Multi-frequency Calderon-Zygmund analysis and connexion to Bochner-Riesz multipliers

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MULTI-FREQUENCY CALDERÓN-ZYGMUND ANALYSIS AND CONNEXION TO BOCHNER-RIESZ MULTIPLIERS

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Abstract. In this work, we describe several results exhibited during a talk at the *El Escorial 2012* conference. We aim to pursue the development of a multi-frequency Calderón-Zygmund analysis introduced in [10]. We set a definition of general multi-frequency Calderón-Zygmund operators. Unweighted estimates are obtained using the corresponding multi-frequency decomposition of [10]. Involving a new kind of maximal sharp function, weighted estimates are obtained.

The so-called Calderón-Zygmund theory and its ramifications have proved to be a powerful tool in many aspects of harmonic analysis and partial differential equations. The main thrust of the theory is provided by

- the Calderón-Zygmund decomposition, whose impact is deep and far-reaching. This decomposition is a crucial tool to obtain weak type $(1,1)$ estimates and consequently L^p bounds for a variety of operators;
- the use of the "local" oscillation $f - \left(\fint_Q f\right)$ (for Q a ball). These oscillations appear in the elementary functions of the "bad part" coming from the Calderón-Zygmund decomposition and in the definition of the maximal sharp function, which allows to get weighted estimates.

The oscillation $f - \left(\fint_Q f\right)$ can be seen as the distance between the function f and the set of constant functions on the ball Q, indeed the average is the best way to locally approximate the function by a constant. By this way, the constant function being associated to the frequency 0, we understand how the classical Calderón-Zygmund theory is related to the frequency 0.

As for example, well-known Calderón-Zygmund operators are the Fourier multipliers associated to a symbol m satisfying Hörmander's condition

$$|\partial^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|} = d(\xi,0)^{-|\alpha|},$$

which encodes regularity assumption of the symbol relatively to the frequency 0.

In this work, we are interested in the extension of this theory with respect to a collection of frequencies and we focus on sharp constants relatively to the number of the considered frequencies.

Such questions naturally arise as soon as we work on a multi-frequency problem:

- Uniform bounds for a Walsh model of the bilinear Hilbert transform (see [12] by Oberlin and Thiele);

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- A variation norm variant of Carleson's theorem (see [11] by Oberlin, Seeger, Tao, Thiele and Wright);
- Such a multi-frequency Calderón-Zygmund was introduced by Nazarov, Oberlin and Thiele in [10] for proving a variation norm variant of a Bourgain's maximal inequality.

Similarly to the fact that a Fourier multiplier with a symbol satisfying Hörmander's condition is a classical Calderón-Zygmund, we may extend this property to a collection of frequencies. More precisely, let $\Theta := (\xi_1, ..., \xi_N)$ be a collection of frequencies and consider a symbol m verifying for all multi-indices α

$$|\partial^\alpha m(\xi)| \lesssim d(\xi, \Theta)^{-|\alpha|},$$

with $d(\xi, \Theta) := \min_{1\leq i\leq N} |\xi - \xi_i|$. Such symbols give rise to Fourier multipliers, which should be the prototype of what we want to call *multi-frequency Calderón-Zygmund operators*.

In the 1-dimensional setting with a collection of frequencies $\Theta := (\xi_1, ..., \xi_N)$ (assumed to be indexed by the increasing order $\xi_1 < \xi_2 < \cdots < \xi_N$), an example is given by the multi-frequency Hilbert transform which corresponds to the symbol

$$m(\xi) = \begin{cases} -1, & \xi < \xi_1 \\ (-1)^{j+1}, & \xi_j < \xi < \xi_{j+1} \\ (-1)^{N+1}, & \xi > \xi_N. \end{cases}$$

Let us now detail a definition of "multi-frequency Calderón-Zygmund" operator:

Definition 0.1. Let $\Theta := (\xi_1, ..., \xi_N)$ be a collection of N frequencies of $\mathbb{R}^n$. An L^2-bounded linear operator T is said to be a Calderón-Zygmund operator relatively to Θ if there exist operators $(T_j)_{j=1,...,N}$ and kernels $(K_j)_{j=1,...,N}$ verifying

- Decomposition: $T = \sum_{j=1}^N T_j$;
- Integral representation of T_j: for every function $f \in L^2$ compactly supported and $x \in \mathrm{supp}(f)^c$,
$$T_j(f)(x) = \int K_j(x,y) f(y);$$
- Regularity of the modulated kernels: for every $x \neq y$
$$\sum_{j=1}^N \left| \nabla_{(x,y)}\ e^{i\xi_j\cdot(x-y)} K_j(x,y) \right| \lesssim |x-y|^{-n-1}.$$

Remark 0.1. As usual, we can weaken the regularity assumption and just require an ϵ-Hölder regularity on the modulated kernels.

Remark 0.2. If the decomposition is assumed to be orthogonal (which means that for $i \neq j$, $T_i T_j^* = 0$) then it follows that each operator T_j is a modulated Calderón-Zygmund operator. Such a multi-frequency Calderón-Zygmund operator can also be pointwisely bounded by a sum of N modulated (classical) Calderón-Zygmund operators and have the same boundedness properties with an implicit constant of order N. The aim is to study how this order can be improved using sharp estimates.

We first obtain unweighted estimates for such operators:

Theorem 0.1. *Let Θ be a collection of N frequencies and T an associated multi-frequency Calderón-Zygmund operator. Then*

- *for* $p \in (1,\infty)$, T *is bounded on* L^p *with*

$$\|T\|_{L^p \to L^p} \lesssim N^{\left|\frac{1}{p}-\frac{1}{2}\right|}.$$

- *for* $p=1$, T *is of weak-type* $(1,1)$ *with*

$$\|T\|_{L^1 \to L^{1,\infty}} \lesssim N^{\frac{1}{2}}.$$

This theorem relies on an adapted Calderón-Zygmund decomposition introduced in [10] by Nazarov, Oberlin and Thiele. We point out that there the constant $N^{\frac{1}{2}}$ is shown to be optimal and this is the same for the previous weak-type estimate.

