A shape optimization formulation of weld pool determination

Abdelkrim Chakib, Abdellatif Ellabib, Abdeljalil Nachaoui, Mourad Nachaoui

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A. Chakib[a], A. Ellabib[b], A. Nachaoui[c], M. Nachaoui[a,c]

[a] *Laboratoire de Mathématiques et Applications Université Sultan Moulay slimane, Faculté des Sciences et Techniques, B.P.523, Béni-Mellal, Maroc.*
[b] *Université Cadi Ayyad, Faculté des Sciences et Techniques, B.P 549, Guéliz Marrakech, Maroc.*
[c] *CNRS UMR6629, Université de Nantes, BP 92208, 44322 Nantes, France.*

Abstract

In this paper, we propose a shape optimization formulation for a problem modeling a process of welding. We show the existence of an optimal solution. The finite element method is used for the discretization of the problem. The discrete problem is solved by an identification technique using a parameterization of the weld pool by Bézier curves and Genetic algorithms.

Keywords: Welding, Shape optimization, Non coercive operator, Bézier curves, Genetic Algorithms.

1. Introduction

The determination of temperature field in a welding process permits the control of mechanical effects (residual stress, distortions, fatigue strength...). Many models are proposed in literature [1, 6].

The approach used here deals only with the solid part of the workpiece. It consists to simplify the physical phenomenon appeared between the welding torch, the workpiece and the liquid pool, by considering that the temperature field on the interface liquid/solid Γ is known.

In the shape optimization formulation that we propose, it appears a state problem governed by a non-coercive equation. This complicates the study of the existence of an optimal solution and more precisely, the uniform extension of the solution of the state problem with respect to domain.
We show the existence of an optimal solution by using recent results on uniform Poincarré inequality [2], and some Sobolev inequality [9], this is reported in section 3. Some numerical results are given in the last section showing the efficiency of our approach.

The welding problem consists in finding Γ the weld pool and T the temperature gradient in the workpiece, solution of:

$$\begin{cases} K\dfrac{\partial T}{\partial x} = \nabla\cdot(\lambda\nabla T) + f \quad \text{in} \quad \Omega \\ \\ \lambda\dfrac{\partial T}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \\ \\ T = T_d \quad \text{on} \quad \Gamma_4, \quad T = T_0 \quad \text{on} \quad \Gamma_0 \quad T = T_f \quad \text{on} \quad \Gamma \end{cases} \tag{1}$$

where Ω denotes the solid part of welded workpiece (see Fig 1); K is a function depending on the density of the

Email addresses: chakib@fstbm.ac.ma (A. Chakib), ellabib@fstg-marrakech.ac.ma (A. Ellabib), Abdeljalil.Nachaoui@univ-nantes.fr (A. Nachaoui), mourad.nachaoui@univ-nantes.fr (M. Nachaoui)

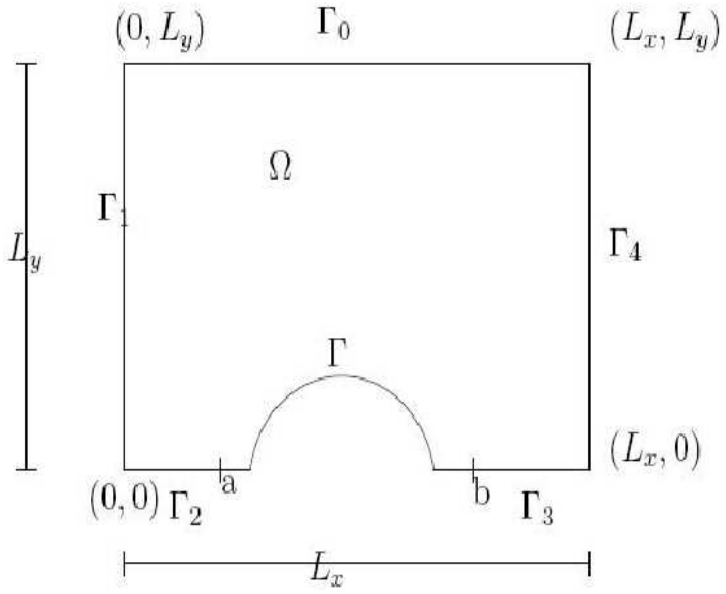

Figure 1: The solid part of the welded workpiece with interface Γ.

