following, we will discuss $\mathrm{Crit}_{\mathrm{per}}(f), \mathrm{Crit}_{\mathrm{p}}(f), \mathrm{Crit}_{\mathrm{n}}(f) \cup \mathrm{Crit}_{\mathrm{r}}(f)$ and $\mathrm{Crit}_{\mathrm{e}}(f)$ successively.

For any $c \in \mathrm{Crit}_{\mathrm{per}}(f)$, by the condition of Proposition 7.3, there are a constant M_c and an integer N_c such that the map H extends to an M_c-qc extension inside $P_n(c)$ for all $n \geq N_c$.

The following lemma can be easily proved by Lemma 7.7.

Lemma 7.10. *Given a point $c_0 \in \mathrm{Crit}_{\mathrm{per}}(f)$ and set $N := \max\{N_c, c \in [c_0]\}$. Let $K_n(c_0) = P_{n+N}(c_0)$ and for every $c \in [c_0] \backslash \{c_0\}$, let $K_n(c) = P_{n+N+l_c}(c)$, where l_c is the smallest positive integer such that $f^{l_c}(\mathbf{K}_f(c)) = \mathbf{K}_f(c_0)$. Then $\cup_{c\in[c_0]} K_n(c)$ is nice for every $n \geq 1$.*

Set

$$b = \#\mathrm{Crit}(f) \text{ and } \delta = \max_{c\in\mathrm{Crit}(f)} \deg_c(f) \tag{2}$$

and $\mathrm{orb}_f([c_0]) = \bigcup_{n\geq 0} \bigcup_{c\in[c_0]} \{f^n(c)\}$ for $c_0 \in \mathrm{Crit}(f)$.

The following theorem is proved in [PQRTY].

Theorem 7.11. *Given a critical point $c_0 \in \mathrm{Crit}_{\mathrm{p}}(f) \backslash \mathrm{Crit}_{\mathrm{per}}(f)$. There are two constants S and $\Delta_0 > 0$, depending on b, δ and $\widehat{\mu}$ (see below), and a nested sequence of critical puzzle pieces $K_n(c_0) \subset\subset K_{n-1}(c_0)$, $n \geq 1$, with $K_0(c_0)$ to be the critical puzzle piece of depth 0, satisfying*

(i) each $K_n(c_0)$, $n \geq 1$, is a pullback of $K_{n-1}(c_0)$, that is $f^{p_n}(K_n(c_0)) = K_{n-1}(c_0)$ for some $p_n \geq 1$, and $\deg(f^{p_n} : K_n(c_0) \to K_{n-1}(c_0)) \leq S$,

(ii) each $K_n(c_0)$, $n \geq 1$, contains a sub-critical piece $K_n^-(c_0)$ such that

$$\mathrm{mod}(K_n(c_0) \backslash \overline{K_n^-(c_0)}) \geq \Delta_0 \text{ and } (K_n(c_0) \backslash \overline{K_n^-(c_0)}) \cap \mathrm{orb}_f([c_0]) = \emptyset.$$

Here

$$\widehat{\mu} = \min\{\mathrm{mod}(P_0(c_0) \backslash \overline{W}) \mid W \text{ a component of } \mathbf{U} \text{ contained in } P_0(c_0)\}. \tag{3}$$

Lemma 7.12. *Given a critical point $c_0 \in \mathrm{Crit}_{\mathrm{p}}(f) \backslash \mathrm{Crit}_{\mathrm{per}}(f)$. Let $(K_n(c_0), K_n^-(c_0))_{n\geq 1}$ be the sequence of pairs of critical puzzle pieces constructed in Theorem 7.11. For $c \in [c_0] \backslash \{c_0\}$, let $K_n(c) := \mathcal{L}_c(K_n(c_0))$. Then*

(1) for every $c \in [c_0]$ and every $n \geq 1$, $H|_{\partial K_n(c)}$ admits a qc extension inside $K_n(c)$ with the maximal dilatation independent of n;

(2) for each $n \geq 1$, $\cup_{c\in[c_0]} K_n(c)$ is nice.

Proof. (1) We first prove that $H|_{\partial K_n(c_0)}$ admits an L'-qc extension inside $K_n(c_0)$ where L' is independent of n. This part is similar to the proof of Proposition 3.1 in [PT].

Since H preserves the degree information, the puzzle piece bounded by $H(\partial K_n(c_0))$ (resp. $H(\partial K_n^-(c_0))$) is a critical piece for $\tilde{f}$, denote it by $\tilde{K}_n(\tilde{c}_0)$ (resp. $H(\partial \tilde{K}_n^-(\tilde{c}_0))$), $\tilde{c}_0 \in \mathrm{Crit}(\tilde{f})$.

Notice that $H|_{\partial K_1(c_0)}$ has a qc extension on a neighborhood of $\partial K_1(c_0)$. It extends thus to an L_1-qc map $K_1(c_0) \to \tilde{K}_1(\tilde{c}_0)$, for some $L_1 \geq 1$ (see e.g. [CT], Lemma C.1).

In the construction of the sequence in Theorem 7.11, the operators $\Gamma, \mathcal{A}, \mathcal{B}$ are used. As they can be read off from the dynamical degree on the boundary of the puzzle pieces, and H preserves this degree information, Theorem 7.11 is valid for the pair of sequences $(\tilde{K}_n(\tilde{c}_0), \tilde{K}_n^-(\tilde{c}_0))_{n\geq 1}$ as well, with the same constant S, and probably a different $\tilde{\Delta}_0$ as a lower bound for $\mathrm{mod}(\tilde{K}_n(\tilde{c}_0)\backslash\overline{\tilde{K}_n^-(\tilde{c}_0)})$.

Recall that for each $i \geq 1$, p_i denotes the integer such that $f^{p_i}(K_i(c_0)) = K_{i-1}(c_0)$. We have $\tilde{f}^{p_i}(\tilde{K}_i(\tilde{c}_0)) = \tilde{K}_{i-1}(\tilde{c}_0)$, and $f^{p_i} : K_i(c_0) \to K_{i-1}(c_0)$ and $\tilde{f}^{p_i} : \tilde{K}_i(\tilde{c}_0) \to \tilde{K}_{i-1}(\tilde{c}_0)$ are proper holomorphic maps of degree S.

Fix now $n \geq 1$.

Set $v_n = c_0$, and then, for $i = n-1, n-2, \cdots, 1$, set consecutively $v_i = f^{p_{i+1}+\ldots+p_n}(c_0)$.