Concerning weighted estimates, it is well-known that linear Calderón-Zygmund operators are bounded on $L^p(\omega)$ for $p \in (1,\infty)$ and every weight ω belonging to the Muckenhoupt's class $\mathbb{A}_p$ (see Definitions 1.1 and 1.2 for more details about Muckenhoupt's class $\mathbb{A}_p$ and Reverse Hölder class RH_s). Similar properties are satisfied by the Hardy-Littlewood maximal operator and some other linear operators as Bochner-Riesz multipliers [15, 4] or non-integral operators (like Riesz transforms) [1]. All these boundedness, obtained by using suitable Fefferman-Stein inequalities related to maximal sharp functions, involve weights belonging to the class $\mathcal{W}^p(p_0,q_0) := \mathbb{A}_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p}\right)'}$ for some exponents $p_0 < q_0$. [1]

As a consequence, it seems that these classes of weights are well-adapted for proving boundedness of linear operators. Following this observation, we will consider a multi-frequency maximal sharp function, in order to prove weighted estimates for our multi-frequency operators:

Theorem 0.2. *Let* Θ *be a collection of* N *frequencies. For* $p \in (1,\infty)$, $s \in (1,p)$ *and* $t \in (1,\infty)$, *then every multi-frequency Calderón-Zygmund operator* T *is bounded on* $L^p(\omega)$ *for every weight* $\omega \in RH_{t'} \cap \mathbb{A}_{\frac{p}{s}}$ *with*

$$\|T\|_{L^p(\omega) \to L^p(\omega)} \lesssim N^{\gamma}$$

and

$$\gamma := \frac{tp}{s\min\{2,s\}} + \left|\frac{1}{2}-\frac{1}{s}\right|.$$

We emphasize that this result is only interesting when $\gamma < 1$.

The current paper is organized as follows: after some preliminaries about weights, examples of multi-frequency operators and the main lemma for the multi-frequency analysis, Theorem 0.1 is proved in Section 2. Then in Section 3, we develop the general approach for weighted estimates, based on a suitable maximal sharp function. In Section 4, we describe how this point of view could be used to Bochner-Riesz multipliers.

[1] From [8], we know that for $r,s>1$,

$$\mathbb{A}_r \cap RH_s = \left\{\omega, \omega^s \in \mathbb{A}_{1+s(r-1)}\right\},$$

so these classes of weights are equivalent to a class of powers of Muckenhoupt's weights.

1. Notations and preliminaries

Let us consider the Euclidean space $\mathbb{R}^n$ equipped with the Lebesgue measure dx and its Euclidean distance $|x-y|$. Given a ball $Q \subset \mathbb{R}^n$ we denote its center by $c(Q)$ and its radius by r_Q. For any $\lambda > 1$, we denote by $\lambda\, Q := B(c(Q), \lambda r_Q)$. We write L^p for $L^p(\mathbb{R}^n, \mathbb{R})$ or $L^p(\mathbb{R}^n, \mathbb{C})$. For a subset $E \subset \mathbb{R}^n$ of finite and non-vanishing measure and f a locally integrable function, the average of f on E is defined by

$$\fint_E f dx := \frac{1}{|E|} \int_E f(x) dx.$$

Let us denote by $\mathcal{Q}$ the collection of all balls in $\mathbb{R}^n$. We write $\mathcal{M}$ for the maximal Hardy-Littlewood function:

$$\mathcal{M}f(x) = \sup_{\substack{Q \in \mathcal{Q} \\ x \in Q}} \fint_Q |f| dx.$$

For $p \in (1, \infty)$, we set $\mathcal{M}_p f(x) = \mathcal{M}(|f|^p)(x)^{1/p}$. The Fourier transform will be denoted by $\mathcal{F}$ as an operator and we make use of the other usual notation $\mathcal{F}(f) = \widehat{f}$ too.

In the current work, we aim to develop a multi-frequency analysis, based on the following lemma:

Lemma 1.1 ([2]). *Let $\Theta \subset \mathbb{R}^n$ be a finite collection of frequencies and Q be a ball. For every function ϕ belonging to the subspace of $L^2(3Q)$, spanned by $(e^{i\xi \cdot})_{\xi \in \Theta}$, we have for $p \in [1, 2]$*

$$\|\phi\|_{L^\infty(Q)} \lesssim (\sharp \Theta)^{\frac{1}{p}} \left(\fint_{3Q} |\phi|^p dx \right)^{\frac{1}{p}}. \tag{1}$$

Remark 1.1. In [2], this lemma is stated and proved in a one-dimensional setting. However, the proof only relies on the additive group structure of the ambient space by using translation operators. So the exact same proof can be extended to a multi-dimensional setting.

Remark 1.2. The question of extending the previous lemma for $p \in (2, \infty)$ is still open in such a general situation. Of course, (1) is true for $p = \infty$ and so it would be reasonable to expect the result for intermediate exponents $p \in (2, \infty)$. Unfortunately, the well-known interpolation theory does not apply here.
However, in some specific situations, we may extend this lemma for $p \geq 2$. Indeed, if $p = 2k$ is an even integer then applying (1) with $p = 2$ and $\Theta^k := \{\theta_{i_1} + ... + \theta_{i_k},\ \theta_i \in \Theta\}$ to ϕ^k yields

$$\begin{aligned} \|\phi\|_{L^\infty(Q)} &\lesssim \|\phi^k\|_{L^\infty(Q)}^{\frac{1}{k}} \\ &\lesssim (\sharp \Theta^k)^{\frac{1}{2k}} \left(\fint_{3Q} |\phi|^{2k} dx \right)^{\frac{1}{2k}} \\ &\simeq (\sharp \Theta^k)^{\frac{1}{p}} \left(\fint_{3Q} |\phi|^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

By this way, we see that an extension of (1) for $p \geq 2$ may be related to sharp combinatorial arguments, to estimate $\sharp \Theta^k$ (a trivial bound is $\sharp \Theta^k \leq (\sharp \Theta)^k$ which does not improve (1)).

