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Welfare Analysis in Games with Substitutabilities

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Abstract

This paper investigates the social optimum in network games of strategic substitutes and identifies how network structure shapes optimal policies. First, we show that the socially optimal profile is obtained through a combination of two opposite network effects, generated by the incoming and the outgoing weighted Bonacich centrality measures. Next, three different policies that restore the social optimum are derived, and the implications of the predecessor(s)-successor(s) relationship between the agents on each policy instrument are explored. Then, the link between optimal taxes and the density of the network is established.

Keywords: network game, social optimum, Bonacich centrality, optimal policy, spectral radius.

JEL: A14, D85, H21.

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1 Introduction

This paper investigates the social optimum in a strategic game played on a directed network, where players' actions are strategic substitutes to their neighbors' actions. This class of games, pioneered among others by Ballester et al. (2006), encompasses various well-known games including the voluntary contribution of public goods (Bramoullé and Kranton, 2007; Bloch and Zenginobuz, 2007; Allouch, 2012). The main contributions of this present paper are (i) to elaborate a structural formula which expresses the socially optimal profile of actions as a combination of two opposite centrality measures, (ii) to derive three different policies that restore optimality and (iii) to highlight the prominent role of network structure on optimal taxes.

In network games of strategic substitutes, equilibrium analysis has traditionally been the primary research subject and great efforts have been made to increase our understanding of equilibrium behaviors and outcomes. Equilibrium existence has been well studied by several authors (Bramoullé and Kranton, 2007; Corbo et al., 2007; Bloch and Zenginobuz, 2007; Ballester and Calvó-Armengol, 2010; Le Breton and Weber, 2011), sufficient conditions on network structure that guarantee the uniqueness of the Nash equilibrium have been derived (Ballester et al., 2006; Corbo et al., 2007; Bloch and Zenginobuz, 2007; Ballester and Calvó-Armengol, 2010; Bramoullé et al., 2011), and the unique Nash equilibrium profile has been characterized in terms of the Bonacich centrality vector (Ballester et al., 2006; Corbo et al., 2007; Ballester and Calvó-Armengol, 2010; İlkiliç, 2010). More recent contributions have been exploring non-linear versions of the problem (Bramoullé et al., 2011; Allouch, 2012).

By contrast, welfare analysis has not been investigated so much, although this subject is crucial to understanding the upper bounds on the network's performance. A natural question already considered in the literature has been to identify the Nash equilibrium maximizing social welfare. In particular, positive and negative effects of removing a player (Ballester et al., 2006), adding a new link (Bramoullé and Kranton, 2007; Bramoullé et al., 2011) or changing the intensity of a link (Bloch and Zenginobuz, 2007) have been analyzed. Despite these attempts, the question of how to reach the Nash equilibrium that leads to the highest overall welfare deserves further investigations.

This present work brings a detailed answer to this question. The focus here is on interior social optima and some intuitions on how the analysis would be affected by relaxing this focus are presented. The agents, arranged in a weighted directed network, consume simultaneously a good that is rival and non-excludable along the directed links (e.g., irrigation water). They benefit only from their action, but their marginal costs, strictly positive and increasing in the action of consumption, depend both on their own action and on the action of their direct predecessors.[1]

The next section introduces the model and provides some equilibrium results that we use for the welfare analysis. We establish a necessary and sufficient condition for a unique interior Nash equilibrium in pure strategies, whenever such an equilibrium exists (Proposition 1). After that, we remind the structural formula found by Ballester and Calvó-Armengol (2010) which expresses the unique equilibrium profile in terms of the incoming Bonacich centrality measure (Proposition 2).

Section 3 examines the social optimum by adopting a standard utilitarian approach. We find that any interior social optimum can be expressed as a combination of the incoming and the outgoing Bonacich centrality measures (Proposition 3). This result directly extends to weighted utilitarian welfare functions (Remark 1) and admits a simple formulation when cost functions are quadratic (Remark 2). It appears that any interior Nash equilibrium may be interpreted as a specific social optimum (Proposition 4). Then, we discuss the case of corner or partially-corner social optima (Remark 3).

In Section 4, we establish three different policies that restore optimality: the optimal quotas program (Proposition 5), the optimal tax plan on benefits (Proposition 6) and the optimal tax plan on costs (Proposition 7). In each case, the network matters, i.e., the optimal intervention involves generally different rates at each site throughout the network. The revenues generated by these policies are compared, and it appears, since the cost of action is elastic, that the highest revenue is always produced by a tax plan on marginal benefits.

Section 5 discusses the relationship between the optimal tax rates and the density of the network. We find structural bounds, in terms of degree centrality, on the optimal tax rates (Proposition 8), and we show these tax rates can be located around the spectral radius of the network adjacency matrix (Proposition 9). Then, it is shown that the optimal tax rate is uniform if and only if the vector of marginal benefits is an eigenvector of the adjacency matrix (Proposition 10). In that case, the optimal uniform tax rate is actually the spectral radius of the network adjacency matrix and reflects therefore precisely the density of the network. Finally, we show how our results simplify with particular network structures (Remark 4).

In the rest of the paper, we will adopt the following notations. The superscript T stands for the transpose of a vector or a matrix. All vectors

[1]The strict convexity of the cost functions reflects the exhaustible feature of the good that is consumed by the agents.

are column vectors, and are denoted by lowercase bold letters, e.g., $\mathbf{v}$. We reserve the use of uppercase bold letters for matrices, e.g., $\mathbf{M}$. By $\mathbf{M}_{k.}$ we denote the k^{th} row-vector of $\mathbf{M}$. We reference the spectral radius of matrix $\mathbf{M}$ by $\rho(\mathbf{M})$. Finally, let $\mathbf{I}$ stand for the identity matrix and $\mathbf{1}$ for the vector of ones.

2 Framework

2.1 Model

There are N individuals connected through a network summarized in the adjacency matrix $\mathbf{\Omega} = [\omega_{kl}]$ with $\omega_{kl} \geq 0$ (and $\omega_{kk} = 0$, by convention).[2] Let $\mathbf{a} \in \mathbb{R}_+^N$ denote the vector of actions (or efforts) of individuals. The utility of individual l is

$$U_l(\mathbf{\Omega}, \mathbf{a}) = p_l a_l - q_l \left(a_l + \sum_{k:k \neq l} \omega_{kl} a_k \right)$$

where q_l is an increasing twice differentiable strictly convex cost function and p_l denote individual marginal benefits. We assume that $q_l'(0) < p_l < q_l'(\infty)$ for all l, and we denote by $\mathbf{q}$ the profile of costs functions. The strategy of utility-maximizing actions involves a simultaneous-move game. This allows us to find Nash equilibria of the game $G(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ by solving the linear complementarity problem (LCP)[3]

$$\begin{aligned} \left(\mathbf{I} + \mathbf{\Omega}^{\mathsf{T}}\right) \hat{\mathbf{a}} &\geq \mathbf{a}^* \\ \hat{\mathbf{a}} &\geq \mathbf{0} \\ \hat{\mathbf{a}}^{\mathsf{T}} \left[\left(\mathbf{I} + \mathbf{\Omega}^{\mathsf{T}}\right) \hat{\mathbf{a}} - \mathbf{a}^*\right] &= \mathbf{0} \end{aligned}$$

where $\hat{\mathbf{a}}$ is the equilibrium profile and $a_l^* = (q_l')^{-1}(p_l) > 0$ for all l. We may call a_l^* agent l's *equilibrium peak* and for ease of exposition, we write $\mathbf{a}^* = (\mathbf{q}')^{-1}(\mathbf{p})$.

