On a functional equation appearing in characterization of distributions by the optimality of an estimate

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On a functional equation appearing in characterization of distributions by the optimality of an estimate[1]

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Let X be a second countable locally compact Abelian group containing no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Let μ be a probability distribution on X such that its characteristic function $\widehat{\mu}(y)$ does not vanish and $\widehat{\mu}(y)$ for some $n \geq 3$ satisfies the equation

$$\prod_{j=1}^{n} \hat{\mu}(y_j + y) = \prod_{j=1}^{n} \hat{\mu}(y_j - y), \quad \sum_{j=1}^{n} y_j = 0, \quad y_1, \dots, y_n, y \in Y.$$

Then μ is a convolution of a Gaussian distribution and a distribution supported in the subgroup of X generated by elements of order 2.

The present note is devoted to study a functional equation on a locally compact Abelian group group which appears in characterization of probability distributions by the optimality of an estimate.

Let X be a second countable locally compact Abelian group, $Y = X^*$ be its character group, (x, y) be the value of a character $y \in Y$ at an element $x \in X$. Denote by $\mathrm{M}^1(X)$ the convolution semigroup of probability distribution on the group X, and denote by

$$\widehat{\mu}(y) = \int_X (x, y) d\mu(x)$$

the characteristic function of a distribution $\mu \in \mathrm{M}^1(X)$. For $\mu \in \mathrm{M}^1(X)$ define the distribution $\bar{\mu} \in \mathrm{M}^1(X)$ by the formula $\bar{\mu}(B) = \mu(-B)$ for all Borel sets of X. Then $\widehat{\bar{\mu}}(y) = \overline{\widehat{\mu}(y)}$.

A distribution $\gamma \in \mathrm{M}^1(X)$ is called Gaussian if its characteristic function is represented in the form

$$\widehat{\gamma}(y) = (x, y) \exp\{-\varphi(y)\}, \tag{1}$$

[1]This research was supported in part by the grant "Network of Mathematical Research 2013–2015"

where $x \in X$, and $\varphi(y)$ is a continuous nonnegative function on the group Y satisfying the equation

$$\varphi(y_1 + y_2) + \varphi(y_1 - y_2) = 2[\varphi(y_1) + \varphi(y_2)], \quad y_1,\ y_2 \in Y. \tag{2}$$

A Gaussian distribution is called symmetric if in (1) $x = 0$. Denote by $\Gamma(X)$ the set of Gaussian distributions on the group X.

Consider a probability space $(X, \mathcal{B}, \mu)$, where $\mathcal{B}$ is a σ-algebra of Borel subsets of X, and $\mu \in \mathrm{M}^1(X)$. Form a family of distributions $\mu_\theta(A) = \mu(A - \theta),\ A \in \mathcal{B},\ \theta \in X$. Denote by Π a class of estimates $f : X^n \mapsto X$ satisfying the condition $f(x_1 + c, \ldots, x_n + c) = f(x_1, \ldots, x_n) + nc$ for all $x_1, \ldots, x_n, c \in X$. According to [1] (see also [2], [3, Ch. 7, §7.10]), an estimate $f_0 \in \Pi$ of a parameter $n\theta$ is called an optimal estimate in the class Π for a sample volume n if for any estimate $f \in \Pi$ and for all $y \in Y$ the inequality

$$\mathbf{E}_\theta |(f_0(\mathbf{x}), y) - (n\theta, y)|^2 \leq \mathbf{E}_\theta |(f(\mathbf{x}), y) - (n\theta, y)|^2$$

holds. It turns out that the existence of an optimal estimate of the papameter $n\theta$ gives the possibility in some cases to describe completely the possible distributions μ.

As has been proved in ([1]), if an estimate f_0 is represented in the form

$$(f_0(\mathbf{x}), y) = (f(\mathbf{x}), y) \frac{\mathbf{E}_0[(f(\mathbf{x}), -y)|\mathbf{z}]}{|\mathbf{E}_0[(f(\mathbf{x}), -y)|\mathbf{z}]|}, \quad y \in Y, \tag{3}$$

