HYPERBOLA METHOD ON TORIC VARIETIES

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Abstract. We develop a very general version of the hyperbola method which extends the known method by Blomer and Brüdern for products of projective spaces to a very large class of toric varieties. We use it to count Campana points of bounded log-anticanonical height on many split toric $\mathbb{Q}$-varieties with torus invariant boundary. We apply the strong duality principle in linear programming to show the compatibility of our results with the conjectured asymptotic.

Contents

1. Introduction

This paper stems from an investigation of the universal torsor method [Sal98, FP16] in relation to the problem of counting Campana points of bounded height on log Fano varieties in the framework of [PSTVA19, Conjecture 1.1]. Campana points are a notion of points that interpolate between rational points and integral points on certain log smooth pairs, or orbifolds, introduced and first studied by Campana [Cam04, Cam11, Cam15]. The study of the distribution of Campana points over number fields was initiated only quite recently and the literature on this topic is still sparse [BVV12, VV12, BY19, PSTVA19]. In this paper we deal with toric varieties, which constitute a fundamental family of examples for the study of the

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distribution of rational points [BT95b, BT95a, BT96, BT98, Sal98, dlB01a], via a combination of the universal torsor method with a very general version of the hyperbola method, which we develop.

We use the universal torsor method, instead of exploiting the toric group structure, because we hope to extend our approach to a larger class of log Fano varieties in the future. Indeed, the hyperbola method is well suited to deal with subvarieties [Sch14, Sch16, BB17, BB18, Mig16, BH19], and all log Fano varieties admit neat embeddings in toric varieties [ADHL15, GOST15] which can be exploited for the universal torsor method.

One of the key technical innovations in this article is the development of a very general form of the hyperbola method, which is motivated by work of Blomer and Brüdern in the case of products of projective spaces [BB18]. With our approach we extend this from products of projective spaces to a very large class of toric varieties with additional flexibility to change the height function.

Let $f : \mathbb{N}^s \to \mathbb{R}_{\geq 0}$ be an arithmetic function for which one has asymptotics for summing the function f over boxes, see Property (I) and Property (II) in Section 4. Let B be a large real parameter, $\mathcal{K}$ a finite index set and $\alpha_{i,k} \geq 0$ for $1 \leq i \leq s$ and $k \in \mathcal{K}$. The goal is then to use this information from sums over boxes to deduce an asymptotic formula for sums of the form

$$S^f := \sum_{\substack{\prod_{i=1}^s y_i^{\alpha_{i,k}} \leq B,\ \forall k \in \mathcal{K} \\ y_i \in \mathbb{N}, 1 \leq i \leq s}} f(\mathbf{y}).$$

We define the polyhedron $\mathcal{P} \subset \mathbb{R}^s$ given by

$$\sum_{i=1}^s \alpha_{i,k} \varpi_i^{-1} t_i \leq 1, \quad k \in \mathcal{K} \tag{1.1}$$

and

$$t_i \geq 0, \quad 1 \leq i \leq s. \tag{1.2}$$

Here the parameters ϖ_i, $1 \leq i \leq s$ are defined in Property I for the function f. The linear function $\sum_{i=1}^s t_i$ takes its maximal value on a face of $\mathcal{P}$ which we call F. We write a for its maximal value.

Theorem 1.1. *Let $f : \mathbb{N}^s \to \mathbb{R}_{\geq 0}$ be a function that satisfies Property I and Property II from Section 4, as well as Condition (4.1) for all subsets $\mathcal{I} \subset \{1, \dots, s\}$ and all vectors $\mathbf{y}_{\mathcal{I}}$.*

Assume that $\mathcal{P}$ is bounded and non-degenerate, and that F is not contained in a coordinate hyperplane of $\mathbb{R}^s$. Let $k = \dim F$. We assume that Assumption 4.14 holds. Then we have

$$S^f = (s-1-k)! C_{f,M} c_P (\log B)^k B^a + O\left(C_{f,E} (\log\log B)^s (\log B)^{k-1} B^a\right),$$

where $C_{f,M}$ and $C_{f,E}$ are the constants in Property I and c_P is the constant in equation (4.6).

The case treated in [BB18] would in this notation correspond to an index set $\mathcal{K}$ with one element where all the $\alpha_{i,k} = \alpha$ for all $1 \leq i \leq s$ and some $\alpha > 0$, and $k = s-1$. Our attack to evaluate the sum S^f starts in a similar way as in [BB18]. We cover the region given by the conditions $\prod_{i=1}^s y_i^{\alpha_{i,k}} \leq B$, $k \in \mathcal{K}$ with boxes of different side lengths on which we can evaluate the function f. One important ingredient in the hyperbola method in [BB18] is a combinatorial identity for the generating series

$$\sum_{\substack{j_1 + \dots + j_s \leq J \\ j_i \geq 0,\ 1 \leq i \leq s}} t^{j_1 + \dots + j_s},$$

which needs to be evaluated for J going to infinity. By induction the authors give a closed expression. For us this part of the argument breaks down, as we have in general more complicated polytopes that arise in the summation condition for S^f and are not aware of comparable combinatorial identities for the tuples $(j_1, \ldots, j_s)$ lying in general convex polytopes. Instead, we approximate the number of integers points in certain intersections of hyperplanes with a convex polytope by lattice point counting arguments and then use asymptotic evaluations for sums of the form

$$\sum_{0 \le m \le M} m^l \theta^m$$

for $0 < \theta < 1$ and $l, M \in \mathbb{N}$. To be able to control the intersections of the underlying polytopes with hyperplanes we need a geometric assumption on the polytope, i.e. on the data $\mathcal{K}$ and $\alpha_{i,k}$, $1 \le i \le s$, $k \in \mathcal{K}$, see Assumption 4.14.

Our main application of Theorem 1.1 is a proof of [PSTVA19, Conjecture 1.1] for many split toric varieties over $\mathbb{Q}$ with the log-anticanonical height:

Theorem 1.2. *Let Σ be the fan of a complete smooth split toric variety X over $\mathbb{Q}$. Let $\{\rho_1, \ldots, \rho_s\}$ be the set of rays of Σ. For each $i \in \{1, \ldots, s\}$ fix a positive integer m_i and denote by D_i the torus invariant divisor corresponding to ρ_i. Assume that $L := \sum_{i=1}^{s} \frac{1}{m_i} D_i$ is ample and satisfies Assumption 6.14. Let H_L be the height defined by L as in Section 6.3. Let $\mathscr{X}$ be the toric scheme defined by Σ over $\mathbb{Z}$, and for each $i \in \{1, \ldots, s\}$, let $\mathscr{D}_i$ be the closure of D_i in $\mathscr{X}$. For every $B > 0$, let $N(B)$ be the number of Campana $\mathbb{Z}$-points on the Campana orbifold $(\mathscr{X}, \sum_{i=1}^{s}(1 - 1/m_i)\mathscr{D}_i)$ (in the sense of* [PSTVA19, Definition 3.4]*) of height H_L at most B. Then for sufficiently large $B > 0$,*

$$N(B) = cB(\log B)^{r-1} + O(B(\log B)^{r-2}(\log\log B)^s), \tag{1.3}$$

where r is the rank of the Picard group of X, and c is a positive constant compatible with the prediction in [PSTVA19, §3.3].

In the situation of Theorem 1.2 the Assumption 6.14, which is the counterpart of Assumption 4.14, is satisfied in many cases, including products of projective spaces, the blow up of $\mathbb{P}^2$ in one point, and all smooth projective toric varieties with Picard rank $r \ge \dim X + 1$. We are not aware of any examples of toric varieties that don't satisfy Assumption 6.14. For all toric varieties that satisfy Assumption 6.14, our application of the hyperbola method recovers Salberger's result [Sal98] and improves on the error term.

Theorem 1.2 could also be deduced from work of de la Bretèche [dlB01b], [dlB01a], who developed a multi-dimensional Dirichlet series approach to count rational points of bounded height on toric varieties. Another approach could be via harmonic analysis of the height zeta function, even though such a proof would probably be more involved than the case of compactifications of vector groups [PSTVA19]. Our proof proceeds via the universal torsor method introduced by Salberger in [Sal98] in combination with Theorem 1.1. One of our main motivations for this approach is that it opens a path to counting Campana points on hypersurfaces in toric varieties.

When we apply Theorem 1.1 to prove Theorem 1.2, we need to verify that both the exponent of B as well as the power of $\log B$ match the prediction in [PSTVA19]. The exponent a in Theorem 1.1 is the result of a linear optimization problem. Similarly, the construction of the height function leading to the exponent one of B in Theorem 1.2 involves another linear optimization problem. We use the strong duality property in linear programming to recognize that the exponents are indeed compatible, and that this holds heuristically also in the more general setting where the height is not necessarily log-anticanonical. For the compatibility of the exponents of $\log B$ we exploit a different duality setup, which involves the Picard group of X.

The paper is organized as follows. In Section 2 we provide some auxiliary estimates on variants of geometric sums which are used later in Section 4. In Section 3 we study volumes of slices of polytopes under small deformations. In Section 4 we develop the hyperbola method and give a proof of Theorem 1.1. Sections 5 and 6 are dedicated to the application of the hyperbola method to prove Theorem 1.2. In Section 5 we study estimates for m-full numbers of bounded size subject to certain divisibility conditions, and we produce the estimates in boxes for the function f associated to the counting problem in Theorem 1.2. In Section 6 we describe the heights associated to semiample $\mathbb{Q}$-divisors on toric varieties over number fields, we study some combinatorial properties of the polytopes that play a prominent role in the application of the hyperbola method, and we show that the heuristic expectations coming from the hyperbola method agree with the prediction in [PSTVA19, Conjecture 1.1] on split toric varieties for Campana points of bounded height, where the height does not need to be anticanonical. We conclude the section with the proof of Theorem 1.2.

1.1. **Notation.** We use $\sharp S$ or $|S|$ to indicate the cardinality of a finite set S. Bold letters denote s-tuples of real numbers, and for given $\mathbf{x} \in \mathbb{R}^s$ we denote by $x_1, \dots, x_s \in \mathbb{R}$ the elements such that $\mathbf{x} = (x_1, \dots, x_s)$. For any subset $S \subset \mathbb{R}^s$ we denote by $\operatorname{cone}(S)$ the cone generated by S.

We denote by $\mathbb{F}_p$ the finite field with p elements and by $\overline{\mathbb{F}}_p$ an algebraic closure. For a number field $\mathbb{K}$, we denote by $\mathcal{O}_{\mathbb{K}}$ the ring of integers, and by $\mathfrak{N}(\mathfrak{a})$ the norm of an ideal $\mathfrak{a}$ of $\mathcal{O}_{\mathbb{K}}$. We denote by $\Omega_{\mathbb{K}}$ the set of places of $\mathbb{K}$, by Ω_f the set of finite places, and by Ω_∞ the set of infinite places. For every place v of $\mathbb{K}$, we denote by $\mathbb{K}_v$ the completion of $\mathbb{K}$ at v, and we define $|\cdot|_v = |N_{\mathbb{K}_v/\mathbb{Q}_{\tilde{v}}}(\cdot)|_{\tilde{v}}$, where $\tilde{v}$ is the place of $\mathbb{Q}$ below v and $|\cdot|_{\tilde{v}}$ is the usual real or p-adic absolute value on $\mathbb{Q}_{\tilde{v}}$. We denote by $|\cdot|$ the usual absolute value on $\mathbb{R}$.

We denote the Picard group and the effective cone of a smooth variety X by $\operatorname{Pic}(X)$ and $\operatorname{Eff}(X)$, respectively. For a divisor D on X we denote by $[D]$ its class in $\operatorname{Pic}(X)$. We say that a $\mathbb{Q}$-divisor D on X is semiample if there exists a positive integer t such that tD has integer coefficients and is base point free.

2. Preliminaries

In the hyperbola method in the next section we need good approximations for finite sums of the form

$$g_l(M, \theta) := \sum_{0 \le m \le M} m^l \theta^m,$$

for some $0 < \theta < 1$ and natural numbers $l, M \ge 0$ (here and in the following we understand $0^0 := 1$). In this subsection we also write $g_l(M)$ for $g_l(M, \theta)$.

We will use the following result.

Lemma 2.1. *For an integer $l \ge 0$ and $0 < \theta < 1$ and a real number $M > l$ we have*

$$(\theta - 1)^{l+1} g_l(M) = (-1)^{l+1} l! + O_l(1 - \theta) + O_l\left(\theta^M M^l\right).$$

Lemma 2.1 can be deduced from the following statement.

Lemma 2.2. *Assume that $M > l \geq 0$ and $\theta > 0$. Then we have*

$$(\theta - 1)^{l+1} g_l(M) = \sum_{0 \leq m < l+1} \theta^m \sum_{h=0}^{m} \binom{l+1}{h} (-1)^{l+1-h} (m-h)^l$$
$$+ \theta^M \sum_{0 < m \leq l+1} \theta^m \sum_{h=m}^{l+1} \binom{l+1}{h} (-1)^{l+1-h} \sum_{k=0}^{l} \binom{l}{k} M^k (m-h)^{l-k}.$$

For the proof of Lemma 2.2 and Lemma 2.1 we need the following identity. For an integer $0 \leq \alpha \leq l$ we have

$$\sum_{h=0}^{l+1} \binom{l+1}{h} (-1)^{l+1-h} h^\alpha = 0. \tag{2.1}$$

To see this consider the identity

$$(t-1)^{l+1} = \sum_{h=0}^{l+1} \binom{l+1}{h} t^h (-1)^{l+1-h}.$$

Now take derivatives with respect to t and then set $t = 1$.

Lemma 2.2 implies Lemma 2.1. We start in observing that

$$\begin{aligned}(\theta - 1)^{l+1} g_l(M) &= \sum_{0 \leq m < l+1} \theta^m \sum_{h=0}^{m} \binom{l+1}{h} (-1)^{l+1-h} (m-h)^l + O_l\left(\theta^M M^l\right) \\ &= \sum_{0 \leq m \leq l} \sum_{h=0}^{m} \binom{l+1}{h} (-1)^{l+1-h} (m-h)^l \\ &\quad + O_l(|\theta - 1|) + O_l\left(\theta^M M^l\right).\end{aligned}$$

We further compute

$$\begin{aligned}(\theta - 1)^{l+1} g_l(M) &= \sum_{0 \leq m \leq l+1} \sum_{h=0}^{m} \binom{l+1}{h} (-1)^{l+1-h} (m-h)^l \\ &\quad - \sum_{h=0}^{l+1} \binom{l+1}{h} (-1)^{l+1-h} (l+1-h)^l \\ &\quad + O_l(|\theta - 1|) + O_l\left(\theta^M M^l\right).\end{aligned}$$

Note that the term in the second line is equal to zero by equation (2.1). Hence we have

$$\begin{aligned}(\theta - 1)^{l+1} g_l(M) &= \sum_{0 \leq m \leq l+1} \sum_{h=0}^{m} \binom{l+1}{h} (-1)^{l+1-h} (m-h)^l \\ &\quad + O_l(|\theta - 1|) + O_l\left(\theta^M M^l\right).\end{aligned}$$

We now switch the summation of m and h to obtain

$$\begin{aligned}(\theta - 1)^{l+1} g_l(M) &= \sum_{0 \leq h \leq l+1} \binom{l+1}{h} (-1)^{l+1-h} \sum_{h \leq m \leq l+1} (m-h)^l \\ &\quad + O_l(|\theta - 1|) + O_l\left(\theta^M M^l\right) \\ &= \sum_{0 \leq h \leq l+1} \binom{l+1}{h} (-1)^{l+1-h} \sum_{0 \leq t \leq l+1-h} t^l \\ &\quad + O_l(|\theta - 1|) + O_l\left(\theta^M M^l\right).\end{aligned}$$

By the Faulhaber formulas $\sum_{0\le t\le l+1-h} t^l$ is a polynomial in h with leading term

$$\frac{(l+1-h)^{l+1}}{l+1} = \frac{(-1)^{l+1}}{l+1}h^{l+1} + \text{ lower order terms in } h.$$

Using equation (2.1) we hence obtain

$$\begin{aligned}(\theta-1)^{l+1}g_l(M) = &\sum_{0\le h\le l+1}\binom{l+1}{h}(-1)^{l+1-h}\frac{(-1)^{l+1}}{l+1}h^{l+1}\\ &+O_l(|\theta-1|)+O_l\left(\theta^M M^l\right)\\ = &\frac{1}{l+1}\sum_{0\le h\le l+1}\binom{l+1}{h}(-1)^h h^{l+1}\\ &+O_l(|\theta-1|)+O_l\left(\theta^M M^l\right).\end{aligned}$$

By equation (1.13) in [Gou72] we have

$$\sum_{0\le h\le l+1}\binom{l+1}{h}(-1)^h h^{l+1} = (-1)^{l+1}(l+1)!$$

Hence we get

$$(\theta-1)^{l+1}g_l(M) = (-1)^{l+1}l! + O_l(|\theta-1|) + O_l\left(\theta^M M^l\right). \qquad \square$$

We finish this section with a proof of Lemma 2.2.

Proof of Lemma 2.2. We compute

$$\begin{aligned}(\theta-1)^{l+1}g_l(M) &= (\theta-1)^{l+1}\sum_{0\le m\le M} m^l\theta^m\\ &= \sum_{h=0}^{l+1}\binom{l+1}{h}\theta^h(-1)^{l+1-h}\sum_{0\le m\le M} m^l\theta^m\\ &= \sum_{h=0}^{l+1}\binom{l+1}{h}(-1)^{l+1-h}\sum_{0\le m\le M} m^l\theta^{m+h}\\ &= \sum_{h=0}^{l+1}\binom{l+1}{h}(-1)^{l+1-h}\sum_{h\le m\le M+h}(m-h)^l\theta^m.\end{aligned}$$

We split the last summation into three ranges depending on the size of m and get

$$\begin{aligned}(\theta-1)^{l+1}g_l(M) &= \sum_{h=0}^{l+1}\binom{l+1}{h}(-1)^{l+1-h}\sum_{l+1\le m\le M}(m-h)^l\theta^m\\ &+\sum_{h=0}^{l}\binom{l+1}{h}(-1)^{l+1-h}\sum_{h\le m<l+1}(m-h)^l\theta^m\\ &+\sum_{h=1}^{l+1}\binom{l+1}{h}(-1)^{l+1-h}\sum_{M<m\le M+h}(m-h)^l\theta^m\\ &=\sum_{l+1\le m\le M}\theta^m\sum_{k=0}^{l}\binom{l}{k}m^k\sum_{h=0}^{l+1}\binom{l+1}{h}(-1)^{l+1-h}(-h)^{l-k}\\ &+\sum_{0\le m<l+1}\theta^m\sum_{h=0}^{m}\binom{l+1}{h}(-1)^{l+1-h}(m-h)^l\\ &+\sum_{M<m\le M+l+1}\theta^m\sum_{h=m-M}^{l+1}\binom{l+1}{h}(-1)^{l+1-h}(m-h)^l\end{aligned}$$

We now use the identity (2.1) for the third last line and deduce that

$$(\theta-1)^{l+1}g_l(M)=\sum_{0\le m<l+1}\theta^m\sum_{h=0}^{m}\binom{l+1}{h}(-1)^{l+1-h}(m-h)^l \tag{2.2}$$

$$+\sum_{0<m\le l+1}\theta^{M+m}\sum_{h=m}^{l+1}\binom{l+1}{h}(-1)^{l+1-h}(M+m-h)^l \tag{2.3}$$

Now the lemma follows in expanding each of the terms $(M+m-h)^l$. □

3. Volumes of certain sections of polytopes

In this section we provide some estimates on the volumes of intersections of convex polytopes with certain hyperplanes. These will be used in the next section in the development of our generalized form of the hyperbola method.

Proposition 3.1. *Let $\mathcal{P}\subseteq\mathbb{R}^s$ be an s-dimensional convex polytope with $s\ge 1$. Let $\mathcal{F}$ be a face of $\mathcal{P}$. Let $H\subseteq\mathbb{R}^s$ be a hyperplane such that $H\cap\mathcal{P}=\mathcal{F}$. Let $w\in\mathbb{R}^s$ such that $\mathcal{P}\subseteq H+\mathbb{R}_{\ge 0}w$. For $\delta>0$, let $H_\delta:=H+\delta w$. Let $k:=\dim\mathcal{F}$. We denote by meas_j the j-dimensional measure induced by the Lebesgue measure on $\mathbb{R}^s$. Then*

(i) for $\delta>0$ sufficiently small,

$$\mathrm{meas}_{s-1}(H_\delta\cap\mathcal{P})=c\delta^{s-1-k}+O(\delta^{s-k}),$$

where c is a positive constant that depends on $\mathcal{P}$, $\mathcal{F}$ and H.