material and the heat capacity and independent of T; λ is the thermal conductivity; f is a given source term. The quantities T_d, T_0 and T_f are the given temperatures.
In the sequel we suppose that the parameters of our problem are such that:
Let $D =]0, L_x[\times]0, L_y[$,
(H_1) $\lambda \in L^\infty(D)$ and $\exists \lambda_0 > 0$ such that $\lambda(x)\xi \cdot \xi \geq \lambda_0 |\xi|^2$ p.p $x \in D$
(H_2) $K \in L^\infty(D)$
(H_3) $f \in L^\infty(D)$

2. The shape Optimization Formulation

The shape optimization formulation of problem (1) that we propose is given by:

$$\left\{\begin{array}{l} \text{find } \Omega^* \in \Theta_{ad} \text{ solution of} \\ J(\Omega^*) = \inf\limits_{\Omega \in \Theta_{ad}} J(\Omega) \\ \text{where } \; J(\Omega) = \frac{1}{2}\int_{\Gamma_0} |T_\Omega(x,y) - T_0|^2 \, d\sigma \\ \text{and } \; T_\Omega \text{ is the solution of} \\ \quad (PE)\left\{\begin{array}{l} K\frac{\partial T}{\partial x} = \nabla \cdot (\lambda \nabla T) + f \text{ in } \Omega \\ \\ \lambda \frac{\partial T}{\partial \nu} = 0 \text{ on } \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \\ \\ T = T_d \text{ on } \Gamma_4, \;\; T = T_f \text{ on } \Gamma \end{array}\right. \end{array}\right. \tag{2}$$

where the set of admissible domains Θ_{ad} is defined by

$$\Theta_{ad} = \{\Omega(\varphi) \;\; / \;\; \varphi \in U_{ad}\}$$

with

$$\Omega(\varphi) =]0, a[\times]0, L_y[\cup \left\{(x, y) \in \mathbb{R}^2 / a \le x \le b, \varphi(x) \le y \le L_y\right\} \cup]b, L_x[\times]0, L_y[$$

and

$$\begin{aligned} U_{ad} \quad = \quad & \Big\{\varphi \in C([a,b]) \ / \ \exists a_\varphi \text{ et } b_\varphi \ , \ \varphi_{|[a,a_\varphi]} = 0 \ , \ \varphi_{|[b_\varphi,b]} = 0 \text{ and} \\ & \left|\varphi(x) - \varphi(x')\right| \le C_0 \left|x - x'\right| \ \ \forall x, x' \in [a,b] \ , \ \ 0 \le \varphi(x) \le L_y \ \forall x \in [a,b]\Big\} \end{aligned}$$

where C_0 is the uniform Lipschitz constant.

In the next section we study the existence of a solution to problem (2).

3. Existence of the optimal solution

From the surjectivity of the trace operator from $H^1(D)$ to $H^{\frac{1}{2}}(\partial D)$,

$$\exists \ V \in H^1(D) \qquad \text{such that} \qquad V = \begin{cases} v \quad \text{on} \quad]b, L_x[\times]0, L_y[\\ \\ T_f \quad \text{on} \quad]0, b[\times]0, L_y[, \end{cases}$$

where $v \in H^1(]b, L_x[\times]0, L_y[)$ such that $v = T_d$ on Γ_4 and $v = T_f$ on $\{b\} \times [0, L_y]$.
Let $\Gamma_D = \Gamma \cup \Gamma_4$, we define the following Sobolev space $H^1_{\Gamma_D}(\Omega) = \left\{u \in H^1(\Omega) \ / \ u|_{\Gamma_D} = 0 \ \right\}$, and take $u = T - V$, then we consider the weak formulation:

$$\begin{cases} \text{find } u \in H^1_{\Gamma_D}(\Omega), \\ \\ \displaystyle\int_\Omega \lambda \nabla u \cdot \nabla \psi dxdy \ + \int_\Omega K \psi \frac{\partial u}{\partial x} dxdy = \langle L, \psi \rangle_{((H^1_{\Gamma_D}(\Omega))', H^1_{\Gamma_D}(\Omega))} \ \ \forall \psi \in H^1_{\Gamma_D}(\Omega), \end{cases} \tag{3}$$

where L is the operator defined by,

$$\langle L, \psi \rangle_{((H^1_{\Gamma_D}(\Omega))', H^1_{\Gamma_D}(\Omega))} = \int_\Omega f \psi dxdy \ - \int_\Omega \lambda \nabla V \cdot \nabla \psi dxdy \ - \int_\Omega K \psi \frac{\partial V}{\partial x} dxdy.$$

