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On the product formula on non-compact Grassmannians

P. Graczyk and P. Sawyer *

Abstract

We study the absolute continuity of the convolution $\delta_{e^X}^{\natural} \star \delta_{e^Y}^{\natural}$ of two orbital measures on the symmetric space $\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$, $q > p$. We prove sharp conditions on $X, Y \in \mathfrak{a}$ for the existence of the density of the convolution measure. This measure intervenes in the product formula for the spherical functions. We show that the sharp criterion developed for $\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$ will also serve for the spaces $\mathbf{SU}(p, q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$ and $\mathbf{Sp}(p, q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q)$, $q > p$. We also apply our results to the study of absolute continuity of convolution powers of an orbital measure $\delta_{e^X}^{\natural}$.

1 Introduction

The spaces $\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$ where $q > p$ (which we will assume throughout the paper), are the noncompact duals of real Grassmannians. They are Riemannian symmetric spaces of noncompact type corresponding to root systems of type B_p . The harmonic analysis on these spaces has been intensely developed in recent years ([1, 13, 14, 15]).

We use throughout the paper the usual notations of the harmonic analysis on Riemannian symmetric spaces. The books [9, 10] constitute a standard reference on these topics.

Let $X, Y \in \mathfrak{a}$ and let m_K denote the Haar measure of the group K . We define $\delta_{e^X}^{\natural} = m_K \star \delta_{e^X} \star m_K$. The question of the absolute continuity of the convolution $\delta_{e^X}^{\natural} \star \delta_{e^Y}^{\natural}$ of two K -invariant orbital measures that we address in our paper has important applications in harmonic analysis itself (the product formula for the spherical functions) and in probability theory (random walks, I_0 characterization of Gaussian measures).

The spherical Fourier transform of the measure $\delta_{e^X}^{\natural}$ is equal to the spherical function $\phi_{\lambda}(e^X)$, where λ is a complex-valued linear form on \mathfrak{a} . Thus the product $\phi_{\lambda}(e^X)\phi_{\lambda}(e^Y)$ is the spherical Fourier transform of the convolution $m_{X,Y} = \delta_{e^X}^{\natural} \star \delta_{e^Y}^{\natural}$. If we denote by $\mu_{X,Y}$ the projection of the measure $m_{X,Y}$ on \mathfrak{a} via the Cartan decomposition $G = KAK$, then

$$\phi_{\lambda}(e^X)\phi_{\lambda}(e^Y) = \int_{\mathfrak{a}} \phi_{\lambda}(e^H) d\mu_{X,Y}(H).$$

Let δ be the density of the invariant measure on \mathfrak{a} in polar coordinates. The existence of a kernel in the last product formula

$$\phi_{\lambda}(e^X)\phi_{\lambda}(e^Y) = \int_{\mathfrak{a}^+} \phi_{\lambda}(e^H) k(H, X, Y) \delta(H) dH \quad (1)$$

is equivalent to the absolute continuity of the measure $\mu_{X,Y}$ with respect to the Lebesgue measure on \mathfrak{a} and to the existence of the density of $m_{X,Y}$ on G , with respect to the invariant measure dg . When the formula (1) holds, we say that we have a product formula for X and $Y \in \mathfrak{a}$. Provided that $X, Y \in \mathfrak{a}^+$, the product formula (1) has been shown previously (see [2] in the rank one case,

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[3] in the complex case and [4] in the general case). In [4] we were able to relax these conditions and show that $\mu_{X,Y}$ is absolutely continuous provided one of X or Y is in \mathfrak{a}^+ as long as the other is nonzero. The density can however exist in some cases when both X and Y are singular. It is a challenging problem to characterize all such pairs X and Y .

This problem was solved in [7] for symmetric spaces with root system of type A_n . We solve it in this paper for the space $\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$: we give a definition of an eligible pair (X, Y) (Definition 3) and next we prove the necessity (Proposition 5) and the sufficiency (Proposition 5 and Theorem 13) of this property for the absolute continuity of $m_{X,Y}$.

=====>By [3, 4], the density $k(H, X, Y)$ exists if and only if $\mathcal{S}_{X,Y} = a(e^X K e^Y)$, the support of the measure $\mu_{X,Y}|_{\mathfrak{a}^+}$, has nonempty interior. Similarly, the density of the measure $m_{X,Y}$ exists if and only if its support $K e^X K e^Y K$ has nonempty interior as seen in [7]. These facts are crucial in the proofs of the results of this paper.

We show in Corollary 15 that the result for the space $\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$ also implies the result for the spaces $\mathbf{SU}(p, q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$ and $\mathbf{Sp}(p, q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q)$. We conclude the paper with two further applications of our main result. One of them is a characterization of an optimal convolution power l of the measure $\delta_{e^X}^\natural$, which is absolutely continuous for any $X \neq 0$, $X \in \mathfrak{a}$. Theorem 17 solves on non-compact Grassmannians a problem raised by Ragozin in [12].
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2 Basic properties

We start by reviewing some useful information on the Lie group $\mathbf{SO}_0(p, q)$, its Lie algebra $\mathfrak{so}(p, q)$ and the corresponding root system. Most of this material was given in [15]. For the convenience of the reader, we gather below the properties we will need in the sequel.

In this paper, E_{ij} is a rectangular matrix with 0's everywhere except at the position (i, j) where it is 1.

Recall that $\mathbf{SO}(p, q)$ is the group of matrices $g \in \mathbf{SL}(p+q, \mathbf{R})$ such that $g^T I_{p,q} g = I_{p,q}$ where $I_{p,q} = \begin{bmatrix} -I_p & 0_{p \times q} \\ 0_{q \times p} & I_q \end{bmatrix}$. Unless otherwise specified, all 2×2 block decompositions in this paper follow the same pattern.

The group $\mathbf{SO}_0(p, q)$ is the connected component of $\mathbf{SO}(p, q)$ containing the identity. The Lie algebra $\mathfrak{so}(p, q)$ of $\mathbf{SO}_0(p, q)$ consists of the matrices

$$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

where A and D are skew-symmetric.

A very important element in our investigations is the Cartan decomposition of $\mathfrak{so}(p, q)$ and $\mathbf{SO}(p, q)$. The maximal compact subgroup K is the subgroup of $\mathbf{SO}(p, q)$ consisting of the matrices

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

of size $(p+q) \times (p+q)$ such that $A \in \mathbf{SO}(p)$ and $D \in \mathbf{SO}(q)$ (hence $K \simeq \mathbf{SO}(p) \times \mathbf{SO}(q)$). If \mathfrak{k}

is the Lie algebra of K and \mathfrak{p} is the set of matrices

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

then the Cartan decomposition is given by $\mathfrak{so}(p, q) = \mathfrak{k} \oplus \mathfrak{p}$ with corresponding Cartan involution $\theta(X) = -X^T$.

The Cartan space $\mathfrak{a} \subset \mathfrak{p}$ is the set of matrices

$$H = \begin{bmatrix} 0_{p \times p} & \mathcal{D}_H & 0_{p \times (q-p)} \\ \mathcal{D}_H & 0_{p \times p} & 0_{p \times (q-p)} \\ 0_{(q-p) \times p} & 0_{(q-p) \times p} & 0_{(q-p) \times (q-p)} \end{bmatrix}$$

where $\mathcal{D}_H = \text{diag}[H_1, \dots, H_p]$. Its canonical basis is given by the matrices

$$A_i := E_{i, p+i} + E_{p+i, i}, \quad 1 \leq i \leq p.$$

The restricted roots and associated root vectors for the Lie algebra $\mathfrak{so}(p, q)$ with respect to \mathfrak{a} are given in Table 1.

root α	multiplicity	root vectors X_α
$\alpha(H) = \pm H_i$ $1 \leq i \leq p$	$q - p$	$X_{ir}^\pm = E_{i, 2p+r} + E_{2p+r, i} \pm (E_{p+i, 2p+r} - E_{2p+r, p+i})$ $r = 1, \dots, q - p$
$\alpha(H) = \pm(H_i - H_j)$ $1 \leq i, j \leq p, i < j$	1	$Y_{ij}^\pm = \pm(E_{ij} - E_{ji} + E_{p+i, p+j} - E_{p+j, p+i}) + E_{i, p+j} + E_{p+j, i}$ $+ E_{j, p+i} + E_{p+i, j}$
$\alpha(H) = \pm(H_i + H_j)$ $1 \leq i, j \leq p, i < j$	1	$Z_{ij}^\pm = \pm(E_{ij} - E_{ji} - E_{p+i, p+j} + E_{p+j, p+i}) - (E_{i, p+j} + E_{p+j, i})$ $+ E_{j, p+i} + E_{p+i, j}$

Table 1: Restricted roots and associated root vectors

The positive roots can be chosen as $\alpha(H) = H_i \pm H_j$, $1 \leq i < j \leq p$ and $\alpha(H) = H_i$, $i = 1, \dots, p$. We therefore have the positive Weyl chamber

$$\mathfrak{a}^+ = \{H \in \mathfrak{a}: H_1 > H_2 > \dots > H_p > 0\}.$$

The simple roots are given by $\alpha_i(H) = H_i - H_{i+1}$, $i = 1, \dots, p - 1$ and $\alpha_p(H) = H_p$.

