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Influence and Social Tragedy in Networks*

Yann Rébillé[†] Lionel Richefort[‡]

Abstract

We model agents in a network game of strategic complements and negative externalities. Sufficient conditions for the existence of a unique Nash equilibrium and of a unique social optimum are established. Under these conditions, we find that players with more vulnerable locations in the network exert more effort at equilibrium, and that the most influential players should exert less effort at efficiency. We then find structural conditions under which each player exerts strictly more effort than her efficient level, whether the social optimum be interior or not.

Keywords: network, strategic complements, equilibrium, efficiency, social tragedy.

JEL: A14, C72, D85.

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1 Introduction

Although social dilemmas are situations in which individual and collective interests are in conflict, their outcome depends heavily on the nature of social (or geographic) interactions. By considering the case where a social dilemma is embedded in a directed and weighted network, this present paper engages the classic question of which social (or geographic) structures are conducives to a “social tragedy” and shows that social (or geographic) structure, pure strategy Nash equilibrium (henceforth PSNE) and social optimum (henceforth SO) are related in a mutually interesting way.

Here we consider a simultaneous-move game in which each player is more likely to exert an effort whenever her neighbors are exerting an effort, in other words, we assume that efforts are local complements. In addition, it is assumed that higher levels of effort by neighbors lower a player’s payoff, i.e., we consider a game in which efforts generate local negative externalities. Our model is thus a network game of strategic complements and negative externalities. Examples of application include: advertising expenditures by firms that produce brand-differentiated substitute goods; investments in training, in infrastructure or in talents by teams in competition for a professional or sporting challenge; campaign commitments and spendings by candidates in an election; research and educational investments by universities and higher education institutions, etc.

The main result is to establish conditions on the structure of the network for a social tragedy, whether the SO be interior or not. First, we find sufficient conditions under which the PSNE exists and we show that in that case, it is unique and interior. These results can be seen as applications of Kennan (2001)’s uniqueness result to the case of games with local complementarities. Our contributions here are to highlight the role of local interactions between neighboring agents and to relate equilibrium behaviors to network centrality measures. Using a modified version of the Bonacich centrality measure, we show that players with more vulnerable locations in the network exert more effort at equilibrium.

Then, we find sufficient conditions under which the SO exists and is unique. Using another modified version of the Bonacich centrality measure, we prove that the efficient profile is a fixed point, we derive a structural condition for the SO to be interior and we show that the most influential players should exert less effort at efficiency. Our proofs are simple, intuitive and based on standard optimization techniques. Then, we show that there is always at least one player that exerts strictly more effort than her efficient level and we find structural conditions under which this is the case for each player. We call this situation a social tragedy. In general, such a situation appears whenever each player has a successor. But when the SO is interior, it appears if and only if each player has a successor *or* a predecessor.

The paper is built on the assumption that the most basic and common mechanism for a social tragedy is the bandwagon effect (or fashion effect): an agent will behave in a certain way because some or all of his neighbors also behave in that way. Hence, the effort level exerted by an agent depends positively on the effort levels exerted by some or all of his neighbors, which depend themselves positively on the effort levels exerted by some or all of their neighbors, and so on. The mechanism's similarity to the contagion of cheating behavior is not really appropriate; we do not consider a threshold above which effort would become illegal, i.e., there is no risk of getting caught, agents are not facing a lottery (see, e.g., Pascual-Ezama et al., 2013). Note also that we do not allow agents to forge alliance with other agents. In our game, all the players are "enemies" (direct or indirect).

The main motivation of the model is collective wasteful behavior towards goods such as power, prestige or status. The welfare losses caused by consumption of such goods have been established by Hirsch (1976) and Frank (1985) and is clearly demonstrated by increased investments made, in some occasions, by rival agents: from the early 1950s through the mid-1960s, the US brewing industry was involved in a "game of market power", the advertising spending per barrel rising from \$5.00 in 1950 to \$8.10 in 1963 (Tremblay and Tremblay, 2007, p. 68); for several years now, US colleges and universities are clearly engaged in a "game of educational prestige", seeking to attract the best students through increased spending or reduced price (Winston, 2000). A structural analysis of collective wasteful behavior towards status goes back at least to Thorstein Veblen, who saw the consumption of some goods or services as "conspicuous expenditures" driven by "the stimulus of an invidious comparison which prompts us to outdo those with whom we are in the habit of classing ourselves" (Veblen, 1899, p. 103). Interestingly, Veblen emphasized the role of the hierarchic structure of social classes (directed network) on the diffusion of the pecuniary standard of living (wasteful behavior).

2 Model

There are n players and the set of players is $N = \{1, \ldots, n\}$. Each player $l \in N$ chooses simultaneously a level of *absolute* effort $x_l \in \mathbb{R}_+$. E.g., the players could be firms producing brand-differentiated substitute goods and x_l could be the strategy of firm l's in advertising expenditures. We assume that the cost of a unit of x_l is w_l. Let $\mathbf{x} = (x_1, \ldots, x_n)$ denote an effort profile of all players.

Players are arranged in a network, which we represent as a weighted directed graph G which consists of a set of nodes (the players), a set of arcs (the directed links between players) and a mapping from the set of arcs to a set of positive weights (the intensities of the directed links). We will use kl to denote an arc directed from node k to node l. If kl is an arc, then we say that node l is a *vulnerable neighbor* of node k, or that node k is an *influential neighbor* of node l. A directed path in G is a sequence of distinct nodes connected by arcs corresponding to the order of the nodes in the sequence. The length of a directed path is its number of arcs. The weight of a directed path is the product of the weights of its arcs. To continue our example, the network could reflect the geographic structure of competition among firms. Two firms compete only when they are geographic neighbors. In that case, these two firms are linked, the direction and the intensity of the link reflecting the balance between their relative market powers (due, for instance, to the quality of the goods being produced).

The basic representation of G is given by its weighted $n \times n$ adjacency matrix $\mathbf{\Lambda} = [\lambda_{kl}] \in \mathbb{R}_+^{n\times n}$ where $\lambda_{kl} > 0$ if kl is an arc and $\lambda_{kl} = 0$ otherwise (by convention $\lambda_{kk} = 0$). Let $\rho(\mathbf{\Lambda})$ denote the spectral radius of the network. For the rest of the paper, we require:

Assumption A0. $\rho(\mathbf{\Lambda}) < 1$.

Since $\mathbf{\Lambda} \geq \mathbf{0}$, it is well-known that A0 holds if and only if $(\mathbf{I} - \mathbf{\Lambda})^{-1}$ exists and is nonnegative (Berman and Plemmons, 1979). Therefore,

$$\mathbf{M} = (\mathbf{I} - \mathbf{\Lambda})^{-1} = \sum_{i=0}^{\infty} \mathbf{\Lambda}^i,$$

hence $\mathbf{M} \geq \mathbf{I} + \mathbf{\Lambda} \geq \mathbf{I}$. The entry m_{kl} counts the total weight of all directed paths in G starting at node k and ending at node l.

