THE QUADRATIC DYNATOMIC CURVES ARE SMOOTH AND IRREDUCIBLE

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Dedicated to John Milnor's 80'th birthday

Abstract. We reprove here the smoothness and the irreducibility of the quadratic dynatomic curves $\{(c,z) \in \mathbb{C}^2 \mid z \text{ is } n\text{-periodic for } z^2+c\}$.

The smoothness is due to Douady-Hubbard. Our proof here is based on elementary calculations on the pushforwards of specific quadratic differentials, following Thurston and Epstein. This approach is a computational illustration of the power of the far more general transversality theory of Epstein.

The irreducibility is due to Bousch, Morton and Lau-Schleicher with different approaches. Our proof is inspired by the proof of Lau-Schleicher. We use elementary combinatorial properties of the kneading sequences instead of internal addresses.

1. Introduction

For $c \in \mathbb{C}$, let $f_c : \mathbb{C} \to \mathbb{C}$ be the quadratic polynomial

$$f_c(z) := z^2 + c.$$

A point $z \in \mathbb{C}$ is periodic for f_c if $f_c^{\circ n}(z) = z$ for some integer $n \geq 1$; it is of period n if $f_c^{\circ k}(z) \neq z$ for $0 < k < n$. For $n \geq 1$, set

$$X_n := \{(c,z) \in \mathbb{C}^2 \mid z \text{ is periodic of period } n \text{ for } f_c\}.$$

The objective of this note is to give new proofs of the following known results.

Theorem 1.1 (Douady-Hubbard)**.** *For every $n \geq 1$, the closure of X_n in $\mathbb{C}^2$ is a smooth affine curve.*

Theorem 1.2 (Bousch, Morton and Lau-Schleicher)**.** *For every $n \geq 1$ the closure of X_n in $\mathbb{C}^2$ is irreducible.*

Milnor [Mi2] reformulated in the language of quadratic differentials a proof of Tsujii showing that the topological entropy of the real quadratic polynomial $x \mapsto x^2 + c$ varies monotonically with respect to the parameter c. Our approach here to prove Theorem 1.1 is similar. We use elementary calculations on quadratic differentials and Thurston's contraction principle (instead of parabolic implosion techniques used in the original proof of Douady-Hubbard). Our calculation is a computational illustration of the far deeper and more conceptual transversality theory of Epstein [E].

Theorem 1.2 has been proved by Bousch [B] and by Morton [Mo] using a combination of algebraic and dynamical arguments, and by Lau and Schleicher [LS, Sc] using dynamical

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arguments only. Our approach here follows essentially [LS, Sc], except we replace their argument on internal addresses [Sc, Lemma 4.5] by a purely combinatorial argument on kneading sequences (Lemma 4.2 below). Also, we make use of a result of Petersen-Ryd [PR] instead of Douady-Hubbard's parabolic implosion theory.

The somewhat similar curve consisting of cubic polynomial maps with periodic critical orbit is smooth of known Euler characteristic, due to the works of Milnor and Bonifant-Kiwi-Milnor ([Mi3, BKM]). But the irreducibility question remains open.

In Section 2 we prove that $\overline{X}_n$ is an affine curve by introducing *dynatomic polynomials* defining the curve, in Section 3 we prove the smoothness while in Section 4 we prove the irreducibility. Sections 3 and 4 can be read independently.

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2. Dynatomic polynomials

In this section, we define the dynatomic polynomials $Q_n \in \mathbb{Z}[c,z]$ (see [Mi1] and [Si]) and show that

$$\overline{X}_n = \big\{(c,z) \in \mathbb{C}^2 \mid Q_n(c,z) = 0\big\}.$$

For $n \geq 1$, let $P_n \in \mathbb{Z}[c,z]$ be the polynomial defined by

$$P_n(c,z) := f_c^{\circ n}(z) - z.$$

The dynatomic polynomials Q_n will be defined so that

$$P_n = \prod_{k|n} Q_k.$$

Example 1. For $n = 1$ and $n = 2$, we have

$$\begin{array}{ll} P_1(c,z) = z^2 - z + c, & P_2(c,z) = z^4 + 2cz^2 - z + c^2 + c, \\ Q_1(c,z) = z^2 - z + c, & Q_2(c,z) = z^2 + z + c + 1 \\ P_1(c,z) = Q_1(c,z), & P_2(c,z) = Q_1(c,z) \cdot Q_2(c,z). \end{array}$$

Further examples may be found in [Si, Table 4.1].

With an abuse of notation, we will identify polynomials in $\mathbb{Z}[c,z]$ and polynomials in $\mathbf{Z}[z]$ with $\mathbf{Z} = \mathbb{Z}[c]$. In particular, we shall write $R(c,z)$ when $R \in \mathbf{Z}[z]$. Note that $P_n \in \mathbf{Z}[z]$ is a monic polynomial (its leading coefficient is 1) of degree 2^n.

Proposition 2.1. *There exists a unique sequence of monic polynomials* $\big(Q_n \in \mathbf{Z}[z]\big)_{n\geq 1}$, *such that for all* $n \geq 1$, *we have* $P_n = \prod_{k|n} Q_k$.

Proof. The proof goes by induction on n. For $n = 1$, it is necessary and sufficient to define

$$Q_1(c,z) := P_1(c,z) = z^2 - z + c.$$

Note that $Q_1 \in \mathbf{Z}[z]$ is indeed monic. Assume now that $n > 1$ and that the polynomials Q_k are defined for $1 \le k < n$. Set

$$A := \prod_{k|n, k<n} Q_k.$$

Since the polynomials $Q_k \in \mathbf{Z}[z]$ are monic, the polynomial $A \in \mathbf{Z}[z]$ is also monic. So, we may perform a Euclidean division to find a monic quotient $Q \in \mathbf{Z}[z]$ and a remainder $R \in \mathbf{Z}[z]$ with $\text{degree}(R) < \text{degree}(A)$, such that $P_n = QA + R$. We need to show that $R = 0$, which enables us to set $Q_n := Q$.

Let $\Delta \in \mathbb{Z}[c]$ be the discriminant of A. We claim that $\Delta(0) \neq 0$. Indeed, for each $k < n$, the polynomial $Q_k(0, z) \in \mathbb{Z}[z]$ divides $P_k(0, z) = z^{2^k} - z$ whose roots are simple. So, the roots of $Q_k(0, z)$ are simple. In addition, a root z_0 of $Q_k(0, z)$ is a periodic point of f_0 whose period m divides k. We have $m = k$ since otherwise,

$$Q_k(0, z) \cdot P_m(0, z) = Q_k(0, z) \cdot \prod_{j|m} Q_j(0, z)$$

would have a double root at z_0 and would at the same time divide

$$P_k(0, z) = \prod_{j|k} Q_j(0, z)$$

whose roots are simple. So, if $1 \le k_1 < k_2 < n$, then $Q_{k_1}(0, z)$ and $Q_{k_2}(0, z)$ do not have common roots. This shows that the roots of $A(0, z)$ are simple, whence $\Delta(0) \neq 0$.

Fix $c_0 \in \mathbb{C}$ such that $\Delta(c_0) \neq 0$ (since Δ does not identically vanish, this holds for every c_0 outside a finite set). Then, the roots of $A(c_0, z) \in \mathbb{Z}[z]$ are simple. Such a root z_0 is a periodic point of f_{c_0}, with period dividing n, whence a root of $P_n(c_0, z)$. As a consequence, $A(c_0, z)$ divides $P_n(c_0, z)$ in $\mathbb{C}[z]$. It follows that $R(c_0, z) = 0$ for all $z \in \mathbb{C}$. Since this is true for every c_0 outside a finite set, we have that $R = 0$ as required. $\square$

Remark 1. The proof we gave shows that the dynatomic polynomials Q_n have no repeated factors (otherwise $Q_n(0, z) \in \mathbb{Z}[z]$ would have a double root) and moreover, if $k_1 \neq k_2$, then Q_{k_1} and Q_{k_2} do not have common factors (otherwise $Q_{k_1}(0, z) \in \mathbb{Z}[z]$ and $Q_{k_2}(0, z) \in \mathbb{Z}[z]$ would have a common root). Those facts will be used later.

Remark 2. The degree of Q_k is at most that of P_k, that is 2^k. It follows that the degree of $A := \prod_{k|n, k<n} Q_k$ is at most $2^n - 2$, and so, the degree of $Q_n = P_n/A$ is at least 2. In particular, for $n \ge 1$, the set X_n is non-empty.

We extensively used the properties of roots of $Q_n(0, z) \in \mathbb{Z}[z]$. We will now study the properties of the roots of $Q_n(c_0, z) \in \mathbb{C}[z]$ for an arbitrary parameter $c_0 \in \mathbb{C}$.

Proposition 2.2. *Let $n \ge 1$ be a positive integer and $c_0 \in \mathbb{C}$ be an arbitrary parameter. Then, $z_0 \in \mathbb{C}$ is a root of $Q_n(c_0, z) \in \mathbb{C}[z]$ if and only if one of the following three exclusive conditions is satisfied:*

(1) *z_0 is periodic for f_{c_0}, the period is n and the multiplier is not 1; in that case $Q_n(c_0, z)$ has a simple root at z_0, or*

(2) *z_0 is periodic for f_{c_0}, the period is n and the multiplier is equal to 1; in that case $Q_n(c_0, z)$ has a double root at z_0, or*
(3) *z_0 is periodic for f_{c_0}, the period $m < n$ is a proper divisor of n and the multiplier of z_0 as a fixed point of $f_{c_0}^{\circ m}$ is a primitive $\frac{n}{m}$-th root of unity; in that case $Q_n(c_0, z)$ has a root of order $\frac{n}{m}$ at z_0.*

Proof. If $Q_n(c_0, z_0) = 0$, then $P_n(c_0, z_0) = 0$ and so, z_0 is periodic for f_{c_0} and the period m divides n. Conversely, if z_0 is periodic of period m for f_{c_0}, then $P_k(c_0, z_0) = 0$ if and only if k is a multiple of m. In particular, if k is not a multiple of m, then $Q_k(c_0, z_0) \neq 0$. Since

$$0 = P_m(c_0, z_0) = \prod_{k|m} Q_k(c_0, z_0),$$

we deduce that $Q_m(c_0, z_0) = 0$.