The action of the Weyl group. The elements of the Weyl group W act as permutations of the diagonal entries of \mathcal{D}_X with eventual sign changes of any number of these entries.

The Lie algebra \mathfrak{k} is generated by the vectors $X_\alpha + \theta X_\alpha$. We will use the notation

$$k_{X_\alpha}^t = e^{t(X_\alpha + \theta X_\alpha)}.$$

The linear space \mathfrak{p} has a basis formed by $A_i \in \mathfrak{a}$, $1 \leq i \leq p$ and by the symmetric matrices $X_\alpha^s := \frac{1}{2}(X_\alpha - \theta X_\alpha)$ which have the following form

$$\begin{aligned} X_{ir} &:= E_{i, 2p+r} + E_{2p+r, i}, & 1 \leq i \leq p, & \quad 1 \leq r \leq q - p; \\ Y_{ij} &:= E_{i, p+j} + E_{j, p+i} + E_{p+j, i} + E_{p+i, j}, & 1 \leq i < j \leq p; \\ Z_{ij} &:= E_{i, p+j} - E_{j, p+i} + E_{p+j, i} - E_{p+i, j}, & 1 \leq i < j \leq p. \end{aligned}$$

If we followed the notation of [7], we should write $(X_{ir}^+)^s$, etc. but we simplify the notation to X_{ir} , Y_{ij} and Z_{ij} . If we write a matrix from the space \mathfrak{p} in the form

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & B_1 & B_2 \\ B_1^T & 0 & 0 \\ B_2^T & 0 & 0 \end{bmatrix}$$

where B_1 is a square $p \times p$ matrix and B_2 is a $p \times (q - p)$ matrix, then the matrices

$$\begin{bmatrix} 0 & B_1 & 0 \\ B_1^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are generated by the vectors A_i (for the diagonal entries of B_1 and B_1^T), Y_{ij} and Z_{ij} (for the non-diagonal entries), whereas the matrices

$$\begin{bmatrix} 0 & 0 & B_2 \\ 0 & 0 & 0 \\ B_2^T & 0 & 0 \end{bmatrix}$$

are spanned by the vectors X_{ir} .

We now recall the useful matrix $S \in \mathbf{SO}(p + q)$ which allows us to diagonalize simultaneously all the elements of \mathfrak{a} . Let

$$S = \begin{bmatrix} \frac{\sqrt{2}}{2} I_p & 0_{p \times (q-p)} & \frac{\sqrt{2}}{2} J_p \\ \frac{\sqrt{2}}{2} I_p & 0_{p \times (q-p)} & -\frac{\sqrt{2}}{2} J_p \\ 0_{(q-p) \times p} & I_{q-p} & 0_{(q-p) \times p} \end{bmatrix}$$

where $J_p = (\delta_{i,p+1-i})$ is a matrix of size $p \times p$. If $H = \begin{bmatrix} 0 & \mathcal{D}_H & 0 \\ \mathcal{D}_H & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with $\mathcal{D}_H = \text{diag}[H_1, \dots, H_p]$

then $S^T H S = \text{diag}[H_1, \dots, H_p, \overbrace{0, \dots, 0}^{q-p}, -H_p, \dots, -H_1]$.

The ‘‘group’’ version of this result is as follows:

$$S^T e^H S = \text{diag}[e^{H_1}, \dots, e^{H_p}, \overbrace{1, \dots, 1}^{q-p}, e^{-H_p}, \dots, e^{-H_1}].$$

Remark 1 *The Cartan projection $a(g)$ on the group $\mathbf{SO}_0(p, q)$, defined as usual by*

$$g = k_1 e^{a(g)} k_2, \quad a(g) \in \overline{\mathfrak{a}^+}, \quad k_1, k_2 \in K$$

is related to the singular values of $g \in \mathbf{SO}(p, q)$ in the following way. Recall that the singular values of g are defined as the non-negative square roots of the eigenvalues of $g^T g$. Let us write $H = a(g)$. We have

$$g^T g = k_2^T e^{2H} k_2 = (k_2^T S) (S^T e^{2H} S) (S^T k_2)$$

where $S^T e^{2H} S$ is a diagonal matrix with nonzero entries $e^{2H_1}, \dots, e^{2H_p}, \overbrace{1, \dots, 1}^{q-p}, e^{-2H_p}, \dots, e^{-2H_1}$. Let us write $a_j = e^{H_j}$. Thus the set of $p+q$ singular values of g contains the value 1 repeated $q-p$ times and the $2p$ values $a_1, \dots, a_p, a_1^{-1}, \dots, a_p^{-1}$.

Hence, in order to determine $a(g)$, we can compute the $p+q$ singular values of $g^T g$ and omit $q-p$ values 1 always appearing among them. The $2p$ remaining singular values may be ordered $a_1 \geq \dots \geq a_p \geq a_p^{-1} \geq \dots \geq a_1^{-1}$ with $a_1 \geq \dots \geq a_p \geq 1$. Then

$$a(g) = \begin{bmatrix} 0 & \mathcal{D}_{a(g)} & 0 \\ \mathcal{D}_{a(g)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{with } \mathcal{D}_{a(g)} = \text{diag}[\log a_1, \dots, \log a_p]$$

Summarizing, if for $g \in \mathbf{SO}(p, q) \subset \mathbf{SL}(p+q)$ the $\mathbf{SL}(p+q)$ -Cartan decomposition writes $g = k_1 e^{\tilde{a}(g)} k_2$, $k_1, k_2 \in \mathbf{SO}(p+q)$, then $\mathcal{D}_{a(g)} = \pi_p(\tilde{a}(g))$, where π_p denotes the projection $\pi_p(\text{diag}[h_1, \dots, h_{p+q}]) = \text{diag}[h_1, \dots, h_p]$.

Singular elements of \mathfrak{a} . In what follows, we will consider singular elements $X, Y \in \partial\mathfrak{a}^+$. As in [7], we need to control the irregularity of X and Y , i.e. consider the simple positive roots annihilating X and Y . A special attention must be paid to the last simple root α_p , different from the roots α_i , $i = 1, \dots, p-1$, that generate a root subsystem of type A_{p-1} . We introduce the following definition of the configuration of $X \in \overline{\mathfrak{a}^+}$.

Definition 2 Let $X \in \overline{\mathfrak{a}^+}$. There exist nonnegative integers $s_1 \geq 1, \dots, s_r \geq 1, u \geq 0$ such that

$$\mathcal{D}_X = \text{diag}[\overbrace{x_1, \dots, x_1}^{s_1}, \overbrace{x_2, \dots, x_2}^{s_2}, \dots, \overbrace{x_r, \dots, x_r}^{s_r}, \overbrace{0, \dots, 0}^u]$$

with $x_1 > x_2 > \dots > x_r > 0$ and $\sum s_i + u = p$. We say that $[s_1, \dots, s_r; u]$ is the **configuration** of X . Writing $\mathbf{s} = (s_1, \dots, s_r)$, we will shorten the notation of the configuration of X to $[\mathbf{s}; u]$. We will also write $X = X[\mathbf{s}; u]$.

Note that $X = 0$ is equivalent to $u = p$ and has configuration $[0; p]$. A regular $X \in \mathfrak{a}^+$ has the configuration $[\mathbf{1}; 0] = [1^p; 0]$. We extend naturally the definition of configuration to any $X \in \mathfrak{a}$, whose configuration is defined as that of the projection $\pi(X)$ of X on $\overline{\mathfrak{a}^+}$.

In what follows, we will write $\max \mathbf{s} = \max_i s_i$ and $\max(\mathbf{s}, u) = \max(\max \mathbf{s}, u)$. We will show that in the case of the symmetric spaces $\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$, $q > p$, the criterion for the existence of the density of the convolution $\delta_{e^X}^{\mathfrak{h}} \star \delta_{e^Y}^{\mathfrak{h}}$ is given by the following definition of an eligible pair X and Y :

Definition 3 Let $X = X[\mathbf{s}; u]$ and $Y = Y[\mathbf{t}; v]$ be two elements of \mathfrak{a} . We say that X and Y are eligible if

$$\max(\mathbf{s}, 2u) + \max(\mathbf{t}, 2v) \leq 2p.$$

Observe that if X and Y are eligible, then $X \neq 0$ and $Y \neq 0$.