We suppose that the players are "enemies" and hence that network externalities are negative. Let e_l denote player l's *effective* effort, defined as the difference between player l's absolute effort level with the absolute effort levels of her influential neighbors multiplied by the corresponding weights. Players value their effective effort according to a twice differentiable strictly concave value function $f_l(e_l)$ defined on $\mathbb{R}$ with $f_l' > 0$ and $f_l'' < 0$ for all l. According to our assumptions stated above, the resulting effective effort is determined according to $e_l = x_l - \sum_{k:k\neq l} \lambda_{kl} x_k$. E.g., $f_l(e_l)$ could be the value (measured in monetary units) to firm l of advertising expenditures $\mathbf{x}$ of all firms.

The payoff function of player l, defined for all effort profile $\mathbf{x} \geq \mathbf{0}$, is given by

$$U_l(\mathbf{x}) = f_l\left(x_l - \sum_{k:k\neq l} \lambda_{kl} x_k\right) - w_l x_l,$$

and we note $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$ the simultaneous-move game with payoffs $U_l : \mathbb{R}_+^n \to \mathbb{R}$ and strategy space $x_l \geq 0$ for each player $l \in N$, where $\mathbf{f}$ is the vector of value functions and $\mathbf{w}$ the vector of marginal costs. Since $\partial U_l / \partial x_k \leq 0$ for all $k \neq l$, this is a game of negative externalities, and since $\partial^2 U_l / \partial x_k \partial x_l \geq 0$ for all $k \neq l$, this is a game of strategic complements.

In our game, the effort space is the positive real line and the payoffs are strictly concave, so every mixed strategy is dominated by its average pure strategy which is available. Then, we study Nash equilibria in pure strategies.

3 Equilibrium Analysis

For the equilibrium analysis, we require:

Assumption A1. For all $l \in N$, $f_l'(0) > w_l$ and $w_l > f_l'(\infty)$.

This assumption, called the *boundary conditions*, guarantees that each player will exert a positive and finite level of absolute effort at equilibrium. Let $x_l^* = (f_l')^{-1}(w_l)$ denote player l's *uninfluenced* equilibrium effort and for ease of exposition, we will write $\mathbf{x}^* = (\mathbf{f}')^{-1}(\mathbf{w})$. Under A1, x_l^* is positive and finite, i.e., a player with no influential neighbors still exerts a positive and finite equilibrium effort level. Hence, the associated reactions functions are linear in strategies:

$$\forall l \in N, \quad x_l(\mathbf{x}_{-l}) = x_l^* + \sum_{k:k\neq l} \lambda_{kl} x_k,$$

where $\mathbf{x}_{-l}$ is the strategy vector $\mathbf{x}$ with player l's strategy removed. This allows us to find PSNE by solving a linear problem and we look for a closed-form solution.

For a weighted adjacency matrix $\mathbf{\Lambda} \in \mathbb{R}_+^{n\times n}$, the Bonacich centrality measure is given by

$$\mathbf{c}(\alpha, \beta, \mathbf{\Lambda}, \mathbf{1}) = \alpha (\mathbf{I} - \beta\mathbf{\Lambda})^{-1} \mathbf{\Lambda}\mathbf{1},$$

where $\mathbf{1}$ is the vector of ones and $\alpha, \beta \in \mathbb{R}$ are two scalars (Bonacich, 1987). Under A0, we have the following power expansion:

$$\mathbf{c}(\alpha, \beta, \mathbf{\Lambda}, \mathbf{1}) = \alpha \sum_{i=0}^{\infty} \beta^i \mathbf{\Lambda}^{i+1} \mathbf{1}.$$

When β is positive, $\mathbf{c}(1, \beta, \mathbf{\Lambda}, \mathbf{1})$ denotes the expected number of influences directly or indirectly caused by a node. The Bonacich centrality measure has proved to be very useful to provide geometric intuitions of how equilibrium efforts are related to network position in a variety of network games (Ballester et al., 2006; Corbo et al., 2007; Ballester and Calvó-Armengol, 2010; Ilkiliç, 2010; Allouch, 2012; Rébillé and Richefort, 2012a; Bramoullé et al., 2013). We introduce a modified version of the Bonacich centrality measure.

Definition 1. Let $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$ be a game. Under A0, the vector

$$\mathbf{b}^{-}(\mathbf{\Lambda}, \mathbf{x}^{*}) = \mathbf{x}^{*} + \mathbf{c}\left(1, 1, \mathbf{\Lambda}^{\mathsf{T}}, \mathbf{x}^{*}\right) = \left(\mathbf{I} - \mathbf{\Lambda}^{\mathsf{T}}\right)^{-1} \mathbf{x}^{*}$$

is called the *vulnerability* measure.

Using the power expansion presented above, we observe that the vulnerability measure of a player l in the network is a sum of her uninfluenced equilibrium effort x_l^* with the total weight of all directed paths that end at her in the network, where a directed path that starts at player $k \neq l$ is weighted by x_k^*, the uninfluenced equilibrium effort of the corresponding player. Hence, having many influential neighbors increases vulnerability, and if one player's influential neighbors themselves have many influential neighbors, vulnerability is increased, and so on.

Example 1 (Acyclic network). There are four firms (players) that produce brand-differentiated substitute goods. Each firm must choose her advertising expenditures. Firm 3 is a geographic neighbor of all the other firms, and all the other firms are only geographic neighbors to firm 3. Firms 1 and 2 produce goods of better quality than firm 3, who produces herself a good of better quality than firm 4. The geographic structure of competition among the four firms is given by the following weighted directed network:

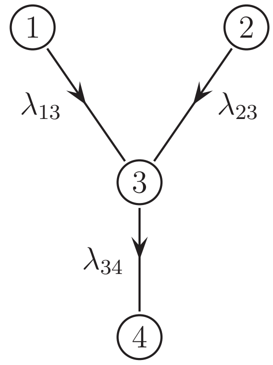

Figure 1: An acyclic network with four players

Since the network is acyclic, $\rho(\mathbf{\Lambda}) = 0$ (Nicholson, 1975). Then, A0 is always satisfied. For all firm $l \in N = \{1, \ldots, 4\}$, let

$$U_l(\mathbf{x}) = 1 - \exp\left(-\left(x_l - \sum_{k:k\neq l} \lambda_{kl} x_k\right)\right) - \frac{1}{2} x_l$$

where x_l is firm l's advertising expenditures and λ_{kl} is the intensity of the influence from firm k to firm l. For ease of exposition, we set the intensity of each influence to 0.5 (i.e., $\lambda_{13} = \lambda_{23} = \lambda_{34} = 0.5$). We get

$$\mathbf{b}^{-}(\mathbf{\Lambda}, \mathbf{x}^*) = \begin{pmatrix} \ln(2) \\ \ln(2) \\ 2\ln(2) \\ 2\ln(2) \end{pmatrix},$$

so firms 3 and 4 have the most vulnerable locations.

Letting $\hat{\mathbf{x}} \geq \mathbf{0}$ denote a PSNE of $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$, we obtain the following result.