Case 1. If the multiplier ρ of z_0 as a fixed point of $f_{c_0}^{\circ m}$ is not a root of unity, then $P_n(c_0, z)$ has a simple root at z_0 whenever n is a multiple of m. In that case, $Q_m(c_0, z)$ is a factor of $P_n(c_0, z)$ and so, no other factor of P_n can vanish at z_0. As a consequence, $Q_n(c_0, z_0)$ vanishes if and only if $n = m$. In addition, $Q_m(c_0, z) \in \mathbb{C}[z]$ has a simple root at z_0.

Next, if the multiplier ρ of z_0 as a fixed point of $f_{c_0}^{\circ m}$ is a primitive s-th root of unity, then the multiplier of z_0 as a fixed point of $f_{c_0}^{\circ mk}$ is ρ^k. It is equal to 1 if and only if k is a multiple of s. In that case, z_0 is a multiple root of $P_{mk}(c_0, z)$ of order $s + 1$. Indeed, f_{c_0} has only one cycle of attracting petals since this cycle must attract the unique critical point of f_{c_0}.

Case 2. If $s = 1$, then $P_n(c_0, z)$ has a double root at z_0 whenever n is a multiple of m. As above, $Q_n(c_0, z_0)$ vanishes if and only if $n = m$, but this time, $Q_m(c_0, z) \in \mathbb{C}[z]$ has a double root at z_0.

Case 3. If $s \geq 2$, then $P_n(c_0, z)$ has a simple root at z_0 whenever n is a multiple of m but not a multiple of ms, and a multiple root at z_0 of order $s + 1$ whenever n is a multiple of ms. So, $Q_n(c_0, z_0)$ vanishes if and only if $n = m$ or $n = ms$; the polynomial $Q_m(c_0, z) \in \mathbb{C}[z]$ has a simple root at z_0 and the polynomial $Q_{ms}(c_0, z) \in \mathbb{C}[z]$ has a root of order s at z_0. □

Proposition 2.3. *For all $n \geq 1$, we have*

$$\overline{X}_n = \big\{(c, z) \in \mathbb{C}^2 \mid Q_n(c, z) = 0\big\}.$$

If $(c, z) \in \overline{X}_n - X_n$, then z is periodic for f_c, its period m is a proper divisor of n, and the multiplier of z as a fixed point of $f_c^{\circ m}$ is a primitive $\frac{n}{m}$-th root of unity.

Proof. Let $Y_n \in \mathbb{C}^2$ be the affine curve defined by Q_n:

$$Y_n := \big\{(c, z) \in \mathbb{C}^2 \mid Q_n(c, z) = 0\big\}.$$

According to the previous Proposition,

- if (c, z) belongs to X_n, then $Q_n(c, z) = 0$ and so, $\overline{X}_n \subseteq Y_n$.

- if $(c,z) \in Y_n - X_n$, then z is periodic for f_c, its period m is a proper divisor of n, and the multiplier of z as a fixed point of $f_c^{\circ m}$ is a primitive $\frac{n}{m}$-th root of unity. In particular, $Q_m(c,z) = 0$. Since Q_n and Q_m do not have common factors, this only occurs for a finite set of points $(c,z) \in Y_n$. Thus, $Y_n \subseteq \overline{X}_n$. $\square$

Remark 3. We saw that $\overline{X}_n - X_n$ is finite. More generally, the set of points $(c_0, z_0) \in \overline{X}_n$ such that $f_{c_0}^{\circ n}(z_0) = z_0$ and $(f_{c_0}^{\circ n})'(z_0) = 1$ is finite. Indeed, for such a point, c_0 is a root of the discriminant of $P_n \in \mathbf{Z}[z]$. Since the roots of $P_n(0,z)$ are simple, this discriminant does not vanish at $c = 0$, and so, its roots form a finite set.

3. Smoothness of the dynatomic curves

Our objective is now to give a proof of Theorem 1.1. We will prove the following more precise version. We shall denote by $\pi_c : \mathbb{C}^2 \to \mathbb{C}$ the projection $(c,z) \mapsto c$ and by $\pi_z : \mathbb{C}^2 \to \mathbb{C}$ the projection $(c,z) \mapsto z$.

Theorem 3.1. *For every $n \geq 1$, the affine curve $\overline{X}_n$ is smooth. More precisely, for $(c_0, z_0) \in \overline{X}_n$, we have:*

1. *if $z_0 \in X_n$ has multiplier different from 1, then $\pi_c : X_n \to \mathbb{C}$ is a local isomorphism;*
2. *if $z_0 \in X_n$ has multiplier 1, then $\pi_z : X_n \to \mathbb{C}$ is a local isomorphism; in addition, $\pi_c : X_n \to \mathbb{C}$ has local degree 2.*
3. *if $z_0 \in \overline{X}_n - X_n$ has multiplier a primitive s-th root of unity, then $\pi_z : \overline{X}_n \to \mathbb{C}$ is a local isomorphism; in addition, $\pi_c : \overline{X}_n \to \mathbb{C}$ has local degree s.*

The idea is to apply the Implicit Function Theorem. In particular, we will prove that $\dfrac{\partial Q_n}{\partial z}(c_0, z_0) \neq 0$ in Case 1 (this is almost immediate), and that $\dfrac{\partial Q_n}{\partial c}(c_0, z_0) \neq 0$ in Cases 2 and 3. This is where we have to work: following Epstein, we will first relate this partial derivative to the coefficient of a quadratic differential of the form $(f_{c_0})_* \mathbf{q} - \mathbf{q}$; we will then show that $(f_{c_0})_* \mathbf{q} \neq \mathbf{q}$ by using (a generalization of) Thurston's Contraction Principle. This approach is fundamentally different from Douady-Hubbard's original proof, where Fatou-Leau's flower theorem on parabolic periodic points as well as Douady-Hubbard's parabolic implosion theory play an essential role.

Once we know that in Cases 2 and 3, the projection $\pi_z : X_n \to \mathbb{C}$ is a local isomorphism, the local degree of the projection $\pi_c : X_n \to \mathbb{C}$ follows from Proposition 2.2. Indeed, if $Q_n(c_0, z) \in \mathbb{C}[z]$ has a root of order ν at z_0 and if $\dfrac{\partial Q_n}{\partial c}(c_0, z_0) \neq 0$, then

$$Q_n(c_0 + \eta, z_0 + \varepsilon) = a\eta + b\varepsilon^\nu + o(\eta) + o(\varepsilon^\nu) \quad \text{with} \quad a \neq 0 \text{ and } b \neq 0$$

and the solutions of $Q_n(c,z) = 0$ are locally of the form $\big(c(z), z\big)$ with

$$c(z_0 + \varepsilon) = c_0 - \frac{b}{a}\varepsilon^\nu + o(\varepsilon^\nu).$$

3.1. Case 1 of Theorem 3.1.

Let $(c_0, z_0) \in X_n$ be such that the multiplier of z_0 as a fixed point of $f_{c_0}^{\circ n}$ is not 1. Then, $Q_n(c_0, z_0) = 0$, for all $k < n$, $Q_k(c_0, z_0) \neq 0$ and

$(f_{c_0}^{\circ n})'(z_0) \neq 1$. Since

$$\prod_{k|n} Q_k(c,z) = P_n(c,z),$$

we have

$$\frac{\partial Q_n}{\partial z}(c_0,z_0)\cdot \prod_{k|n,k<n} Q_k(c_0,z_0) = \frac{\partial P_n}{\partial z}(c_0,z_0) = (f_{c_0}^{\circ n})'(z_0) - 1 \neq 1.$$

As a consequence, $\dfrac{\partial Q_k}{\partial z}(c_0,z_0) \neq 0$. By the Implicit Function Theorem, X_n is smooth near (c_0,z_0) and the projection $\pi_c : X_n \to \mathbb{C}$ is a local isomorphism.

3.2. **Case 2 of Theorem 3.1.** Let $(c_0,z_0) \in X_n$ be such that the multiplier of z_0 as a fixed point of $f_{c_0}^{\circ n}$ is 1. As previously, since $Q_n(c_0,z_0) = 0$ and

$$\prod_{k|n} Q_k(c,z) = P_n(c,z),$$

we have

$$\frac{\partial Q_n}{\partial c}(c_0,z_0)\cdot \prod_{k|n,k<n} Q_k(c_0,z_0) = \frac{\partial P_n}{\partial c}(c_0,z_0).$$

Since for all $k<n$, $Q_k(c_0,z_0) \neq 0$ it is enough to prove that

$$\frac{\partial P_n}{\partial c}(c_0,z_0) \neq 0.$$

We shall use the following notations: for $n \geq 0$, we let $\zeta_n : \mathbb{C} \to \mathbb{C}$ be defined by

$$\zeta_n(c) := f_c^{\circ n}(z_0),$$

and we set

$$z_n := \zeta_n(c_0) = f_{c_0}^{\circ n}(z_0) \quad \text{and} \quad \delta_n := f'_{c_0}(z_n) = 2z_n.$$

Since $P_n(c,z_0) = f_c^{\circ n}(z_0) - z_0 = \zeta_n(c) - z_0$, we have

$$\frac{\partial P_n}{\partial c}(c_0,z_0) = \zeta'_n(c_0).$$

Lemma 3.2 (Compare with [Mi2]). *We have*

$$\zeta'_n(c_0) = 1 + \delta_{n-1} + \delta_{n-1}\delta_{n-2} + \ldots + \delta_{n-1}\delta_{n-2}\cdots\delta_1.$$

Proof. The function ζ_0 is constant (equal to z_0). From $\zeta_n(c) = \big(\zeta_{n-1}(c)\big)^2 + c$, we obtain

$$\zeta'_n(c_0) = 1 + \delta_{n-1}\zeta'_{n-1}(c_0) \quad \text{with} \quad \zeta'_0(c_0) = 0.$$

The result follows by induction. □

In order to prove that

$$1 + \delta_{n-1} + \delta_{n-1}\delta_{n-2} + \ldots + \delta_{n-1}\delta_{n-2}\cdots\delta_1 \neq 0$$

we shall now work with meromorphic quadratic differentials.