We aim to obtain weighted estimates, involving Muckenhoupt's weights.

Definition 1.1. A weight ω is a non-negative locally integrable function. We say that a weight $\omega \in \mathbb{A}_p$, $1 < p < \infty$, if there exists a positive constant C such that for every ball Q,

$$\left(\fint_Q \omega \, dx\right)\left(\fint_Q \omega^{1-p'} \, dx\right)^{p-1} \leq C.$$

For $p = 1$, we say that $\omega \in \mathbb{A}_1$ if there is a positive constant C such that for every ball Q,

$$\fint_Q \omega \, dx \leq C\, \omega(y), \qquad \text{for a.e. } y \in Q.$$

We write $\mathbb{A}_\infty = \cup_{p \geq 1} \mathbb{A}_p$.

We just recall that for $p \in (1, \infty)$, the maximal function $\mathcal{M}$ is bounded on $L^p(\omega)$ if and only if $\omega \in \mathbb{A}_p$. We also need to introduce the reverse Hölder classes.

Definition 1.2. A weight $\omega \in RH_p$, $1 < p < \infty$, if there is a constant C such that for every ball Q,

$$\left(\fint_Q \omega^p \, dx\right)^{1/p} \leq C\left(\fint_Q \omega \, dx\right).$$

It is well known that $\mathbb{A}_\infty = \cup_{r>1} RH_r$. Thus, for $p = 1$ it is understood that $RH_1 = \mathbb{A}_\infty$.

1.1. **Examples of multi-frequency Calderón-Zygmund operators.** Let us detail particular situations where such multi-frequency operators appear.

The multi-frequency Hilbert transform. As explained in the introduction, an example of such multi-frequency operators in the 1-dimensional setting is the multi-frequency Hilbert transform. In $\mathbb{R}$, consider an arbitrary collection of frequencies $\Theta := (\xi_1, ..., \xi_N)$ (assumed to be indexed by the increasing order $\xi_1 < \xi_2 < \cdots < \xi_N$). The associated multi-frequency Hilbert transform is the Fourier multiplier corresponding to the symbol

$$m(\xi) = \begin{cases} -1, & \xi < \xi_1 \\ (-1)^{j+1}, & \xi_j < \xi < \xi_{j+1} \\ (-1)^{N+1}, & \xi > \xi_N. \end{cases}$$

Associated to Θ, we have a collection of disjoint intervals $\Delta := \{(-\infty, \xi_1), (\xi_1, \xi_2), ..., (\xi_N, \infty)\}$. It is well-known by Rubio de Francia's work [13] that for $q \in (1, 2]$, the functional

$$f \to \left(\sum_{\omega \in \Delta} \left|\mathcal{F}^{-1}[\mathbf{1}_\omega \mathcal{F} f]\right|^q\right)^{\frac{1}{q}} \tag{2}$$

is bounded on L^p for $p \in (q', \infty)$.

The boundedness of the multi-frequency Hilbert transform is closely related to the understanding of (2) for $q \to 1$.

We point out that in Rubio de Francia's result, the obtained estimates do not depend on the collection of intervals Δ. More precisely, excepted the end-point $p = q'$, the range (q', ∞) is optimal for a uniform (with respect to the collection Δ) L^p-boundedness of (2). So it is natural that for $q \to 1$ things are more difficult, which is illustrated by our multi-frequency Calderón-Zygmund analysis. Indeed, for example if one considers the particular case $\Theta := (1, ..., N)$, then following the notations of Remark 1.2, we have $\Theta^k = \{k, ..., kN\}$ and so $\sharp\Theta^k = k(N-1)+1 \simeq kN$.

Hence, in this situation we have observed (see Remark 1.2) that we can extend Lemma 1.1 to exponents $p \in [1,\infty]$ (the implicit constant appearing in (1) is only depending on p). By this way, Theorem 0.2 can be improved and we obtain a better exponent

$$\gamma = \frac{tp}{s^2} + \left|\frac{1}{2} - \frac{1}{s}\right|.$$

Consequently, it seems that for the L^p-boundedness of the multi-frequency Hilbert transform, the collection Θ could play an important role (which was not the case for the ℓ^q-functional (2) with $q' < p$).

Multi-frequency operators coming from a covering of the frequency space. Let $(Q_j)_{j=1,...,N}$ be a family of disjoint cubes and ϕ_j a smooth function with $\widehat{\phi_j}$ supported and adapted to Q_j. Then consider the linear operator given by

$$T(f) = \sum_{j=1}^{N} \phi_j * f.$$

It is easy to check that T is a multi-frequency Calderón-Zygmund operator, associated to the collection $\Theta := (\xi_1, ..., \xi_N)$ where for every j, $\xi_j := c(Q_j)$ is the center of the ball Q_j. With r_j the radius of Q_j, we have the regularity estimate

$$\sum_{j=1}^{N} \left|\nabla_{(x,y)}\ e^{i\xi_j\cdot(x-y)}\phi_j(x-y)\right| \lesssim |x-y|^{-n-1} \sum_{j=1}^{N} \frac{(r_j|x-y|)^{n+1}}{(1+r_j|x-y|)^M},$$

for every integer $M > 0$.