Since $(K_i(c_0)\backslash K_i^-(c_0)) \cap \mathrm{orb}_f([c_0]) = \emptyset$, all the critical values of $f^{p_{i+1}}|_{K_{i+1}(c_0)}$, as well as v_i, are contained in $K_i^-(c_0)$, $1 \leq i \leq n-1$.

Let $\psi_i : (K_i(c_0), v_i) \to (\mathbb{D}, 0)$ be a bi-holomorphic uniformization, $i = 1, \cdots, n$. For $i = 2, \cdots, n$, let $g_i = \psi_{i-1} \circ f^{p_i} \circ \psi_i^{-1}$. These maps fix the point 0, are proper holomorphic maps of degree at most S, with the critical values contained in $\psi_{i-1}(K_{i-1}^-(c_0))$.

Let $\psi_i(K_i^-(c_0)) = \Omega_i$. Since $\mathrm{mod}(\mathbb{D} \setminus \overline{\Omega_i}) = \mathrm{mod}(K_i(c_0))\backslash\overline{K_i^-(c_0)}) \geq \Delta_0 > 0$ and $\Omega_i \ni \psi_i(v_i) = 0$, $1 \leq i \leq n$, these domains are contained in some disk $\mathbb{D}_s$ with $s = s(\Delta_0) < 1$. So the critical values of g_i are contained in $\Omega_{i-1} \subset \mathbb{D}_s$, $2 \leq i \leq n$.

The corresponding objects for $\tilde{f}$ will be marked with tilde. The same assertions hold for $\tilde{g}_i$. Then all the maps g_i and $\tilde{g}_i$ satisfy the assumptions of Lemma 7.9, with $d \leq S$, and $\rho = \max\{s, \tilde{s}\}$.

$$
\begin{array}{ccccccc}
(\mathbb{D},0) & \xleftarrow{\psi_n} & (K_n(c_0),v_n) & & (\tilde{K}_n(\tilde{c}_0),\tilde{v}_n) & \xrightarrow{\tilde{\psi}_n} & (\mathbb{D},0) \\
g_n \downarrow & & \downarrow f^{p_n} & & \tilde{f}^{p_n} \downarrow & & \downarrow \tilde{g}_n \\
(\mathbb{D},0) & \xleftarrow{\psi_{n-1}} & (K_{n-1}(c_0),v_{n-1}) & & (\tilde{K}_{n-1}(\tilde{c}_0),\tilde{v}_{n-1}) & \xrightarrow{\tilde{\psi}_{n-1}} & (\mathbb{D},0) \\
g_{n-1} \downarrow & & \downarrow f^{p_{n-1}} & & \tilde{f}^{p_{n-1}} \downarrow & & \downarrow \tilde{g}_{n-1} \\
\vdots & & \vdots & & \vdots & & \vdots \\
g_3 \downarrow & & \downarrow f^{p_3} & & \tilde{f}^{p_3} \downarrow & & \downarrow \tilde{g}_3 \\
(\mathbb{D},0) & \xleftarrow{\psi_2} & (K_2(c_0),v_2) & & (\tilde{K}_2(\tilde{c}_0),\tilde{v}_2) & \xrightarrow{\tilde{\psi}_2} & (\mathbb{D},0) \\
g_2 \downarrow & & \downarrow f^{p_2} & & \tilde{f}^{p_2} \downarrow & & \downarrow \tilde{g}_2 \\
(\mathbb{D},0) & \xleftarrow{\psi_1} & (K_1(c_0),v_1) & & (\tilde{K}_1(\tilde{c}_0),\tilde{v}_1) & \xrightarrow{\tilde{\psi}_1} & (\mathbb{D},0)
\end{array}
$$

Note that each of $\psi_i, \tilde{\psi}_i$ extends to a homeomorphism from the closure of the puzzle piece to $\overline{\mathbb{D}}$.

Let us consider homeomorphisms $\eta_i : \mathbb{T} \to \mathbb{T}$ given by $\eta_i = \tilde{\psi}_i \circ H|_{\partial K_i(c_0)} \circ \psi_i^{-1}$. They are equivariant with respect to the g-actions, i.e., $\eta_{i-1} \circ g_i = \tilde{g}_i \circ \eta_i$.

Due to the qc extension of $H|_{\partial K_1(c_0)}$, we know that η_1 extends to an L_1-qc map $\mathbb{D} \to \mathbb{D}$. Then η_1 is a L_1-quasisymmetric map. Fix some r with $\rho < r < 1$. We conclude that η_1 extends to an L-qc map $\xi_1 : \mathbb{D} \to \mathbb{D}$ which is the identity on $\mathbb{D}_r$, where L depends on L_1, ρ and r.

Let $L_0 = L_0(\rho, r, S)$ be as in Lemma 7.9, and let $L' = \max\{L, L_0\}$. For $i = 2, 3, \cdots, n$, apply consecutively Lemma 7.9 to the following left diagram (from below to top):

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\eta_n} & \mathbb{T} \\ g_n \downarrow & & \downarrow \tilde{g}_n \\ \mathbb{T} & \xrightarrow{\eta_{n-1}} & \mathbb{T} \\ g_{n-1} \downarrow & & \downarrow \tilde{g}_{n-1} \\ \vdots & & \vdots \\ \mathbb{T} & \xrightarrow{\eta_2} & \mathbb{T} \\ g_2 \downarrow & & \downarrow \tilde{g}_2 \\ \mathbb{T} & \xrightarrow{\eta_1} & \mathbb{T} \end{array}, \quad \text{we get} \quad \begin{array}{ccc} (\mathbb{D},0) & \xrightarrow{\xi_n} & (\mathbb{D},0) \\ g_n \downarrow & & \downarrow \tilde{g}_n \\ (\mathbb{D},0) & \xrightarrow{\xi_{n-1}} & (\mathbb{D},0) \\ g_{n-1} \downarrow & & \downarrow \tilde{g}_{n-1} \\ \vdots & & \vdots \\ (\mathbb{D},0) & \xrightarrow{\xi_2} & (\mathbb{D},0) \\ g_2 \downarrow & & \downarrow \tilde{g}_2 \\ (\mathbb{D},0) & \xrightarrow{\xi_1} & (\mathbb{D},0) \end{array}$$

so that for $i = 2, \ldots, n$, the map η_i admits an L'-qc extension $\xi_i : \mathbb{D} \to \mathbb{D}$ which is the identity on $\mathbb{D}_r$. The desired extension of $H|_{\partial K_n(c_0)}$ inside $K_n(c_0)$ is now obtained by taking $\tilde{\psi}_n^{-1} \circ \xi_n \circ \psi_n$.