3.2.1. *Quadratic differentials.* A meromorphic quadratic differential $\mathbf{q}$ on $\mathbb{C}$ takes the form $\mathbf{q} = q\,\mathrm{dz}^2$ with q a meromorphic function on $\mathbb{C}$. We use $\mathcal{Q}(\mathbb{C})$ to denote the set of meromorphic quadratic differentials on $\mathbb{C}$ whose poles (if any) are all simple. If $\mathbf{q} = q\,\mathrm{dz}^2 \in \mathcal{Q}(\mathbb{C})$ and U is a bounded open subset of $\mathbb{C}$, the norm

$$\|\mathbf{q}\|_U := \iint_U \big|q(x+\mathrm{i}y)\big|\;\mathrm{d}x\mathrm{d}y$$

is well defined and finite.

Example 2.

$$\left\|\frac{\mathrm{dz}^2}{z}\right\|_{D(0,R)} = \int_0^{2\pi}\int_0^R \frac{1}{r} r\,\mathrm{d}r\mathrm{d}\theta = 2\pi R\ .$$

3.2.2. *Pushforward.* For $f:\mathbb{C}\to\mathbb{C}$ a non-constant polynomial and $\mathbf{q} = q\,\mathrm{dz}^2$ a meromorphic quadratic differential on $\mathbb{C}$, the pushforward $f_*\mathbf{q}$ is defined by

$$f_*\mathbf{q} := Tq\,\mathrm{dz}^2 \quad \text{with} \quad Tq(z) := \sum_{f(w)=z} \frac{q(w)}{f'(w)^2}.$$

If $\mathbf{q}\in\mathcal{Q}(\mathbb{C})$, then $f_*\mathbf{q}\in\mathcal{Q}(\mathbb{C})$ also.

Lemma 3.3 (Compare with [Mi2] or [L])**.** *For $f = f_c$, we have*

$$\begin{cases} f_*\left(\dfrac{\mathrm{dz}^2}{z}\right) = 0 \\ f_*\left(\dfrac{\mathrm{dz}^2}{z-a}\right) = \dfrac{1}{f'(a)}\left(\dfrac{\mathrm{dz}^2}{z-f(a)} - \dfrac{\mathrm{dz}^2}{z-c}\right) \quad \textit{if } a\neq 0. \end{cases} \tag{3.1}$$

Proof. If $f(w) = z$, then $w = \pm\sqrt{z-c}$ and

$$\mathrm{dw}^2 = \frac{\mathrm{dz}^2}{4(z-c)}.$$

We then have

$$\begin{aligned} f_*\left(\frac{\mathrm{dz}^2}{z-a}\right) &= \frac{\mathrm{dz}^2}{4(z-c)}\left(\frac{1}{\sqrt{z-c}-a} + \frac{1}{-\sqrt{z-c}-a}\right) \\ &= \frac{a\,\mathrm{dz}^2}{2(z-c)(z-f(a))}. \end{aligned}$$

If $a = 0$, we get the first equality in (3.1). Otherwise, we get the second equality using

$$\frac{1}{f'(a)}\left(\frac{1}{z-f(a)} - \frac{1}{z-c}\right) = \frac{f(a)-c}{2a(z-c)(z-f(a))} = \frac{a}{2(z-c)(z-f(a))}. \qquad \square$$

3.2.3. *A particular quadratic differential.* Consider the quadratic differential $\mathbf{q}\in\mathcal{Q}(\mathbb{C})$ defined by

$$\mathbf{q} := \sum_{k=0}^{n-1} \frac{\rho_k}{z-z_k}\,\mathrm{dz}^2 \quad \text{with} \quad \rho_k = \delta_{n-1}\delta_{n-2}\cdots\delta_k.$$

The multiplier of z_0 as a periodic point of f_{c_0} is $\rho_0 = 1$. So,

$$\frac{\rho_{n-1}}{\delta_{n-1}} = 1 = \rho_0.$$

In addition, for $0 \le k \le n-2$, we have

$$\frac{\rho_k}{\delta_k} = \rho_{k+1}.$$

Applying Lemma 3.3, and writing f for f_{c_0}, we obtain

$$f_*\mathbf{q} = \sum_{k=0}^{n-1} \frac{\rho_k}{\delta_k}\left(\frac{\mathrm{d}z^2}{z - z_{k+1}} - \frac{\mathrm{d}z^2}{z - c_0}\right) = \mathbf{q} - \left(\sum_{k=0}^{n-1} \rho_k\right) \cdot \frac{\mathrm{d}z^2}{z - c_0}.$$

As mentioned earlier,

$$\frac{\partial P_n}{\partial c}(c_0, z_0) = \zeta_n'(c_0) = \sum_{k=0}^{n-1} \rho_k.$$

It is therefore enough to prove that $f_*\mathbf{q} \ne \mathbf{q}$. This is done in the next paragraph using a Contraction Principle.

3.2.4. *Contraction Principle.* The following lemma is a weak version of Thurston's contraction principle (which applies to the setting of rational maps on $\mathbb{P}^1$).

Lemma 3.4 (Contraction Principle)**.** *For a non-constant polynomial f and a round disk V of radius large enough so that $U := f^{-1}(V)$ is relatively compact in V, we have*

$$\|f_*\mathbf{q}\|_V \le \|\mathbf{q}\|_U < \|\mathbf{q}\|_V, \quad \forall\, \mathbf{q} \in \mathcal{Q}(\mathbb{C}).$$

Proof. The strict inequality on the right is a consequence of the fact that U is relatively compact in V. The inequality on the left comes from

$$\begin{aligned}\|f_*\mathbf{q}\|_V &= \iint_{x+iy\in V} \left| \sum_{f(w)=x+iy} \frac{q(w)}{f'(w)^2} \right| \mathrm{d}x\mathrm{d}y \\ &\le \iint_{x+iy\in V} \sum_{f(w)=x+iy} \left| \frac{q(w)}{f'(w)^2} \right| \mathrm{d}x\mathrm{d}y = \iint_{u+iv\in U} \big|q(u+iv)\big| \,\mathrm{d}u\mathrm{d}v = \|\mathbf{q}\|_U. \quad \square\end{aligned}$$

Corollary 3.5. *If $f : \mathbb{C} \to \mathbb{C}$ is a polynomial and if $\mathbf{q} \in \mathcal{Q}(\mathbb{C})$, then $f_*\mathbf{q} \ne \mathbf{q}$.*

3.3. **Case 3 of Theorem 3.1.** Let $(c_0, z_0) \in \overline{X}_n - X_n$ be such that z_0 is periodic for f_{c_0} with period $m < n$ dividing n and multiplier ρ, a primitive s-th root of unity with $s := \frac{n}{m}$.

According to Proposition 2.2 point 3, the polynomial $Q_n(c_0, z) \in \mathbb{C}[z]$ has a root of order $s \ge 2$ at z_0, so that

$$\frac{\partial Q_n}{\partial z}(c_0, z_0) = 0.$$

We want to show that

$$\frac{\partial Q_n}{\partial c}(c_0, z_0) \ne 0.$$

Let us write

$$P_n(c,z) = P_m(c,z)\cdot R(c,z) \quad \text{with} \quad R(c,z) = \prod_{k|n, k\nmid m} Q_k(c,z). \tag{3.2}$$

On the first hand, since $Q_n(c_0,z_0)=0$, we have

$$\frac{\partial R}{\partial c}(c_0,z_0) = \frac{\partial Q_n}{\partial c}(c_0,z_0)\cdot \prod_{k|n, k\nmid m, k<n} Q_k(c_0,z_0).$$

Since for all $k<n$ with $k\neq m$, $Q_k(c_0,z_0)\neq 0$, it is enough to prove that

$$\frac{\partial R}{\partial c}(c_0,z_0)\neq 0.$$

3.3.1. *Variation along X_m.* Note that $(c_0,z_0)\in X_m$ and the multiplier ρ of z_0 as a fixed point of $f_{c_0}^{\circ m}$ is $\rho\neq 1$. Thus, according to Case 1, X_m is locally the graph of a function $\zeta(c)$ defined and holomorphic near c_0 with $\zeta(c_0)=z_0$. The point $\zeta(c)$ is periodic of period m for f_c. We denote by ρ_c its multiplier and set

$$\dot\rho := \frac{d\rho_c}{dc}\Big|_{c_0}.$$

Lemma 3.6. *We have*

$$\frac{\partial R}{\partial c}(c_0,z_0) = \frac{s\dot\rho}{\rho(\rho-1)}.$$

Proof. Differentiating Equation (3.2) with respect to z, and evaluating at $\big(c,\zeta(c)\big)$, we get:

$$\rho_c^s - 1 = (\rho_c-1)\cdot R\big(c,\zeta(c)\big) + \underbrace{P_m\big(c,\zeta(c)\big)}_{=0}\cdot\frac{\partial R}{\partial z}\big(c,\zeta(c)\big) = (\rho_c-1)\cdot R\big(c,\zeta(c)\big).$$

Differentiating with respect to c and evaluating at c_0, we get:

$$s\rho^{s-1}\dot\rho = \dot\rho\underbrace{R(c_0,z_0)}_{=0} + (\rho-1)\frac{\partial R}{\partial c}(c_0,z_0) + (\rho-1)\underbrace{\frac{\partial R}{\partial z}(c_0,z_0)}_{=0}\zeta'(c_0) = (\rho-1)\frac{\partial R}{\partial c}(c_0,z_0).$$

The result follows since $\rho^s=1$ and so, $\rho^{s-1}=1/\rho$. $\square$

Thus, we are left with proving that $\dot\rho\neq 0$. This will be done by using a particular meromorphic quadratic differential having double poles along the cycle of z_0.