3 Necessity of the eligibility condition

In the proof of the necessity of the eligibility condition we will use the result stated in [8, Step 1, page 1767]:

Lemma 4 *Let $U = \text{diag}([\overbrace{u_0, \dots, u_0}^r, u_1, \dots, u_{N-r}])$ and $V = \text{diag}([\overbrace{v_0, \dots, v_0}^{N-s}, v_1, \dots, v_s])$. where $s+1 \leq r < N$, $s \geq 1$, and the u_i 's and v_j 's are arbitrary. Then each element of $\tilde{a}(e^U \mathbf{SU}(N, \mathbf{F}) e^V)$ has at least $r - s$ entries equal to $u_0 + v_0$.*

We will use Lemma 4 with $N = p + q$ in the proofs of Proposition 5 and Theorem 17.

Proposition 5 *If $X[\mathbf{s}; u]$ and $Y[\mathbf{t}, v]$ are not eligible then the measure $\mu_{X,Y}$ is not absolutely continuous with respect to the Lebesgue measure on \mathfrak{a} .*

Proof: Suppose $\max(\mathbf{s}, 2u) + \max(\mathbf{t}, 2v) > 2p$ and consider the matrices $a(e^X k e^Y)$, $k \in \mathbf{SO}(p) \times \mathbf{SO}(q)$. Applying Remark 1, the diagonal $p \times p$ matrix $\mathcal{D}_{a(e^X k e^Y)}$ contains the p biggest diagonal terms of the matrix

$$\tilde{a}(e^X k e^Y) = \tilde{a}(\underbrace{(S^T e^X S)}_{e^{s^T X s}} \underbrace{(S^T k S)}_{\in \mathbf{SO}(p+q)} \underbrace{(S^T e^Y S)}_{e^{s^T Y s}})$$

If $u + v > p$ then there are $r - s = r + (N - s) - N = (2u + q - p) + (2v + q - p) - (p + q) = 2(u + v - p) + (q - p)$ repetitions of $0 + 0 = 0$ in coefficients of $\tilde{a}(e^X k e^Y)$. Therefore 0 occurs at least $u + v - p > 0$ times as a diagonal entry of \mathcal{D}_H for every $H \in a(e^X K e^Y)$ which implies that $a(e^X K e^Y)$ has empty interior.

If $2u + \max(\mathbf{t}) > 2p$ denote $t = \max(\mathbf{t})$. Let $Y_i \neq 0$ be repeated t times in \mathcal{D}_Y . Then there are $r - s = r + (N - s) - N = (2u + q - p) + t - (p + q) = 2u + t - 2p$ repetitions of $Y_i + 0$ in coefficients of $\tilde{a}(e^X k e^Y)$. Therefore Y_i occurs at least $2u + t - 2p > 0$ times as a diagonal entry of \mathcal{D}_H for every $H \in a(e^X K e^Y)$ which implies that $a(e^X K e^Y)$ has empty interior. ■

4 Sufficiency of the eligibility condition

We use basic ideas and some results and notations of [7, Section 3].

Proposition 6 (i) *The density of the measure $m_{X,Y}$ exists if and only if its support $Ke^X Ke^Y K$ has nonempty interior.*

(ii) *Consider the analytic map $T: K \times K \times K \rightarrow \mathbf{SO}_0(p, q)$ defined by*

$$T(k_1, k_2, k_3) = k_1 e^X k_2 e^Y k_3.$$

If the derivative of T is surjective for some choice of $\mathbf{k} = (k_1, k_2, k_3)$, then the set $T(K \times K \times K) = Ke^X Ke^Y K$ contains an open set.

Proof: Part (i) follows from arguments explained in [4] in the case of the support of the measure $\mu_{X,Y}$, equal to $a(e^X K e^Y)$. Part (ii) is justified for example in [10, p. 479]. ■

Proposition 7 Let $U_Z = \mathfrak{k} + \text{Ad}(Z)\mathfrak{k}$. If there exists $k \in K$ such that

$$U_{-X} + \text{Ad}(k)U_Y = \mathfrak{g} \quad (2)$$

then the measure $m_{X,Y}$ is absolutely continuous.

Proof: We want to show that the derivative of T is surjective for some choice of $\mathbf{k} = (k_1, k_2, k_3)$.

Let $A, B, C \in \mathfrak{k}$. The derivative of T at \mathbf{k} in the direction of (A, B, C) equals

$$\begin{aligned} dT_{\mathbf{k}}(A, B, C) &= \left. \frac{d}{dt} \right|_{t=0} e^{tA} k_1 e^{tX} e^{tB} k_2 e^{tY} e^{tC} k_3 \\ &= A k_1 e^X k_2 e^Y k_3 + k_1 e^X B k_2 e^Y k_3 + k_1 e^X k_2 e^Y C k_3 \end{aligned} \quad (3)$$

We now transform the space of all matrices of the form (3) without modifying its dimension:

$$\begin{aligned} &\dim\{A k_1 e^X k_2 e^Y k_3 + k_1 e^X B k_2 e^Y k_3 + k_1 e^X k_2 e^Y C k_3 : A, B, C \in \mathfrak{k}\} \\ &= \dim\{k_1^{-1} A k_1 e^X k_2 e^Y + e^X B k_2 e^Y + e^X k_2 e^Y C : A, B, C \in \mathfrak{k}\} \\ &= \dim\{A e^X k_2 e^Y + e^X B k_2 e^Y + e^X k_2 e^Y C : A, B, C \in \mathfrak{k}\} \\ &= \dim\{e^{-X} A e^X + B + k_2 e^Y C e^{-Y} k_2^{-1} : A, B, C \in \mathfrak{k}\} \end{aligned}$$

The space in the last line equals $\mathfrak{k} + \text{Ad}(e^{-X})(\mathfrak{k}) + \text{Ad}(k_2)(\text{Ad}(e^Y)(\mathfrak{k})) = U_{-X} + \text{Ad}(k_2)U_Y$. ■

In order to apply the condition (2), we will consider convenient root vectors and their symmetrizations. For $Z \in \mathfrak{a}$, we define the space

$$V_Z = \text{span}\{X_\alpha^s \mid \alpha(Z) \neq 0\},$$

where $X_\alpha^s = X_\alpha - \theta X_\alpha$. Note that this space would be called V_Z^S in the notation of [7].

Lemma 8 Let $Z \in \mathfrak{a}$. The vector space $U_Z = \mathfrak{k} + \text{Ad}(e^Z)(\mathfrak{k})$ contains the root vectors X_α for which $\alpha(Z) \neq 0$. \implies Consequently, $V_Z = V_{-Z} \subset U_{\pm Z}$. \Leftarrow

Proof: \implies Suppose α is a root such that $\alpha(Z) \neq 0$. \Leftarrow Note that $[Z, X_\alpha] = \alpha(Z) X_\alpha$ and $[Z, \theta(X_\alpha)] = -\alpha(Z) \theta(X_\alpha)$. Let $U = X_\alpha + \theta(X_\alpha) \in \mathfrak{k}$. Now,

$$\begin{aligned} \text{Ad}(e^Z)U &= e^{\text{ad } Z}(X_\alpha + \theta(X_\alpha)) = \sum_{k=0}^{\infty} \frac{(\text{ad } Z)^k}{k!} (X_\alpha + \theta(X_\alpha)) \\ &= \sum_{k=0}^{\infty} \frac{(\text{ad } Z)^k}{k!} X_\alpha + \sum_{k=0}^{\infty} \frac{(\text{ad } Z)^k}{k!} \theta(X_\alpha) \\ &= \sum_{k=0}^{\infty} \frac{(\alpha(Z))^k}{k!} X_\alpha + \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha(Z))^k}{k!} \theta(X_\alpha) \\ &= e^{\alpha(Z)} X_\alpha + e^{-\alpha(Z)} \theta(X_\alpha). \end{aligned}$$

Therefore $X_\alpha = (e^{\alpha(Z)} - e^{-\alpha(Z)})^{-1} (-e^{-\alpha(Z)} U + \text{Ad}(e^Z)U) \in \mathfrak{k} + \text{Ad}(e^Z)(\mathfrak{k}) = U_Z$. The vector θX_α is a root vector for the root $-\alpha$, so we also have $\theta X_\alpha \in U_Z$. ■

\implies

Proposition 9 *If there exists $k \in K$ such that*

$$V_X + \text{Ad}(k) V_Y = \mathfrak{p} \quad (4)$$

then the measure $m_{X,Y}$ is absolutely continuous.