Now we show that for $c \in [c_0] \backslash \{c_0\}$, for each $n \geq 1$, $H|_{\partial K_n(c)}$ admits an $\tilde{L}'$-qc extension inside $K_n(c)$ with the constant $\tilde{L}'$ independent of n.

Fix $n \geq 1$.

Let $f^{q_n}(K_n(c)) = K_n(c_0)$. Since $K_n(c_0)) \backslash K_n^-(c_0) \cap \mathrm{orb}_f([c_0]) = \emptyset$, all the critical values of $f^{q_n}|_{K_n(c)}$ are contained in $K_n^-(c_0)$.

Let $\varphi_n : (K_n(c_0), f^{q_n}(c)) \to (\mathbb{D}, 0)$ and $\lambda_n : (K_n(c), c) \to (\mathbb{D}, 0)$ be bi-holomorphic uniformizations. Set $\pi_n = \varphi_n \circ f^{q_n} \circ \lambda_n^{-1}$. This map fixes the point 0, are proper holomorphic maps of degree at most δ^b, with the critical values contained in $\varphi_n(K_n^-(c_0))$.

Since

$$\mathrm{mod}(\mathbb{D} \setminus \overline{\varphi_n(K_n^-(c_0))}) = \mathrm{mod}(K_n(c_0) \backslash \overline{K_n^-(c_0)}) \geq \Delta_0 > 0$$

and $\varphi_n(f^{q_n}(c)) = 0$ belongs to $\varphi_n(K_n^-(c_0))$, the set $\varphi_n(K_n^-(c_0))$ is contained in the disk $\mathbb{D}_s$ (here s is exactly the number defined for the case of c_0 in this proof). So the critical values of π_n are contained in $\varphi_n(K_n^-(c_0)) \subset \mathbb{D}_s$.

$$\begin{array}{ccc} (\mathbb{D},0) & \xleftarrow{\lambda_n} & (K_n(c), c) \\ \pi_n \downarrow & & \downarrow f^{q_n} \\ (\mathbb{D},0) & \xleftarrow{\varphi_n} & (K_n(c_0), f^{q_n}(c)) \end{array} \qquad \begin{array}{ccc} (\tilde{K}_n(\tilde{c}), \tilde{c}) & \xrightarrow{\tilde{\lambda}_n} & (\mathbb{D},0) \\ \tilde{f}^{q_n} \downarrow & & \downarrow \tilde{\pi}_n \\ (\tilde{K}_n(\tilde{c}_0), \tilde{f}^{q_n}(\tilde{c})) & \xrightarrow{\tilde{\varphi}_n} & (\mathbb{D},0) \end{array}$$

Let $\tilde{K}_n(\tilde{c})$, $\tilde{c} \in \mathrm{Crit}(\tilde{f})$ be the puzzle piece bounded by $H(\partial K_n(c))$. The corresponding objects for $\tilde{f}$ will be marked with tilde. The same assertions hold for $\tilde{\pi}_n$. Then all the maps π_n and $\tilde{\pi}_n$ satisfy the assumptions of Lemma 7.9, with $d \leq \delta^b$, and $\rho = \max\{s, \tilde{s}\}$.

Note that each of $\varphi_n, \tilde{\varphi}_n, \lambda_n, \tilde{\lambda}_n$ extends to a homeomorphism from the closure of the puzzle piece to $\overline{\mathbb{D}}$.

Let us consider homeomorphisms $\alpha_n : \mathbb{T} \to \mathbb{T}$ and $\beta_n : \mathbb{T} \to \mathbb{T}$ given by $\alpha_n = \tilde{\varphi}_n \circ H|_{\partial K_n(c_0)} \circ \varphi_n^{-1}$ and $\beta_n = \tilde{\lambda}_n \circ H|_{\partial K_n(c)} \circ \lambda_n^{-1}$. Then $\beta_n \circ \pi_n = \pi_n \circ \alpha_n$.

Due to the L'-qc extension of $H|_{\partial K_n(c_0)}$, we know that α_n extends to an L'-qc map $\mathbb{D} \to \mathbb{D}$. We still fix the number r with $\rho < r < 1$. We can extend α_n to be an $\tilde{L}$-qc map $\mu_n : \mathbb{D} \to \mathbb{D}$ which is the identity on $\mathbb{D}_r$, where $\tilde{L}$ depends on L', ρ and r.

Let $\tilde{L}_0 = \tilde{L}_0(\rho, r, \delta^b)$ be as in Lemma 7.9, and let $\tilde{L}' = \max\{\tilde{L}, \tilde{L}_0\}$. We apply Lemma 7.9 to the following left diagram:

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\beta_n} & \mathbb{T} \\ \pi_n \downarrow & & \downarrow \tilde{\pi}_n \\ \mathbb{T} & \xrightarrow{\alpha_n} & \mathbb{T} \end{array}, \quad \text{we get} \quad \begin{array}{ccc} (\mathbb{D}, 0) & \xrightarrow{\nu_n} & (\mathbb{D}, 0) \\ \pi_n \downarrow & & \downarrow \tilde{\pi}_n \\ (\mathbb{D}, 0) & \xrightarrow{\mu_n} & (\mathbb{D}, 0) \end{array}$$

so that the map β_n admits an $\tilde{L}'$-qc extension $\nu_n : \mathbb{D} \to \mathbb{D}$ which is the identity on $\mathbb{D}_r$. The desired extension of $H|_{\partial K_n(c)}$ inside $K_n(c)$ is obtained by taking $\tilde{\lambda}_n^{-1} \circ \nu_n \circ \lambda_n$.

(2) The set $\cup_{c \in [c_0]} K_n(c)$ is nice follows directly from Lemma 7.7. $\square$

Lemma 7.13. *Given $c_0 \in \mathrm{Crit_n}(f) \cup \mathrm{Crit_r}(f)$. Then there exist a puzzle piece P of depth n_0, a topological disk $T \subset\subset P$, and a nested sequence of puzzle pieces containing c, denoted by $\{K_n(c)\}_{n \geq 1}$, for each $c \in \mathrm{Crit}(f)$ with $c = c_0$ or $c \to c_0$, satisfying the following properties.*

(1) Every $K_n(c)$ is a pullback of P, that is $f^{s_n}(K_n(c)) = P$ for some $s_n \geq 1$, $\deg(f^{s_n} : K_n(c) \to P) \leq \delta^{b+1}$, and all critical values of the map $f^{s_n} : K_n(c) \to P$ are contained in T.