where $f \in \Pi$, and $\mathbf{z} = (x_2 - x_1, \ldots, x_n - x_1)$, then $f_0 \in \Pi$, f_0 does not depend on the choice of f and f_0 is an optimal estimate. It follows from (3) that f_0 is an optimal estimate if and only if $\arg \mathbf{E}_0[(f_0(\mathbf{x}), y)|\mathbf{z}] = 0$. When $f_0(\mathbf{x}) = \sum_{j=1}^n x_j$ it follows from this that the characteristic function $\widehat{\mu}(y)$ satisfies the equation

$$\prod_{j=1}^n \hat{\mu}(y_j + y) = \prod_{j=1}^n \hat{\mu}(y_j - y), \quad \sum_{j=1}^n y_j = 0, \quad y_1, \ldots, y_n, y \in Y, \tag{4}$$

and $\widehat{\mu}^n(y) > 0$. When $n \geq 3$ this implies that if a group X contains no elements of order 2, then $\mu \in \Gamma(X)$ (see [1]).

This note is devoted to solving of equation (4) in a general case when X is a locally compact Abelian group. Let us fix the notation. Denote by $f_2 : X \mapsto X$ the endomorphism of X defined by the formula $f_2(x) = 2x$. Put $X_{(2)} = \mathrm{Ker}\ f_2$, $X^{(2)} = \mathrm{Im}\ f_2$. Denote by $\mathbb{T}$ the circle group, and by $\mathbb{Z}$ the group of integers.

Let $\psi(y)$ be an arbitrary function on the group Y and $h \in Y$. Denote by Δ_h the finite difference operator

$$\Delta_h \psi(y) = \psi(y + h) - \psi(y), \quad y \in Y.$$

A continuous function $\psi(y)$ on the group Y is called a polynomial if

$$\Delta_h^{m+1} \psi(y) = 0 \tag{5}$$

for some m and for all $y, h \in Y$. The minimal m for which (5) holds is called the degree of the polynomial $\psi(y)$.

From analytical point of view the result proved in [1] can be reformulated in the following way. Let $\mu \in \mathrm{M}^1(X)$, the characteristic function $\widehat{\mu}(y)$ satisfy equation (4) for some $n \geq 3$ and $\widehat{\mu}^n(y) > 0$. Then if the group X contains no elements of order 2, then $\mu \in \Gamma(X)$.

It is easy to see that if γ is a symmetric Gaussian distribution on the group X and $\pi \in \mathrm{M}^1(X_{(2)})$, then the characteristic functions $\widehat{\gamma}(y)$ and $\widehat{\pi}(y)$ satisfy equation (4), and hence the characteristic function of the distribution $\mu = \gamma * \pi$ also satisfies equation (4). Describe first the groups X for which the converse statement is true.

Theorem 1. *Let X be a second countable locally compact Abelian group, $\mu \in \mathrm{M}^1(X)$. Let the characteristic function $\widehat{\mu}(y)$ satisfy equation* (4) *for some $n \geq 3$ and $\widehat{\mu}(y) \neq 0$. Assume that the following condition holds:* (*i*) *the group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Then $\mu = \gamma * \pi$, where $\gamma \in \Gamma(X)$ and $\pi \in \mathrm{M}^1(X_{(2)})$.*

Proof. Set $\nu = \mu * \bar{\mu}$. Then $\widehat{\nu}(y) = |\widehat{\mu}(y)|^2 > 0$. Put $\psi(y) = -\ln \widehat{\nu}(y)$. Equation (4) is equivalent to the equation

$$\sum_{j=1}^{n} \psi(y_j + y) = \sum_{j=1}^{n} \psi(y_j - y), \quad \sum_{j=1}^{n} y_j = 0, \quad y_1, \dots, y_n, y \in Y. \tag{6}$$

We also note that

$$\psi(-y) = \psi(y), \quad y \in Y. \tag{7}$$

Substituting in (6) $y_3 = -y_1 - y_2, \quad y_4 = \dots = y_n = 0$ and taking into account (7), we get

$$\psi(y_1 + y_2 + y) - \psi(y_1 + y_2 - y) = \psi(y_1 + y) - \psi(y_1 - y)$$

$$+\psi(y_2 + y) - \psi(y_2 - y), \quad y_1, y_2, \ y \in Y. \tag{8}$$

Setting successively $y = y_1 + y_2$, $y = y_1$, $y = y_2$, we find from (8) that

$$\psi(2y_1 + 2y_2) = \psi(2y_1) + 2\psi(y_1 + y_2) - 2\psi(y_1 - y_2) + \psi(2y_2), \quad y_1, y_2 \in Y.$$

This implies that

$$\psi(2y_1 + 2y_2) + \psi(2y_1 - 2y_2) = 2[\psi(2y_1) + \psi(2y_2)], \quad y_1, y_2 \in Y,$$

i.e. the function $\psi(y)$ satisfies equation (2) on the subgroup $Y^{(2)}$, and hence, the function $\psi(y)$ satisfies equation (2) on the subgroup $\overline{Y^{(2)}}$. Denote by $\varphi_0(y)$ the restriction of the function $\psi(y)$ to the subgroup $\overline{Y^{(2)}}$.