Remark 1. *Note that according to the assumptions $(H_1) - (H_3)$, we have $L \in (H^1_{\Gamma_D}(D))'$ and that there exists $\delta > 0$ such that $\|L\|_{(H^1_{\Gamma_D}(\Omega))'} \le \delta \ \forall \Omega \in \Theta_{ad}$.*

Define the space $\mathcal{F}$

$$\mathcal{F} = \{(\Omega, u(\Omega)) \ | \ \Omega \in \Theta_{ad} \ \text{ and } \ u(\Omega) \quad \text{solution of (3) in } \Omega\}. \tag{4}$$

and consider the following shape optimization problem

$$\text{Minimize } J(\Omega, u(\Omega)) \quad \text{ for all } (\Omega, u(\Omega)) \in \mathcal{F}. \tag{5}$$

Note that $T = u + V$ is solution of (PE) for each u solution of (3). Thus if $(\Omega, u(\Omega))$ is solution of (5) then $(\Omega, T(\Omega))$ is solution of the problem (2).

The existence of an optimal solution of (5), requires the definition of a topology on $\mathcal{F}$, which ensure the compactness of $\mathcal{F}$ and the Lower semicontinuity of J on $\mathcal{F}$.
For this, let $\Omega_n = \Omega(\varphi_n)$, $\Omega = \Omega(\varphi)$, $u_n = u(\Omega_n)$ and $u = u(\Omega)$, and define the convergence of Ω_n to Ω by

$$\Omega_n \longrightarrow \Omega \iff \varphi_n \longrightarrow \varphi \ \text{ uniformly on } \ [a,b] \ \text{ when } n \longrightarrow \infty. \tag{6}$$

Then we consider on $\mathcal{F}$ the topology defined by the following convergence:

$$(\Omega_n, u(\Omega_n)) \longrightarrow (\Omega, u(\Omega)) \iff \begin{cases} \Omega_n \longrightarrow \Omega \\ \tilde{u}_n \rightharpoonup \tilde{u} \quad \text{weakly in } H^1(D) \end{cases} \qquad \text{when } n \longrightarrow \infty, \tag{7}$$

where $\tilde{u}$ is a uniform extension in $H^1(D)$ of $u \in H^1(\Omega)$ (see D.Chenais [3]).

Then we have the following result

Theorem 1. *Under the assumptions* $(H_1) - (H_3)$*, the problem* (5) *is well posed and admits at least one solution in* $\mathcal{F}$

The proof of this theorem is based on the following lemmas.

Lemma 1. *Under assumptions* $(H_1) - (H_3)$*, the state problem* (3) *admits a unique solution.*

The presence of the term $\frac{\partial T}{\partial x}$ in the state problem equation, does not allow to have the coercivity, which is necessary for the application of the classical result of Lax-Milgram, without restriction on the physical parameters of the problem (K and λ). To overcome this problem, we use Leray Schauder topological degree [4]. To show this lemma, we consider the following application:

$$\begin{array}{ccc} \mathcal{G} : H^1_{\Gamma_D}(\Omega) & \mapsto & H^1_{\Gamma_D}(\Omega) \\ \bar{u} & \mapsto & u \end{array}$$

where u is the unique solution of problem:

$$\int_\Omega \lambda \nabla u \cdot \nabla \psi dxdy = \langle L, \psi \rangle_{((H^1_{\Gamma_D}(\Omega))', H^1_{\Gamma_D}(\Omega))} - \int_\Omega K \psi \frac{\partial \bar{u}}{\partial x} dxdy \qquad \forall \psi \in H^1_{\Gamma_D}(\Omega). \tag{8}$$

It's easy to see that $\mathcal{G}$ is well defined.

