Graph-Based Trust Model for Evaluating Trust Using Subjective Logic

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Abstract. Before using a digital system, it is necessary to evaluate it according to different parameters. Lately *trust* emerged as a momentous aspect of evaluation. Evaluating trust in a system is a complex issue that becomes more challenging when systems use distributed architectures. In a previous work, we proposed SOCIOTRUST, a trust model that is based on probability theory to evaluate trust in a system for an activity. In SOCIOTRUST, trust values are considered as the probability, by which a trustor believes that a trustee behaves as expected. A limitation of using traditional probability is that users cannot express their uncertainties about some actors of their activity. In real situations, not everyone is in possession of all the necessary information to provide a dogmatic opinion about something or someone. Subjective logic thus emerged to facilitate the expression of trust as a subjective opinion with degrees of uncertainty. In this paper, we propose SUBJECTIVETRUST, a graph-based trust model to evaluate trust in a system for an activity using subjective logic. The distinctive features of our proposal are *(i)* user's uncertainties are taken into account in trust evaluation and *(ii)* besides taking into account the trust in the different entities the user depends on to perform an activity, it takes into consideration the architecture of the system to determine its trust level.

1 Introduction

When users need to choose a system to perform an activity, they are faced with a lot of available options. To choose a system, they evaluate it considering many criteria: functionality, ease of use, QoS, economical aspects, *etc*. Trust is also a key factor of choice [14,18]. However, evaluating this trustworthiness is a challenging issue due to the system complexity. We argue that studying trust in the separate entities that compose a system does not give a picture of how trustworthy a system is as a whole. Indeed, the trust in a system depends on its architecture, more precisely, on the way the entities which the users depends on to do their activities, are organized.

Trust has been widely studied in several aspects of daily life [4,5,6,20,22]. In the trust management community, graph-based trust [7,8,12,13,15,16,17] is a way to derive trust that has been used a lot recently. The main idea in graph-based trust is to estimate two levels of granularity so-called *trust in a path* and *trust in a graph or a target node in the graph* [1].

SOCIOTRUST [3] is a graph-based trust model, based on probability theory, to evaluate trust in a system for an activity. In SOCIOTRUST, trust values are considered as the probability by which a trustor believes that a trustee behaves as expected. A limitation of SocioTrust is that users cannot express their uncertainties about a proposition. In real situations, no one is in possession of all the necessary information to provide a dogmatic opinion.

Subjective logic [11] is suitable for dealing with trust because trust can be expressed as subjective opinions with degrees of uncertainty. In this study, we extend SocioTrust to use subjective logic. The main contribution of this paper is proposing a generic method, named SubjectiveTrust, for evaluating trust in a system for an activity. The system definition is based on SocioPath [2] which allows to model the architecture of a system by taking into account entities of the social and the digital world involved in an activity. To focus on the trust in the system, the SocioPath model is abstracted in a graph-based view. Levels of trust are then defined for each node in the graph. By combining trust values, we are able to estimate two different granularities of trust, namely, *trust in a path* and *trust in a system*, both for an activity to be performed by a person.

This paper is organized as follows. Section 2 gives a quick overview of subjective logic and SocioPath. In Section 3, we propose SubjectiveTrust for evaluating trust in a system for an activity using subjective logic. Section 4 presents the experiments that validate the proposed approach. Section 5 presents some related works before concluding in Section 6.

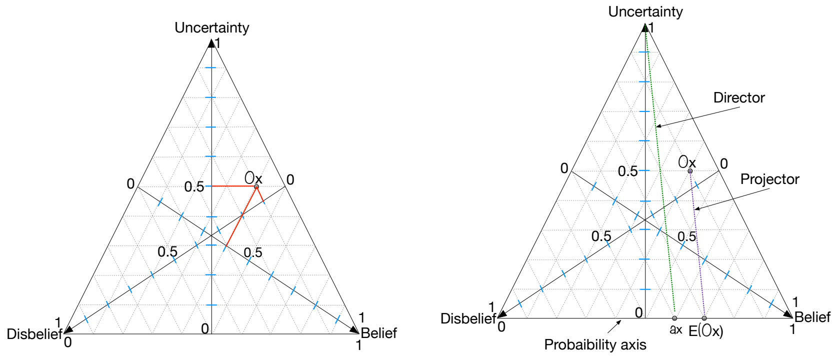
2 Background and preliminaries

2.1 Overview of subjective logic

A lot of trust metrics has been proposed to evaluate trust like binary [9], simple [7] or probabilistic metrics [3]. In previous metrics, a given person can not express her ignorance or her degree of uncertainty. In another words, she cannot say "I do not know" or "I am not sure". In real world situations, no one can determine an absolute certainty about a proposition. This philosophical idea leads researchers to look for a mathematical formalism that can express the uncertainty.

Subjective logic [11], which is an extension of classical probability, is a good candidate to solve this problem. Subjective logic is a probabilistic logic that uses opinions as input and output variables. Opinions explicitly express uncertainty about probability values, and can express degrees of ignorance about a subject matter such as trust. In the terminology of subjective logic, an opinion held by an individual P about a proposition x is the ordered quadruple $O_x = (b_x, d_x, u_x, a_x)$ where b_x (belief) is the belief that the x is true, d_x (disbelief) is the belief that the x is false, and u_x (uncertainty) is the amount of uncommitted belief, $b_x, d_x, u_x \in [0..1]$ and $b_x + d_x + u_x = 1$. The last value $a_x \in [0..1]$ is called the base rate, it is the *priori* probability in the absence of evidence and is used for computing an opinion's probability expectation value that can be determined as $E(O_x) = b_x + a_x u_x$. More precisely, a_x determines how uncertainty shall contribute to the probability expectation value $E(O_x)$. The latter can be interpreted as a probability measure indicating how x is expected to behave in the future.

An opinion O_x can be defined as a point in the triangle shown in Figure 1(a). The belief axis b_x, the disbelief axis d_x and the uncertainty axis u_x run from the middle



(a) (b)

Fig. 1. Opinion triangle [11]

point of one edge to the opposite corner. In Figure 1(b), the horizontal bottom line between the belief and disbelief corners represents opinion's probability expectation $E(O_x)$. The base rate is represented as a point on the probability axis. The line joining the top corner of the triangle and the base rate point is called the director. The value $E(O_x)$ is formed by projecting the opinion point onto the probability axis in parallel to the base rate director line. For instance, in Figure 1, the point O_x represents the opinion $O_x = (0.4, 0.1, 0.5, 0.6)$ and $E(O_x) = 0.7$.

In subjective logic, a mapping between the opinion parameters and the number of positive and negative observations is provided [11,12,13]. Let r and s express the number of positive and negative past observations about x respectively, then an opinion about x can be determined as:

$$\begin{cases} b_x = \frac{r}{r+s+2} \\ d_x = \frac{s}{r+s+2} \\ u_x = \frac{2}{r+s+2} \end{cases} \iff \begin{cases} r = \frac{2b_x}{u_x} \\ s = \frac{2d_x}{u_x} \\ 1 = b_x + d_x + u_x \end{cases} \quad (1)$$

Subjective Logic is directly compatible with traditional mathematical frameworks as we show in the following:

- if $b = 1$ is equivalent to binary logic TRUE,
- if $d = 1$ is equivalent to binary logic FALSE,
- if $b + d = 1$ is equivalent to a traditional probability,

if $b + d < 1$ expresses degrees of uncertainty, $b + d = 0$ expresses total uncertainty.

2.2 Operators in subjective logic

In subjective logic, a set of standard logical operations like the conjunction and disjunction, and non-standard logical operations like the consensus and the discounting are defined to combine the opinions. In this section, we present the most important operators in subjective logic.

– Conjunction operator: the conjunction operator represents the opinion of a person toward several propositions. Let $O_x^P = (b_x^P, d_x^P, u_x^P, a_x^P)$ be P's opinion about x and $O_y^P = (b_y^P, d_y^P, u_y^P, a_y^P)$ be P's opinion about y, $O_{x\wedge y}^P$ represents P's opinion about both x and y and can be calculated with the following relations:

$$O_x^P \wedge O_y^P = O_{x\wedge y}^P = \begin{cases} b_{x\wedge y}^P = b_x^P b_y^P \\ d_{x\wedge y}^P = d_x^P + d_y^P - d_x^P d_y^P \\ u_{x\wedge y}^P = b_x^P u_y^P + u_x^P b_y^P + u_x^P u_y^P \\ a_{x\wedge y}^P = \frac{b_x^P u_y^P a_y^P + b_y^P u_x^P a_x^P + u_x^P a_x^P u_y^P a_y^P}{b_x^P u_y^P + u_x^P b_y^P + u_x^P u_y^P} \end{cases} \quad (2)$$

$$E(O_x^P \wedge O_y^P) = E(O_{x\wedge y}^P) = E(O_x^P)E(O_y^P) \quad (3)$$

– Disjunction operator: the disjunction operator represents the opinion of a person toward one of the propositions or any union of them. Let $O_x^P = (b_x^P, d_x^P, u_x^P, a_x^P)$ be P's opinion about x and $O_y^P = (b_y^P, d_y^P, u_y^P, a_y^P)$ be P's opinion about y, $O_{x\vee y}^P$ represents P's opinion about x or y or both and can be calculated with the following relations:

$$O_x^P \vee O_y^P = O_{x\vee y}^P = \begin{cases} b_{x\vee y}^P = b_x^P + b_y^P - b_x^P b_y^P \\ d_{x\vee y}^P = d_x^P d_y^P \\ u_{x\vee y}^P = d_x^P u_y^P + u_x^P d_y^P + u_x^P u_y^P \\ a_{x\vee y}^P = \frac{u_x^P a_x^P + u_y^P a_y^P - b_x^P u_y^P a_y^P - b_y^P u_x^P a_x^P - u_x^P a_x^P u_y^P a_y^P}{u_x^P + u_y^P - b_x^P u_y^P - b_y^P u_x^P - u_x^P u_y^P} \end{cases} \quad (4)$$

$$E(O_x^P \vee O_y^P) = E(O_{x\vee y}^P) = E(O_x^P) + E(O_y^P) - E(O_x^P)E(O_y^P) \quad (5)$$

– Discounting operator: the discounting operator represents the transitivity of the opinions. Let $O_B^P = (b_B^P, d_B^P, u_B^P, a_B^P)$ be P's opinion about B's advice, and $O_x^B = (b_x^B, d_x^B, u_x^B, a_x^B)$ be B's opinion about x, O_x^{PB}=$O_B^P \otimes O_x^B$ represents P's opinion about x as a result of B's advice to P:

$$O_x^{PB} = O_B^P \otimes O_x^B = \begin{cases} b_x^{PB} = b_B^P b_x^B \\ d_x^{PB} = b_B^P d_x^B \\ u_x^{PB} = d_B^P + u_B^P + b_B^P u_x^B \\ a_x^{PB} = a_x^B \end{cases} \quad (6)$$

– Consensus operator: the consensus operator represents the consensus of the opinions of different persons. Let $O_x^A = (b_x^A, d_x^A, u_x^A, a_x^A)$ be A's opinion about x, and $O_x^B = (b_x^B, d_x^B, u_x^B, a_x^B)$ be B's opinion about x, $O_x^{A,B}$=$O_x^A \oplus O_x^B$ represents the opinion of an imaginary person $[A, B]$ about x .

$$O_x^{A,B} = O_x^A \oplus O_x^B = \begin{cases} b_x^{A,B} = \frac{b_x^A u_x^B + b_x^B u_x^A}{u_x^A + u_x^B - u_x^A u_x^B} \\ d_x^{A,B} = \frac{d_x^A u_x^B + d_x^B u_x^A}{u_x^A + u_x^B - u_x^A u_x^B} \\ u_x^{A,B} = \frac{u_x^A u_x^B}{u_x^A + u_x^B - u_x^A u_x^B} \\ a_x^{A,B} = \frac{u_x^A a_x^B + u_x^B a_x^A - (a_x^A + a_x^B) u_x^A u_x^B}{u_x^A + u_x^B - 2u_x^A u_x^B} \end{cases} \quad (7)$$

It is important to mention that conjunction and disjunction are commutative and associative.

$$O_x^P \wedge O_y^P = O_y^P \wedge O_x^P$$
$$O_x^P \vee O_y^P = O_y^P \vee O_x^P$$
$$(O_x^P \wedge O_y^P) \wedge O_z^P = O_x^P \wedge (O_y^P \wedge O_z^P)$$
$$(O_x^P \vee O_y^P) \vee O_z^P = O_x^P \vee (O_y^P \vee O_z^P)$$

But the conjunction over the disjunction is not distributive. This is due to the fact that opinions must be assumed to be independent, whereas distribution always introduces an element of dependence.

$$O_x^P \wedge (O_y^P \vee O_z^P) \neq (O_x^P \wedge O_y^P) \vee (O_x^P \wedge O_z^P)$$

For the same reason, the discounting over the consensus is not distributive.

$$O_x^P \otimes (O_y^P \oplus O_z^P) \neq (O_x^P \otimes O_y^P) \oplus (O_x^P \otimes O_z^P)$$

Let us now present an overview of SOCIOPATH.

2.3 Overview of SOCIOPATH

The SOCIOPATH meta-model [2] allows to describe a system in terms of the entities that exist in *(i)* the *social world*[1], where *persons* own *physical resources* and *data*, and in *(ii)* the *digital world*, where *instances of data* (including application programs) are stored and *artifacts* (software) are running. SOCIOPATH also allows to describe the relations between the different entities of the two worlds. Figure 2 shows a graphical representation of SOCIOPATH. Enriched with deduction rules, the SOCIOPATH meta-model allows to underline and discover chains of *access* relations between *artifacts*, and *control* relations between *persons* and *digital resources* in a system. The main concepts defined in SOCIOPATH are:

- *minimal path* ($\widehat{\sigma}$); a list that begins from an *actor*, ends with a *data instance* and contains *artifacts* in between. Between each two consecutive elements in this list, there is a relation *access*. A *minimal path* describes a straight way an *actor* achieves an *activity* without passing by cycles.
- *activity* (ω); a task like editing a document by a user, where some restrictions are considered to impose the presence of particular elements in the path. For instance, if a user wants to read a `.doc` document, she must use an *artifact* that can *understand* this type of document (*e.g.,* Microsoft Word or LibreOffice Writer).

Each *artifact* in a path is controlled by at least one *person* and supported by at least one *physical resource*. In SOCIOPATH, the persons who *control* an *artifact* are the persons who *own* a *physical resource* that *supports* the *artifact* or who *own* some *data* represented by a *data instance* that *supports* the *artifact* (the *providers*).

[1] The words in italic in this section refer to keywords of Figure 2, describing the SOCIOPATH meta-model [2].

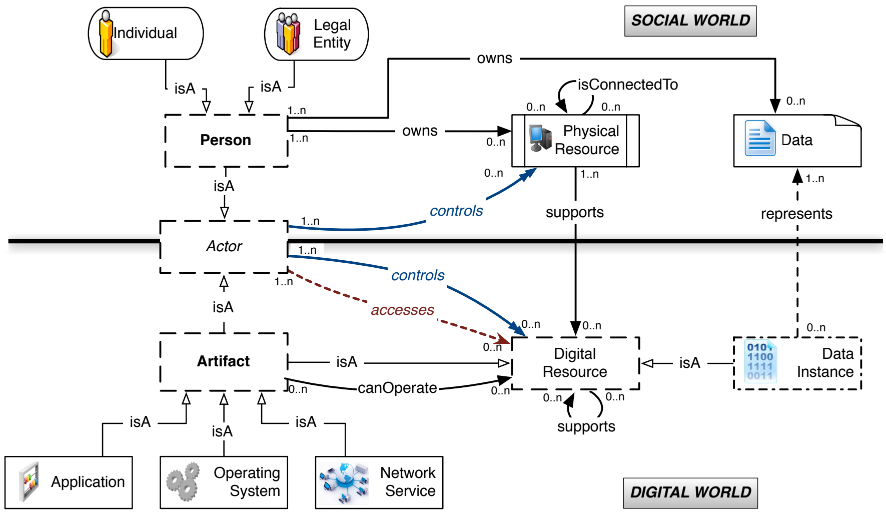


Fig. 2. Graphical view of SocioPath as a UML class diagram.

Figure 3 presents a graphical representation of a simple system drawn by applying SocioPath. Consider that a person `John` wants to achieve the activity "accessing the document `toto` using `GoogleDocs`". In the social world, the person `John` owns some `Data`, a `PC` and an `iPad`. `Microsoft`, `Google` and `Apple` are moral persons who provide resources and artifacts. `Renater`, `Orange` and `SFR` are French telco companies. John's `iPad` is connected to `SFR Servers` and `Renater Servers` as well as John's `PC` is connected to `Orange Servers`. On the other hand, in the digital world, the operating system `Windows` is running on John's PC. `Windows` supports `IExplorer`. John's `iPad` supports the running `iOS`, which supports the application `Safari`. John's `data` are represented in the digital world by the document `toto` which is supported by the physical resources owned by Google. For sake of simplicity, we consider `Google Cloud` as the storage system used by the application `GoogleDocs`. By applying the SocioPath rules on this example, we obtain the relations of *access* and *control* shown in Figure 3 where John has the following minimal paths to access `toto`:

$\widehat{\sigma}_1 =$ {`John, Windows, IExplorer, ADSL Network, Google Cloud, GoogleDocs, toto`}.

$\widehat{\sigma}_2 =$ {`John, iOS, Safari, SFR Network, Google Cloud, GoogleDocs, toto`}.

$\widehat{\sigma}_3 =$ {`John, iOS, Safari, Professional Network, Google Cloud, GoogleDocs, toto`}.

For sake of simplicity, in the current paper we voluntary limit the digital activities to those that can be represented using a straight path. We do not consider activities that need multiple paths in parallel to be achieved. Of course, an activity can be achieved through several paths and each path represents a different way to perform it. Most of the popular activities can be illustrated this way like connecting to a search engine, consulting a web page, publishing a picture, editing a document, *etc.* In the next sections, "accessing a document" embodies our illustrative activity.

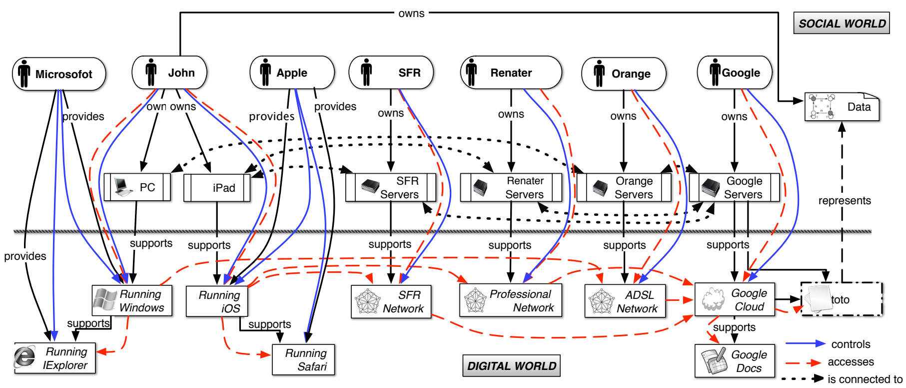


Fig. 3. Graphical representation of a system for the activity "John *accesses* a document `toto` on `GoogleDoc`" using SocioPath.