Using this formulation, Ballester and Calvó-Armengol (2010) shows that $G(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ admits a unique Nash equilibrium whenever the spectral radius of the network adjacency matrix $\rho(\mathbf{\Omega})$ is small enough.[4] If the network is

[2] We may refer to ω_{kl} as the *weight* of the link from k to l.

[3] See Cottle et al. (1992) for a comprehensive treatment devoted to the LCP. See Corbo et al. (2007), Ballester and Calvó-Armengol (2010) or İlkiliç (2010) for the application of LCP to the problem of finding Nash equilibria in network games.

[4] The spectral radius of a network adjacency matrix is a standard measure of the network density. The higher the spectral radius, the denser is the network. See Cvetković et al. (1997).

undirected, the adjacency matrix is symmetric and a sharper condition has been established by Bramoullé et al. (2011).[5] In fact, at the root of the uniqueness problem is the matrix $\mathbf{I}+\mathbf{\Omega}$. The following result, which highlights the crucial role played by this matrix in directed networks, elaborates a necessary and sufficient condition for a unique interior Nash equilibrium, whenever such an equilibrium exists.

Proposition 1. *Let $G(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game. Assume there exists an interior Nash equilibrium. Then, $G(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ admits a unique interior Nash equilibrium if and only if $\mathbf{I}+\mathbf{\Omega}$ is invertible.*

Proof. Let $\hat{\mathbf{a}}$ be an interior Nash equilibrium. Then, $\hat{\mathbf{a}}$ is solution of the LCP

$$\begin{aligned} \left(\mathbf{I}+\mathbf{\Omega}^{\mathsf{T}}\right)\hat{\mathbf{a}} &\geq \mathbf{a}^* \\ \hat{\mathbf{a}} &>> \mathbf{0} \\ \hat{\mathbf{a}}^{\mathsf{T}}\left[\left(\mathbf{I}+\mathbf{\Omega}^{\mathsf{T}}\right)\hat{\mathbf{a}}-\mathbf{a}^*\right] &= \mathbf{0} \end{aligned}$$

which is equivalent to

$$\begin{aligned} \left(\mathbf{I}+\mathbf{\Omega}^{\mathsf{T}}\right)\hat{\mathbf{a}} &= \mathbf{a}^* \\ \hat{\mathbf{a}} &>> \mathbf{0} \end{aligned}$$

Assume $\mathbf{I}+\mathbf{\Omega}$ is invertible. So $(\mathbf{I}+\mathbf{\Omega})^{\mathsf{T}} = \left(\mathbf{I}+\mathbf{\Omega}^{\mathsf{T}}\right)$ is invertible. Then,

$$\hat{\mathbf{a}} = \left(\mathbf{I}+\mathbf{\Omega}^{\mathsf{T}}\right)^{-1}\mathbf{a}^*$$

is the unique solution of the above LCP.

(Only if). Let $\hat{\mathbf{a}}$ be an interior Nash equilibrium. Assume $\mathbf{I}+\mathbf{\Omega}$ is not invertible. So $(\mathbf{I}+\mathbf{\Omega})^{\mathsf{T}}$ is not invertible. There exists $\bar{\mathbf{a}} \neq \mathbf{0}$ such that $(\mathbf{I}+\mathbf{\Omega})^{\mathsf{T}}\bar{\mathbf{a}} = \mathbf{0}$ or equivalently, $\bar{\mathbf{a}}^{\mathsf{T}}(\mathbf{I}+\mathbf{\Omega}) = \mathbf{0}$. There exists $\alpha > 0$ small enough such that

$$\mathbf{0} << \hat{\mathbf{a}} + \alpha\bar{\mathbf{a}}$$

and

$$\begin{aligned} (\hat{\mathbf{a}}+\alpha\bar{\mathbf{a}})^{\mathsf{T}}(\mathbf{I}+\mathbf{\Omega}) &= \hat{\mathbf{a}}^{\mathsf{T}}(\mathbf{I}+\mathbf{\Omega}) + \alpha\bar{\mathbf{a}}^{\mathsf{T}}(\mathbf{I}+\mathbf{\Omega}) \\ &= \hat{\mathbf{a}}^{\mathsf{T}}(\mathbf{I}+\mathbf{\Omega}) \\ &= \mathbf{a}^*, \end{aligned}$$

so $(\hat{\mathbf{a}}+\alpha\bar{\mathbf{a}})^{\mathsf{T}}$ is an interior Nash equilibrium. Thus, there exists two interior Nash equilibria. □

[5]Bramoullé et al. (2011) show that $G(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ admits a unique Nash equilibrium whenever the lowest eigenvalue of the network adjacency matrix is high enough. That is, there is a unique equilibrium whenever the network is sufficiently tight (or sparse). However, this result does not hold when the network is directed (because in that case, the adjacency matrix is asymmetric and its eigenvalues are generally complex numbers).

2.2 Characterization of the unique equilibrium

For a network adjacency matrix $\mathbf{\Omega}$, let

$$\left(\mathbf{I} - \mathbf{\Omega}^{\mathsf{T}}\right)^{-1} = \sum_{k=0}^{\infty} \left(\mathbf{\Omega}^{\mathsf{T}}\right)^{k}$$

be well-defined and nonnegative. Therefore, its kl-entries count the total weight of all directed paths[6] in the network, beginning at player l and ending at player k. Let us introduce the Bonacich centrality measure.

Definition 1 (Bonacich, 1987)**.** Let $\mathbf{\Omega}$ be a network adjacency matrix. If $\left(\mathbf{I} - \mathbf{\Omega}^{\mathsf{T}}\right)^{-1}$ exists and is nonnegative, the vector

$$\mathbf{b}^{-}(\mathbf{\Omega}, \mathbf{1}) = \left(\mathbf{I} - \mathbf{\Omega}^{\mathsf{T}}\right)^{-1} \mathbf{1}$$

is called the *incoming weighted Bonacich centrality measure*[7] applied to $\mathbf{1}$.

Following this definition, let

$$\mathbf{b}^{-}_{\text{alt}}(\mathbf{\Omega}, \mathbf{1}) = \left(\mathbf{I} + \mathbf{\Omega}^{\mathsf{T}}\right)^{-1} \mathbf{1}$$

be the *alternate incoming weighted Bonacich centrality measure* applied to $\mathbf{1}$, provided that $\left(\mathbf{I} + \mathbf{\Omega}^{\mathsf{T}}\right)^{-1}$ exists. If $\rho(\mathbf{\Omega}) < 1$, we have the following algebraic identity:

$$\mathbf{b}^{-}_{\text{alt}}(\mathbf{\Omega}, \mathbf{1}) = \sum_{k=0}^{\infty} \left(\mathbf{\Omega}^{\mathsf{T}}\right)^{2k} \mathbf{1} - \sum_{k=0}^{\infty} \left(\mathbf{\Omega}^{\mathsf{T}}\right)^{2k+1} \mathbf{1}.$$

Then, its l-entry measures the difference between (i) the total weight of even length directed paths[8] that end at player l in the network and (ii) the total weight of odd length directed paths that end at him. Moreover, the alternate incoming weighted Bonacich can be recovered from the incoming weighted Bonacich of the squared adjacency matrix, i.e.,

$$\mathbf{b}^{-}_{\text{alt}}(\mathbf{\Omega}, \mathbf{1}) = \left(\mathbf{I} - \mathbf{\Omega}^{\mathsf{T}}\right) \mathbf{b}^{-}(\mathbf{\Omega}^{\mathbf{2}}, \mathbf{1})$$

Ballester and Calvó-Armengol (2010) shows that $\hat{\mathbf{a}} = \left(\mathbf{I} - \mathbf{\Omega}^{\mathsf{T}}\right) \mathbf{b}^{-}(\mathbf{\Omega}^{\mathbf{2}}, \mathbf{a}^{*})$ whenever the equilibrium is unique and interior. Then, $\hat{\mathbf{a}}$ is given by the alternate incoming weighted Bonacich centrality measure applied to the profile of equilibrium peaks.