Proposition 1. *Let $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$ be a game. Under A0 and A1, the PSNE exists, is unique and is interior. Moreover,*

$$\hat{\mathbf{x}} = \mathbf{b}^{-}(\mathbf{\Lambda}, \mathbf{x}^*) = \mathbf{b}^{-}\left(\mathbf{\Lambda}, (\mathbf{f}')^{-1}(\mathbf{w})\right) \geq \mathbf{x}^*$$

and for all $l \in N$,

$$U_l(\hat{\mathbf{x}}) = f_l(x_l^*) - w_l \hat{x}_l.$$

The interpretation is straightforward. When the boundary conditions are met and when the spectral radius of the network is sufficiently low, the PSNE exists, is unique, is interior and is such that players with a more vulnerable location in the network exert a higher level of absolute effort (if the spectral radius is too high, no PSNE exists; see, e.g., Ballester et al., 2006; Corbo et al., 2007). When the network is undirected, the spectral condition can be interpreted as follows (see, e.g., Ballester and Calvó-Armengol, 2010): the game $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$ admits a unique PSNE whenever the network is not too much *dense*. When the network is directed, things are less clear, although A0 still carries details on the topological structure of the network (for a survey of results on the spectra of graphs, see Cvetkovic and Rowlinson, 1990, and for a survey of results on the spectra of directed graphs, see Brualdi, 2010). We now turn to examine social optima.

4 Welfare Analysis

The social welfare or efficiency of a strategy profile in $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$ is defined as the sum of payoffs of all players. Let SW be the social welfare function defined for all $\mathbf{x} \geq \mathbf{0}$ by

$$SW(\mathbf{x}) = \sum_{l} f_l \left(x_l - \sum_{k:k\neq l} \lambda_{kl} x_k \right) - w_l x_l.$$

An effort profile is said to be socially optimal or efficient if it maximizes the social welfare function. Thus, the efficient behavior of a player depends on how she is impacted by her predecessors' behavior and how she impacts her successors' behavior. Consequently, we need an opposite version of the vulnerability measure for getting a characterization of the efficient profile.

Definition 2. Let $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$ be a game. Under A0, the vector

$$\mathbf{b}^{+}(\mathbf{\Lambda}, \mathbf{w}) = \mathbf{w} + \mathbf{c}(1, 1, \mathbf{\Lambda}, \mathbf{w}) = (\mathbf{I} - \mathbf{\Lambda})^{-1} \mathbf{w}$$

is called the *influence* measure.

Using the power expansion presented in the previous section, we observe that the influence measure of a player l in the network is a sum of her marginal cost of absolute effort w_l with the total weight of all directed paths that start at her in the network, where a directed path that ends at player $k \neq l$ is weighted by w_k, the marginal cost of absolute effort of the corresponding player. Hence, having many vulnerable neighbors increases influence, and if one player's vulnerable neighbors themselves have many vulnerable neighbors, influence is increased, and so on.

Example 2 (Example 1 continued). We get

$$\mathbf{b}^{+}(\mathbf{\Lambda}, \mathbf{w}) = \begin{pmatrix} 7/8 \\ 7/8 \\ 3/4 \\ 1/2 \end{pmatrix},$$

so firms 1 and 2 are the most influential firms.

We prove that SW is strictly concave whenever A0 is met and reaches a maximum on a compact and non-empty set if A1 is satisfied. This allows us to derive an existence and uniqueness result for the SO. In addition, we show that the efficient effort profile is a fixed point when the SO is corner. Let $D = \{l : \tilde{x}_l > 0\}$, $\bar{D}$ its complement and $\tilde{\mathbf{x}} \geq \mathbf{0}$ the SO of $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$.

Theorem 1. *Let $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$ be a game. Under A0 and A1, the SO exists, is unique and is a fixed point, i.e.,*

$$\tilde{\mathbf{x}}_D = \mathbf{b}^- \left(\mathbf{\Lambda}_{D\times D}, (\mathbf{f}')^{-1} \left(\mathbf{b}^+ \left(\mathbf{\Lambda}_{D\times D}, \boldsymbol{\phi}\left(\tilde{\mathbf{x}}_D\right)\right)\right)\right),$$

where $\boldsymbol{\phi}(\tilde{\mathbf{x}}_D) = \mathbf{w}_D + \mathbf{\Lambda}_{D\times\bar{D}}\ \mathbf{f}'(-\mathbf{\Lambda}_{D\times\bar{D}}{}^{\mathsf{T}}\ \tilde{\mathbf{x}}_D)$.

This formula provides intuitions on how efficient behavior is related to centrality. When $D \neq N$, there are inactive players at efficiency and the efficient efforts of the active players can be computed by solving the fixed point equation. When $D = N$, the SO is interior. In that case, $\boldsymbol{\phi}(\tilde{\mathbf{x}}_D) = \mathbf{w}_D$. The efficient efforts of the active players are constant and given by a combination of the vulnerability and the influence measures. In that case, the connection between PSNE and SO can be established. Then, we now look for structural conditions under which the SO is interior. We require:

Assumption A2. For all $l \in N$, $f_l'(0) > b_l^+(\mathbf{\Lambda}, \mathbf{w})$ and $w_l > f_l'(\infty)$.

This assumption, called the *modified boundary conditions*, guarantees that each player should exert a positive and finite level of absolute effort at efficiency (note that since $b_l^+(\mathbf{\Lambda}, \mathbf{w}) \geq w_l$, A1 holds whenever A2 holds). Let $\tilde{x}_l^* = (f_l')^{-1}(b_l^+(\mathbf{\Lambda}, \mathbf{w}))$ denote player l's *uninfluenced* efficient effort and for ease of exposition, we will write $\tilde{\mathbf{x}}^* = (\mathbf{f}')^{-1}(\mathbf{b}^+(\mathbf{\Lambda}, \mathbf{w}))$. Under A2, $\tilde{x}_l^*$ is positive and finite, i.e., a player with no influential neighbors still has a positive and finite efficient effort level. Hence, the first order conditions become linear in efforts:

$$\forall l \in N, \quad x_l\left(\mathbf{x}_{-l}\right) = \tilde{x}_l^* + \sum_{k:k\neq l} \lambda_{kl} x_k.$$

We obtain the following result.

Proposition 2. *Let $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$ be a game. Under A0 and A2, the SO exists, is unique and is interior. Moreover,*

$$\tilde{\mathbf{x}} = \mathbf{b}^- \left(\mathbf{\Lambda}, \tilde{\mathbf{x}}^*\right) = \mathbf{b}^- \left(\mathbf{\Lambda}, (\mathbf{f}')^{-1} \left(\mathbf{b}^+ \left(\mathbf{\Lambda}, \mathbf{w}\right)\right)\right) \geq \tilde{\mathbf{x}}^*$$

and for all $l \in N$,

$$U_l\left(\tilde{\mathbf{x}}\right) = f_l\left(\tilde{x}_l^*\right) - w_l \tilde{x}_l.$$

The interpretation is as follows. When the modified boundary conditions are met and when the spectral radius of the network is sufficiently low, the SO exists, is unique, is interior and is such that players with a more influential location in the network should exert a lower level of absolute effort. Then,

$\mathbf{b}^{-}(\mathbf{\Lambda}, \tilde{\mathbf{x}}^{*})$ can be interpreted as the *social* vulnerability measure: players with a more socially vulnerable location in the network should exert a higher level of absolute effort at efficiency. Interestingly, this result is in line with the closed-form expression obtained for the SO in games with substitutabilities (Rébillé and Richefort, 2012a and 2012b). Moreover, it holds even if we consider social welfare functions that are weighted sums of individual payoffs.