3 Inferring trust in a system for an activity

In order to evaluate the trust level of a particular user in a system for a particular activity, we first obtain a coarse-grained view of the system, from a SocioPath model, as a weighted directed acyclic graph (WDAG) (*cf.* Section 3.1). This graph represents the system allowed to perform the digital activity of the user. We then apply subjective logic on this graph to obtain the user's trust in a system for an activity achieved through the different paths in the graph (*cf.* Section 3.2).

3.1 A SocioPath model as a weighted directed acyclic graph

We simplify the representation of SocioPath by using only *access* and *control* relations derived from SocioPath rules. We combine an artifact, the set of persons controlling it and the set of physical resources supporting it in one unique component. These merged components are represented by nodes in a WDAG. Moreover, edges in this WDAG represent the relations *access*. A user achieves an activity by passing through several successive *access* relations of the graph, so-called a *path*[2]. Each node is associated with the user's opinion about this node. To summarize, a system that enable a user to achieve an activity ω can be formally modeled as a tuple:
$\alpha_{\omega,P} =< \mathbb{N}_\omega, \mathbb{A}_\omega, w_\omega >$ where:

- ω: the activity the user wants to achieve.
- P: the user who wants to achieve an activity.
- $\mathbb{N}_\omega$: the set of usable nodes in a system for an activity. Each node aggregates one artifact, the persons who control it and the physical resources that support it.
- $\mathbb{A}_\omega \in \mathbb{N}_\omega \times \mathbb{N}_\omega$: the set of edges in a system. From the rules of SocioPath and the aggregation we made for a node, our WDAG exhibits only the relation *access*.

[2] If there is no ambiguity, we denote a minimal path through the WDAG by simply a path σ.

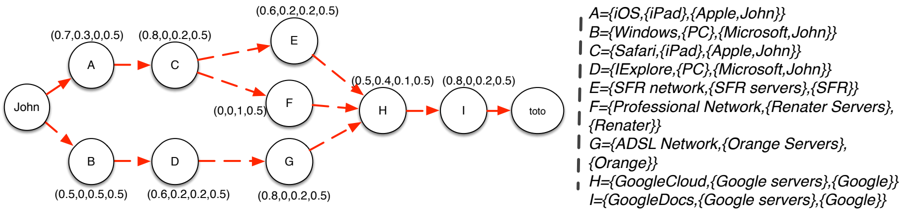


Fig. 4. The activity "John accesses a document `toto` on `GoogleDoc`" as a WDAG.

- $w_\omega : \mathbb{N} \to ([0,1],[0,1],[0,1],[0,1])$: a function that assigns to each node an opinion. In this paper, an opinion about a node N is denoted by w_N for simplicity.

Figure 4 shows the same system presented in Figure 3 as a merged WDAG where each node represents an artifact with all additional informations as physical resources it depends on and persons who control it and each edge represents the relation *accesses*. The associated values on the node represents John's opinion about this node. The paths that enable John to access `toto` become: σ_1 ={`A, C, E, H, I`}; σ_2 ={`A, C, F, H, I`}; σ_3 ={`B, D, G, H, I`}.

3.2 SUBJECTIVETRUST: an approach to infer trust in a system with subjective logic

Trust can be modeled through a graph that contains a source node and a target node and intermediates nodes. The edges in the graph represent the relations between nodes. Trust values are usually associated to the graph's node or edges. Trust evaluation in graph-based trust models has three phases:

- *Trust evaluation in a node*: the evaluation of a node's trust value differs from one person to another. There are several ways to construct this trust level. We can figure out different objective and subjective factors that impact this trust level like the reputation or the personal experience with this node. In this paper, we depend on the user's local binary observations of a node to build the user's trust in a node *(c.f. Section 3.2.1)*.
- *Trust evaluation in a path*: this phase is based on concatenating the node's/edge's trust value along a path between a source and a target *(c.f. Section 3.2.2)*.
- *Trust evaluation in a graph*: this phase is based on aggregating the different path's trust values in the graph *(c.f. Section 3.2.3)*.

The main problem that faces this type of trust evaluation is the common nodes/edges between paths. Since the trust evaluation is firstly based on concatenation the trust value along a path then aggregating the trust value of all paths, the trust values of the common nodes/edges between paths are multi-counted which leads to non-accurate result in evaluating trust [12,13].

In our approach, we propose several methods which are based on graph simplification and edge splitting to obtain a graph that has independent paths to resolve the previous problem *(c.f. Section 3.2.3-2)*.

3.2.1 Opinion about a node: opinion about a node depends on user's negative or positive observations r, s and is computed by Relations 1. Since our work focuses on local trust, local observations are considered. A trust survey shown in Appendix A.6 allows to collect the users' observations in order to build an opinion about a node. The proposed questions in this survey collect information about the user's usage of a node to estimate the uncertainty u. The negative observations of using a node is also demanded to estimate the value of d for a node. The value of b is computed by the relation $b = 1 - d - u$.

3.2.2 Opinion about a path: after building an opinion about a node, an opinion about a path that contains several nodes can be computed. If a person needs to achieve an activity through a path, she needs to pass by all the nodes composing this path. Hence, an opinion about a path is the opinion of all the nodes composing this path.

The conjunction operator in subjective logic represents the opinion of a person about several propositions. If $O_x^P = (b_x^P, d_x^P, u_x^P, a_x^P)$ is P's opinion about x and $O_y^P = (b_y^P, d_y^P, u_y^P, a_y^P)$ is P's opinion about y, $O_{x\wedge y}^P$ represents P's opinion about both x and y. Thus, the conjunction operator is the appropriate operator to compute an opinion about a path from the opinions about the nodes.

Let $\sigma = \{N_1, N_2, \ldots, N_n\}$ be a path that enables a user P to achieve an activity. P's opinion about the nodes $\{N_i\}_{i\in[1..n]}$ for an activity are denoted by $O_{N_i} = (b_{N_i}, d_{N_i}, u_{N_i}, a_{N_i})$. P's opinion about the path σ for achieving an activity, denoted by $O_\sigma = (b_\sigma, d_\sigma, u_\sigma, a_\sigma)$ can be derived by the conjunction of P's opinions about $\{N_i\}_{i\in[1..n]}$. $O_{\sigma=\{N_1,\ldots,N_n\}} = \bigwedge\{O_{N_i}\}_{i\in[1..n]}$. Given the following relations from [11];

$$O_{x\wedge y} = \begin{cases} b_{x\wedge y} = b_x b_y \\ d_{x\wedge y} = d_x + d_y - d_x d_y \\ u_{x\wedge y} = b_x u_y + u_x b_y + u_x u_y \\ a_{x\wedge y} = \frac{b_x u_y a_y + b_y u_x a_x + u_x a_x u_y a_y}{b_x u_y + u_x b_y + u_x u_y} \end{cases}$$

We obtain the following generalization:

$$O_{\sigma=\{N_1,\ldots,N_n\}} = \begin{cases} b_{\sigma=\{N_1,\ldots,N_n\}} = b_{\bigwedge\{N_i\}_{i\in[1..n]}} = \prod_{i=1}^n b_{N_i} \\ d_{\sigma=\{N_1,\ldots,N_n\}} = d_{\bigwedge\{N_i\}_{i\in[1..n]}} = 1 - \prod_{i=1}^n (1 - d_{N_i}) \\ u_{\sigma=\{N_1,\ldots,N_n\}} = u_{\bigwedge\{N_i\}_{i\in[1..n]}} = \prod_{i=1}^n (b_{N_i} + u_{N_i}) - \prod_{i=1}^n (b_{N_i}) \\ a_{\sigma=\{N_1,\ldots,N_n\}} = a_{\bigwedge\{N_i\}_{i\in[1..n]}} = \frac{\prod_{i=1}^n (b_{N_i} + u_{N_i} a_{N_i}) - \prod_{i=1}^n (b_{N_i})}{\prod_{i=1}^n (b_{N_i} + u_{N_i}) - \prod_{i=1}^n (b_{N_i})} \end{cases} \quad (8)$$

The proofs of Relations 8 in Appendix A.1 and the verifications of the relations: $b_\sigma + d_\sigma + u_\sigma = 1$, $0 < b_\sigma < 1$, $0 < d_\sigma < 1$, $0 < u_\sigma < 1$ and $0 < a_\sigma < 1$ are in Appendix A.1.1.

3.2.3 Opinion about a system after building an opinion about a path, an opinion about a system that contains several paths can be built. An opinion about a system is the opinion of a person about one of the paths or any union of them.

The disjunction operator in subjective logic represents the opinion of a person in one proposition or several. If $O_x^P = (b_x^P, d_x^P, u_x^P, a_x^P)$ is P's opinion about x and $O_y^P =$

$(b_y^P, d_y^P, u_y^P, a_y^P)$ is P's opinion about y, $O_{x\vee y}^P$ represents P's opinion about x or y or both. Thus, the disjunction operator is the appropriate operator to evaluate an opinion about a system. In the following, we show how to build an opinion about a system when *(i)* there are not common nodes between paths and *(ii)* there are some common nodes between paths.

1. **Opinion about a system having independent paths:**
   Let $\{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ be the paths that enable a user P to achieve an activity. The user opinion about the paths $\{\sigma_i\}_{i\in[1..m]}$ for an activity are denoted by $O_{\sigma_i} = (b_{\sigma_i}, d_{\sigma_i}, u_{\sigma_i}, a_{\sigma_i})$. The user opinion about the system α for achieving the activity, denoted by $O_\alpha = (b_\alpha, d_\alpha, u_\alpha, a_\alpha)$ can be derived by the disjunction of P's opinions about $\{\sigma_i\}_{i\in[1..m]}$. $O_\alpha = \bigvee\{O_{\sigma_i}\}_{i\in[1..m]}$. Given the following relations from [11];

$$O_{x\vee y} = \begin{cases} b_{x\vee y} = b_x + b_y - b_x b_y \\ d_{x\vee y} = d_x d_y \\ u_{x\vee y} = d_x u_y + u_x d_y + u_x u_y \\ a_{x\vee y} = \frac{u_x a_x + u_y a_y - b_x u_y a_y - b_y u_x a_x - u_x a_x u_y a_y}{u_x + u_y - b_x u_y - b_y u_x - u_x u_y} \end{cases}$$

   We obtain the following generalization:

$$O_{\alpha=\{\sigma_1,\ldots,\sigma_m\}} = \begin{cases} b_{\alpha=\{\sigma_1,\ldots,\sigma_m\}} = b_{\bigvee\{\sigma_i\}} = 1 - \prod_{i=1}^m (1 - b_{\sigma_i}) \\ d_{\alpha=\{\sigma_1,\ldots,\sigma_m\}} = d_{\bigvee\{\sigma_i\}} = \prod_{i=1}^m d_{\sigma_i} \\ u_{\alpha=\{\sigma_1,\ldots,\sigma_m\}} = u_{\bigvee\{\sigma_i\}} = \prod_{i=1}^m (d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^m (d_{\sigma_i}) \\ a_{\alpha=\{\sigma_1,\ldots,\sigma_m\}} = a_{\bigvee\{\sigma_i\}} = \frac{\prod_{i=1}^m (d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^m (d_{\sigma_i} + u_{\sigma_i} - u_{\sigma_i} a_{\sigma_i})}{\prod_{i=1}^m (d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^m (d_{\sigma_i})} \end{cases} \quad (9)$$

   The proofs of relations 9 are in Appendix A.2 and the verifications of the relations: $b_\alpha + d_\alpha + u_\alpha = 1$, $0 < b_\alpha < 1$, $0 < d_\alpha < 1$, $0 < u_\alpha < 1$ and $0 < a_\alpha < 1$ are in Appendix A.2.1.

2. **Opinion about a system having dependent paths:**
   In subjective logic the disjunction is not distributive over the conjunction, *ie.* we have $O_x \wedge (O_y \vee O_z) \neq (O_x \wedge O_y) \vee (O_y \wedge O_z)$. Then when there are common nodes between paths, the Relations 8, 9 can not be applied directly. In order to apply subjective logic for evaluating trust in a system, we propose to transform a graph having dependent paths to a graph having independent paths. Once this transformation is made, we can apply the Relations 8, 9. Three methods are proposed:
   - *Method 1 (M1)*: this method is achieved by simplifying a graph with common nodes between paths into a graph having only independent paths. This is made by removing the dependent paths which have high value of uncertainty as proposed in [13]. The principle of this method focuses on maximizing certainty, and not on deriving the most positive or negative trust value.
     An issue of of M1 is that the graph simplification cause information loss. To minimize the loss of information, we propose other methods that are based on splitting common nodes into several nodes to obtain a graph that contains independent paths as follows.

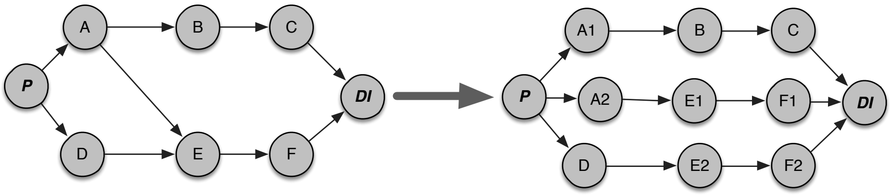


Fig. 5. Graph transformation using node splitting.

– *Method 2 (M2)*: this method is achieved by splitting a dependent node into several different nodes as illustrated in Figure 5. The left side of this figure shows an example of a graph that has dependent paths. The source node is S and the target node is T, the dependent paths are: $\sigma_1 = \{A, B, C\}, \sigma_2 = \{A, E, F\}, \sigma_3 = \{D, E, F\}$. The common nodes are A, E and F. For instance, A is a common node between σ_1 and σ_2. A is split into A_1, A_2 where $A_1 \in \sigma'_1 = \{A1, B, C\}$ and $A_2 \in \sigma'_2 = \{A2, E, F\}$ in the new graph, so is the case for the nodes E and F. The right part of Figure 5 shows the new graph after splitting the common nodes. The new graph contains the paths $\sigma'_1 = \{A1, B, C\}$, $\sigma'_2 = \{A2, E1, F1\}$ and $\sigma'_3 = \{D, E2, F2\}$.
  Node splitting should be followed by opinion splitting. In M2, we keep the same opinion associated to the original node on the split nodes. This method is based on the idea that the new produced path σ' maintains the same opinion of the original path σ. For instance, if A is a common node between σ_1 and σ_2 and the opinion about A is O_A, we split the node A into $A_1 \in \sigma'_1$ and $A_2 \in \sigma'_2$ in a new graph, and the opinion about A_1 and A_2 remains the same of A, that is O_A. In this case $O_{\sigma_1} = O_{\sigma'_1}$ and $O_{\sigma_2} = O_{\sigma'_2}$. This method is formalized in Algorithm 1.

Find all the paths $\sigma_{i:i\in[1..n]}$ for an activity ω performed by a person P
foreach $\sigma_{i:i\in[1..n]}$ **do**
  foreach $N_{j:j\in[2..length(\sigma_i)-1]} \in \sigma_i$ **do**
    if $N_j \in \sigma_{k:k\neq j}$ **then**
      Create a node N_{jk}
      Replace N_j by N_{jk} in σ_k
      Initialize $O_{N_{jk}} \leftarrow O_{N_j}$
    end
  end
end

Algorithm 1: M2 algorithm.

– *Method 3 (M3)*: this method follows the process of M2 in splitting dependent nodes to obtain a graph that has independent paths as shown in Figure 5. In order to maintain the opinion about a system, we split the opinion on the dependent node into independent opinions, such that when they are disjunct, they produce the original opinion. Formally speaking, if the node A is in common between σ_1 and σ_2 and the opinion about it is O_A, A is split into $A_1 \in \sigma'_1$ and $A_2 \in \sigma'_2$ and the opinion O_A is also split into O_{A_1} and O_{A_2} where O_{A_1} and O_{A_2} satisfy the following relations: $O_{A_1} = O_{A_2}$ and $O_{A_1} \vee O_{A_2} = O_A$. The following are the obtained split opinion in two cases, the case of splitting an opinion into two independent opinions and the case of splitting an opinion into n independent opinions.

- Splitting a dependent opinion into two independent opinions:

$$\bigwedge \begin{cases} O_{A_1} \vee O_{A_2} = O_A \\ O_{A_1} = O_{A_2} \end{cases} \Leftrightarrow \begin{cases} b_{A_1} \vee b_{A_2} = b_A \\ d_{A_1} \vee d_{A_2} = d_A \\ u_{A_1} \vee u_{A_2} = u_A \\ a_{A_1} \vee a_{A_2} = a_A \end{cases} \bigwedge \begin{cases} b_{A_1} = b_{A_2} \\ d_{A_1} = d_{A_2} \\ u_{A_1} = u_{A_2} \\ a_{A_1} = a_{A_2} \end{cases} \Rightarrow$$

$$\begin{cases} b_{A_1} = b_{A_2} = 1 - \sqrt{1 - b_A} \\ d_{A_1} = d_{A_2} = \sqrt{d_A} \\ u_{A_1} = u_{A_2} = \sqrt{d_A + u_A} - \sqrt{d_A} \\ a_{A_1} = a_{A_2} = \frac{\sqrt{1-b_A} - \sqrt{1-b_A-a_A u_A}}{\sqrt{d_A+u_A} - \sqrt{d_A}} \end{cases} \quad (10)$$

- Splitting a dependent opinion into n independent opinions:

$$\bigwedge \begin{cases} O_{A_1} \vee O_{A_2} \vee \ldots \vee O_{A_n} = O_A \\ O_{A_1} = O_{A_2} = \ldots = O_{A_n} \end{cases} \Leftrightarrow$$

$$\begin{cases} b_{A_1} \vee b_{A2} \vee \ldots \vee b_{A_n} = b_A \\ d_{A_1} \vee d_{A2} \vee \ldots \vee d_{A_n} = d_A \\ u_{A_1} \vee u_{A2} \vee \ldots \vee u_{A_n} = u_A \\ a_{A_1} \vee a_{A2} \vee \ldots \vee a_{A_n} = a_A \end{cases} \bigwedge \begin{cases} b_{A_1} = b_{A2} = \ldots = b_{A_n} \\ d_{A_1} = d_{A2} = \ldots = d_{A_n} \\ u_{A_1} = u_{A2} = \ldots = u_{A_n} \\ a_{A_1} = a_{A2} = \ldots = a_{A_n} \end{cases} \Rightarrow$$

$$\begin{cases} b_{A1} = b_{A2} = \ldots = b_{A_n} = 1 - (1 - b_A)^{\frac{1}{n}} \\ d_{A1} = d_{A2} = \ldots = d_{A_n} = d_A^{\frac{1}{n}} \\ u_{A1} = u_{A2} = \ldots = u_{A_n} = (d_A + u_A)^{\frac{1}{n}} - d_A^{\frac{1}{n}} \\ a_{A1} = a_{A2} = \ldots = a_{A_n} = \frac{(1-b_A)^{\frac{1}{n}} - (1-b_A-a_A u_A)^{\frac{1}{n}}}{(d_A+u_A)^{\frac{1}{n}} - d_A^{\frac{1}{n}}} \end{cases} \quad (11)$$

The proofs of Relations 10 and 11 are in Appendix A.3. M3 is formalized in Algorithm 2.