[6] The total weight of a directed path is the sum of the weights of its links.

[7] See, for instance, Ballester et al. (2006), Corbo et al. (2007), Ballester and Calvó-Armengol (2010) or İlkiliç (2010) for the generalization of the Bonacich centrality to weighted networks.

[8] The length of a directed path is the number of links that compose the path.

Proposition 2 (Ballester and Calvó-Armengol, 2010). *Let $G(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game, and $\left(\mathbf{I} + \mathbf{\Omega}^{\mathsf{T}}\right)^{-1}$ exists. Assume the Nash equilibrium profile is interior. Then,*

$$\hat{\mathbf{a}} = \mathbf{b}^{-}_{\text{alt}}\left(\mathbf{\Omega}, \mathbf{a}^{*}\right) = \mathbf{b}^{-}_{\text{alt}}\left(\mathbf{\Omega}, \left(\mathbf{q}'\right)^{-1}(\mathbf{p})\right).$$

Proposition 2 entails that the equilibrium action of a player is positively related with the weight of even length directed paths and negatively related with the weight of odd length directed paths that end at him.[9] The actions of players who have a directed path of even length between them are strategic complements, whereas their actions are strategic substitutes if there is a directed path of odd length between them.

3 Social Optimum

To characterize the socially optimal profile of actions, we adopt a standard utilitarian approach. Given $\mathbf{\Omega}$, the maximum social welfare, SW, can be determined by solving,

$$SW\left(\mathbf{\Omega}, \mathbf{p}, \mathbf{q}\right) = \max_{\mathbf{a} \geq \mathbf{0}} \sum_{l} \left[p_l a_l - q_l\left(a_l + \sum_{k:k \neq l} \omega_{kl} a_k\right)\right].$$

We say a profile is *socially optimal* for a given network if, and only if, there is no other profile that leads to a strictly higher social welfare.

For all l, q_l is strictly convex and therefore, U_l is strictly concave. Then, W is a strictly concave function so there always exists a unique socially optimal profile $\tilde{\mathbf{a}} \in \mathbb{R}^N_+$. At social optimum, we have the following first order conditions:

$$\forall l,\ \tilde{a}_l > 0 \Longrightarrow p_l - q_l'\left(\tilde{a}_l + \sum_{k:k \neq l} \omega_{kl} \tilde{a}_k\right) - \sum_{j:j \neq l} \omega_{lj} q_j' \left(\tilde{a}_j + \sum_{i:i \neq j} \omega_{ij} \tilde{a}_i\right) = 0,$$

[9] It is well-known that Proposition 2 is not only valid for the case of no inactive agents (or free riders). Let's take a unique equilibrium with inactive agents $\hat{\mathbf{a}}$ and let $C = \{i : \hat{a}_i > 0\}$. Consider the subnetwork obtained by deleting all the inactive agents, and let $\mathbf{\Omega}_{C \times C}$ be its corresponding adjacency matrix. The subvector $\hat{\mathbf{a}}_C$ consisting of all the active agents in the original game is also a Nash equilibrium for the subgame obtained. Moreover, there are no inactive agents in this subgame, hence $\hat{\mathbf{a}}_C$ can be expressed as a function of the incoming Bonacich centrality measure of the subnetwork obtained after deleting the inactive agents, provided that $\mathbf{I} + \mathbf{\Omega}_{C \times C}$ is invertible, i.e.,

$$\hat{\mathbf{a}}_C = \mathbf{b}^{-}_{\text{alt}}\left(\mathbf{\Omega}_{C \times C}, \left(\mathbf{q}'\right)^{-1}(\mathbf{p}_C)\right),$$

where $\mathbf{p}_C$ denote the subvector of marginal benefits obtained after deleting all the inactive agents in the original game.

otherwise $\tilde{a}_l = 0$.

Definition 2. Let $\mathbf{\Omega}$ be a network adjacency matrix. If $(\mathbf{I} - \mathbf{\Omega})^{-1}$ exists and is nonnegative, the vector

$$\mathbf{b}^{+}(\mathbf{\Omega}, \mathbf{1}) = (\mathbf{I} - \mathbf{\Omega})^{-1}\,\mathbf{1}$$

is called the *outgoing weighted Bonacich centrality measure* applied to $\mathbf{1}$.

Following this definition, let

$$\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega}, \mathbf{1}) = (\mathbf{I} + \mathbf{\Omega})^{-1}\,\mathbf{1}$$

be the *alternate outgoing weighted Bonacich centrality measure* applied to $\mathbf{1}$, provided that $(\mathbf{I} + \mathbf{\Omega})^{-1}$ exists. By definition,

$$\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega}, \mathbf{1}) = \mathbf{b}^{-}_{\text{alt}}(\mathbf{\Omega}^{\mathsf{T}}, \mathbf{1}).$$

Therefore, its l-entry measures the difference between (i) the total weight of even directed paths that begin at player l in the network and (ii) the total weight of odd directed paths that begin at him.

It appears that the socially optimal profile of game $G(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ can be expressed as the alternate incoming weighted Bonacich centrality measure applied to a function of the alternate outgoing weighted Bonacich centrality measure applied to the profile of marginal benefits.

Proposition 3. *Let $G(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game, and $(\mathbf{I} + \mathbf{\Omega})^{-1}$ exists. Assume the social optimum is interior. Then,*

$$\tilde{\mathbf{a}} = \mathbf{b}^{-}_{\text{alt}}\left(\mathbf{\Omega}, (\mathbf{q}')^{-1}\left(\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega}, \mathbf{p})\right)\right),$$

and

$$SW(\mathbf{\Omega}, \mathbf{p}, \mathbf{q}) = \sum_{l}\left[p_l \tilde{a}_l - q_l\left((q'_l)^{-1}\left(\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega}, \mathbf{p})\right)_l\right)\right].$$

Proof. Since $\tilde{a}_l > 0$ for all l, at social optimum we have the following first order conditions:

$$\forall l,\;\; p_l - q'_l\left(\tilde{a}_l + \sum_{k:k\neq l}\omega_{kl}\tilde{a}_k\right) - \sum_{j:j\neq l}\omega_{lj}q'_j\left(\tilde{a}_j + \sum_{i:i\neq j}\omega_{ij}\tilde{a}_i\right) = 0.$$