3.3.2. *Quadratic differentials with double poles.*

Lemma 3.7 (Compare with [L])**.** *For $f=f_c$, we have*

$$f_*\left(\frac{\mathrm{d}z^2}{(z-a)^2}\right) = \frac{\mathrm{d}z^2}{(z-f(a))^2} - \frac{1}{2a^2}\left(\frac{\mathrm{d}z^2}{z-f(a)} - \frac{\mathrm{d}z^2}{z-c}\right) \text{ if } a\neq 0.$$

Proof. If $f(w) = z$, then $w = \pm\sqrt{z-c}$ and

$$\mathrm{dw}^2 = \frac{\mathrm{dz}^2}{4(z-c)}.$$

Then

$$\begin{aligned} f_*\left(\frac{\mathrm{dz}^2}{(z-a)^2}\right) &= \frac{\mathrm{dz}^2}{4(z-c)}\left(\frac{1}{(\sqrt{z-c}-a)^2} + \frac{1}{(-\sqrt{z-c}-a)^2}\right) \\ &= \frac{(z-c+a^2)\ \mathrm{dz}^2}{2(z-c)(z-c-a^2)^2} = \frac{(z-c+a^2)\ \mathrm{dz}^2}{2(z-c)(z-f(a))^2}. \end{aligned}$$

Decomposing the last expression into partial fractions gives

$$\frac{z-c+a^2}{2(z-c)(z-f(a))^2} = \frac{A}{(z-f(a))^2} + \frac{B}{z-f(a)} + \frac{C}{z-c}$$

with

$$A = \frac{f(a)-c+a^2}{2(f(a)-c)} = \frac{2a^2}{2a^2} = 1, \quad C = \frac{c-c+a^2}{2(c-f(a))^2} = \frac{a^2}{2a^4} = \frac{1}{2a^2}$$

and

$$B = -C = -\frac{1}{2a^2}.$$ □

Set $f := f_{c_0}$,

$$z_k := f^{\circ k}(z_0), \quad \delta_k := f'(z_k) = 2z_k, \quad \zeta_k(c) := f_c^{\circ k}\big(\zeta(c)\big) \quad \text{and} \quad \dot\zeta_k := \zeta_k'(c_0).$$

Then

$$\zeta_{k+1}(c) = f_c\big(\zeta_k(c)\big) \quad \text{and} \quad \zeta_n = \zeta_0.$$

Since

$$\delta_0\delta_1\cdots\delta_{m-1} = \rho \neq 1,$$

there is a unique m-tuple $(\mu_0,\ldots,\mu_{m-1})$ such that

$$\mu_{k+1} = \frac{\mu_k}{2z_k} - \frac{1}{2z_k^2},$$

where the indices are considered to be modulo m.

Now consider the quadratic differential $\mathbf{q}$ (with double poles) defined by

$$\mathbf{q} := \sum_{k=0}^{m-1}\left(\frac{1}{(z-z_k)^2} + \frac{\mu_k}{z-z_k}\right)\ \mathrm{dz}^2.$$

Lemma 3.8 (Levin)**.** *We have*

$$f_*\mathbf{q} = \mathbf{q} - \frac{\dot\rho}{\rho}\cdot\frac{\mathrm{dz}^2}{z-c_0}.$$

Proof. By construction of $\mathbf{q}$ and the calculation of $f_*\mathbf{q}$ in Lemma 3.3, the polar parts of $\mathbf{q}$ and $f_*\mathbf{q}$ along the cycle of z_0 are identical. But $f_*\mathbf{q}$ has an extra simple pole at the critical value c_0 with coefficient

$$\sum_{k=0}^{m-1}\left(-\frac{\mu_k}{2z_k} + \frac{1}{2z_k^2}\right) = -\sum_{k=0}^{m-1}\mu_{k+1}.$$

We need to show that this coefficient is equal to $-\frac{\dot\rho}{\rho}$.

Using $\zeta_{k+1}(c) = \big(\zeta_k(c)\big)^2 + c$, we get

$$\dot\zeta_{k+1} = 2z_k\dot\zeta_k + 1.$$

It follows that

$$\dot\zeta_{k+1}\mu_{k+1} - \mu_{k+1} = 2z_k\dot\zeta_k\mu_{k+1} = \dot\zeta_k\mu_k - \frac{\dot\zeta_k}{z_k}.$$

Therefore

$$\sum_{k=0}^{m-1}\mu_{k+1} = \sum_{k=0}^{m-1}\left(\dot\zeta_{k+1}\mu_{k+1} - \dot\zeta_k\mu_k + \frac{\dot\zeta_k}{z_k}\right) = \sum_{k=0}^{m-1}\frac{\dot\zeta_k}{z_k} = \frac{\dot\rho}{\rho},$$

where last equality is obtained by evaluating at c_0 of the logarithmic derivative of

$$\rho_c := \prod_{k=0}^{m-1} 2\zeta_k(c). \qquad \square$$

To complete the proof that $\dot\rho \neq 0$, we will use a generalization of the Contraction Principle due to Epstein.

Lemma 3.9 (Epstein)**.** *We have $f_*\mathbf{q} \neq \mathbf{q}$.*

Proof. The proof rests again on the contraction principle, but we can not apply directly Lemma 3.4 since $\mathbf{q}$ is not integrable near the cycle $\langle z_0, \ldots, z_{m-1}\rangle$. Consider a sufficiently large round disk V so that $U := f^{-1}(V)$ is relatively compact in V. Given $\varepsilon > 0$, we set

$$V_\varepsilon := \bigcup_{k=1}^{m} f^{\circ k}\big(D(z_0,\varepsilon)\big) \quad \text{and} \quad U_\varepsilon := f^{-1}(V_\varepsilon).$$

When ε tends to 0, we have

$$\|f_*\mathbf{q}\|_{V-V_\varepsilon} \le \|\mathbf{q}\|_{U-U_\varepsilon} = \|\mathbf{q}\|_{V-V_\varepsilon} - \|\mathbf{q}\|_{V-U} + \|\mathbf{q}\|_{U_\varepsilon-V_\varepsilon} - \|\mathbf{q}\|_{V_\varepsilon-U_\varepsilon}.$$

If we had $f_*\mathbf{q} = \mathbf{q}$, we would have

$$0 < \|\mathbf{q}\|_{V-U} \le \|\mathbf{q}\|_{U_\varepsilon-V_\varepsilon}.$$

However, $\|\mathbf{q}\|_{U_\varepsilon-V_\varepsilon}$ tends to 0 as ε tends to 0, which is a contradiction. Indeed, $\mathbf{q} = q\,\mathrm{d}z^2$, the meromorphic function q being equivalent to $\frac{1}{(z-z_0)^2}$ as z tends to z_0. In addition, since the multiplier of z_0 has modulus 1,

$$D(z_0,\varepsilon) \subset U_\varepsilon - V_\varepsilon \subset D(z_0,\varepsilon') \quad \text{with} \quad \frac{\varepsilon'}{\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} 1.$$

Therefore,

$$\|\mathbf{q}\|_{U_\varepsilon-V_\varepsilon} \le \int_0^{2\pi}\int_\varepsilon^{\varepsilon'} \frac{1+o(1)}{r^2} r\mathrm{d}r\mathrm{d}\theta = 2\pi\big(1+o(1)\big)\log\frac{\varepsilon'}{\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} 0. \qquad \square$$

4. Irreducibility of the dynatomic curves

Our objective is now to give a proof of Theorem 1.2. Note that since the affine curve $\overline{X}_n$ is defined by a polynomial Q_n which has no repeated factors, this proves that Q_n is irreducible.

Since $\overline{X}_n$ is smooth, we may equivalently prove the following result.

Theorem 4.1. *For every $n \geq 1$, the set $\overline{X}_n$ is connected.*

4.1. **Kneading sequences.** Set $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and let $\tau : \mathbb{T} \to \mathbb{T}$ be the angle doubling map

$$\tau : \mathbb{T} \ni \theta \mapsto 2\theta \in \mathbb{T}.$$

By abuse of notation we shall often identify an angle $\theta \in \mathbb{T}$ with its representative in $[0, 1[$. In particular, the angle $\theta/2 \in \mathbb{T}$ is the element of $\tau^{-1}(\theta)$ with representative in $[0, 1/2[$ and the angle $(\theta + 1)/2$ is the element of $\tau^{-1}(\theta)$ with representative in $[1/2, 1[$.

Every angle $\theta \in \mathbb{T}$ has an associated kneading sequence $\nu(\theta) = \nu_1\nu_2\nu_3 \ldots$ defined by

$$\nu_k = \begin{cases} 1 & \text{if } \tau^{\circ(k-1)}(\theta) \in \left]\dfrac{\theta}{2}, \dfrac{\theta+1}{2}\right[, \\ 0 & \text{if } \tau^{\circ(k-1)}(\theta) \in \mathbb{T} - \left[\dfrac{\theta}{2}, \dfrac{\theta+1}{2}\right], \\ \star & \text{if } \tau^{\circ(k-1)}(\theta) \in \left\{\dfrac{\theta}{2}, \dfrac{\theta+1}{2}\right\}. \end{cases}$$

For example, $\nu(1/7) = \overline{11\star}$ and $\nu(7/31) = \overline{1100\star}$.

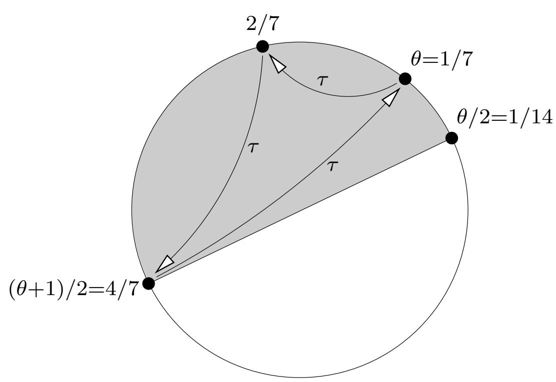


Figure 1. The kneading sequence of $\theta = 1/7$ is $\nu(1/7) = \overline{11\star}$.

We shall say that an angle $\theta \in \mathbb{T}$, periodic under τ, is *maximal in its orbit* if its representative in $[0, 1)$ is maximal among the representatives of $\tau^{\circ j}(\theta)$ in $[0, 1)$ for all $j \geq 1$. If the period is n and the binary expansion of θ is $.\overline{\varepsilon_1 \ldots \varepsilon_n}$, then θ is maximal in its orbit if and only if the periodic sequence $\overline{\varepsilon_1 \ldots \varepsilon_n}$ is maximal (in the lexigographic order) among its iterated shifts, where the shift of a sequence $\varepsilon_1\varepsilon_2\varepsilon_3$ indexed by $\mathbb{N}$ is

$$\sigma(\varepsilon_1\varepsilon_2\varepsilon_3 \ldots) = \varepsilon_2\varepsilon_3\varepsilon_4 \ldots.$$

Example 3. The angle $7/31 = .\overline{00111}$ is not maximal in its orbit but $28/31 = .\overline{11100}$ is maximal in the same orbit.