So boundedness of T (Theorem 0.1) yields the inequality

$$\left\|\sum_{j=1}^{N} \phi_j * f\right\|_{L^p} \lesssim C(r_1, ..., r_N) N^{\left|\frac{1}{p}-\frac{1}{2}\right|} \|f\|_{L^p}, \tag{3}$$

with

$$C(r_1, ..., r_N) := \sup_{t>0} \sum_{j=1}^{N} \frac{(r_j t)^{n+1}}{(1+r_j t)^M}.$$

Let us examine some particular situations:

- If the cubes $(Q_j)_j$ have an equal side-length, then as for Proposition 4.1, simple arguments imply (3) for $p \in [1,\infty]$ without the constant $C(r_1, ..., r_N)$.
- If the collection $(Q_j)_j$ is dyadic: it exists a point ξ_0, $d(Q_j, \xi_0) \simeq r_{Q_j} \simeq 2^j$ then Littlewood-Paley theory implies (3) without the factor $N^{|\frac{1}{p}-\frac{1}{2}|}$ (in this case $C(r_1, ..., r_N) \simeq 1$).
- If the cubes (Q_j) have only the dyadic scale: $r_{Q_j} \simeq 2^j$ (but no assumptions on the centers of the balls) then Littlewood-Paley theory cannot be used. However, our previous results can be applied in this situation and so (3) holds and $C(r_1, ..., r_N) \simeq 1$.

We aim to use the new multi-frequency Calderón-Zygmund analysis to extend these inequalities with replacing the convolution operators by more general Calderón-Zygmund operators, still satisfying some orthogonality properties.

Multi-frequency operators coming from variation norm estimates. As explained in the introduction, the multi-frequency Calderón-Zygmund analysis has been first developed for proving a variation norm variant of a Bourgain's maximal inequality. So our results can be adapted in such a framework. For example, in [7] Grafakos, Martell and Soria have studied maximal inequalities of the form

$$\left\| \sup_{j=1,\dots,N} \left| T(e^{i\theta_j \cdot} f) \right| \right\|_{L^p} \lesssim \|f\|_{L^p}$$

where $(\theta_j)_{j=1,\dots,N}$ is a collection of frequencies and T a fixed Calderón-Zygmund operator.

We can ask the same question, for a variation norm variant: for $q \in [1, \infty)$ consider

$$\left(\sum_{j=1}^{N} \left| T(e^{i\theta_j \cdot} f) \right|^q \right)^{\frac{1}{q}}$$

and study its boundedness on L^p, with a sharp control of the behaviour with respect to N. By a linearization argument (involving Rademacher's functions), this ℓ^q-functional can be realized as an average of modulated Calderón-Zygmund operators, associated to the collection $\Theta := (\theta_j)_j$.

2. Unweighted estimates for multi-frequency Calderón-Zygmund operators

In this section, we aim to prove the weak L^1-estimate for a multi-frequency Calderón-Zygmund operator, then Theorem 0.1 will easily follow from interpolation and duality.

Proposition 2.1. *Let $\Theta = (\xi_1, ..., \xi_N)$ be a collection of N frequencies as above and T be a Calderón-Zygmund operator relatively to Θ. Then T is of weak type $(1,1)$ with (uniformly with respect to N)*

$$\|T\|_{L^1 \to L^{1,\infty}} \lesssim N^{\frac{1}{2}}.$$

Proof. Consider f a function in L^1 and $\lambda > 0$, we use the Calderón-Zygmund decomposition[2] of [10] related to the collection of frequencies Θ. So the function f can be decomposed $f = g + \sum_{J \in \mathbf{J}} b_J$ with the following properties:

- $\mathbf{J}$ is a collection of balls and $(3J)_{J\in\mathbf{J}}$ has a bounded overlap;
- for each $J \in \mathbf{J}$, b_J is supported in $3J$;
- we have

$$\sum_{J \in \mathbf{J}} |J| \lesssim \sqrt{N} \|f\|_{L^1} \lambda^{-1}; \tag{4}$$

- the "good part" g satisfies

$$\|g\|_{L^2}^2 \lesssim \|f\|_{L^1} \sqrt{N} \lambda; \tag{5}$$

- the cubes J satisfy

$$\|f\|_{L^1(J)} \lesssim |J| \lambda N^{-\frac{1}{2}}, \quad \|f - b_J\|_{L^2(J)} \lesssim \sqrt{|J|}\lambda; \tag{6}$$

[2]In [10], the multi-frequency Calderón-Zygmund decomposition is only described in $\mathbb{R}$. The proof is a combination of Lemma 1.1 and the usual Calderón-Zygmund decomposition. Since both of them can be extended in a multi-dimensional framework, the multi-frequency Calderón-Zygmund decomposition performed in [10] still holds in $\mathbb{R}^n$.

- we have cancellation for all the frequencies of Θ: for all $j=1,...,N$ and $J\in\mathbf{J}$, $\widehat{b_J}(\xi_j)=0$.