Let $e_j = q'_j\left(\tilde{a}_j + \sum_{i:i\neq j}\omega_{ij}\tilde{a}_i\right)$ for all j. Then, the first order conditions may be written:

$$\forall l,\;\; p_l = e_l + \sum_{j:j\neq l}\omega_{lj}e_j = ([\mathbf{I} + \mathbf{\Omega}]\,\mathbf{e})_l = (\mathbf{I} + \mathbf{\Omega})_{l.}\,\mathbf{e}.$$

In matrix notation,

$$\mathbf{p} = (\mathbf{I} + \mathbf{\Omega})\, \mathbf{e}.$$

Since $\mathbf{I} + \mathbf{\Omega}$ is invertible, we obtain

$$\mathbf{e} = (\mathbf{I} + \mathbf{\Omega})^{-1}\, \mathbf{p}.$$

We have specified $e_j = q'_j \left(\tilde{a}_j + \sum_{i:i\neq j} \omega_{ij} \tilde{a}_i \right)$ for all j. Thus,

$$\forall j, \;\; \left(q'_j\right)^{-1}(e_j) = \tilde{a}_j + \sum_{i:i\neq j} \omega_{ij} \tilde{a}_i = \left(\mathbf{I} + \mathbf{\Omega}^\top\right)_{j.}\, \tilde{\mathbf{a}}.$$

Hence,

$$\left(\mathbf{q}'\right)^{-1}(\mathbf{e}) = \left(\mathbf{I} + \mathbf{\Omega}^\top\right) \tilde{\mathbf{a}},$$

and therefore,

$$\tilde{\mathbf{a}} = \left(\mathbf{I} + \mathbf{\Omega}^\top\right)^{-1} \left(\mathbf{q}'\right)^{-1}(\mathbf{e}) = \left(\mathbf{I} + \mathbf{\Omega}^\top\right)^{-1} \left(\mathbf{q}'\right)^{-1} \left(\left[\mathbf{I} + \mathbf{\Omega}\right]^{-1} \mathbf{p}\right).$$

□

At social optimum, the incoming weighted Bonacich centrality measure that determines the equilibrium profile (see Proposition 2) is counterbalanced by the outgoing weighted Bonacich centrality vector. While at equilibrium, the action of a player depends only upon how he is impacted by the actions of all his direct and indirect predecessors, Proposition 3 indicates that the socially optimal action of a player also reflects how he impacts the actions of all his direct and indirect successors.

Remark 1 (Weighted utilitarian welfare functions). More generally, the utilitarian welfare function could be weighted, reflecting the interest of the social planner for the various players with respect to their location. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) >> 0$ be social weights and consider the welfare function

$$\sum_l \alpha_l \left[p_l a_l - q_l \left(a_l + \sum_{k:k\neq l} \omega_{kl} a_k \right) \right].$$

Then, the social optimum is given by

$$\tilde{\mathbf{a}} = \mathbf{b}^-_{\text{alt}} \left(\mathbf{\Omega}, \left(\mathbf{q}'\right)^{-1} \left(\frac{\mathbf{1}}{\boldsymbol{\alpha}}\, \mathbf{b}^+_{\text{alt}} \left(\mathbf{\Omega}, \boldsymbol{\alpha}\mathbf{p}\right) \right) \right),$$

where $\left(\frac{\mathbf{1}}{\boldsymbol{\alpha}}\right)_l = \frac{1}{\alpha_l}$ for all l.

Remark 2 (Quadratic costs). Assume $q_l(a_l) = \frac{c_l}{2} a_l^2$ for all l with $c_l > 0$. Then,

$$\tilde{\mathbf{a}} = \mathbf{b}^{-}_{\text{alt}} \left(\mathbf{\Omega}, \mathbf{b}^{+}_{\text{alt}} \left(\mathbf{\Omega}, \frac{\mathbf{p}}{\mathbf{c}} \right) \right)$$

where $\left(\frac{\mathbf{p}}{\mathbf{c}}\right)_l = \frac{p_l}{c_l}$ for all l. Since the incoming and outgoing weighted Bonacich centrality measures are linear transformations, the social optimum is also a linear transformation of marginal benefits. In particular, if $c_l = c$ for all l, then

$$\tilde{\mathbf{a}} = \frac{1}{c}\, \mathbf{b}^{-}_{\text{alt}} \left(\mathbf{\Omega}, \mathbf{b}^{+}_{\text{alt}} \left(\mathbf{\Omega}, \mathbf{p} \right) \right).$$

The interior socially optimal profile of game $G(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ can be interpreted as a Nash equilibrium of another game with modified equilibrium peaks. It is the alternate incoming weighted Bonacich centrality measure applied to the profile of *efficient peaks* $\tilde{\mathbf{a}}^*$, which may itself be expressed in terms of the alternate outgoing weighted Bonacich centrality measure applied to $\mathbf{p}$, i.e.,

$$\tilde{\mathbf{a}} = \mathbf{b}^{-}_{\text{alt}} \left(\mathbf{\Omega}, \tilde{\mathbf{a}}^* \right),$$

where

$$\tilde{\mathbf{a}}^* = (\mathbf{q}')^{-1} \left(\mathbf{b}^{+}_{\text{alt}} \left(\mathbf{\Omega}, \mathbf{p} \right) \right).$$

Following the proof of Proposition 1, $\tilde{\mathbf{a}}$ is therefore the unique solution of the linear problem given by

$$\begin{array}{rcl} \left(\mathbf{I} + \mathbf{\Omega}^{\mathsf{T}} \right) \tilde{\mathbf{a}} & = & \tilde{\mathbf{a}}^* \\ \tilde{\mathbf{a}} & >> & \mathbf{0} \end{array}$$

It follows that any interior Nash equilibrium can be interpreted as a social optimum with a specific "price" system.

Proposition 4. *Let $G(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game, and $(\mathbf{I} + \mathbf{\Omega})^{-1}$ exists. Assume there is an interior Nash equilibrium, i.e., $\hat{\mathbf{a}} >> \mathbf{0}$, then $\hat{\mathbf{a}}$ is an interior social optimum of the network game $G(\mathbf{\Omega}, \tilde{\mathbf{p}}, \mathbf{q})$ where $\tilde{\mathbf{p}} = (\mathbf{I} + \mathbf{\Omega})\, \mathbf{p}$.*

Proof. Since $\hat{a}_l > 0$ for all l, at equilibrium the following first order conditions are satisfied:

$$\forall l, \;\; p_l = q_l' \left(\hat{a}_l + \sum_{k:k \neq l} \omega_{kl} \hat{a}_k \right).$$

Hence, for all l,

$$\begin{array}{rcl} \tilde{p}_l = (\mathbf{I} + \mathbf{\Omega})_l & = & p_l + \sum_{j:j \neq l} \omega_{lj} p_j \\ & = & q_l' \left(\hat{a}_l + \sum_{k:k \neq l} \omega_{kl} \hat{a}_k \right) + \sum_{j:j \neq l} \omega_{lj} q_j' \left(\hat{a}_j + \sum_{i:i \neq j} \omega_{ij} \hat{a}_i \right) \end{array}$$

and these conditions are precisely the first order conditions for an interior social optimum. □