Proof: We want to prove formula (2). By Lemma 8, we know that $V_X = V_{-X} \subset U_{-X}$ and $V_Y \subset U_Y$. As $\mathfrak{k} \subset U_X$, we see that the formula (4) implies (2). ■

←=====

Later in this section, (Theorem 13), we will show that the hypotheses of the last Proposition are always satisfied for X and Y eligible. For technical reasons, in order to make an induction proof work, we will show more, i.e. that a “better” matrix $k \in K$ exists such that the formula (4) holds. The meaning of a “better” k will be similar to the notion of a total matrix given in Definition 10. Here is a definition and a lemma about total matrices in K . The reasoning of the proof of this lemma will be used in a more general setting in Steps 2 and 3 of the proof of Theorem 13.

Definition 10 *We say that a square $n \times n$ matrix k is total if by removing any $r < n$ rows and r columns of k we always obtain a nonsingular matrix.*

Note that this definition of totality is more restrictive than in [7, Definition 3.7].

Lemma 11 *The set of matrices in $\mathbf{SO}(n)$ which are total is dense and open in $\mathbf{SO}(n)$.*

Proof: Consider first the set $M_{I,J} = M_{\{i_1, \dots, i_j\}, \{j_1, \dots, j_r\}} \subset \mathbf{SO}(n)$ of orthogonal matrices which remain nonsingular once the rows of indices i_1, \dots, i_j and the columns of indices j_1, \dots, j_r are removed. To see that such matrices exist, take the identity matrix (whose determinant is 1 if we remove, say, the first r rows and columns). By taking convenient permutations of the rows and columns of the identity matrix, we obtain an element of $M_{I,J}$. Given that $\mathbf{SO}(n) \setminus M_{I,J}$ corresponds to the set of zeros of a certain determinant function, it must be closed and nowhere dense in $\mathbf{SO}(n)$.

To conclude, it suffices to notice that the set of total matrices in $\mathbf{SO}(n)$ is the finite intersection of all the sets $M_{I,J}$. ■

In the proof of the main Theorem 13 we will need the following technical lemma.

Lemma 12 (i) *For the root vectors X_{ir}^+ , Z_{ij}^+ , Y_{ij}^+ , we have*

$$\begin{aligned} \text{Ad}(e^{t(X_{ir}^+ + \theta X_{ir}^+)}) (X_{ir}) &= \cos(2t) X_{ir} + 2 \sin(2t) A_i, \\ \text{Ad}(e^{t(Y_{ij}^+ + \theta Y_{ij}^+)}) (Y_{ij}) &= \cos(4t) Y_{ij} + 2 \sin(4t) (A_i - A_j), \\ \text{Ad}(e^{t(Z_{ij}^+ + \theta Z_{ij}^+)}) (Z_{ij}) &= \cos(4t) Z_{ij} + 2 \sin(4t) (A_i + A_j). \end{aligned}$$

(ii) *The functions $\text{Ad}(e^{t(X_{ir}^+ + \theta X_{ir}^+)})$, $\text{Ad}(e^{t(Y_{ij}^+ + \theta Y_{ij}^+)})$ and $\text{Ad}(e^{t(Z_{ij}^+ + \theta Z_{ij}^+)})$ applied to the other symmetrized root vectors do not produce any components in \mathfrak{a} .*

Proof: It is just a matter of carefully evaluating

$$\text{Ad}(e^{t(Z+\theta Z)})(W) = e^{t \text{ad}(Z+\theta Z)}(W) = \sum_{k=0}^{\infty} (\text{ad}(Z + \theta Z))^k(W) \frac{t^k}{k!}.$$

For (ii), use the well known properties of the root system: $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \in \mathfrak{g}_{\alpha+\beta}$ and $[X_\alpha, \theta X_\alpha] \in \mathfrak{a}$. ■

By Proposition 9, in order to justify the sufficiency of the eligibility condition, it is enough to prove the following theorem. This is the main result of this section.

Theorem 13 *Let $G = \mathbf{SO}_0(p, q)$ and let $X, Y \in \mathfrak{a}$. If X and Y are eligible then there exists a matrix $k \in K$ such that*

$$V_X + \text{Ad}(k) V_Y = \mathfrak{p}. \quad (5)$$

Proof: We will assume that $X = X[\mathfrak{s}; u]$ and $Y = Y[\mathfrak{t}; v]$. Observe that the spaces V_X and V_Y depend on the Weyl chambers where X and Y belong. However, see [7, Lemma 3.3 and Reduction 1, p. 759], the property (5) is equivalent to $V_{w_1 X} + \text{Ad}(k') V_{w_2 Y} = \mathfrak{p}$ for any $w_1, w_2 \in W$ and a convenient $k' \in K$. Throughout the proof we will assume that the diagonal entries of \mathcal{D}_X and \mathcal{D}_Y are non-negative and we will arrange (permute) them conveniently.

To lighten the notation, for a matrix c of size $p \times q$, we will consider the $(p+q) \times (p+q)$ symmetric matrix

$$c^s = \begin{bmatrix} 0 & c \\ c^T & 0 \end{bmatrix} \in \mathfrak{p}.$$

The proof will be organized in the following way:

1. Proof for $q = p + 1$ using induction on p
 - (a) Proof for $p = 2$ and $q = 3$
 - (b) Proof of the induction step
 - i. Proof in the case $u > 0$ or $v > 0$
 - ii. Proof in the case $X[p; 0], Y[p; 0]$
2. Proof that the case (p, q) implies the case $(p, q + 1)$.

1. Proof for $q = p + 1$ using induction on p

(a) Proof for $p = 2$ and $q = 3$

This corresponds to the space $\mathbf{SO}_0(2, 3)$. Only two configurations $[2; 0]$ and $[1; 1]$ may be realized by singular non-zero X and Y . When $Z \in \bar{\mathfrak{a}}^+$, we have $\mathcal{D}_{Z[1;1]} = \text{diag}[z, 0]$, and $\mathcal{D}_{Z[2;0]} = \text{diag}[z, z]$, $z \neq 0$. It is easy to check that in all 3 possible cases:

$$(i) X[2; 0], Y[2; 0] \quad (ii) X[2; 0], Y[1; 1] \text{ or } X[1; 1], Y[2; 0] \quad (iii) X[1; 1], Y[1; 1],$$

X and Y are eligible. Note that

$$\begin{aligned} \mathfrak{p} &= \left\{ \begin{bmatrix} h_1 & a & b \\ c & h_2 & d \end{bmatrix}^s : h_1, h_2, a, b, c, d \in \mathbf{R} \right\}, \\ V_{Z[2;0]} &= \left\{ \begin{bmatrix} 0 & a & b \\ -a & 0 & c \end{bmatrix}^s : a, b, c \in \mathbf{R} \right\}, \\ V_{Z[1;1]} &= \left\{ \begin{bmatrix} 0 & a & c \\ b & 0 & 0 \end{bmatrix}^s : a, b, c \in \mathbf{R} \right\}. \end{aligned}$$

$$\text{If } k_1 = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \text{ then}$$

$$\text{Ad}(k_1) V_{Z[2;0]} = \left\{ \begin{bmatrix} \sqrt{2}a/2 & (a-b+c)/2 & (a+b-c)/2 \\ -\sqrt{2}a/2 & (a-b-c)/2 & (a+b+c)/2 \end{bmatrix}^s : a, b, c \in \mathbf{R} \right\},$$

and

$$\text{Ad}(k_1) V_{Z[1;1]} = \left\{ \begin{bmatrix} -\sqrt{2}b/2 & (a-c)/2 & (a+c)/2 \\ \sqrt{2}b/2 & (a-c)/2 & (a+c)/2 \end{bmatrix}^s : a, b, c \in \mathbf{R} \right\}$$

$$\text{If } k_2 = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix} \text{ then}$$

$$\text{Ad}(k_2) V_{Z[1;1]} = \left\{ \begin{bmatrix} -(b+c)/2 & \sqrt{2}a/2 & (-b+c)/2 \\ (b-c)/2 & \sqrt{2}a/2 & (b+c)/2 \end{bmatrix}^s : a, b, c \in \mathbf{R} \right\}.$$

We verify easily that in the cases (i) and (iii) we have $V_X + \text{Ad}(k_1) V_Y = \mathfrak{p}$. For $X[2;0]$ and $Y[1;1]$, we can see that $V_X + \text{Ad}(k_2) V_Y = \mathfrak{p}$.