Remark 2. *Note that for all* $\bar{u} \in H^1_{\Gamma_D}(\Omega)$*, the existence of the unique solution of the problem (8) is obtained thanks to the Lax-Milgram result.*

A fixed point of $\mathcal{G}$ is solution of (3). To prove the existence of fixed point of $\mathcal{G}$, we have to show that $\mathcal{G}$ is compact and continuous, and find $R > 0$ such that $\forall t \in [0, 1]$, there exists no solution of $u - t\mathcal{G}(u) = 0$ satisfying $\|u\|_{1,\Omega} = R$. For the compactness of $\mathcal{G}$ it suffices to show, using $\psi = \mathcal{G}(\bar{u}_n) = \bar{u}_n$ as a test function in (3), that if $(\bar{u}_n)_n$ is bounded in $H^1_{\Gamma_D}(\Omega)$ then $(u_n)_n$ is a Cauchy sequence in $H^1_{\Gamma_D}(\Omega)$ and converges in this space. It's easy to see that $\mathcal{G}$ is continuous.
For the last point we show that there exists $C > 0$ such that $\|u\|_{1,\Omega} < C$ then we take $R = C + 1$.
For the uniqueness of the solution, since (3) is a linear problem, we show that the only solution of (3) with $L = 0$ is the null one.

The compactness of $\mathcal{F}$ for the topology defined in (7) requires the compactness of Θ_{ad}, which follows from the Ascoli Arzelà theorem, and the continuity of the state problem based on the following lemmas.

Lemma 2. *Under the assumptions* $(H_1)-(H_3)$*, we have: for all* $u \in H^1_{\Gamma_D}(\Omega)$ *solution of* (3) *in* Ω*, there exists a uniform extension* $\tilde{u} \in H^1(D)$ *of* u *and* $M > 0$ *independent of* $\Omega \in \Theta_{ad}$ *such that:*

$$\|\tilde{u}\|_{1,D} \leq M. \tag{9}$$

Proof

Note that the uniform cone property [3] is satisfied for all Ω in Θ_{ad}, thus for all $u \in H^1_{\Gamma_D}(\Omega)$, there exists $\tilde{u} \in H^1_{\Gamma_D}(D)$ and a constant $c > 0$ independent of Ω such that $\|\tilde{u}\|_{1,D} \leq c \|u\|_{1,\Omega}$.

The main difficulty of this work is to show that $\|u\|_{1,\Omega}$ is uniformly bounded with respect to Ω. For this we use the two following inequalities (see [2, 9])
- There exists $C_0 > 0$ independent of Ω such that $\forall u \in H^1_{\Gamma_D}(\Omega)$

$$C_0 \|u\|^2_{H^1(\Omega)} \leq \int_\Omega |\nabla u|^2 dxdy. \tag{10}$$

- There exists $C > 0$ independent of Ω such that

$$\|u\|_{L^4(\Omega)} \leq C|\Omega|^{\frac{1}{4}} \|u\|_{H^1(\Omega)}. \tag{11}$$

Then we define the set $A_k = \{x \in \Omega, |u(x)| > k\}$ and the functions $h_k(u) = max(-k, min(u, k))$ and $\psi_k(u) = u - h_k(u)$. First we show the following uniform estimation of $\psi_k(u)$:

$$(C_0 - C|A_k|^{\frac{1}{4}}) \|\psi_k(u)\|^2_{H^1(\Omega)} \leq | < L, \psi_k(u) >_{((H^1_{\Gamma_D}(\Omega))', H^1_{\Gamma_D}(\Omega))} | \tag{12}$$

To show that the constante $(C_0 - C|A_k|^{\frac{1}{4}})$ is positive, we use an idea of Droniou and Gallouet [5]. We start by showing the uniform control of Lebesgue measure of A_k, using Tchebycheff inquality and the uniform estimate of $\ln(1 + |u|)$, i.e. there exists $C_2 > 0$ independent of Ω such that

$$|A_k| = |\{(x, y) \in \Omega / \ln(1 + |w|)^2 \geq \ln(1 + k)^2\}| \leq \frac{1}{\ln(1 + k)^2} \|\ln(1 + |w|)\|_{L^2(\Omega)} \leq \frac{C_2}{\ln(1 + k)^2} \tag{13}$$