Remark 3 (Corner and partially-corner social optima). Consider a social optimum where some agents are inactive. Let $C = \{i : \tilde{a}_i > 0\}$ and NC its complement. Then, the first order conditions of social welfare maximization for the active agents are: $\forall l \in C$,

$$p_l - q_l' \left(\tilde{a}_l + \sum_{k:k\neq l,k\in C} \omega_{kl}\tilde{a}_k \right) - \sum_{j:j\neq l,j\in C} \omega_{lj} q_j' \left(\tilde{a}_j + \sum_{i:i\neq j,i\in C} \omega_{ij}\tilde{a}_i \right) - \sum_{g:g\neq l,g\in NC} \omega_{lg} q_g' \left(0 + \sum_{h:h\neq g,h\in C} \omega_{hg}\tilde{a}_h \right) = 0.$$

Let $e_i = q_i'(.)$ for all i. We note $\mathbf{e}_C$ (resp. $\mathbf{e}_{NC}$) the subvector of $\mathbf{e}$ obtained after deletion of all the inactive (resp. active) agents. In matrix notation, the first order conditions for the active agents write

$$\mathbf{p}_C - (\mathbf{I} + \mathbf{\Omega}_{C\times C})\, \mathbf{e}_C - \mathbf{\Omega}_{C\times NC}\ \mathbf{e}_{NC} = \mathbf{0},$$

where $\mathbf{p}_C$ denote the subvector of marginal benefits obtained after deletion of all the inactive agents, $\mathbf{\Omega}_{C\times C}$ the adjacency matrix of the subnetwork obtained after deletion of all the inactive agents, and $\mathbf{\Omega}_{C\times NC}$ the (possibly rectangular) submatrix of $\mathbf{\Omega}$ consisting of rows with all the active agents and columns with all the inactive agents. Assuming that $\mathbf{I} + \mathbf{\Omega}_{C\times C}$ is invertible, and letting $\tilde{\mathbf{a}}_C$ be the subvector of socially optimal actions consisting of all the active agents (then, $\tilde{\mathbf{a}}_{NC} = \mathbf{0}$), we obtain

$$\tilde{\mathbf{a}}_C = \mathbf{b}^-_{\text{alt}} \left(\mathbf{\Omega}_{C\times C}, (\mathbf{q}')^{-1} \left(\mathbf{b}^+_{\text{alt}} \left(\mathbf{\Omega}_{C\times C}, \bar{\mathbf{p}} \right) \right) \right),$$

where

$$\bar{\mathbf{p}} = \mathbf{p}_C - \mathbf{\Omega}_{C\times NC}\ \mathbf{e}_{NC}.$$

Therefore, $\tilde{\mathbf{a}}_C$ is a fixed point (since $\bar{\mathbf{p}}$ is a function of $\tilde{\mathbf{a}}_C$).

4 Optimality-restoring policies

4.1 Optimal quotas

We introduce individual action quotas in order to restore the social optimum in games with substitutabilities. The utility of an agent is then given by

$$U_l (\mathbf{\Omega}, \mathbf{a}) = p_l a_l - q_l \left(a_l + \sum_{k:k\neq l} \omega_{kl} a_k \right),$$

with $a_l \leq k_l$ for all l, where k_l is agent l's action quota. As it turns out, the vector of optimal quotas is actually the socially optimal profile, and is expressed as a combination of the incoming and the outgoing alternate weighted Bonacich centrality measures.

Proposition 5. *Let $G(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game, and $(\mathbf{I} + \mathbf{\Omega})^{-1}$ exists. Assume the social optimum is interior, then the optimal quota vector is*

$$\tilde{\mathbf{k}} = \tilde{\mathbf{a}} = \mathbf{b}^{-}_{\text{alt}} \left(\mathbf{\Omega}, (\mathbf{q}')^{-1} \left(\mathbf{b}^{+}_{\text{alt}} (\mathbf{\Omega}, \mathbf{p}) \right) \right).$$

Proof. Since $\tilde{a}_l > 0$ for all l, at social optimum we have the following first order conditions:

$$\forall l, \;\; p_l - q_l' \left(\tilde{a}_l + \sum_{k:k \neq l} \omega_{kl} \tilde{a}_k \right) - \sum_{j:j \neq l} \omega_{lj} q_j' \left(\tilde{a}_j + \sum_{i:i \neq j} \omega_{ij} \tilde{a}_i \right) = 0.$$

Then, by strict convexity of the cost functions, $p_l - q_l' \left(\tilde{a}_l + \sum_{k:k \neq l} \omega_{kl} \tilde{a}_k \right) \geq 0$.

Now, we show that the socially optimal profile $\tilde{\mathbf{a}}$ is also the Nash equilibrium of a game where each player is constrained to exert an action at most equal to his socially optimal level. Given $\mathbf{\Omega}$ and $\tilde{\mathbf{a}}_{-l}$, an agent l's maximization program is:

$$\max_{a_l} \; p_l a_l - q_l(a_l + \sum_{k:k \neq l} \lambda_{kl} \tilde{a}_k)$$
$$\text{s.t.} \quad a_l \in [0, \tilde{a}_l].$$

By assumption, $U_l(\mathbf{\Omega}, \tilde{\mathbf{a}}_{-l}, a_l) = p_l a_l - q_l \left(a_l + \sum_{k:k \neq l} \lambda_{kl} \tilde{a}_k \right)$ is a strictly concave utility function and

$$U_l'(\mathbf{\Omega}, \tilde{\mathbf{a}}_{-l}, a_l) = p_l - q_l' \left(a_l + \sum_{k:k \neq l} \omega_{kl} \tilde{a}_k \right).$$

Then, for $a_l = \tilde{a}_l$, we have

$$U_l'(\mathbf{\Omega}, \tilde{\mathbf{a}}) = p_l - q_l' \left(\tilde{a}_l + \sum_{k:k \neq l} \omega_{kl} \tilde{a}_k \right) \geq 0$$

because $\tilde{\mathbf{a}}$ is the socially optimal profile of game $G(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$. Since U_l is strictly concave and $U_l'(\mathbf{\Omega}, \tilde{\mathbf{a}}) \geq 0$, $a_l = \tilde{a}_l$ is player l's best reply. By Proposition 3, we obtain the result. □

4.2 Optimal taxes

We now investigate how taxes can be used to restore optimality. We distinguish between taxes on (marginal) benefits and taxes on costs. First, we introduce the optimal tax plan on benefits. The utility of a player is then given by

$$U_l(\mathbf{\Omega}, \mathbf{a}) = p_l(1-\tau_l)\, a_l - q_l\left(a_l + \sum_{k:k\neq l}\omega_{kl}a_k\right).$$

where τ_l is agent l's tax rate on benefits. The following result shows that the optimal tax rate imposed to a player is negatively related with his outgoing weighted Bonacich centrality measure applied to the vector of marginal benefits. Therefore, the optimal tax plan on benefits to achieve the social optimum involves generally different tax rates at each site throughout the network.