In the following, we evaluate our approach with several experiments.

4 Experimental evaluation

Evaluating this proposal faces multiple issues. First, the variations that we focus on for studying our approach is the percentage of dependent nodes or the percentage of dependent paths in a graph. Thus studying our approach on one single graph is not that useful, each graph has its own topology and its own characteristics that can not been

```
Find all the paths σ_{i:i∈[1..n]} for an activity ω performed by a person P
foreach σ_{i:i∈[1..n]} do
    foreach N_{j:j∈[2..length(σ_i)−1]} ∈ σ_i do
        if N_j ∈ σ_{k:k≠j} then
            Create a node N_{jk}
            Replace N_j by N_{jk} in σ_k
            O_{N_{jk}} ← opinion resulted from Relations 11
        end
    end
end
```

Algorithm 2: M3 algorithm.

generalized on one single graph. Second, the proposed methods are approximate methods for evaluating trust in a system. There is not a referenced method to be compared with the proposed methods. Third, subjective logic is relatively new, there are not yet circulating opinions that we can extract to test our approach on a real case study.

To overcome the previous difficulties, we decide to treat them one by one on separated experiments that have the following objectives:

- Comparing the proposed methods.
- Evaluating the accuracy of the proposed methods.
- Confronting this approach with real users.

Next sections present the different experiments, their results, analysis and how we deal with previous difficulties.

4.1 Comparing M1, M2, M3

In order to compare the different graph transformation methods, we conducted experiments on several simple graphs having different topologies (*see Table 1*). Besides the topology, these graphs vary in the percentage of dependent nodes and dependent paths they have.

To make this comparison, random opinions $O_N = (b_N, d_N, u_N, a_N)$ are associated to each node, and the opinion's probability expectation value of the graph, $E(O_\alpha) = b_\alpha + a_\alpha u_\alpha$ is computed using the methods M1, M2, M3. This step is repeated 50 times for each graph where each time represents random opinions of a person towards the different nodes the compose the graph. Table 2 shows the obtained results.

Conclusion of the first experiment: all methods have the same probability expectation value $E(O_\alpha)$ when the graph has independent paths (α_1). The difference between the methods becomes more clear when the percentage of the dependent nodes or paths increases. M2 is much more optimist than M3 and M1, it always give higher opinion's probability expectation than the ones given by M3 or M1. The mathematical proofs that M2 is more optimisic than M3 and M1 are in Appendices A.4, A.5. In general M1, M2 and M3 show the same behavior, when the $E(O_\alpha)$ increases in one method it increases in the other methods, and vice versa.

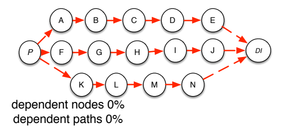
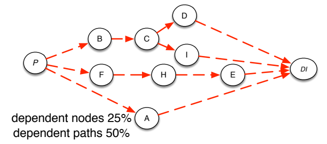
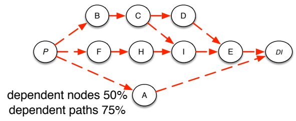
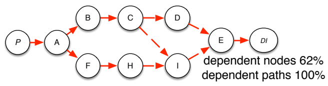
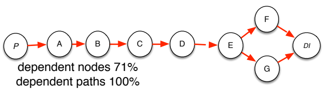
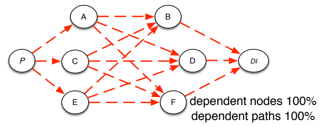
α_1		α_2	
α_3		α_4	
α_5		α_6	

Table 1: Graphs that have different topologies.

4.2 Studying the accuracy of M1, M2, M3

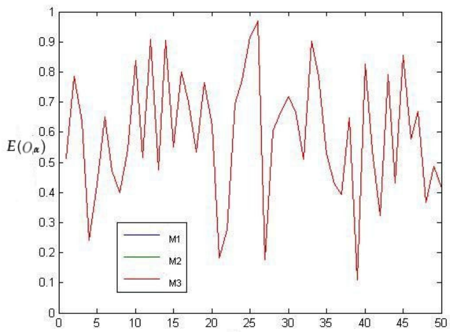
The three methods of SUBJECTIVETRUST are approximate methods. Comparing these methods to each other does not show which of them is the most accurate. In order to evaluate their accuracy, we propose to compare this approach to the SOCIOTRUST approach proposed in [3]. However, the building component in the latter approach is a single value that represents a node's trust value. In M1, M2,M3, the building component is a node's opinion $O_N = (b_N, d_N, u_N, a_N)$. Thus, such a comparison is not possible directly.

Subjective logic returns back to traditional probability when $b+d = 1$ such that $u = 0$, *i.e.* the value of uncertainty is equal to 0. When $u = 0$, the operations in subjective logic are directly compatible with the operations of the traditional probability. In this case the value of $E(O) = b + au = b$ corresponds to the value of probability.

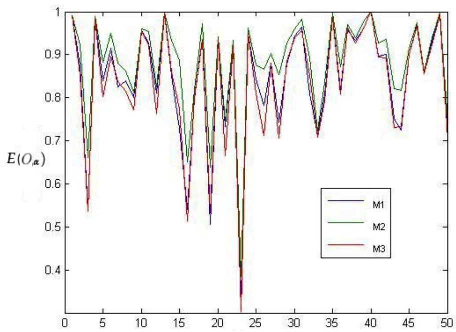
Since SOCIOTRUST is based on probability theory, the results obtained by applying subjective logic should be equal to the obtained results using probability theory if $u = 0$. SOCIOTRUST is a proposition that has no approximations, thus we can evaluate the accuracy of these methods by choosing $u = 0$ and compare the value of $b_\alpha = E(O_\alpha)$ resulted from applying the three methods of SUBJECTIVETRUST to the system's trust value obtained by applying SOCIOTRUST.

The experiments are conducted on the graphs shown in Table 1. Random opinions $O_N = (b_N, d_N, 0, a_N)$ are associated to each node, and opinion's probability expectation of the graph $E(O_\alpha) = b_\alpha + a_\alpha u_\alpha = b_\alpha$ is computed. This step is repeated 10000 times for each graph. For simplicity, the notations T, T_{M1}, T_{M2}, T_{M3} respectively denote system's trust value resulting from applying SOCIOTRUST and system's opinion probability expectation resulting from applying M1, M2 and M3. Table 3 shows the results of comparing system's opinion probability expectation T_{M1},T_{M2},T_{M3} and the trust value T resulting of using probability theory on the graphs shown in Table 1. The values of $T - T_{M1}$, $T - T_{M2}$, $T - T_{M3}$ are computed and the average of these values

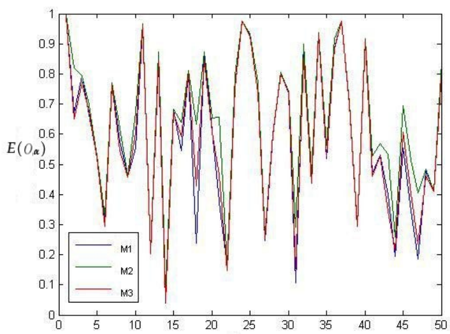
α_1



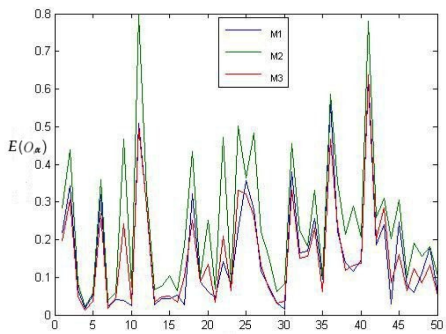
α_2



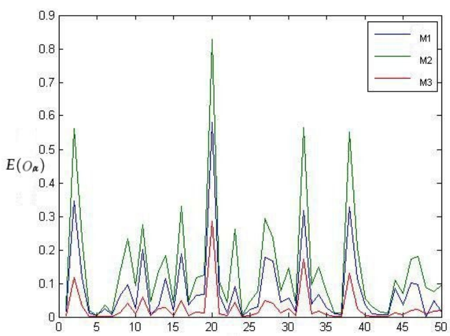
α_3



α_4



α_5



α_6

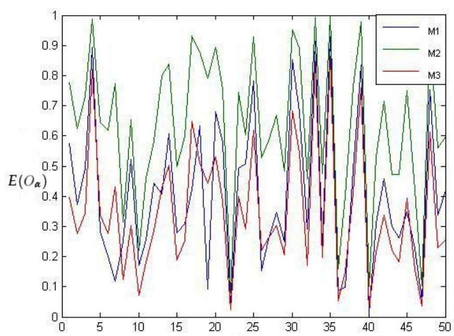


Table 2: Different graphs and their value of $E(O_\alpha)$ for 50 persons using the three methods M1, M2 and M3.

$\frac{\sum(T-T_{M1})}{10000}$, $\frac{\sum(T-T_{M2})}{10000}$, $\frac{\sum(T-T_{M3})}{10000}$ are computed as an indication of the accuracy of the methods of SUBJECTIVETRUST.

Conclusion of the second experiment: in the graph that has independent paths α_1, all results are equal as expected. M2 is the method that gives the closest results to the results obtained by applying probability theory, which is an indication that this method gives the nearest result to the exact result. M2 is more optimistic than the probability theory, its results is always higher than the obtained results in probability theory. M2 shows a high accuracy, between 0 (in a graph that has total independent nodes and paths) and 0.078327 (in a graph that has 100% dependent nodes and 100% dependent paths).

M1, M3 are more pessimistic than the probability theory, their results are always lower than the obtained results in probability theory. The accuracy of M1 is between 0 (in a graph that has total independent nodes and paths) and 0.12486 (in a graph that has 100% dependent nodes and 100% dependent paths). The accuracy of M3 is between 0 (in a graph that has total independent nodes and paths) and 0.20546 (in a graph that has 100% dependent nodes and 100% dependent paths).

The system's trust value obtained by using probability theory is always between the system's opinion probability expectation value obtained by M2 and M1, or by M2 and M3.

4.3 Social evaluation: a real case

In order to study our approach on a real case study, we need the following data: a real system modeled by using SOCIOPATH and real opinions held by real users who use the entities of the system.

We modeled a subpart of the `LINA` research laboratory system[3] using SOCIOPATH. We applied the rules of SOCIOPATH on this system for the activity "a user accesses a document `toto` that is stored on the `SVN` server of `LINA`". Figure 8 presents the WDAG for this activity, with renamed nodes A, B, C, D, E, F, G for privacy issues. For sack of clarity, we simplify this graph as much as possible.

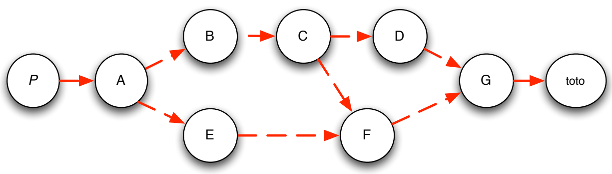


Fig. 6. `LINA`'s WDAG for the activity "accessing a document `toto` on the SVN".

Since subjective logic is relatively new, users are not used to build an opinion directly using this logic. We build this opinion ourselves from users' positive or negative

[3] https://www.lina.univ-nantes.fr/

observations. Since our work focuses on local trust, local observations are considered. To do that, the survey introduced in Appendix A.6 helps us to collect the observations of LINA users about the nodes. A local opinion in each entity is built for each user, few examples are shown in Appendix A.6 as well. The opinion and the opinion's probability expectation of the system are then computed using M2 for each user. The results are shown in Table 4.

Conclusion of the third experiment: in this experiment, depending on the local observations, opinions in the nodes are built and a local opinion in a system is computed. We asked each user for a feedback about their opinion in the nodes and in the system. LINA users were satisfied of the obtained results whereas, in [3], some users were not satisfied of the results. In the latter approach, when users do not have enough knowledge about a node, they vote with the value 0.5, which are considered for them as neutral value. That led to incorrect data that gave a false trust value in a system. In SUBJECTIVETRUST, uncertainties are expressed for the opinions in the nodes and computing an opinion in a system is made considering these uncertainties. That shows that, in uncertain environment, it is more suitable to use subjective logic than other metrics for trust evaluations.

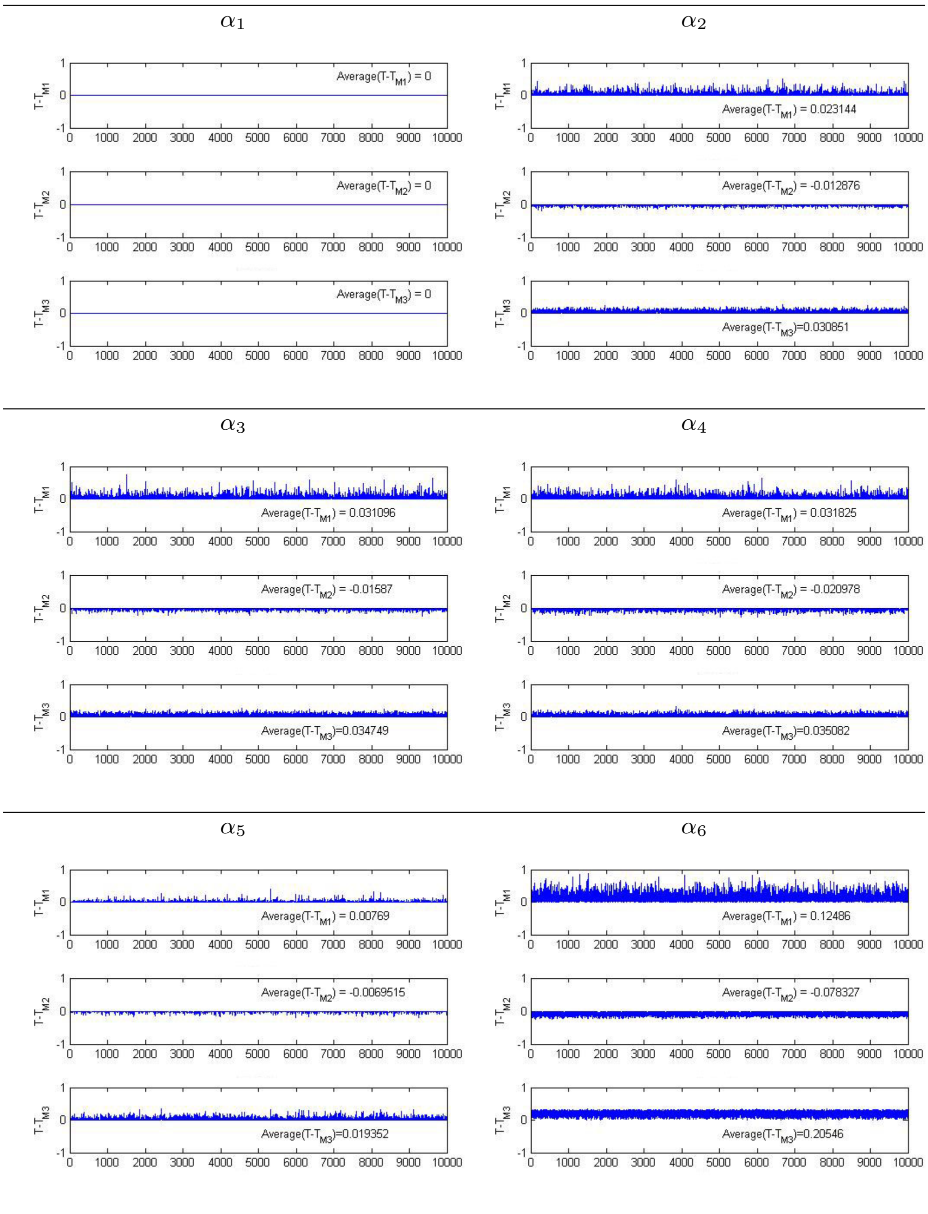


Table 3: The difference between the opinion's probability expectation of a graph $E(O_\alpha)$ using SUBJECTIVETRUST and the trust value resulting of using SOCIOTRUST.