(ii) If $s\ge 2$, for $\delta>0$ sufficiently small we have

$$\mathrm{meas}_{s-2}(\partial(H_\delta\cap\mathcal{P}))=\begin{cases}c'\delta^{s-2-k}+O(\delta^{s-1-k}), & \text{if } k\le s-2,\\ c'+O(\delta), & \text{if } k=s-1,\end{cases}$$

where c' is a positive constant that depends on $\mathcal{P}$, $\mathcal{F}$ and H.

(iii) Let $T\subseteq\mathbb{R}^s$ be a hyperplane and $u\in\mathbb{R}^s$ a vector such that $\mathcal{P}\subseteq T+\mathbb{R}_{\ge 0}u$ and $T\cap\mathcal{P}$ is a face of $\mathcal{P}$. For $\kappa>0$, let $T_\kappa=T+[0,\kappa]u$. Then for δ and

κ sufficiently small and positive,

$$\operatorname{meas}_{s-1}(H_\delta \cap \mathcal{P} \cap T_\kappa) \ll \begin{cases} \kappa\delta^{s-1-k} & \text{if } \mathcal{F} \not\subseteq T, \\ \min\{\kappa,\delta\}\delta^{s-2-k} & \text{if } \mathcal{F} \subseteq T, \end{cases}$$

where the implicit constant is independent of κ and δ.

Proof. For parts (i) and (ii) we proceed by induction on k. If $k = 0$, then $\mathcal{F}$ is a vertex of $\mathcal{P}$. Up to a translation, which is a volume preserving automorphism of $\mathbb{R}^s$, we can assume that $\mathcal{F}$ is the origin of $\mathbb{R}^s$. We denote by Q the cone with vertex $\mathcal{F}$ generated by $\mathcal{P}$. Then, for δ small enough, $H_\delta \cap \mathcal{P} = H_\delta \cap Q = \delta(H_1 \cap Q)$ and $\partial(H_\delta \cap \mathcal{P}) = \delta\partial(H_1 \cap Q)$. Thus $\operatorname{meas}_{s-1}(H_\delta \cap \mathcal{P}) = \delta^{s-1}\operatorname{meas}_{s-1}(H_1 \cap Q)$ and $\operatorname{meas}_{s-2}(\partial(H_\delta \cap \mathcal{P})) = \delta^{s-2}\operatorname{meas}_{s-2}(\partial(H_1 \cap Q))$.

Now we assume that $s \geq 2$ and $k \geq 1$. Let $\mathcal{P}_1, \dots, \mathcal{P}_N \subseteq \mathbb{R}^s$ be simplices of dimension s such that $\mathcal{P} = \bigcup_{i=1}^N \mathcal{P}_i$ is a triangulation of $\mathcal{P}$. Then $\operatorname{meas}_{s-1}(H_\delta \cap \mathcal{P}) = \sum_{i=1}^N \operatorname{meas}_{s-1}(H_\delta \cap \mathcal{P}_i)$. For δ small enough, we have $H_\delta \cap \mathcal{P}_i \neq \emptyset$ if and only if $\mathcal{P}_i \cap \mathcal{F} \neq \emptyset$. For $i \in \{1, \dots, N\}$ such that $\mathcal{P}_i \cap \mathcal{F} \neq \emptyset$ and $\dim(\mathcal{P}_i \cap \mathcal{F}) < k$ we have $\operatorname{meas}_{s-1}(H_\delta \cap \mathcal{P}_i) = O(\delta^{s-k})$ by the induction hypothesis. Since $\bigcup_{i=1}^N (\mathcal{P}_i \cap \mathcal{F}) = \mathcal{F}$, there is at least one index $i \in \{1, \dots, N\}$ such that $\dim(\mathcal{P}_i \cap \mathcal{F}) = k$. Therefore, in order to conclude the proof of (i) it suffices to prove the desired asymptotic formula in the case where $\mathcal{P}$ is a simplex.

From now on we assume that $\mathcal{P}$ is a simplex. We denote by $v_0, \dots, v_k$ the vertices of $\mathcal{F}$ and by $v_{k+1}, \dots, v_s$ the vertices of $\mathcal{P}$ not contained in $\mathcal{F}$. Up to a translation, which is a volume preserving automorphism of $\mathbb{R}^s$, we can assume that v_0 is the origin of $\mathbb{R}^s$. We observe that $v_1, \dots, v_s$ form a basis of the vector space $\mathbb{R}^s$. Let C be the cone (with vertex v_0) generated by $v_{k+1}, \dots, v_s$. We observe that $C \cap H = \{v_0\}$ as $H \cap \mathcal{P} = \mathcal{F}$ and C is contained in the cone (with vertex v_0) generated by $\mathcal{P}$. Let $Q := \mathcal{F} + C$. Then $\mathcal{P} \subseteq Q$ and $H \cap Q = \mathcal{F}$. Let $L \subseteq \mathbb{R}^s$ be the hyperplane that contains $v_1, \dots, v_s$. Then $L^+ = L + \mathbb{R}_{\leq 0}v_1$ is the halfspace with boundary L that contains v_0, and $\mathcal{P} = Q \cap L^+$. Let $L^- = L + \mathbb{R}_{\geq 0}v_1$. Then $\operatorname{meas}_{s-1}(H_\delta \cap \mathcal{P}) = \operatorname{meas}_{s-1}(H_\delta \cap Q) - \operatorname{meas}_{s-1}(H_\delta \cap Q \cap L^-)$. Let $H_1^+ = H_1 + \mathbb{R}_{\leq 0}w$ be the half space with boundary H_1 that contains $\mathcal{F}$. Then $H_1^+ \cap Q$ is bounded, and for δ small enough, $H_\delta \cap Q \cap L^- = H_\delta \cap (H_1^+ \cap Q \cap L^-)$ and $H_1^+ \cap Q \cap L^-$ is an s-dimensional polytope that intersects H in the $(k-1)$-dimensional face with vertices $v_1, \dots, v_k$. Hence, by induction hypothesis we have $\operatorname{meas}_{s-1}(H_\delta \cap Q \cap L^-) = O(\delta^{s-k})$. We observe that $\operatorname{meas}_{s-1}(H_\delta \cap Q) = \operatorname{meas}_{s-1}((H_\delta \cap Q) - \delta w)$, and

$$(H_\delta \cap Q) - \delta w = H \cap (\mathcal{F} + C - \delta w) = \mathcal{F} + (H \cap (C - \delta w)).$$

Hence, there is a positive constant a (which is the determinant of the matrix of a suitable linear change of variables in H) such that

$$\operatorname{meas}_{s-1}(H_\delta \cap Q) = a \operatorname{meas}_k(\mathcal{F}) \operatorname{meas}_{s-k-1}(H \cap (C - \delta w)).$$

We conclude the proof of (i) as

$$\operatorname{meas}_{s-k-1}(H \cap (C - \delta w)) = \operatorname{meas}_{s-k-1}(H_\delta \cap C) = \delta^{s-k-1}\operatorname{meas}_{s-k-1}(H_1 \cap C).$$

If $s = 2$, part (ii) holds. Hence, it remains to prove it for $s \geq 3$ and $k \geq 1$. Let $\mathcal{F}_1, \dots, \mathcal{F}_M$ be the faces of $\mathcal{P}$, then $\partial(H_\delta \cap \mathcal{P}) = \bigcup_{i=1}^M (H_\delta \cap \mathcal{F}_i)$. For δ small enough, we have $H_\delta \cap \mathcal{F}_i \neq \emptyset$ if and only if $\mathcal{F}_i \cap \mathcal{F} \neq \emptyset$. Moreover, for δ small enough we can assume that H_δ does not contain any vertex of $\mathcal{P}$. Since $\operatorname{meas}_{s-2}(H_\delta \cap \mathcal{F}_i) = 0$ whenever $H_\delta \cap \mathcal{F}_i$ has dimension strictly smaller than $s-2$, we have $\operatorname{meas}_{s-2}(\partial(H_\delta \cap \mathcal{P})) = \sum_{i=1}^M \operatorname{meas}_{s-2}(H_\delta \cap \mathcal{F}_i)$, where the sum actually runs over the maximal faces of $\mathcal{P}$ that intersect $\mathcal{F}$. Let $\widetilde{\mathcal{F}}$ be an $(s-1)$-dimensional face of $\mathcal{P}$ that intersects $\mathcal{F}$ such that $\widetilde{\mathcal{F}} \neq \mathcal{F}$, and let $\tilde{k} = \dim(\widetilde{\mathcal{F}} \cap \mathcal{F})$. By part (i) applied replacing $\mathcal{P}$ by $\widetilde{\mathcal{F}}$, we have $\operatorname{meas}_{s-2}(H_\delta \cap \widetilde{\mathcal{F}}) = \widetilde{c}\delta^{s-2-\tilde{k}} + O(\delta^{s-1-\widetilde{k}})$.

If $k \leq s-2$, there is an $(s-1)$-dimensional face $\widetilde{\mathcal{F}}$ of $\mathcal{P}$ that contains $\mathcal{F}$, hence $\tilde{k} = k$, and we conclude. If $k = s-1$, then there is an $(s-1)$-dimensional face $\widetilde{\mathcal{F}}$ of $\mathcal{P}$ such that $\tilde{k} = s-2$, hence $\mathrm{meas}_{s-2}(\partial(H_\delta \cap \mathcal{P})) = c + O(\delta)$.

For part (iii), we observe that if $\mathcal{F} \cap T = \emptyset$, then for δ and κ small enough we have $H_\delta \cap \mathcal{P} \cap T_\kappa = \emptyset$. Hence we can assume that $\mathcal{F} \cap T \neq \emptyset$. Let $\mathcal{P} = \bigcup_{i=1}^N \mathcal{P}_i$ be the triangulation of $\mathcal{P}$ in the proof of part (i). Up to reordering we can assume that there is $N' \leq N$ such that $\dim(\mathcal{P}_i \cap \mathcal{F}) = k$ and $\mathcal{P}_i \cap \mathcal{F} \cap T \neq \emptyset$ if and only if $i \leq N'$. Then

$$\mathrm{meas}_{s-1}(H_\delta \cap \mathcal{P}_\kappa) = \sum_{i=1}^{N'} \mathrm{meas}_{s-1}(H_\delta \cap \mathcal{P}_i \cap T_\kappa) + O(\delta^{s-k}).$$

Fix $i \in \{1, \dots, N'\}$. Let $v_{0,i}, \dots, v_{s,i}$ be the vertices of $\mathcal{P}_i$ such that $v_{0,i} \in \mathcal{F} \cap T$ and $v_{0,i}, \dots, v_{k,i}$ are the vertices of $\mathcal{F} \cap \mathcal{P}_i$. Since the Lebesgue measure is invariant under translation, we can assume without loss of generality that $v_{0,i}$ is the origin of $\mathbb{R}^s$. Let C_i be the cone (with vertex $v_{0,i}$) generated by $v_{k+1,i}, \dots, v_{s,i}$, and $Q_i := (\mathcal{F} \cap \mathcal{P}_i) + C_i$. Then $\mathrm{meas}_{s-1}(H_\delta \cap \mathcal{P}_i \cap T_\kappa) = \mathrm{meas}_{s-1}(H_\delta \cap Q_i \cap T_\kappa) + O(\delta^{s-k})$ and

$$\mathrm{meas}_{s-1}(H_\delta \cap Q_i \cap T_\kappa) \ll \mathrm{meas}_k(\mathcal{F} \cap \mathcal{P}_i \cap T_\kappa)\, \mathrm{meas}_{s-1-k}(H \cap ((C_i \cap T_\kappa) - \delta w)),$$

where the implicit constant is independent of κ and δ. Since $\mathcal{F} \cap \mathcal{P}_i$ is a simplex (it is a face of a simplex) and $v_{0,i} \in \mathcal{F} \cap \mathcal{P}_i \cap T$, there is $j_i \in \{1, \dots, s\}$ such that $(\mathbb{R}_{\geq 0} v_{j_i,i}) \cap T_\kappa$ is bounded, i.e. $(\mathbb{R}_{\geq 0} v_{j_i,i}) \cap T_\kappa = [0,1]\kappa a_{j_i,i} v_{j_i,i}$ for some $a_{j_i,i} > 0$ independent of κ. Let $T_{\kappa,i,j_i} := \{\sum_{i=1}^s \lambda_i v_i : 0 \leq \lambda_{j_i} \leq \kappa a_{j_i,i}\}$.

If $F \not\subseteq T$, then $\mathcal{F} \cap \mathcal{P}_i \not\subseteq T$ and we can choose $j_i \leq k$. Then $\mathcal{F} \cap \mathcal{P}_i \cap T_\kappa \subseteq \mathcal{F} \cap \mathcal{P}_i \cap T_{\kappa,i,j_i}$. Let $F_{j_i,i}$ be the maximal face of $\mathcal{F} \cap \mathcal{P}_i$ that does not contain $v_{j_i,i}$. Then $\mathrm{meas}_k(\mathcal{F} \cap \mathcal{P}_i \cap T_\kappa) \ll \kappa a_{j_i,i} \mathrm{meas}_{k-1}(F_{j_i,i}) \ll \kappa$, and $\mathrm{meas}_{s-1-k}(H \cap ((C_i \cap T_\kappa) - \delta w)) \leq \mathrm{meas}_{s-1-k}(H \cap (C_i - \delta w)) \ll \delta^{s-1-k}$, where the implicit constants are independent of κ and δ.

If $F \subseteq T$, then $j_i \geq k+1$, and $C_i \cap T_\kappa \subseteq C_i \cap T_{\kappa,i,j_i}$. Hence,

$$\begin{aligned}
&\mathrm{meas}_{s-1-k}(H \cap ((C_i \cap T_\kappa) - \delta w)) \\
&\qquad \leq \delta^{s-1-k} \mathrm{meas}_{s-1-k}(H_1 \cap C_i \cap (\delta^{-1} T_{\kappa,i,j_i})) \ll \delta^{s-2-k} \min\{\delta, \kappa\}. \quad \square
\end{aligned}$$

4. Hyperbola method

We consider a function $f : \mathbb{N}^s \to \mathbb{R}_{\geq 0}$ with the following properties.

Property I: Assume that there are non-negative real constants $C_{f,M} \leq C_{f,E}$ and $\Delta > 0$ and $\varpi_i > 0$, $1 \leq i \leq s$ such that for all $B_1, \dots, B_s \in \mathbb{R}_{\geq 1}$ we have

$$\sum_{1 \leq y_i \leq B_i,\ 1 \leq i \leq s} f(\mathbf{y}) = C_{f,M} \prod_{i=1}^s B_i^{\varpi_i} + O\left(C_{f,E} \prod_{i=1}^s B_i^{\varpi_i} \left(\min_{1 \leq i \leq s} B_i \right)^{-\Delta} \right)$$

where the implied constant is independent of f.

Property II: Assume that there are positive real numbers D and ν such that the following holds. Let $\mathcal{I} \subsetneq \{1, \dots, s\}$ be a non-empty subset of indices and fix some $(y_i)_{i \in \mathcal{I}} \in \mathbb{N}^{|\mathcal{I}|}$. Write $\mathbf{y}_\mathcal{I}$ for the vector $(y_i)_{i \in \mathcal{I}}$ and $|\mathbf{y}_\mathcal{I}|$ for its maximums norm. Then there is a non-negative constant $C_{f,M,\mathcal{I}}(\mathbf{y}_\mathcal{I})$ such that for all $B_i \in \mathbb{R}_{\geq 1}$, $i \in \{1, \dots, s\} \setminus \mathcal{I}$ one has

$$\sum_{1 \leq y_i \leq B_i,\ i \notin \mathcal{I}} f(\mathbf{y}) = C_{f,M,\mathcal{I}}(\mathbf{y}_\mathcal{I}) \prod_{i \notin \mathcal{I}} B_i^{\varpi_i} + O(C_{f,E} |\mathbf{y}_\mathcal{I}|^D \prod_{i \notin \mathcal{I}} B_i^{\varpi_i} (\min_{i \notin \mathcal{I}} B_i)^{-\Delta}),$$

uniformly in $|\mathbf{y}_\mathcal{I}| \leq (\prod_{i \notin \mathcal{I}} B_i)^\nu$.

Remark 4.1. Assume that $f:\mathbb{N}^s\to\mathbb{R}_{\geq 0}$ satisfies Property I and Property II. Fix a vector $\mathbf{y}_\mathcal{I}$ and set

$$f_{\mathbf{y}_\mathcal{I}}(\mathbf{y}')=f(\mathbf{y}_\mathcal{I},\mathbf{y}'),$$

for $\mathbf{y}'=(y_i)_{i\notin I}$. Assume furthermore that

$$C_{f,M,\mathcal{I}}(\mathbf{y}_\mathcal{I})\leq C_{f,E}|\mathbf{y}_\mathcal{I}|^D. \tag{4.1}$$

Then the function $f_{\mathbf{y}_\mathcal{I}}$ in $|\mathcal{I}^c|$ variables satisfies Property I with constants

$$\widetilde{C_{f,M}}=C_{f,M,\mathcal{I}}(\mathbf{y}_\mathcal{I})$$

and

$$\widetilde{C_{f,E}}=C_{f,E}|\mathbf{y}_\mathcal{I}|^{D'},$$

for some D' sufficiently large, depending on ν, Δ and ϖ_i, $i\notin\mathcal{I}$.

Next we need an analogous statement to Lemma 2.3 in [BB18].

Lemma 4.2. *Assume that $f:\mathbb{N}^s\to\mathbb{R}_{\geq 0}$ satisfies Property I and Property II above. Let $B_i\in\mathbb{R}_{\geq 1}$ for $i\in\mathcal{I}$. Then one has*

$$\sum_{1\leq y_i\leq B_i,\ i\in\mathcal{I}}C_{f,M,\mathcal{I}}(\mathbf{y}_\mathcal{I})=C_{f,M}\prod_{i\in\mathcal{I}}B_i^{\varpi_i}+O\left(C_{f,E}\prod_{i\in\mathcal{I}}B_i^{\varpi_i}(\min_{i\in\mathcal{I}}B_i)^{-\Delta}\right).$$

Proof. Let $Z>1$ be a real parameter. Then by Property II

$$\sum_{1\leq y_i\leq Z,\ i\notin\mathcal{I}}f(\mathbf{y})=C_{f,M,\mathcal{I}}(\mathbf{y}_\mathcal{I})Z^{\sum_{i\notin\mathcal{I}}\varpi_i}+O\left(C_{f,E}|\mathbf{y}_\mathcal{I}|^DZ^{\sum_{i\notin\mathcal{I}}\varpi_i}Z^{-\Delta}\right) \tag{4.2}$$

uniformly in $|\mathbf{y}_\mathcal{I}|\leq Z^{\nu|\mathcal{I}^c|}$. Now assume that $B_i\in\mathbb{R}_{\geq 1}$, $i\in\mathcal{I}$, are real parameters with $B_i\leq Z^{\nu|\mathcal{I}^c|}$, for all $i\in\mathcal{I}$. Summing the relation (4.2) over $y_i\leq B_i$ for $i\in\mathcal{I}$ we obtain that

$$\sum_{\substack{1\leq y_i\leq Z\ \ i\notin\mathcal{I}\\ 1\leq y_i\leq B_i\ \ i\in\mathcal{I}}}f(\mathbf{y})=Z^{\sum_{i\notin\mathcal{I}}\varpi_i}\sum_{1\leq y_i\leq B_i,\ i\in\mathcal{I}}C_{f,M,\mathcal{I}}(\mathbf{y}_\mathcal{I})$$
$$+O\left(C_{f,E}(\prod_{i\in\mathcal{I}}B_i)|\max_i B_i|^DZ^{\sum_{i\notin\mathcal{I}}\varpi_i}Z^{-\Delta}\right).$$

On the other hand, if we evaluate the sum on the left hand side using Property I we obtain (if we take ν sufficiently small such that $\nu|\mathcal{I}^c|\leq 1$)

$$\sum_{\substack{1\leq y_i\leq Z\ \ i\notin\mathcal{I}\\ 1\leq y_i\leq B_i\ \ i\in\mathcal{I}}}f(\mathbf{y})=C_{f,M}Z^{\sum_{i\notin\mathcal{I}}\varpi_i}\prod_{i\in\mathcal{I}}B_i^{\varpi_i}+O\left(C_{f,E}Z^{\sum_{i\notin\mathcal{I}}\varpi_i}\prod_{i\in\mathcal{I}}B_i^{\varpi_i}(\min_{i\in\mathcal{I}}B_i)^{-\Delta}\right).$$

Comparing these two asymptotics implies that

$$\sum_{1\leq y_i\leq B_i,\ i\in\mathcal{I}}C_{f,M,\mathcal{I}}(\mathbf{y}_\mathcal{I})=C_{f,M}\prod_{i\in\mathcal{I}}B_i^{\varpi_i}$$
$$+O\left(C_{f,E}(\prod_{i\in\mathcal{I}}B_i)|\max_i B_i|^DZ^{-\Delta}\right)+O\left(C_{f,E}\prod_{i\in\mathcal{I}}B_i^{\varpi_i}(\min_{i\in\mathcal{I}}B_i)^{-\Delta}\right).$$

The lemma follows in taking Z sufficiently large. ☐

Let B be a large real parameter. Let $\mathcal{K}$ be a finite index set and $s\in\mathbb{N}$. Let $\alpha_{i,k}\geq 0$ for $1\leq i\leq s$ and $k\in\mathcal{K}$.