Remark 1 (Weighted social welfare functions)*.* More generally, the social welfare function could be weighted, reflecting the interest of the social planner for the various players with respect to their location in the network. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) >> \mathbf{0}$ be social weights and consider the social welfare function

$$\sum_{l} \alpha_l \left(f_l \left(x_l - \sum_{k:k\neq l} \lambda_{kl} x_k \right) - w_l x_l \right).$$

Then, the SO is given by

$$\tilde{\mathbf{x}} = \mathbf{b}^{-} \left(\mathbf{\Lambda}, (\mathbf{f}')^{-1} \left(\frac{\mathbf{1}}{\boldsymbol{\alpha}} \, \mathbf{b}^{+} \left(\mathbf{\Lambda}, \boldsymbol{\alpha} \mathbf{w} \right) \right) \right).$$

where $(\mathbf{1}/\boldsymbol{\alpha})_l = 1/\alpha_l$ and $(\boldsymbol{\alpha w})_l = \alpha_l w_l$ for all l, provided that A2 holds (that is, provided that $\alpha_l f_l'(0) > b_l^{+}(\mathbf{\Lambda}, \boldsymbol{\alpha w})$ and $w_l > f_l'(\infty)$ for all l).

5 Social Tragedy in Networks

From now on, we assume that $\mathbf{\Lambda} \neq \mathbf{0}$. The fact that in network games (of strategic complements or strategic substitutes), the PSNE is always inefficient is not surprising because players do not take into account the (positive or negative) externalities which they generate on their successors. However, network games of strategic complements differed from network games of strategic substitutes in that the best response of each player is increasing in efforts of her direct predecessors. Then, when externalities are negative, we shall prove the following result.

Theorem 2. *Let $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$ be a game. Under A0 and A1, each player exerts as much or more effort than her efficient level with at least one player exerting strictly more effort, i.e.,* $\tilde{\mathbf{x}} < \hat{\mathbf{x}}$.

This result entails that when a PSNE exists, too much collective effort is always exerted at equilibrium, i.e., $\sum_l \tilde{x}_l < \sum_l \hat{x}_l$, whether the SO be interior or not. There may exist, however, some players whose equilibrium effort level is equal to their efficient effort level. Next, we consider the situation in which each player exerts strictly more effort than her efficient level.

Definition 3. Let $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$ be a game. Under A0 and A1, a situation in which

$$\tilde{\mathbf{x}} << \hat{\mathbf{x}}$$

is called a *social tragedy.*

A player who has no successors, i.e., a *sink player*, does not generate externalities. Such a player has the particularity that her uninfluenced efficient effort is equal to her uninfluenced equilibrium effort. This observation provides the intuition for our next result, that shows that network games without sink players always lead to a social tragedy, even if the SO is corner.

Corollary 1. *Let $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$ be a game. Under A0 and A1, there exists a social tragedy whenever there are no sink players.*

Typical examples of network games without sink players are games played in undirected networks (with uniform or non-uniform weights attached to the undirected links). The proof clearly shows, however, that the no sink property is only sufficient. In other words, there exists games with complementarities which lead to a social tragedy even if there are sink players. The following example illustrates this point.

Example 3 (Example 1 modified). Consider the same network structure as in example 1, but with the following modified payoffs. For all $l \in N = \{1, \ldots, 4\}$, let

$$U_l(\mathbf{x}) = 1 - \exp\left(-\left(x_l - \sum_{k:k \neq l} \lambda_{kl} x_k\right)\right) - \frac{3}{5} x_l.$$

Assumptions A0 and A1 are met while A2 is not (we still assume that $\lambda_{13} = \lambda_{23} = \lambda_{34} = 0.5$). We get

$$\hat{\mathbf{x}} = \begin{pmatrix} \ln(5/3) \\ \ln(5/3) \\ 2\ln(5/3) \\ 2\ln(5/3) \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{x}} = \begin{pmatrix} 0 \\ 0 \\ \ln(10/9) \\ 1/2\ \ln(2/3) + 3/2\ \ln(5/3) \end{pmatrix}.$$

Firm 4 is a sink player, but the game still leads to a social tragedy, $\tilde{\mathbf{x}} << \hat{\mathbf{x}}$.

A player who has neither successors nor predecessors, i.e., an *isolated player*, does not generate and is not impacted by externalities. Such a player has the particularity that her efficient effort is equal to her equilibrium effort. Clearly, the absence of isolated players is a necessary condition for the appearance of a social tragedy. Moreover, the SO is interior under A0 and A2.

In that case, the efficient profile of $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$ can be interpreted as a PSNE of another game with "modified" uninfluenced equilibrium efforts. That is, we have

$$\tilde{\mathbf{x}} = \mathbf{b}^{-}\left(\mathbf{\Lambda}, \tilde{\mathbf{x}}^{*}\right),$$

where

$$\tilde{\mathbf{x}}^{*} = \left(\mathbf{f}'\right)^{-1}\left(\mathbf{b}^{+}\left(\mathbf{\Lambda}, \mathbf{w}\right)\right) \leq \left(\mathbf{f}'\right)^{-1}\left(\mathbf{w}\right) = \mathbf{x}^{*}.$$

This last observation leads us directly to our next result, that shows that when the SO is interior, the absence of isolated players becomes a sufficient condition for the appearance of a social tragedy.

Theorem 3. *Let $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$ be a game. Under A0 and A2, there exists a social tragedy if and only if there are no isolated players.*

It is worth noting that under A0 and A2, the global over effort and the resulting loss in welfare can be computed, thanks to Proposition 2. The global over effort is given by

$$\hat{\mathbf{x}} - \tilde{\mathbf{x}} = \mathbf{b}^{+}\left(\mathbf{\Lambda}, \mathbf{x}^{*} - \tilde{\mathbf{x}}^{*}\right) >> \mathbf{0}$$

and the loss in welfare is given by

$$\sum_{l} U_l\left(\tilde{\mathbf{x}}\right) - U_l\left(\hat{\mathbf{x}}\right) \in \left(0, \sum_{l} w_l\left(\hat{x}_l - \tilde{x}_l\right)\right).$$

These measures indicate the performance of the network and interestingly, they both depend on network centrality measures.