(b) Proof of the induction step

(i) Proof in the case $u > 0$ or $v > 0$

We consider the space $\mathbf{SO}_0(p, p+1)/\mathbf{SO}(p) \times \mathbf{SO}(p+1)$ with $p > 2$ and the case when $X[\mathbf{s}; u]$ and $Y[\mathbf{t}; v]$ are such that $u > 0$ or $v > 0$. We assume $u \geq v$. We choose the predecessors in $\mathbf{SO}_0(p-1, p)/\mathbf{SO}(p-1) \times \mathbf{SO}(p)$ in the following way:

$$X' = X'[\mathbf{s}; u-1], \quad Y' = Y'[\mathbf{t}'; v]$$

where \mathbf{t}' means that we suppress one term from the longest block of size $\max \mathbf{t}$. Note that if $p > 2$ then \mathbf{t}' is not the zero partition (otherwise, \mathbf{t} would have been the partition $[1]$ meaning that $u \geq v = p-1$ which would make X and Y ineligible).

We arrange X, X', Y, Y' in the following way.

1. The first diagonal entry of \mathcal{D}_X is zero and all the other zeros are at the end. The diagonal entries of $\mathcal{D}_{X'}$ are those of \mathcal{D}_X without the first zero:

$$\mathcal{D}_X = \text{diag}[0, \overbrace{x_1, \dots, x_{p-u}}^{\neq 0}, \overbrace{0, \dots, 0}^{u-1}], \quad \mathcal{D}_{X'} = \text{diag}[\overbrace{x_1, \dots, x_{p-u}}^{\neq 0}, \overbrace{0, \dots, 0}^{u-1}]$$

2. We put a longest block of size t of equal diagonal entries y_1 of \mathcal{D}_Y in the beginning of Y . The diagonal entries of $\mathcal{D}_{Y'}$ are those of \mathcal{D}_Y with the first entry omitted:

$$\mathcal{D}_Y = \text{diag}[\overbrace{y_1, \dots, y_1}^t, y_2, \dots, y_s], \quad \mathcal{D}_{Y'} = \text{diag}[\overbrace{y_1, \dots, y_1}^{t-1}, y_2, \dots, y_s].$$

It is easy to check that if X, Y are eligible in $\mathbf{SO}_0(p, p+1)$ then X', Y' are eligible in $\mathbf{SO}_0(p-1, p)$.

Step 1. By the induction hypothesis, there exists a matrix $k_0 \in \mathbf{SO}(p-1) \times \mathbf{SO}(p)$ such that

$$V_{X'} + \text{Ad}(k_0)V_{Y'} = \mathfrak{p}'. \quad (6)$$

We embed $K' = \mathbf{SO}(p-1) \times \mathbf{SO}(p)$ in $\mathbf{SO}(p) \times \mathbf{SO}(p+1)$ in the following way

$$K' = \begin{bmatrix} 1 & & & \\ & \mathbf{SO}(p-1) & & \\ & & 1 & \\ & & & \mathbf{SO}(p) \end{bmatrix} \subset \begin{bmatrix} \mathbf{SO}(p) & \\ & \mathbf{SO}(p+1) \end{bmatrix}.$$

Hence, we have (taking the natural embedding of \mathfrak{p}' into \mathfrak{p})

$$V_1 := V_{X'} + \text{Ad}(k_0)V_{Y'} = \mathfrak{p}' = \begin{bmatrix} 0 & B' \\ B'^T & 0 \end{bmatrix} \quad (7)$$

where $B' = \left[\begin{array}{c|c} 0_{1 \times 1} & 0_{1 \times p} \\ \hline 0_{p \times 1} & B''_{(p-1) \times p} \end{array} \right]$ and B'' is arbitrary (note that \mathfrak{p}' is of dimension $(p-1)p$). We must show that for some $k \in K$, the space $V_X + \text{Ad}(k)V_Y = \mathfrak{p}$, i.e. that

- (i) $V_X + \text{Ad}(k)V_Y$ contains \mathfrak{p}' embedded into \mathfrak{p} as in (7).
- (ii) $V_X + \text{Ad}(k)V_Y$ contains all the matrices of the form

$$C = \left[\begin{array}{c|c} * & * \dots * \\ * & \\ \vdots & \\ * & 0_{(p-1) \times p} \end{array} \right]^s.$$

New vectors in V_X and V_Y . In order to prove the induction conclusion, we must now use the elements of V_X and V_Y which do not come from $V_{X'}$ or $V_{Y'}$. They appear by the interaction of, respectively, the first diagonal entry of \mathcal{D}_X with the others of \mathcal{D}_X and the interaction of the first

entry of \mathcal{D}_Y with the others of \mathcal{D}_Y . We see that the new independent root vectors in V_X and V_Y are respectively

$$N_X = \{Y_{1j}, Z_{1j}, j = 2, \dots, p+1-u\}, \quad N_Y = \{X_1, Y_{1i}, i = t+1, \dots, p, Z_{1j}, j = 2, \dots, p\}$$

where $t = \max \mathbf{t} > 1$ and we wrote X_1 for X_{11} . Note that N_X has $2p - 2u$ elements while N_Y has $2p - t$.

Step 2. We show that there exists $k'_0 \in \mathbf{SO}(p-1) \times \mathbf{SO}(p)$ for which (6) holds, and with the following property:

The space $V_2 := \text{Ad}(k'_0)\text{span}(N_Y)$ is of dimension $2p - t$ and its elements can be written in the form

$$\left[\begin{array}{c|cccc} 0 & \sigma_1 & \dots & \sigma_r & a_1 & \dots & a_{p-r} \\ \hline \tau_1 & & & & & & \\ \vdots & & & & & & \\ \tau_s & & & & & 0 & \\ a_{p-r+1} & & & & & & \\ \vdots & & & & & & \\ a_{2p-t} & & & & & & \end{array} \right]^s \quad (8)$$

with $r = [(t-1)/2]$, $s = t-1-r$, $a_i \in \mathbf{R}$ arbitrary, $\sigma_i = \sigma_i(a_1, \dots, a_{2p-t})$ and $\tau_j = \tau_j(a_1, \dots, a_{2p-t})$, $i \leq r, j \leq s$.

We will not need to write explicitly the functions σ_i and τ_j . Note that $s = r$ if t is odd and $s = r + 1$ if t is even.

To justify Step 2, we write

$$k_0 = \begin{bmatrix} 1 & & & \\ & k_{01} & & \\ & & 1 & \\ & & & k_{02} \end{bmatrix}$$

where $k_{01} \in \mathbf{SO}(p-1)$ and $k_{02} \in \mathbf{SO}(p)$. Let $\alpha_1, \dots, \alpha_{p-1}$ be the columns of the matrix k_{01} and β_1, \dots, β_p the columns of the matrix k_{02} . A simple block multiplication to compute the action of $\text{Ad}(k_0)$ on the elements of N_Y gives the linearly independent matrices

$$\begin{aligned} \text{Ad}(k_0)X_1 &= \left[\begin{array}{c|c} 0 & \beta_p^T \\ \hline 0 & 0 \end{array} \right]^s, \quad \text{Ad}(k_0)Y_{1i} = \left[\begin{array}{c|c} 0 & \beta_{i-1}^T \\ \hline \alpha_{i-1} & 0 \end{array} \right]^s, \quad i = t+1, \dots, p, \\ \text{Ad}(k_0)Z_{1i} &= \left[\begin{array}{c|c} 0 & \beta_{i-1}^T \\ \hline -\alpha_{i-1} & 0 \end{array} \right]^s, \quad i = 2, \dots, p. \end{aligned} \quad (9)$$

Let us write β'_i for a column β_i from which we have removed the first r entries and α'_i for a column α_i with the first s entries omitted. In order to prove the statement of Step 2, we must show that the matrices obtained by replacing β_i by β'_i and α_i by α'_i in (9) are still linearly independent. This is equivalent to the linear independence of the matrices

$$\left[\begin{array}{c|c} 0 & \beta_i'^T \\ \hline -\alpha'_i & 0 \end{array} \right]^s, \quad i = 1, \dots, t-1, \quad \left[\begin{array}{c|c} 0 & \beta_i'^T \\ \hline 0 & 0 \end{array} \right]^s, \quad i = t, \dots, p, \quad \left[\begin{array}{c|c} 0 & 0 \\ \hline \alpha'_i & 0 \end{array} \right]^s, \quad i = t, \dots, p-1. \quad (10)$$

We will reason in the same way as in Lemma 11.

It is enough to show that there exists at least a choice of matrices k_{01} and k_{02} such that the matrices in (10) are linearly independent. Then, as in Lemma 11, it will follow that such matrices form a dense open subset in $\mathbf{SO}(p-1) \times \mathbf{SO}(p)$. By choosing k'_0 with the matrices in (10) linearly independent and close enough to k_0 , the property (6) will be preserved for k'_0 .