It is well known that we can associate to each function $\varphi(y)$ satisfying equation (2) a symmetric 2-additive function

$$\Phi(u, v) = \frac{1}{2}[\varphi(u + v) - \varphi(u) - \varphi(v)], \quad u, v \in Y.$$

Then $\varphi(y) = \Phi(y, y)$. Using this representation it is not difficult to verify that the function $\psi(y)$ on the subgroup $\overline{Y^{(2)}}$ satisfies the equation

$$\Delta_k^2 \Delta_{2h} \, \psi(y) = 0, \quad k, \ y \in \overline{Y^{(2)}}, \quad h \in Y. \tag{9}$$

Return to equation (8) and apply the finite difference method to solve it. Let h_1 be an arbitrary element of the group Y. Put $k_1 = h_1$. Substitute $y_2 + h_1$ for y_2 and $y + k_1$ for y in equation (8). Subtracting equation (8) from the resulting equation we obtain

$$\Delta_{2h_1}\ \psi(y_1 + y_2 + y) = \Delta_{h_1}\ \psi(y_1 + y) - \Delta_{-h_1}\ \psi(y_1 - y) + \Delta_{2h_1}\ \psi(y_2 + y). \qquad (10)$$

Next, let h_2 be an arbitrary element of the group Y. Put $k_2 = -h_2$. Substitute $y_2 + h_2$ for y_2 and $y + k_2$ for y in equation (10). Subtracting equation (10) from the resulting equation we get

$$\Delta_{-h_2}\Delta_{h_1}\ \psi(y_1 + y) - \Delta_{h_2}\Delta_{-h_1}\ \psi(y_1 - y) = 0.$$

Reasoning similarly we find from this

$$\Delta_{2h_3}\Delta_{-h_2}\Delta_{h_1}\ \psi(y_1 + y) = 0,$$

and finally

$$\Delta_{2h_3}\Delta_{-h_2}\Delta_{h_1}\Delta_{y_1}\ \psi(y) = 0. \qquad (11)$$

Note that h_j, y_1, y are arbitrary elements of the group Y. Setting in (11) $h_3 = h$, $-h_2 = h_1 = y_1 = k$, we find

$$\Delta_k^3\Delta_{2h}\ \psi(y) = 0, \quad k,\ h,\ y \in Y. \qquad (12)$$

Fix $h \in Y$. On the one hand, it follows from (12) that the function $\Delta_{2h}\psi(y)$ is a polynomial of degree ≤ 2 on the group Y. On the other hand, as follows from (9) the function $\Delta_{2h}\psi(y)$ is a polynomial of degree ≤ 1 on the subgroup $\overline{Y^{(2)}}$. Then as not difficult to verify, the function $\Delta_{2h}\psi(y)$ must be a polynomial of degree ≤ 1 on the group Y, i.e.

$$\Delta_k^2\Delta_{2h}\ \psi(y) = 0, \quad k,\ h,\ y \in Y. \qquad (13)$$

Theorem 1 follows now from the following lemma.

Lemma 1 ([5, Prop. 1]). *Let X be a second countable locally compact Abelian group containing no subgroup topologically isopmorphic to the circle group $\mathbb{T}$. Let $\mu \in \mathrm{M}^1(X)$, $\nu = \mu * \bar{\mu}$ and*

$$\hat{\nu}(y) = \exp\{-\psi(y)\},$$

where the function $\psi(y)$ satisfies equation (13). *Then $\mu = \gamma * \pi$, where $\gamma \in \Gamma(X)$ and $\pi \in \mathrm{M}^1(X_{(2)})$.*

Remark 1. Obviously, the above mentioned Rukhin's theorem follows directly from Theorem 1.