Then, a situation in which $\tilde{x}_l < \hat{x}_l$ for all l appears if and only if each player has at least one (influential or vulnerable) neighbor, provided that the SO is interior. The reason is that players, who are rational and selfish, are trapped in a continuous prisoner's dilemma. Many continuous prisoner's dilemmas can be found in the literature, and a large majority of them have been developed from an underlying social dilemma (Eaton, 2004; Acharya and Ramsay, 2013). Depending on the nature of the interactions among players, social dilemma games can be classified into four groups (i.e., games of strategic substitutes/complements and negative/positive externalities; see Eaton, 2004).

A famous multi-player generalization of the prisoner's dilemma is the commons dilemma, which leads to what has become known as the "tragedy of the commons" (Gordon, 1954; Hardin, 1960). Our results are closely related to such a situation, since the commons dilemma is generally represented as a game of negative externalities (see, e.g., Hardin, 1971; Cornes and Sandler, 1983; Gardner et al., 1994). With the exception of Ilkiliç (2010), this literature typically assumes that players are arranged in a complete, undirected network (even if the assumption on the completeness of the network is generally not explicitly stated). Taking our previous example on firms producing brand-differentiated substitute goods, Corollary 2 and Theorem 3 may then be interpreted as a "tragedy of the commons in networks", with x_l representing the share of a common-pool resource consumed by firm l (individual consumptions are strategic complements when users imitate each other, and generate negative externalities when the consumption by one user reduces the quantity available for other users). The following example illustrates a social tragedy when the SO is interior.

Example 4 (Example 1 continued). Since $f_l(e_l) = 1 - \exp(-e_l)$ and $w_l = 1/2$ for all l, Assumption A2 is satisfied. So the PSNE and the SO are both unique and interior. We get

$$\hat{\mathbf{x}} = \begin{pmatrix} \ln(2) \\ \ln(2) \\ 2\ln(2) \\ 2\ln(2) \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{x}} = \begin{pmatrix} \ln(8/7) \\ \ln(8/7) \\ \ln(32/21) \\ 1/2 \ \ln(128/21) \end{pmatrix},$$

so $\tilde{\mathbf{x}} << \hat{\mathbf{x}}$, i.e., a social tragedy always appears.

Though this example focus on acyclic networks, it is worth noting that our results are not restricted to such network structures; they apply to each kind of network, i.e., with or withtout cycles, directed or undirected, weighted or unweighted, connected or disconnected, etc. Moreover, it is interesting to note that, whatever the network structure, a social tragedy always appears for people belonging to a "community".

Corollary 2. *Let $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$ be a game. Under A0 and A2, there exists a* social tragedy for non-isolated players*, i.e., $\tilde{x}_l < \hat{x}_l$ if and only if l is not isolated, otherwise $\tilde{x}_l = \hat{x}_l = x_l^*$.*

In the economic theory of negative externalities, the focus is on how to share and reduce the cost due to the external effects (Montgomery, 1972; Baumol and Oates, 1988). One well known way to perform this goal is to impose a tax mechanism which achieves the efficient profile (this is the so-called "Pigouvian" tax when negative externalities are pollution). The mechanism punishes players for their deviations from the efficient effort level and hence, players prefer to exert the efficient effort level. In networks, it appears that each player should pay a tax that is precisely equal to the difference between her influence measure and her marginal cost of absolute effort. Thus, the more influential a player, the higher her optimal tax.

Remark 2 (Optimal tax). The SO of game $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$ may be restored through a tax plan on marginal costs. Assume A2 holds (with $\mathbf{w}$). Following Proposition 2, we obtain

$$\tilde{\mathbf{x}}\left(\mathbf{\Lambda}, \mathbf{w}\right) = \mathbf{b}^{-}\left(\mathbf{\Lambda}, \left(\mathbf{f}'\right)^{-1}\left(\mathbf{b}^{+}\left(\mathbf{\Lambda}, \mathbf{w}\right)\right)\right).$$

Let us introduce the tax rate as follows

$$\boldsymbol{\tau} = \mathbf{b}^{+}\left(\mathbf{\Lambda}, \mathbf{w}\right) - \mathbf{w} = \mathbf{\Lambda}\mathbf{b}^{+}\left(\mathbf{\Lambda}, \mathbf{w}\right) \geq \mathbf{0}.$$

Then, A2 holds (with $\mathbf{w} + \boldsymbol{\tau}$) and by Proposition 1, the PSNE of game $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w} + \boldsymbol{\tau})$ is given by

$$\hat{\mathbf{x}}\left(\mathbf{\Lambda}, \mathbf{w} + \boldsymbol{\tau}\right) = \mathbf{b}^{-}\left(\mathbf{\Lambda}, \left(\mathbf{f}'\right)^{-1}\left(\mathbf{w} + \boldsymbol{\tau}\right)\right) = \tilde{\mathbf{x}}\left(\mathbf{\Lambda}, \mathbf{w}\right).$$

Therefore, the network matters to restore optimality in game $\mathcal{G}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$, i.e., the optimal tax plan involves different tax rate at each location throughout the network.

Finally, it is worth noting that our results hold even if we consider weighted social welfare functions.

Remark 3 (Weighted social welfare functions continued). Let us remind that a PSNE is an ordinal property. Thus, whatever the social weights $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) >> \mathbf{0}$ chosen to satisfy A2, the corresponding SO will also satisfy $\tilde{\mathbf{x}}(\boldsymbol{\alpha}) << \hat{\mathbf{x}}$ if and only if there are no isolated players, leading therefore to a social tragedy.

6 Discussion

This work is related to the branch of the literature on social dilemmas that study games of strategic complements and negative externalities. This line of research has been pioneered by Augustin Cournot (1838, ch. IX), who developed a model where two component goods are produced to form a composite good (e.g., copper and zinc to form brass). This model is referred by Eaton (2004) as the "Cournot model of price competition". In a way, we extend this model by adding a (social or geographic) network structure between the players. We now discuss the main assumptions we use for its analysis.

In our game, we assume that efforts are unbounded. When efforts are bounded from above, there always exists a unique PSNE which can not be characterized through the Bonacich centrality measure (Belhaj et al., 2012). When efforts are unbounded, we show that there exists a unique PSNE whenever the spectral radius of the network is sufficiently low (A0). This condition is known to guarantee the uniqueness of PSNE in network games of strategic complements or strategic substitutes (Ballester et al., 2006; Corbo et al., 2007; Ballester and Calvó-Armengol, 2010). Recent research on the uniqueness of PSNE in network games of strategic substitutes has moved beyond this spectral condition, showing that the lowest eigenvalue is key to PSNE outcomes when the network is undirected (Bramoullé et al., 2013; Rébillé and Richefort, 2013). In network games of strategic complements, there is no equilibrium whenever A0 is not met (Ballester et al., 2006; Corbo et al., 2007; Ballester and Calvó-Armengol, 2010). Interestingly, our paper is the first, to our knowledge, to highlight the role of A0 for the uniqueness and the characterization of SO in such games, and for the appearance of a social tragedy.