(2) $H|_{\partial K_n(c)}$ admits a qc extension inside $K_n(c)$ with the maximal dilatation independent of n.

(3) For every $n \geq 1$, $\bigcup_{c \in [c_0]} K_n(c)$ is a nice set.

Proof. (1) Suppose $c_0 \in \mathrm{Crit_n}(f)$ and then $[c_0] = \{c_0\}$. In $\mathcal{T}(c_0) \backslash \{$0-th column$\}$, every vertex is non-critical. So for each $n \geq 1$,

$$\deg(f^n : P_n(c_0) \to P_0(f^n(c_0))) = \deg_{c_0}(f) \leq \delta.$$

Since there are finitely many puzzle pieces of the same depth, we can take a subsequence $\{u_n\}_{n \geq 1}$ such that $f^{u_n}(P_{u_n}(c_0)) = P$ for some fixed puzzle piece P of depth 0.

Given $c \to c_0, c \in \mathrm{Crit}(f)$, let $f^{v_n}(\mathcal{L}_c(P_{u_n}(c_0)) = P_{u_n}(c_0)$. Then

$$\begin{aligned} & \deg(f^{v_n + u_n} : \mathcal{L}_c(P_{u_n}(c_0)) \to P) \\ = \; & \deg(f^{v_n} : \mathcal{L}_c(P_{u_n}(c_0)) \to P_{u_n}(c_0)) \cdot \deg(f^{u_n} : P_{u_n}(c_0) \to P) \\ \leq \; & \delta^b \cdot \delta \end{aligned}$$

For $c_0 \in \mathrm{Crit}(f)$, we set $K_n(c_0) = P_{u_n}(c_0)$ and $s_n = u_n$. For $c \to c_0, c \in \mathrm{Crit}(f)$, set $K_n(c) = \mathcal{L}_c(P_{u_n}(c_0))$ and $s_n = v_n + u_n$.

Now suppose $c_0 \in \mathrm{Crit_r}(f)$ and $c \to c_0, c \in \mathrm{Crit}(f)$. Since $\mathcal{T}(c_0)$ is reluctantly recurrent, there exist an integer $n_0 \geq 0$, $c_1, c_2 \in [c_0]$ and infinitely many integers $l_n \geq 1$ such that $\{P_{n_0 + l_n}(c_2)\}_{n \geq 1}$ are children of $P_{n_0}(c_1)$ and then $\deg(f^{l_n} : P_{n_0 + l_n}(c_2) \to P_{n_0}(c_1)) = \deg_{c_2}(f) \leq \delta$. Suppose $f^{k_n}(\mathcal{L}_c(P_{n_0 + l_n}(c_2))) = P_{n_0 + l_n}(c_2)$ for $k_n \geq 1$. Then

$$\begin{aligned} & \deg(f^{l_n + k_n} : P_{n_0 + l_n + k_n}(c) \to P_{n_0}(c_1)) \\ = \; & \deg(f^{k_n} : \mathcal{L}_c(P_{n_0 + l_n}(c_2)) \to P_{n_0 + l_n}(c_2)) \cdot \deg(f^{l_n} : P_{n_0 + l_n}(c_2) \to P_{n_0}(c_1)) \\ \leq \; & \delta^b \cdot \delta. \end{aligned}$$

Take a strictly increasing subsequence of l_n, still denoted by l_n such that $\{P_{n_0 + l_n}(c_2)\}_{n \geq 1}$ is a nested sequence of puzzle pieces containing c_2 and then $\{P_{n_0 + l_n + k_n}(c)\}_{n \geq 1}$ is a nested sequence of puzzle pieces containing c. Set $P = P_{n_0}(c_1)$. For $c = c_2$, set $s_n = l_n$ and $K_n(c) = P_{n_0+l_n}(c)$; for $c \neq c_2$, set $s_n = k_n + l_n$ and $K_n(c) = P_{n_0+k_n+l_n}(c)$.

Now fix $c_0 \in \mathrm{Crit_n}(f) \cup \mathrm{Crit_r}(f)$. Take a topological disk $T \subset\subset P$ such that T contains all the puzzle pieces of depth n_0+1. For each $c = c_0$ or $c \to c_0$, $c \in \mathrm{Crit}(f)$, each $n \geq 1$, all critical values of $f^{s_n}|_{K_n(c)}$ are contained in the union of puzzle pieces of depth $n_0 + 1$ because $\mathrm{Crit}(f) \subset \mathbf{K}_f$ and $f(\mathbf{K}_f) = \mathbf{K}_f$. Consequently, the set T contains all the critical values of the map $f^{s_n}|_{K_n(c)}$.

(2) We will use Lemma 7.9 to construct the qc extension.

Fix $c = c_0$ or $c \to c_0$ and $n \geq 1$.

Let $\tilde{P}$ and $\tilde{K}_n(\tilde{c})$ be the puzzle pieces for $\tilde{f}$ bounded by $H(\partial P)$ and $H(\partial K_n(c))$ respectively. Since H preserves the degree information, for the map $\tilde{f}$, we also have the similar statement as for f in (1), more precisely, $\tilde{f}^{s_n}(\tilde{K}_n(\tilde{c})) = \tilde{P}$, $\deg(\tilde{f}^{s_n} : \tilde{K}_n(\tilde{c}) \to \tilde{P}) \leq \delta^{b+1}$, and all critical values of the map $\tilde{f}^{s_n} : \tilde{K}_n(\tilde{c}) \to \tilde{P}$ are contained in $\tilde{T}$, where $\tilde{T}$ is a topological disk in $\tilde{P}$ containing all puzzle pieces for $\tilde{f}$ of depth $n_0 + 1$ in $\tilde{P}$.

Let $\mathrm{mod}(P \backslash \overline{T}) = \Delta_1$ and $\mathrm{mod}(\tilde{P} \backslash \overline{\tilde{T}}) = \tilde{\Delta}_1$.

Let $\iota_n : (P, f^{s_n}(c)) \to (\mathbb{D}, 0)$ and $\theta_n : (K_n(c), c) \to (\mathbb{D}, 0)$ be bi-holomorphic uniformizations. Let $h_n = \iota_n \circ f^{s_n} \circ \theta_n^{-1}$. Then h_n fixes the point 0, is a proper holomorphic map of degree at most δ^{b+1}, with all the critical values contained in $\iota_n(T)$.