Our goal is to evaluate the sum (if finite)

$$S^f=\sum_{\substack{\prod_{i=1}^s y_i^{\alpha_{i,k}}\leq B,\ \forall k\in\mathcal{K}\\ y_i\in\mathbb{N},1\leq i\leq s}}f(\mathbf{y}).$$

More generally, let $B_k \in \mathbb{R}_{\geq 1}$, $k \in \mathcal{K}$ be real parameters and write $\mathbf{B} = (B_k)_{k\in\mathcal{K}}$. We define

$$S^f(\mathbf{B}) := \sum_{\substack{\prod_{i=1}^s y_i^{\alpha_{i,k}} \leq B_k,\ \forall k\in\mathcal{K} \\ y_i\in\mathbb{N}, 1\leq i\leq s}} f(\mathbf{y}).$$

Let

$$b_k := \log B_k / \log B, \quad k \in \mathcal{K}.$$

We assume that there are absolute positive constants C_3 and C_4 such that

$$0 < C_3 \leq b_k \leq C_4, \quad k \in \mathcal{K}. \tag{4.3}$$

In applications we can later choose C_3 and C_4 very close to 1.

We start with a heuristic for the expected growth of the sum S^f. Let B be a large real parameter. Consider the contribution to the sum S^f from a dyadic box where each $y_i \sim B^{t_i \varpi_i^{-1}}$ say for real parameters $t_i \geq 0$ (for example we could think of $\frac{1}{2} B^{t_i \varpi_i^{-1}} \leq y_i \leq B^{t_i \varpi_i^{-1}}$, $1 \leq i \leq s$). Such a box is expected to contribute about

$$\sum_{y_i \sim B^{t_i \varpi_i^{-1}}, 1\leq i\leq s} f(\mathbf{y}) \sim B^{\sum_{i=1}^s t_i}$$

to the sum S^f. In order for such a box to lie in the summation range we roughly speaking need

$$\sum_{i=1}^s \alpha_{i,k} \varpi_i^{-1} t_i \leq b_k, \quad k \in \mathcal{K} \tag{4.4}$$

and

$$t_i \geq 0, \quad 1 \leq i \leq s. \tag{4.5}$$

The system of equations (4.4) and (4.5) define a polyhedron $\mathcal{P} \subset \mathbb{R}^s$.

Assumption 4.3. We assume that $\mathcal{P}$ is bounded and non-degenerate in a sense that it is not contained in a $s-1$ dimensional subspace of $\mathbb{R}^s$.

The linear function

$$\sum_{i=1}^s t_i$$

takes its maximum on a face of $\mathcal{P}$. We call the maximal value a and assume that this maximum is obtained on a k-dimensional face of $\mathcal{P}$.

Let H_δ be the hypersurface given by

$$\sum_{i=1}^s t_i = a - \delta.$$

It comes equipped with an $s-1$ dimensional measure which is obtained from the pull-back of the standard Lebesgue measure to any of its coordinate plane projections. In the following we write meas for this measure.

Assumption 4.4. We assume that there is a constant c_P such that for $\delta > 0$ which are sufficiently small (in terms of $\mathcal{P}$) we have

$$|\operatorname{meas}_{s-1}(H_\delta \cap \mathcal{P}) - c_P \delta^{s-1-k}| \leq C\delta^{s-k}, \tag{4.6}$$

for a sufficiently large constant C, depending only on $\mathcal{P}$. Moreover, we assume that the measure of the boundary of the polytope $H_\delta \cap \mathcal{P}$ is bounded by

$$\operatorname{meas}_{s-2}(\partial(H_\delta \cap \mathcal{P})) \ll 1. \tag{4.7}$$

Here we mean by $\operatorname{meas}_{s-2}$ the $s-2$ dimensional boundary measure.

We note that Assumption 4.4 is a consequence of Proposition 3.1 parts (i) and (ii).

Assumption 4.5. The face F on which the function $\sum_{i=1}^s t_i$ takes its maximum on $\mathcal{P}$ is not contained in a coordinate hyperplane of $\mathbb{R}^s$ (with coordinates t_i).

Let $\mathbf{l} \in \mathbb{Z}_{\geq 0}^s$ and $1 < \theta < 2$ a parameter to be chosen later. We define box counting functions

$$B_f(\mathbf{l}, \theta) = \sum_{y_i \in [\theta_i^{l_i}, \theta_i^{l_i+1}), 1 \leq i \leq s} f(\mathbf{y}),$$

where

$$\theta_i = \theta^{\varpi_i^{-1}}, \quad 1 \leq i \leq s.$$

Assume that f satisfies Property I and Property II. We use Property I and inclusion-exclusion to evaluate

$$\begin{aligned} B_f(\mathbf{l}, \theta) &= C_{f,M} \prod_{i=1}^s (\theta_i^{\varpi_i(l_i+1)} - \theta_i^{\varpi_i l_i}) + O(C_{f,E} \prod_{i=1}^s \theta_i^{l_i \varpi_i} (\min_i \theta_i^{l_i})^{-\Delta}) \\ &= C_{f,M} (\theta - 1)^s \prod_{i=1}^s \theta^{l_i} + O(C_{f,E} \prod_{i=1}^s \theta^{l_i} (\min_i \theta_i^{l_i})^{-\Delta}). \end{aligned}$$

We deduce that

$$B_f(\mathbf{l}, \theta) = C_{f,M} (\theta - 1)^s \theta^{\sum_{i=1}^s l_i} + O(C_{f,E} \theta^{\sum_{i=1}^s l_i} (\min_i \theta^{\varpi_i^{-1} l_i})^{-\Delta}).$$

Recall that we assumed in Property I that

$$C_{f,M} \leq C_{f,E}.$$

Hence we have

$$B_f(\mathbf{l}, \theta) \ll C_{f,E} \theta^{\sum_{i=1}^s l_i}. \tag{4.8}$$

We note that the sum

$$E_2^f := \sum_{\substack{\prod_{i=1}^s y_i^{\alpha_{i,k}} \leq B_k, \ k \in \mathcal{K} \\ y_i \in \mathbb{N}, \ 1 \leq i \leq s \\ \prod_{i=1}^s y_i^{\varpi_i} > B^a}} f(\mathbf{y})$$

is empty.

Let $\tilde{A}$ be a large natural number, which we view as a parameter to be specified later.

We set

$$S_{1,f} := \sum_{\substack{\prod_{i=1}^s y_i^{\alpha_{i,k}} \leq B_k, \ k \in \mathcal{K} \\ y_i \in \mathbb{N}, \ 1 \leq i \leq s \\ y_i \geq (\log B)^{\tilde{A}} \forall 1 \leq i \leq s}} f(\mathbf{y}),$$

and note that

$$S_{1,f} = \sum_{\substack{\prod_{i=1}^s y_i^{\alpha_{i,k}} \leq B_k, \ k \in \mathcal{K} \\ y_i \in \mathbb{N}, \ 1 \leq i \leq s \\ \prod_i y_i^{\varpi_i} \leq B^a \\ y_i \geq (\log B)^{\tilde{A}} \forall 1 \leq i \leq s}} f(\mathbf{y}).$$

Let $1 < \theta < 2$ be a parameter to be chosen later. We now cover the sum $S_{1,f}$ with boxes of the form $B_f(\mathbf{l}, \theta)$. Let $\mathcal{L}^+$ be the set of $\mathbf{l} \in \mathbb{Z}_{\geq 0}^s$ such that the following inequalities hold

$$\sum_{i=1}^{s} \alpha_{i,k}\varpi_i^{-1} l_i \leq \frac{\log B_k}{\log\theta}, \quad k \in \mathcal{K}$$

$$l_i + 1 \geq \frac{\varpi_i \tilde{A}\log\log B}{\log\theta}, \quad 1 \leq i \leq s.$$

Similarly, let $\widetilde{\mathcal{L}}^-$ be the set of $\mathbf{l} \in \mathbb{Z}_{\geq 0}^s$ such that the following inequalities hold

$$\sum_{i=1}^{s} \alpha_{i,k}\varpi_i^{-1}(l_i+1) \leq \frac{\log B_k}{\log\theta}, \quad k \in \mathcal{K}$$

$$l_i \geq \frac{\varpi_i \tilde{A}\log\log B}{\log\theta}, \quad 1 \leq i \leq s.$$

Let C_5 be a positive constant such that

$$\sum_{i=1}^{s} \alpha_{i,k}\varpi_i^{-1} \leq C_5 \frac{\log B_k}{\log B}, \quad k \in \mathcal{K}.$$

We define $\mathcal{L}^-$ to be the set of $\mathbf{l} \in \mathbb{Z}_{\geq 0}^s$ such that the following inequalities hold

$$\sum_{i=1}^{s} \alpha_{i,k}\varpi_i^{-1} l_i \leq \frac{\log B_k}{\log\theta} - C_5\frac{\log B_k}{\log B}, \quad k \in \mathcal{K}$$

$$l_i \geq \frac{\varpi_i \tilde{A}\log\log B}{\log\theta}, \quad 1 \leq i \leq s.$$

Then we have

$$\sum_{\mathbf{l}\in\mathcal{L}^-} B_f(\mathbf{l},\theta) = S_{1,f}^- \leq S_{1,f} \leq S_{1,f}^+ = \sum_{\mathbf{l}\in\mathcal{L}^+} B_f(\mathbf{l},\theta),$$

where we read the last line as a definition for $S_{1,f}^-$ and $S_{1,f}^+$. Note that the coverings into boxes do not depend on the function f but only on the summation conditions on the variables y_i, $1 \leq i \leq s$.

Let $r^+(l)$ (resp. $r^-(l)$) be the set of $\mathbf{l} \in \mathcal{L}^+$ (resp. $\mathcal{L}^-$) such that

$$\sum_{i=1}^{s} l_i = l.$$

We recall that

$$B_f(\mathbf{l},\theta) = C_{f,M}(\theta-1)^s\theta^{\sum_{i=1}^s l_i} + O(C_{f,E}\theta^{\sum_{i=1}^s l_i}(\min_i \theta^{\varpi_i^{-1}l_i})^{-\Delta}).$$

This leads to

$$\sum_{\mathbf{l}\in\mathcal{L}^-} B_f(\mathbf{l},\theta) = C_{f,M}\sum_{\mathbf{l}\in\mathcal{L}^-}\theta^{\sum_{i=1}^s l_i}(\theta-1)^s + O\left(C_{f,E}\sum_{\mathbf{l}\in\mathcal{L}^-}\theta^{\sum_{i=1}^s l_i}(\log B)^{-\Delta\tilde{A}}\right)$$

$$= (\theta-1)^s C_{f,M}\sum_{l\leq a\log B/\log\theta} r^-(l)\theta^l + O\left(C_{f,E}\sum_{\mathbf{l}\in\mathcal{L}^-}\theta^{\sum_{i=1}^s l_i}(\log B)^{-\Delta\tilde{A}}\right)$$

Every vector $\mathbf{l} \in \mathcal{L}^-$ satisfies the bound

$$\sum_{i=1}^{s} l_i \leq a\frac{\log B}{\log\theta},$$

and hence

$$l_i \leq a\frac{\log B}{\log\theta} \quad 1 \leq i \leq s.$$

This leads to the bound

$$\sum_{\mathbf{l}\in\mathcal{L}^-}\theta^{\sum_{i=1}^s l_i}\ll\left(\frac{\log B}{\log\theta}\right)^s B^a.$$

We deduce that

$$\sum_{\mathbf{l}\in\mathcal{L}^-}B_f(\mathbf{l},\theta)=(\theta-1)^s C_{f,M}\sum_{l\leq a\log B/\log\theta}r^-(l)\theta^l$$
$$+O\left(C_{f,E}\left(\frac{\log B}{\log\theta}\right)^s B^a(\log B)^{-\Delta\tilde{A}}\right).$$

Let $\tilde{r}(l)$ be the number of $\mathbf{l}\in\mathbb{Z}_{\geq 0}^s$ such that $\sum_{i=1}^s l_i=l$ and the following inequalities hold

$$\sum_{i=1}^s\alpha_{i,k}\varpi_i^{-1}l_i\leq\frac{\log B_k}{\log\theta}-C_5\frac{\log B_k}{\log B},\quad k\in\mathcal{K}.$$

Note that $\tilde{r}(l)$ is the number of lattice points in the polytope given by

$$\sum_{i=1}^s t_i=l,$$
$$\sum_{i=1}^s\alpha_{i,k}\varpi_i^{-1}t_i\leq\frac{\log B_k}{\log\theta}-C_5\frac{\log B_k}{\log B},\quad k\in\mathcal{K}$$
$$t_i\geq 0,\quad 1\leq i\leq s.$$

Finally, for $1\leq i_0\leq s$ let $r_{\tilde{A},i_0}(l)$ be the number of $\mathbf{l}\in\mathbb{Z}_{\geq 0}^s$ such that $\sum_{i=1}^s l_i=l$ and

$$l_{i_0}\leq\frac{\varpi_i\tilde{A}\log\log B}{\log\theta},$$

and

$$\sum_{i=1}^s\alpha_{i,k}\varpi_i^{-1}l_i\leq\frac{\log B_k}{\log\theta},\quad k\in\mathcal{K}.$$

Note that we have

$$r^-(l)=\tilde{r}(l)+O\left(\sum_{1\leq i_0\leq s}r_{\tilde{A},i_0}(l)\right).$$

We now stop a moment to introduce some more auxiliary polytopes. We recall that $\mathcal{P}\subset\mathbb{R}^s$ is the polytope given by the system of equations (4.4) and (4.5).

For $1\leq i_0\leq s$ and $\kappa>0$ we introduce the polytope $\mathcal{P}_{i_0,\kappa}$ given by the system of equations

$$\sum_{i=1}^s\alpha_{i,k}\varpi_i^{-1}t_i\leq\log B_k/\log B,\quad k\in\mathcal{K}$$

and

$$t_i\geq 0,\quad 1\leq i\leq s,$$

and

$$t_{i_0}\leq\kappa.$$

I.e. $\mathcal{P}_{i_0,\kappa}$ is obtained from intersecting $\mathcal{P}$ with the halfspace $t_{i_0}\leq\kappa$. Let H_δ be defined as before, i.e. the hyperplane given by

$$\sum_{i=1}^s t_i=a-\delta.$$

Assumption 4.6. Let $0 \leq k \leq s-1$. Assume that $\kappa \leq \epsilon$ for ϵ sufficiently small as well as $\delta > 0$ sufficiently small in terms of the data describing $\mathcal{P}$. Then we have

$$\text{meas}(H_\delta \cap \mathcal{P}_{i_0,\kappa}) \ll \kappa \delta^{s-1-k}.$$

Moreover, we have

$$\text{meas}\, \partial(H_\delta \cap \mathcal{P}_{i_0,\kappa}) \ll 1.$$

Remark 4.7. Note that in the case $k=0$ and where the maximal face F is not contained in a coordinate hyperplane, the intersection $H_\delta \cap \mathcal{P}_{i_0,\kappa}$ is empty for ϵ and δ sufficiently small.

Remark 4.8. Assumption 4.6 is a consequence of Assumption 4.5 together with Proposition 3.1.

In our applications it would already be sufficient to take κ of size $\kappa \ll \frac{\log\log B}{\log B}$. We next evaluate the function $r^-(l)$ asymptotically.

Lemma 4.9. *Assume that* $0 \leq k \leq s-1$. *Assume that Assumption 4.5 holds. Let* l *be an integer with*

$$(a-\delta)\left(\frac{\log B}{\log \theta} - C_5\right) \leq l \leq a \left(\frac{\log B}{\log \theta} - C_5\right).$$

Then we have

$$\begin{aligned} r^-(l) = c_P \left(\frac{\log B}{\log \theta}\right)^{s-1} \left(a - \frac{\log \theta}{\log B} l\right)^{s-1-k} \\ + O\left(\left(\frac{\log B}{\log \theta}\right)^{s-2} + \left(\frac{\log B}{\log \theta}\right)^{s-1}\left(a - \frac{\log \theta}{\log B} l\right)^{s-k}\right) \\ + O\left(\frac{\log\log B}{\log B}\left(\frac{\log B}{\log \theta}\right)^{s-1}\left(a - \frac{\log \theta}{\log B} l\right)^{s-1-k}\right). \end{aligned}$$

Here we read $0^0 = 1$.

If

$$l > a\left(\frac{\log B}{\log \theta} - C_5\right),$$

then we have $r^-(l) = 0$.