The quasi-linear preferences we consider, although restrictive, help us focus on the effects of network structure at equilibrium and at efficiency. This class of preferences allows us to formulate the equilibrium and the efficient efforts in terms of the directed paths that end and start at each player in the network. These results are in line with previous PSNE characterizations in various network games using bilinear payoffs (Ballester et al., 2006; Corbo et al., 2007) or quasi-linear payoffs (Ballester and Calvó-Armengol, 2010; Bramoullé et al., 2013). In our paper, the relevance to consider quasi-linear preferences to study a game of strategic complements is that it allows us to *compare* PSNE to SO and to *quantify* this comparison (such a comparison is an open issue in network games of strategic substitutes because directed paths of even and odd length have opposing signs in the closed-form expressions of PSNE and SO; see Rébillé and Richefort, 2012a and 2012b).

In our model, preferences are taken to be cardinal, thus admitting only an interpretation in terms of benefits and costs. A useful direction for further research would be to consider payoffs that increase as players improve their position in some ordinal ranking produced by the game, where orderings of the players could represent how well they have done in the game relative to their neighbors (Brandt et al., 2009; Hopkins and Kornienko, 2009). In this regard, a status loss function should be added to individual payoffs (see, e.g., Immorlica et al., 2012). This will be the next step on our research agenda.

Appendix

Proof of Proposition 1. Let $l \in N$ and $\mathbf{x} \in \mathbb{R}^n$. Player l maximizes her payoff function without constraint given others efforts $\mathbf{x}_{-l}$. The first order conditions are sufficient and are the following:

$$U_l'(x_l, \mathbf{x}_{-l}) = f_l'\left(x_l - \sum_{k:k\neq l} \lambda_{kl} x_k\right) - w_l = 0,$$

equivalently,

$$x_l - \sum_{k:k\neq l} \lambda_{kl} x_k = \left(f_l'\right)^{-1}(w_l) = x_l^*,$$

and in matrix notation,

$$\left(\mathbf{I} - \mathbf{\Lambda}^\mathsf{T}\right)\mathbf{x} = \mathbf{x}^*.$$

Now, $\mathbf{I} - \mathbf{\Lambda}$ has a nonnegative inverse, thus the unique PSNE $\hat{\mathbf{x}}$ is obtained by

$$\hat{\mathbf{x}} = \left(\mathbf{I} - \mathbf{\Lambda}^\mathsf{T}\right)^{-1}\mathbf{x}^* \geq \mathbf{x}^* >> \mathbf{0}.$$

□

Proof of Theorem 1. (Existence). We shall build a sufficiently large box where the maximum is reached. Let $\mathbf{x} \geq \mathbf{0}$. We have, by increasingness of the f_l's,

$$SW(\mathbf{x}) \leq \sum_{l=1}^{n} f_l(x_l) - w_l x_l.$$

Under A1, for all l, $f_l'(\infty) < w_l < f_l'(0)$, so there exists $\overline{x}_l > 0$ such that $f_l'(\overline{x}_l) < w_l$. Since f_l is strictly concave and differentiable for all l, the convexity inequality entails

$$f_l(x_l) \leq f_l(\overline{x}_l) + f_l'(\overline{x})(x_l - \overline{x}_l)$$

and also

$$f_l(0) < f_l(\overline{x}_l) - f_l'(\overline{x})\overline{x}_l.$$

Thus,

$$\begin{aligned} SW(\mathbf{x}) &\leq \sum_{l=1}^{n} f_l(\overline{x}_l) + f_l'(\overline{x})(x_l - \overline{x}_l) - w_l x_l \\ &= \sum_{l=1}^{n} f_l(\overline{x}_l) - f_l'(\overline{x})\overline{x}_l - (w_l - f_l'(\overline{x}))x_l \\ &\leq A - B\left(\sum_{i=1}^{n} x_l\right) \end{aligned}$$

where $A = \sum_{l=1}^{n} f_l(\overline{x}_l) - f_l'(\overline{x})\overline{x}_l$ and $B = \min_l w_l - f_l'(\overline{x})$ (> 0). So by the convexity inequality, $A > \sum_{l=1}^{n} f_l(0) = SW(\mathbf{0})$.

Hence, if $\sum_{l=1}^{n} x_l > C = (A - SW(\mathbf{0}))/B$ (> 0) then $SW(\mathbf{x}) < SW(\mathbf{0})$. So, being continuous, SW reaches a maximum on $\Delta = \{\mathbf{x} \geq \mathbf{0} : \sum_{l=1}^{n} x_l \leq C\}$ which is compact and non-empty ($\mathbf{0} \in \Delta$), hence on $\mathbb{R}_+^n$ since $SW(\mathbf{x}) < SW(\mathbf{0})$ whenever $\mathbf{x} \notin \Delta$.

(Uniqueness). Let us prove that SW is strictly concave. We may ignore the linear part, we shall show that the following function F is strictly concave where

$$F(\mathbf{x}) = \sum_l f_l \left(x_l - \sum_{k:k \neq l} \lambda_{kl} x_k \right), \ \mathbf{x} \geq \mathbf{0} \ .$$

Let $\mathbf{x}', \mathbf{x}'' \geq \mathbf{0}$ and $\theta \in (0,1)$ with $\mathbf{x}' \neq \mathbf{x}''$. Under A0, $\mathbf{I} - \mathbf{\Lambda}$ is invertible so $\mathbf{I} - \mathbf{\Lambda}^{\mathsf{T}}$ is invertible too. Now, there exists some l_0 such that

$$\left(\left(\mathbf{I} - \mathbf{\Lambda}^{\mathsf{T}} \right) \mathbf{x}' \right)_{l_0} \neq \left(\left(\mathbf{I} - \mathbf{\Lambda}^{\mathsf{T}} \right) \mathbf{x}'' \right)_{l_0},$$

that is,

$$x'_{l_0} - \sum_{k:k \neq l_0} \lambda_{kl_0} x'_k \neq x''_{l_0} - \sum_{k:k \neq l_0} \lambda_{kl_0} x''_k .$$

By strict concavity of f_{l_0} and concavity of f_l for $l \neq l_0$ it comes

$$\begin{aligned} & f_{l_0} \left(\theta \left(x'_{l_0} - \sum_{k:k \neq l_0} \lambda_{kl_0} x'_k \right) + (1-\theta) \left(x''_{l_0} - \sum_{k:k \neq l_0} \lambda_{kl_0} x''_k \right) \right) \\ > \ & \theta f_{l_0} \left(x'_{l_0} - \sum_{k:k \neq l_0} \lambda_{kl_0} x'_k \right) + (1-\theta) f_{l_0} \left(x''_{l_0} - \sum_{k:k \neq l_0} \lambda_{kl_0} x''_k \right) \end{aligned}$$

and

$$\begin{aligned} & f_l \left(\theta \left(x'_l - \sum_{k:k \neq l} \lambda_{kl} x'_k \right) + (1-\theta) \left(x''_l - \sum_{k:k \neq l} \lambda_{kl} x''_k \right) \right) \\ \geq \ & \theta f_l \left(x'_l - \sum_{k:k \neq l} \lambda_{kl} x'_k \right) + (1-\theta) f_l \left(x''_l - \sum_{k:k \neq l} \lambda_{kl} x''_k \right) . \end{aligned}$$

Summing these inequalities over i, we obtain

$$F(\theta \mathbf{x}' + (1-\theta) \mathbf{x}'') > \theta F(\mathbf{x}') + (1-\theta) F(\mathbf{x}'')$$

and this establishes strong concavity of F, thus strong concavity of SW. Therefore SW's maximum is unique.