	O_A (b,d,u,a)	O_B (b,d,u,a)	O_C (b,d,u,a)	O_D (b,d,u,a)	O_E (b,d,u,a)	O_F (b,d,u,a)	O_G (b,d,u,a)	O_α (b,d,u,a)	$E(O_\alpha)$
P_1	$(1,0,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.99,0.01,0,0.5)$	$(0.83,0,0.17,0.5)$	$(0.99,0.01,0,0.5)$	$(0.83,0,0.17,0.5)$	$(0.6820,0,0.3810,0.8389)$	0.9488
P_2	$(1,0,0,0.5)$	$(1,0,0,0.5)$	$(1,0,0,0.5)$	$(1,0,0,0.5)$	$(0.83,0,0.17,0.5)$	$(1,0,0,0.5)$	$(1,0,0,0.5)$	$(1,0,0,\ )$	1
P_3	$(0.99,0.01,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.99,0.01,0,0.5)$	$(0,0,1,0.5)$	$(0.99,0.01,0,0.5)$	$(0.6,0,0.4,0.5)$	$(0,0,1,0.7753)$	0.7753
P_4	$(1,0,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(1,0,0,0.5)$	$(0.83,0,0.17,0.5)$	$(0.99,0.01,0,0.5)$	$(0.96,0,0.04,0.5)$	$(0.7888,0,0.2112,0.8604)$	0.9705
P_5	$(0.99,0.01,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.99,0.01,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.5,0,0.5,0.5)$	$(0,0,1,0.7500)$	0.75
P_6	$(0.99,0.01,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.9,0.1,0,0.5)$	$(0.5,0,0.5,0.5)$	$(1,0,0,0.5)$	$(0.6,0,0.4,0.5)$	$(0.2970,0,0.7030,0.7755)$	0.8422
P_7	$(0.99,0.01,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.9,0.1,0,0.5)$	$(0.83,0,0.17,0.5)$	$(1,0,0,0.5)$	$(1,0,0,0.5)$	$(0.8217,0,0.1783,0.8522)$	0.9736
P_8	$(1,0,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.9,0.1,0,0.5)$	$(0.5,0,0.5,0.5)$	$(1,0,0,0.5)$	$(1,0,0,0.5)$	$(0.5000,0,0.5000,0.8625)$	0.9313
P_9	$(1,0,0,0.5)$	$(1,0,0,0.5)$	$(1,0,0,0.5)$	$(0.95,0.05,0,0.5)$	$(0,0,1,0.5)$	$(0.99,0.01,0,0.5)$	$(0.96,0,0.04,0.5)$	$(0.9956,0,0.0044,0.7583)$	0.9989
P_{10}	$(0.99,0.01,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.9,0.1,0,0.5)$	$(0,0,1,0.5)$	$(0.8,0.2,0,0.5)$	$(0.98,0,0.02,0.5)$	$(0,0.0047,0.9953,0.7972)$	0.7934
P_{11}	$(0.99,0.01,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.95,0.05,0,0.5)$	$(0.72,0,0.28,0.5)$	$(0.99,0.01,0,0.5)$	$(0.96,0,0.04,0.5)$	$(0.6774,0.0001,0.3225,0.8489)$	0.9512
P_{12}	$(1,0,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.95,0.05,0,0.5)$	$(0.83,0,0.17,0.5)$	$(0.95,0.05,0,0.5)$	$(1,0,0,0.5)$	$(0.7885,0,0001,0.2114,0.8301)$	0.9640
P_{13}	$(1,0,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.95,0.05,0,0.5)$	$(0.83,0,0.17,0.5)$	$(0.95,0.05,0,0.5)$	$(0.83,0,0.17,0.5)$	$(0.6545,0.0001,0.3545,0.8110)$	0.9346
P_{14}	$(1,0,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.99,0.01,0,0.5)$	$(0.72,0,0.28,0.5)$	$(0.99,0.01,0,0.5)$	$(0.72,0,0.28,0.5)$	$(0.5132,0,0.4868,0.8186)$	0.9117
P_{15}	$(1,0,0,0.5)$	$(1,0,0,0.5)$	$(1,0,0,0.5)$	$(0.99,0.01,0,0.5)$	$(0.72,0,0.28,0.5)$	$(0.99,0.01,0,0.5)$	$(0.83,0,0.17,0.5)$	$(0.9870,0,0.0130,0.8492)$	0.9980
P_{16}	$(0.99,0.01,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.9,0.1,0,0.5)$	$(0.5,0,0.5,0.5)$	$(1,0,0,0.5)$	$(0.72,0,0.28,0.5)$	$(0.3564,0,0.6436,0.8011)$	0.8719
P_{17}	$(1,0,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.99,0.01,0,0.5)$	$(0.83,0,0.17,0.5)$	$(1,0,0,0.5)$	$(0.83,0,0.17,0.5)$	$(0.6889,0,0.3111,0.8447)$	0.9517
P_{18}	$(1,0,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.99,0.01,0,0.5)$	$(0.83,0,0.17,0.5)$	$(1,0,0,0.5)$	$(1,0,0,0.5)$	$(0.8300,0,0.1700,0.8737)$	0.9785
P_{19}	$(1,0,0,0.5)$	$(1,0,0,0.5)$	$(1,0,0,0.5)$	$(0.99,0.01,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.72,0,0.28,0.5)$	$(0.9196,0,0.0804,0.8525)$	0.9811
P_{20}	$(1,0,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.95,0.05,0,0.5)$	$(0,0,1,0.5)$	$(1,0,0,0.5)$	$(0.83,0,0.17,0.5)$	$(0,0,1,0.8836)$	0.8336

Table 4: Users' opinions in the system for the activity "accassing a document on the server `SVN` in `LINA`".

5 Related work

This paper proposes an approach to evaluate the system trust value for an activity as a combination of several trust values through a graph. This work is principally related to the trust propagation problem in social networks [10,21].

Usually, a social network is represented as a graph where the nodes are persons and the edges reflect the trust relations between these persons. The values associated to the edges represent the values of trust between these persons. Trust propagation problem in social network focuses on finding a trust value toward a defined person or resource through the multiple paths that relate the trustor with the trustee.

In our work, a system for an activity is represented by a graph where the nodes are the entities that compose the system and the edges are the relations of access between the nodes. The relationship of trust are the directed relations applied from the source node, which is the user who achieves an activity, towards the nodes that build the system. The graph-based trust models in social network aim to propagate trust between two nodes in a graph, whereas the objective of our work is evaluating trust in the whole graph that reflects the activity achieved through it. Figure 7 shows the difference between a trust graph in a social network and our work.

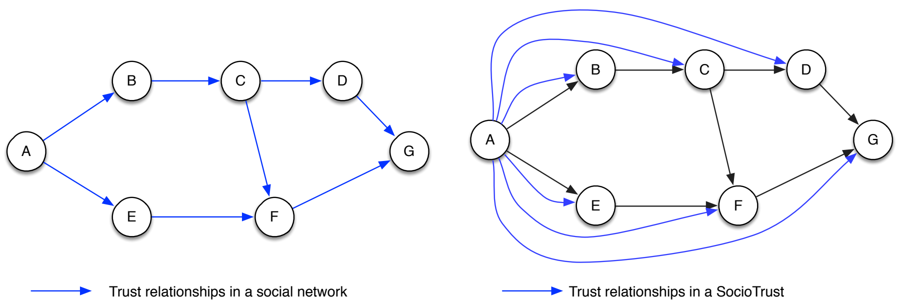


Fig. 7. Difference between trust relationships in a social network and SocioPath.

A lot of metrics have been proposed to propagate trust through a social network like binary metrics [9] or simple metrics (average, weighted average, *etc.*) [7,19] or probabilistic metrics [3,17], subjective logic is also used for the trust propagation problem as a type of probabilistic metrics [12,13]. In [12,13], the graph-based trust model is divided into two phases, trust propagation through a path and trust propagation through a graph. In the first phase, the proposed methods focus on the transitivity of trust, the methods in the second phase focus on the consensus of trust through the multiple paths. Our work converges with these works in the phases of evaluating trust through a graph, but it diverges in the proposed formalisms and algorithms in the two phases due to the differences in the trust relationships represented in the graph, see Figure 7 and the different objective of our work. That imposes us to propose different relations and algorithms and not straightly adopted what exists in this domain.

6 Conclusion and perspectives

This paper presents a graph-based trust model to evaluate user's trust in a system for an activity using subjective logic. The problem of dependent paths in evaluating trust using a weighted directed acyclic graph is illustrated and some methods are proposed to resolve this problem, all by using subjective logic. Some experiments are conducted to validate the proposed methods and a real case study is made to confront this approach to real users. The limitation of this approach that the proposed methods are approximate methods. The way that has been proposed to evaluate their accuracy is limited, because it is made for the case where the value of uncertainty is equal to 0.

In this paper, building an opinion about a node is based on the user's binary observations towards this node, either positive or negative observation. In future works, we plan to use a degree of strength for the observations. Thus, they should be represented as a range of discrete values, in this case, a user observation is a value that belongs to a discrete interval like [0..10]. This imposes using a different mapping function between the observation and the opinion parameters.

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A Appendix

A.1 Opinion about a path (mathematical proof):

$$O_{x\wedge y} = \begin{cases} b_{x\wedge y} = b_x b_y \\ d_{x\wedge y} = d_x + d_y - d_x d_y \\ u_{x\wedge y} = b_x u_y + u_x b_y + u_x u_y \\ a_{x\wedge y} = \frac{b_x u_y a_y + b_y u_x a_x + u_x a_x u_y a_y}{b_x u_y + u_x b_y + u_x u_y} \end{cases} \Rightarrow$$

$$O_{\sigma=\{N_1,\ldots,N_n\}} = \begin{cases} b_{\sigma=\{N_1,\ldots,N_n\}} = b_{\bigwedge\{N_i\}_{i\in[1..n]}} = \prod_{i=1}^n b_{N_i} \\ d_{\sigma=\{N_1,\ldots,N_n\}} = d_{\bigwedge\{N_i\}_{i\in[1..n]}} = 1 - \prod_{i=1}^n (1 - d_{N_i}) \\ u_{\sigma=\{N_1,\ldots,N_n\}} = u_{\bigwedge\{N_i\}_{i\in[1..n]}} = \prod_{i=1}^n (b_{N_i} + u_{N_i}) - \prod_{i=1}^n (b_{N_i}) \\ a_{\sigma=\{N_1,\ldots,N_n\}} = a_{\bigwedge\{N_i\}_{i\in[1..n]}} = \frac{\prod_{i=1}^n (b_{N_i} + u_{N_i} a_{N_i}) - \prod_{i=1}^n (b_{N_i})}{\prod_{i=1}^n (b_{N_i} + u_{N_i}) - \prod_{i=1}^n (b_{N_i})} \end{cases}$$

1. The mathematical proof of the relation b_σ:

   Lemma 1. $b_{\sigma=\{N_1,\ldots,N_n\}} = b_{\bigwedge\{N_i\}_{i\in[1..n]}} = \prod_{i=1}^n b_{N_i}$

   Proof. We prove by induction that, for all $n \in Z^+$,

   $$b_{\bigwedge\{N_i\}_{i\in[1..n]}} = \prod_{i=1}^n b_{N_i}$$

   Base case. When $n = 2$:

   $$b_{N_1\wedge N_2} = b_{N_1} b_{N_2} = \prod_{i=1}^2 b_{N_i}$$

   Induction step. Let $k \in Z^+$ be given and suppose that Lemma 1 is true for $n = k$. Then

   $$b_{\bigwedge\{N_i\}_{i\in[1..k+1]}} = b_{\{\wedge\{N_i\}_{i\in[1..k]}\}\wedge N_{k+1}} = \prod_{i=1}^k b_{N_i} b_{N_{k+1}} = \prod_{i=1}^{k+1} b_{N_i}$$

   Thus, Lemma 1 holds for $n = k + 1$. By the principle of induction, Lemma 1 is true for all $n \in Z^+$. □

   □

2. The mathematical proof of the relation d_σ:

   Lemma 2. $d_{\sigma=\{N_1,\ldots,N_n\}} = d_{\bigwedge\{N_i\}_{i\in[1..n]}} = 1 - \prod_{i=1}^n (1 - d_{N_i})$

   Proof. We prove by induction that, for all $n \in Z^+$,

   $$d_{\bigwedge\{N_i\}_{i\in[1..n]}} = 1 - \prod_{i=1}^n (1 - d_{N_i})$$

Base case. When $n = 2$:

$$\begin{aligned} d_{N_1 \wedge N_2} &= d_{N_2} + d_{N_1} - d_{N_1} d_{N_2} \\ &= 1 - (1 - d_{N_2} - d_{N_1} + d_{N_1} d_{N_2}) \\ &= 1 - (1 - d_{N_1})(1 - d_{N_2}) \\ &= 1 - \prod_{i=1}^{2} (1 - d_{N_i}) \end{aligned}$$

Induction step. Let $k \in Z^+$ be given and suppose that Lemma 2 is true for $n = k$. Then

$$\begin{aligned} d_{\bigwedge \{N_i\}_{i \in [1..k+1]}} &= d_{\{\wedge \{N_i\}_{i \in [1..k]}\} \wedge N_{k+1}} \\ &= d_{\wedge \{N_i\}_{i \in [1..k]}} + d_{N_{k+1}} - d_{\wedge \{N_i\}_{i \in [1..k]}} d_{N_{k+1}} \\ &= \left[1 - \prod_{i=1}^{k} (1 - d_{N_i})\right] + d_{N_{k+1}} - \left[1 - \prod_{i=1}^{k} (1 - d_{N_i})\right] d_{N_{k+1}} \\ &= 1 - \prod_{i=1}^{k} (1 - d_{N_i}) + d_{N_{k+1}} - d_{N_{k+1}} + d_{N_{k+1}} \prod_{i=1}^{k} (1 - d_{N_i}) \\ &= 1 - \prod_{i=1}^{k} (1 - d_{N_i}) + d_{N_{k+1}} \prod_{i=1}^{k} (1 - d_{N_i}) \\ &= 1 - \left[\prod_{i=1}^{k} (1 - d_{N_i})\right] [1 - d_{N_{k+1}}] \\ &= 1 - \prod_{i=1}^{k+1} (1 - d_{N_i}) \end{aligned}$$

Thus, Lemma 2 holds for $n = k + 1$. By the principle of induction, Lemma 2 is true for all $n \in Z^+$. □

□

3. The mathematical proof of the relation u_σ:

Lemma 3. $u_{\sigma=\{N_1,\ldots,N_n\}} = u_{\bigwedge \{N_i\}_{i \in [1..n]}} = \prod_{i=1}^{n} (b_{N_i} + u_{N_i}) - \prod_{i=1}^{n} (b_{N_i})$

Proof. We prove by induction that, for all $n \in Z^+$,

$$u_{\bigwedge \{N_i\}_{i \in [1..n]}} = \prod_{i=1}^{n} (b_{N_i} + u_{N_i}) - \prod_{i=1}^{n} (b_{N_i})$$

Base case. When $n = 2$:

$$\begin{aligned} u_{N_1 \wedge N_2} &= b_{N_1} u_{N_2} + u_{N_1} b_{N_2} + u_{N_1} u_{N_2} \\ &= b_{N_1} u_{N_2} + u_{N_1} b_{N_2} + u_{N_1} u_{N_2} + b_{N_1} b_{N_2} - b_{N_1} b_{N_2} \\ &= (b_{N_1} + u_{N_1})(b_{N_2} + u_{N_2}) - b_{N_1} b_{N_2} \\ &= \prod_{i=1}^{2} (b_{N_i} + u_{N_i}) - \prod_{i=1}^{2} (b_{N_i}) \end{aligned}$$

Induction step. Let $k \in Z^+$ be given and suppose that Lemma 3 is true for $n = k$. Then

$$\begin{aligned} u_{\bigwedge \{N_i\}_{i \in [1..k+1]}} &= u_{\{\wedge \{N_i\}_{i \in [1..k]}\} \wedge N_{k+1}} \\ &= b_{\wedge \{N_i\}_{i \in [1..k]}} u_{N_{k+1}} + u_{\wedge \{N_i\}_{i \in [1..k]}} b_{N_{k+1}} + u_{\wedge \{N_i\}_{i \in [1..k]}} u_{N_{k+1}} \\ &= \left[\prod_{i=1}^{k} b_{N_i} \right] u_{N_{k+1}} + \left[\prod_{i=1}^{k} (b_{N_i} + u_{N_i}) - \prod_{i=1}^{k} (b_{N_i}) \right] b_{N_{k+1}} \\ &+ \left[\prod_{i=1}^{k} (b_{N_i} + u_{N_i}) - \prod_{i=1}^{k} (b_{N_i}) \right] u_{N_{k+1}} \\ &= \prod_{i=1}^{k} b_{N_i} u_{N_{k+1}} + \prod_{i=1}^{k} (b_{N_i} + u_{N_i}) b_{N_{k+1}} - \prod_{i=1}^{k} (b_{N_i}) b_{N_{k+1}} \\ &+ \prod_{i=1}^{k} (b_{N_i} + u_{N_i}) u_{N_{k+1}} - \prod_{i=1}^{k} (b_{N_i}) u_{N_{k+1}} \\ &= \left[\prod_{i=1}^{k} (b_{N_i} + u_{N_i}) \right] (b_{N_{k+1}} + u_{N_{k+1}}) - \prod_{i=1}^{k+1} (b_{N_i}) \\ &= \prod_{i=1}^{k+1} (b_{N_i} + u_{N_i}) - \prod_{i=1}^{k+1} (b_{N_i}) \end{aligned}$$

Thus, Lemma 3 holds for $n = k + 1$. By the principle of induction, Lemma 3 is true for all $n \in Z^+$. □

□

4. The mathematical proof of the relation a_σ:

Lemma 4. $a_{\sigma=\{N_1,\ldots,N_n\}} = a_{\bigwedge\{N_i\}_{i\in[1..n]}} = \frac{\prod_{i=1}^{n}(b_{N_i}+u_{N_i}a_{N_i})-\prod_{i=1}^{n}(b_{N_i})}{\prod_{i=1}^{n}(b_{N_i}+u_{N_i})-\prod_{i=1}^{n}(b_{N_i})}$

Proof. We prove by induction that, for all $n \in Z^+$,

$$a_{\bigwedge \{N_i\}_{i \in [1..n]}} = \frac{\prod_{i=1}^{n} (b_{N_i} + u_{N_i} a_{N_i}) - \prod_{i=1}^{n} (b_{N_i})}{\prod_{i=1}^{n} (b_{N_i} + u_{N_i}) - \prod_{i=1}^{n} (b_{N_i})}$$

Base case. When $n = 2$:

$$
\begin{aligned}
a_{N_1 \wedge N_2} &= \frac{b_{N_1} u_{N_2} a_{N_2} + b_{N_2} u_{N_1} a_{N_1} + u_{N_1} a_{N_1} u_{N_2} a_{N_2}}{b_{N_1} u_{N_2} + u_{N_1} b_{N_2} + u_{N_1} u_{N_2}} \\
&= \frac{(b_{N_1} u_{N_2} a_{N_2} + b_{N_2} u_{N_1} a_{N_1} + u_{N_1} a_{N_1} u_{N_2} a_{N_2} + b_{N_1} b_{N_2}) - b_{N_1} b_{N_2}}{u_{N_1 \wedge N_2}} \\
&= \frac{(b_{N_1} + u_{N_1} a_{N_1})(b_{N_2} + u_{N_2} a_{N_2}) - (b_{N_1} b_{N_2})}{u_{N_1 \wedge N_2}} \\
&= \frac{\prod_{i=1}^{2}(b_{N_i} + u_{N_i} a_{N_i}) - \prod_{i=1}^{2}(b_{N_i})}{\prod_{i=1}^{2}(b_{N_i} + u_{N_i}) - \prod_{i=1}^{2}(b_{N_i})}
\end{aligned}
$$

Induction step. Let $k \in Z^+$ be given and suppose that Lemma 4 is true for $n = k$. Then

$$
a_{\bigwedge\{N_i\}_{i\in[1..k+1]}} = a_{\{\wedge\{N_i\}_{i\in[1..k]}\}\wedge N_{k+1}} = \frac{b_{\wedge\{N_i\}_{i\in[1..k]}} u_{N_{k+1}} a_{N_{k+1}} + b_{N_{k+1}} u_{\wedge\{N_i\}_{i\in[1..k]}} a_{\wedge\{N_i\}_{i\in[1..k]}} + u_{\wedge\{N_i\}_{i\in[1..k]}} a_{\wedge\{N_i\}_{i\in[1..k]}} u_{N_{k+1}} a_{N_{k+1}}}{b_{\wedge\{N_i\}_{i\in[1..k]}} u_{N_{k+1}} + u_{\wedge\{N_i\}_{i\in[1..k]}} b_{N_{k+1}} + u_{\wedge\{N_i\}_{i\in[1..k]}} u_{N_{k+1}}}
$$