Since $\mathrm{mod}(\mathbb{D} \setminus \overline{\iota_n(T)}) = \mathrm{mod}(P \backslash \overline{T}) = \Delta_1 > 0$ and $\iota_n(T) \ni \iota_n(f^{s_n}(c)) = 0$, we have $\iota_n(T) \subset \mathbb{D}_t$ with $t = t(\Delta_1) < 1$. So the critical values of h_n are contained in $\iota_n(T) \subset \mathbb{D}_t$.

The corresponding objects for $\tilde{f}$ will be marked with tilde. The same assertions hold for $\tilde{h}_n$. Then the maps h_n and $\tilde{h}_n$ satisfy the assumptions of Lemma 7.9, with $d \leq \delta^{b+1}$, and $\rho = \max\{t, \tilde{t}\}$.

$$\begin{array}{ccccccc}
(\mathbb{D}, 0) & \xleftarrow{\theta_n} & (K_n(c), c) & & (\tilde{K}_n(\tilde{c}), \tilde{c}) & \xrightarrow{\tilde{\theta}_n} & (\mathbb{D}, 0) \\
h_n \downarrow & & \downarrow f^{s_n} & & \tilde{f}^{s_n} \downarrow & & \downarrow \tilde{h}_n \\
(\mathbb{D}, 0) & \xleftarrow{\iota_n} & (P, f^{s_n}(c)) & & (\tilde{P}, \tilde{f}^{s_n}(\tilde{c})) & \xrightarrow{\tilde{\iota}_n} & (\mathbb{D}, 0)
\end{array}$$

Note that each of $\iota_n, \tilde{\iota}_n, \theta_n, \tilde{\theta}_n$ extends to a homeomorphism from the closure of the puzzle piece to $\overline{\mathbb{D}}$.

Let us consider homeomorphisms $\kappa_n : \mathbb{T} \to \mathbb{T}$ and $\sigma_n : \mathbb{T} \to \mathbb{T}$ given by $\kappa_n = \tilde{\iota}_n \circ H|_{\partial P} \circ \iota_n^{-1}$ and $\sigma_n = \tilde{\theta}_n \circ H|_{\partial K_n(c)} \circ \theta_n^{-1}$ respectively. Then $\kappa_n \circ h_n = \tilde{h}_n \circ \sigma_n$.

Notice that $H|_{\partial P}$ has a K_1-qc extension in P for some $K_1 \geq 1$. The number K_1 is independent of n because the choice of P does not depend on n. Fix some r with $\rho < r < 1$. We conclude that κ_n extends to a K-qc map $\omega_n : \mathbb{D} \to \mathbb{D}$ which is the identity on $\mathbb{D}_r$, where K depends on K_1, ρ and r.

Let $K_0 = K_0(\rho, r, \delta^{b+1})$ be as in Lemma 7.9 and let $K' = \max\{K, K_0\}$. Apply Lemma 7.9 to the following left diagram :

$$\begin{array}{ccc}
\mathbb{T} & \xrightarrow{\sigma_n} & \mathbb{T} \\
h_n \downarrow & & \downarrow \tilde{h}_n \\
\mathbb{T} & \xrightarrow{\kappa_n} & \mathbb{T}
\end{array}, \quad \text{we get} \quad \begin{array}{ccc}
(\mathbb{D}, 0) & \xrightarrow{\zeta_n} & (\mathbb{D}, 0) \\
h_n \downarrow & & \downarrow \tilde{h}_n \\
(\mathbb{D}, 0) & \xrightarrow{\omega_n} & (\mathbb{D}, 0)
\end{array}$$

so that the map σ_n admits a K'-qc extension $\zeta_n : \mathbb{D} \to \mathbb{D}$ which is the identity on $\mathbb{D}_r$. The desired extension of $H|_{\partial K_n(c)}$ inside $K_n(c)$ is now obtained by taking $\tilde{\theta}_n^{-1} \circ \zeta_n \circ \theta_n$.

(3) Fix $n \geq 1$.

For $c_0 \in \mathrm{Crit_n}(f)$, since $[c_0] = \{c_0\}$, we know that $\cup_{c\in[c_0]} K_n(c) = K_n(c_0)$ and obviously $K_n(c_0)$ is nice.

For the case that $c_0 \in \mathrm{Crit_r}(f)$, we apply Lemma 7.7 to

$$\bigcup_{c\in[c_0]} K_n(c) = \cup_{c\in[c_0]\setminus\{c_2\}} \mathcal{L}_c(P_{n_0+l_n}(c_2)) \bigcup P_{n_0+l_n}(c_2)$$

and easily get the conclusion. □

Lemma 7.14. *Suppose $c_0 \in \mathrm{Crit_e}(f)$. Then*

(1) there is a nested sequence of puzzle pieces containing c_0, denoted by $\{K_n(c_0)\}_{n\geq 1}$, such that for each $n \geq 1$, $H|_{\partial K_n(c_0)}$ admits a qc extension inside $K_n(c_0)$ with the maximal dilatation independent of n,

(2) for every $n \geq 1$, $\bigcup_{c\in[c_0]} K_n(c)$ is a nice set.

Proof. Suppose $c_0 \in \mathrm{Crit_e}(f)$. Recall that

$$\mathrm{Forw}(c_0) = \{c \in \mathrm{Crit}(f) \mid c_0 \to c\}$$

and in Lemma 7.6 (ii), we distinguish three cases for $\mathrm{Forw}(c_0)$. In the following, we will discuss the three cases.

In Case 1, i.e., $\mathrm{Forw}(c_0)\cap(\mathrm{Crit_n}(f)\cup\mathrm{Crit_r}(f)) = \emptyset$, by using Lemma 7.13, we can get a nested sequence of puzzle pieces containing c_0, denoted by $\{K_n(c_0)\}_{n\geq 1}$, and $H|_{\partial K_n(c)}$ admits a qc extension inside $K_n(c_0)$ whose maximal dilatation independent of n.

We divide Case 2 ($\mathrm{Forw}(c_0) \subset \mathrm{Crit_p}(f)$) into two subcases.

Subcase 1. There is a critical point $c_1 \in \mathrm{Forw}(c_0) \cap (\mathrm{Crit_p}(f)\setminus\mathrm{Crit_{per}}(f))$. Let $(K_n(c_1), K_n^-(c_1))_{n\geq 1}$ be the sequence of pairs of critical puzzle pieces constructed in Theorem 7.11.