Remark 2. Let X be a second countable locally compact Abelian group containing a subgroup topologically isomorphic to the circle group $\mathbb{T}$. Then we can consider any distribution μ on the circle group $\mathbb{T}$ as a distribution on X. Note that $\mathbb{Z}$ is the character group of $\mathbb{T}$. Following to [1] consider on the group $\mathbb{Z}$ the function

$$f(m) = \begin{cases} \exp\{-m^2\}, & \text{if} \quad m \in \mathbb{Z}^{(2)}, \\ \exp\{-m^2 + \varepsilon\}, & \text{if} \quad m \notin \mathbb{Z}^{(2)}, \end{cases}$$

where $\varepsilon > 0$ is small enough. Then

$$\rho(t) = \sum_{m=-\infty}^{\infty} f(m)e^{-imt} > 0.$$

Let μ be a distribution on $\mathbb{T}$ with density $\rho(t)$ with respect to the Lebesque measure. Then $f(m)$ is the characteristic function of a distribution μ on the circle group $\mathbb{T}$. Considering μ as a distribution on the group X, we see that $\widehat{\mu}(y) > 0$ and the characteristic function $\widehat{\mu}(y)$ satisfies equation (4), but as easily seen, $\mu \notin \Gamma(X) * \mathrm{M}^1(X_{(2)})$. This example shows that condition (i) in Theorem 1 is sharp.

Remark 3. Let X be a second countable locally compact Abelian group. In the articles [4] and [5] (see also [5, §16]) were studied group analogs of the well-known Heyde theorem, where a Gaussian distribution is characterized by the symmetry of the conditional distribution of a linear form $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ of independent random variables ξ_j given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ (coefficients of the forms are topological automorphisms of the group X). Let $\widehat{\mu}_j(y)$ be the characteristic function of the random variable ξ_j. It is interesting to remark that if the number of independent random variables $n = 2$, then the functions $\psi_j(y) = -\ln|\widehat{\mu}_j(y)|^2$ also satisfy equation (13). For the groups X containing no subgroup topologically isomorphic to the circle group $\mathbb{T}$, and also for the two-dimensional torus $X = \mathbb{T}^2$ this implies that all $\mu_j \in \Gamma(X) * \mathrm{M}^1(X_{(2)})$.

We use Theorem 1 to prove the following statement, a significant part of which refers to the case when the group X contains a subgroup topologically isomorphic to the circle group $\mathbb{T}$.

Theorem 2. *Let X be a second countable locally compact Abelian group. Let $\mu \in \mathrm{M}^1(X)$, let the characteristic function $\widehat{\mu}(y)$ satify equation* (4) *for some odd n, and $\widehat{\mu}^n(y) > 0$. Assume that the group X satisfies the condition:* (i) *the subgroup $X_{(2)}$ is finite. Then $\mu = \gamma_0 * \pi$, where $\gamma_0 \in \Gamma(X)$, and π is a signed measure on $X_{(2)}$.*

Proof. Put $\psi(y) = -\ln|\widehat{\mu}(y)|$. Then the function $\psi(y)$ satisfies equation (13). As has been proved in [5] in this case the function $\psi(y)$ is represented in the form

$$\psi(y) = \varphi(y) + r_\alpha, \quad y \in y_\alpha + \overline{Y^{(2)}},$$

where $\varphi(y)$ is a continuous function satisfying equation (2), and $Y = \bigcup_\alpha (y_\alpha + \overline{Y^{(2)}})$ is a decomposition of the group Y with respect to the subgroup $\overline{Y^{(2)}}$. Since $X_{(2)}$ is a finite subgroup, it is easy to see that the function $g(y) = \exp\{-r_\alpha\}, \quad y \in y_\alpha + \overline{Y^{(2)}}$, is the chracteristic function of a signed measure π on the subgroup $X_{(2)}$. It follows from this that

$$|\hat{\mu}(y)| = \hat{\gamma}(y)\hat{\pi}(y),$$

where $\gamma \in \Gamma(X)$ and $\widehat{\gamma}(y) = \exp\{-\varphi(y)\}$.

Set $l(y) = \widehat{\mu}(y)/|\hat{\mu}(y)|$ and check that the function $l(y)$ is a character of the group Y. Hence, Theorem 2 will be proved.