We denote the numerator with γ, and the denominator with β.

$$
a_{\bigwedge\{N_i\}_{i\in[1..k+1]}} = \frac{\gamma}{\beta}
$$

$$
\begin{aligned}
\gamma &= b_{\wedge\{N_i\}_{i\in[1..k]}} u_{N_{k+1}} a_{N_{k+1}} + b_{N_{k+1}} u_{\wedge\{N_i\}_{i\in[1..k]}} a_{\wedge\{N_i\}_{i\in[1..k]}} \\
&\quad + u_{\wedge\{N_i\}_{i\in[1..k]}} a_{\wedge\{N_i\}_{i\in[1..k]}} u_{N_{k+1}} a_{N_{k+1}} \\
\gamma &= \left[\prod_{i=1}^{k} b_{N_i}\right] u_{N_{k+1}} a_{N_{k+1}} \\
&\quad + \left[\prod_{i=1}^{k}(b_{N_i} + u_{N_i}) - \prod_{i=1}^{k}(b_{N_i})\right]\left[\frac{\prod_{i=1}^{k}(b_{N_i} + u_{N_i} a_{N_i}) - \prod_{i=1}^{k}(b_{N_i})}{\prod_{i=1}^{k}(b_{N_i} + u_{N_i}) - \prod_{i=1}^{k}(b_{N_i})}\right] b_{N_{k+1}} \\
&\quad + \left[\prod_{i=1}^{k}(b_{N_i} + u_{N_i}) - \prod_{i=1}^{k}(b_{N_i})\right]\left[\frac{\prod_{i=1}^{k}(b_{N_i} + u_{N_i} a_{N_i}) - \prod_{i=1}^{k}(b_{N_i})}{\prod_{i=1}^{k}(b_{N_i} + u_{N_i}) - \prod_{i=1}^{k}(b_{N_i})}\right] u_{N_{k+1}} a_{N_{k+1}} \\
\gamma &= \left[\prod_{i=1}^{k} b_{N_i} u_{N_{k+1}} a_{N_{k+1}}\right] + \left[\prod_{i=1}^{k}(b_{N_i} + u_{N_i} a_{N_i}) - \prod_{i=1}^{k}(b_{N_i})\right] b_{N_{k+1}} \\
&\quad + \left[\prod_{i=1}^{k}(b_{N_i} + u_{N_i} a_{N_i}) - \prod_{i=1}^{k}(b_{N_i})\right] u_{N_{k+1}} a_{N_{k+1}} \\
\gamma &= \left[\prod_{i=1}^{k} b_{N_i} u_{N_{k+1}} a_{N_{k+1}}\right] + \left[\prod_{i=1}^{k}(b_{N_i} + u_{N_i} a_{N_i}) - \prod_{i=1}^{k}(b_{N_i})\right]\left[b_{N_{k+1}} + u_{N_{k+1}} a_{N_{k+1}}\right] \\
\gamma &= \left[\prod_{i=1}^{k} b_{N_i} u_{N_{k+1}} a_{N_{k+1}}\right] + \left[\prod_{i=1}^{k}(b_{N_i} + u_{N_i} a_{N_i})\right]\left[b_{N_{k+1}} + u_{N_{k+1}} a_{N_{k+1}}\right] \\
&\quad - \prod_{i=1}^{k}(b_{N_i})\left[b_{N_{k+1}} + u_{N_{k+1}} a_{N_{k+1}}\right]
\end{aligned}
$$

$$\gamma = \left[\prod_{i=1}^{k} b_{N_i} u_{N_{k+1}} a_{N_{k+1}}\right] + \left[\prod_{i=1}^{k+1}(b_{N_i} + u_{N_i} a_{N_i})\right] - \prod_{i=1}^{k}(b_{N_i}) b_{N_{k+1}} - \prod_{i=1}^{k}(b_{N_i}) u_{N_{k+1}} a_{N_{k+1}}$$

$$\gamma = \left[\prod_{i=1}^{k+1}(b_{N_i} + u_{N_i} a_{N_i})\right] - \prod_{i=1}^{k+1}(b_{N_i})$$

$$\beta = b_{\wedge\{N_i\}_{i\in[1..k]}} u_{N_{k+1}} + u_{\wedge\{N_i\}_{i\in[1..k]}} b_{N_{k+1}} + u_{\wedge\{N_i\}_{i\in[1..k]}} u_{N_{k+1}}$$

$$\beta = \left[\prod_{i=1}^{k} b_{N_i}\right] u_{N_{k+1}} + \left[\prod_{i=1}^{k}(b_{N_i} + u_{N_i}) - \prod_{i=1}^{k}(b_{N_i})\right] b_{N_{k+1}}$$
$$+ \left[\prod_{i=1}^{k}(b_{N_i} + u_{N_i}) - \prod_{i=1}^{k}(b_{N_i})\right] u_{N_{k+1}}$$

$$\beta = \prod_{i=1}^{k} b_{N_i} u_{N_{k+1}} + \prod_{i=1}^{k}(b_{N_i} + u_{N_i}) b_{N_{k+1}} - \prod_{i=1}^{k}(b_{N_i}) b_{N_{k+1}}$$
$$+ \prod_{i=1}^{k}(b_{N_i} + u_{N_i}) u_{N_{k+1}} - \prod_{i=1}^{k}(b_{N_i}) u_{N_{k+1}}$$

$$\beta = \left[\prod_{i=1}^{k}(b_{N_i} + u_{N_i})\right](b_{N_{k+1}} + u_{N_{k+1}}) - \prod_{i=1}^{k+1}(b_{N_i})$$
$$= \prod_{i=1}^{k+1}(b_{N_i} + u_{N_i}) - \prod_{i=1}^{k+1}(b_{N_i})$$

$$a_{\bigwedge\{N_i\}_{i\in[1..k+1]}} = \frac{\gamma}{\beta} = \frac{\left[\prod_{i=1}^{k+1}(b_{N_i} + u_{N_i} a_{N_i})\right] - \prod_{i=1}^{k+1}(b_{N_i})}{\prod_{i=1}^{k+1}(b_{N_i} + u_{N_i}) - \prod_{i=1}^{k+1}(b_{N_i})}$$

Thus, Lemma 4 holds for $n = k + 1$. By the principle of induction, Lemma 4 is true for all $n \in Z^{+}$. □

□

A.1.1 Verifications: in this section we verify the following relations:

$$\begin{cases} 0 < b_\sigma = \prod_{i=1}^{n} b_{N_i} < 1 \\ 0 < d_\sigma = 1 - \prod_{i=1}^{n}(1 - d_{N_i}) < 1 \\ 0 < u_\sigma = \prod_{i=1}^{n}(b_{N_i} + u_{N_i}) - \prod_{i=1}^{n}(b_{N_i}) < 1 \\ 0 < a_\sigma = \frac{\prod_{i=1}^{n}(b_{N_i} + u_{N_i} a_{N_i}) - \prod_{i=1}^{n}(b_{N_i})}{\prod_{i=1}^{n}(b_{N_i} + u_{N_i}) - \prod_{i=1}^{n}(b_{N_i})} < 1 \\ b_\sigma + d_\sigma + u_\sigma = 1 \end{cases}$$

Lemma 5. $0 < b_\sigma = \prod_{i=1}^{n} b_{N_i} < 1$

Proof.

$$\forall i \in [1..n] : 0 < b_{N_i} < 1$$

The multiplications of several values between 0 and 1 is between 0 and 1 $\Rightarrow$

$$0 < \prod_{i=1}^{n} b_{N_i} < 1 \Rightarrow 0 < b_\sigma < 1$$

□ □

Lemma 6. $0 < d_\sigma = 1 - \prod_{i=1}^{n}(1 - d_{N_i}) < 1$

Proof.

$$\begin{aligned}
&\forall i \in [1..n]: \\
&0 < d_{N_i} < 1 \Rightarrow \\
&1 > 1 - d_{N_i} > 0 \Rightarrow \\
&1 > \prod_{i=1}^{n}(1 - d_{N_i}) > 0 \Rightarrow \\
&0 < 1 - \prod_{i=1}^{n}(1 - d_{N_i}) < 1 \Rightarrow \\
&0 < d_\sigma < 1
\end{aligned}$$

□ □

Lemma 7. $0 < u_\sigma = \prod_{i=1}^{n}(b_{N_i} + u_{N_i}) - \prod_{i=1}^{n}(b_{N_i}) < 1$

Proof. The left side:

$$\begin{aligned}
&\forall i \in [1..n] : u_{N_i} > 0 \Rightarrow \\
&b_{N_i} + u_{N_i} > b_{N_i} \Rightarrow \\
&\prod_{i=1}^{n}(b_{N_i} + u_{N_i}) > \prod_{i=1}^{n}(b_{N_i}) \Rightarrow \\
&\prod_{i=1}^{n}(b_{N_i} + u_{N_i}) - \prod_{i=1}^{n}(b_{N_i}) > 0 \Rightarrow \\
&u_\sigma > 0
\end{aligned}$$

The right side:

$$\begin{aligned}
&\forall i \in [1..n] : b_{N_i} + u_{N_i} < 1 \Rightarrow \\
&\prod_{i=1}^{n}(b_{N_i} + u_{N_i}) < 1 \Rightarrow \\
&\prod_{i=1}^{n}(b_{N_i} + u_{N_i}) - \prod_{i=1}^{n}(b_{N_i}) < 1 \Rightarrow \\
&u_\sigma < 1
\end{aligned}$$

□ □

Lemma 8. $0 < a_\sigma = \frac{\prod_{i=1}^{n}(b_{N_i} + u_{N_i} a_{N_i}) - \prod_{i=1}^{n}(b_{N_i})}{\prod_{i=1}^{n}(b_{N_i} + u_{N_i}) - \prod_{i=1}^{n}(b_{N_i})} < 1$

Proof. The left side:

$$\left.\begin{array}{l}
\forall i \in [1..n] : \\
b_{N_i} + u_{N_i} a_{N_i} > b_{N_i} \Rightarrow \prod_{i=1}^{n}(b_{N_i} + u_{N_i} a_{N_i}) - \prod_{i=1}^{n}(b_{N_i}) > 0 \\
b_{N_i} + u_{N_i} > b_{N_i} \Rightarrow \prod_{i=1}^{n}(b_{N_i} + u_{N_i}) - \prod_{i=1}^{n}(b_{N_i}) > 0
\end{array}\right\} \Rightarrow$$

$$\frac{\prod_{i=1}^{n}(b_{N_i}+u_{N_i}a_{N_i})-\prod_{i=1}^{n}b_{N_i})}{\prod_{i=1}^{n}(b_{N_i}+u_{N_i})-\prod_{i=1}^{n}(b_{N_i})}>0\Rightarrow a_\sigma>0$$

The right side:

$$\begin{aligned}
&\forall i\in[1..n]:0<a_{N_i}<1\Rightarrow\\
&u_{N_i}a_{N_i}<u_{N_i}\Rightarrow\\
&b_{N_i}+u_{N_i}a_{N_i}<b_{N_i}+u_{N_i}\Rightarrow\\
&\prod_{i=1}^{n}(b_{N_i}+u_{N_i}a_{N_i})<\prod_{i=1}^{n}(b_{N_i}+u_{N_i})\Rightarrow\\
&\prod_{i=1}^{n}(b_{N_i}+u_{N_i}a_{N_i})-\prod_{i=1}^{n}(b_{N_i})<\prod_{i=1}^{n}(b_{N_i}+u_{N_i})-\prod_{i=1}^{n}(b_{N_i})\Rightarrow\\
&\frac{\prod_{i=1}^{n}(b_{N_i}+u_{N_i}a_{N_i})-\prod_{i=1}^{n}(b_{N_i})}{\prod_{i=1}^{n}(b_{N_i}+u_{N_i})-\prod_{i=1}^{n}(b_{N_i})}<1\Rightarrow\\
&a_\sigma<1
\end{aligned}$$

□ □

Lemma 9. $b_\sigma+d_\sigma+u_\sigma=1$

Proof.

$$\begin{cases}b_\sigma=\prod_{i=1}^{n}b_{N_i}\\ d_\sigma=1-\prod_{i=1}^{n}(1-d_{N_i})\\ u_\sigma=\prod_{i=1}^{n}(b_{N_i}+u_{N_i})-\prod_{i=1}^{n}(b_{N_i})\end{cases}\Rightarrow$$

$$\begin{aligned}
&b_\sigma+d_\sigma+u_\sigma=\prod_{i=1}^{n}b_{N_i}+1-\prod_{i=1}^{n}(1-d_{N_i})+\prod_{i=1}^{n}(b_{N_i}+u_{N_i})-\prod_{i=1}^{n}(b_{N_i})\\
&b_\sigma+d_\sigma+u_\sigma=\prod_{i=1}^{n}b_{N_i}+1-\prod_{i=1}^{n}(b_{N_i}+u_{N_i})+\prod_{i=1}^{n}(b_{N_i}+u_{N_i})-\prod_{i=1}^{n}(b_{N_i})\\
&b_\sigma+d_\sigma+u_\sigma=1
\end{aligned}$$

□ □

A.2 Opinion about a system (mathematical proof):

$$O_{x\vee y}=\begin{cases} b_{x\vee y}=b_x+b_y-b_xb_y\\ d_{x\vee y}=d_xd_y\\ u_{x\vee y}=d_xu_y+u_xd_y+u_xu_y\\ a_{x\vee y}=\frac{u_xa_x+u_ya_y-b_xu_ya_y-b_yu_xa_x-u_xa_xu_ya_y}{u_x+u_y-b_xu_y-b_yu_x-u_xu_y}\end{cases}\Rightarrow$$

$$O_{\alpha=\{\sigma_1,\ldots,\sigma_m\}}=\begin{cases} b_{\alpha=\{\sigma_1,\ldots,\sigma_m\}}=b_{\bigvee\{\sigma_i\}}=1-\prod_{i=1}^{m}(1-b_{\sigma_i})\\ d_{\alpha=\{\sigma_1,\ldots,\sigma_m\}}=d_{\bigvee\{\sigma_i\}}=\prod_{i=1}^{m}d_{\sigma_i}\\ u_{\alpha=\{\sigma_1,\ldots,\sigma_m\}}=u_{\bigvee\{\sigma_i\}}=\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i})\\ a_{\alpha=\{\sigma_1,\ldots,\sigma_m\}}=a_{\bigvee\{\sigma_i\}}=\frac{\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i})}{\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i})}\end{cases}$$

1. The mathematical proof of the relation b_α:

Lemma 10. $b_{\alpha=\{\sigma_1,\ldots,\sigma_m\}}=b_{\vee\{\sigma_i\}_{i\in[1..m]}}=1-\prod_{i=1}^{m}(1-b_{\sigma_i})$

Proof. We prove by induction that, for all $m\in Z^+$,

$$b_{\vee\{\sigma_i\}_{i\in[1..m]}}=1-\prod_{i=1}^{m}(1-b_{\sigma_i}) \quad (12)$$

Base case. When $m=2$:

$$\begin{aligned} b_{\sigma_1\vee\sigma_2}&=b_{\sigma_2}+b_{\sigma_1}-b_{\sigma_1}b_{\sigma_2}\\ &=1-(1-b_{\sigma_2}-b_{\sigma_1}+b_{\sigma_1}b_{\sigma_2})\\ &=1-(1-b_{\sigma_1})(1-b_{\sigma_2})\\ &=1-\prod_{i=1}^{2}(1-b_{\sigma_i})\end{aligned}$$

Induction step. Let $k\in Z^+$ be given and suppose that Lemma 10 is true for $m=k$. Then

$$
\begin{aligned}
b_{\vee\{\sigma_i\}_{i\in[1..k+1]}} &= b_{\vee\{\{\sigma_i\}_{i\in[1..k]}\}\vee\sigma_{k+1}} \\
&= b_{\{\sigma_i\}_{i\in[1..k]}} + b_{\sigma_{k+1}} - b_{\{\sigma_i\}_{i\in[1..k]}} b_{\sigma_{k+1}} \\
&= \left[1 - \prod_{i=1}^{k}(1-b_{\sigma_i})\right] + b_{\sigma_{k+1}} - \left[1 - \prod_{i=1}^{k}(1-b_{\sigma_i})\right] b_{\sigma_{k+1}} \\
&= 1 - \prod_{i=1}^{k}(1-b_{\sigma_i}) + b_{\sigma_{k+1}} - b_{\sigma_{k+1}} + b_{\sigma_{k+1}} \prod_{i=1}^{k}(1-b_{\sigma_i}) \\
&= 1 - \prod_{i=1}^{k}(1-b_{\sigma_i}) + b_{\sigma_{k+1}} \prod_{i=1}^{k}(1-b_{\sigma_i}) \\
&= 1 - \left[\prod_{i=1}^{k}(1-b_{\sigma_i})\right][1-b_{\sigma_{k+1}}] \\
&= 1 - \prod_{i=1}^{k+1}(1-b_{\sigma_i})
\end{aligned}
$$

Thus, Lemma 10 holds for $m = k + 1$. By the principle of induction, Lemma 10 is true for all $m \in Z^+$. □

□

2. The mathematical proof of the relation d_α:

Lemma 11. $d_{\alpha=\{\sigma_1,\ldots,\sigma_m\}} = d_{\bigvee\{\sigma_i\}} = \prod_{i=1}^{m} d_{\sigma_i}$

Proof. We prove by induction that, for all $m \in Z^+$,

$$d_{\bigvee\{\sigma_i\}_{i\in[1..m]}} = \prod_{i=1}^{m} d_{\sigma_i}$$

Base case. When $m = 2$:

$$d_{\sigma_1\vee\sigma_2} = d_{\sigma_1} d_{\sigma_2} = \prod_{i=1}^{2} d_{\sigma_i}$$

Induction step. Let $k \in Z^+$ be given and suppose that Lemma 11 is true for $m = k$. Then

$$d_{\vee\{\sigma_i\}_{i\in[1..k+1]}} = d_{\vee\{\{\sigma_i\}_{i\in[1..k]}\}\vee\sigma_{k+1}} = d_{\vee\{\{\sigma_i\}_{i\in[1..k]}\}} d_{\sigma_{k+1}} = \prod_{i=1}^{k} d_{\sigma_i} d_{\sigma_{k+1}} = \prod_{i=1}^{k+1} d_{\sigma_i}$$

Thus, Lemma 11 holds for $m = k + 1$. By the principle of induction, Lemma 11 is true for all $m \in Z^+$. □

□

3. The mathematical proof of the relation u_α:

Lemma 12. $u_{\alpha=\{\sigma_1,\ldots,\sigma_m\}} = u_{\bigvee\{\sigma_i\}} = \prod_{i=1}^{m}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{m}(d_{\sigma_i})$