For $n \geq 1$, set

$$K_n^-(c_0) = \mathcal{L}_{c_0}(K_n^-(c_1)), f^{r_n}(K_n^-(c_0)) = K_n^-(c_1) \text{ and } K_n(c_0) = \mathrm{Comp}_{c_0}(f^{-r_n}(K_n(c_1))).$$

Clearly, $(K_n(c_0)\setminus K_n^-(c_0)) \cap \mathrm{Crit}(f) = \emptyset$. Since $(K_n(c_1)\setminus K_n^-(c_1)) \cap \mathrm{orb}_f([c_1]) = \emptyset$ and $\mathrm{Forw}(c_1) = [c_1]$, by Lemma 7.8, we conclude that for all $1 \leq i < r_n$, $(f^i(K_n(c_0))\setminus f^i(K_n^-(c_0))) \cap \mathrm{Forw}(c_1) = \emptyset$.

We claim that for every $n \geq 1$ and every $1 \leq i \leq r_n$,

$$(f^i(K_n(c_0))\setminus f^i(K_n^-(c_0))) \bigcap (\mathrm{Crit}(f)\setminus\mathrm{Forw}(c_1)) = \emptyset.$$

If not, there is some n such that

$$\{f(K_n(c_0))\setminus f(K_n^-(c_0)), \cdots, f^{r_n-1}(K_n(c_0))\setminus f^{r_n-1}(K_n^-(c_0))\}$$

meets some critical point, say $c_2 \in \mathrm{Crit}(f)\setminus\mathrm{Forw}(c_1)$. See Figure 5.

Then $c_0 \to c_2 \to c_1$. Since $\mathrm{Forw}(c_0) \subset \mathrm{Crit_p}(f)$, we have $c_2 \in \mathrm{Crit_p}(f)$ and then $\mathrm{Forw}(c_2) = [c_2]$. So $c_1 \to c_2$. It contradicts $c_2 \notin \mathrm{Forw}(c_1)$.

Hence for every $n \geq 1$ and every $0 \leq i < r_n$,

$$(f^i(K_n(c_0))\setminus f^i(K_n^-(c_0))) \bigcap \mathrm{Crit}(f) = \emptyset. \tag{4}$$

From the equation (4) and $K_n^-(c_0) = \mathcal{L}_{c_0}(K_n^-(c_1))$, we conclude that for $n \geq 1$,

$$\deg(f^{r_n} : K_n(c_0) \to K_n(c_1)) \leq \delta^b.$$

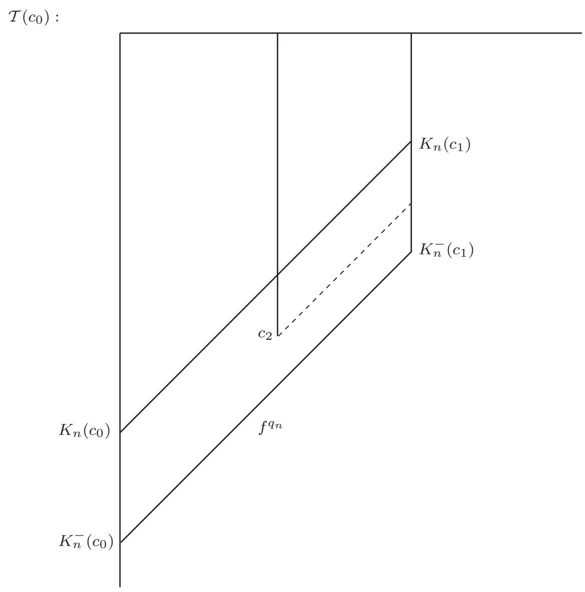


Figure 5

Again by the equation (4), we know that all critical values of the map $f^{r_n} : K_n(c_0) \to K_n(c_1)$ are contained in $K_n^-(c_1)$.

Using the similar method as in the proof of Lemma 7.12 (1), we could obtain the uniformly qc extension of $H|_{K_n(c_0)}$ inside $K_n(c_0)$. We omit the details here.

Subcase 2. Suppose $\mathrm{Forw(c_0)} \subset \mathrm{Crit}_{\mathrm{per}}(f)$.

If $f^l(\mathbf{K}_f(c_0))$ is periodic for some $l \geq 1$, then there is some critical periodic component in the periodic cycle of it. By the condition of Proposition 7.3, there is an integer N_{c_0} such that $P_{N_{c_0}+n}(c_0)$ has an M_{c_0} extension, where M_{c_0} is independent of n. Set $K_n(c_0) := P_{N_{c_0}+n}(c_0)$. It is done.

Now we suppose $\mathbf{K}_f(c_0)$ is wandering. For each $\hat{c} \in \mathrm{Forw(c_0)}$, by the condition of Proposition 7.3, there are a constant $M_{\hat{c}}$ and an integer $N_{\hat{c}}$ such that the map H extends to an $M_{\hat{c}}$-qc extension inside $P_n(c)$ for all $n \geq N_{\hat{c}}$. Set $N := \max\{N_{\hat{c}}, \hat{c} \in \mathrm{Forw(c_0)}\}$.

We claim that

Claim 1. *There exist a point $c_1 \in \mathrm{Forw(c_0)}$, a topological disk $Z \subset\subset P_N(c_1)$ and a nested sequence of puzzle pieces containing c_0, denoted by $\{K_n(c_0)\}_{n\geq 1}$, satisfying that for every $n \geq 1$, $f^{w_n}(K_n(c_0)) = P_N(c_1)$ for some $w_n \geq 1$, $\deg(f^{w_n} : K_n(c_0) \to P_N(c_1)) \leq \delta^b$ and all critical values of the map $f^{w_n}|_{K_n(c_0)}$ are contained in the set Z.*

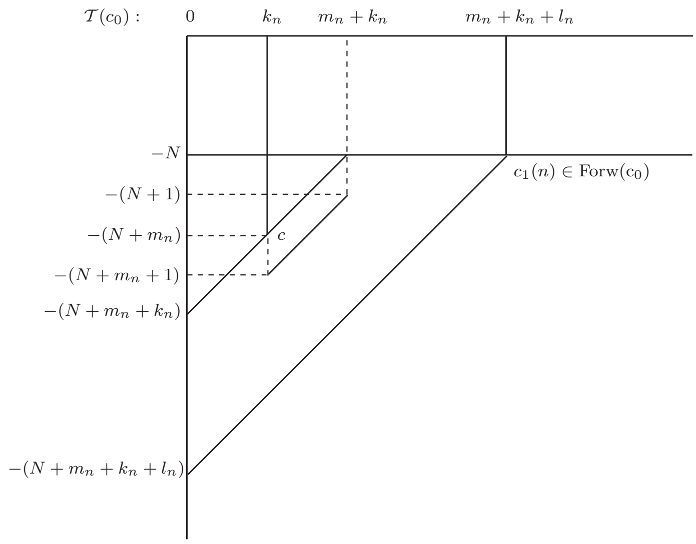
Proof. Suppose $c \in \mathrm{Forw}(c_0)$. Refer to the following figure for the proof.