Note that the function $l(y)$ satisfies equation (4) and

$$l(-y) = \overline{l(y)}, \quad l^n(y) = 1, \quad y \in Y. \tag{14}$$

Put in (4) $y_2 = -y_1$, $y_3 = \cdots = y_n = 0$. We get

$$l^{n-2}(y)l(y_1 + y)l(-y_1 + y) = l^{n-2}(-y)l(y_1 - y)l(-y_1 - y), \quad y,\ y_1,\ y_2 \in Y.$$

Taking into account (14), it follows from this that

$$l^2(y + y_1)l^2(y - y_1) = l^4(y), \quad y,\ y_1 \in Y.$$

Set $m(y) = l^2(y)$. Then the function $m(y)$ satisfies the equation

$$m(u + v)m(u - v) = m^2(u), \quad u,\ v \in Y. \tag{15}$$

We find by induction from (15) that

$$m(py) = m^p(y), \quad p \in \mathbb{Z},\ y \in Y. \tag{16}$$

Now we formulate as a lemma the following statement.

Lemma 2. *Let $Y = Y_1 + Y_2$, let a continuous function $m(y)$ on Y satisfy equation* (15) *and $m^n(y) = 1$ for some odd n. Then, if the restriction of the function $m(y)$ to Y_j is a character of the group Y_j, $j = 1, 2$, then $m(y)$ is a character of the group Y.*

Proof. Denote by $y = (y_1, y_2)$, $y_1 \in Y_1, y_2 \in Y_2$ elements of the group Y. Put $a(y_1, y_2) = m(y_1, 0)m(0, y_2)$, $b(y_1, y_2) = m(y_1, y_2)/a(y_1, y_2)$. Then $b(y_1, 0) = b(0, y_2) = 1$, $y_1 \in Y_1, y_2 \in Y_2$. It is obvious that the function $b(y_1, y_2)$ also satisfies equation (15). Substitute in (15) $u = (y_1, 0)$, $v = (y_1, y_2)$. We have

$$b(2y_1, y_2)b(0, -y_2) = b^2(y_1, 0), \quad y_1 \in Y_1, y_2 \in Y_2.$$

This implies that $b(2y_1, y_2) = 1$ for $y_1 \in Y_1, y_2 \in Y_2$. In particular, $b(2y_1, 2y_2) = 1$. But it follows from (16) that $b(2y_1, 2y_2) = b^2(y_1, y_2)$. Hence, $b(y_1, y_2) = \pm 1$. Since $b^n(y_1, y_2) = 1$ and n is odd, we have $b(y_1, y_2) = 1$ for $y_1 \in Y_1, y_2 \in Y_2$, i.e. $m(y_1, y_2) = a(y_1, y_2)$ is a character of the group Y.

Continue the proof of Theorem 2. Since, by the assumption, $X_{(2)}$ is a finite subgroup, there exist $q \geq 0$ such that the group X contains a subgroup topologically isomorphic to the group $\mathbb{T}^q$, but X does not contain a subgroup topologically isomorphic to the group $\mathbb{T}^{q+1}$. It is well known that a subgroup of X topologically isomorphic to a group of the form $\mathbb{T}^k$ is a topologically direct summand in X. For this reason the group X is represented in the form $X = \mathbb{T}^q + G$, where the group G contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. We have $Y \cong \mathbb{Z}^q + H$, $H = G^*$. It follows from Lemma 2 and (16) by induction that the function $m(y)$ on the group $\mathbb{Z}^q$, satisfying equation (15) and the condition $m^n(y) = 1$ is a character of the group $\mathbb{Z}^q$. By Theorem 1 the restriction of the function $\widehat{\mu}(y)$ to H is a product of the characteristic function of a Gaussian distribution on the group G and the characteristic function of a distribution on the subgroup $G_{(2)}$. Taking into account that the characteristic function of any distribution on $G_{(2)}$ takes only real values, it follows from the equality

$$\widehat{\mu}^2(y) = |\widehat{\mu}(y)|^2 m(y), \quad y \in Y,$$

that the restriction of the function $m(y)$ to H is a character of the subgroup H. Applying again Lemma 2 to the group Y, we obtain that $m(y)$ is a character of the group Y.

Since n is odd, we have $2r + ns = 1$ for some integers r and s. Taking into account (14) this implies that $l(y) = (l(y))^{2r+ns} = (m(y))^r$ is a character of the group Y. Theorem 2 is completely proved.