We aim to estimate the measure of the level-set

$$\Upsilon_\lambda := \{x, |T(f)(x)|>\lambda\}.$$

With $b=\sum_J b_J$, we have

$$\begin{aligned} |\Upsilon_\lambda| & \leq |\{x, |T(g)(x)|>\lambda/2\}| + |\{x, |T(b)(x)|>\lambda/2\}| \\ & \lesssim \lambda^{-2}\|T(g)\|_{L^2}^2 + |\{x, |T(b)(x)|>\lambda/2\}| \\ & \lesssim \lambda^{-1}\sqrt{N}\|f\|_{L^1} + |\{x, |T(b)(x)|>\lambda/2\}|, \end{aligned}$$

where we used the L^2-boundedness of T. So it remains us to study the last term. Since (4), we get

$$\left|\bigcup_{J\in\mathbf{J}} 4J\right| \lesssim \sum_J |J| \lesssim \sqrt{N}\|f\|_{L^1}\lambda^{-1}.$$

Consequently, it only remains to estimate the measure of the set

$$O_\lambda := \left\{ x\in \left(\bigcup_{J\in\mathbf{J}} 4J\right)^c, \quad |T(b)(x)|>\lambda/2 \right\}.$$

Since

$$|O_\lambda| \lesssim \lambda^{-1}\sum_J \|T(b_J)\|_{L^1((2J)^c)}, \tag{7}$$

it is sufficient to estimate the L^1-norms. Consider K the kernel of T and a point $x_0\in\left(\bigcup_{J\in\mathbf{J}} 4J\right)^c$. Then, we can use the integral representation and we have

$$T(b)(x_0) = \int K(x_0,y)b(y)dy = \sum_J \int_{3J} K(x_0,y)b_J(y)dy.$$

To each J, we aim to take advantage of the cancellation properties of b_J, so we subtract the projection of $[y\to K(x_0,y)]$ on the space, spanned by $(e^{iy\cdot\eta})_{\eta\in\Theta}$. So we have

$$\begin{aligned} T(b)(x_0) & = \sum_J\sum_{j=1}^N \int_{3J}\left[K_j(x_0,y) - e^{i\xi_j\cdot c(J)}K_j(x_0,c(J))e^{-i\xi_j\cdot y}\right] b_J(y)dy \\ & = \sum_J\sum_{j=1}^N \int_{3J}\left[\widetilde{K}_j(x_0,y) - \widetilde{K}_j(x_0,c(J))\right] e^{i\xi_j\cdot(x_0-y)} b_J(y)dy \end{aligned}$$

where $c(J)$ is the center of J and $\widetilde{K}_j(x,y) := K_j(x,y)e^{-i\xi_j\cdot(x-y)}$. We then write

$$T_j(b)(x_0) := \int \left[\widetilde{K}_j(x_0,y) - \widetilde{K}_j(x_0,c(J))\right] e^{i\xi_j\cdot(x_0-y)} b(y)dy.$$

such that $T(b)=\sum_j T_j(b)$. Due to the regularity assumption on K (and so on $\widetilde{K}_j$), it comes for $y\in J$ and $x_0\in(2J)^c$

$$\sum_{j=1}^N \left|\widetilde{K}_j(x_0,y) - \widetilde{K}_j(x_0,c(J))\right| \lesssim \frac{r_J}{|x_0-y|^{n+1}}. \tag{8}$$

So we have

$$\|T(b_J)\|_{L^1((2J)^c)} \lesssim \iint_{|x-y|\geq r_J} \frac{r_J}{|x-y|^{n+1}} |b_J(y)| dx dy \lesssim \|b_J\|_{L^1} \lesssim |J|\lambda.$$

Finally, we obtain with (7) that

$$|O_\lambda| \lesssim \sum_J |J| \lesssim \sqrt{N} \|f\|_{L^1} \lambda^{-1},$$

which concludes the proof. □

Remark 2.1. Following [10], the bound of order $N^{\frac{1}{2}}$ is optimal for the multi-frequency decomposition and for the weak-L^1 estimate.

3. Weighted estimates for multi-frequency Calderón-Zygmund operators

Aiming to obtain weighted estimates on such multi-frequency operators (using *Good-lambda inequalities*), we also have to define a suitable maximal sharp function, associated to a collection of frequencies.

Definition 3.1 (Maximal sharp function). Let Θ be a collection of N frequencies and $s \in [1, \infty)$. Consider a ball Q, we denote by $\mathbb{P}_{\Theta,Q}$ the projection operator (in the L^s-sense) on the subspace of $L^s(3Q)$, spanned by $(\exp i\xi\cdot)_{\xi\in\Theta}$. Let us specify this projection operator: consider E the finite dimensional sub-space of $L^s(3Q)$, spanned by $(e^{i\xi\cdot})_{\xi\in\Theta}$ and equipped with the $L^s(3Q)$-norm. Since E is of finite dimension, then for every $f \in L^s(Q)$ there exists $v := \mathbb{P}_{\Theta,Q}(f) \in E$ such that

$$\|f - v\|_{L^s(3Q)} = \inf_{\phi\in E} \|f - \phi\|_{L^s(3Q)}.$$

This projection operator may depend on s, which is not important for our purpose so this is implicit in the notation and we forget it.

Since $0 \in E$, we obviously have

$$\|\mathbb{P}_{\Theta,Q}(f)\|_{L^s(3Q)} \leq 2\|f\|_{L^s(Q)}. \tag{9}$$

Then, we may define the maximal sharp function

$$\mathcal{M}^\sharp_{s,\Theta}(f)(x_0) := \sup_{x_0\in Q} \left(\fint_Q |f - \mathbb{P}_{\Theta,Q}(f\mathbf{1}_Q)|^s \, dx \right)^{\frac{1}{s}}.$$

Note that the usual sharp maximal function is the one obtained for $\Theta := \{0\}$ and in this situation it is well-known that the maximal sharp function satisfies a so-called Fefferman-Stein inequality (see [6]). We first prove an equivalent property for this generalised maximal sharp function:

Proposition 3.1. *Let $s \in (1, \infty)$, $t \in [1, \infty)$ and $p \in (s, \infty)$ be fixed. Then for every function $f \in L^s$ and every weight $\omega \in RH_{t'}$, we have for every $p \geq s$*

$$\|f\|_{L^p(\omega)} \lesssim N^{\frac{tp}{s} \max\{\frac{1}{2}, \frac{1}{s}\}} \left\| \mathcal{M}^\sharp_{s,\Theta}(f) \right\|_{L^p(\omega)}.$$

The proof relies on a *Good-lambda inequality* and Lemma 1.1.