Remark 4.10. Note that exactly the same asymptotic also holds for $r^+(l)$, but then in the range

$$(a-\delta)\frac{\log B}{\log \theta} \leq l \leq a \frac{\log B}{\log \theta}.$$

Proof. We recall that $\tilde{r}(l)$ counts lattice points in the polytope $P(l, B, \theta)$ given by

$$\sum_{i=1}^s t_i = l,$$

$$\sum_{i=1}^s \alpha_{i,k} \varpi_i^{-1} t_i \leq \frac{\log B_k}{\log \theta} - C_5 \frac{\log B_k}{\log B}, \quad k \in \mathcal{K},$$

$$t_i \geq 0, \quad 1 \leq i \leq s.$$

We observe that $P(l, B, \theta)$ is equal to the polytope $\left(\frac{\log B}{\log \theta} - C_5\right)\mathcal{P}$, i.e. the polytope $\mathcal{P}$ blown up by a factor of $\frac{\log B}{\log \theta} - C_5$, intersected with the hyperplane $\left(\frac{\log B}{\log \theta} - C_5\right) H_{\delta'}$ given by

$$\sum_{i=1}^s t_i = \left(\frac{\log B}{\log \theta} - C_5\right)(a - \delta'),$$

where $\delta' \geq 0$ is chosen such that

$$l = \left(\frac{\log B}{\log \theta} - C_5\right)(a - \delta').$$

If we assume the estimate (4.6) then we have

$$\operatorname{meas} P(l, B, \theta) = c_P \left(\frac{\log B}{\log \theta} - C_5\right)^{s-1} \left(a - \left(\frac{\log B}{\log \theta} - C_5\right)^{-1} l\right)^{s-1-k}$$
$$+ O\left(\left(\frac{\log B}{\log \theta}\right)^{s-1} \left(a - \left(\frac{\log B}{\log \theta} - C_5\right)^{-1} l\right)^{s-k}\right).$$

We can rewrite this as

$$\operatorname{meas} P(l, B, \theta) = c_P \left(\frac{\log B}{\log \theta}\right)^{s-1} \left(a - \frac{\log \theta}{\log B} l\right)^{s-1-k}$$
$$+ O\left(\left(\frac{\log B}{\log \theta}\right)^{s-1} \left(a - \frac{\log \theta}{\log B} l\right)^{s-k} + \left(\frac{\log B}{\log \theta}\right)^{s-2}\right).$$

If we assume the estimate (4.7), then the measure of the boundary of $P(l, B, \theta)$ is bounded by

$$\ll \left(\frac{\log B}{\log \theta}\right)^{s-2}.$$

As our lattice is of fixed shape we deduce that for δ sufficiently small

$$\tilde{r}(l) = c_P \left(\frac{\log B}{\log \theta}\right)^{s-1} \left(a - \frac{\log \theta}{\log B} l\right)^{s-1-k}$$
$$+ O\left(\left(\frac{\log B}{\log \theta}\right)^{s-2} + \left(\frac{\log B}{\log \theta}\right)^{s-1} \left(a - \frac{\log \theta}{\log B} l\right)^{s-k}\right).$$

Finally, by Assumption 4.6 we have for any $1 \leq i_0 \leq s$ that

$$r_{\tilde{A}, i_0}(l) \ll_{\tilde{A}} \frac{\log \log B}{\log B} \left(\frac{\log B}{\log \theta}\right)^{s-1} \left(a - \frac{\log \theta}{\log B} l\right)^{s-1-k} + \left(\frac{\log B}{\log \theta}\right)^{s-2}.$$

□

We write

$$S_{1,f}^- = M_{1,f}^- + E_{3,f}^- + E_{4,f}^-$$

with

$$M_{1,f}^- = (\theta - 1)^s C_{f,M} \sum_{(a-\delta)(\log B/\log - C_5)\theta \leq l \leq a(\log B/\log \theta - C_5)} r^-(l) \theta^l$$

and

$$E_{3,f}^- \ll C_{f,E} \left(\frac{\log B}{\log \theta}\right)^s B^a (\log B)^{-\Delta \tilde{A}}$$

and

$$E_{4,f}^- \ll (\theta - 1)^s C_{f,M} \sum_{l \leq (a-\delta)(\log B/\log \theta - C_5)} r^-(l) \theta^l.$$

First we bound the error term $E_{4,f}^-$. For this we observe that if $l \leq (a - \delta) \log B / \log \theta$, then

$$\theta^l \ll B^{a-\delta}.$$

Moreover, as each of the l_i in the counting function $r^-(l)$ is bounded by $\ll \frac{\log B}{\log \theta}$ we have

$$\sum_{l \le a \log B/\log \theta} r^-(l) \ll \left(\frac{\log B}{\log \theta}\right)^s.$$

This gives the estimate

$$E_{4,f}^- \ll (\theta-1)^s C_{f,M} \left(\frac{\log B}{\log \theta}\right)^s B^{a-\delta}.$$

We conclude that

$$S_{1,f}^- = M_{1,f}^- + O(E_{3,f}^-).$$

We now use Lemma 4.9 to first evaluate the main term $M_{1,f}^-$. We have

$$\begin{aligned}
M_{1,f}^- &= (\theta-1)^s C_{f,M} \sum_{(a-\delta)(\log B/\log\theta - C_5) \le l \le a(\log B/\log\theta - C_5)} r^-(l)\theta^l \\
&= (\theta-1)^s C_{f,M} \sum_{(a-\delta)(\log B/\log\theta - C_5) \le l \le a(\log B/\log\theta - C_5)} c_P \left(\frac{\log B}{\log\theta}\right)^{s-1} \left(a - \frac{\log\theta}{\log B} l\right)^{s-1-k} \theta^l \\
&+ O\left(C_{f,M}(\theta-1)^s \sum_{l \le a \log B/\log\theta} \left(\frac{\log B}{\log\theta}\right)^{s-2} \theta^l\right) \\
&+ O\left(C_{f,M}(\theta-1)^s \sum_{l \le a \log B/\log\theta} \left(\frac{\log B}{\log\theta}\right)^{s-1} \left(a - \frac{\log\theta}{\log B} l\right)^{s-k} \theta^l\right) \\
&+ O\left(\frac{\log\log B}{\log B}(\theta-1)^s C_{f,M} \sum_{l \le a \log B/\log\theta} c_P \left(\frac{\log B}{\log\theta}\right)^{s-1} \left(a - \frac{\log\theta}{\log B} l\right)^{s-1-k} \theta^l\right).
\end{aligned}$$

We can also write this as

$$\begin{aligned}
M_{1,f}^- &= (\theta-1)^s C_{f,M} \sum_{(a-\delta)(\log B/\log\theta - C_5) \le l \le a(\log B/\log\theta - C_5)} c_P \left(\frac{\log B}{\log\theta}\right)^{k} \left(a\frac{\log B}{\log\theta} - l\right)^{s-1-k} \theta^l \\
&+ O\left(C_{f,M}(\theta-1)^s \sum_{l \le a \log B/\log\theta} \left(\frac{\log B}{\log\theta}\right)^{s-2} \theta^l\right) \\
&+ O\left(C_{f,M}(\theta-1)^s \sum_{l \le a \log B/\log\theta} \left(\frac{\log B}{\log\theta}\right)^{k-1} \left(a\frac{\log B}{\log\theta} - l\right)^{s-k} \theta^l\right) \\
&+ O\left(\frac{\log\log B}{\log B}(\theta-1)^s C_{f,M} \sum_{l \le a \log B/\log\theta} \left(\frac{\log B}{\log\theta}\right)^{k} \left(a\frac{\log B}{\log\theta} - l\right)^{s-1-k} \theta^l\right)
\end{aligned}$$

or

$$\begin{aligned}
M_{1,f}^{-} &= C_{f,M} c_P (\log B)^k \left(\frac{\theta-1}{\log\theta}\right)^k (\theta-1)^{s-k} \\
&\quad \sum_{(a-\delta)(\log B/\log\theta - C_5)\leq l\leq a(\log B/\log\theta - C_5)} \left(a\frac{\log B}{\log\theta} - l\right)^{s-1-k} \theta^l \\
&\quad + O\left(C_{f,M}(\theta-1)^{s-1}\left(\frac{\log B}{\log\theta}\right)^{s-2} B^a\right) \\
&\quad + O\left(C_{f,M}(\log B)^{k-1}\left(\frac{\theta-1}{\log\theta}\right)^{k-1}(\theta-1)^{s-k+1}\sum_{0\leq l\leq a\log B/\log\theta}\left(\frac{\log B}{\log\theta}a - l\right)^{s-k}\theta^l\right) \\
&\quad + O\left(\frac{\log\log B}{\log B} C_{f,M}(\log B)^k\left(\frac{\theta-1}{\log\theta}\right)^k(\theta-1)^{s-k}\sum_{0\leq l\leq a\log B/\log\theta}\left(a\frac{\log B}{\log\theta} - l\right)^{s-1-k}\theta^l\right)
\end{aligned}$$

In the second line we computed the geometric series.

Assumption 4.11. Assume that for B large we choose θ in a way such that $a\frac{\log B}{\log\theta}$ is an integer.

Under Assumption 4.11 we have

$$\begin{aligned}
M_{1,f}^{-} &= C_{f,M} c_P (\log B)^k \left(\frac{\theta-1}{\log\theta}\right)^k (\theta-1)^{s-k} \sum_{aC_5\leq m\leq (a-\delta)C_5+\delta\log B/\log\theta} m^{s-1-k}\theta^{a\log B/\log\theta - m} \\
&\quad + O\left(C_{f,M}(\theta-1)^{s-1}\left(\frac{\log B}{\log\theta}\right)^{s-2} B^a\right) \\
&\quad + O\left(C_{f,M}(\log B)^{k-1}\left(\frac{\theta-1}{\log\theta}\right)^{k-1}(\theta-1)^{s-k+1}\sum_{0\leq m\leq a\log B/\log\theta} m^{s-k}\theta^{a\log B/\log\theta - m}\right) \\
&\quad + O\left(C_{f,M}\frac{\log\log B}{\log B}(\log B)^k\left(\frac{\theta-1}{\log\theta}\right)^k(\theta-1)^{s-k}\sum_{0\leq m\leq a\log B/\log\theta} m^{s-1-k}\theta^{a\log B/\log\theta - m}\right).
\end{aligned}$$

We further rewrite this as

$$\begin{aligned}
M_{1,f}^{-} &= C_{f,M} c_P (\log B)^k B^a \left(\frac{\theta-1}{\log\theta}\right)^k (\theta-1)^{s-k} \sum_{aC_5\leq m\leq (a-\delta)C_5+\delta\log B/\log\theta} m^{s-1-k}\theta^{-m} \\
&\quad + O\left(C_{f,M}(\theta-1)^{s-1}\left(\frac{\log B}{\log\theta}\right)^{s-2} B^a\right) \\
&\quad + O\left(C_{f,M}(\log B)^{k-1}B^a\left(\frac{\theta-1}{\log\theta}\right)^{k-1}(\theta-1)^{s-k+1}\sum_{0\leq m\leq a\log B/\log\theta} m^{s-k}\theta^{-m}\right) \\
&\quad + O\left(C_{f,M}\frac{\log\log B}{\log B}(\log B)^k B^a\left(\frac{\theta-1}{\log\theta}\right)^k(\theta-1)^{s-k}\sum_{0\leq m\leq a\log B/\log\theta} m^{s-1-k}\theta^{-m}\right)
\end{aligned}$$

We recall the notation

$$g_l(M,\theta) = \sum_{0\leq m\leq M} m^l\theta^m.$$

With this notation we have

$$\begin{aligned}
M_{1,f}^{-} &= C_{f,M} c_P (\log B)^k B^a \left(\frac{\theta-1}{\log\theta}\right)^k (\theta-1)^{s-k} g_{s-1-k}((a-\delta)C_5 + \delta \log B/\log\theta, \theta^{-1}) \\
&+ O\left(C_{f,M}(\log B)^k B^a \left(\frac{\theta-1}{\log\theta}\right)^k (\theta-1)^{s-k}\right) \\
&+ O\left(C_{f,M}(\theta-1)^{s-1}\left(\frac{\log B}{\log\theta}\right)^{s-2} B^a\right) \\
&+ O\left(C_{f,M}(\log B)^{k-1} B^a \left(\frac{\theta-1}{\log\theta}\right)^{k-1} (\theta-1)^{s-k+1} g_{s-k}(a\log B/\log\theta, \theta^{-1})\right) \\
&+ O\left(C_{f,M}\frac{\log\log B}{\log B}(\log B)^k B^a \left(\frac{\theta-1}{\log\theta}\right)^k (\theta-1)^{s-k} g_{s-1-k}(a\log B/\log\theta, \theta^{-1})\right).
\end{aligned}$$

We now apply Lemma 2.1 and obtain

$$\begin{aligned}
M_{1,f}^{-} &= C_{f,M} c_P (\log B)^k B^a \left(\frac{\theta-1}{\log\theta}\right)^k \theta^{s-k}(s-1-k)! \\
&+ O\left(C_{f,M}(\theta-1)^{s-1}\left(\frac{\log B}{\log\theta}\right)^{s-2} B^a\right) \\
&+ O\left(C_{f,M}(\log B)^{k-1}B^a\right) \\
&+ O\left(C_{f,M}B^a(\log B)^k(\theta-1)\right) \\
&+ O\left(C_{f,M}B^{a-\delta}(\log B)^k\left(\frac{\log B}{\log\theta}\right)^{s-1-k} + C_{f,M}(\log B)^{k-1}\left(\frac{\log B}{\log\theta}\right)^{s-k}\right) \\
&+ O\left(C_{f,M}\frac{\log\log B}{\log B}\left((\log B)^k B^a + B^{a-\delta}(\log B)^k\left(\frac{\log B}{\log\theta}\right)^{s-1-k}\right)\right).
\end{aligned}$$

Next we need to choose θ. We assume that

$$1 + \frac{1}{(\log B)^{10A}} < \theta < 1 + \frac{1}{(\log B)^A}, \tag{4.9}$$

where $A > s$ is a fixed parameter. Then we have

$$\left(\frac{\theta-1}{\log\theta}\right)^k \theta^{s-k} = 1 + O\left(\frac{1}{(\log B)^A}\right),$$

and

$$(\theta-1)\left(\frac{\theta-1}{\log\theta}\right)^{s-2} = O\left(\frac{1}{(\log B)^A}\right).$$

We deduce that

$$M_{1,f}^{-} = (s-1-k)! C_{f,M} c_P (\log B)^k B^a + O\left(C_{f,M}B^a(\log B)^{k-1}\log\log B\right).$$

We now turn to the treatment of the error term. Recall that we have

$$E_{3,f}^{-} \ll C_{f,E}\left(\frac{\log B}{\log\theta}\right)^s B^a(\log B)^{-\Delta\tilde{A}}$$

We now observe under the assumption of equation (4.9) that we have that

$$\frac{1}{\log\theta} \ll (\log B)^{10A},$$

and

$$E_{3,f}^{-} \ll C_{f,E}(\log B)^{s+10As}B^a(\log B)^{-\Delta\tilde{A}}.$$

Let $A^\dagger$ be a positive real parameter. If we take $\tilde{A}$ sufficiently large depending on A, $A^\dagger$, s and Δ, then we get

$$E_{3,f}^- \ll C_{f,E} B^a (\log B)^{-A^\dagger}.$$

Observe that the same calculations are also valid for $S_{1,f}^+$ in place of $S_{1,f}^-$. We deduce that

$$S_{1,f}^- = S_{1,f}^+ = S_{1,f}$$

and

$$\begin{aligned} S_{1,f} &= (s-1-k)! C_{f,M} c_P (\log B)^k B^a \\ &\quad + O\left(C_{f,M} (\log\log B)(\log B)^{k-1} B^a\right) + O\left(C_{f,E} B^a (\log B)^{-A^\dagger}\right). \end{aligned}$$

We recall that we have made the assumption that

$$a \log B / \log \theta$$

is integral. Hence we need to show that for every B sufficiently large there is a θ in the range (4.9) such that this expression is integral. Note that the conditions on θ in (4.9) translate into saying that

$$a \frac{\log B}{\log(1+(\log B)^{-A})} < a \frac{\log B}{\log \theta} < a \frac{\log B}{\log(1+(\log B)^{-10A})}.$$

or

$$\log B (\log B)^A \ll a \frac{\log B}{\log \theta} \ll (\log B)^{10A} (\log B),$$

which for B growing certainly contains an integer.

More generally, let $W_i > 0$, $1 \le i \le s$ be parameters such that there exists $N > 0$ with

$$(\log B)^{\tilde{A}} < W_i < (\log B)^N, \quad 1 \le i \le s.$$

Set

$$S_{1,f}(\mathbf{W}, \mathbf{B}) := \sum_{\substack{\prod y_i^{\alpha_{i,k}} \le B_k,\ k \in \mathcal{K} \\ y_i \in \mathbb{N}, 1 \le i \le s \\ y_i \ge W_i\ \forall 1 \le i \le s}} f(\mathbf{y}).$$

We rephrase our findings in the following lemma.

Lemma 4.12. *Let $f : \mathbb{N}^s \to \mathbb{R}_{\ge 0}$ be a function satisfying Property I. Assume that Assumption 4.3 and Assumption 4.5 hold and that (4.3) holds. Then, for any real $A^\dagger > 0$ one has*

$$\begin{aligned} S_{1,f}(\mathbf{W}, \mathbf{B}) &= (s-1-k)! C_{f,M} c_P (\log B)^k B^a \\ &\quad + O\left(C_{f,M} (\log\log B)(\log B)^{k-1} B^a\right) + O\left(C_{f,E} B^a (\log B)^{-A^\dagger}\right). \end{aligned}$$

where the implied constants may depend on N, $A^\dagger$ and $\tilde{A}$ and the polytope $\mathcal{P}$.

Next we turn to the treatment of the contributions where some variables in the sum S^f can be small. Let $J \subsetneq \{1, \dots, s\}$ and Let $(V_i)_{i \notin J}$ and $(W_j)_{j \in J}$ be real parameters all bounded by $\ll (\log B)^N$ for some N sufficiently large and

$$W_i \ge (\log B)^{\tilde{A}}, \quad i \in J.$$

We consider the sum

$$S_{1,f}^J(\mathbf{V}, \mathbf{W}) := \sum_{\substack{\prod_{i=1}^s y_i^{\alpha_{i,k}} \le B,\ k \in \mathcal{K} \\ y_i \in \mathbb{N},\ 1 \le i \le s \\ y_i \le V_i\ \ \forall i \notin J \\ y_i > W_i\ \ \forall i \in J}} f(\mathbf{y}).$$

If $J=\emptyset$ then we have

$$S_{1,f}^{\emptyset}(\mathbf{V},\mathbf{W}) \ll \sum_{y_i\le V_i \ \forall 1\le i\le s} f(\mathbf{y}) \ll C_{f,E}(\log B)^{N\sum_{i=1}^s \varpi_i},$$

by Property I and the upper bound for each V_i, $1\le i\le s$. As $a>0$ by Assumption 4.5 we have

$$S_{1,f}^{\emptyset}(\mathbf{V},\mathbf{W}) \ll C_{f,E}B^a(\log B)^{-A^\dagger}, \tag{4.10}$$

for any fixed $A^\dagger>0$.

We now follow the approach in section 2.6 in [BB18]. Let U be a real parameter to be chosen later that is a power of $\log B$ and $R\in\mathbb{N}$ a large natural number also to be chosen later. We set

$$U_0=1,\quad U_1=U,\quad U_j=U_{j-1}^R,\quad 2\le j\le s,\quad U_{s+1}=B^{a\max_i \varpi_i^{-1}}.$$

In particular,

$$U_s=U^{R^{s-1}}$$

is bounded by a power of $\log B$. We now have the intervals

$$\mathcal{U}_0=[U_0,U_1],\quad \mathcal{U}_j=(U_j,U_{j+1}],\quad 1\le j\le s.$$

These are $s+1$ disjoint intervals. If we consider a vector $\mathbf{y}$ counted by the sum S_1, then there exists at least one such interval $\mathcal{U}_j$ which does not contain any of the coordinates y_i, $1\le i\le s$. By the bound (4.10) and the non-negativity of f it is now sufficient to show that terms of the form

$$S_{1,f}^{J}(\mathbf{V},\mathbf{W})$$

are bounded by

$$S_{1,f}^{J}(\mathbf{V},\mathbf{W}) \ll C_{f,E}(\log\log B)^s(\log B)^{k-1}B^a,$$

where $J\subsetneq\{1,\dots,s\}$ with $J\neq\emptyset$ and each of the V_i and W_j is equal to one of $U_0,\dots,U_s$ and where we may assume that

$$\left(\max_{i\notin J} V_i\right)^R \le \min_{i\in J} W_i.$$

I.e. we divide the coordinates $i=1,\dots s$ into small and large coordinates and may assume that there is a gap between them.

Remark 4.13. The terms where all coordinates y_i are considered large can either be treated as in section 2.6 in [BB18] or we can observe that their sum is equal to the main term where we have the restrictions $y_i\ge U_1$ for all $1\le i\le s$.

We write

$$S_{1,f}^{J}(\mathbf{V},\mathbf{W}) = \sum_{y_i\le V_i \ \forall i\notin J} \ \sum_{\substack{\prod_{i=1}^s y_i^{\alpha_{i,k}}\le B,\ k\in\mathcal{K}\\ y_i\in\mathbb{N},\ 1\le i\le s\\ y_i>W_i\ \forall i\in J}} f(\mathbf{y}).$$

We next aim to apply Lemma 4.12 to the inner sum

$$S^J((y_i)_{i\notin J},\mathbf{W}) := \sum_{\substack{\prod_{i\in J} y_i^{\alpha_{i,k}}\le B\prod_{i\notin J} y_i^{-\alpha_{i,k}},\ k\in\mathcal{K}\\ y_i\in\mathbb{N},\ \forall i\in J\\ y_i>W_i\ \forall i\in J}} f(\mathbf{y}).$$

Note that the function $f((y_i)_{i\in J})=f(\mathbf{y})$ in the variables y_i, $i\in J$, satisfies Property I by Remark 4.1 if we assume that f satisfies Assumption 4.1 for all vectors $\mathbf{y}_{\mathcal{I}}$. We may assume that

$$y_i\le(\log B)^N,\quad i\notin J, \tag{4.11}$$

for some sufficiently large N (which is independent of B).