We note that the example given in Remark 2 shows that a signed measure π needs not be a measure.

Remark 4. Consider the infinite-dimensional torus $X = \mathbb{T}^{\aleph_0}$. Then $Y \cong \mathbb{Z}^{\aleph_0 *}$, where $\mathbb{Z}^{\aleph_0 *}$ is the group of all sequences of integers such that in each sequence only finite number of members are not equal to zero.

Consider on the group $\mathbb{Z}$ the sequence of the functions

$$f_k(m) = \begin{cases} \exp\{-a_k m^2\}, & \text{if} \quad m \in \mathbb{Z}^{(2)}, \\ \exp\{-a_k m^2 + k\}, & \text{if} \quad m \notin \mathbb{Z}^{(2)}, \end{cases}$$

where $k = 1, 2, \dots$. Put

$$f(m_1, \dots, m_l, 0, \dots) = \prod_{k=1}^{l} f_k(m_k), \quad (m_1, \dots, m_l, 0, \dots) \in \mathbb{Z}^{\aleph_0 *}.$$

Take $a_k > 0$ such that

$$\sum_{(m_1, \dots, m_l, 0, \dots) \in \mathbb{Z}^{\aleph_0 *}} f(m_1, \dots, m_l, 0, \dots) < 2.$$

Then

$$\rho(t_1, \dots, t_l, \dots) = \sum_{(m_1, \dots, m_l, 0, \dots) \in \mathbb{Z}^{\aleph_0 *}} f(m_1, \dots, m_l, 0, \dots) e^{-i(m_1 t_1 + \dots + m_l t_l + \dots)} > 0, \quad t_j \in \mathbb{R}.$$

It follows from this that $f(m_1, \dots, m_l, 0, \dots)$ is the characteristic function of a distribution $\mu \in \mathrm{M}^1(\mathbb{T}^{\aleph_0})$ such that $\widehat{\mu}(y) > 0$ and $\widehat{\mu}(y)$ satisfies equation (4), but μ can not be represented as a convolution $\mu = \gamma * \pi$, where $\gamma \in \Gamma(\mathbb{T}^{\aleph_0})$, and π a signed measure on the group $\mathbb{T}^{\aleph_0}_{(2)}$. The subgroup $\mathbb{T}^{\aleph_0}_{(2)}$ is infinite in this case. This example shows that condition (i) in Theorem 2 is sharp.

Remark 5. We assumed in Theorem 2 that n is odd. This condition can not be omitted even for the circle group $X = \mathbb{T}$. Indeed, let $n = 4$. Take a in such a way that the function

$$f(m) = \exp\{-am^2 + i\frac{\pi}{2}m^3\}, \quad m \in \mathbb{Z}$$

be the characteristic function of a distribution $\mu \in \mathrm{M}^1(\mathbb{T})$. On the one hand, it is ibvious that the function $f(m)$ satisfies equation (4) and $f^4(m) > 0$, $m \in \mathbb{Z}$. On the other hand, the distribution μ can not be represented in the form $\mu = \gamma * \pi$, where $\gamma \in \Gamma(\mathbb{T})$, and π is a signed measure on $\mathbb{T}_{(2)}$.

This example also shows that a function $f(y)$ satisfying equation (4), generally speaking, needs not be real.

References

[1] *A.L. Rukhin.* Certain statistical and probability problems on groups. Proceedings of the Steklov Institute of Mathematics, 1970, Vol. 111, p. 52–109.

[2] *A.L. Rukhin.* Strongly symmetric families. Dokl. Akad. Nauk SSSR, 1970, Vol. 190, No 2, p. 280–282.

[3] *A. M. Kagan, Yu. V. Linnik, C.R. Rao.* Characterization problems in mathematical statistics, Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, New York-London-Sydney, 1973, 656 p.

[4] *G. M. Feldman.* On a characterization theorem for locally compact abelian groups. Probab. Theory Relat. Fields, 2005, Vol. 133, p. 345–357.

[5] *G. M. Feldman.* The Heyde theorem for locally compact Abelian groups. J. of Functional Analysis, 2010, Vol. 258, No 12, p. 3977–3987.

[6] *G. M. Feldman.* Functional equations and characterization problems on locally compact Abelian groups. EMS Tracts in Mathematics, Vol. 5, Zurich. European Mathematical Society (EMS), 2008, 268 p.