Then there exists $k_0 \in \mathbb{N}^*$, such that

$$\forall k \geq k_0 \qquad\qquad C|A_k|^{\frac{1}{4}} \leq \frac{C_0}{2}. \tag{14}$$

Taking $k = k_0$, we show that there exists $C_3 > 0$ independent of Ω such that

$$\left\|\psi_{k_0}(u)\right\|_{H^1(\Omega)} \leq C_3. \tag{15}$$

Finally, using the fact that $h_{k_0}(u)u \geq (h_{k_0}(u))^2$, $\nabla h_{k_0}(u) = \chi_{A_{k_0}} \nabla u$ and the inequality (10), we show that there exists $C_4 > 0$ independent of Ω such that

$$\left\|h_{k_0}(u)\right\|_{H^1(\Omega)} \leq C_4. \tag{16}$$

To conclude, we show this result:

Lemma 3. (**i**) *Let $u_n \in H^1_{\Gamma_D}(\Omega_n)$ be the solution of* (3) *on Ω_n, there exists $\tilde{u}_n$ a uniform extension of u_n which converges weakly in $H^1(D)$ to a limit denoted W, such that $u = W|\Omega^*$ is the solution of* (3) *in Ω^*, where Ω^* is the limit of Ω_n for the topology defined by* (6).
(**ii**) *The cost functional J is lower semicontinuous on $\mathcal{F}$.*

Proof

(**i**) Using the Lemma 2, for a sequence $(u_n)_n$, such that $u_n \in H^1_{\Gamma_D}(\Omega_n)$, we can extract a subsequence of $(\tilde{u}_n)_n$, where $\tilde{u}_n$ is the uniform extension of u_n, which converges weakly to W in $H^1(D)$. To show that $u = W|_{\Omega^*}$ is solution of equation (3) on Ω^*, it's easy to see that $u|_{\Gamma_4} = 0$ and according to the compactness of the trace operator from $H^1(D)$ into $L^2(\Gamma^*)$, we show that $u \in H^1{}_{\Gamma_D}(\Omega^*)$. Now, it suffices to show that u is solution of the weak formulation of the equation (3) on Ω^*. Indeed, let $\psi \in H^1_{\Gamma_D}(\Omega^*)$, and denoted by $\tilde{\psi} \in H^1(D)$ an extension of ψ defined by

$$\tilde{\psi} = \begin{cases} \psi & \text{in } \Omega \\ 0 & \text{in } D \setminus \Omega. \end{cases}$$

Then we can construct a sequence $(\psi_j)_j$, $\psi_j \in \mathcal{D}(\bar{D})$, such that,

$$dist(supp\ \psi_j, \overline{\Gamma_D}) > 0 \quad \forall j \in \mathbf{N} \quad \text{and} \quad \psi_j \to \tilde{\psi} \ \text{ in } H^1(D), j \to \infty.$$

Let $j \in \mathbf{N}$, since $\Omega_n \to \Omega^*$, there exists n_0 such that $\psi_j|_{\Omega_n} \in H^1_{\Gamma_D}(\Omega_n), \forall n \geq n_0$.
For all $n \geq n_0$, we have

$$\int_D \chi_{\Omega_n} \lambda \nabla \tilde{u}_n \cdot \nabla \psi_j dxdy \ + \int_D K \chi_{\Omega_n} \psi_j \frac{\partial \tilde{u}_n}{\partial x} dxdy \ = \langle L, \chi_{\Omega_n} \psi_j \rangle_{((H^1_{\Gamma_D}(\Omega))', H^1_{\Gamma_D}(\Omega))} \tag{17}$$

Using the convergence of characteristic functions χ_{Ω_n} to χ_{Ω^*} in $L^2(D)$, the weak convergence of $\tilde{u}_n$ to $\tilde{u}$ in $H^1(D)$, the convergence of ψ_j to $\tilde{\psi}$ in $H^1(D)$ and by passing to the limit in equation (17), we obtain that u solution of a weak formulation (3) in Ω^*.