For a non-empty subset $J \subsetneq \{1, \ldots, s\}$ and a vector $(y_i)_{\notin J}$ with $y_i \in \mathbb{N}$, for $i \notin J$ we introduce the polytope $\mathcal{P}^J((y_i)_{i\notin J}) \subset \mathbb{R}^{|J|}$ by

$$\sum_{i\in J} \alpha_{i,k} \varpi_i^{-1} t_i \leq \log \left(B \prod_{i\notin J} y_i^{-\alpha_{i,k}} \right) / \log B, \quad k \in \mathcal{K}$$

and

$$t_i \geq 0, \quad i \in J.$$

We note that Assumption 4.3 implies that the polytope $\mathcal{P}^J((y_i)_{i\notin J}) \subset \mathbb{R}^{|J|}$ is bounded.

Now consider the linear form $\sum_{i\in J} t_i$. It takes its maximal value on a face $F^J((y_i)_{i\notin J})$ of the polytope $\mathcal{P}^J((y_i)_{i\notin J})$, say that this maximal value equals $a((y_i)_{i\notin J})$.

If we have $a((y_i)_{i\notin J}) < a - C_{10} \frac{\log \log B}{\log B}$, then by a dyadic intersection into boxes, the contribution of those terms $S^J((y_i)_{i\notin J}, \mathbf{W})$ to the sum $S^J_{1,f}(\mathbf{V}, \mathbf{W})$ is bounded by

$$\ll C_{f,E} (\log B)^{Ns} B^{a - C_{10} \frac{\log \log B}{\log B}} \ll C_{f,E} B^a (\log B)^{Ns - C_{10}}.$$

Hence for $C_{10} > Ns$ they are of acceptable size. Let C_{10} in the following be a fixed constant with $C_{10} > Ns$.

Assumption 4.14. If

$$a((y_i)_{i\notin J}) \geq a - C_{10} \frac{\log \log B}{\log B}$$

then we assume the following. The dimension $k_J((y_i)_{i\notin J})$ of the face $F^J((y_i)_{i\notin J})$ is at most $k - 1$ if $J \subsetneq \{1, \ldots, s\}$, and this holds uniformly in the range (4.11).

Proposition 3.1 implies that the following estimates hold. Let $H^J_\delta((y_i)_{i\notin J})$ be the hyperplane given by

$$\sum_{i\in J} t_i = a((y_i)_{i\notin J}) - \delta.$$

Then for $\delta > 0$ sufficiently small, there is a constant $C > 0$ with

$$|\operatorname{meas}_{|J|-1}(H^J_\delta((y_i)_{i\notin J}) \cap \mathcal{P}^J((y_i)_{i\notin J})) - c_J((y_i)_{i\notin J}) \delta^{|J|-1-k_J((y_i)_{i\notin J})}| \leq C \delta^{|J|-k_J((y_i)_{i\notin J})}, \tag{4.12}$$

for some real constant $c_J((y_i)_{i\notin J})$. Moreover, if $|J| \geq 2$, we assume that the measure of the boundary of the polytope $H^J_\delta((y_i)_{i\notin J}) \cap \mathcal{P}^J((y_i)_{i\notin J})$ is bounded by

$$\operatorname{meas}_{|J|-2}(\partial(H^J_\delta((y_i)_{i\notin J}) \cap \mathcal{P}^J((y_i)_{i\notin J}))) \ll 1. \tag{4.13}$$

We observe that all the constants involved in these estimates are uniformly bounded in the range (4.11) as $\mathcal{P}$ is bounded.

For a vector $(y_i)_{i\notin J}$ satisfying (4.11), $i_0 \in J$ and $\kappa > 0$ we introduce the polytope $\mathcal{P}^J_{i_0,\kappa}((y_i)_{i\notin J})$ given by the system of equations

$$\sum_{i\in J} \alpha_{i,k} \varpi_i^{-1} t_i \leq \log \left(B \prod_{i\notin J} y_i^{-\alpha_{i,k}} \right) / \log B, \quad k \in \mathcal{K},$$

$$t_i \geq 0, \quad i \in J,$$

and

$$t_{i_0} \leq \kappa.$$

Assumption 4.15. We assume that the following holds uniformly in the range (4.11). For $\delta > 0$ and $\kappa > 0$ sufficiently small we have for $|J| \geq 2$ the bound

$$\operatorname{meas}_{|J|-1}(H^J_\delta((y_i)_{i\notin J}) \cap \mathcal{P}^J_{i_0,\kappa}((y_i)_{i\notin J})) \ll \min\{\kappa, \delta\} \delta^{|J|-2-k_J((y_i)_{i\notin J})},$$

and

$$\operatorname{meas}_{|J|-2} \partial(H^J_\delta \cap \mathcal{P}^J_{i_0,\kappa}) \ll 1.$$

We note that Assumption 4.15 is a consequence of Proposition 3.1 parts (iii), and (ii) and of the boundedness of $\mathcal{P}$ for the uniformity.

Moreover, we observe that if $k_J((y_i)_{i\notin J}) \le |J|-1$, then the bound

$$\mathrm{meas}_{|J|-1}(H_\delta^J((y_i)_{i\notin J}) \cap \mathcal{P}_{i_0,\kappa}^J((y_i)_{i\notin J})) \ll \delta^{|J|-1-k_J((y_i)_{i\notin J})}$$

leads to

$$\begin{aligned} S^J((y_i)_{i\notin J}, \mathbf{W}) \ll\ & C_{f,M,J}((y_i)_{i\notin J})(\log B)^{k_J((y_i)_{i\notin J})} B^{a((y_i)_{i\notin J})} \\ & + C_{f,E,J}((y_i)_{i\notin \mathcal{J}}) B^{a((y_i)_{i\notin J})} (\log B)^{-A^\dagger}, \end{aligned}$$

and the bound

$$\mathrm{meas}_{|J|-1}(H_\delta^J((y_i)_{i\notin J}) \cap \mathcal{P}_{i_0,\kappa}^J((y_i)_{i\notin J})) \ll \kappa\delta^{|J|-2-k_J((y_i)_{i\notin J})}$$

leads to

$$\begin{aligned} S^J((y_i)_{i\notin J}, \mathbf{W}) \ll\ & C_{f,M,J}((y_i)_{i\notin J}) \frac{\log\log B}{\log B} (\log B)^{k_J((y_i)_{i\notin J})+1} B^{a((y_i)_{i\notin J})} \\ & + C_{f,E,J}((y_i)_{i\notin \mathcal{J}}) B^{a((y_i)_{i\notin J})} (\log B)^{-A^\dagger}, \end{aligned}$$

which are both sufficient for our purposes. Note that here we extended the above calculations to the case where possibly $k = s$.

If $|J| \le k_J((y_i)_{i\notin J})$, then a dyadic intersection leads to

$$\begin{aligned} S^J((y_i)_{i\notin J}, \mathbf{W}) & \ll C_{f,M,J}((y_i)_{i\notin J})(\log B)^{|J|} B^{a((y_i)_{i\notin J})} \\ & \ll C_{f,M,J}((y_i)_{i\notin J})(\log B)^{k_J((y_i)_{i\notin J})} B^{a((y_i)_{i\notin J})}. \end{aligned}$$

Assume that (4.1) holds. We now apply Lemma 4.12 and its proof strategy and find that under Assumption 4.14 and Assumption 4.15 we have

$$\begin{aligned} S^J((y_i)_{i\notin J}, \mathbf{W}) \ll\ & C_{f,M,J}((y_i)_{i\notin J})(\log\log B)(\log B)^{k-1} B^{a((y_i)_{i\notin J})} \\ & + C_{f,E,J}((y_i)_{i\notin \mathcal{J}}) B^{a((y_i)_{i\notin J})} (\log B)^{-A^\dagger}. \end{aligned}$$

Next, we observe that

$$\prod_{i\notin J} y_i^{\varpi_i} B^{a((y_i)_{i\notin J})} \le B^a$$

by definition of the optimization problem defining $a((y_i)_{i\notin J})$. We make use of Remark 4.1 and bound

$$\begin{aligned} S_{1,f}^J(\mathbf{V}, \mathbf{W}) \ll & \sum_{y_i \le V_i,\ \forall i\notin J} C_{f,M,J}((y_i)_{i\notin J}) \prod_{i\notin J} y_i^{-\varpi_i} (\log\log B)(\log B)^{k-1} B^a \\ & + \sum_{y_i \le V_i,\ \forall i\notin J} C_{f,E} |(y_i)_{i\notin \mathcal{J}}|^{D'} \prod_{i\notin J} y_i^{-\varpi_i} B^a (\log B)^{-A^\dagger}. \end{aligned}$$

Now we use Lemma 4.2 together with the non-negativity of the constants $C_{f,M,J}((y_i)_{i\notin J})$ and dyadic intersections of the ranges for y_i, $i \notin J$ to deduce that

$$\begin{aligned} S_{1,f}^J(\mathbf{V}, \mathbf{W}) \ll\ & C_{f,E}(\log\log B)^s (\log B)^{k-1} B^a \\ & + C_{f,E}(\max_i V_i)^{s+D'} B^a (\log\log B)^s (\log B)^{-A^\dagger}. \end{aligned}$$

Hence for $A^\dagger$ sufficiently large we obtain the bound

$$S_{1,f}^J(\mathbf{V}, \mathbf{W}) \ll C_{f,E}(\log\log B)^s (\log B)^{k-1} B^a.$$

This completes the proof of Theorem 1.1.

4.1. **The technical assumption.** We conclude the section by discussing Assumption 4.14.

We recall that $\mathcal{P} \subseteq \mathbb{R}^s$ is the polyhedron defined by (4.4) and (4.5). Fix $\varepsilon > 0$. For every nonempty subset $J \subsetneq \{1, \dots, s\}$, let $J^c := \{1, \dots, s\} \smallsetminus J$. For every vector $\tau = (\tau_i)_{i \in J^c}$, with $0 \le \tau_i \le \varepsilon$ for every $i \in J^c$, we define $\mathcal{P}^J_\tau = \mathcal{P} \cap \{t_i = \tau_i \ \forall i \in J^c\}$. We recall that a is the maximal value of $\sum_{i=1}^s t_i$ on $\mathcal{P}$ and k is the dimension of the face F of $\mathcal{P}$ where $\sum_{i=1}^s t_i = a$. We denote by a^J_τ the maximal value of $\sum_{i=1}^s t_i$ on $\mathcal{P}^J_\tau$ and by k^J_τ the dimension of the face F^J_τ of $\mathcal{P}^J_\tau$ where $\sum_{i=1}^s t_i = a^J_\tau$. If $\tau_i = 0$ for all $i \in J^c$ we write 0 instead of τ: $\mathcal{P}^J_0$, F^J_0, a^J_0, k^J_0.

Assumption 4.16. There exists $\eta > 0$ such that for sufficiently small $\varepsilon > 0$, the following holds for every J and τ as above: if $a^J_\tau \ge a - \eta$ then $k^J_\tau \le k - 1$.

We observe that if Assumption 4.16 is satisfied, then Assumption 4.14 holds for $\mathcal{P}$. The next two lemmas present two typical situations where Assumption 4.14 holds.

Lemma 4.17. *Assume that $\mathcal{P}$ is bounded and nondegenerate, and that F is not contained in a coordinate hyperplane. Then the following statements hold.*

(i) If $F \cap \{t_i = \tau_i \ \forall i \in J^c\} \ne \emptyset$, then $k^J_\tau \le k - 1$.
(ii) If $|J| \le k$, then $k^J_\tau \le k - 1$.
(iii) For sufficiently small $\varepsilon > 0$, if $|J| \ge s - k$ and $F \cap \{t_i = 0 \ \forall i \in J^c\} \ne \emptyset$, then $k^J_\tau \le k - 1$.
(iv) If $F \cap \{t_i = 0 \ \forall i \in J^c\} = \emptyset$, there exists $\eta > 0$ such that for sufficiently small $\varepsilon > 0$ we have $a^J_\tau \le a - \eta$.
(v) If $s \le 2k + 1$, then $\mathcal{P}$ satisfies Assumption 4.14.

Proof. Part (ii) holds simply as $k^J_\tau \le |J| - 1$. For part (i) we observe that if $F \cap P^J_\tau \ne \emptyset$ then $F^J_\tau = F \cap P^J_\tau$ and $k^J_\tau \le k-1$ as F is not contained in any coordinate hyperplane. For part (iii) let $\langle F \rangle$ be the affine space spanned by F. We observe that $\langle F \rangle \cap \{t_i = 0 \ \forall i \in J^c\} \ne \emptyset$ implies $\langle F \rangle \cap \{t_i = \tau_i \ \forall i \in J^c\} \ne \emptyset$ by dimension reasons. The for sufficiently small $\varepsilon > 0$, we have $F \cap \{t_i = \tau_i \ \forall i \in J^c\} \ne \emptyset$, and we can conclude by part (i). It remains to prove part (iv). If $F \cap \{t_i = 0 \ \forall i \in J^c\} = \emptyset$ then $a^J_0 < a$. Fix a positive real number $\eta < a - a^J_0$. Since $P^J_0 \cap \{\sum_{i=1}^s t_i \ge a - \eta\} = \emptyset$, for sufficiently small $\varepsilon > 0$ we have also $P^J_\tau \cap \{\sum_{i=1}^s t_i \ge a - \eta\} = \emptyset$, and hence, $a^J_\tau \le a - \eta$. Part (v) is a consequence of the previous ones. □

Lemma 4.18. *Assume that $\mathcal{P}$ is bounded and nondegenerate, and that F is not contained in a coordinate hyperplane. Assume that $\mathcal{P}$ has $s + |\mathcal{K}|$ distinct facets. If $\mathcal{P}$ is simple (i.e., every vertex of $\mathcal{P}$ is contained in exactly s facets of $\mathcal{P}$) then Assumption 4.14 holds (for N and B sufficiently large).*

Proof. It suffices to show that there exists $\eta > 0$ such that for sufficiently small $\varepsilon > 0$, the following holds for every J and τ as above: if $a^J_\tau \ge a - \eta$ then $k^J_\tau \le k - 1$.

Fix $\varepsilon > 0$, J and τ as above. By Lemma 4.17 (iv) we can assume without loss of generality that $F \cap \{t_i = 0 \ \forall i \in J^c\} \ne \emptyset$. Let $\eta_J > 0$ such that every facet of $\mathcal{P}^J_0$ that does not intersect F^J_0 is contained in the half space defined by $\sum_{i \in J} t_i \le a - \eta_J$.

For every $k \in \mathcal{K}$, let $\alpha_k(\mathbf{t}) := \sum_{i \in J} \alpha_{i,k} \varpi_i^{-1} t_i$. For every $\mathcal{K}' \subset \mathcal{K}$ and every $k \in \mathcal{K}$, we define $\beta_{k,\tau,\mathcal{K}'} = 1 - \sum_{i \in J^c} \alpha_{i,k} \varpi_i^{-1} \tau_i$ if $k \in \mathcal{K}'$ and $\beta_{k,\tau,\mathcal{K}'} = 1$ otherwise. For every subset $\mathcal{K}' \subset \mathcal{K}$, let $\mathcal{P}^J_{\tau,\mathcal{K}'} = \mathcal{P}^J_0 \cap \{\alpha_k(\mathbf{t}) \le b_{k,\tau,\mathcal{K}'} \ \forall k \in \mathcal{K}'\}$. We denote by $a^J_{\tau,\mathcal{K}'}$ the maximal value of $\sum_{i \in J} t_i$ on $\mathcal{P}^J_{\tau,\mathcal{K}'}$, and by $k^J_{\tau,\mathcal{K}'}$ the dimension of the face $F^J_{\tau,\mathcal{K}'}$ of $\mathcal{P}^J_{\tau,\mathcal{K}'}$ where $\sum_{i \in J} t_i = a^J_{\tau,\mathcal{K}'}$. We want to show that $k^J_{\tau,\mathcal{K}'} \le k - 1$ for every $\mathcal{K}' \subseteq \mathcal{K}$. We proceed by induction on the cardinality of $\mathcal{K}'$. If $\mathcal{K}' = \emptyset$, then we are done as $k^J_0 \le k - 1$ by Lemma 4.17(i). Assume that $|\mathcal{K}'| \ge 1$ and that $k^J_{\tau,\mathcal{K}''} \le k - 1$ for a subset $\mathcal{K}''$ of $\mathcal{K}'$ of cardinality $|\mathcal{K}'| - 1$. Let $\tilde{k}$ be the unique element in $\mathcal{K}' \setminus \mathcal{K}''$. Let $\widetilde{F} = \mathcal{P}^J_{\tau,\mathcal{K}''} \cap \{\alpha_{\tilde{k}}(\mathbf{t}) = \beta_{\tilde{k},\tau,\mathcal{K}''}\}$ be the facet corresponding to $\tilde{k}$. If $\widetilde{F}$ does not contain $F^J_{\tau,\mathcal{K}''}$, then for $\varepsilon > 0$ small enough, $\mathcal{P}^J_{\tau,\mathcal{K}'} \cap F^J_{\tau,\mathcal{K}''} \neq \emptyset$, hence $F^J_{\tau,\mathcal{K}'} \subseteq F^J_{\tau,\mathcal{K}''}$. Hence, we can assume that $\widetilde{F}$ contains $F^J_{\tau,\mathcal{K}''}$. Let $\mathcal{K}'''$ be the set of indices $k \in \mathcal{K}$ such that $\alpha_k(\mathbf{t}) - \beta_{k,\tau,\mathcal{K}''}$ vanishes on $F^J_{\tau,\mathcal{K}''}$. Let $J' \subseteq J$ be the set of indices i such that t_i vanishes on $F^J_{\tau,\mathcal{K}''}$. Since $\mathcal{P}$ is simple, then also $\mathcal{P}^J_0$, $\mathcal{P}^J_{\tau,\mathcal{K}'}$, and $\mathcal{P}^J_{\tau,\mathcal{K}''}$ are simple by construction. Then $|\mathcal{K}'''| + |J'| = |J| - k^J_{\tau,\mathcal{K}''}$, and the affine space $< F^J_{\tau,\mathcal{K}''} >$ spanned by $F^J_{\tau,\mathcal{K}''}$ is the set of solutions of the full rank linear system

$$t_j = 0 \ \forall j \in J', \quad \alpha_k = \beta_{k,\tau,\mathcal{K}''} \ \forall k \in \mathcal{K}'''. \tag{4.14}$$

Let $\mathcal{P}'$ be the convex polyhedron given by

$$t_j \geq 0 \ \forall j \in J', \quad \alpha_k(\mathbf{t}) \leq \beta_{k,\tau,\mathcal{K}''} \ \forall k \in \mathcal{K}'''. \tag{4.15}$$

By construction, the hyperplane $\sum_{i\in J} t_i = a^J_{\tau,\mathcal{K}''}$ intersects $\mathcal{P}'$ exactly in the face $< F^J_{\tau,\mathcal{K}''} >$, and hence defines a supporting hyperplane of $\mathcal{P}'$. So we have $\sum_{i\in J} t_i = \sum_{j\in J'} \lambda_j t_j + \sum_{k\in\mathcal{K}'''} \mu_k \alpha_k(\mathbf{t})$ with $\lambda_j \leq 0$ for all $j \in J'$ and $\mu_k \geq 0$ for all $k \in \mathcal{K}'''$. Now we observe that since the linear system (4.14) has full rank $|\mathcal{K}'''| + |J'|$, the linear system

$$t_j = 0 \ \forall j \in J', \quad \alpha_k(\mathbf{t}) = \beta_{k,\tau,\mathcal{K}'} \ \forall k \in \mathcal{K}'''$$

has an affine space of solutions F' of dimension $|\mathcal{K}'''| + |J'|$. If $\varepsilon > 0$ is small enough, we have $F' \cap \mathcal{P}^J_{\tau,\mathcal{K}'} \neq \emptyset$. Since $\sum_{i\in J} t_i \leq \sum_{k\in\mathcal{K}'''} \mu_k \beta_{k,\tau,\mathcal{K}'}$ on $\mathcal{P}^J_{\tau,\mathcal{K}'}$, and $\sum_{i\in J} t_i - \sum_{k\in\mathcal{K}'''} \mu_k \beta_{k,\tau,\mathcal{K}'}$ vanishes on F', we get $F^J_{\tau,\mathcal{K}'} \subseteq F'$, and hence $k^J_{\tau,\mathcal{K}'} \leq k - 1$. We can take $\eta = \min_{\emptyset \neq J \subseteq \{1,\ldots,s\}} \eta_J$. □

We observe that for the applications of Theorem 1.1 we can assume without loss of generality that $\mathcal{P}$ has $s + |\mathcal{K}|$ distinct facets. Otherwise, some of the conditions indexed by the set $\mathcal{K}$ are redundant and can be removed.