Proposition 6. *Let $G(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game, and $(\mathbf{I}+\mathbf{\Omega})^{-1}$ exists. Assume the social optimum is interior. Then, the optimal tax rate on benefits is*

$$\forall l,\ \ \tilde{\tau}_l = \frac{p_l - \left(\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega}, \mathbf{p})\right)_l}{p_l}.$$

Proof. Since $\tilde{a}_l > 0$ for all l, at social optimum we have the following first order conditions:

$$\forall l,\ \ p_l - q_l'\left(\tilde{a}_l + \sum_{k:k\neq l}\omega_{kl}\tilde{a}_k\right) - \sum_{j:j\neq l}\omega_{lj}q_j'\left(\tilde{a}_j + \sum_{i:i\neq j}\omega_{ij}\tilde{a}_i\right) = 0.$$

Let $\tau_l = \frac{1}{p_l}\sum_{j:j\neq l}\omega_{lj}q_j'\left(\tilde{a}_j + \sum_{i:i\neq j}\omega_{ij}\tilde{a}_i\right)$ for all l. The first order conditions may be written:

$$\forall l,\ \ p_l(1-\tau_l) = q_l'\left(\tilde{a}_l + \sum_{k:k\neq l}\omega_{kl}\tilde{a}_k\right).$$

Then, the socially optimal profile $\tilde{\mathbf{a}}$ is also a Nash equilibrium of a game where, for all l,

$$U_l(\mathbf{\Omega}, \mathbf{a}) = p_l(1-\tau_l)\, a_l - q_l\left(a_l + \sum_{k:k\neq l}\omega_{kl}a_k\right).$$

Let $e_j = q_j'\left(\tilde{a}_j + \sum_{i:i\neq j}\omega_{ij}\tilde{a}_i\right)$ for all j. Then,

$$\forall l,\ \ \tau_l = \frac{1}{p_l}\sum_{j:j\neq l}\omega_{lj}e_j = \frac{1}{p_l}\,\mathbf{\Omega}_{l.}\mathbf{e}.$$

Since $\mathbf{e} = (\mathbf{I} + \mathbf{\Omega})^{-1} \mathbf{p}$ (see the proof of Proposition 3), it follows that

$$\forall l, \;\; \tau_l = \frac{1}{p_l} \mathbf{\Omega}_{l.} (\mathbf{I} + \mathbf{\Omega})^{-1} \mathbf{p} = \frac{1}{p_l} \left(\mathbf{\Omega} [\mathbf{I} + \mathbf{\Omega}]^{-1} \mathbf{p} \right)_l .$$

Finally, we note that

$$\mathbf{\Omega} [\mathbf{I} + \mathbf{\Omega}]^{-1} = \mathbf{I} - [\mathbf{I} + \mathbf{\Omega}]^{-1}$$

so

$$\tau_l = \frac{p_l - \left(\mathbf{b}^{+}_{\text{alt}} (\mathbf{\Omega}, \mathbf{p}) \right)_l}{p_l}$$

□

Following Proposition 6, the higher the outgoing weighted Bonacich centrality of a player, the lower is his optimal tax rate on benefits. Then, the optimal tax rate imposed to a player is positively related with the weight of odd length directed paths and negatively related with the weight of even length directed paths that begin at him, where directed paths that end at j in the network are weighted by p_j. In other words, the optimal tax plan reflects both the marginal damages and the marginal benefits a player produces on other players at the socially optimal profile.[10]

Next, we introduce the optimal tax plan on costs. The utility of a player is then given by

$$U_l (\mathbf{\Omega}, \mathbf{a}) = p_l a_l - (1 + t_l) q_l \left(a_l + \sum_{k: k \neq l} \omega_{kl} a_k \right).$$

where t_l is agent l's tax rate on costs. The following result shows that there is a direct relationship between the optimal tax plan on benefits and the optimal tax plan on costs.

Proposition 7. *Let $G (\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game, and $(\mathbf{I} + \mathbf{\Omega})^{-1}$ exists. Assume the social optimum is interior. Then, the optimal tax rate on costs is*

$$\forall l, \;\; \tilde{t}_l = \frac{p_l - \left(\mathbf{b}^{+}_{\text{alt}} (\mathbf{\Omega}, \mathbf{p}) \right)_l}{\left(\mathbf{b}^{+}_{\text{alt}} (\mathbf{\Omega}, \mathbf{p}) \right)_l}.$$

[10] Remind that the actions of players who have a directed path of even (resp. odd) length between them are strategic complements (resp. substitutes). Therefore, the higher the total weight of even length directed paths that begin at a player, the higher marginal benefits this player brings to other players in the network, and the lower is his optimal tax rate. Moreover, the higher the total weight of odd length directed paths that begin at a player, the higher marginal damages this player inflicts to other players in the network, and the higher is his optimal tax rate.

Proof. Let us deduce the optimal tax level from Proposition 6. First note that a Nash equilibrium is an ordinal property, so we may consider instead the utility functions

$$V_l(\mathbf{\Omega}, \mathbf{a}) = \frac{1}{1+t_l} U_l(\mathbf{\Omega}, \mathbf{a}) = \frac{1}{1+t_l} p_l a_l - q_l \left(a_l + \sum_{k:k \neq l} \omega_{kl} a_k \right).$$

So in order to obtain $\tilde{\mathbf{a}}$ as a Nash equilibrium we may choose $\mathbf{t}$ such that

$$\forall l, \ \ \frac{1}{1+t_l} = 1 - \tau_l$$

so according to Proposition 6, $\tilde{\mathbf{a}}$ can be decentralized. The optimal tax rate on costs is therefore

$$\forall l, \ \ \tilde{t}_l = \frac{\tilde{\tau}_l}{1-\tilde{\tau}_l} = \frac{p_l - \left(\mathbf{b}^+_{\text{alt}}(\mathbf{\Omega}, \mathbf{p})\right)_l}{\left(\mathbf{b}^+_{\text{alt}}(\mathbf{\Omega}, \mathbf{p})\right)_l}$$

□

4.3 Policy revenues comparison

The implementation of a policy based on optimal taxes will encourage players to choose the socially optimal profile and will generate a tax revenue for the social planner. Let

$$T_p = \sum_l p_l \tilde{\tau}_l \tilde{a}_l$$

and

$$T_q = \sum_l \tilde{t}_l q_l \left(\tilde{a}_l + \sum_{k:k \neq l} \omega_{kl} \tilde{a}_k \right) = \sum_l \frac{\tilde{\tau}_l}{1-\tilde{\tau}_l} q_l \left(\tilde{a}_l + \sum_{k:k \neq l} \omega_{kl} \tilde{a}_k \right)$$

denote, respectively, the tax revenue generated by a tax plan on benefits and by a tax plan on costs. Then,

$$T_p - T_q = \sum_l \tilde{\tau}_l p_l \tilde{a}_l \left[1 - \frac{q_l \left(\tilde{a}_l + \sum_{k:k \neq l} \omega_{kl} \tilde{a}_k \right)}{\tilde{a}_l p_l (1-\tilde{\tau}_l)} \right].$$

According to the first order conditions of the decentralized Nash equilibrium in Proposition 6 we have:

$$\forall l, \ \ p_l (1-\tilde{\tau}_l) = q_l' \left(\tilde{a}_l + \sum_{k:k \neq l} \omega_{kl} \tilde{a}_k \right),$$

so

$$T_p - T_q = \sum_l \tilde{\tau}_l p_l \tilde{a}_l \left[1 - \frac{q_l\left(\tilde{a}_l + \sum_{k:k\neq l} \omega_{kl}\tilde{a}_k\right)}{\tilde{a}_l q_l'\left(\tilde{a}_l + \sum_{k:k\neq l} \omega_{kl}\tilde{a}_k\right)} \right]$$

$$= \sum_l \tilde{\tau}_l p_l \tilde{a}_l \left[1 - \frac{1}{\varepsilon_l} \right]$$

where

$$\varepsilon_l = \frac{\tilde{a}_l q_l'\left(\tilde{a}_l + \sum_{k:k\neq l} \omega_{kl}\tilde{a}_k\right)}{q_l\left(\tilde{a}_l + \sum_{k:k\neq l} \omega_{kl}\tilde{a}_k\right)}$$

denotes the cost elasticity of action. Since q_l is strictly convex and positive for all l, the cost of action is elastic, i.e., $\varepsilon_l > 1$. Then, $T_p - T_q > 0$ and therefore, the highest tax revenue is always generated by a tax plan on benefits. That is,

$$T_p > T_q \geq 0 = T_k,$$

since an optimal quota policy will generate no revenue for the social planner, i.e., $T_k = 0$.