(**ii**) The continuity of J on $\mathcal{F}$ is based on the weak convergence of $\tilde{u}_n$ to $\tilde{u}$ in $H^1(D)$ and the compactness of the trace operator from $H^1(D)$ into $L^2(\Gamma_0)$.

4. Numerical results

The shape optimization problem is approached by the P_1 finite element method. The free boundary is parameterized by piecewise spline approximation locally realized by quadratic Bézier functions. These allow to have a smooth domains and in the same time they are defined by a finite number of parameters. The corresponding discrete problem is solved by the genetic algorithms. Genetic algorithms (GA), primarily developed by Holland [8], have been successfully applied to various optimizations problems. It is essentially a searching method based on the Darwinian principles of biological evolution. It offer a good robustness, since they do not impose any regularity requirements on objective functions. Moreover, as (GA) are global optimization methods they can find new innovative designs instead of traditional designs corresponding to local minima. The GA is summarized in the following algorithm see [10].

begin

$t \leftarrow 0$

initialize a population $P(t)$

evaluate $P(t)$

while (not termination-condition) **do**

begin

$t \leftarrow t + 1$

select $P(t)$ from $P(t-1)$

alter $P(t)$

evaluate $P(t)$

end

end

To test the efficiency of our algorithm, we present an approximation of the exact solution $u = exp(x+y)$ and the exact boundary Γ parameterized by the half circle with center $(0.5, 0.0)$ and radius $r = 0.15$ (for $L_x = 1, L_y = 1, K = 1$ and $\lambda = 1$).

The following figures show that the cost decreases with respect to the number of iterations. The obtained numerical results are found to be in good agreement with the exact solution.

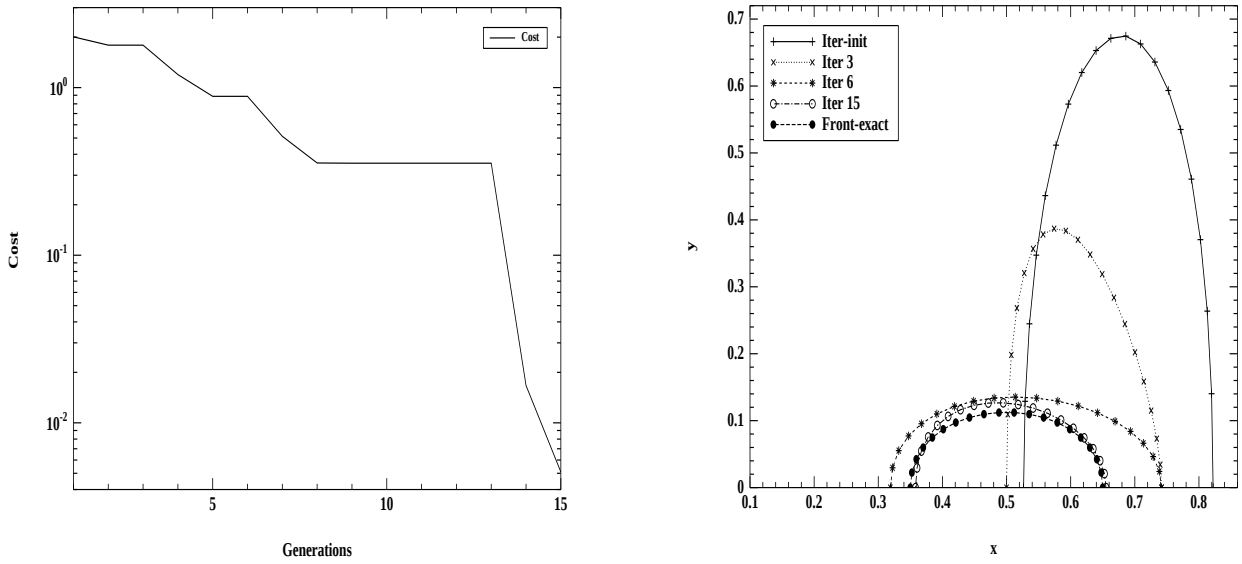

Figure 2: Cost functional and boundary evolution

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