Proof. We make use on the abstract theory developed in [1] by Auscher and Martell. We also follow notations of [1, Theorem 3.1]. Indeed, for each ball $Q\subset\mathbb{R}^n$ we have the following

$$F(x):=|f(x)|^s \lesssim |f(x)-\mathbb{P}_{\Theta,Q}(f\mathbf{1}_Q)(x)|^s+|\mathbb{P}_{\Theta,Q}(f\mathbf{1}_Q)(x)|^s := G_Q(x)+H_Q(x).$$

By definition, it comes

$$\fint_Q G_Q dx \leq \inf_Q \mathcal{M}^\sharp_{s,\Theta}(f)^s$$

and following Lemma 1.1 (with (9))

$$\begin{aligned}\sup_{x\in Q} H_Q = \|\mathbb{P}_{\Theta,Q}(f\mathbf{1}_Q)\|^s_{L^\infty(Q)} &\lesssim N^{s\max\{\frac{1}{2},\frac{1}{s}\}}\left(\fint_{3Q}|\mathbb{P}_{\Theta,Q}(f\mathbf{1}_Q)|^s dx\right)\\ &\lesssim N^{s\max\{\frac{1}{2},\frac{1}{s}\}}\left(\fint_Q |f|^s dx\right)\lesssim N^{s\max\{\frac{1}{2},\frac{1}{s}\}}\inf_Q \mathcal{M}F.\end{aligned}$$

So we can apply [1, Theorem 3.1] (with $q=\infty$ and $a\simeq N^{s\max\{\frac{1}{2},\frac{1}{s}\}}$) and by checking the behaviour of the constants with respect to "a" in its proof, we obtain for every $p\geq 1$

$$\|\mathcal{M}_s(f)^s\|_{L^p(\omega)} \lesssim N^{spt\max\{\frac{1}{2},\frac{1}{s}\}}\left\|\mathcal{M}^\sharp_{s,\Theta}(f)^s\right\|_{L^p(\omega)},$$

which yields the desired result. □

Then, we evaluate a multi-frequency Calderón-Zygmund operator via this new maximal sharp function.

Proposition 3.2. *Let T be a Calderón-Zygmund operator relatively to Θ and $s\in(1,\infty)$. Then, we have the following pointwise estimate:*

$$\mathcal{M}^\sharp_{s,\Theta}(T(f)) \lesssim N^{|\frac{1}{s}-\frac{1}{2}|}\mathcal{M}_s(f).$$

Proof. We follow the well-known proof for usual Calderón-Zygmund operators and adapt the arguments to the current situation. So consider a point x_0 and a ball $Q\subset\mathbb{R}^n$ containing x_0, we have to estimate

$$\left(\fint_Q |T(f)-\mathbb{P}_{\Theta,Q}(T(f)\mathbf{1}_Q)|^s\, dx\right)^{\frac{1}{s}}.$$

We split the function into a local part f_0 and an off-diagonal part f_∞:

$$f=f_0+f_\infty := f\mathbf{1}_{10Q}+f\mathbf{1}_{(10Q)^c}.$$

By definition of the projection operator, we know that

$$\begin{aligned}\left(\fint_Q |T(f)-\mathbb{P}_{\Theta,Q}(T(f)\mathbf{1}_Q)|^s\, dx\right)^{\frac{1}{s}} &\leq \left(\fint_Q |T(f)-\mathbb{P}_{\Theta,Q}(T(f_\infty)\mathbf{1}_Q)|^s\, dx\right)^{\frac{1}{s}}\\ &\leq \left(\fint_Q |T(f_0)|^s\, dx\right)^{\frac{1}{s}}+\left(\fint_Q |T(f_\infty)-\mathbb{P}_{\Theta,Q}(T(f_\infty)\mathbf{1}_Q)|^s\, dx\right)^{\frac{1}{s}}.\end{aligned}$$

For the local part, we use boundedness in L^s of the operator T (Proposition 2.1), hence

$$\left(\fint_Q |T(f_0)|^s \, dx\right)^{\frac{1}{s}} \lesssim |Q|^{-\frac{1}{s}} \|T(f_0)\|_{L^s(Q)} \lesssim N^{\left(\frac{1}{2}-\frac{1}{s}\right)} \left(|Q|^{-\frac{1}{s}} \|f_0\|_{L^s}\right)$$
$$\lesssim N^{\left|\frac{1}{2}-\frac{1}{s}\right|} \mathcal{M}_s(f)(x_0).$$

Then let us focus on the second part, involving f_∞.
We use the decomposition (with an integral representation) since we are in the off-diagonal case: for $x \in Q$

$$T(f_\infty)(x) = \sum_{j=1}^N \int K_j(x,y) f_\infty(y) dy.$$

Consider the following function, defined on $3Q$ by (where $c(Q)$ is the center of Q)

$$\Phi := x \in 3Q \to \sum_{j=1}^N \int e^{i\xi_j \cdot (x - c(Q))} K_j(c(Q), y) f_\infty(y) dy.$$

So $\Phi \in E$ (see Definition 3.1) and hence

$$\left(\fint_Q |T(f_\infty) - \mathbb{P}_{\Theta,Q}(T(f_\infty)\mathbf{1}_Q)|^s \, dx\right)^{\frac{1}{s}} \leq \left(\fint_Q |T(f_\infty) - \Phi|^s \, dx\right)^{\frac{1}{s}}. \tag{10}$$