Since $\mathbf{K}_f(c_0)$ is wandering, in $\mathcal{T}(c_0)$, there are infinitely many vertices $\{(-(N+m_n), k_n)\}_{n\geq 1}$ such that $(-(N+m_n), k_n)$ is the first vertex being c on the m_n-th row, $(-(N+m_n+1), k_n)$ is not critical and $\lim_{n\to\infty} m_n = \infty$. Then

$$f^{k_n}(\mathcal{L}_{c_0}(P_{N+m_n}(c))) = P_{N+m_n}(c)$$

and $\deg(f^{k_n} : P_{N+m_n+k_n}(c_0) \to P_{N+m_n}(c)) \leq \delta^b$.

Let p be the period of $\mathbf{K}_f(c)$. Then in $\mathcal{T}(c)$, for every $0 < j < p$, either $(-(N+m_n-j), j)$ is not critical or $(-(N+m_n+1-j), j)$ is critical. Using (Rule 3) several times, we conclude that there are no critical vertices on the diagonal starting



from the vertex at $(-(N+m_n+1), k_n)$ to the vertex at $(-(N+1), m_n+k_n)$. From the vertex $(-N, m_n+k_n)$, march horizontally $l_n \geq 1$ steps until the first hit of some $c_1(n)$ vertex for some $c_1(n) \in \mathrm{Forw}(c_0)$. Then there is no critical vertex on the diagonal starting from the vertex $(-(N+m_n+k_n+l_n-1), 1)$ to the vertex $(-(N+1), m_n+k_n+l_n-1)$. Therefore

$$\deg(f^{m_n+k_n+l_n} : P_{N+m_n+k_n+l_n}(c_0) \to P_N(c_1(n))) \leq \delta^b.$$

Since $c_1(n)$ belongs to the finite set Forw(c_0) and $m_n \to \infty$ as $n \to \infty$, we could find a subsequence of n, say itself, such that $\{P_{N+m_n+k_n+l_n}(c_0)\}_{n\geq 1}$ form a nested sequence and $c_1(n) \equiv c_1$. Set $w_n = N+m_n+k_n+l_n$.

Similarly to the proof of Lemma 7.13 (1), one take a topological disk $Z \subset\subset P_N(c_1)$ such that all the puzzle pieces of depth $N+1$ are contained in Z and particularly, all of the critical values of $f^{m_n+k_n+l_n}|_{P_{N+m_n+k_n+l_n}(c_0)}$ are contained in Z. $\square$

Using the similar method in the proof of Lemma 7.13 (2), we could show that $H|_{\partial K_n(c_0)}$ admits a qc extension inside $K_n(c_0)$ whose maximal dilatation is independent of n.

In Case 3, we will first draw the similar conclusion to Lemma 7.13 (1).

Take arbitrarily a point $c \in \mathrm{Crit_e}(f) \cap \mathrm{Forw}(c_0)$ such that Frow(c) contains some point in $\mathrm{Crit_p}(f)$. In $T(c_0)$, let $\{(0, t_n)\}_{n\geq 1}$ be all the c-vertices on the 0-th row with $1 \leq t_1 < t_2 < \cdots$.

Since $c_0 \in \mathrm{Crit_e}(f)$ and then $c_0 \not\to c_0$, by the assumption $(**)$, the c_0-vertex will not appear in $\mathcal{T}(c_0)\backslash\{0\text{-th column}\}$, in particular, for each $n \geq 1$, there are no c_0-vertices on the diagonal starting from the vertex $(-(t_n-1), 1)$ and ending at the vertex $(-1, t_n-1)$. Denote that diagonal by J_n. Since $c \not\to c$, by the assumption $(**)$, there are no c-vertices on the diagonal J_n.

We claim that for every $n \geq 1$, the diagonal J_n meets every point in $\mathrm{Crit}(f)\backslash\{c_0, c\}$ at most once. In fact, if not, then there is some n' and some $c' \in \mathrm{Crit}(f)$ such that the diagonal J_n' meets c' at least twice. By the assumption $(**)$, we can conclude that $c_0 \to c' \to c' \to c$. See Figure 6. By the condition of Case 3 and $c' \to c'$, we know that $c' \in \mathrm{Crit_p}(f)$ and then $\mathrm{Forw}(c') = [c']$ by Lemma 7.5. Thus $c \to c'$ and then $c \to c$. This contradicts $c \in \mathrm{Crit_e}(f)$.

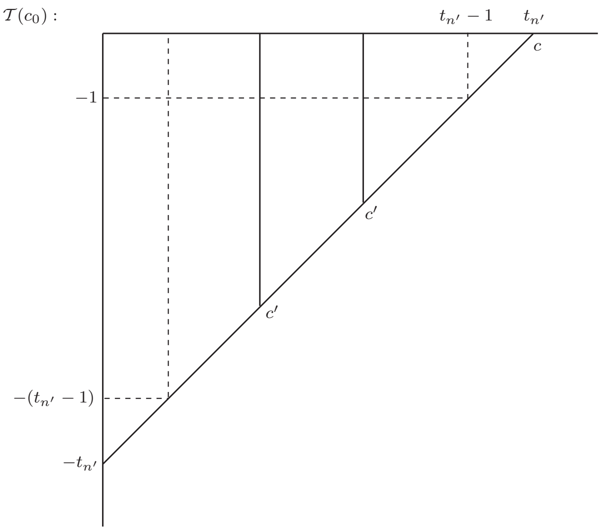


Figure 6

By the argument above, one easily find that $\{P_{t_n}(c_0)\}_{n\geq 1}$ is a nested sequence of puzzle pieces containing c_0 and $\deg(f^{t_n} : P_{t_n}(c_0) \to P_0(c)) \leq \delta^{b-1}$. Let T_2 be a topological disk compactly contained in $P_0(c)$ such that T_2 contains all the puzzle pieces of depth 1 in $P_0(c)$. Notice that all the critical values are contained in the union of all the puzzle pieces of depth 1 in $P_0(c)$, so they are also contained in T_2.