The following lemma indicates cases where the binary expansion and the kneading sequence coincide.

Lemma 4.2. *Let $\theta \in \mathbb{T}$ be a periodic angle which is maximal in its orbit and let $.\overline{\varepsilon_1 \ldots \varepsilon_n}$ be its binary expansion. Then, $\varepsilon_n = 0$ and the kneading sequence $\nu(\theta)$ is equal to $\overline{\varepsilon_1 \ldots \varepsilon_{n-1}\star}$.*

For example,

$$\frac{28}{31} = .\overline{11100} \quad \text{and} \quad \nu(\theta) = \overline{1110\star}.$$

Proof. Since θ is maximal in its orbit under τ, the orbit of θ is disjoint from $]\theta/2, 1/2] \cup]\theta, 1]$. It follows that the orbit $\tau^{\circ j}(\theta)$, $j = 0, 1, \ldots, n-2$ have the same itinerary relative to the two partitions $\mathbb{T} - \left\{0, \dfrac{1}{2}\right\}$ and $\mathbb{T} - \left\{\dfrac{\theta}{2}, \dfrac{\theta+1}{2}\right\}$. The first one gives the binary expansion whereas the second gives the kneading sequence. Therefore, the kneading sequence of θ is $\overline{\varepsilon_1 \ldots \varepsilon_{n-1}\star}$. Since $\tau^{\circ(n-1)}(\theta) \in \tau^{-1}(\theta) = \left\{\dfrac{\theta}{2}, \dfrac{\theta+1}{2}\right\}$ and since $\dfrac{\theta+1}{2} \in]\theta, 1]$, we must have $\tau^{\circ(n-1)}(\theta) = \dfrac{\theta}{2} < \dfrac{1}{2}$. So ε_n, as the first digit of $\tau^{\circ(n-1)}(\theta)$, must be equal to 0. □

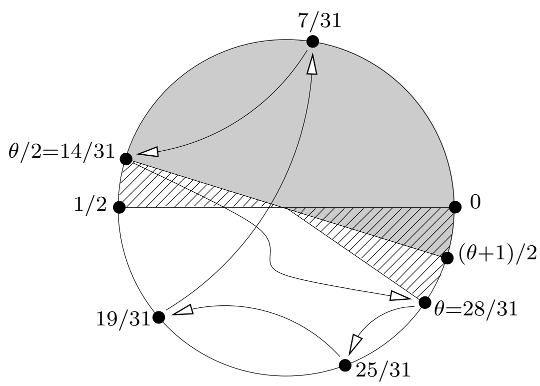


FIGURE 2. The kneading sequence of $\theta := 28/31 = .\overline{11100}$ is $\nu(28/31) = \overline{1110\star}$.

4.2. **Filled-in Julia sets and the Mandelbrot set.** We will use results proved by Douady and Hubbard in the Orsay Notes [DH] (see also [PR] for simpler proofs of some results) that we now recall.

For $c \in \mathbb{C}$, we denote by K_c the filled-in Julia set of f_c, that is the set of points $z \in \mathbb{C}$ whose orbit under f_c is bounded. We denote by M the Mandelbrot set, that is the set of parameters $c \in \mathbb{C}$ for which the critical point 0 belongs to K_c.

If $c \in M$, then K_c is connected. There is a conformal isomorphism $\phi_c : \mathbb{C} - K_c \to \mathbb{C} - \overline{\mathbb{D}}$ which satisfies $\phi_c \circ f_c = f_0 \circ \phi_c$. The dynamical ray of angle $\theta \in \mathbb{T}$ is

$$R_c(\theta) := \big\{z \in \mathbb{C} - K_c \mid \arg\big(\phi_c(z)\big) = 2\pi\theta\big\}.$$

If θ is rational, then as r tends to 1 from above, $\phi_c^{-1}(re^{2\pi i\theta})$ converges to a point $\gamma_c(\theta) \in K_c$. We say that $R_c(\theta)$ lands at $\gamma_c(\theta)$. We have $f_c \circ \gamma_c = \gamma_c \circ \tau$ on $\mathbb{Q}/\mathbb{Z}$. In particular, if θ is periodic under τ, then $\gamma_c(\theta)$ is periodic under f_c. In addition, $\gamma_c(\theta)$ is either repelling (its multiplier has modulus > 1) or parabolic (its multiplier is a root of unity).

If $c \notin M$, then K_c is a Cantor set. There is a conformal isomorphism $\phi_c : U_c \to V_c$ between neighborhoods of ∞ in $\mathbb{C}$, which satisfies $\phi_c \circ f_c = f_0 \circ \phi_c$ on U_c. We may choose U_c so that U_c contains the critical value c and V_c is the complement of a closed disk. For each $\theta \in \mathbb{T}$, there is an infimum $r_c(\theta) \geq 1$ such that ϕ_c^{-1} extends analytically along $R_0(\theta) \cap \{z \in \mathbb{C} \mid r_c(\theta) < |z|\}$. We denote by ψ_c this extension and by $R_c(\theta)$ the dynamical ray

$$R_c(\theta) := \psi_c\Big(R_0(\theta) \cap \big\{z \in \mathbb{C} \mid r_c(\theta) < |z|\big\}\Big).$$

As r tends to $r_c(\theta)$ from above, $\psi_c(re^{2\pi i\theta})$ converges to a point $x \in \mathbb{C}$. If $r_c(\theta) > 1$, then $x \in \mathbb{C} - K_c$ is an iterated preimage of 0 and we say that $R_c(\theta)$ bifucates at x. If $r_c(\theta) = 1$, then $\gamma_c(\theta) := x$ belongs to K_c and we say that $R_c(\theta)$ lands at $\gamma_c(\theta)$. Again, $f_c \circ \gamma_c = \gamma_c \circ \tau$ on the set of θ such that $R_c(\theta)$ does not bifurcate. In particular, if θ is periodic under τ and $R_c(\theta)$ does not bifurcate, then $\gamma_c(\theta)$ is periodic under f_c.

The Mandelbrot set is connected. The map

$$\phi_M : \mathbb{C} - M \ni c \mapsto \phi_c(c) \in \mathbb{C} - \overline{\mathbb{D}}$$

is a conformal isomorphism. For $\theta \in \mathbb{T}$, the parameter ray $R_M(\theta)$ is

$$R_M(\theta) := \big\{c \in \mathbb{C} - M \mid \arg\big(\phi_M(c)\big) = 2\pi\theta\big\}.$$

It is known that if θ is rational, then as r tends to 1 from above, $\phi_M^{-1}(re^{2\pi i\theta})$ converges to a point $\gamma_M(\theta) \in M$. We say that $R_M(\theta)$ lands at $\gamma_M(\theta)$.

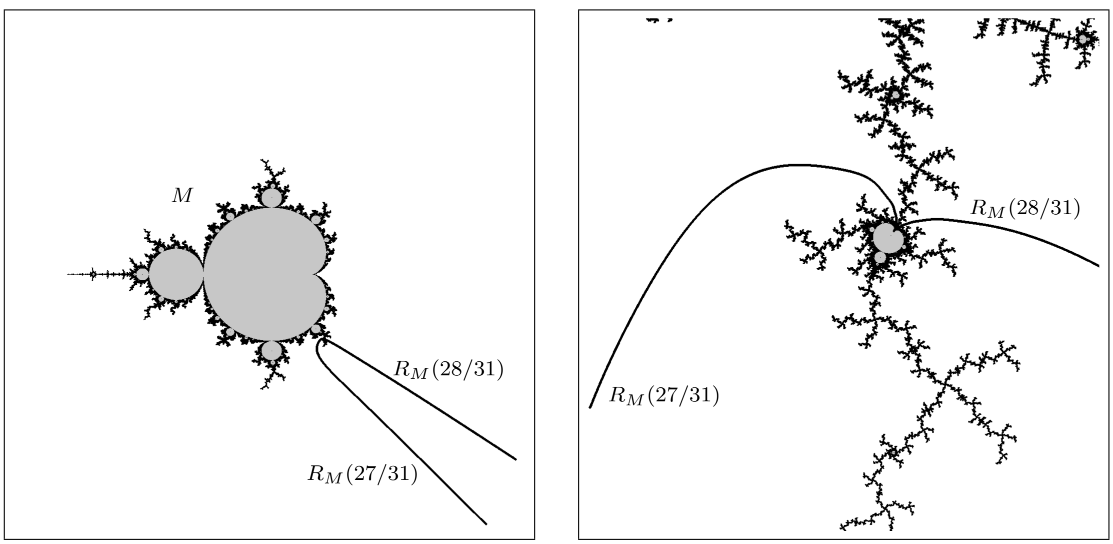


FIGURE 3. The parameter rays $R_M(27/31)$ and $R_M(28/31)$ land at a common root of a primitive hyperbolic component.

If θ is periodic for τ of exact period n and if $c = \gamma_M(\theta)$, then the point $\gamma_c(\theta)$ is periodic for f_c with period dividing n and multiplier a root of unity. If the period of $\gamma_c(\theta)$ for f_c is exactly n then the multiplier is 1, $\gamma_c(\theta)$ disconnects K_c in exactly two connected components and c is the root of a *primitive hyperbolic component* of M.