5 Optimal taxes and network density

We establish structural bounds on the optimal tax rates. It appears that the well-known inequalities relating the spectral radius and the maximal and minimal row sum of a matrix is appropriate to *localize* the optimal tax rates.[11]

Proposition 8. *Let* $G(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ *be a network game, and* $(\mathbf{I} + \mathbf{\Omega})^{-1}$ *exists. Assume the social optimum is interior. Then, the optimal tax rates satisfy*

$$s(\mathbf{\Omega}) \leq \tilde{t}_l \leq S(\mathbf{\Omega})$$

where $s(\mathbf{\Omega}) = \min_k \sum_l \omega_{kl}, \; S(\mathbf{\Omega}) = \max_k \sum_l \omega_{kl}$.

Proof. We shall proove the first inequality. The second can be obtained similarly. Let us write $\forall l, \; b_l = \left(\mathbf{b}^+_{\text{alt}}(\mathbf{\Omega}, \mathbf{p})\right)_l$ as a shorthand. We have,

$$\min_l \frac{b_l}{p_l} \, \mathbf{1} \leq \frac{\mathbf{b}}{\mathbf{p}} \leq \frac{1}{\min_l p_l} \, \mathbf{b}$$

[11] See Theorem 2.2.35 p.37 in Berman and Plemmons (1994), that states $\rho(\mathbf{\Omega}) \in [s(\mathbf{\Omega}), S(\mathbf{\Omega})]$, where $s(\mathbf{\Omega}) = \min_k \sum_l \omega_{kl}$ and $S(\mathbf{\Omega}) = \max_k \sum_l \omega_{kl}$.

since $\mathbf{I}+\mathbf{\Omega} > \mathbf{0}$, by composition, it comes

$$\min_l \tfrac{b_l}{p_l} (\mathbf{I}+\mathbf{\Omega})\, \mathbf{1} \quad \leq (\mathbf{I}+\mathbf{\Omega}) \tfrac{\mathbf{b}}{\mathbf{p}} \quad \leq \tfrac{1}{\min_l p_l} (\mathbf{I}+\mathbf{\Omega})\, \mathbf{b}$$

$$= \tfrac{1}{\min_l p_l} \; \mathbf{p}.$$

Taking the minimum componentwise over l, we have

$$\min_l \frac{b_l}{p_l} \; [1+s(\mathbf{\Omega})] \leq 1$$

thus

$$\min_l \frac{b_l}{p_l} \leq [1+s(\mathbf{\Omega})]^{-1} .$$

Now since $(x \mapsto \frac{1-x}{x})$ is decreasing on $(0, 1]$ we have for all l,

$$\tilde{t}_l = \frac{1-b_l/p_l}{b_l/p_l} \geq \frac{1-[1+s(\mathbf{\Omega})]^{-1}}{[1+s(\mathbf{\Omega})]^{-1}} = s(\mathbf{\Omega})$$

and the first inequality is obtained.

Finally, the second can be obtained starting from

$$\max_l \frac{b_l}{p_l} \; \mathbf{1} \geq \frac{\mathbf{b}}{\mathbf{p}} \geq \frac{1}{\max_l p_l} \; \mathbf{b}.$$

By composition by $\mathbf{I}+\mathbf{\Omega}$, it comes

$$\max_l \frac{b_l}{p_l} (\mathbf{I}+\mathbf{\Omega})\, \mathbf{1} \geq \frac{1}{\max_l p_l} \; \mathbf{p}.$$

Then, taking the maximum componentwise, we have

$$\max_l \frac{b_l}{p_l} [1+S(\mathbf{\Omega})] \geq 1 \iff \max_l \frac{b_l}{p_l} \geq [1+S(\mathbf{\Omega})]^{-1}$$

so for all l,

$$\tilde{t}_l \leq \frac{1-[1+S(\mathbf{\Omega})]^{-1}}{[1+S(\mathbf{\Omega})]^{-1}} = S(\mathbf{\Omega}).$$

□

Proposition 8 provides bounds on the optimal tax rates in terms of the *weighted out-degree* measure.[12] It entails that the optimal tax rate imposed to

[12]See Freeman (1978) for a review and clarification of research on centrality measures for unweighted networks. See Barrat et al. (2004) for the generalization of degree centrality to weighted networks.

a player would never be less than the minimal weighted out-degree and never be higher than the maximal weighted out-degree of the network. Moreover, these tax rates can be located around the spectral radius of the adjacency matrix. Thus, they can not be all located below or all located above the spectral radius. In fact, the tax rates are *centered* on the spectral radius.

Proposition 9. *Let $G(\boldsymbol{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game, and $(\mathbf{I}+\boldsymbol{\Omega})^{-1}$ exists. Assume the social optimum is interior. Then, the optimal tax rates satisfy the following inequalities*

$$\tilde{t}_{\min} \leq \rho(\boldsymbol{\Omega}) \leq \tilde{t}_{\max}$$

where $\tilde{t}_{\min} = \min_l\{\tilde{t}_l\}$ *and* $\tilde{t}_{\max} = \max_l\{\tilde{t}_l\}$.

Proof. From Proposition 7, we have for all l that

$$\tilde{t}_{\min} \leq \frac{p_l - \left(\mathbf{b}^+_{\text{alt}}(\boldsymbol{\Omega}, \mathbf{p})\right)_l}{\left(\mathbf{b}^+_{\text{alt}}(\boldsymbol{\Omega}, \mathbf{p})\right)_l} \leq \tilde{t}_{\max}$$

or equivalently

$$\left(1+\tilde{t}_{\min}\right)\left(\mathbf{b}^+_{\text{alt}}(\boldsymbol{\Omega}, \mathbf{p})\right)_l \leq p_l \leq \left(1+\tilde{t}_{\max}\right)\left(\mathbf{b}^+_{\text{alt}}(\boldsymbol{\Omega}, \mathbf{p})\right)_l .$$

Since $\mathbf{I}+\boldsymbol{\Omega}$ is nonnegative,

$$\left(1+\tilde{t}_{\min}\right)\mathbf{p} \leq (\mathbf{I}+\boldsymbol{\Omega})\,\mathbf{p} \leq \left(1+\tilde{t}_{\max}\right)\mathbf{p}$$

so

$$\tilde{t}_{\min}\mathbf{p} \leq \boldsymbol{\Omega}\mathbf{p} \leq \tilde{t}_{\max}\mathbf{p}.$$

Then, by Theorem 2.1.11 p.28 in Berman and Plemmons (1994), since $\mathbf{p} >> \mathbf{0}$ we have

$$\tilde{t}_{\min} \leq \rho(\boldsymbol{\Omega}) \leq \tilde{t}_{\max}.$$

□

Since $\rho(\boldsymbol{\Omega})$ measures the density of the network, Proposition 9 entails that the denser the network, the higher might be the maximal optimal tax rate. Conversely, the tighter the network, the lower might be the minimal optimal tax rate. Moreover, if $\rho(\boldsymbol{\Omega}) = 0$, there exists l such that $\tilde{t}_l = 0$. If $\rho(\boldsymbol{\Omega}) > 0$, there exists l such that $\tilde{t}_l > 0$. Hence, whenever the optimal tax rate is uniform, it must coincide with the spectral radius $\rho(\boldsymbol{\Omega})$. The next proposition elaborates a necessary and sufficient condition for uniform optimal tax rates.