Pick $k_{01} = I_{p-1}$ which implies that $\alpha'_i = \mathbf{0}$ for $i = 1, \dots, s$ and $\alpha'_i = \mathbf{e}_{i-s}$ for $i > s$. With this choice, (10) becomes

$$\left[\begin{array}{c|c} 0 & \beta_i'^T \\ \hline 0 & 0 \end{array} \right]^s, \quad i = 1, \dots, s, \quad \left[\begin{array}{c|c} 0 & \beta_i'^T \\ \hline -\mathbf{e}_{i-s} & 0 \end{array} \right]^s, \quad i = s+1, \dots, t-1, \quad \left[\begin{array}{c|c} 0 & \beta_i'^T \\ \hline 0 & 0 \end{array} \right]^s, \quad i = t, \dots, p, \\ \left[\begin{array}{c|c} 0 & 0 \\ \hline \mathbf{e}_{i-s} & 0 \end{array} \right]^s \quad i = t, \dots, p-1$$

which are linearly independent provided that

$$\left[\begin{array}{c|c} 0 & \beta_i'^T \\ \hline 0 & 0 \end{array} \right]^s, \quad i = 1, \dots, s, \quad \left[\begin{array}{c|c} 0 & \beta_i'^T \\ \hline 0 & 0 \end{array} \right]^s, \quad i = t, \dots, p$$

are linearly independent. This is the case for a total matrix $k_{02} \in \mathbf{SO}(p)$ or by taking convenient permutations of the rows and columns of the identity matrix I_p .

Step 3. We show that there exists a proper subset N'_X of N_X such that, if

$$V_3 := \text{span}(N'_X) + V_{X'} + \text{Ad}(k'_0)V_Y = \text{span}(N'_X) + V_1 + V_2$$

then $\dim V_3 = pq - 1 = \dim \mathfrak{p} - 1$ and V_3 is given by

$$V_3 = \left\{ \left[\begin{array}{c|cccc} 0 & a_1 & a_2 & \dots & a_p \\ \hline a_{p+1} & & & & \\ \vdots & & & & \\ a_{2p-1} & & & & \end{array} \right]^s, \quad a_1, \dots, a_{2p-1} \in \mathbf{R} \right\}. \quad (11)$$

Note that in matrices from the space V_2 , there are $r = \lfloor (t-1)/2 \rfloor$ pairs (σ_i, τ_i) plus possibly an extra τ_s if t is even and therefore $s = r + 1$. Note also that $t + 2u \leq 2p$ implies that $p - u \geq s \geq r$. For $j \leq r \leq p - u$, each pair $Y_{1j} = \left[\begin{array}{c|c} 0 & \mathbf{e}_{j-1}^T \\ \hline \mathbf{e}_{j-1} & 0 \end{array} \right]^s$, $Z_{1j} = \left[\begin{array}{c|c} 0 & \mathbf{e}_{j-1}^T \\ \hline -\mathbf{e}_{j-1} & 0 \end{array} \right]^s$ of elements of N_X allows us to replace σ_j and τ_j by independent variables. If t is odd, all the σ_j 's and τ_j 's will be taken care off and at least 2 elements of N_X will remain off N'_X . If t is even, all the σ_j and τ_j 's, $1 \leq j \leq r$ will be replaced by independent variables and only τ_s will remain. Now, letting the coefficient a_1 "vis-à-vis" the remaining τ_s be equal to 1 and all the other variables a_i equal to 0, either $\tau_s = 1$ or -1 or $\tau_s \neq \pm 1$. If $\tau_s = 1$ then Z_{1s} allows us to introduce the missing independent variable, if $\tau_s = -1$ then adding Y_{1s} to N'_X will do the trick. In the case $\tau_s \neq \pm 1$ we choose indifferently between Y_{1s} and Z_{1s} . In all cases the set $N_X \setminus N'_X$ has at least one element.

Step 4. Let v_1 be the positive root vector corresponding to an element of $N_X \setminus N'_X$. We denote $k_1^t = k_{v_1}^t$. There exists $\epsilon > 0$ such that for $t \in (0, \epsilon)$

$$V_4^t := \text{Ad}(k_1^t)(\text{span}(N'_X) + V_{X'}) + \text{Ad}(k'_0)V_Y = V_3.$$

Observe that v_1 is equal to Z_{1j}^+ or Y_{1j}^+ for one of the remaining Z_{1j} or Y_{1j} that was not used in the preceding step. The space $V_4^t \subset V_4^0$ for all t according to Lemma 12 (ii).

Recall the definition of $k_{X_\alpha}^t = e^{t(X_\alpha + \theta X_\alpha)}$, $t > 0$. Let $d(t) = \dim V_4^t$; for $t = 0$ we have $k_1^0 = Id$, and $\text{Ad}(k_1^0)(\text{span}(N'_X) + V_{X'}) + \text{Ad}(k'_0)V_Y = V_3$ is of dimension $pq - 1$, so $d(0) = pq - 1$. The equality $d(t) = d(0)$ is equivalent to non-nullity of an appropriate determinant continuous in t . Thus $d(t) = pq - 1$ holds for $t \in (0, \epsilon)$ for some $\epsilon > 0$. As $V_4^t \subset V_4^0$, the statement of Step 4 follows.

Step 5. *Generation of A_1 .* By Lemma 12, we have $\text{Ad}(k_1^t)v_1^s = a_tv_1^s + b_tA_1 + c_tA_j$ with $j \neq 1$ and $b_t \neq 0$ for $t \in (0, \epsilon)$ with ϵ small enough. Consequently

$$\text{Ad}(k_1^t)\text{span}(v_1^s) + V_4^t = \mathfrak{p}.$$

Conclusion. We have $\mathfrak{p} = \text{Ad}(k_1^t)(\mathbf{R}v_1^s + \text{span}N'_X + V_{X'}) + \text{Ad}(k'_0)V_Y \subset \text{Ad}(k_1^t)V_X + \text{Ad}(k'_0)V_Y$, so $\text{Ad}(k_1^t)V_X + \text{Ad}(k'_0)V_Y = \mathfrak{p}$. It follows that

$$V_X + \text{Ad}((k_1^t)^{-1}k'_0)V_Y = \mathfrak{p}.$$

(ii) Proof in the case $X[p; 0], Y[p; 0]$

This case must be treated separately because the predecessors X', Y' and consequently the sets N_X and N_Y are different than in case (i). The structure of the induction proof is identical as in (i), with the Steps 2 and 3 executed together.

We choose both predecessors $X'[p-1; 0], Y'[p-1; 0]$ and arrange X, X', Y, Y' in the same way as we did in the first part of the proof with $Y[\mathbf{t}; v]$ and $Y'[\mathbf{t}'; v]$. In that case, $N_X = \{X_1, Z_{12}, \dots, Z_{1p}\} = N_Y$ and the space $\text{Ad}(k'_0)(N_Y)$ is generated by

$$\left[\begin{array}{c|c} 0 & \beta_i^T \\ \hline -\alpha_i & 0 \end{array} \right]^s, \quad i = 1, \dots, p-1 \quad \text{and} \quad \left[\begin{array}{c|c} 0 & \beta_p^T \\ \hline 0 & 0 \end{array} \right]^s. \quad (12)$$

Recall that

$$Z_{1j} = \left[\begin{array}{c|c} 0 & \mathbf{e}_{j-1}^T \\ \hline -\mathbf{e}_{j-1} & 0 \end{array} \right]^s, \quad j = 2, \dots, p. \quad (13)$$

We want to show that the matrices in (12) together with those of (13) are linearly independent for a $k'_0 \in \mathbf{SO}(p-1) \times \mathbf{SO}(p)$ for which the equality (6) holds. Note that if $k'_0 = \begin{bmatrix} -I_{p-1} & 0 \\ 0 & I_p \end{bmatrix}$

(p odd) or $k'_0 = \begin{bmatrix} I_{p-1} & 0 \\ 0 & -I_p \end{bmatrix}$ (p even) then the matrices (12) and (13) are linearly independent.

Using once more the reasoning in Lemma 11, this implies that the set of matrices k'_0 for which this is true, is open and dense in $\mathbf{SO}(p-1) \times \mathbf{SO}(p)$.

We conclude that if $N'_X = N_X \setminus \{X_1\}$ then $\text{span}(N'_X + V_{X'}) + \text{Ad}(k'_0)V_Y$ has the form given in (11).

We reproduce the previous Step 4 and Step 5 using $v_1 = X_1^+$. The rest follows.

2. Proof that the case (p, q) implies the case $(p, q+1)$

=====>We will show by induction that for any $q > p$, there exists a matrix $k \in K$ such that (5) holds.<=====<We know by the first part of the proof that this is true for $\mathbf{SO}_0(p, p+1)$.