If we set $\widetilde{K}_j(x,z) := K_j(x,z) e^{-i\xi_j \cdot (x-z)}$, then

$$T(f_\infty)(x) - \Phi(x) = \sum_j \int \left[\widetilde{K}_j(x,y) - \widetilde{K}_j(c(Q),y)\right] e^{i\xi_j (x-y)} f_\infty(y) dy.$$

From the regularity assumption on the kernels K_j's, we have for $y \in (10Q)^c$

$$\sum_j \left|\widetilde{K}_j(x,y) - \widetilde{K}_j(c(Q),y)\right| \lesssim r_Q \sup_{z \in Q} \sum_j \left|\nabla_x \widetilde{K}_j(z,y)\right| \lesssim r_Q^{-n} \left(1 + \frac{d(y,Q)}{r_Q}\right)^{-n-1}. \tag{11}$$

We also have (since $y \in (10Q)^c$ and $x, c(Q) \in Q$)

$$|T(f_\infty)(x) - \Phi(x)| \lesssim \int_{|z| \geq 10 r_Q} r_Q^{-n} \left(1 + \frac{|x - c(Q) - z|}{r_Q}\right)^{-n-1} |f(c(Q)+z)| dz$$
$$\lesssim \int_{|z| \geq 5 r_Q} r_Q^{-n} \left(1 + \frac{|z|}{r_Q}\right)^{-n-1} |f(x_0 + z)| dz$$
$$\lesssim \mathcal{M}(f)(x_0),$$

which concludes the proof. □

We obtain the following corollary:

Corollary 3.3. *Let Θ be a collection of N frequencies. For $p \in (2,\infty)$, $s \in [2,p)$ and $t \in (1,\infty)$, a multi-frequency Calderón-Zygmund operator T is bounded on $L^p(\omega)$ for every weight $\omega \in RH_{t'} \cap \mathbb{A}_{\frac{p}{s}}$ with*

$$\|T\|_{L^p(\omega) \to L^p(\omega)} \lesssim N^{\frac{tp}{2s} + \left(\frac{1}{2} - \frac{1}{s}\right)}.$$

Proof. Using Propositions 3.1 and 3.2, it follows that for $p > s \geq 2$ (assuming $\omega \in \mathbb{A}_{\frac{p}{s}}$)

$$\begin{aligned} \|T(f)\|_{L^p(\omega)} &\lesssim N^{\frac{tp}{2s}} \left\| \mathcal{M}^{\sharp}_{s,\Theta}[T(f)] \right\|_{L^p(\omega)} \\ &\lesssim N^{\frac{tp}{2s}+\left(\frac{1}{2}-\frac{1}{s}\right)} \|\mathcal{M}_s(f)\|_{L^p(\omega)} \\ &\lesssim N^{\frac{tp}{2s}+\left(\frac{1}{2}-\frac{1}{s}\right)} \|f\|_{L^p(\omega)}, \end{aligned}$$

where we used weighted boundedness of the maximal function since $\omega \in \mathbb{A}_{\frac{p}{s}}$. □

As explained in the introduction, this estimate is only interesting when the exponent $\frac{tp}{2s}+\left(\frac{1}{2}-\frac{1}{s}\right)$ is lower than 1.

4. Connexion to Bochner-Riesz multipliers

In this section, we aim to describe how such arguments could be applied to generalized Bochner-Riesz multipliers. Weighted estimates for Bochner-Riesz multipliers has been initiated in [15, 5, 4]. We first emphasize that we do not pretend to obtain new weighted estimates for Bochner-Riesz multipliers. But we only want to describe here a new point of view and a new approach for such estimates, which will be the subject of a future investigation. Such an application is a great motivation for pursuing the study of a multi-frequency Calderón-Zygmund analysis.

Consider also Ω a bounded open subset of $\mathbb{R}^n$ such that its boundary $\Gamma := \overline{\Omega} \setminus \Omega$ is an hyper-manifold of Hausdorff dimension $n-1$. For $\delta > 0$, we then define the generalized Bochner-Riesz multiplier, given by

$$R_{\Omega,\delta}(f)(x) := \int_{\Omega} e^{ix\cdot\xi} \widehat{f}(\xi) m_\delta d\xi,$$

where m_δ is a smooth symbol supported in $\overline{\Omega}$ and satisfying in Ω

$$|\partial^\alpha m_\delta(\xi)| \lesssim d(\xi,\Gamma)^{\delta-|\alpha|}.$$

We first use a Whitney covering $(O_i)_i$ of Ω. That is a collection of sub-balls such that

- the collection $(O_i)_i$ covers Ω and has a bounded overlap;
- the radius r_{O_i} is equivalent to $d(O_i,\Gamma)$.

Associated to this collection, we build a partition of the unity $(\chi_i)_i$ of smooth functions such that χ_i is supported on O_i with

$$\sum_i \chi_i(\xi) = \mathbf{1}_\Omega(\xi)$$

and $\|\partial^\alpha \chi_i\|_\infty \lesssim r_{O_i}^{-|\alpha|}$.
Then, R_δ may be written as

$$R_\delta(f)(x) = \sum_{j=-\infty}^{\infty} T_j(f)(x),$$

with

$$T_j(f)(x) := \sum_{\substack{l, \\ 2^j \leq r_{O_l} < 2^{j+1}}} \int_\Omega e^{ix\cdot\xi} \widehat{f}(\xi) m_\delta(\xi) \chi_l(\xi) d\xi$$

$$= 2^{j\delta} U_j(f)(x), \tag{12}$$

where we set

$$U_j(f)(x) := \sum_{\substack{l, \\ 2^j \leq r_{O_l} < 2^{j+1}}} \int_\Omega e^{ix\cdot\xi} \widehat{f}(\xi) (2^{-j\delta} m_\delta(\xi)) \chi_l(\xi) d\xi.$$