(Characterization.) Let $D = \{l : \tilde{x}_l > 0\}$ and $\bar{D}$ its complement. For the inactive players, we have $\tilde{\mathbf{x}}_{\bar{D}} = \mathbf{0}$. For the active players, the first order conditions of social welfare maximization are: $\forall l \in D$,

$$f_l'\left(\tilde{x}_l - \sum_{k:k\neq l, k\in D} \lambda_{kl}\tilde{x}_k\right) - w_l - \sum_{j:j\neq l, j\in D} \lambda_{lj} f_j'\left(\tilde{x}_j - \sum_{i:i\neq j, i\in D} \lambda_{ij}\tilde{x}_i\right) - \sum_{g:g\neq l, g\in \bar{D}} \lambda_{lg} f_g'\left(0 - \sum_{h:h\neq g, h\in D} \lambda_{hg}\tilde{x}_h\right) = 0.$$

Let $e_i = f_i'(.)$ for all i. We note $\mathbf{\Lambda}_{I\times J}$ the submatrix of $\mathbf{\Lambda}$ with rows in $I \subseteq N$ and columns in $J \subseteq N$. In matrix notation, the first order conditions for the active players write

$$(\mathbf{I} - \mathbf{\Lambda})_{D\times D}\ \mathbf{e}_D - \mathbf{\Lambda}_{D\times \bar{D}}\ \mathbf{e}_{\bar{D}} - \mathbf{w}_D = \mathbf{0},$$

where $\mathbf{\Lambda}_{D\times \bar{D}}$ denotes the (possibly rectangular) submatrix of $\mathbf{\Lambda}$ consisting of rows with all the active players and columns with all the inactive players. We obtain

$$\tilde{\mathbf{x}}_D = \mathbf{b}^-\left(\mathbf{\Lambda}_{D\times D}, (\mathbf{f}')^{-1}\left(\mathbf{b}^+\left(\mathbf{\Lambda}_{D\times D}, \boldsymbol{\phi}\left(\tilde{\mathbf{x}}_D\right)\right)\right)\right),$$

where $\boldsymbol{\phi}(\tilde{\mathbf{x}}_D) = \mathbf{w}_D + \mathbf{\Lambda}_{D\times \bar{D}}\ \mathbf{e}_{\bar{D}}$ and $\mathbf{e}_{\bar{D}} = \mathbf{f}'(-(\mathbf{\Lambda}_{D\times \bar{D}})^\top\ \tilde{\mathbf{x}}_D)$. □

Proof of Proposition 2. Let us prove that the SO is interior. The first order conditions provide the following inequalities:

$$\forall l,\ \ f_l'\left(\tilde{x}_l - \sum_{k:k\neq l} \lambda_{kl}\tilde{x}_k\right) - \sum_{j:j\neq l} \lambda_{lj} f_j'\left(\tilde{x}_j - \sum_{i:i\neq j} \lambda_{ij}\tilde{x}_i\right) \leq w_l,$$

that is,

$$(\mathbf{I} - \mathbf{\Lambda})\left(\mathbf{f}'\left(\left(\mathbf{I} - \mathbf{\Lambda}^\top\right)\tilde{\mathbf{x}}\right)\right) \leq \mathbf{w},$$

or,

$$\mathbf{f}'\left(\left(\mathbf{I} - \mathbf{\Lambda}^\top\right)\tilde{\mathbf{x}}\right) \leq (\mathbf{I} - \mathbf{\Lambda})^{-1}\mathbf{w}.$$

By assumption, $\mathbf{f}'(\mathbf{0}) >> (\mathbf{I} - \mathbf{\Lambda})^{-1}\mathbf{w} \geq \mathbf{w}$, thus $((\mathbf{I} - \mathbf{\Lambda})^{-1}\mathbf{w})_l \in [w_l, f_l'(0))$. Hence,

$$\tilde{\mathbf{x}} \geq \left(\mathbf{I} - \mathbf{\Lambda}^\top\right)\tilde{\mathbf{x}} \geq (\mathbf{f}')^{-1}\left((\mathbf{I} - \mathbf{\Lambda})^{-1}\mathbf{w}\right) >> \mathbf{0}\ .$$

Since the SO is interior, the first order conditions provide actually the following equalities,

$$(\mathbf{I} - \mathbf{\Lambda})\left(\mathbf{f}'\left(\left(\mathbf{I} - \mathbf{\Lambda}^\top\right)\tilde{\mathbf{x}}\right)\right) = \mathbf{w},$$

thus

$$\tilde{\mathbf{x}} = \left(\mathbf{I} - \mathbf{\Lambda}^\top\right)^{-1}(\mathbf{f}')^{-1}\left((\mathbf{I} - \mathbf{\Lambda})^{-1}\mathbf{w}\right).$$

□

Proof of Theorem 2. At equilibrium, the FOCs are for all l,

$$f_l'\left(\hat{x}_l - \sum_{k:k\neq l} \lambda_{kl}\hat{x}_k\right) = w_l,$$

since the PSNE is interior. Under A0 and A1, the SO may be corner. Let $D = \{l : \tilde{x}_l > 0\}$. For $l \in D$, the FOCs for the efficient profile are

$$f_l'\left(\tilde{x}_l - \sum_{k:k\neq l, k\in D} \lambda_{kl}\tilde{x}_k\right) - w_l - \sum_{j:j\neq l} \lambda_{lj} f_j'\left(\tilde{x}_j - \sum_{i:i\neq j} \lambda_{ij}\tilde{x}_i\right) = 0,$$

and since $f_l' > 0$, we have

$$f_l'\left(\tilde{x}_l - \sum_{k:k\neq l, k\in D} \lambda_{kl}\tilde{x}_k\right) \geq w_l,$$

with a strict inequality if $\lambda_{lj} > 0$ for some j, i.e., if l is not a sink player. Now,

$$f_l'\left(\hat{x}_l - \sum_{k:k\neq l, k\in D} \lambda_{kl}\hat{x}_k\right) \leq f_l'\left(\hat{x}_l - \sum_{k:k\neq l} \lambda_{kl}\hat{x}_k\right) = w_l,$$

so,

$$f_l'\left(\hat{x}_l - \sum_{k:k\neq l, k\in D} \lambda_{kl}\hat{x}_k\right) \leq f_l'\left(\tilde{x}_l - \sum_{k:k\neq l, k\in D} \lambda_{kl}\tilde{x}_k\right).$$