Assume that X and Y are eligible in $\mathbf{SO}_0(p, q+1)$. Their configurations are eligible in $\mathbf{SO}_0(p, q)$. We write X', Y' when we work in $\mathbf{SO}_0(p, q)$.

We embed $K' = \mathbf{SO}(p) \times \mathbf{SO}(q)$ in $K = \mathbf{SO}(p) \times \mathbf{SO}(q+1)$ in the following way

$$K' = \begin{bmatrix} \mathbf{SO}(p) & & \\ & \mathbf{SO}(q) & \\ & & 1 \end{bmatrix} \subset \begin{bmatrix} \mathbf{SO}(p) & & \\ & \mathbf{SO}(q+1) & \\ & & 1 \end{bmatrix}.$$

The space \mathfrak{p}' is formed by matrices

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix},$$

where B are $p \times q$ matrices. We embed \mathfrak{p}' in \mathfrak{p} by adding a last column of zeros to B .

Step 1. We suppose that there exists a matrix $k_0 \in K'$ such that

$$V_{X'} + \text{Ad}(k_0)V_{Y'} = \mathfrak{p}'. \quad (14)$$

Then, by [7, Lemma 3.3], for any permutations s_1 and s_2 of the diagonal entries of $\mathcal{D}_{X'}$ and $\mathcal{D}_{Y'}$, there exists $k_0 \in K'$ such that

$$V_{s_1 X'} + \text{Ad}(k_0)V_{s_2 Y'} = \mathfrak{p}'$$

so we can permute the elements of X' and Y' in a convenient way and still have the equality (14). We will arrange them in the following way (where the stars denote nonzero entries):

$$\mathcal{D}_{X'} = \text{diag}[\overbrace{0, \dots, 0}^u, \star, \dots, \star], \quad \mathcal{D}_{Y'} = \text{diag}[\star, \dots, \star, \overbrace{0, \dots, 0}^v].$$

Let us denote $k_{01} \in \mathbf{SO}(p)$ and $k_{02} \in \mathbf{SO}(q)$ the matrices compositing k_0 corresponding in (14) to such X' and Y' . We can suppose that the matrix k_{01} is total.

By the eligibility of X and Y , $u + v \leq p$, so no two zeros in $\mathcal{D}_{X'}$ and $\mathcal{D}_{Y'}$ are at the same position.

Let $N = \{X_{i,q+1}\}_{i=1}^p$. We set $N_X := V_X \cap N = \{X_{u+1,q+1}, \dots, X_{p,q+1}\}$, $N_Y := V_Y \cap N = \{X_{1,q+1}, \dots, X_{p-v,q+1}\}$. We have $p - v \geq u$.

Step 2. Let

$$k_1 = \begin{bmatrix} k_{01} & & \\ & k_{02} & \\ & & 1 \end{bmatrix}$$

where $k_{01} \in \mathbf{SO}(p)$ and $k_{02} \in \mathbf{SO}(q)$ are the blocks compositing k_0 . We then have

$$V_{X'} + \text{Ad}(k_1)V_{Y'} = [\mathfrak{p}' \quad \mathbf{0}]^s. \quad (15)$$

The space $V_X + \text{Ad}(k_1)V_Y$ contains, in addition to the matrices in (15), the linear span of $N_X + \text{Ad}(k_1)N_Y$.

Denote the columns of the matrix k_{01} by $\mathbf{c}_1, \dots, \mathbf{c}_p$. By block multiplication in $\mathbf{SO}_0(p, q + 1)$, we obtain

$$\text{Ad}(k_1)X_{j,q+1} = \begin{bmatrix} 0_{p \times q} & \mathbf{c}_j \end{bmatrix}^s.$$

This implies that the linear span of $N_X + \text{Ad}(k_1)N_Y$ contains the following symmetric matrices:

$$\begin{bmatrix} 0_{p \times q} & \mathbf{c}_1 \end{bmatrix}^s, \dots, \begin{bmatrix} 0_{p \times q} & \mathbf{c}_u \end{bmatrix}^s, \begin{bmatrix} 0_{p \times q} & \mathbf{e}_{u+1} \end{bmatrix}^s, \dots, \begin{bmatrix} 0_{p \times q} & \mathbf{e}_p \end{bmatrix}^s,$$

which are linearly independent by the totality of k_{01} . Thus $V_X + \text{Ad}(k_1)V_Y = \mathfrak{p}$. ■

=====>

We conclude this section with an example to illustrate our proof.

Example 14 Consider $X = X[2; 1], Y = Y[1, 1; 1]$ in $\mathfrak{so}(3, 4)$. We write X and Y in such a way that $\mathcal{D}_X = \text{diag}[0, a, a]$ and $\mathcal{D}_Y = \text{diag}[b, c, 0]$. Their predecessors in $\mathfrak{so}(2, 3)$ are X' and Y' such that $\mathcal{D}_{X'} = \text{diag}[a, a]$ and $\mathcal{D}_{Y'} = \text{diag}[c, 0]$.

Note that X and Y form an eligible pair and so are $X' = X[2; 0]$ and $Y' = Y'[1; 1]$. In Step 1,

we show that there exists a matrix $k_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & k_{0,1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k_{0,2} \end{bmatrix}$ with $k_{01} \in \mathbf{SO}(2)$ and $k_{0,2} \in \mathbf{SO}(3)$

such that

$$V_{X'} + \text{Ad}(k_0)V_{Y'} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}^s$$

where $*$ designates an arbitrary element. We have

$$N_X = \{Z_{12}, Y_{12}, Z_{13}, Y_{13}\}, \quad N_Y = \{X_1, Z_{12}, Y_{12}, Z_{13}, Y_{13}\}.$$

In Step 2, we observe that

$$\text{Ad}(k_0)\text{span}(N_Y) = \left\{ \left[\begin{array}{c|ccc} 0 & a_1 & a_2 & a_3 \\ a_4 & & & \\ a_5 & & & 0 \end{array} \right]^s, \quad a_1, \dots, a_6 \in \mathbf{R} \right\}$$

since the matrices

$$\left[\begin{array}{c|c} 0 & \beta_1^T \\ -\alpha_1 & 0 \end{array} \right]^s, \left[\begin{array}{c|c} 0 & \beta_1^T \\ \alpha_1 & 0 \end{array} \right]^s, \left[\begin{array}{c|c} 0 & \beta_2^T \\ -\alpha_2 & 0 \end{array} \right]^s, \left[\begin{array}{c|c} 0 & \beta_2^T \\ \alpha_2 & 0 \end{array} \right]^s, \left[\begin{array}{c|c} 0 & \beta_3^T \\ 0 & 0 \end{array} \right]^s$$

are linearly independent. Note that in this case, there are no σ_i and no τ_i .

Now, $V_X = \text{span} \left\{ \overbrace{Z_{12}, Y_{12}, Z_{13}, Y_{13}}^{N_X} \right\} \cup V_{X'}$ while $V_Y = \text{span} \left\{ \overbrace{X_1, Z_{12}, Y_{12}, Z_{13}, Y_{13}}^{N_Y} \right\} \cup V_{Y'}$. We can show that

$$\begin{aligned} \text{Ad}(e^{t(Z_{1,2}^+ + \theta Z_{1,2}^+)}) (V_{X'}) + \text{Ad}(k_0) (\text{span } N_Y \cup V_{Y'}) \\ = \begin{bmatrix} 0 & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}^s \end{aligned}$$

for t small enough (with t small enough, the dimension will not decrease). Now,

$$Ad(e^{t(Z_{1,2}^+ + \theta Z_{1,2}^+)}) \overbrace{(\text{span}\{Z_{12}\} \cup V_{X'})}^{\subset V_X} + Ad(k_0) \overbrace{(\text{span} N_Y \cup V_{Y'})}^{V_Y} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}^s = \mathfrak{p}$$

if t close to 0 since $Ad(e^{t(Z_{1,2}^+ + \theta Z_{1,2}^+)}) (Z_{12}) = \cos(4t) Z_{12} + 2 \sin(4t) (A_1 + A_2)$. Therefore,

$$V_X + Ad(\overbrace{e^{-t(Z_{1,2}^+ + \theta Z_{1,2}^+)}}^k k_0) V_Y = Ad(e^{-t(Z_{1,2}^+ + \theta Z_{1,2}^+)}) \mathfrak{p} = \mathfrak{p}$$

which means that the density exists.