The parameter ray $R_M(0)$ lands at 1/4 and this is the only ray landing at 1/4. Let us now assume that $c \in \mathbb{C} - \{1/4\}$ is the root of a hyperbolic component of M, that is f_c has a parabolic cycle. Then there are exactly two parameter rays $R_M(\theta)$ and $R_M(\eta)$ landing

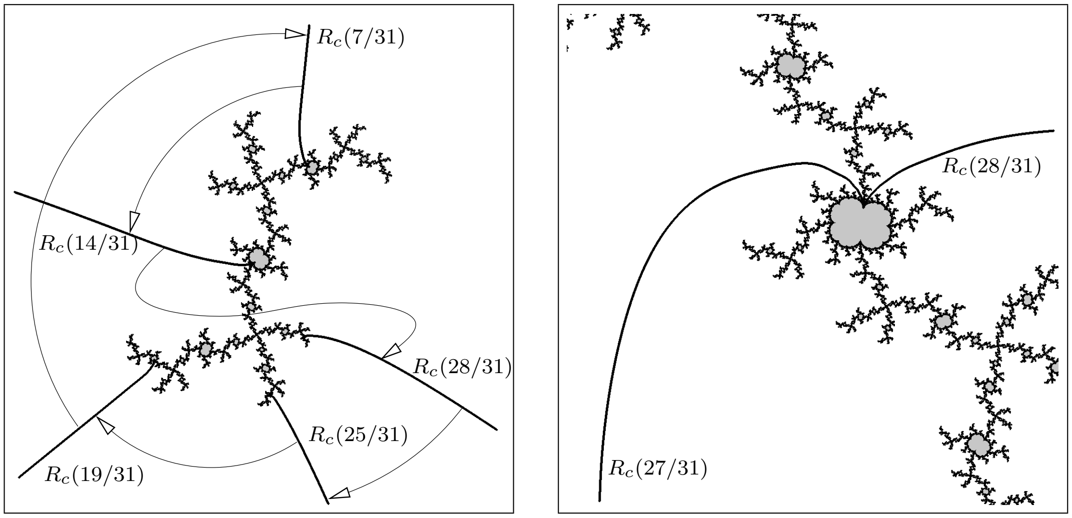


FIGURE 4. The filled-in Julia set K_c and a cycle of dynamical rays for $c := \gamma_M(28/31)$. The dynamical rays $R_c(28/31)$ and $R_c(27/31)$ both land at the same point.

at c. We say that θ and η are *companion angles*. Both θ and η are periodic under τ with the same period. The hyperbolic component is primitive if and only if the orbits of θ and η under τ are distinct. Otherwise, the orbits are equal.

The dynamical rays $R_c(\theta)$ and $R_c(\eta)$ land at a common point $x_1 := \gamma_c(\theta) = \gamma_c(\eta)$. This point x_1 is the point of the parabolic cycle whose immediate basin contains the critical value c. The dynamical rays $R_c(\theta)$ and $R_c(\eta)$ are adjacent to the Fatou component containing c. The curve $R_c(\theta) \cup R_c(\eta) \cup \{x_1\}$ is a Jordan arc that cuts the plane in two connected components. One component, denoted V_0, contains the dynamical ray $R_c(0)$ and all the points of the parabolic cycle, except x_1. The other component, denoted V_1, contains the critical value c.

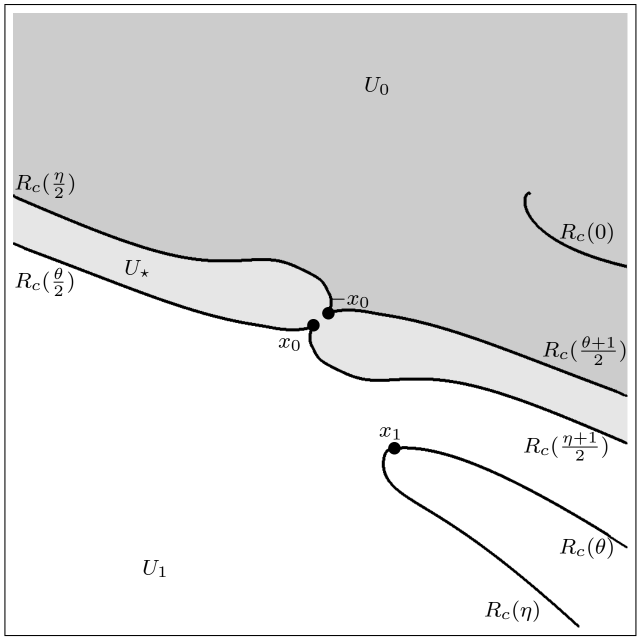
Since V_1 contains the critical value, its preimage $U_\star := f_c^{-1}(V_1)$ is connected and contains the critical point 0. It is bounded by the dynamical rays $R_c(\theta/2)$, $R_c(\eta/2)$, $R_c\big((\theta+1)/2\big)$ and $R_c\big((\eta+1)/2\big)$. Two of those dynamical rays land at the point x_0 of the parabolic cycle whose immediate basin contains the critical point 0. The two other dynamical rays land at $-x_0$. Since V_0 does not contain the critical value, its preimage has two connected components. One component, denoted U_0, contains the dynamical ray $R_c(0)$. The other component is denoted U_1.

Lemma 4.3. *Let $\theta \in \mathbb{T}$ be a periodic angle which is maximal in its orbit. Then,*

- *either $\theta = .\overline{11\ldots10}$,*
- *or $\gamma_M(\theta)$ is the root of a primitive hyperbolic component.*

Proof. If $\theta = 0$, then $\gamma_M(0) = 1/4$ is the root of a hyperbolic component. So, without loss of generality, we may assume that $\theta \neq 0$. Let $n \geq 2$ be the period of θ under τ, let η be the companion angle of θ and let U_0 and U_1 and $U_\star$ be defined as above.

Since θ is maximal in its orbit, $\tau^{\circ(n-1)}(\theta) = \theta/2$ (see Lemma 4.2). So, $R_c(\theta/2)$ lands on x_0. One of the two rays $R_c(\eta/2)$ and $R_c\big((\eta+1)/2\big)$ lands on x_0. Since $U_\star$ is connected



and contains dynamical rays with angles in between $\eta/2$ and $\theta/2$ and dynamical rays with angles in between $(\eta+1)/2$ and $(\theta+1)/2$, the ray landing on x_0 has to be $R_c\big((\eta+1)/2\big)$. It follows that $(\eta+1)/2$ is in the orbit of η under τ.

Since $\theta/2 < \theta < (\theta+1)/2$ and since $R_c(\theta)$ avoids $U_\star$, we have $\theta \le (\eta+1)/2$. On the one hand, if $\theta < (\eta+1)/2$, then the orbit of θ under τ does not contain $(\eta+1)/2$ since otherwise θ would not be maximal in its orbit. In that case, the orbits of θ and η are disjoint and $\gamma_M(\theta)$ is the root of a primitive hyperbolic component. On the other hand, if $\theta = (\eta+1)/2$, then the rays $R_c(\theta)$ and $R_c\big((\eta+1)/2\big)$ are equal. In that case, their landing point is the same, so $x_0 = x_1 = f_c(x_0)$ is a fixed point of f_c. The rays landing at this fixed point are permuted cyclically. The dynamical rays $R_c(\theta)$ and $R_c(\eta)$ are consecutive among the rays landing at x_0, $\eta < (\eta+1)/2 = \theta$ and $R_c(\theta)$ is mapped to $R_c(2\theta) = R_c(\eta)$. It follows that each dynamical ray landing at x_0 is mapped to the one which is once further clockwise. Consequently, the kneading sequence of θ is $\overline{1\ldots1\star}$ and according to Lemma 4.2, the binary expansion of θ is $.\overline{1\ldots10}$. □

4.3. **Outside the Mandelbrot set.** The projection $\pi_c : \overline{X}_n \to \mathbb{C}$ is a ramified covering. According to Proposition 3.1, the critical points are the points $(c,z) \in \overline{X}_n$ such that $f_c^{\circ n}(z) = z$ and $(f_c^{\circ n})'(z) = 1$. So, the critical values are precisely the roots of the polynomial $\Delta_n \in \mathbb{Z}[c]$ which is the discriminant of $P_n \in \mathbf{Z}[z]$. Those critical values are contained in the Mandelbrot set since a parabolic cycle for f_c attracts the critical point of f_c.

The open set

$$W := \mathbb{C} - \big(M \cup R_M(0)\big)$$

is simply connected. It avoids the critical values of the ramified covering $\pi_c : \overline{X}_n \to \mathbb{C}$. Let $W_n \subset X_n$ be the preimage of W by $\pi_c : \overline{X}_n \to \mathbb{C}$. It follows from the previous comment that $\pi_c : W_n \to W$ is a (unramified) cover, which is trivial since W is simply connected: each connected component of W_n maps isomorphically to W by π_c.

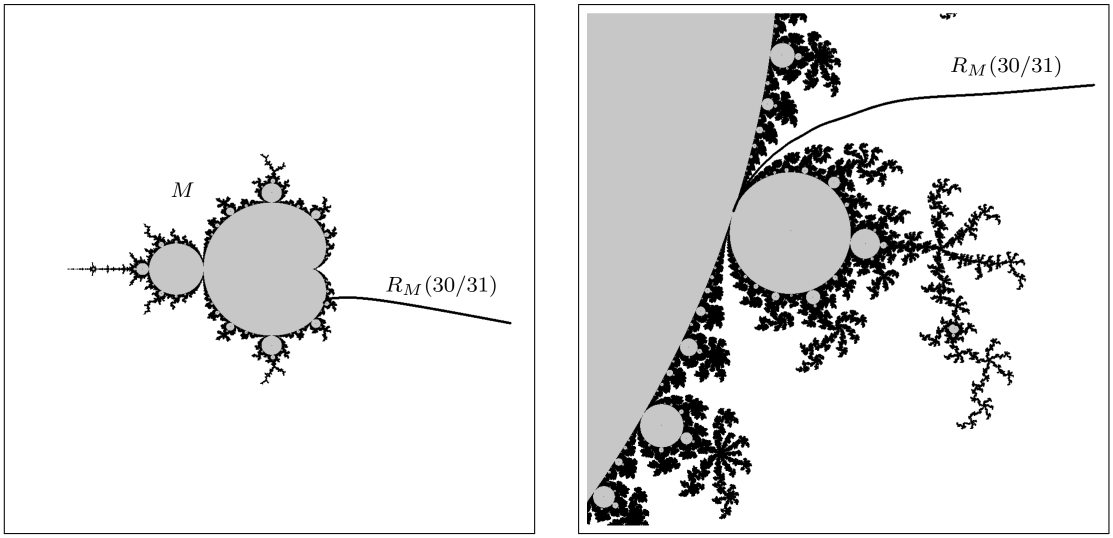


FIGURE 5. We have $30/31 = .\overline{11110}$ and the parameter ray $R_M(30/31)$ lands on the boundary of the main cardioid.