The function f_l' being decreasing, we have

$$\tilde{x}_l - \sum_{k:k\neq l, k\in D} \lambda_{kl}\tilde{x}_k \leq \hat{x}_l - \sum_{k:k\neq l, k\in D} \lambda_{kl}\hat{x}_k,$$

that is, in matrix notation,

$$\left(\mathbf{I} - \mathbf{\Lambda}^{\mathsf{T}}\right)_{D\times D} \tilde{\mathbf{x}}_D \leq \left(\mathbf{I} - \mathbf{\Lambda}^{\mathsf{T}}\right)_{D\times D} \hat{\mathbf{x}}_D.$$

Then, since $\left(\mathbf{I} - \mathbf{\Lambda}^{\mathsf{T}}\right)_{D\times D}$ has also a nonnegative inverse with

$$\left(\left(\mathbf{I} - \mathbf{\Lambda}^{\mathsf{T}}\right)_{D\times D}\right)^{-1} \geq \mathbf{I}_D,$$

it comes by composition

$$\tilde{\mathbf{x}}_D \leq \hat{\mathbf{x}}_D,$$

and $\tilde{x}_l < \hat{x}_l$ whenever l is not a sink player. If $D \neq N$, then $\tilde{\mathbf{x}} < \hat{\mathbf{x}}$ since $\hat{\mathbf{x}}_{N\setminus D} >> \mathbf{0} = \tilde{\mathbf{x}}_{N\setminus D}$. If $D = N$, there is a player which is not a sink player (since $\mathbf{\Lambda} \neq \mathbf{0}$), so $\tilde{\mathbf{x}} < \hat{\mathbf{x}}$. □

Proof of Corollary 1. If there are no sink players, then $\tilde{x}_l < \hat{x}_l$ for all l (see the proof of Theorem 2). Hence, $\tilde{\mathbf{x}} << \hat{\mathbf{x}}$, i.e., a social tragedy appears. □

Proof of Theorem 3. (If). According to Propositions 1 and 2, a social tragedy holds if

$$\mathbf{0} << \left(\mathbf{I} - \mathbf{\Lambda}^\mathsf{T}\right)^{-1} \left((\mathbf{f}')^{-1} (\mathbf{w}) - (\mathbf{f}')^{-1} (\mathbf{I} - \mathbf{\Lambda})^{-1} \mathbf{w} \right).$$

Put $\mathbf{X} = (\mathbf{f}')^{-1}(\mathbf{w}) - (\mathbf{f}')^{-1}(\mathbf{I} - \mathbf{\Lambda})^{-1}\mathbf{w}$. Then, the condition is equivalent to,

$$\forall l,\ \exists k \ / \ \left(\mathbf{I} - \mathbf{\Lambda}^\mathsf{T}\right)^{-1}_{lk} \times X_k > 0$$

or equivalently,

$$\forall l,\ \exists k, \exists p \geq 0 \ / \ \left(\left(\mathbf{\Lambda}^\mathsf{T}\right)^p\right)_{lk} > 0 \text{ and } \left((\mathbf{I} - \mathbf{\Lambda})^{-1} \mathbf{w}\right)_k > w_k$$

that is,

$$\forall l,\ \exists k, \exists p \geq 0,\ \exists q \geq 1 \ / \ \left(\left(\mathbf{\Lambda}^\mathsf{T}\right)^p\right)_{lk} > 0 \text{ and } \left(\mathbf{\Lambda}^q \mathbf{w}\right)_k > 0$$

since $(\mathbf{I} - \mathbf{\Lambda})^{-1} - \mathbf{I} = \sum_{i=1}^{\infty} \mathbf{\Lambda}^i$. By transposition,

$$\forall l,\ \exists k, \exists p \geq 0,\ \exists q \geq 1 \ / \ \left(\mathbf{\Lambda}^p\right)_{kl} > 0 \text{ and } \left(\mathbf{\Lambda}^q \mathbf{w}\right)_k > 0$$

or

$$\forall l,\ \exists k, \exists p \geq 0,\ \exists q \geq 1 \ / \ \left(\mathbf{\Lambda}^p\right)_{kl} > 0 \text{ and } \left(\mathbf{\Lambda}^q\right)_{k.} \mathbf{w} > 0$$

or

$$\forall l,\ \exists k, \exists p \geq 0,\ \exists q \geq 1 \ / \ \left(\mathbf{\Lambda}^p\right)_{kl} > 0 \text{ and } \left(\mathbf{\Lambda}^q\right)_{k.} \neq \mathbf{0}^\mathsf{T}$$

since $\mathbf{w} >> \mathbf{0}$. Assume l is not an isolated player. Then either l has an influential neighbor or a vulnerable neighbor. If l admits an influential neighbor l', then take $k = l'$, $p = 1$ and $q = 1$ so $(\mathbf{\Lambda}^1)_{l'l} = \lambda_{l'l} > 0$ and $(\mathbf{\Lambda}^1)_{l'.} \neq \mathbf{0}^\mathsf{T}$. Otherwise, l admits a vulnerable neighbor l'', then take $k = l$, $p = 0$ and $q = 1$ so $(\mathbf{\Lambda}^0)_{ll} = 1 > 0$ and $(\mathbf{\Lambda}^1)_{l.} \neq \mathbf{0}^\mathsf{T}$ since $(\mathbf{\Lambda}^1)_{ll''} = \lambda_{ll''} > 0$.

(Only if). Assume there exists some isolated player l. As part of a PSNE $\hat{x}_l = x_l^*$ since l has no interaction with other players. And as part of SO $\tilde{x}_l = x_l^*$ since the social welfare function is separable with respect to x_l. Thus, $\hat{x}_l = \tilde{x}_l = x_l^*$. □

Proof of Corollary 2. (If). Let $I = \{\text{isolated players}\}$ and $I^c = \{\text{non-isolated players}\}$. Consider the subgame $\mathcal{G}_{I^c}(\mathbf{\Lambda}, \mathbf{f}, \mathbf{w})$ played by non-isolated players in the subnetwork $\mathbf{\Lambda}_{I^c \times I^c}$ with vertex I^c. Then, A0 and A2 hold since $\mathbf{\Lambda}_{I^c \times I^c}$ has positive inverse too, and $\mathbf{b}^+(\mathbf{\Lambda}_{I^c \times I^c}, .) = (\mathbf{b}^+(\mathbf{\Lambda}, .))_{I^c \times I^c}$. By Theorem 3, a social tragedy appears with $\tilde{\mathbf{x}}(I^c) << \hat{\mathbf{x}}(I^c)$. Since isolated player do not intervene at PSNE nor at SO we have $\hat{\mathbf{x}}_{I^c} = \hat{\mathbf{x}}(I^c)$ and $\tilde{\mathbf{x}}_{I^c} = \tilde{\mathbf{x}}(I^c)$ hence $\tilde{\mathbf{x}}_{I^c} << \hat{\mathbf{x}}_{I^c}$. And $\tilde{\mathbf{x}}_I = \hat{\mathbf{x}}_I = \mathbf{x}^*{}_I$, so a social tragedy for non-isolated players appears.

(Only if). If l is isolated, then $\tilde{x}_l = \hat{x}_l = x_l^*$ (see Theorem 3). □

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