5. m-full numbers

Let $m \geq 1$ be a natural number. We recall that an integer y is called m-full if for each prime divisor p of y, we have that p^m divides y. We introduce the function that counts the number of m-full natural numbers less than B

$$\begin{aligned} F_m(B) :&= \sharp\{1 \leq y \leq B, v_p(y) \in \{0\} \cup \mathbb{Z}_{\geq m} \forall p\} \\ &= \sharp\{1 \leq y \leq B, y \text{ is m-full}\}. \end{aligned}$$

Lemma 5.1 ([ES34, BG58])**.** *For each $m \geq 1$ and $B > 0$ we have*

$$F_m(B) = C_m B^{\frac{1}{m}} + O_m\left(B^{\kappa_m}\right), \tag{5.1}$$

where $C_1 = 1$, $\kappa_1 = 0$, and for $m \geq 2$,

$$C_m = \prod_p \left(1 + \sum_{j=m+1}^{2m-1} p^{-\frac{j}{m}}\right) \qquad \kappa_m = \frac{1}{m+1}. \tag{5.2}$$

For a square-free positive integer d, we define

$$F_m(B,d) := \sharp\{1 \leq y \leq B, d \mid y, \ y \text{ is m-full}\}. \tag{5.3}$$

In this section we prove an asymptotic formula for the function $F_m(B,d)$. We will first do it for the case that d is a prime. We will then inductively on the number of prime factors of d provide a general asymptotic formula. First we provide a form of inclusion-exclusion lemma, which expresses $F_m(B,p)$ for a prime number p in terms of sums of the function $F_m(B)$. Before we state the lemma, we introduce a convenient piece of notation. For $r \geq 1$ and $k \in \mathbb{Z}$ let

$$\rho_m(k,r) := \sharp\{1 \leq k_1, \ldots, k_r \leq (m-1) : \ k = k_1 + \ldots + k_r\}.$$

Note that $\rho_m(k,r)$ is zero, unless $r \leq k \leq r(m-1)$.

Lemma 5.2. *For $m \geq 2$ one has*

$$\begin{aligned} F_m(B,p) =& F_m(Bp^{-m}) + \sum_{r=1}^{\infty} \sum_{k=2r}^{2r(m-1)} \rho_m(k,2r) F_m(Bp^{-(2r+1)m-k}) \\ &+ \sum_{r=1}^{\infty} \sum_{k=2r-1}^{(2r-1)(m-1)} \rho_m(k,2r-1) F_m(Bp^{-(2r-1)m-k}) \\ &- \sum_{r=1}^{\infty} \sum_{k=2r-1}^{(2r-1)(m-1)} \rho_m(k,2r-1) F_m(Bp^{-2rm-k}) \\ &- \sum_{r=1}^{\infty} \sum_{k=2r}^{2r(m-1)} \rho_m(k,2r) F_m(Bp^{-2rm-k}). \end{aligned}$$

Note that the summations in Lemma 5.2 are in fact finite, as $F_m(P) = 0$ if $P < 1$.

Proof. We start the proof in reinterpreting terms of the shape $F_m(Bp^{-K})$ for some $K > 0$ as

$$F_m(Bp^{-K}) = \sharp\{1 \leq p^K l \leq B, l \text{ is m-full}\}.$$

Then the right hand side in the identity in Lemma 5.2 becomes

$$\begin{aligned} RHS =& \sharp\{1 \leq p^m l \leq B, l \text{ is m-full}\} \\ &+ \sum_{r=1}^{\infty} \sum_{k=2r}^{2r(m-1)} \rho_m(k,2r) \sharp\{1 \leq p^{(2r+1)m+k} l \leq B, l \text{ is m-full}\} \\ &+ \sum_{r=1}^{\infty} \sum_{k=2r-1}^{(2r-1)(m-1)} \rho_m(k,2r-1) \sharp\{1 \leq p^{(2r-1)m+k} l \leq B, l \text{ is m-full}\} \\ &- \sum_{r=1}^{\infty} \sum_{k=2r-1}^{(2r-1)(m-1)} \rho_m(k,2r-1) \sharp\{1 \leq p^{2rm+k} l \leq B, l \text{ is m-full}\} \\ &- \sum_{r=1}^{\infty} \sum_{k=2r}^{2r(m-1)} \rho_m(k,2r) \sharp\{1 \leq p^{2rm+k} l \leq B, l \text{ is m-full}\}. \end{aligned} \tag{5.4}$$

For any $K \geq 0$ we use the identity

$$\begin{aligned} \sharp\{1 \leq p^K l \leq B, l \text{ is m-full}\} =& \sharp\{1 \leq p^K l \leq B, l \text{ is m-full}, p \nmid l\} \\ &+ \sharp\{1 \leq p^{K+m} l \leq B, l \text{ is m-full}\} \\ &+ \sum_{k=1}^{m-1} \sharp\{1 \leq p^{K+m+k} l \leq B, l \text{ is m-full}, p \nmid l\} \end{aligned} \tag{5.5}$$

We use this identity for the terms in the third line in (5.4). We observe that the terms counting $1 \leq p^{2rm+k} l \leq B$ with l m-full identically cancel with the fourth line in (5.4). Hence we obtain

$$\begin{aligned} RHS =& \sharp\{1 \le p^m l \le B, l \text{ is m-full}\} \\ &+ \sum_{r=1}^{\infty} \sum_{k=2r}^{2r(m-1)} \rho_m(k, 2r) \sharp\{1 \le p^{(2r+1)m+k} l \le B, l \text{ is m-full}\} \\ &+ \sum_{r=1}^{\infty} \sum_{k=2r-1}^{(2r-1)(m-1)} \rho_m(k, 2r-1) \sharp\{1 \le p^{(2r-1)m+k} l \le B, l \text{ is m-full}, p \nmid l\} \\ &+ \sum_{r=1}^{\infty} \sum_{k=2r-1}^{(2r-1)(m-1)} \sum_{k_r=1}^{m-1} \rho_m(k, 2r-1) \sharp\{1 \le p^{2rm+k+k_r} l \le B, l \text{ is m-full}, p \nmid l\} \\ &- \sum_{r=1}^{\infty} \sum_{k=2r}^{2r(m-1)} \rho_m(k, 2r) \sharp\{1 \le p^{2rm+k} l \le B, l \text{ is m-full}\}. \end{aligned}$$

Recalling the definition of the functions $\rho_m(k, r)$ we can further rewrite this as

$$\begin{aligned} RHS =& \sharp\{1 \le p^m l \le B, l \text{ is m-full}\} \\ &+ \sum_{r=1}^{\infty} \sum_{k=2r}^{2r(m-1)} \rho_m(k, 2r) \sharp\{1 \le p^{(2r+1)m+k} l \le B, l \text{ is m-full}\} \\ &+ \sum_{r=1}^{\infty} \sum_{k=2r-1}^{(2r-1)(m-1)} \rho_m(k, 2r-1) \sharp\{1 \le p^{(2r-1)m+k} l \le B, l \text{ is m-full}, p \nmid l\} \\ &+ \sum_{r=1}^{\infty} \sum_{k=2r}^{2r(m-1)} \rho_m(k, 2r) \sharp\{1 \le p^{2rm+k} l \le B, l \text{ is m-full}, p \nmid l\} \\ &- \sum_{r=1}^{\infty} \sum_{k=2r}^{2r(m-1)} \rho_m(k, 2r) \sharp\{1 \le p^{2rm+k} l \le B, l \text{ is m-full}\}. \end{aligned} \tag{5.6}$$

We now use the identity (5.5) for the terms in the last line in equation (5.6). The resulting terms with $p^{2rm+k} l \le B$ and $p \nmid l$ cancel with the terms in the fourth line. Moreover, the terms with $1 \le p^{(2r+1)m+k} l \le B$ and l m-full identically cancel with the terms in the second line in (5.6). Hence we obtain

$$\begin{aligned} RHS =& \sharp\{1 \le p^m l \le B, l \text{ is m-full}\} \\ &+ \sum_{r=1}^{\infty} \sum_{k=2r-1}^{(2r-1)(m-1)} \rho_m(k, 2r-1) \sharp\{1 \le p^{(2r-1)m+k} l \le B, l \text{ is m-full}, p \nmid l\} \\ &- \sum_{r=1}^{\infty} \sum_{k=2r}^{2r(m-1)} \sum_{k_{2r+1}=1}^{m-1} \rho_m(k, 2r) \sharp\{1 \le p^{(2r+1)m+k+k_{2r+1}} l \le B, l \text{ is m-full}, p \nmid l\}. \end{aligned}$$

Again using the definition of the functions $\rho_m(k, 2r)$ we can rewrite this as

$$\begin{aligned} RHS =& \sharp\{1 \le p^m l \le B, l \text{ is m-full}\} \\ &+ \sum_{r=1}^{\infty} \sum_{k=2r-1}^{(2r-1)(m-1)} \rho_m(k, 2r-1) \sharp\{1 \le p^{(2r-1)m+k} l \le B, l \text{ is m-full}, p \nmid l\} \\ &- \sum_{r=1}^{\infty} \sum_{k=2r+1}^{(2r+1)(m-1)} \rho_m(k, 2r+1) \sharp\{1 \le p^{(2r+1)m+k} l \le B, l \text{ is m-full}, p \nmid l\}. \end{aligned} \tag{5.7}$$

The last two sums in (5.7) cancel except for the terms with $r = 1$ in the second line. Hence we get

$$\begin{aligned} RHS =& \sharp\{1 \leq p^m l \leq B, l \text{ is m-full}\} \\ &+ \sum_{k=1}^{m-1} \sharp\{1 \leq p^{m+k} l \leq B, l \text{ is m-full}, p \nmid l\}. \end{aligned}$$

Similarly as in (5.5) we now observe that on the right hand side we count exactly all $1 \leq l \leq B$ such that $p \mid l$ and l is m-full, which completes the proof of the lemma. □

Lemma 5.2 now allows us to deduce an asymptotic formula for $F_m(B,p)$ given that we know (5.1). We recall that the sums in Lemma 5.2 are all finite and hence we can first reorder them to take into account cancellation between different sums and then complete the resulting series to infinity. First we rewrite the expression for $F_m(B,p)$ in Lemma 5.2 as

$$F_m(B,p) = F_m(Bp^{-m}) + \sum_{\mu=m+1}^{\infty} a_m(\mu) F_m(Bp^{-\mu}), \tag{5.8}$$

with coefficients $a_m(\mu)$ that are given by

$$\begin{aligned} a_m(\mu) =& \sum_{r=1}^{\infty} \sum_{k=2r}^{2r(m-1)} \rho_m(k,2r)\, \mathbf{1}_{[\mu=(2r+1)m+k]} \\ &+ \sum_{r=1}^{\infty} \sum_{k=2r-1}^{(2r-1)(m-1)} \rho_m(k,2r-1)\, \mathbf{1}_{[\mu=(2r-1)m+k]} \\ &- \sum_{r=1}^{\infty} \sum_{k=2r-1}^{(2r-1)(m-1)} \rho_m(k,2r-1)\, \mathbf{1}_{[\mu=2rm+k]} \\ &- \sum_{r=1}^{\infty} \sum_{k=2r}^{2r(m-1)} \rho_m(k,2r)\, \mathbf{1}_{[\mu=2rm+k]}\,. \end{aligned}$$

Note that all the appearing sums are in fact finite and hence we can reorder them freely. Our next goal is to get more understanding on the coefficients $a_m(\mu)$ (in particular their size) and hence we group them in a generating series. Define

$$G_m(x) := \sum_{\mu=m+1}^{\infty} a_m(\mu) x^{\mu}.$$

A first rough bound on the coefficients $a_m(\mu)$ can be obtained via the estimate

$$a_m(\mu) \leq 2 \sum_{1 \leq t \leq \frac{\mu}{m}} \sum_{k=t}^{t(m-1)} \rho_m(k,t) = 2 \sum_{1 \leq t \leq \frac{\mu}{m}} (m-1)^t.$$

For $m \geq 3$ we obtain

$$a_m(\mu) \leq 2(m-1)^{\frac{\mu}{m}+1},$$

whereas for $m = 2$ we have the estimate

$$a_m(\mu) \leq 2\frac{\mu}{m}.$$

In particular we deduce that there is some constant R_m only depending on m such that the power series $G_m(x)$ is absolutely convergent for $|x| < R_m$. Moreover, in choosing R_m sufficiently small we can also assume that the sum

$$\sum_{t=1}^{\infty}\sum_{k=t}^{t(m-1)} \rho_m(k,t)x^{tm+k}$$

is absolutely convergent. Our next goal is to write the generating series $G_m(x)$ as a fractional function and in this way realise that it has a larger radius of absolute convergence than the bound that is obtained from the very rough estimate on $a_m(\mu)$. For this we observe that for $|x| < R_m$ we can express $G_m(x)$ as

$$\begin{aligned} G_m(x) =& \sum_{r=1}^{\infty}\sum_{k=2r}^{2r(m-1)} \rho_m(k,2r)x^{(2r+1)m+k} \\ &+ \sum_{r=1}^{\infty}\sum_{k=2r-1}^{(2r-1)(m-1)} \rho_m(k,2r-1)x^{(2r-1)m+k} \\ &- \sum_{r=1}^{\infty}\sum_{k=2r-1}^{(2r-1)(m-1)} \rho_m(k,2r-1)x^{2rm+k} \\ &- \sum_{r=1}^{\infty}\sum_{k=2r}^{2r(m-1)} \rho_m(k,2r)x^{2rm+k}. \end{aligned}$$

We can now compute the generating function $G_m(x)$ as

$$\begin{aligned} G_m(x) =& \sum_{r=1}^{\infty} x^{(2r+1)m}\left(\sum_{k=1}^{m-1}x^k\right)^{2r} + \sum_{r=1}^{\infty} x^{(2r-1)m}\left(\sum_{k=1}^{m-1}x^k\right)^{2r-1} \\ &- \sum_{r=1}^{\infty} x^{2rm}\left(\sum_{k=1}^{m-1}x^k\right)^{2r-1} - \sum_{r=1}^{\infty} x^{2rm}\left(\sum_{k=1}^{m-1}x^k\right)^{2r}. \end{aligned}$$

In the area of absolute convergence one may reorder the sums as

$$\begin{aligned} G_m(x) =& \sum_{t=1}^{\infty}(-1)^t x^{(t+1)m}\left(\sum_{k=1}^{m-1}x^k\right)^{t} + \sum_{t=1}^{\infty}(-1)^{t+1} x^{tm}\left(\sum_{k=1}^{m-1}x^k\right)^{t} \\ =&(x^m-1)(-1)x^m\left(\sum_{k=1}^{m-1}x^k\right)\left(1-(-1)x^m\left(\sum_{k=1}^{m-1}x^k\right)\right)^{-1} \\ =&(1-x^m)x^m x\frac{x^{m-1}-1}{x-1}\left(1+x^{m+1}\frac{x^{m-1}-1}{x-1}\right)^{-1} \\ =&(1-x^m)x^{m+1}\frac{x^{m-1}-1}{x-1}(x-1)\left(x-1+x^{m+1}(x^{m-1}-1)\right)^{-1} \\ =&(1-x^m)x^{m+1}(x^{m-1}-1)\left(x-1+x^{m+1}(x^{m-1}-1)\right)^{-1} \\ =&(1-x^m)x^{m+1}(x^{m-1}-1)\left(x-1+x^{2m}-x^{m+1}\right)^{-1} \end{aligned}$$

We observe that

$$x^{2m}-x^{m+1}+x-1 = (x^m-1)(x^m+1)-x(x^m-1) = (x^m-1)(x^m-x+1).$$

Hence we obtain

$$G_m(x) = -x^{m+1}(x^{m-1}-1)\left(x^m-x+1\right)^{-1}.$$

In the interval $x \in (0,1)$ the function $x^m - x$ takes its minimum at $x = m^{-\frac{1}{m-1}}$, and at this point $x^m - x = m^{-\frac{1}{m-1}}(m^{-1}-1)$. In particular we observe that the Taylor series for $G_m(x)$ is absolutely convergent in the interval $x \in (0,1)$.

We can now deduce an asymptotic for $F_m(B,p)$.

Lemma 5.3. *Let $m \geq 2$. Let p be a prime number. Then we have*

$$F_m(B,p) = C_m B^{\frac{1}{m}} \left(p^{-1} + G_m(p^{-\frac{1}{m}})\right) + O_m\left(B^{\kappa_m} p^{-m\kappa_m}\right).$$

Here the implicit constant is independent of p.

Proof. We start in recalling equation (5.8)

$$F_m(B,p) = F_m(Bp^{-m}) + \sum_{\mu=m+1}^{\infty} a_m(\mu) F_m(Bp^{-\mu}).$$

From the asymptotic formula in (5.1) we deduce that

$$\begin{aligned} F_m(B,p) =& C_m (Bp^{-m})^{\frac{1}{m}} + C_m \sum_{\mu=m+1}^{\infty} a_m(\mu)(Bp^{-\mu})^{\frac{1}{m}} \\ &+ O_m\left((Bp^{-m})^{\kappa_m} + \sum_{\mu=m+1}^{\infty} |a_m(\mu)|(Bp^{-\mu})^{\kappa_m}\right), \\ =& C_m B^{\frac{1}{m}} \left(p^{-1} + G_m(p^{-\frac{1}{m}})\right) \\ &+ O_m\left(B^{\kappa_m} p^{-m\kappa_m} + B^{\kappa_m} p^{-m\kappa_m} \sum_{\mu=m+1}^{\infty} |a_m(\mu)| 2^{(-\mu+m)\kappa_m}\right). \end{aligned}$$

The last sum is absolutely convergent (consider the generating function $x^{-m}G_m(x)$ at the point $x = 2^{-\kappa_m}$) and hence we have established the asymptotic

$$F_m(B,p) = C_m B^{\frac{1}{m}} \left(p^{-1} + G_m(p^{-\frac{1}{m}})\right) + O_m\left(B^{\kappa_m} p^{-m\kappa_m}\right). \qquad \square$$

Next we aim to generalize Lemma 5.3 to obtain an asymptotic formula for $F_m(B,d)$ for a general square-free number d. For this we start with a generalization of Lemma 5.2.

Lemma 5.4. *Let $d > 0$ be a square-free integer and p a prime with $p \mid d$. Write $d' = d/p$. For $m \geq 2$ one has*

$$\begin{aligned} F_m(B,d) =& F_m(Bp^{-m}, d') + \sum_{r=1}^{\infty} \sum_{k=2r}^{2r(m-1)} \rho_m(k,2r) F_m(Bp^{-(2r+1)m-k}, d') \\ &+ \sum_{r=1}^{\infty} \sum_{k=2r-1}^{(2r-1)(m-1)} \rho_m(k,2r-1) F_m(Bp^{-(2r-1)m-k}, d') \\ &- \sum_{r=1}^{\infty} \sum_{k=2r-1}^{(2r-1)(m-1)} \rho_m(k,2r-1) F_m(Bp^{-2rm-k}, d') \\ &- \sum_{r=1}^{\infty} \sum_{k=2r}^{2r(m-1)} \rho_m(k,2r) F_m(Bp^{-2rm-k}, d'). \end{aligned}$$

The proof of Lemma 5.4 is exactly the same as the proof of Lemma 5.2 where the condition l is m-full is replaced by the condition that l is m-full and $d' \mid l$. Moreover, as in equation (5.8) one can rewrite the identity from Lemma 5.4 as

$$F_m(B,d) = F_m(Bp^{-m}, d') + \sum_{\mu=m+1}^{\infty} a_m(\mu) F_m(Bp^{-\mu}, d'). \tag{5.9}$$

Via induction on the number of prime factors of d we now establish the following lemma.