←=====

5 Applications

=====> We now extend our results to the complex and quaternion cases. ←=====

Recall that $\mathbf{SU}(p, q)$ is the subgroup of $\mathbf{SL}(p + q, \mathbf{C})$ such that $g^* I_{p,q} g = I_{p,q}$ while $\mathbf{Sp}(p, q)$ is the subgroup of $\mathbf{SL}(p + q, \mathbf{H})$ such that $g^* I_{p,q} g = I_{p,q}$. Their respective maximal compact subgroups are $\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$ and $\mathbf{Sp}(p) \times \mathbf{Sp}(q) \equiv \mathbf{SU}(p, \mathbf{H}) \times \mathbf{SU}(q, \mathbf{H})$.

Their subspaces \mathfrak{p} can be described as $\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$ where B is an arbitrary complex (respectively quaternionic) matrix of size $p \times q$. The Cartan subalgebra \mathfrak{a} is chosen in the same way as for $\mathfrak{so}(p, q)$.

Corollary 15 Consider the symmetric spaces $\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$, $\mathbf{SU}(p, q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$ and $\mathbf{Sp}(p, q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q)$, $q > p$.

Let $X, Y \in \mathfrak{a}$. Then the measure $\delta_{e^X}^{\mathfrak{h}} \star \delta_{e^Y}^{\mathfrak{h}}$ is absolutely continuous if and only if X and Y are eligible, as defined in Definition 3.

Proof: Let $X, Y \in \mathfrak{a}$. If they are eligible then since

$$a(e^X (\mathbf{SO}(p) \times \mathbf{SO}(q)) e^Y) \subset a(e^X \mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q)) e^Y) \subset a(e^X (\mathbf{Sp}(p) \times \mathbf{Sp}(q)) e^Y),$$

it follows from Theorem 13 that these sets have nonempty interior. Hence the density exists in all three cases.

On the other hand, given Lemma 4, one can reproduce Proposition 5 using $\mathbf{F} = \mathbf{C}$ and $\mathbf{F} = \mathbf{H}$ to show that the eligibility condition is necessary in the complex and quaternionic cases. ■

We will conclude this paper with two further applications.

Proposition 16 Let X and $Y \in \mathfrak{a}$ be such that $(\delta_{e^X}^{\mathfrak{h}})^{*2}$ and $(\delta_{e^Y}^{\mathfrak{h}})^{*2}$ are absolutely continuous. Then $\delta_{e^X}^{\mathfrak{h}} \star \delta_{e^Y}^{\mathfrak{h}}$ is absolutely continuous.

Proof: \implies Let $X = X[\mathbf{s}; u]$ and $Y = Y[\mathbf{t}; v]$. \Leftarrow We know that the couple (X, X) is eligible; therefore

$$2 \max\{\mathbf{s}, 2u\} \leq 2p.$$

In the same manner, $\max\{\mathbf{t}, 2v\} \leq p$. Hence,

$$\max\{\mathbf{s}, 2u\} + \max\{\mathbf{t}, 2v\} \leq p + p = 2p$$

which means that X and Y are eligible. Consequently, $\delta_{e^X}^{\mathfrak{h}} * \delta_{e^Y}^{\mathfrak{h}}$ is absolutely continuous. \blacksquare

If $X \in \mathfrak{a}$ and $X \neq 0$, it is important to know for which convolution powers l the measure $(\delta_{e^X}^{\mathfrak{h}})^l$ is absolutely continuous. This problem is equivalent to the study of the absolute continuity of convolution powers of uniform orbital measures $\delta_g^{\mathfrak{h}} = m_K * \delta_g * m_K$ for $g \notin K$.

It was proved in [8, Corollary 7] that it is always the case for $l \geq r + 1$, where r is the rank of the symmetric space G/K . It was also conjectured ([8, Conjecture 10]) that $r + 1$ is optimal for this property, which was effectively proved for symmetric spaces of type A_n ([8, Corollary 18]). In the following theorem, the conjecture is shown not to hold on symmetric spaces of type B_p , where $r = p$. Thanks to the rich structure of the root system B_p , already all p -th powers of orbital measures are absolutely continuous and p is optimal for this property.

Theorem 17 *On symmetric spaces $\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$, $\mathbf{SU}(p, q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$ and $\mathbf{Sp}(p, q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q)$, $q > p$, for every nonzero $X \in \mathfrak{a}$, the measure $(\delta_{e^X}^{\mathfrak{h}})^p$ is absolutely continuous. Moreover, p is the smallest value for which this is true: if X has a configuration $[1; p - 1]$ then the measure $(\delta_{e^X}^{\mathfrak{h}})^{p-1}$ is singular.*

Proof: We will write S_X^l for the set $a(e^X K \dots K e^X)$ where the factor e^X appears l times. Note that $(\delta_{e^X}^{\mathfrak{h}})^l$ is absolutely continuous if and only if S_X^l has nonempty interior.

We prove first that for $l < p$, the measure $(\delta_{e^X}^{\mathfrak{h}})^l$ may not be absolutely continuous. Let $X = X[1; p - 1]$. Using Lemma 4 repeatedly, as in the proof of Proposition 5, we show that for $l < p$, there are at least $p - l$ diagonal entries of \mathcal{D}_H which are equal to 0 for every $H \in S_X^l$. Consequently, S_X^l has empty interior and $(\delta_{e^X}^{\mathfrak{h}})^l$ is not absolutely continuous when $l \leq p - 1$.

We will now show that $(\delta_{e^X}^{\mathfrak{h}})^p$ has a density for every $X \neq 0$.

If $X = X[\mathbf{s}; 0]$ then the measure $(\delta_{e^X}^{\mathfrak{h}})^2$ is already absolutely continuous (the couple (X, X) is eligible). Suppose then that $X = X[\mathbf{s}; u] \in \overline{\mathfrak{a}^+}$, $u > 0$.

Remark that if $H \in S_X^l$ then $a(e^X K e^H) \subset S_X^{l+1}$. Indeed, we have $e^X k_1 E^X \dots k_{l-1} e^X = k_a e^H k_b$ and therefore $a(e^X K e^H) = a(e^X K k_a e^H k_b) = a(e^X K e^X k_1 \dots k_{l-1} e^X) \subset S_X^{l+1}$.

We claim that there exists $H \in S_X^{p-1}$ such that $H = H[1^{p-1}; 1]$ or $H \in \mathfrak{a}^+$.

We prove the claim using induction on p . If $p = 2$ then $S_X^{p-1} = \{X\}$ and the result follows (in that case, u cannot be higher than 1 for $X \neq 0$).

Suppose that the claim is true for $p - 1 \geq 2$. Let $K_0 = \begin{bmatrix} \mathbf{SO}(p-1) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{SO}(q-1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Consider the set $B = a(e^X K_0 e^X \dots e^X)$ with $p - 1$ factors e^X . By the induction hypothesis, there exists $H_0 \in B$ with $H_0 = H_0[1^{p-2}; 2]$ or $H_0 = H_0[1^{p-1}; 1]$. In the second case, we are done.

If $H_0 = H_0[1^{p-2}; 2] \in B$, we can assume that the diagonal entries which are 0 in \mathcal{D}_X and in H_0 are at the end. We note that X and H_0 considered without their last entries are eligible in $\mathbf{SO}_0(p-1, q-1)$, their configurations being $[\mathbf{s}; u-1]$ and $[1^{p-2}; 1]$ respectively. Hence $a(e^X K_0 e^{H_0})$ has nonempty interior in the subspace $\overline{\mathfrak{a}^+} \cap \{H_p = 0\}$. Therefore, there exists $H \in a(e^X K_0 e^{H_0}) \subset S_X^{p-1}$ with $H = H[1^{p-1}; 1]$ which proves the claim.

To conclude, we take $H \in S_X^{p-1}$ with $H \in \mathfrak{a}^+$ or $H = H[1^{p-1}; 1]$. In both cases, X and H are eligible, so by Corollary 15 the set $a(e^X K e^H)$ has nonempty interior. As $a(e^X K e^H) \subset S_X^p$, this ends the proof. ■

6 Conclusion

With this paper and with [7], we have now obtained sharp criteria on singular X and Y for the existence of the density of $\delta_{e^X}^{\mathfrak{h}} \star \delta_{e^Y}^{\mathfrak{h}}$ for the root systems of type A_n and type B_p . Thanks to [8] and Theorem 17 of the present paper, sharp criteria are now given for the l -th convolution powers $(\delta_{e^X}^{\mathfrak{h}})^l$ to be absolutely continuous for any $X \neq 0$, $X \in \mathfrak{a}$.

Although there is considerable similarity between the criteria for both type of spaces, a characterization of eligibility that would be applicable for all Riemannian symmetric spaces of non-compact type has yet to emerge. The solution of the second problem in Theorem 17 seems to indicate that the answer may depend on the type of the symmetric space.

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