Note that each connected component of $\overline{X}_n$ is unbounded (because $\overline{X}_n$ is an affine curve), and so, intersects W_n. Thus, in order to prove that $\overline{X}_n$ is connected, it is enough to prove that the closure $\overline{W}_n$ of W_n in $\overline{X}_n$ is connected. We shall say that two components of W_n are adjacent if they have a common boundary point in $\overline{X}_n$.

4.4. **Labeling components of W_n.** Here, we explain how the components of W_n may be labelled dynamically.

A parameter $c \in W$ belongs to a parameter ray $R_M(\theta)$ with $\theta \neq 0$ not necessarily periodic. The dynamical rays $R_c(\theta/2)$ and $R_c\big((\theta+1)/2\big)$ bifurcate on the critical point. The Jordan curve $R_c(\theta/2) \cup R_c\big((\theta+1)/2\big) \cup \{0\}$ separates the complex plane in two connected components. We denote by $U_0(c)$ the component containing the dynamical ray $R_c(0)$ and by $U_1(c)$ the other component.

The orbit of a point $z \in K_c$ has an itinerary with respect to this partition. In other words, to each $z \in K_c$, we can associate a sequence $\iota_c(z) \in \{0,1\}^{\mathbb{N}}$ whose j-th term is equal to 0 if $f_c^{\circ(j-1)}(z) \in U_0$ and is equal to 1 if $f_c^{\circ(j-1)}(z) \in U_1$. A point $z \in K_c$ is periodic for f_c if and only if the itinerary $\iota_c(z)$ is periodic for the shift with the same period. The map $\iota_c : K_c \to \{0,1\}^{\mathbb{N}}$ is a bijection.

Let us define $\iota_n : W_n \to \{0,1\}^{\mathbb{N}}$ by

$$\iota_n(c,z) := \iota_c(z).$$

As c varies in W, the periodic points of f_c, the dynamical ray $R_c(0)$ and the Jordan curve $R_c(\theta/2) \cup R_c\big((\theta+1)/2\big) \cup \{0\}$ move continuously. As a consequence, the map $\iota_n : W_n \to \{0,1\}^{\mathbb{N}}$ is locally constant, whence constant on each connected component of W_n. So, each connected component V of W_n may be labelled by the itinerary $\iota_n(V)$. Since $\iota_c : K_c \to \{0,1\}^{\mathbb{N}}$ is injective, distinct components have distinct labels. Since $\iota_c : K_c \to \{0,1\}^{\mathbb{N}}$ is surjective, each periodic itinerary of period n is the label of a

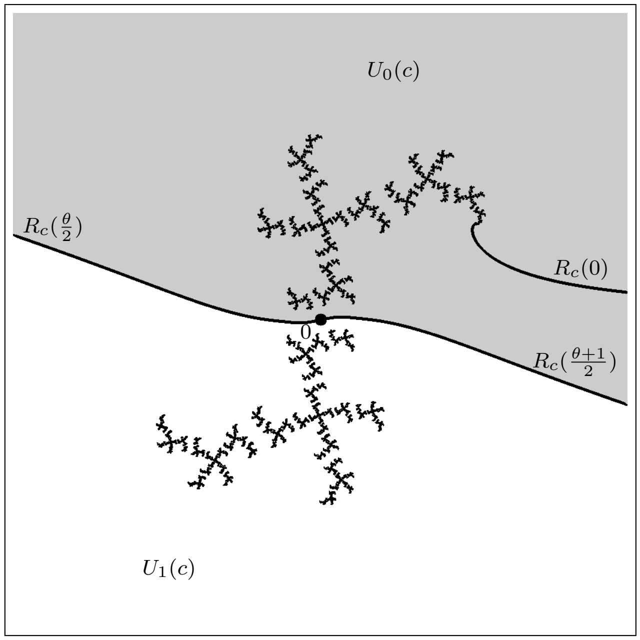


FIGURE 6. The regions U_0 and U_2 for a parameter c belonging to $R_M(28/31)$.

component of W_n. It follows that the number of connected components of W_n is equal to the number of n-periodic sequences in $\{0,1\}^{\mathbb{N}}$.

4.5. **Turning around critical points.** We now exhibit connected components of W_n which have common boundary points.

Proposition 4.4. *Let $\overline{\varepsilon_1 \ldots \varepsilon_{n-1}\star}$ be the kneading sequence of an angle $\theta \in \mathbb{T} - \{0\}$ which is periodic of period n. If $\gamma_M(\theta)$ is the root of a primitive hyperbolic component and if one follows continuously the periodic points of period n of f_c as c makes a small turn around $\gamma_M(\theta)$, then the periodic points with itineraries $\overline{\varepsilon_1 \ldots \varepsilon_{n-1}0}$ and $\overline{\varepsilon_1 \ldots \varepsilon_{n-1}1}$ get exchanged.*

Proof. Set $c_0 := \gamma_M(\theta)$. Since c_0 is the root of a primitive hyperbolic component, the periodic point $x_1 := \gamma_{c_0}(\theta)$ has period n and multiplier 1. According to Theorem 3.1 Case 2 (see also [DH, Exposé XIV, Proposition 3]), X_n is smooth at (c_0, x_1), the projection to the first coordinate has degree 2 and the projection to the second coordinate has degree 1. So, in a neighborhood of (c_0, x_1) in $\mathbb{C}^2$, X_n can be written as

$$\big\{(c_0+\delta^2, x(\delta)), (c_0+\delta^2, x(-\delta))\big\}$$

where $x : (\mathbb{C}, 0) \to (\mathbb{C}, x_1)$ is a holomorphic germ with $x'(0) \neq 0$. In particular, as c moves away from c_0, the periodic point x_1 of f_{c_0} splits into a pair of nearby periodic points $x(\pm\sqrt{c-c_0})$ for f_c, that get exchanged when c makes a small turn around c_0. So, it is enough to show that for $c \in \mathbb{C} - M$ close to c_0, those two periodic points have itineraries $\overline{\varepsilon_1 \ldots \varepsilon_{n-1}0}$ and $\overline{\varepsilon_1 \ldots \varepsilon_{n-1}1}$.

Let us denote by V_0, V_1, U_0, U_1 and $U_\star$ the open sets defined in Section 4.2. For $j \geq 0$, set $x_j := f_{c_0}^{\circ j}(x_0)$ and observe that for $j \in [1, n-1]$, we have $x_j \in U_{\varepsilon_j}$.

For $c \in R_M(\theta)$, consider the following compact subsets of the Riemann sphere $\mathbb{C}\cup\{\infty\}$:

$$R(c) := R_c(\theta) \cup \{c, \infty\} \quad \text{and} \quad S(c) := R_c(\theta/2) \cup R_c\big((\theta+1)/2\big) \cup \{0, \infty\}.$$

Denote by $U_0(c)$ the component of $\mathbb{C} - S(c)$ containing $R_c(0)$ and by $U_1(c)$ the other component. From any sequence $c_j \in R_M(\theta)$ converging to c_0, we can extract a subsequence so that $R(c_j)$ and $S(c_j)$ converge respectively, for the Hausdorff topology on compact subsets of $\mathbb{C} \cup \{\infty\}$, to connected compact sets R and S. Since $S(c) = f_c^{-1}\big(R(c)\big)$, we have $S = f_{c_0}^{-1}(R)$. According to [PR, Sections 2 and 3], $R \cap (\mathbb{C} - K_{c_0}) = R_{c_0}(\theta)$, the intersection of R with the boundary of K_{c_0} is reduced to $\{x_1\}$, and the intersection of R with the interior of K_{c_0} is contained in the immediate basin of x_1, whence in V_1. It follows that as $c \in R_M(\theta)$ tends to c_0, any Hausdorff accumulation value of the family of compact sets $R(c)$ is contained in $\overline{V}_1$ and so, any accumulation value of the family of compact sets $S(c)$ is contained in $\overline{U}_\star$. In other words, any compact subset of $\mathbb{C} - \overline{U}_\star$ is contained in $\mathbb{C} - S(c)$ for $c \in R_M(\theta)$ close enough to c_0. More precisely, every compact subset of U_0 is contained in a connected compact set $L \subset U_0$ whose interior intersects $R_{c_0}(0)$; for $c \in R_M(\theta)$ close enough to c_0, L intersects $R_c(0)$ and is contained in $\mathbb{C} - S(c)$, whence in $U_0(c)$. As a consequence, every compact subset of U_0 is contained in $U_0(c)$ for $c \in R_M(\theta)$ close enough to c_0. Similarly, every compact subset of U_1 is contained in $U_1(c)$ for $c \in R_M(\theta)$ close enough to c_0.

Fix $j \in [1, n-1]$ and let D_j be a sufficiently small disk around x_j so that

$$\overline{D}_j \subset U_{\varepsilon_j} \subset \mathbb{C} - \overline{U}_\star.$$

According to the previous discussion, if $c \in R_M(\theta)$ is close enough to c_0, we have

$$f_c^{\circ(j-1)}\big(x(\pm\sqrt{c-c_0})\big) \subset D_j \subset U_{\varepsilon_j}(c).$$

So, the itineraries of $x(\pm\sqrt{c-c_0})$ are of the form $\overline{\varepsilon_1 \ldots \varepsilon_{n-1}\varepsilon^\pm}$ with $\varepsilon^\pm \in \{0,1\}$ and $\varepsilon^+ \neq \varepsilon^-$ (each itinerary corresponds to a unique point in K_c). The result follows. □

Corollary 4.5. *Let $\overline{\varepsilon_1 \ldots \varepsilon_{n-1}\star}$ be the kneading sequence of an angle $\theta \in \mathbb{T} - \{0\}$ which is periodic of period n. If $\gamma_M(\theta)$ is the root of a primitive hyperbolic component, then, the components of W_n with labels $\overline{\varepsilon_1 \ldots \varepsilon_{n-1}0}$ and $\overline{\varepsilon_1 \ldots \varepsilon_{n-1}1}$ are adjacent.*

Proof. According to the previous proposition, the closures of those components both contain the point (c_0, x_1) with $c_0 := \gamma_M(\theta)$ and $x_1 := \gamma_{c_0}(\theta)$. □

Proposition 4.6. *Let $\theta = 1/(2^n-1) = .\overline{1\ldots10}$ be periodic of period $n \geq 2$. If one follows continuously the periodic points of period n of f_c as c makes a small turn around $\gamma_M(\theta)$, then the periodic points in the cycle of $\iota_c^{-1}(\overline{1\ldots10})$ get permuted cyclically.*

Proof. Set $c_0 := \gamma_M(\theta)$. As mentioned earlier, all the dynamical rays $R_{c_0}\big(\tau^{\circ j}(\theta)\big)$ land on a common fixed point x_0. This fixed point is parabolic and each ray landing at x_0 is mapped to the one which is once further clockwise. It follows that the multiplier of f_{c_0} at x_0 is $\omega := \mathrm{e}^{-2\pi\mathrm{i}/n}$.