Proposition 10. *Let $G(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game, and $(\mathbf{I}+\mathbf{\Omega})^{-1}$ exists. Assume the social optimum is interior. If the optimal tax rate is uniform, then* $\mathbf{p}$ *is an eigenvector of* $\mathbf{\Omega}$*.*

Conversely, if $\mathbf{p}$ *is an eigenvector of* $\mathbf{\Omega}$ *and* $t_{\mathbf{p}}$ *its associated eigenvalue, then* $t_{\mathbf{p}} \geq 0$ *and* $t_{\mathbf{p}}$ *is the uniform optimal tax rate.*

Moreover, the optimal uniform tax rate is given by $\tilde{t} = t_{\mathbf{p}} = \rho(\mathbf{\Omega})$*.*

Proof. Assume the optimal tax rate is uniform, i.e., $\forall l,\ \tilde{t}_l = \tilde{t}$. Then,

$$\begin{aligned} \tilde{t} &= \frac{p_l - \left(\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega},\mathbf{p})\right)_l}{\left(\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega},\mathbf{p})\right)_l},\ \forall l \\ \iff \left(1+\tilde{t}\right)\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega},\mathbf{p}) &= \mathbf{p} \\ \iff \left(1+\tilde{t}\right)\mathbf{p} &= (\mathbf{I}+\mathbf{\Omega})\,\mathbf{p} \\ \iff \tilde{t}\mathbf{p} &= \mathbf{\Omega}\mathbf{p} \end{aligned}$$

(Conversely). If $\mathbf{\Omega} = \mathbf{0}$, then 0, the unique eigenvalue, is clearly an optimal uniform tax rate. If $\mathbf{\Omega} \neq \mathbf{0}$, assume $\mathbf{p}$ is an eigenvector of $\mathbf{\Omega}$ and let $t_{\mathbf{p}} \in \mathbb{C}$ be its associated eigenvalue. Since $t_{\mathbf{p}}\mathbf{p} = \mathbf{\Omega}\mathbf{p}$, $\mathbf{\Omega} \geq \mathbf{0}$ and $\mathbf{p} >> 0$, for some l, $t_{\mathbf{p}} p_l > 0$, thus $t_{\mathbf{p}} > 0$. Following the same lines than in the previous statement, we obtain

$$\forall l,\ t_{\mathbf{p}} = \frac{p_l - \left(\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega},\mathbf{p})\right)_l}{\left(\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega},\mathbf{p})\right)_l} = \tilde{t}_l$$

which is the optimal tax rate.

(Moreover). If $\mathbf{p}$ is an eigenvector of $\mathbf{\Omega}$, then by Corollary 2.1.12 p.28 in Berman and Plemmons (1994), $\mathbf{p}$ corresponds to $\rho(\mathbf{\Omega})$ since $\mathbf{p} >> \mathbf{0}$. So, $t_{\mathbf{p}} = \rho(\mathbf{\Omega})$. □

If a game with substitutabilities admits an optimal tax rate which is uniform, then the denser (resp. the tighter) the network, the higher (resp. the lower) is the optimal tax rate. But, generically, $\mathbf{p}$ is not an eigenvector of $\mathbf{\Omega}$. Then, Proposition 10 highlights the low probability for the optimal tax rate to be uniform.

Remark 4 (Sub-stochastic matrices). Assume $\mathbf{\Omega}$ has constant rowsums (i.e., all agents have the same weighted out-degree) smaller than 1, that is $s(\mathbf{\Omega}) = S(\mathbf{\Omega}) = \rho(\mathbf{\Omega}) < 1$ and that there is a *market price* $p > 0$, that is $\mathbf{p} = p\mathbf{1}$. Let $\tilde{\mathbf{a}}$ be the interior social optimum. Then, since $\mathbf{1}$ is an eigenvector corresponding to $\rho(\mathbf{\Omega})$, the optimal tax is uniform and equals $\rho(\mathbf{\Omega})$ and

$$\tilde{\mathbf{a}} = \mathbf{b}^{-}_{\text{alt}}\left(\mathbf{\Omega}, (\mathbf{q}')^{-1}\left(\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega},\mathbf{p})\right)\right) = \mathbf{b}^{-}_{\text{alt}}\left(\mathbf{\Omega}, (\mathbf{q}')^{-1}\left(\frac{p}{1+\rho(\mathbf{\Omega})}\,\mathbf{1}\right)\right).$$

If, $q_l(a_l) = \frac{c}{2} a_l^2$ for all l with $c > 0$, then,

$$\tilde{\mathbf{a}} = \frac{p}{c\left[1 + \rho(\mathbf{\Omega})\right]} \; \mathbf{b}^{-}_{\text{alt}}\left(\mathbf{\Omega}, \mathbf{1}\right).$$

Moreover, when $\mathbf{\Omega}$ is doubly sub-stochastic, i.e., $\mathbf{\Omega}$ has constant rowsums and columnsums, that is $s(\mathbf{\Omega}) = S(\mathbf{\Omega}) = s(\mathbf{\Omega}^{\mathsf{T}}) = S(\mathbf{\Omega}^{\mathsf{T}}) = \rho(\mathbf{\Omega})$ (e.g., $\mathbf{\Omega}$ is symmetric) then,

$$\tilde{\mathbf{a}} = \frac{p}{c\left[1 + \rho(\mathbf{\Omega})\right]^2} \; \mathbf{1}.$$

Hence, individual action is increasing in p, decreasing in c and also with respect to the density of the network through $\rho(\mathbf{\Omega})$. One may note also that all individuals have the same optimal action, no matter their specific locations in the network.

6 Conclusion

This paper brings a social welfare analysis to games with substitutabilities. It is worth noting that this work presents similarities with the literature on the river sharing problem initiated by Ambec and Sprumont (2002).

In particular, Ni and Wang (2007) have explored the implications of the upstream-downstream relationship between the agents on cost sharing methods when the river carries pollutants, and they show how the structure of a river network can shape cooperative behaviors within a group. Although we have not used the metaphor of polluted water flows to describe the network[13], our work contributes to this literature by showing the implications of the predecessor(s)-successor(s) relationship between the agents on policy instruments, such as taxes and quotas, to achieve the social optimum when players do not cooperate.

Then, a useful direction for further research would be to investigate how to restore optimality in network games of strategic substitutes when players cooperate. In this regard, the definition of rights owned by the agents is crucial, but the optimal policies designed in this paper raise issues as to the how property rights might be defined when players are connected through a directed network. A further issue for investigation is how to redistribute the revenue generated by taxes. Finally, it would also be pertinent to test the robustness of our results to more general specifications of preferences. The case of additive utility functions could be a reasonable first step towards this goal.

[13] Note, however, that river networks are weighted directed acyclic networks, and are only one particular case of the networks we consider in this paper.

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