Lemma 5.5. *Let $d > 0$ be a square-free integer. Write $\omega(d)$ for the number of prime divisors of d. Then for each integer $m \geq 2$ there exists a positive constant K_m such that we have*

$$F_m(B, d) = C_m B^{\frac{1}{m}} \prod_{p|d} \left(p^{-1} + G_m(p^{-\frac{1}{m}})\right) + O_m\left(K_m^{\omega(d)} B^{\kappa_m} d^{-m\kappa_m}\right).$$

Here the implicit constant is independent of d and

$$p^{-1} + G_m(p^{-\frac{1}{m}}) = \frac{1}{1 + p - p^{\frac{m-1}{m}}}. \tag{5.10}$$

For $m = 1$ the asymptotic holds with $C_1 = 1$, $K_1 = 1$, $\kappa_1 = 0$ (and $G_1 = 0$).

Proof. For $m = 1$ the statement is immediate. Let us assume that $m \geq 2$. If d is prime, then the statement follows from Lemma 5.3 (or note that if $d = 1$ then the statement reduces to the assumption in (5.1)). Let $d > 0$ be squarefree and q a prime with $q \mid d$. Assume that we have established the asymptotic

$$F_m(B, d') = C_m B^{\frac{1}{m}} \prod_{p|d'} \left(p^{-1} + G_m(p^{-\frac{1}{m}})\right) + O_m\left(K_m^{\omega(d')} B^{\kappa_m} d'^{-m\kappa_m}\right),$$

with a constant K_m given by

$$K_m := 1 + \sum_{\mu=m+1}^{\infty} |a_m(\mu)| 2^{-\kappa_m(\mu-m)}.$$

Note that K_m is indeed a convergent sum. Then by Lemma 5.4 and equation (5.9) we deduce that

$$\begin{aligned} F_m(B, d) =& F_m(Bq^{-m}, d') + \sum_{\mu=m+1}^{\infty} a_m(\mu) F_m(Bq^{-\mu}, d') \\ =& C_m B^{\frac{1}{m}} \prod_{p|d'} \left(p^{-1} + G_m(p^{-\frac{1}{m}})\right) \left(q^{-1} + \sum_{\mu=m+1}^{\infty} a_m(\mu) q^{-\frac{\mu}{m}}\right) \\ &+ O_m\left(K_m^{\omega(d')} B^{\kappa_m} q^{-m\kappa_m} d'^{-m\kappa_m} \left(1 + \sum_{\mu=m+1}^{\infty} |a_m(\mu)| q^{-\kappa_m(\mu-m)}\right)\right). \end{aligned}$$

By definition of K_m we obtain

$$\begin{aligned} F_m(B, d) =& C_m B^{\frac{1}{m}} \prod_{p|d} \left(p^{-1} + G_m(p^{-\frac{1}{m}})\right) \\ &+ O_m\left(K_m^{\omega(d')} B^{\kappa_m} q^{-m\kappa_m} d'^{-m\kappa_m} \left(1 + \sum_{\mu=m+1}^{\infty} |a_m(\mu)| 2^{-\kappa_m(\mu-m)}\right)\right) \\ =& C_m B^{\frac{1}{m}} \prod_{p|d} \left(p^{-1} + G_m(p^{-\frac{1}{m}})\right) + O_m\left(K_m^{\omega(d')+1} B^{\kappa_m} (qd')^{-m\kappa_m}\right) \\ =& C_m B^{\frac{1}{m}} \prod_{p|d} \left(p^{-1} + G_m(p^{-\frac{1}{m}})\right) + O_m\left(K_m^{\omega(d)} B^{\kappa_m} d^{-m\kappa_m}\right). \end{aligned}$$

□

Next we given an upper bounds for the leading constant in Lemma 5.5. For squarefree d we introduce the notation

$$c_{m,d} = C_m \prod_{p|d} \left(p^{-1} + G_m(p^{-\frac{1}{m}})\right). \tag{5.11}$$

We observe that $c_{1,d} = 1/d$ and we recall that

$$p^{-1} + G_m(p^{-\frac{1}{m}}) = \frac{1}{1 + p - p^{\frac{m-1}{m}}}. \tag{5.12}$$

For every fixed $m \geq 1$ there is a positive constant $c_2(m) < 1$ such that

$$p^{\frac{m-1}{m}} \leq c_2(m)p,$$

holds for all primes $p \geq 2$. Hence we deduce from equation (5.12) that there exists a constant $c_3(m)$, only depending on m, such that

$$\prod_{p|d} \left(p^{-1} + G_m(p^{-\frac{1}{m}})\right) \leq c_3(m)^{\omega(d)} \frac{1}{d}. \tag{5.13}$$

Hence, we get

$$c_{m,d} \ll C_m c_3(m)^{\omega(d)} \frac{1}{d} \ll_{m,\varepsilon} d^{-1+\varepsilon}. \tag{5.14}$$

Moreover, for $m \geq 2$ we have

$$mk_m = \frac{m}{m+1} = 1 - \frac{1}{m+1} \geq \frac{2}{3}. \tag{5.15}$$

5.1. **An auxiliary counting function.** For any s-tuple of positive integers $\mathbf{m} = (m_1, \dots, m_s)$ and any s-tuple of squarefree positive integers $\mathbf{d} = (d_1, \dots, d_s)$, let $f_{\mathbf{m},\mathbf{d}} : \mathbb{Z}_{>0}^s \to \mathbb{R}$ be the function defined by

$$f_{\mathbf{m},\mathbf{d}}(y_1, \dots, y_s) = \begin{cases} 1 & \text{if } d_i \mid y_i \text{ and } y_i \text{ is } m_i\text{-full } \forall i \in \{1, \dots, s\}, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.6. *Let $\mathbf{m} = (m_1, \dots, m_s) \in \mathbb{Z}_{>0}^s$. For every s-tuple of squarefree positive integers $\mathbf{d} = (d_1, \dots, d_s)$ and every $\varepsilon > 0$ we have*

$$\sum_{1 \leq y_i \leq B_i, 1 \leq i \leq s} f_{\mathbf{m},\mathbf{d}}(y_1, \dots, y_s)$$
$$= \left(\prod_{i=1}^s c_{m_i,d_i}\right) \prod_{i=1}^s B_i^{\frac{1}{m_i}} + O_{\mathbf{m},\varepsilon}\left(\left(\prod_{i=1}^s d_i\right)^{-\frac{2}{3}+\varepsilon} \left(\prod_{i=1}^s B_i^{\frac{1}{m_i}}\right) \left(\min_{1 \leq i \leq s} B_i\right)^{-\delta}\right)$$

for all $B_1, \dots, B_s > 0$, where $\delta = \min\{1/3, \min_{1 \leq i \leq s} 1/(m_i(m_i+1))\}$.

Proof. We observe that

$$\sum_{1 \leq y_i \leq B_i, 1 \leq i \leq s} f_{\mathbf{m},\mathbf{d}}(y_1, \dots, y_s) = \prod_{i=1}^s F_{m_i}(B_i, d_i).$$

For $i \in \{1, \dots, s\}$ such that $m_i \geq 2$ apply Lemma 5.5, for $i \in \{1, \dots, s\}$ such that $m_i = 1$ use the estimate $F_1(B_i, d_i) = B_i/d_i + O((B_i/d_i)^{2/3})$. Then apply (5.14) and (5.15) to estimate the error term. $\square$

Lemma 5.6 implies that the function $f_{\mathbf{m},\mathbf{d}}$ satisfies Property I with the constants

$$C_{f,M} = \prod_{i=1}^s c_{m_i,d_i}, \quad C_{f,E} = \left(\prod_{i=1}^s d_i\right)^{-\frac{2}{3}+\varepsilon},$$

and

$$\varpi_i = \frac{1}{m_i}, \quad 1 \leq i \leq s$$

and

$$\varDelta = \min\{1/3, \min_{1 \leq i \leq s} 1/(m_i(m_i+1))\}.$$

Similarly, Lemma 5.6 implies that $f_{\mathbf{m},\mathbf{d}}$ satisfies Property II. If $d_i \mid y_i$ and y_i is m_i-full for $i \in \mathcal{I}$ then

$$C_{f,M,\mathcal{I}}(\mathbf{y}_{\mathcal{I}}) = \prod_{i \notin \mathcal{I}} c_{m_i,d_i}$$

and $C_{f,M,\mathcal{I}}(\mathbf{y}_{\mathcal{I}}) = 0$ otherwise. Moreover one can for example take $D = 2s/3$ and $\nu > 0$ some arbitrary positive constant. Moreover, up to multiplying $C_{f,E}$ by a positive constant depending only on $\mathbf{m}$ and ε, with this choice of parameters Condition (4.1) holds.

6. Campana points on toric varieties

6.1. **Toric varieties over number fields.** Let X be a complete smooth split toric variety over a number field $\mathbb{K}$. Let $T \subseteq X$ be the dense torus. Let Σ be the fan that defines X. We denote by $\{\rho_1, \ldots, \rho_s\}$ the set of rays of Σ and by Σ_{max} the set of maximal cones of Σ. For every maximal cone σ we define $\mathscr{J}_\sigma$ to be the set of indices $i \in \{1, \ldots, s\}$ such that the ray ρ_i belongs to the cone σ, and we set $\mathscr{I}_\sigma = \{1, \ldots, s\} \smallsetminus \mathscr{J}_\sigma$. Then we have $|\mathscr{J}_\sigma| = n$ and $|\mathscr{I}_\sigma| = r$ for every maximal cone σ of Σ, where n is the dimension of X and r is the rank of the Picard group of X. In particular, $s = n + r$. For each $i \in \{1, \ldots, s\}$, we denote by D_i the prime toric invariant divisor corresponding to the ray ρ_i. We fix a canonical divisor $K_X := -\sum_{i=1}^s D_i$.

By [Cox95] the Cox ring of X is $\mathbb{K}[y_1, \ldots, y_s]$ where the degree of the variable y_i is the class of the divisor D_i in $\operatorname{Pic}(X)$. For every $\mathbf{y} = (y_1, \ldots, y_s) \in \mathbb{C}^s$ and every $D = \sum_{i=1}^s a_i D_i$, let

$$\mathbf{y}^D := \prod_{i=1}^s y_i^{a_i}.$$

Let $Y \to X$ be the universal torsor of X as in [Sal98, §8]. We recall that the variety Y is an open subset of $\mathbb{A}^s_{\mathbb{K}}$ whose complement is defined by $\mathbf{y}^{D_\sigma} = 0$ for all maximal cones σ, where $D_\sigma := \sum_{i \in \mathscr{I}_\sigma} D_i$ for all $\sigma \in \Sigma_{max}$.

The integral model $\pi : \mathscr{Y} \to \mathscr{X}$ of the universal torsor $Y \to X$ as in [Sal98, Remarks 8.6] gives a parameterization of the rational points on X via integral points in $\mathcal{O}_{\mathbb{K}}^s = \mathbb{A}^s(\mathcal{O}_{\mathbb{K}})$ as follows. Let $\mathcal{C}$ be a set of ideals of $\mathcal{O}_{\mathbb{K}}$ that form a system of representatives for the class group of $\mathbb{K}$. We fix a basis of $\operatorname{Pic}(X)$, and for every divisor D on X we write $\mathfrak{c}^D := \prod_{i=1}^r \mathfrak{c}_i^{b_i}$ where $[D] = (b_1, \ldots, b_r)$ with respect to the fixed basis of $\operatorname{Pic}(X)$. Then, as in [Pie16, §2],

$$X(\mathbb{K}) = X(\mathcal{O}_{\mathbb{K}}) = \bigsqcup_{\mathfrak{c} \in \mathcal{C}^r} \pi^{\mathfrak{c}}\left(\mathscr{Y}^{\mathfrak{c}}(\mathcal{O}_{\mathbb{K}})\right),$$

where $\pi^{\mathfrak{c}} : \mathscr{Y}^{\mathfrak{c}} \to \mathscr{X}$ is the twist of π defined in [FP16, Theorem 2.7]. The fibers of $\pi|_{\mathscr{Y}^{\mathfrak{c}}(\mathcal{O}_{\mathbb{K}})}$ are all isomorphic to $(\mathcal{O}_{\mathbb{K}}^\times)^r$, and $\mathscr{Y}^{\mathfrak{c}}(\mathcal{O}_{\mathbb{K}}) \subseteq \mathbb{A}^s(\mathcal{O}_{\mathbb{K}})$ is the subset of points $\mathbf{y} \in \bigoplus_{i=1}^s \mathfrak{c}^{D_i}$ that satisfy

$$\sum_{\sigma \in \Sigma_{\max}} \mathbf{y}^{D_\sigma} \mathfrak{c}^{-D_\sigma} = \mathcal{O}_{\mathbb{K}}. \tag{6.1}$$

Let N be the lattice of cocharacters of X. Then $\Sigma \subseteq N \otimes_{\mathbb{Z}} \mathbb{R}$. For every $i \in \{1, \ldots, s\}$, let ν_i be the unique generator of $\rho_i \cap N$. For every torus invariant divisor $D = \sum_{i=1}^s a_i D_i$ of X and for every $\sigma \in \Sigma_{max}$, let $u_{\sigma,D}$ be the character of N determined by $u_{\sigma,D}(\nu_j) = a_j$ for all $j \in \mathscr{J}_\sigma$, and define $D(\sigma) := D - \sum_{i=1}^s u_{\sigma,D}(\nu_i) D_i$. Then D and $D(\sigma)$ are linearly equivalent. For every $i, j \in \{1, \ldots, s\}$, let $\beta_{\sigma,i,j} := -u_{\sigma,D_j}(\nu_i)$. Then, for every $i, j \in \{1, \ldots, s\}$, we have

$\beta_{\sigma,i,j} = 0$ whenever $j \in \mathscr{I}_\sigma$, and whenever $i \neq j$ are both in $\mathscr{J}_\sigma$. Hence,

$$D_j(\sigma) = \begin{cases} D_j & \text{if } j \in \mathscr{I}_\sigma, \\ \sum_{i\in\mathscr{I}_\sigma} \beta_{\sigma,i,j} D_i & \text{if } j \in \mathscr{J}_\sigma. \end{cases} \tag{6.2}$$

Lemma 6.1. *For every $i, j \in \{i, \dots, s\}$ and $\sigma, \sigma' \in \Sigma_{max}$ we have*

$$\beta_{\sigma,i,j} = - \sum_{l\in\mathscr{J}_{\sigma'}} \beta_{\sigma',i,l}\beta_{\sigma,l,j}.$$

Proof. From the equality $D_j(\sigma') = (D_j(\sigma))(\sigma')$ we get

$$\begin{aligned} 0 &= \sum_{l=1}^{s} u_{\sigma,D_j}(\nu_l) D_l(\sigma') = \sum_{l=1}^{s} u_{\sigma,D_j}(\nu_l)(D_l - \sum_{i=1}^{s} u_{\sigma',D_l}(\nu_i) D_i) \\ &= \sum_{i=1}^{s} \left(u_{\sigma,D_j}(\nu_i) - \sum_{l=1}^{s} u_{\sigma,D_j(\nu_l)} u_{\sigma',D_l}(\nu_i) D_i \right) D_i. \end{aligned}$$ □

6.2. **Polytopes.** In this section, we fix a semiample $\mathbb{Q}$-divisor $L = \sum_{i=1}^{s} a_i D_i$, and we study a number of polytopes associated to L. The content of this section is purely combinatorial, in particular, it does not depend on the base field $\mathbb{K}$ where X is defined.

For each $\sigma \in \Sigma_{\max}$, we write $L(\sigma) = \sum_{i=1}^{s} \alpha_{i,\sigma} D_i$. Then $\alpha_{i,\sigma} = 0$ for all $i \in \mathscr{J}_\sigma$ by construction.

Remark 6.2. Since L is semiample, $L(\sigma)$ is effective for all $\sigma \in \Sigma_{\max}$ by [CLS11, Proposition 6.1.1]; that is, $\alpha_{i,\sigma} \geq 0$ for all $i \in \{1, \dots, s\}$ and all $\sigma \in \Sigma_{\max}$. If, moreover, L is ample, then $\alpha_{i,\sigma} > 0$ for all $i \in \mathscr{I}_\sigma$ and all $\sigma \in \Sigma_{\max}$ by [CLS11, Theorem 6.1.14].

Assumption 6.3. We assume that for every $i \in \{1, \dots, s\}$ there exists $\sigma \in \Sigma_{\max}$ such that $\alpha_{i,\sigma} > 0$.

We observe that Assumption 6.3 is satisfied if L is ample by Remark 6.2, or if L is linearly equivalent to an effective divisor $\sum_{i=1}^{s} b_i D_i$ with $b_i > 0$ for all $i \in \{1, \dots, s\}$ by Lemma 6.4 below.

6.2.1. *The polytope P_L.* We describe a classical polytope associated to L that we use in Section 6.3 to study the height function defined by L.

We denote by M the lattice of characters of T, dual to N, and by $M_\mathbb{R}$ the vector space $M \otimes_\mathbb{Z} \mathbb{R}$. Similarly, we set $\operatorname{Pic}(X)_\mathbb{R} := \operatorname{Pic}(X) \otimes_\mathbb{Z} \mathbb{R}$. We recall that there is an exact sequence (e.g. [CLS11, Theorem 4.2.1])

$$0 \to M_\mathbb{R} \to \bigoplus_{i=1}^{s} \mathbb{R} D_i \xrightarrow{\varphi} \operatorname{Pic}(X)_\mathbb{R} \to 0,$$

such that the effective cone $\operatorname{Eff}(X)$ of X is the image under φ of the cone generated by the effective torus invariant divisors

$$C := \left\{ \sum_{i=1}^{s} a_i D_i : a_1, \dots, a_s \geq 0 \right\} \subseteq \bigoplus_{i=1}^{s} \mathbb{R} D_i.$$

Since L is semiample,

$$P_L := \{ m \in M_\mathbb{R} : m(\nu_i) + a_i \geq 0 \ \forall i \in \{1, \dots, s\} \}.$$

is a polytope with vertices $\{-u_{\sigma,L} : \sigma \in \Sigma_{\max}\}$ by [CLS11, Proposition 4.3.8, Theorem 6.1.7]. In particular,

$$P_L = \left\{ \sum_{\sigma\in\Sigma_{\max}} -\lambda_\sigma u_{\sigma,L} : (\lambda_\sigma)_{\sigma\in\Sigma_{\max}} \in \mathbb{R}_{\geq 0}^{\Sigma_{\max}}, \sum_{\sigma\in\Sigma_{\max}} \lambda_\sigma = 1 \right\}. \tag{6.3}$$

Lemma 6.4. *For every $t \in \mathbb{R}_{\geq 0}$,*

$$(tL + M_{\mathbb{R}}) \cap C = \left\{ \sum_{\sigma \in \Sigma_{\max}} \lambda_\sigma L(\sigma) : (\lambda_\sigma)_{\sigma \in \Sigma_{\max}} \in (\mathbb{R}_{\geq 0})^{\Sigma_{\max}}, \sum_{\sigma \in \Sigma_{\max}} \lambda_\sigma = t \right\}.$$

Proof. By (6.3) $L + P_L$ is the polytope with vertices $\{L(\sigma) : \sigma \in \Sigma_{\max}\}$. Since $(tL + M_{\mathbb{R}}) \cap C = tL + P_{tL} = t(L + P_L)$ by [CLS11, Exercise 4.3.2], the statement follows. □

Remark 6.5. If L is ample, then there exists a positive integer t such that $[t^{-1}L] = \left[\sum_{i=1}^{s} \frac{1}{m_i} D_i\right]$ with $m_1, \dots, m_s \in \mathbb{Z}_{>0}$. Indeed, $[L]$ has at least one representative of the form $\sum_{i=1}^{s} b_i D_i$ with $b_1, \dots, b_s \in \mathbb{Q}_{>0}$ by Lemma 6.4 and Remark 6.2. Hence, it suffices to choose any positive integer t such that $tb_i^{-1} \in \mathbb{Z}$ for all $i \in \{1, \dots, s\}$.

6.2.2. *The polytope $\widetilde{P}$.* We investigate some polytopes associated to $[L]$ that we use for the application of the hyperbola method in Sections 6.5–6.7.