According to Theorem 3.1 Case 3, $\overline{X}_n$ is smooth at (c_0, x_0), the projection to the first coordinate has local degree n and the projection to the second coordinate has local degree 1. It follows that in a neighborhood of (c_0, x_0) in $\mathbb{C}^2$, $\overline{X}_n$ can be written as

$$\big\{(c_0+\delta^n, x(\delta)), (c_0+\delta^n, x(\omega\delta)), \ldots, (c_0+\delta^n, x(\omega^{n-1}\delta))\big\}$$

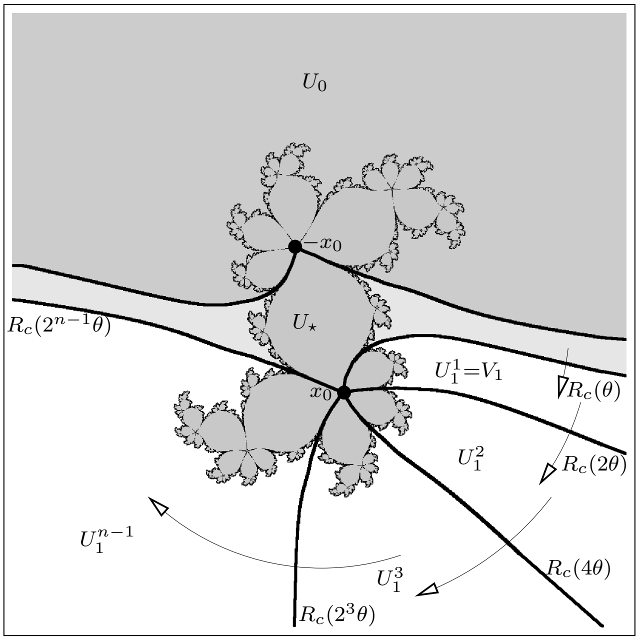


FIGURE 7. For $c_0 := \gamma_M(.\overline{11110})$, the dynamical rays $R_{c_0}\big(\tau^{\circ j}(\theta)\big)$ land on a common fixed point x_0.

where $x : (\mathbb{C}, 0) \to (\mathbb{C}, x_0)$ is a holomorphic germ satisfying $x'(0) \neq 0$. In addition,

$$f_{c_0+\delta^n}\big(x(\delta)\big) = x(\omega\delta).$$

So, for c close to c_0, the set $x\{\sqrt[n]{c-c_0})\}$ is a cycle of period n of f_c, and when c makes a small turn around c_0, the periodic points in the cycle $x\{\sqrt[n]{c-c_0})\}$ get permuted cyclically. So, it is enough to show that for $c \in \mathbb{C} - M$ close enough to c_0, the point $\iota_c^{-1}(\overline{1\ldots 10})$ belongs to $x\{\sqrt[n]{c-c_0}\}$.

Equivalently, we must show that there is a sequence $c_j \in \mathbb{C} - M$ converging to c_0, such that the sequence of periodic point $y_j = \iota_{c_j}^{-1}(\overline{1\ldots 10})$ converges to x_0. Let $c_j \in R_M(\theta)$ converge to c_0. As in the previous proof, consider the following compact subsets of the Riemann sphere $\mathbb{C} \cup \{\infty\}$:

$$R(c_j) := R_{c_j}(\theta) \cup \{c_j, \infty\} \quad \text{and} \quad S(c_j) := R_{c_j}(\theta/2) \cup R_{c_j}\big((\theta+1)/2\big) \cup \{0, \infty\}.$$

Denote by $U_0(c_j)$ the component of $\mathbb{C} - S(c_j)$ containing $R_{c_j}(0)$ and by $U_1(c_j)$ the other component. Without loss of generality, extracting a subsequence if necessary, we may assume that the sequence y_j converges to a point y, and that the sequence $R(c_j)$ and $S(c_j)$ have Hausdorff limits R and S. Passing to the limit on $f_{c_j}^{\circ n}(y_j) = y_j$, we see that $f_{c_0}^{\circ n}(y) = y$, and so, y is periodic for f_{c_0} with period dividing n. In particular, it is contained in the boundary of K_{c_0}. We must show that $y = \{x_0\}$.

It follows from [PR, Sections 2 and 3] that $R \cap (\mathbb{C} - K_{c_0}) = R_{c_0}(\theta)$, the intersection of R with the boundary of K_{c_0} is reduced to $\{x_0\}$ and the intersection of R with the interior of K_{c_0} is contained in the immediate basin of x_0. We cannot quite conclude that L is contained in $\overline{U}_\star$, but rather that it is contained in $\overline{U}_\star \cup \overset{\circ}{K}_{c_0}$. As in the previous proof, it follows that the Hausdorff limit of $\overline{U}_1(c_j)$ is contained in $\overline{U}_\star \cup U_1 \cup \overset{\circ}{K}_{c_0}$. Since $\iota_{c_j}(y_j) = \overline{1\ldots 10}$, we know that y_j, $f_{c_j}(y_j)$, $\ldots$, $f_{c_j}^{\circ(n-2)}(y_j)$ belong to $U_1(c_j)$. So the points

y, $f_{c_0}(y)$, …, $f_{c_0}^{\circ(n-2)}(y)$ belong to $\overline{U}_\star \cup U_1 \cup \overset{\circ}{K}_{c_0}$. Since y is in the boundary of K_{c_0}, we deduce that y, $f_{c_0}(y)$, …, $f_{c_0}^{\circ(n-2)}(y)$ belong to $\overline{U}_\star \cup U_1$.

The dynamical rays landing at x_0 divide U_1 in $n-1$ connected components U_1^j labelled clockwise so that

$$V_1 = U_1^1 \xrightarrow{f_{c_0}} U_1^2 \xrightarrow{f_{c_0}} \cdots \xrightarrow{f_{c_0}} U_1^{n-1} \xrightarrow{f_{c_0}} \mathbb{C} - \overline{U}_1.$$

The component $U_\star$ maps with degree 2 to $U_1^1 = V_1$ (see Figure 7).

Now, we claim that the orbit of y intersects $\overline{U}_\star$. Indeed, either y itself is in $\overline{U}_\star$, or y is in U_1^j for some $j \geq 1$. Then, $f_{c_0}^{\circ(n-j)}(y) \in \mathbb{C} - \overline{U}_1$. Since it cannot be in U_0, it belongs to $\overline{U}_\star$.

The map $f_{c_0}^{\circ n} : U_\star \to \mathbb{C} - \overline{U}_1$ is a proper map of degree 2 and $U_\star \subset \mathbb{C} - \overline{U}_1$. Note that $x_0 \in \overline{U}_\star$ is a multiple fixed point of $f_{c_0}^{\circ n}$ and that there is an attracting petal contained in $U_\star$. It follows from a version of the Lefschetz fixed point formula (see [GM, Lemma 3.7]) that x_0 is the only fixed point of $f_{c_0}^{\circ n}$ contained in $\overline{U}_\star$.

As a consequence, the orbit of the periodic point y contains x_0, and since x_0 is a fixed point, we have $y = x_0$ as required. □

Corollary 4.7. *The components of W_n whose labels contain a single 0 are adjacent.*

Proof. Let $\theta := .\overline{1\dots10}$ and $(c_0, x_0) := (\gamma_M(\theta), \gamma_{c_0}(\theta))$. By the above proposition every component of W_n whose label is a shift of $\overline{1\dots10}$ contains (c_0, x_0) in its boundary. But every periodic label of period n containing a single 0 is indeed a shift of $\overline{1\dots10}$. □

4.6. **Proof of Theorem 4.1.** We will finally deduce that $\overline{W}_n$ is connected. According to Corollary 4.7, components of W_n whose label contain a single 0 have a common boundary point. So, it is enough to show that component of W_n whose label has at least two 0 has a common boundary point with a component of W_n whose label has one less 0.

The map $F : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$F(c,z) := \big(c, f_c(z)\big)$$

restricts to an isomorphism $F : X_n \to X_n$. It permutes the components of W_n as follows: the label of $F(C)$ is the shift of the label of C. In addition, two components C_1 and C_2 of W_n are adjacent if and only if $F(C_1)$ and $F(C_2)$ are adjacent.

Let C be a connected component of W_n whose label ι contains at least two 0. Let $\overline{\varepsilon_1 \dots \varepsilon_n} = \sigma^{\circ k}(\iota)$ be maximal (in the lexigographic order) among the iterated shifts of ι. Then, the angle $\theta := .\overline{\varepsilon_1 \dots \varepsilon_n}$ is maximal in its orbit. According to Lemma 4.2, $\varepsilon_n = 0$ and the kneading sequence $\nu(\theta)$ is $\overline{\varepsilon_1 \dots \varepsilon_{n-1}\star}$. According to Lemma 4.3, $\gamma_M(\theta)$ is the root of a primitive hyperbolic component. According to Corollary 4.5, the component $F^{\circ k}(C)$ which is labeled $\overline{\varepsilon_1 \dots \varepsilon_{n-1}0}$ is adjacent to the component C' of W_n which is labeled $\overline{\varepsilon_1 \dots \varepsilon_{n-1}1}$. Then, $F^{\circ(n-k)}(C')$ is a component of W_n adjacent to $F^{\circ(n-k)}\big(F^{\circ k}(C)\big) = C$, and its label contains one less 0 than the label of C.

This completes the proof of Theorem 4.1. □

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