Proof. We prove by induction that, for all $m \in Z^+$,

$$u_{\bigvee\{\sigma_i\}_{i\in[1..m]}} = \prod_{i=1}^{m}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{m}(d_{\sigma_i})$$

Base case. When $m = 2$:

$$\begin{aligned} u_{\sigma_1 \vee \sigma_2} &= d_{\sigma_1}u_{\sigma_2} + u_{\sigma_1}d_{\sigma_2} + u_{\sigma_1}u_{\sigma_2} \\ &= d_{\sigma_1}u_{\sigma_2} + u_{\sigma_1}d_{\sigma_2} + u_{\sigma_1}u_{\sigma_2} + d_{\sigma_1}d_{\sigma_2} - d_{\sigma_1}d_{\sigma_2} \\ &= (d_{\sigma_1} + u_{\sigma_1})(d_{\sigma_2} + u_{\sigma_2}) - d_{\sigma_1}d_{\sigma_2} \\ &= \prod_{i=1}^{2}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{2}(d_{\sigma_i}) \end{aligned}$$

Induction step. Let $k \in Z^+$ be given and suppose that Lemma 12 is true for $m = k$. Then

$$\begin{aligned} u_{\vee\{\sigma_i\}_{i\in[1..k+1]}} &= u_{\vee\{\{\sigma_i\}_{i\in[1..k]}\}\vee\sigma_{k+1}} \\ &= d_{\{\sigma_i\}_{i\in[1..k]}}u_{\sigma_{k+1}} + u_{\{\sigma_i\}_{i\in[1..k]}}d_{\sigma_{k+1}} + u_{\{\sigma_i\}_{i\in[1..k]}}u_{\sigma_{k+1}} \\ &= \left[\prod_{i=1}^{k} d_{\sigma_i}\right] u_{\sigma_{k+1}} + \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i})\right] d_{\sigma_{k+1}} \\ &+ \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i})\right] u_{\sigma_{k+1}} \\ &= \prod_{i=1}^{k} d_{\sigma_i}u_{\sigma_{k+1}} + \prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i})d_{\sigma_{k+1}} - \prod_{i=1}^{k}(d_{\sigma_i})d_{\sigma_{k+1}} \\ &+ \prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i})u_{\sigma_{k+1}} - \prod_{i=1}^{k}(d_{\sigma_i})u_{\sigma_{k+1}} \\ &= \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i})\right](d_{\sigma_{k+1}} + u_{\sigma_{k+1}}) - \prod_{i=1}^{k+1}(d_{\sigma_i}) \\ &= \prod_{i=1}^{k+1}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k+1}(d_{\sigma_i}) \end{aligned}$$

Thus, Lemma 12 holds for $m = k + 1$. By the principle of induction, Lemma 12 is true for all $m \in Z^+$. □

□

4. The mathematical proof of the relation a_α:

Lemma 13. $a_\alpha = \frac{\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i})}{\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i})}$

Proof. We prove by induction that, for all $m \in Z^+$,

$$a_{\bigvee\{\sigma_i\}} = \frac{\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i})}{\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i})}$$

Base case. When $m = 2$:

$$a_{\sigma_1\vee\sigma_2} = \frac{u_{\sigma_1}a_{\sigma_1}+u_{\sigma_2}a_{\sigma_2}-b_{\sigma_1}u_{\sigma_2}a_{\sigma_2}-b_{\sigma_2}u_{\sigma_1}a_{\sigma_1}-u_{\sigma_1}a_{\sigma_1}u_{\sigma_2}a_{\sigma_2}}{u_{\sigma_1}+u_{\sigma_2}-b_{\sigma_1}u_{\sigma_2}-b_{\sigma_2}u_{\sigma_1}-u_{\sigma_1}u_{\sigma_2}}$$

We denote the numerator with γ, and the denominator with β.

$$a_{\sigma_1\vee\sigma_2} = \frac{\gamma}{\beta}$$

$$\begin{aligned}
\gamma &= u_{\sigma_1}a_{\sigma_1}+u_{\sigma_2}a_{\sigma_2}-b_{\sigma_1}u_{\sigma_2}a_{\sigma_2}-b_{\sigma_2}u_{\sigma_1}a_{\sigma_1}-u_{\sigma_1}a_{\sigma_1}u_{\sigma_2}a_{\sigma_2}\\
&= (1-b_{\sigma_1}-b_{\sigma_2}+b_{\sigma_1}b_{\sigma_2})-(1-b_{\sigma_1}-b_{\sigma_2}+b_{\sigma_1}b_{\sigma_2})\\
&+u_{\sigma_1}a_{\sigma_1}+u_{\sigma_2}a_{\sigma_2}-b_{\sigma_1}u_{\sigma_2}a_{\sigma_2}-b_{\sigma_2}u_{\sigma_1}a_{\sigma_1}-u_{\sigma_1}a_{\sigma_1}u_{\sigma_2}a_{\sigma_2}\\
&= (1-b_{\sigma_1}-b_{\sigma_2}+b_{\sigma_1}b_{\sigma_2})-(1-b_{\sigma_1}-b_{\sigma_2}+b_{\sigma_1}b_{\sigma_2}\\
&-u_{\sigma_1}a_{\sigma_1}-u_{\sigma_2}a_{\sigma_2}+b_{\sigma_1}u_{\sigma_2}a_{\sigma_2}+b_{\sigma_2}u_{\sigma_1}a_{\sigma_1}+u_{\sigma_1}a_{\sigma_1}u_{\sigma_2}a_{\sigma_2})\\
&= (1-b_{\sigma_1})(1-b_{\sigma_2})-(1-b_{\sigma_1}-u_{\sigma_1}a_{\sigma_1}-b_{\sigma_2}+b_{\sigma_1}b_{\sigma_2}+b_{\sigma_2}u_{\sigma_1}a_{\sigma_1}\\
&-u_{\sigma_2}a_{\sigma_2}+b_{\sigma_1}u_{\sigma_2}a_{\sigma_2}+u_{\sigma_1}a_{\sigma_1}u_{\sigma_2}a_{\sigma_2})\\
&= (1-b_{\sigma_1})(1-b_{\sigma_2})\\
&-[(1-b_{\sigma_1}-u_{\sigma_1}a_{\sigma_1})-b_{\sigma_2}(1-b_{\sigma_1}-u_{\sigma_1}a_{\sigma_1})-u_{\sigma_2}a_{\sigma_2}(1-b_{\sigma_1}-u_{\sigma_1}a_{\sigma_1})]\\
&= (1-b_{\sigma_1})(1-b_{\sigma_2})-[(1-b_{\sigma_1}-u_{\sigma_1}a_{\sigma_1})(1-b_{\sigma_2}-u_{\sigma_2}a_{\sigma_2})]\\
&= (d_{\sigma_1}+u_{\sigma_1})(d_{\sigma_2}+u_{\sigma_2})-[(d_{\sigma_1}+u_{\sigma_1}-u_{\sigma_1}a_{\sigma_1})(d_{\sigma_2}+u_{\sigma_2}-u_{\sigma_2}a_{\sigma_2})]\\
&= \prod_{i=1}^{2}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{2}(d_{\sigma_i}+u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i})
\end{aligned}$$

$$\begin{aligned}
\beta &= u_{\sigma_1}+u_{\sigma_2}-b_{\sigma_1}u_{\sigma_2}-b_{\sigma_2}u_{\sigma_1}-u_{\sigma_1}u_{\sigma_2}\\
&= u_{\sigma_1}+u_{\sigma_2}-(1-d_{\sigma_1}-u_{\sigma_1})u_{\sigma_2}-(1-d_{\sigma_2}-u_{\sigma_2})u_{\sigma_1}-u_{\sigma_1}u_{\sigma_2}\\
&= d_{\sigma_1}u_{\sigma_2}+u_{\sigma_1}u_{\sigma_2}+d_{\sigma_2}u_{\sigma_1}+u_{\sigma_2}u_{\sigma_1}-u_{\sigma_1}u_{\sigma_2}\\
&= d_{\sigma_1}u_{\sigma_2}+d_{\sigma_2}u_{\sigma_1}+u_{\sigma_2}u_{\sigma_1}\\
&= u_{\sigma_1\wedge\sigma_2}\\
&= \prod_{i=1}^{2}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{2}(d_{\sigma_i})
\end{aligned}$$

$$a_{\sigma_1\vee\sigma_2} = \frac{\gamma}{\beta} = \frac{\prod_{i=1}^{2}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{2}(d_{\sigma_i}+u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i})}{\prod_{i=1}^{2}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{2}(d_{\sigma_i})}$$

Induction step. Let $k \in Z^+$ be given and suppose that Lemma 13 is true for $m = k$. Then

$$a_{\vee\{\sigma_i\}_{i\in[1..k+1]}} = a_{\vee\{\{\sigma_i\}_{i\in[1..k]}\}\vee\sigma_{k+1}} = \frac{\gamma}{\beta}:$$

$$\gamma = u_{\{\sigma_i\}_{i\in[1..k]}} a_{\{\sigma_i\}_{i\in[1..k]}} + u_{\sigma_{k+1}} a_{\sigma_{k+1}} - b_{\{\sigma_i\}_{i\in[1..k]}} u_{\sigma_{k+1}} a_{\sigma_{k+1}}$$
$$- b_{\sigma_{k+1}} u_{\{\sigma_i\}_{i\in[1..k]}} a_{\{\sigma_i\}_{i\in[1..k]}} - u_{\{\sigma_i\}_{i\in[1..k]}} a_{\{\sigma_i\}_{i\in[1..k]}} u_{\sigma_{k+1}} a_{\sigma_{k+1}}$$
$$\beta = u_{\{\sigma_i\}_{i\in[1..k]}} + u_{\sigma_{k+1}} - b_{\{\sigma_i\}_{i\in[1..k]}} u_{\sigma_{k+1}} - b_{\sigma_{k+1}} u_{\{\sigma_i\}_{i\in[1..k]}} - u_{\{\sigma_i\}_{i\in[1..k]}} u_{\sigma_{k+1}}$$

$$
\begin{aligned}
\gamma &= u_\alpha a_\alpha + u_{\sigma_{k+1}} a_{\sigma_{k+1}} - b_\alpha u_{\sigma_{k+1}} a_{\sigma_{k+1}} - b_{\sigma_{k+1}} u_\alpha a_\alpha - u_\alpha a_\alpha u_{\sigma_{k+1}} a_{\sigma_{k+1}} \\
&= \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i})\right] \left[\frac{\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i} - u_{\sigma_i} a_{\sigma_i})}{\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i})}\right] \\
&+ u_{\sigma_{k+1}} a_{\sigma_{k+1}} - \left[1 - \prod_{i=1}^{k}(1 - b_{\sigma_i})\right] u_{\sigma_{k+1}} a_{\sigma_{k+1}} \\
&- b_{\sigma_{k+1}} \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i})\right] \left[\frac{\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i} - u_{\sigma_i} a_{\sigma_i})}{\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i})}\right] \\
&- u_{\sigma_{k+1}} a_{\sigma_{k+1}} \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i})\right] \left[\frac{\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i} - u_{\sigma_i} a_{\sigma_i})}{\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i})}\right] \\
\gamma &= \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i} - u_{\sigma_i} a_{\sigma_i})\right] \\
&+ u_{\sigma_{k+1}} a_{\sigma_{k+1}} - \left[1 - \prod_{i=1}^{k}(1 - b_{\sigma_i})\right] u_{\sigma_{k+1}} a_{\sigma_{k+1}} \\
&- b_{\sigma_{k+1}} \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i} - u_{\sigma_i} a_{\sigma_i})\right] \\
&- u_{\sigma_{k+1}} a_{\sigma_{k+1}} \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i} - u_{\sigma_i} a_{\sigma_i})\right] \\
\gamma &= \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i} - u_{\sigma_i} a_{\sigma_i})\right] \left[1 - b_{\sigma_{k+1}} - u_{\sigma_{k+1}} a_{\sigma_{k+1}}\right] \\
&+ u_{\sigma_{k+1}} a_{\sigma_{k+1}} \left[1 - 1 + \prod_{i=1}^{k}(1 - b_{\sigma_i})\right] \\
\gamma &= \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i} - u_{\sigma_i} a_{\sigma_i})\right] \left[d_{\sigma_{k+1}} + u_{\sigma_{k+1}} - u_{\sigma_{k+1}} a_{\sigma_{k+1}}\right] \\
&+ u_{\sigma_{k+1}} a_{\sigma_{k+1}} \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i})\right] \\
\gamma &= \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i})\right] \left[d_{\sigma_{k+1}} + u_{\sigma_{k+1}} - u_{\sigma_{k+1}} a_{\sigma_{k+1}}\right] - \left[\prod_{i=1}^{k+1}(d_{\sigma_i} + u_{\sigma_i} - u_{\sigma_i} a_{\sigma_i})\right] \\
&+ \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i})\right] u_{\sigma_{k+1}} a_{\sigma_{k+1}} \\
\gamma &= \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i})\right] \left[d_{\sigma_{k+1}} + u_{\sigma_{k+1}}\right] - \left[\prod_{i=1}^{k+1}(d_{\sigma_i} + u_{\sigma_i} - u_{\sigma_i} a_{\sigma_i})\right] \\
\gamma &= \left[\prod_{i=1}^{k+1}(d_{\sigma_i} + u_{\sigma_i})\right] - \left[\prod_{i=1}^{k+1}(d_{\sigma_i} + u_{\sigma_i} - u_{\sigma_i} a_{\sigma_i})\right]
\end{aligned}
$$

$$\beta = u_\alpha + u_{\sigma_{k+1}} - b_\alpha u_{\sigma_{k+1}} - b_{\sigma_{k+1}} u_\alpha - u_\alpha u_{\sigma_{k+1}}$$
$$\beta = \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i})\right] + u_{\sigma_{k+1}} - \left[1 - \prod_{i=1}^{k}(1 - b_{\sigma_i})\right] u_{\sigma_{k+1}}$$
$$- b_{\sigma_{k+1}} \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i})\right] - \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i})\right] u_{\sigma_{k+1}}$$
$$\beta = \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i})\right] + u_{\sigma_{k+1}} - u_{\sigma_{k+1}} + \prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) u_{\sigma_{k+1}}$$
$$- \left[1 - d_{\sigma_{k+1}} - u_{\sigma_{k+1}}\right] \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i})\right] - \prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) u_{\sigma_{k+1}} + \prod_{i=1}^{k}(d_{\sigma_i}) u_{\sigma_{k+1}}$$
$$\beta = \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i})\right]$$
$$- \left[1 - d_{\sigma_{k+1}} - u_{\sigma_{k+1}}\right] \left[\prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i})\right] + \prod_{i=1}^{k}(d_{\sigma_i}) u_{\sigma_{k+1}}$$
$$\beta = \prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k}(d_{\sigma_i})$$
$$- \left[1 - d_{\sigma_{k+1}} - u_{\sigma_{k+1}}\right] \prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) + \left[1 - d_{\sigma_{k+1}} - u_{\sigma_{k+1}}\right] \prod_{i=1}^{k}(d_{\sigma_i}) + \prod_{i=1}^{k}(d_{\sigma_i}) u_{\sigma_{k+1}}$$
$$\beta = \prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) \left[1 - 1 + d_{\sigma_{k+1}} + u_{\sigma_{k+1}}\right] - \prod_{i=1}^{k}(d_{\sigma_i}) \left[1 - 1 + d_{\sigma_{k+1}} + u_{\sigma_{k+1}} - u_{\sigma_{k+1}}\right]$$
$$= \prod_{i=1}^{k}(d_{\sigma_i} + u_{\sigma_i}) \left[d_{\sigma_{k+1}} + u_{\sigma_{k+1}}\right] - \prod_{i=1}^{k}(d_{\sigma_i}) \left[d_{\sigma_{k+1}}\right]$$
$$= \prod_{i=1}^{k+1}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k+1}(d_{\sigma_i})$$

$$a_{\vee\{\sigma_i\}_{i\in[1..k+1]}} = \frac{\gamma}{\beta} = \frac{\left[\prod_{i=1}^{k+1}(d_{\sigma_i} + u_{\sigma_i})\right] - \left[\prod_{i=1}^{k+1}(d_{\sigma_i} + u_{\sigma_i} - u_{\sigma_i} a_{\sigma_i})\right]}{\prod_{i=1}^{k+1}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{k+1}(d_{\sigma_i})}$$

Thus, Lemma 13 holds for $m = k + 1$. By the principle of induction, Lemma 13 is true for all $m \in Z^+$. □

□

A.2.1 Verifications: in this section we verify the following relations:

$$\begin{cases} 0 < b_\alpha = 1 - \prod_{i=1}^{m}(1 - b_{\sigma_i}) < 1 \\ 0 < d_\alpha = \prod_{i=1}^{m} d_{\sigma_i} < 1 \\ 0 < u_\alpha = \prod_{i=1}^{m}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{m}(d_{\sigma_i}) < 1 \\ 0 < a_\alpha = \frac{\prod_{i=1}^{m}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{m}(d_{\sigma_i} + u_{\sigma_i} - u_{\sigma_i} a_{\sigma_i})}{\prod_{i=1}^{m}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{m}(d_{\sigma_i})} < 1 \\ b_\alpha + d_\alpha + u_\alpha = 1 \end{cases}$$

Lemma 14. $0 < b_\alpha = 1 - \prod_{i=1}^{m}(1 - b_{\sigma_i}) < 1$

Proof.

$$
\begin{aligned}
&\forall i \in [1..m]: \\
&0 < b_{\sigma_i} < 1 \Rightarrow \\
&1 > 1 - b_{\sigma_i} > 0 \Rightarrow \\
&1 > \prod_{i=1}^{m}(1 - b_{\sigma_i}) > 0 \Rightarrow \\
&0 < 1 - \prod_{i=1}^{m}(1 - b_{\sigma_i}) < 1 \Rightarrow \\
&0 < b_\alpha < 1
\end{aligned}
$$

□ □

Lemma 15. $0 < d_\alpha = \prod_{i=1}^{m} d_{\sigma_i} < 1$

Proof.

$$\forall i \in [1..m] : 0 < d_{\sigma_i} < 1$$

The multiplication of several values between 0 and 1 is between 0 and 1 $\Rightarrow$

$$0 < \prod_{i=1}^{m} d_{\sigma_i} < 1 \Rightarrow 0 < d_\alpha < 1$$

□ □

Lemma 16. $0 < u_\alpha = \prod_{i=1}^{m}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{m}(d_{\sigma_i}) < 1$