Observation : The main idea is to observe that the operator U_j is a multi-frequency Calderón-Zygmund operator associated to the collection

$$\Theta_j := \{c(O_l),\ 2^j \leq r_{O_l} < 2^{j+1}\} \qquad \text{with} \qquad \sharp\Theta_j \simeq 2^{-j(n-1)}.$$

However, these operators have specific properties, one of them is that the considered balls have equivalent radius, which means that these operators have only one scale 2^j. For example, this observation allows us to easily prove some boundedness:

Proposition 4.1. *Uniformly with $j \lesssim 0$, the multiplier U_j is a convolution operation with a kernel K_j satisfying*

$$\|K_j\|_{L^1} \lesssim 2^{-j\frac{n-1}{2}}.$$

Hence, it follows that U_j is bounded on Lebesgue space L^p for every $p \in [1, \infty]$. Moreover for every $s \in [1, 2]$, $p \in (s, \infty)$ and every weight $\omega \in \mathbb{A}_{\frac{p}{s}}$, U_j is bounded on $L^p(\omega)$ with

$$\|U_j\|_{L^p(\omega)\to L^p(\omega)} \lesssim 2^{-j\frac{n-1}{s}}.$$

Proof. The operator U_j is a Fourier multiplier, associated to the symbol

$$\sigma_j(\xi) := \sum_{\substack{l, \\ 2^j \leq r_{O_l} < 2^{j+1}}} (2^{-j\delta} m_\delta(\xi)) \chi_l(\xi).$$

Since the considered balls $(O_l)_l$ are almost disjoint, it comes that

$$\|\sigma_j\|_{L^2} \lesssim |\{\xi, d(\xi, \partial\Omega) \simeq 2^j\}|^{\frac{1}{2}} \lesssim 2^{\frac{j}{2}}.$$

Moreover, using regularity assumptions on m_δ, we deduce that for every α

$$\|\partial^\alpha \sigma_j\|_{L^2} \lesssim 2^{-j|\alpha|} |\{\xi, d(\xi, \partial\Omega) \simeq 2^j\}|^{\frac{1}{2}} \lesssim 2^{j(\frac{1}{2}-|\alpha|)}.$$

So with $K_j := \mathcal{F}(\sigma_j)$, it follows that for any integer M

$$\left\|(1 + 2^j|\cdot|)^M K_j\right\|_{L^2} \lesssim 2^{\frac{j}{2}}. \tag{13}$$

Hence

$$\|K_j\|_{L^1} \lesssim 2^{-j\frac{n-1}{2}}.$$

Using Minkowski inequality, we deduce that for every $p \in [1, \infty]$

$$\|U_j\|_{L^p \to L^p} \lesssim \|K_j\|_{L^1} \lesssim 2^{-j\frac{n-1}{2}}.$$

Let us now focus on the second claim about weighted estimates. Using integrations by parts for computing the kernel K_j, it comes for any integer M

$$\left\|(1+2^j|\cdot|)^M K_j\right\|_{L^\infty} \lesssim 2^j. \tag{14}$$

By interpolation with (13), for $s\in[1,2]$ we get

$$\left\|(1+2^j|\cdot|)^M K_j\right\|_{L^{s'}} \lesssim 2^{\frac{j}{s}}, \tag{15}$$

which gives

$$U_j(f) \lesssim 2^{-j\frac{n-1}{s}} \mathcal{M}_s(f).$$

Hence, for every $p>s$ and every weight $\omega \in \mathbb{A}_{\frac{p}{s}}$

$$\|U_j\|_{L^p(\omega)\to L^p(\omega)} \lesssim 2^{-j\frac{n-1}{s}}.$$

□

In this context, $\sharp\Theta_j \simeq 2^{-j(n-1)}$, so the constant $2^{-j\frac{n-1}{s}}$ is equivalent to $(\sharp\Theta_j)^{\frac{1}{s}}$ and this is a better constant than the one obtained in Corollary 3.3 (for a subclass of $\mathbb{A}_{\frac{p}{s}}$ weights).
So improving these "easy bounds" means to obtain inequalities such as

$$\|U_j\|_{L^p(\omega)\to L^p(\omega)} \lesssim (\sharp\Theta_j)^\gamma$$

for some better exponent $\gamma<\frac{1}{s}$.

Let us finish by suggesting how could we get improvements of our approach to get interesting results for Bochner-Riesz multipliers:

Question : The general approach, developed in the previous section, only allows to get an exponent

$$\gamma = \frac{tp}{2s} + \left(\frac{1}{2}-\frac{1}{s}\right)$$

(with some $s\in[2,p)$) which is bigger than $\frac{1}{2}$ (since $p>s\geq 2$ and $t>1$). So to improve this exponent γ, two things seem to be crucial:

- to extend the use of Lemma 1.1 for $p\geq 2$ which would allow us to get an exponent $\frac{tp}{s^2}$ instead of $\frac{tp}{2s}$;
- to use the geometry of the boundary Γ to get better exponents, even for the unweighted estimates. Indeed, for example for the unit ball (using its non vanishing curvature), we know that (see [9, 14])

$$\|U_j\|_{L^p\to L^p} \lesssim 2^{-j\delta(p)}$$

with if $n=2$

$$\delta(p) := \max\left\{2\left|\frac{1}{2}-\frac{1}{p}\right| - \frac{1}{2}, 0\right\}.$$

and if $n\geq 3$ and $p\geq \frac{2(n+2)}{n}$ or $p\leq \frac{2(n+2)}{n+4}$

$$\delta(p) := \max\left\{n\left|\frac{1}{2}-\frac{1}{p}\right| - \frac{1}{2}, 0\right\}.$$

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