Set $K_n(c_0) := P_{t_n}(c_0)$ and then we get similar objects as in Lemma 7.13 (1). Using the similar method in the proof of Lemma 7.13 (2), we could also show that $H|_{\partial K_n(c_0)}$ admits a qc extension inside $K_n(c_0)$ whose maximal dilatation is independent of n.

(2) Since $c_0 \in \mathrm{Crit}_\mathrm{e}(f)$ and then $[c_0] = \{c_0\}$, we know that $\cup_{c\in[c_0]}K_n(c) = K_n(c_0)$ and obviously $K_n(c_0)$ is nice. $\square$

Summarizing Lemmas 7.12, 7.13 and 7.14, we have proved Proposition 7.3.

A An application of Theorem 2.3

Cui and Peng proved the following result in [CP] (see Theorem 1.1 in [CP]).

Theorem A.1. *Let U be a multiply-connected fixed (super)attracting Fatou component of a rational map f. Then there exist a rational map g and a completely invariant Fatou component V of g, such that*
(1) (f, U) and (g, V) are holomorphically conjugate, i.e., there is a conformal map mapping from U onto V and conjugating f to g,
(2) each Julia component of g consisting of more than one point is a quasi-circle which bounds an eventually superattracting Fatou component of g containing at most one postcritical point of g.
Moreover, g is unique up to a holomorphic conjugation.

We call (g, V) is a holomorphic model for (f, U).

To show the uniqueness of the model (g, V), they divided the proof into two parts. First they proved the following proposition (see also Proposition 1.3 in [CP]).

Proposition A.2. *Suppose g, $\tilde{g}$ are two rational maps and $V, \tilde{V}$ are two completely invariant Fatou components of g and $\tilde{g}$ respectively satisfying the conditions (1) and (2) in Theorem A.1. Then there is a qc map from the Riemann sphere $\overline{\mathbb{C}}$ onto itself conjugating g and $\tilde{g}$ on $\overline{\mathbb{C}}$ and this conjugation is conformal on the Fatou set of g.*

The other part is to show the Julia set of the model g carries no invariant line fields (see Proposition 1.4 in [CP]).

In this appendix, we will apply Theorem 2.3 (b) to give an another proof of Proposition A.2.

Proof of Proposition A.2. First starting from (g, V) and $(\tilde{g}, \tilde{V})$, we construct $(g : \mathbf{U} \to \mathbf{V})$ and $(\tilde{g} : \tilde{\mathbf{U}} \to \tilde{\mathbf{V}})$ satisfying the conditions as in the set-up.

We may assume that the fixed points of g and $\tilde{g}$ in V and $\tilde{V}$ are the infinity point. By Koenigs Linearization Theorem and Böttcher Theorem, we can take a disk $D_0 = \{|z| > r\} \subset V$ such that
(1)$D_0 \subset\subset g^{-1}(D_0)$,
(2)$\partial D_0 \bigcap (\cup_{n\geq 1} \cup_{c\in \mathrm{Crit}(g)} \{f^n(c)\}) = \emptyset$,
where $\mathrm{Crit}(g)$ denotes the critical point set of g.

Let D_n be the connected component of $g^{-n}(D_0)$ containing D_0 for each $n \geq 1$. Then $D_n \subset\subset D_{n+1}$ and $V = \bigcup_{n=0}^{\infty} D_n$. There is an integer N_0 satisfying that for any $n \geq 0$, $g^{-n}(D_{N_0})$ has only one component, the set $\mathrm{Crit}(g)$ is contained in $g^{-n}(D_{N_0})$ and every component of $\overline{\mathbb{C}}\backslash\overline{D}_{N_0}$ contains at most one component of $\overline{\mathbb{C}}\backslash V$ having critical points.

By Theorem A.1 (1), there is a conformal map $H : V \to \tilde{V}$ with $\tilde{g} \circ H = H \circ g$ on V. Set $\mathbf{V} := \overline{\mathbb{C}}\backslash\overline{D}_{N_0}$, $\tilde{\mathbf{V}} := \overline{\mathbb{C}}\backslash H(\overline{D}_{N_0})$ and $\mathbf{U} := g^{-1}(\mathbf{V})$ and $\tilde{\mathbf{U}} := \tilde{g}^{-1}(\tilde{\mathbf{V}})$. One can check that $(g : \mathbf{U} \to \mathbf{V})$ and $(\tilde{g} : \tilde{\mathbf{U}} \to \tilde{\mathbf{V}})$ satisfying the conditions as in the set-up.

Clearly, $\mathbf{K}_g = \overline{\mathbb{C}}\backslash V$ and $\mathbf{K}_{\tilde{g}} = \overline{\mathbb{C}}\backslash\tilde{V}$. Since $H : V \to \tilde{V}$ is a qc conjugacy from g to $\tilde{g}$ and $\mathbf{V} := \overline{\mathbb{C}}\backslash\overline{D}_{N_0}$, $\tilde{\mathbf{V}} := \overline{\mathbb{C}}\backslash H(\overline{D}_{N_0})$, we know that $H : \mathbf{V}\backslash\mathbf{K}_g \to \tilde{\mathbf{V}}\backslash\mathbf{K}_{\tilde{g}}$ is a qc conjugacy off $\mathbf{K}_g$. Let E be a periodic critical component of $\mathbf{K}_g$ which is mapped to a periodic critical component of $\mathbf{K}_g$ under some forward iterate of g. According to Theorem A.1 (2), E is a quasi-circle which bounds an eventually superattracting Fatou component containing a critical point c. In the proof of Proposition 4.4 in [CP], a qc map h_E is constructed. That map is defined on a puzzle piece $P_{n_E}(c)$ containing E. From the definition of that map, one can easily check that $h_E|_{\partial P_{n_E+n}}(c) = H$ for all $n \geq 0$. Set $N_c := n_E$ and let M_c be the maximal dilatation of the map h_E. Then by Theorem 2.3 (2), H extends to a qc conjugacy off $\mathrm{int}\mathbf{K}_g$. Notice that every component of $\mathrm{int}\mathbf{K}_g$ is a bounded eventually superattracting Fatou component and vice versa. So H can extend to a conformal map in every component of $\mathrm{int}\mathbf{K}_g$ which is again a conjugacy (refer to the proof of Claim 4.1). Hence H extends to a qc conjugacy on $\mathbf{V}$, that is, H extends to a qc conjugacy on $\overline{\mathbb{C}}$ which is conformal on the Fatou set of g. □

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