Proof. The left side:

$$
\begin{aligned}
&\forall i \in [1..m] : u_{\sigma_i} > 0 \Rightarrow \\
&d_{\sigma_i} + u_{\sigma_i} > d_{\sigma_i} \Rightarrow \\
&\prod_{i=1}^{m}(d_{\sigma_i} + u_{\sigma_i}) > \prod_{i=1}^{m}(d_{\sigma_i}) \Rightarrow \\
&\prod_{i=1}^{m}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{m}(d_{\sigma_i}) > 0 \Rightarrow \\
&u_\alpha > 0
\end{aligned}
$$

The right side:

$$
\begin{aligned}
&\forall i \in [1..m] : d_{\sigma_i} + u_{\sigma_i} < 1 \Rightarrow \\
&\prod_{i=1}^{m}(d_{\sigma_i} + u_{\sigma_i}) < 1 \Rightarrow \\
&\prod_{i=1}^{m}(d_{\sigma_i} + u_{\sigma_i}) - \prod_{i=1}^{m}(d_{\sigma_i}) < 1 \Rightarrow \\
&u_\alpha < 1
\end{aligned}
$$

□

□

Lemma 17. $0 < a_\alpha = \frac{\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i})}{\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i})} < 1$

Proof. The left side:

$$\left.\begin{array}{l}\forall i \in [1..m]:\\ d_{\sigma_i}+u_{\sigma_i} > d_{\sigma_i}+u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i} \Rightarrow \prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i}) > 0\\ d_{\sigma_i}+u_{\sigma_i} > d_{\sigma_i} \Rightarrow \prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i}) > 0\end{array}\right\} \Rightarrow$$

$$a_\alpha = \frac{\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i})}{\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i})} > 0$$

The right side:

$$\begin{aligned}\forall i \in [1..m]: 0 < a_{\sigma_i} < 1 \Rightarrow\\ & u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i} > 0 \Rightarrow\\ & d_{\sigma_i}+u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i} > d_{\sigma_i} \Rightarrow\\ & \prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i}) > \prod_{i=1}^{m}(d_{\sigma_i}) \Rightarrow\\ & -\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i}) < -\prod_{i=1}^{m}(d_{\sigma_i}) \Rightarrow\\ & -\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i}) < -\prod_{i=1}^{m}(d_{\sigma_i}) \Rightarrow\\ & \prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i}) < \prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i}) \Rightarrow\\ & \frac{\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i})}{\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i})} < 1\end{aligned}$$

□

□

Lemma 18. $b_\alpha + d_\alpha + u_\alpha = 1$

Proof.

$$\begin{cases} b_\alpha = 1-\prod_{i=1}^{m}(1-b_{\sigma_i})\\ d_\alpha = \prod_{i=1}^{m} d_{\sigma_i}\\ u_\alpha = \prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i})\end{cases} \Rightarrow$$

$$\begin{aligned} b_\alpha + d_\alpha + u_\alpha &= 1-\prod_{i=1}^{m}(1-b_{\sigma_i})+\prod_{i=1}^{m} d_{\sigma_i}+\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i})\\ b_\alpha + d_\alpha + u_\alpha &= 1-\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})+\prod_{i=1}^{m} d_{\sigma_i}+\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i})\\ b_\alpha + d_\alpha + u_\alpha &= 1\end{aligned}$$

□ □

A.3 Relations of M3 (mathematical proof):

- The case of splitting a dependent opinion into two independent opinions:

$$\begin{cases} b_{A_1} \vee b_{A_2} = b_A \\ d_{A_1} \vee d_{A_2} = d_A \\ u_{A_1} \vee u_{A_2} = u_A \\ a_{A_1} \vee a_{A_2} = a_A \end{cases} \bigwedge \begin{cases} b_{A_1} = b_{A_2} \\ d_{A_1} = d_{A_2} \\ u_{A_1} = u_{A_2} \\ a_{A_1} = a_{A_2} \end{cases} \Rightarrow \begin{cases} b_{A_1} = b_{A_2} = 1 - \sqrt{1 - b_A} \\ d_{A_1} = d_{A_2} = \sqrt{d_A} \\ u_{A_1} = u_{A_2} = \sqrt{d_A + u_A} - \sqrt{d_A} \\ a_{A_1} = a_{A_2} = \frac{\sqrt{1-b_A} - \sqrt{1-b_A-a_A u_A}}{\sqrt{d_A+u_A} - \sqrt{d_A}} \end{cases} \quad (13)$$

- The case of splitting a dependent opinion into n independent opinions:

$$\begin{cases} b_{A_1} \vee b_{A2} \vee \ldots \vee b_{A_n} = b_A \\ d_{A_1} \vee d_{A2} \vee \ldots \vee d_{A_n} = d_A \\ u_{A_1} \vee u_{A2} \vee \ldots \vee u_{A_n} = u_A \\ a_{A_1} \vee a_{A2} \vee \ldots \vee a_{A_n} = a_A \end{cases} \bigwedge \begin{cases} b_{A_1} = b_{A2} = \ldots = b_{A_n} \\ d_{A_1} = d_{A2} = \ldots = d_{A_n} \\ u_{A_1} = u_{A2} = \ldots = u_{A_n} \\ a_{A_1} = a_{A2} = \ldots = a_{A_n} \end{cases} \Rightarrow$$

$$\begin{cases} b_{A1} = b_{A2} = \ldots = b_{A_n} = 1 - (1 - b_A)^{\frac{1}{n}} \\ d_{A1} = d_{A2} = \ldots = d_{A_n} = d_A^{\frac{1}{n}} \\ u_{A1} = u_{A2} = \ldots = u_{A_n} = (d_A + u_A)^{\frac{1}{n}} - d_A^{\frac{1}{n}} \\ a_{A1} = a_{A2} = \ldots = a_{A_n} = \frac{(1-b_A)^{\frac{1}{n}} - (1-b_A-a_A u_A)^{\frac{1}{n}}}{(d_A+u_A)^{\frac{1}{n}} - d_A^{\frac{1}{n}}} \end{cases} \quad (14)$$

In the following sections we denote $O_A = (b_A, d_A, u_A, a_A)$ with $O_A = (b, d, u, a)$ for simplicity.

A.3.1 The mathematical proof of Relations 13:

1. The mathematical proof of the relation b in case of splitting a node into two independent nodes:

Lemma 19. $(b_{A1} \vee b_{A2} = b) \wedge (b_{A1} = b_{A2}) \Rightarrow b_{A_1} = b_{A_2} = 1 - \sqrt{1 - b}$

Proof.

$$\begin{aligned} &b_{A1} \vee b_{A2} = b \Rightarrow \\ &b_{A1} + b_{A2} - b_{A1} b_{A2} = b \Rightarrow \\ &b_{A1} + b_{A1} - b_{A1} b_{A1} = b \Rightarrow \\ &b_{A1}^2 - 2b_{A1} + b = 0 \end{aligned}$$

This is an equation from the second degree

$$\begin{aligned} &\Delta = 4 - 4b = 4(1 - b) > 0 \Rightarrow \\ &b_{A1} = b_{A2} = \frac{2 + 2\sqrt{(1-b)}}{2} = 1 + \sqrt{(1-b)} > 1 (refused) \\ &b_{A1} = b_{A2} = \frac{2 - 2\sqrt{(1-b)}}{2} = 1 - \sqrt{(1-b)} \\ &0 < b_{A1} = b_{A2} = 1 - \sqrt{(1-b)} < 1 \quad (accepted\ solution) \end{aligned}$$

□ □

2. The mathematical proof of the relation d in case of splitting a node into two independent nodes:

Lemma 20. $(d_{A1} \vee d_{A2} = d) \wedge (d_{A1} = d_{A2}) \Rightarrow d_{A_1} = d_{A_2} = \sqrt{d}$

Proof.

$$\begin{aligned}
&d_{A1} \vee d_{A2} = d \Rightarrow \\
&d_{A1} d_{A2} = d \Rightarrow \\
&d_{A1}^2 = d \Rightarrow \\
&d_{A1} = d_{A2} = -\sqrt{d} < 0 (refused) \\
&d_{A1} = d_{A2} = \sqrt{d} \\
&0 < d_{A1} = d_{A2} = \sqrt{d} < 1 \quad (accepted\ solution)
\end{aligned}$$

□ □

3. The mathematical proof of the relation u in case of splitting a node into two independent nodes:

Lemma 21. $(u_{A1} \vee u_{A2} = u) \wedge (u_{A1} = u_{A2}) \Rightarrow u_{A_1} = u_{A_2} = \sqrt{d+u} - \sqrt{d}$

Proof.

$$\begin{aligned}
&u_{A1} \vee u_{A2} = u \Rightarrow \\
&d_{A1} u_{A2} + d_{A2} u_{A1} + u_{A1} u_{A2} = u \Rightarrow \\
&d_{A1} u_{A1} + d_{A1} u_{A1} + u_{A1} u_{A1} = u \Rightarrow \\
&2 d_{A1} u_{A1} + u_{A1}^2 = u \Rightarrow \\
&u_{A1}^2 + 2 d_{A1} u_{A1} - u = 0 \Rightarrow \\
&u_{A1}^2 + 2\sqrt{d} u_{A1} - u = 0 \Rightarrow
\end{aligned}$$

This is an equation from the second degree

$$\begin{aligned}
&\Delta = 4d + 4u = 4(d+u) > 0 \Rightarrow \\
&u_{A1} = u_{A2} = \frac{-2\sqrt{d} - 2\sqrt{(d+u)}}{2} = -\sqrt{d} - \sqrt{(d+u)} < 0 (refused) \\
&u_{A1} = u_{A2} = \frac{-2\sqrt{d} + 2\sqrt{(d+u)}}{2} = -\sqrt{d} + \sqrt{(d+u)} \\
&0 < u_{A1} = u_{A2} = -\sqrt{d} + \sqrt{(d+u)} < 1 \quad (accepted\ solution)
\end{aligned}$$

□ □

4. The mathematical proof of the relation a in case of splitting a node into 2 independent nodes:

Lemma 22. $(a_{A1} \vee a_{A2} = a) \wedge (a_{A1} = a_{A2}) \Rightarrow a_{A_1} = a_{A_2} = \frac{\sqrt{1-b} - \sqrt{1-b-au}}{\sqrt{d+u} - \sqrt{d}}$

Proof.

$$
\begin{aligned}
& a_{A1} \vee a_{A2} = a \Rightarrow \\
& \frac{u_{A_1}a_{A_1} + u_{A_2}a_{A_2} - b_{A_1}u_{A_2}a_{A_2} - b_{A_2}u_{A_1}a_{A_1} - u_{A_1}a_{A_1}u_{A_2}a_{A_2}}{u_{A_1} + u_{A_2} - b_{A_1}u_{A_2} - b_{A_2}u_{A_1} - u_{A_1}u_{A_2}} = a \Rightarrow \\
& \frac{u_{A_1}a_{A_1} + u_{A_1}a_{A_1} - b_{A_1}u_{A_1}a_{A_1} - b_{A_1}u_{A_1}a_{A_1} - u_{A_1}a_{A_1}u_{A_1}a_{A_1}}{u_{A_1} + u_{A_1} - b_{A_1}u_{A_1} - b_{A_1}u_{A_1} - u_{A_1}u_{A_1}} = a \Rightarrow \\
& \frac{[2 - 2b_{A_1}]u_{A_1}a_{A_1} - u_{A_1}^2 a_{A_1}^2}{2u_{A_1} - 2b_{A_1}u_{A_1} - u_{A_1}^2} = a \Rightarrow \\
& \frac{2[1-(1-\sqrt{(1-b)})][-\sqrt{d}+\sqrt{(d+u)}]a_{A_1} - [-\sqrt{d}+\sqrt{(d+u)}]^2 a_{A_1}^2}{2[-\sqrt{d}+\sqrt{(d+u)}] - 2[1-\sqrt{(1-b)}][-\sqrt{d}+\sqrt{(d+u)}] - [-\sqrt{d}+\sqrt{(d+u)}]^2} = a \Rightarrow \\
& \frac{2[\sqrt{(d+u)}][-\sqrt{d}+\sqrt{(d+u)}]a_{A_1} - [-\sqrt{d}+\sqrt{(d+u)}]^2 a_{A_1}^2}{[-\sqrt{d}+\sqrt{(d+u)}][2-2[1-\sqrt{(d+u)}] - [-\sqrt{d}+\sqrt{(d+u)}]} = a \Rightarrow \\
& \frac{2[\sqrt{(d+u)}]a_{A_1} - [-\sqrt{d}+\sqrt{(d+u)}]a_{A_1}^2}{[2-2[1-\sqrt{(d+u)}] - [-\sqrt{d}+\sqrt{(d+u)}]} = a \Rightarrow \\
& \frac{[\sqrt{d}-\sqrt{(d+u)}]a_{A_1}^2 + 2[\sqrt{(d+u)}]a_{A_1}}{[2\sqrt{(d+u)}] + [\sqrt{d}-\sqrt{(d+u)}]} = a \Rightarrow \\
& [\sqrt{d}-\sqrt{(d+u)}]a_{A_1}^2 + 2[\sqrt{(d+u)}]a_{A_1} = [\sqrt{(d+u)}+\sqrt{d}]a \Rightarrow \\
& [\sqrt{d}-\sqrt{(d+u)}]a_{A_1}^2 + 2[\sqrt{(d+u)}]a_{A_1} - [\sqrt{(d+u)}+\sqrt{d}]a = 0 \Rightarrow
\end{aligned}
$$

This is an equation from the second degree

$$
\begin{aligned}
\Delta &= 4(d+u) - 4[\sqrt{d}-\sqrt{(d+u)}][-\sqrt{(d+u)}-\sqrt{d}]a \\
\Delta &= 4d + 4u + 4[d-d-u]a \\
\Delta &= 4[d+u-ua] \\
\Delta &= 4[1-b-ua]
\end{aligned}
$$

$$
\begin{aligned}
a_{A1} = a_{A2} &= \frac{-2[\sqrt{(d+u)}] + 2\sqrt{(1-b-ua)}}{2[\sqrt{d}-\sqrt{(d+u)}]} \\
&= \frac{[\sqrt{(1-b)}] + \sqrt{(1-b-ua)}}{[\sqrt{(d+u)}-\sqrt{d}]} > 1(refused) \\
a_{A1} = a_{A2} &= \frac{-2[\sqrt{(d+u)}] - 2\sqrt{(1-b-ua)}}{2[\sqrt{d}-\sqrt{(d+u)}]} \\
&= \frac{[\sqrt{(1-b)}] - \sqrt{(1-b-ua)}}{[\sqrt{(d+u)}-\sqrt{d}]} \\
0 < a_{A1} = a_{A2} &= \frac{[\sqrt{(1-b)}] - \sqrt{(1-b-ua)}}{[\sqrt{(d+u)}-\sqrt{d}]} < 1 \ \ (accepted\ solution)
\end{aligned}
$$

□ □

A.3.2 The mathematical proof of Relations 14: Relations 14 can be proved by induction. In this section, we will limit ourselves to verfying the following relations.

$$\begin{cases} b_{A1} \vee b_{A2} \vee \ldots \vee b_{A_n} = b \\ 0 \leq 1 - (1-b)^{\frac{1}{n}} \leq 1 \\ d_{A_1} \vee d_{A_2} \vee \ldots \vee d_{A_n} = d \\ 0 \leq d^{\frac{1}{n}} \leq 1 \\ u_{A_1} \vee u_{A_2} \vee \ldots \vee u_{A_n} = u \\ 0 \leq (d+u)^{\frac{1}{n}} - d^{\frac{1}{n}} \leq 1 \\ a_{A_1} \vee a_{A_2} \vee \ldots \vee a_{A_n} = a \\ 0 \leq a_{A_i} = \frac{(1-b)^{\frac{1}{n}} - (1-b-au)^{\frac{1}{n}}}{(d+u)^{\frac{1}{n}} - d^{\frac{1}{n}}} \leq 1 \\ b_{A_i} + d_{A_i} + u_{A_i} = 1 \end{cases}$$

Lemma 23. $b_{A1} \vee b_{A2} \vee \ldots \vee b_{A_n} = b$

Proof.

$$\begin{cases} From\ Relations\ \ 9 : b_{\bigvee\{A_i\}} = 1 - \prod_{i=1}^{n}(1 - b_{A_i}) \\ From\ Relations\ \ 13 : b_{A_i} = 1 - (1-b)^{\frac{1}{n}} \end{cases} \Rightarrow$$

$$\begin{aligned} b_{\bigvee\{A_i\}} &= 1 - \prod_{i=1}^{n}(1 - (1 - (1-b)^{\frac{1}{n}})) \\ &= 1 - \prod_{i=1}^{n}((1-b)^{\frac{1}{n}}) \\ &= 1 - (1-b) \\ &= b \end{aligned}$$

□ □

Lemma 24. $0 \leq 1 - (1-b)^{\frac{1}{n}} \leq 1$

Proof.

$$\begin{aligned} &0 \leq b \leq 1 \Rightarrow \\ &0 \leq 1 - b \leq 1 \Rightarrow \\ &0 \leq (1-b)^{\frac{1}{n}} \leq 1 \Rightarrow \\ &0 \leq 1 - (1-b)^{\frac{1}{n}} \leq 1 \end{aligned}$$

□ □

Lemma 25. $d_{A_1} \vee d_{A_2} \vee \ldots \vee d_{A_n} = d$

Proof.

$$\begin{cases} From\ Relations\ \ 9 : d_{\bigvee\{A_i\}} = \prod_{i=1}^{n} d_{A_i} \\ From\ Relations\ \ 13 : d_{A_i} = d^{\frac{1}{n}} \end{cases} \Rightarrow$$

$$d_{\bigvee\{A_i\}} = \prod_{i=1}^{n} d^{\frac{1}{n}}$$
$$= d$$

□ □

Lemma 26. $0 \leq d^{\frac{1}{n}} \leq 1$

Proof.

$$0 \leq d \leq 1 \Rightarrow$$
$$0 \leq d^{\frac{1}{n}} \leq 1$$

□ □

Lemma 27. $u_{A_1} \vee u_{A_2} \vee \ldots \vee u_{A_n} = u$

Proof.

$$\begin{cases} From\ Relations\ \ 9 : u_{\bigvee\{A_i\}} = \prod_{i=1}^{n}(d_{A_i} + u_{A_i}) - \prod_{i=1}^{n}(d_{A_i}) \\ From\ Relations\ \ 13 : \begin{cases} d_{A_i} = d^{\frac{1}{n}} \\ u_{A_i} = (d+u)^{\frac{1}{n}} - d^{\frac{1}{n}} \end{cases} \end{cases} \Rightarrow$$

$$u_{\bigvee\{A_i\}} = \prod_{i=1}^{n}[(d^{\frac{1}{n}}) + ((d+u)^{\frac{1}{n}} - d^{\frac{1}{n}}))] - \prod_{i=1}^{n} d^{\frac{1}{n}}$$
$$= \prod_{i=1}^{n}(d+u)^{\frac{1}{n}} - \prod_{i=1}^{n} d^{\frac{1}{n}}$$
$$= d + u - d$$
$$= u$$

□ □

Lemma 28. $0 \leq (d+u)^{\frac{1}{n}} - d^{\frac{1}{n}} \leq 1$

Proof. The left side:

$$d \leq d + u \Rightarrow$$
$$d^{\frac{1}{n}} \leq (d+u)^{\frac{1}{n}} \Rightarrow$$
$$0 \leq (d+u)^{\frac{1}{n}} - d^{\frac{1}{n}}$$

The right side:

$$d + u = 1 - b$$

$0 \leq b \leq 1 \Rightarrow$

$$\begin{aligned}
&d + u \leq 1 \Rightarrow \\
&(d+u)^{\frac{1}{n}} \leq 1 \Rightarrow \\
&(d+u)^{\frac{1}{n}} \leq 1 + d^{\frac{1}{n}} \Rightarrow \\
&(d+u)^{\frac{1}{n}} - d^{\frac{1}{n}} \leq 1 \Rightarrow \\
&u \leq 1
\end{aligned}$$

□ □

Lemma 29. $a_{A_1} \vee a_{A_2} \vee \ldots \vee a_{A_n} = a$

Proof.

$$\begin{cases} From\ Relations\ \ 9: u_{\bigvee\{A_i\}} = \prod_{i=1}^{n}(d_{A_i} + u_{A_i}) - \prod_{i=1}^{n}(d_{A_i}) \\ From\ Relations\ \ 13: \begin{cases} a_{A_i} = \frac{(1-b)^{\frac{1}{n}} - (1-b-au)^{\frac{1}{n}}}{(d+u)^{\frac{1}{n}} - d^{\frac{1}{n}}} \\ d_{A_i} = d^{\frac{1}{n}} \\ u_{A_i} = (d+u)^{\frac{1}{n}} - d^{\frac{1}{n}} \end{cases} \end{cases} \Rightarrow$$

$$a_{\bigvee\{A_i\}} = \frac{\prod_{i=1}^{n}(d^{\frac{1}{n}} + ((d+u)^{\frac{1}{n}} - d^{\frac{1}{n}})) - \prod_{i=1}^{n}[(d^{\frac{1}{n}} + ((d+u)^{\frac{1}{n}} - d^{\frac{1}{n}}) - ((d+u)^{\frac{1}{n}} - d^{\frac{1}{n}})(\frac{(1-b)^{\frac{1}{n}} - (1-b-au)^{\frac{1}{n}}}{(d+u)^{\frac{1}{n}} - d^{\frac{1}{n}}}))]}{\prod_{i=1}^{n}(d^{\frac{1}{n}} + ((d+u)^{\frac{1}{n}} - d^{\frac{1}{n}})) - \prod_{i=1}^{n}(d^{\frac{1}{n}})}$$

$$\begin{aligned}
a_{\bigvee\{A_i\}} &= \frac{\prod_{i=1}^{n}(d+u)^{\frac{1}{n}} - \prod_{i=1}^{n}[(d+u)^{\frac{1}{n}} - ((1-b)^{\frac{1}{n}} - (1-b-au)^{\frac{1}{n}})]}{\prod_{i=1}^{n}(d+u)^{\frac{1}{n}} - \prod_{i=1}^{n}(d^{\frac{1}{n}})} \\
&= \frac{\prod_{i=1}^{n}(d+u)^{\frac{1}{n}} - \prod_{i=1}^{n}[(1-b-au)^{\frac{1}{n}}]}{\prod_{i=1}^{n}(d+u)^{\frac{1}{n}} - \prod_{i=1}^{n}(d^{\frac{1}{n}})} \\
&= \frac{\prod_{i=1}^{n}(1-b)^{\frac{1}{n}} - \prod_{i=1}^{n}[(1-b-au)^{\frac{1}{n}}]}{\prod_{i=1}^{n}(d+u)^{\frac{1}{n}} - \prod_{i=1}^{n}(d^{\frac{1}{n}})} \\
&= a
\end{aligned}$$

□ □

Lemma 30. $0 \leq a_{A_i} = \frac{(1-b)^{\frac{1}{n}} - (1-b-au)^{\frac{1}{n}}}{(d+u)^{\frac{1}{n}} - d^{\frac{1}{n}}} \leq 1$

Proof. $(d+u)^{\frac{1}{n}} - d^{\frac{1}{n}} = u \wedge 0 \leq u \leq 1$. Thus, we just have o prove that: $0 \leq (1-b)^{\frac{1}{n}} - (1-b-au)^{\frac{1}{n}} \leq 1$ with the condition that $u \neq 0$

The left side:

$$\begin{aligned}0 \leq a \leq 1 \wedge 0 \leq u \leq 1 \Rightarrow 0 \leq au \leq 1 \Rightarrow \\ 1-b \geq 1-b-au \Rightarrow \\ (1-b)^{\frac{1}{n}} \geq (1-b-au)^{\frac{1}{n}} \Rightarrow \\ (1-b)^{\frac{1}{n}} - (1-b-au)^{\frac{1}{n}} \geq 0\end{aligned}$$

The right side:

$$b+au = E(O) \Rightarrow 0 \leq b+au \leq 1 \Rightarrow 1 \geq 1-(b+au) \geq 0 \Rightarrow 1 \geq (1-b-au)^{\frac{1}{n}} \geq 0$$

$$0 \leq b \leq 1 \Rightarrow 1 \geq 1-b \geq 0 \Rightarrow 1 \geq (1-b)^{\frac{1}{n}} \geq 0$$

The substraction of two values that are less than one is less than one $\Rightarrow$

$$(1-b)^{\frac{1}{n}} - (1-b-au)^{\frac{1}{n}} \leq 1$$

□ □

Lemma 31. $b_{A_i} + d_{A_i} + u_{A_i} = 1$

Proof.

$$\begin{cases} b_{A_i} = 1-(1-b)^{\frac{1}{n}} \\ d_{A_i} = d^{\frac{1}{n}} \\ u_{A_i} = (d+u)^{\frac{1}{n}} - d^{\frac{1}{n}} \end{cases} \Rightarrow$$

$$\begin{aligned} b+d+u &= 1-(1-b)^{\frac{1}{n}} + d^{\frac{1}{n}} + (d+u)^{\frac{1}{n}} - d^{\frac{1}{n}} \\ &= 1-(d+u)^{\frac{1}{n}} + d^{\frac{1}{n}} + (d+u)^{\frac{1}{n}} - d^{\frac{1}{n}} \\ &= 1 \end{aligned}$$

□ □

A.4 Comparing M1 to M2

Let m be the number of paths in a graph α that have some common nodes between the paths. if we follow the M2, and we denote the opinion about α by $O_\alpha(M2)$, we have:

$$O_\alpha(M2) = \begin{cases} b_\alpha(M2) = b_{\bigvee\{\sigma_i\}} = 1-\prod_{i=1}^{m}(1-b_{\sigma_i}) \\ d_\alpha(M2) = d_{\bigvee\{\sigma_i\}} = \prod_{i=1}^{m} d_{\sigma_i} \\ u_\alpha(M2) = u_{\bigvee\{\sigma_i\}} = \prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i}) - \prod_{i=1}^{m}(d_{\sigma_i}) \\ a_\alpha(M2) = a_{\bigvee\{\sigma_i\}} = \frac{\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i})}{\prod_{i=1}^{m}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{m}(d_{\sigma_i})} \end{cases}$$

If we follow M1, and we denote the opinion toward α by $O_\alpha(M1)$, we have:

$$O_\alpha(M1) = \begin{cases} b_\alpha(M1) = b_{\bigvee\{\sigma_i\}} = 1-\prod_{i=1}^{l}(1-b_{\sigma_i}) \\ d_\alpha(M1) = d_{\bigvee\{\sigma_i\}} = \prod_{i=1}^{l} d_{\sigma_i} \\ u_\alpha(M1) = u_{\bigvee\{\sigma_i\}} = \prod_{i=1}^{l}(d_{\sigma_i}+u_{\sigma_i}) - \prod_{i=1}^{l}(d_{\sigma_i}) \\ a_\alpha(M1) = a_{\bigvee\{\sigma_i\}} = \frac{\prod_{i=1}^{l}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{l}(d_{\sigma_i}+u_{\sigma_i}-u_{\sigma_i}a_{\sigma_i})}{\prod_{i=1}^{l}(d_{\sigma_i}+u_{\sigma_i})-\prod_{i=1}^{l}(d_{\sigma_i})} \end{cases}$$

In this section, we show that M2 is more optimistic than M1 by proving the following relations:

$$\begin{cases} b_\alpha(M2) \geq b_\alpha(M1) \\ d_\alpha(M2) \leq d_\alpha(M1) \\ E(O_\alpha(M2)) \geq E(O_\alpha(M1)) \end{cases}$$

1. **Comparing b_α in M1 and M2:**
   $b_\alpha(M2) = 1 - \prod_{i=1}^{m}(1 - b_{\sigma_i})$
   $b_\alpha(M1) = 1 - \prod_{i=1}^{l}(1 - b_{\sigma_i})$

   Lemma 32. $b_\alpha(M1) \leq b_\alpha(M2)$

   Proof. $m \geq l$ because M1 is based on deleting uncertain paths thus the number of paths in M2 is greater than M1. The opinions about the remaining paths are the same because in M2, the split nodes have the same opinion of the original node.

$$\begin{aligned} &(m \geq l) \wedge (0 \leq 1 - b_{\sigma_i} \leq 1) \Rightarrow \\ &\prod_{i=1}^{m}(1 - b_{\sigma_i}) \leq \prod_{i=1}^{l}(1 - b_{\sigma_i}) \Rightarrow \\ &1 - \prod_{i=1}^{m}(1 - b_{\sigma_i}) \geq 1 - \prod_{i=1}^{l}(1 - b_{\sigma_i}) \Rightarrow \\ &b_\alpha(M2) \geq b_\alpha(M1) \end{aligned}$$

□
□

2. **Comparing d_α in M1 and M2:**
   $d_\alpha(M2) = \prod_{i=1}^{m} d_{\sigma_i}$
   $d_\alpha(M1) = \prod_{i=1}^{l} d_{\sigma_i}$

   Lemma 33. $d_\alpha(M2) \leq d_\alpha(M1)$

   Proof.

$$\begin{aligned} &(m \geq l) \wedge (0 \leq d_{\sigma_i} \leq 1) \Rightarrow \\ &\prod_{i=1}^{m} d_{\sigma_i} \leq \prod_{i=1}^{l} d_{\sigma_i} \Rightarrow \\ &d_\alpha(M2) \leq d_\alpha(M1) \end{aligned}$$

□
□

3. **Comparing $E(O_\alpha)$ in M1 and M2:**
$E(O_x) = b_x + a_x u_x$
From [11]: $E(O_x \vee y) = E(O_x) + E(O_y) - E(O_x)E(O_y) \Rightarrow$
$E(O_\alpha) = E(O_{\bigvee\{\sigma_i\}}) = 1 - \prod_{i=1}^{m} (1 - E(O_{\sigma_i}))$

Lemma 34. $E(O_\alpha(M1)) \leq E(O_\alpha(M2))$

Proof.

$$
\begin{aligned}
&(m \geq l) \wedge (0 \leq 1 - E(O_{\sigma_i}) \leq 1) \Rightarrow \\
&\prod_{i=1}^{m} (1 - E(O_{\sigma_i})) \leq \prod_{i=1}^{1} (1 - E(O_{\sigma_i})) \\
&1 - \prod_{i=1}^{m} (1 - E(O_{\sigma_i})) \geq 1 - \prod_{i=1}^{1} (1 - E(O_{\sigma_i})) \Rightarrow \\
&E(O_\alpha(M2)) \geq E(O_\alpha(M1))
\end{aligned}
$$

□
□

A.5 Comparing M2 to M3

In this section, we show that M2 is more optimistic than M3 by proving the following relations:

$$
\begin{cases}
b_\alpha(M2) \geq b_\alpha(M3) \\
d_\alpha(M2) \leq d_\alpha(M3) \\
E(O_\alpha(M2)) \geq E(O_\alpha(M3))
\end{cases}
$$

To prove the previous relations, it is enough to prove that $b_{A_1}, d_{A_1}, E(O_{A_1})$ associated to the split node $A1$ from the original node A in M3 satisfy the following relations:

$$
\begin{cases}
b_{A_1} = 1 - \sqrt{1 - b_A} \leq b_A \\
d_{A_1} = \sqrt{d_A} \geq d_A \\
E(O_{A_1}) = b_{A_1} + u_{A_1} a_{A_1} = 1 - \sqrt{1 - b_A} + [\sqrt{d_A + u_A} - \sqrt{d_A}][\frac{\sqrt{1-b_A} - \sqrt{1-b_A-a_A u_A}}{\sqrt{d_A+u_A} - \sqrt{d_A}}] \\
= 1 - \sqrt{1 - b_A - a_A u_A} \leq E(O_A) = b_A + a_A u_A
\end{cases}
$$

1. **Comparing b_α in M2 and M3:**
Lemma 35. $1 - \sqrt{1 - b_A} \leq b_A$

Proof.

$$
\begin{aligned}
&0 \leq 1 - b_A \leq 1 \Rightarrow \\
&\sqrt{1 - b_A} \geq 1 - b_A \Rightarrow \\
&1 - \sqrt{1 - b_A} \leq b_A \Rightarrow \\
&b_\alpha(M3) \leq b_\alpha(M2)
\end{aligned}
$$

□
□

2. **Comparing d_α in M2 and M3:**

   Lemma 36. $\sqrt{d_A} \geq d_A$

   Proof.

$$\begin{aligned} &0 \leq d_A \leq 1 \Rightarrow \\ &\sqrt{d_A} \geq d_A \Rightarrow \\ &d_\alpha(M3) \geq d_\alpha(M2) \end{aligned}$$

□
□

3. **Comparing $E(O_\alpha)$ in M2 and M3:**

   Lemma 37. $1 - \sqrt{1 - b_A - a_A u_A} \leq b_A + a_A u_A$

   Proof.

$$\begin{aligned} &0 \leq b_A + a_A u_A \leq 1 \Rightarrow \\ &\sqrt{1 - b_A - a_A u_A} \geq 1 - b_A - a_A u_A \Rightarrow \\ &1 - \sqrt{1 - b_A - a_A u_A} \leq b_A + a_A u_A \Rightarrow \\ &E_{O_\alpha}(M3) \leq E_{O_\alpha}(M2) \end{aligned}$$

□
□

A.6 Proposed Survey

Opinion about a node depends on user's negative or positive observations r, s and is computed by Relations 1. Since our work focuses on local trust, local observations are considered. This survey allows to collect the users' observations in order to build an opinion about a node. The proposed questions in this survey collect information about the user's usage of a node to estimate the uncertainty u. The negative observations of using a node is also demanded to estimate the value of d for a node. The value of b is computed by the relation $b = 1 - d - u$.

A node in the graph represents an artifact which is controlled by persons and is supported by physical resources.

In the following, we show the proposed survey that helps to build an opinion about each node.

1. Have you ever used Node N?
   - Yes
   - No
     If the answer is "No" we conclude that the user's opinion about this node is (0,0,1,0.5).

   If the answer of Question 1 is "Yes":
2. At which frequency do you use Node N?
   - Everyday
   - From time to time
     If the answer is "Everyday" we conclude that the $u = 0$ since the user is used to use the node.

   If the answer of Question 2 is "From time to time":
3. More precisely, how many times have you used Node N?
   - 1 time $\Rightarrow u = 0.67$
   - 2 times $\Rightarrow u = 0.5$
   - 3 times $\Rightarrow u = 0.4$
   - Around 5 times $\Rightarrow u = 0.28$
   - Around 10 times $\Rightarrow u = 0.17$
   - Around 20 times $\Rightarrow u = 0.09$
   - Around 50 times $\Rightarrow u = 0.04$
   - Around 100 times $\Rightarrow u = 0.02$
   - More than 100 times $\Rightarrow u = 0$

   The answer of this questions allow to conclude the value of u from the relation $u = 2/$(total observations$+2$) in subjective logic.
   By the end of these questions we conclude the value of u, now we need to conclude d or b. Since users remember their negative observations more than their positive ones, the following questions are launched:

   If the answer of Question 2 is "From time to time" or "Everyday":
4. Have you ever had a problem with the Node N?
   - Yes
   - No
     If the answer is "No" we conclude that the $d = 0$ since the user has not negative observations.

If the answer of Question 4 is "Yes" and the answer of Question 2 is "Everyday":

5. How much do you estimate that you had problems with this node comparing to the total times you use this node (the answer should be given as a percentage)?
The given value is the value of d
If the answer of Question 4 is "Yes" and the answer of Question 2 is "From time to time":
6. How many times you had a problem with the node N?
d *can be computed from the relation* $d = \frac{the\ value\ given\ by\ Question\ 4}{((The\ value\ given\ by\ Question\ 3)+2))}$
By the end of these questions we have the value of d *and* $u \Rightarrow b$ *can be computed from the relation* $b = 1 - d - u$

We modeled a subpart of the `LINA` research laboratory system[4] using SocioPath. We applied the rules of SocioPath on this system for the activity "a user accesses a document `toto` that is stored on the `SVN` server of `LINA`". Figure 8 presents the WDAG for this activity, with renamed nodes A, B, C, D, E, F, G for privacy issues. For sack of clarity, we simplify this graph as much as possible.

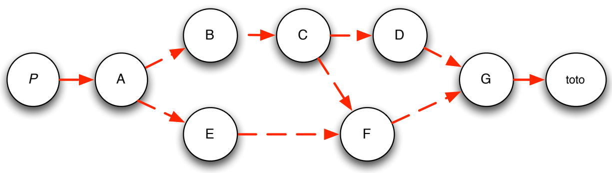

Fig. 8. `LINA`'s WDAG for the activity "accessing a document `toto` on the SVN"

20 members of LINA participated and answered these questions for each node. The value of a is equal to 0.5 for each node since it is a prior probability in the absence of the evidence. For instance, P_1 answers the questions about G as following:

- Question 1: Yes.
- Question 2: From time to time.
- Question 3: Around 10 times $\Rightarrow u = 0.17$.
- Question 4: No $\Rightarrow d = 0 \Rightarrow b = 0.83$.

Thus, $O_G = (0.83, 0, 0.17, 0.5)$ for P_1.

Another example is the opinion of P_{10} about G. P_{10} answers the questions about G as following:

- Question 1: Yes.
- Question 2: Everyday $\Rightarrow u = 0$.
- Question 4: Yes.
- Question 5: 20% $\Rightarrow d = 0.2 \Rightarrow b = 0.8$.

Thus, $O_G = (0.8, 0.2, 0, 0.5)$ for P_{10}.

[4] https://www.lina.univ-nantes.fr/