We identify $\mathbb{R}^s$ with the space of linear functions on $\bigoplus_{i=1}^{s} \mathbb{R} D_i$ by defining $\mathbf{t}(D_i) = t_i$ for all $i \in \{1, \dots, s\}$ and all $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$. Under this identification, the dual of $\operatorname{Pic}(X)$ is the linear subspace $\widetilde{H}$ of $\mathbb{R}^s$ defined by

$$t_j = \sum_{i \in \mathscr{I}_\sigma} \beta_{\sigma,i,j} t_i \quad \forall j \in \mathscr{J}_\sigma, \tag{6.4}$$

for one, or equivalently all, $\sigma \in \Sigma_{\max}$ (cf. Lemma 6.1).

Let $\widetilde{P} \subseteq \mathbb{R}^s$ be the polyhedron defined by

$$t_i \geq 0 \ \forall i \in \{1, \dots, s\} \quad \text{and} \quad \sum_{i=1}^{s} \alpha_{i,\sigma} t_i \leq 1 \ \forall \sigma \in \Sigma_{\max}. \tag{6.5}$$

Then $\widetilde{P}$ is a full dimensional convex polytope by Remark 6.2 and Assumption 6.3. Moreover, $\operatorname{cone}(\widetilde{P})$ is dual to the cone C defined above. For every $\sigma \in \Sigma_{\max}$, let

$$\widetilde{P}_\sigma := \widetilde{P} \cap \operatorname{cone}\left(\widetilde{P} \cap \left\{ \sum_{i=1}^{s} \alpha_{i,\sigma} t_i = 1 \right\} \right),$$

so that

$$\bigcup_{\sigma \in \Sigma_{\max}} \widetilde{P}_\sigma = \widetilde{P}. \tag{6.6}$$

We observe that the polytopes $\widetilde{P}$ and $\widetilde{P}_\sigma$ depend only on the class of L in $\operatorname{Pic}(X)$ and not on the chosen representative $\sum_{i=1}^{s} a_i D_i$.

Lemma 6.6. *(i) For every $\sigma, \sigma' \in \Sigma_{\max}$,*

$$\widetilde{P}_\sigma \cap \widetilde{P}_{\sigma'} = \widetilde{P}_\sigma \cap \left\{ t_j = \sum_{i \in \mathscr{I}_\sigma} \beta_{\sigma,i,j} t_i \quad \forall j \in \{l \in \mathscr{J}_\sigma : \alpha_{l,\sigma'} \neq 0\}. \right\}.$$

(ii) Under Assumption 6.3 we have $\bigcap_{\sigma \in \Sigma_{\max}} \widetilde{P}_\sigma = \widetilde{P} \cap \widetilde{H}$.

(iii) If L is ample, then

$$\widetilde{P}_\sigma = \widetilde{P} \cap \left\{ t_j \leq \sum_{i \in \mathscr{I}_\sigma} \beta_{\sigma,i,j} t_i \quad \forall j \in \mathscr{J}_\sigma \right\}$$

for every $\sigma \in \Sigma_{\max}$. In particular, $\widetilde{P}_\sigma$ is the polytope in $\mathbb{R}^s$ defined by

$$t_1, \dots, t_s \geq 0, \quad t_j \leq \sum_{i \in \mathscr{I}_\sigma} \beta_{\sigma,i,j} t_i \quad \forall j \in \mathscr{J}_\sigma, \quad \sum_{i \in \mathscr{I}_\sigma} \alpha_{i,\sigma} t_i \leq 1.$$

Proof. By definition, $\widetilde{P}_\sigma$ is the set of elements $\mathbf{t} \in \widetilde{P}$ such that $\mathbf{t}(L(\sigma')) \leq \mathbf{t}(L(\sigma))$ for all $\sigma' \in \Sigma_{\max}$. By (6.2) we have

$$\mathbf{t}(L(\sigma) - L(\sigma')) = \mathbf{t}\left(\sum_{j=1}^{s} \alpha_{j,\sigma'}(D_j(\sigma) - D_j)\right) = \sum_{j \in \mathscr{J}_\sigma} \alpha_{j,\sigma'}\left(\left(\sum_{i \in \mathscr{I}_\sigma} \beta_{\sigma,i,j} t_i\right) - t_j\right)$$

Hence, (i) and the inclusion $\supseteq$ in (iii) follow. For the reverse inclusion in (iii) we fix $j \in \mathscr{J}_\sigma$. By [Sal98, Lemma 8.9] there is $\sigma' \in \Sigma_{\max}$ such that $\mathscr{J}_\sigma \cap \mathscr{J}_{\sigma'} = \mathscr{J}_\sigma \smallsetminus \{j\}$. Then

$$\alpha_{j,\sigma'}\left(t_j - \sum_{i \in \mathscr{I}_\sigma} \beta_{\sigma,i,j} t_i\right) = \mathbf{t}(L(\sigma') - L(\sigma)) \leq 0$$

for all $\mathbf{t} \in \widetilde{P}_\sigma$ and $\alpha_{j,\sigma'} > 0$ by Remark 6.2. Part (ii) follows from (i), as $\bigcap_{\sigma \in \Sigma_{\max}} \widetilde{P}_\sigma = \widetilde{P}_{\sigma'} \cap \widetilde{H}$ for every $\sigma' \in \Sigma_{\max}$ by (i) together with Assumption 6.3, and we conclude by (6.6). □

Lemma 6.7. *Assume that L is ample. Let $\varpi = (\varpi_1, \ldots, \varpi_s) \in \mathbb{R}^s_{>0}$. Let $\widetilde{F}$ be the face of $\widetilde{P}$ where the maximum value $a(L, \varpi)$ of $\sum_{i=1}^s \varpi_i t_i$ is attained. Then*

(i) $a(L, \varpi) > 0$ and $\widetilde{F} \subseteq \widetilde{H}$.
(ii) If, additionally, $[L] = [\sum_{i=1}^s \varpi_i D_i]$ in $\operatorname{Pic}(X)_\mathbb{R}$, then $a(L, \varpi) = 1$ and $\widetilde{F} = \widetilde{H} \cap \{\sum_{i=1}^s \varpi_i t_i = 1\}$. In particular, $\widetilde{F} \cap \{t_1, \ldots, t_s > 0\} \neq \emptyset$.

Proof. Since $s \geq 1$ and $\widetilde{P}$ is full dimensional, we have $a(L, \varpi) > 0$. For every $\sigma \in \Sigma_{\max}$, let $\widetilde{F}_\sigma := \widetilde{F} \cap \widetilde{P}_\sigma$. Fix $\sigma \in \Sigma_{\max}$. Let $\mathbf{t} \in \widetilde{P}_\sigma \smallsetminus \widetilde{H}$. By (6.4) and Lemma 6.6 there exists $j \in \mathscr{J}_\sigma$ such that $t_j < \sum_{i \in \mathscr{I}_\sigma} \beta_{\sigma,i,j} t_i$. Let $t'_j := \sum_{i \in \mathscr{I}_\sigma} \beta_{\sigma,i,j} t_i$. For each $i \in \{1, \ldots, s\} \smallsetminus \{j\}$ let $t'_i := t_i$. Then $(t'_1, \ldots, t'_s) \in \widetilde{P}_\sigma$, and $\sum_{i=1}^s \varpi_i t'_i > \sum_{i=1}^s \varpi_i t_i$ by construction. Hence $\mathbf{t} \notin \widetilde{F}_\sigma$. Thus $\widetilde{F}_\sigma \subseteq \widetilde{H}$, which implies $\widetilde{F}_\sigma \subseteq \widetilde{F}_{\sigma'}$ for all $\sigma' \in \Sigma_{\max}$. Since this proof works for every $\sigma \in \Sigma_{\max}$, we conclude that $\widetilde{F}_\sigma = \widetilde{F}_{\sigma'}$ for all $\sigma, \sigma' \in \Sigma_{\max}$. Now (i) follows, because $\widetilde{F} = \bigcup_{\sigma \in \Sigma_{\max}} \widetilde{F}_\sigma$.

For (ii) we recall that $\alpha_{i,\sigma} = \varpi_i + \sum_{j \in \mathscr{J}_\sigma} \varpi_j \beta_{\sigma,i,j}$ for all $i \in \mathscr{I}_\sigma$. Hence, $\sum_{i \in \mathscr{I}_\sigma} \alpha_{i,\sigma} t_i = \sum_{i=1}^s \varpi_i t_i$ for all $\mathbf{t} \in \widetilde{H}$ and for all $\sigma \in \Sigma_{\max}$. Since $\widetilde{H}$ is the subspace of $\mathbb{R}^s$ dual to $\operatorname{Pic}(X)_\mathbb{R}$, a torus invariant divisor D satisfies $\mathbf{t}(D) = 0$ for all $\mathbf{t} \in \widetilde{H}$ if and only if D is a principal divisor. Since $D_1, \ldots, D_s$ are not principal divisors, then $\widetilde{H} \cap \{t_1, \ldots, t_s > 0\} \neq \emptyset$. Let $\mathbf{t} \in \widetilde{H}$ with $t_1, \ldots, t_s > 0$, up to rescaling $\mathbf{t}$ by a positive real number we can assume that $\sum_{i=1}^s \varpi_i t_i = 1$, and hence $\mathbf{t} \in \widetilde{F}$. □

6.2.3. *The geometric constant.* We compute certain volumes of polytopes that appear in the leading constant of the asymptotic formula 1.3.

Fix $\sigma \in \Sigma_{\max}$. Since X is smooth, we know that $\operatorname{Pic}(X) = \bigoplus_{i \in \mathscr{I}_\sigma} \mathbb{Z}[D_i]$. We identify $\mathbb{R}^r$ with the space of linear functions on $\bigoplus_{i \in \mathscr{I}_\sigma} \mathbb{R}[D_i]$ by defining $\mathbf{z}\left(\sum_{i \in \mathscr{I}_\sigma} a_i [D_i]\right) := \sum_{i \in \mathscr{I}_\sigma} a_i z_i$ for all $\mathbf{z} = (z_i)_{i \in \mathscr{I}_\sigma} \in \mathbb{R}^r$. Let $\lambda_{[L]} : \mathbb{R}^r \to \mathbb{R}$ be the evaluation at $[L]$; that is, $\lambda_{[L]}(\mathbf{z}) = \sum_{i \in \mathscr{I}_\sigma} \alpha_{i,\sigma} z_i$. Fix $\tilde{i} \in \mathscr{I}_\sigma$ such that $\alpha_{\tilde{i},\sigma} \neq 0$. The change of variables $x = \lambda_{[L]}(\mathbf{z})$, $\mathrm{d}x = \alpha_{\tilde{i},\sigma} \mathrm{d}z_{\tilde{i}}$, gives

$$\int_{\mathbb{R}^r} g \prod_{i \in \mathscr{I}_\sigma} \mathrm{d}z_i = \int_{\mathbb{R}} \left(\int_{z_{\tilde{i}} = \left(x - \sum_{i \in \mathscr{I}_\sigma, i \neq \tilde{i}} \alpha_{i,\sigma} z_i\right)/\alpha_{\tilde{i},\sigma}} g\ \alpha_{\tilde{i},\sigma}^{-1} \prod_{i \in \mathscr{I}_\sigma, i \neq \tilde{i}} \mathrm{d}z_i \right) \mathrm{d}x$$

for all integrable functions $g : \mathbb{R}^r \to \mathbb{R}$.

Lemma 6.8. *The volume*

$$\alpha(L) := \int_{\mathrm{Eff}(X)^*\cap\lambda_{[L]}^{-1}(1)} \alpha_{\tilde{i},\sigma}^{-1} \prod_{i\in\mathscr{I}_\sigma, i\neq\tilde{i}} \mathrm{d}z_i$$

is positive and independent of the choice of $\tilde{i}$ *and of the choice of* σ.

Proof. The transversal intersection $\mathrm{Eff}(X)^*\cap\lambda_{[L]}^{-1}(1)$ is an $(r-1)$-dimensional polytope, hence the volume is positive. The independence of the choice of $\tilde{i}$ is clear. The independence of the choice of σ is a consequence of Lemma 6.1. □

Lemma 6.9. *Assume that* L *is ample and* $[L] = [\sum_{i=1}^s \varpi_i D_i]$ *with* $\varpi_1,\dots,\varpi_s > 0$. *For* $\delta \geq 0$, *let* H_δ *be the hyperplane defined by* $\sum_{i=1}^s \varpi_i t_i = 1-\delta$. *Then for* $\delta > 0$ *small enough,* $\mathrm{meas}_{s-1}(H_\delta\cap\widetilde{P}) = c\delta^{s-r} + O(\delta^{s-r+1})$, *where* meas_{s-1} *is the* $(s-1)$-*dimensional measure on* H_δ *given by* $\prod_{1\leq i\leq s, i\neq\tilde{i}}(\varpi_i \mathrm{d}t_i)$ *for any choice of* $\tilde{i}\in\{1,\dots,s\}$, *and*

$$c = \frac{\alpha(L)}{(s-r)!}\sum_{\sigma\in\Sigma_{\max}}\prod_{i\in\mathscr{I}_\sigma}\varpi_i.$$

Proof. Since L is ample, the decomposition (6.6) and Lemma 6.6 give

$$\mathrm{meas}_{s-1}(H_\delta\cap\widetilde{P}) = \sum_{\sigma\in\Sigma_{\max}}\mathrm{meas}_{s-1}(H_\delta\cap\widetilde{P}_\sigma).$$

Fix $\sigma\in\Sigma_{\max}$. Let $V_{\delta,\sigma} := \mathrm{meas}_{s-1}(H_\delta\cap\widetilde{P}_\sigma)$. By the choice of L we have $\alpha_{i,\sigma} = \varpi_i + \sum_{j\in\mathscr{J}_\sigma}\varpi_j\beta_{\sigma,i,j}$ for all $i\in\mathscr{I}_\sigma$, and hence,

$$\sum_{i=1}^s \varpi_i t_i = \sum_{i\in\mathscr{I}_\sigma}\alpha_{i,\sigma}t_i - \sum_{j\in\mathscr{J}_\sigma}\varpi_j\Big(\sum_{i\in\mathscr{I}_\sigma}\beta_{\sigma,i,j}t_i - t_j\Big)$$

for every $\mathbf{t}\in\mathbb{R}^s$. Then $H_0\cap\widetilde{P}_\sigma\subseteq\widetilde{H}$ by Lemma 6.6(iii). Fix $\xi = (\xi_1,\dots,\xi_s)\in H_0\cap\widetilde{P}_\sigma$, and fix $\tilde{i}\in\mathscr{I}_\sigma$. Then $V_{\delta,\sigma} = \mathrm{meas}_{s-1}((\widetilde{P}_\sigma\cap H_0)+\delta\xi)$ is the volume of the polytope given by

$$t_i \geq \delta\xi_i\ \forall i\in\{1,\dots,s\},\ t_j \leq \sum_{i\in\mathscr{I}_\sigma}\beta_{\sigma,i,j}t_i\ \forall j\in\mathscr{J}_\sigma,\ \sum_{i\in\mathscr{I}_\sigma}\alpha_{i,\sigma}t_i \leq 1+\delta,\ \sum_{i=1}^s\varpi_i t_i = 1,$$

with respect to the measure $\prod_{1\leq i\leq s, i\neq\tilde{i}}(\varpi_i\mathrm{d}t_i)$.

For all $i\in\mathscr{I}_\sigma$, let $u_i = t_i$. For all $j\in\mathscr{J}_\sigma$, let $u_j = (\sum_{i\in\mathscr{I}_\sigma}\beta_{\sigma,i,j}t_i - t_j)/\delta$, i.e.,

$$u_j = \delta^{-1}\left(\sum_{i\in\mathscr{I}_\sigma, i\neq\tilde{i}}\beta_{\sigma,i,j}t_i + \frac{\beta_{\sigma,\tilde{i},j}}{\varpi_{\tilde{i}}}\left(1 - \sum_{1\leq i\leq s, i\neq\tilde{i}}\varpi_i t_i\right) - t_j\right).$$

Let $g(\mathbf{u}) := \sum_{j\in\mathscr{J}_\sigma}\varpi_j u_j$ and $h(\mathbf{u}) := 1 - \sum_{i\in\mathscr{I}_\sigma, i\neq\tilde{i}}\alpha_{i,\sigma}u_i$. Then $\delta^{r-s}V_{\delta,\sigma}$ is the volume of the polytope given by

$$u_j \geq 0\ \forall j\in\mathscr{J}_\sigma,\quad g(\mathbf{u})\leq 1,$$
$$u_i \geq \delta\xi_i\ \forall i\in\mathscr{I}_\sigma\smallsetminus\{\tilde{i}\},\qquad \sum_{i\in\mathscr{I}_\sigma, i\neq\tilde{i}}\alpha_{i,\sigma}u_i \leq 1+\delta g(\mathbf{u}),$$
$$\sum_{i\in\mathscr{I}_\sigma, i\neq\tilde{i}}\beta_{\sigma,i,j}u_i + \frac{\beta_{\sigma,\tilde{i},j}}{\alpha_{\tilde{i},\sigma}}h(\mathbf{u}) \geq \delta\left(u_j + \xi_j - \frac{\beta_{\sigma,\tilde{i},j}}{\alpha_{\tilde{i},\sigma}}g(\mathbf{u})\right)\ \forall j\in\mathscr{J}_\sigma,$$

with respect to the measure $\varpi_{\tilde{i}}\alpha_{\tilde{i},\sigma}^{-1}\prod_{1\leq i\leq s, i\neq\tilde{i}}(\varpi_i \mathrm{d}u_i)$, as

$$
\begin{aligned}
|\det((\partial u_j/\partial t_l)_{1\leq j,l\leq s, j\neq\tilde{i}, l\neq\tilde{i}})| &= \delta^{r-s}\det((\beta_{\sigma,\tilde{i},l}\varpi_l/\varpi_{\tilde{i}}+\delta_{l,j})_{j,l\in\mathscr{J}_\sigma})\\
&= \delta^{r-s}\left(1+\sum_{j\in\mathscr{J}_\sigma}\beta_{\sigma,\tilde{i},j}\varpi_j/\varpi_{\tilde{i}}\right) = \delta^{r-s}\alpha_{\tilde{i},\sigma}/\varpi_{\tilde{i}},
\end{aligned}
$$

where $\delta_{l,j}=0$ if $l\neq j$ and $\delta_{j,j}=1$. By dominated convergence we can compute $c=\lim_{\delta\to 0}\delta^{r-s}V_{\delta,\sigma}$ as the volume of the polytope given by

$$
\begin{gathered}
u_j\geq 0\ \forall j\in\mathscr{J}_\sigma,\quad g(\mathbf{u})\leq 1,\\
u_i\geq 0\ \forall i\in\mathscr{I}_\sigma,\quad \sum_{i\in\mathscr{I}_\sigma}\beta_{\sigma,i,j}u_i\geq 0\ \forall j\in\mathscr{J}_\sigma,\quad \sum_{i\in\mathscr{I}_\sigma}\alpha_{i,\sigma}u_i=1,
\end{gathered}
$$

with respect to the measure $\varpi_{\tilde{i}}\alpha_{\tilde{i},\sigma}^{-1}\prod_{1\leq i\leq s, i\neq\tilde{i}}(\varpi_i \mathrm{d}u_i)$. We conclude as

$$
\int_{u_j\geq 0\ \forall j\in\mathscr{J}_\sigma, g(\mathbf{u})\leq 1}\prod_{j\in\mathscr{J}_\sigma}(\varpi_j \mathrm{d}u_j)=\frac{1}{(s-r)!}. \qquad \square
$$

6.3. **Heights.** Now we study the height associated to a semiample $\mathbb{Q}$-divisor L on X. Let t be a positive integer such that tL has integer coefficients and is base point free. By [CLS11, Proposition 4.3.3] we have $H^0(X,tL)=\bigoplus_{m\in P_{tL}\cap M}\mathbb{K}\chi^m$, where P_{tL} and M are defined in Section 6.2 and $\chi^m\in\mathbb{K}[T]$ is the character of T corresponding to m.

Let $H_{tL}: X(\mathbb{K})\to\mathbb{R}_{\geq 0}$ be the pullback of the exponential Weil height under the morphism $X\to\mathbb{P}(H^0(X,tL))$ defined by the basis of $H^0(X,tL)$ corresponding to $P_{tL}\cap M$. We define $H_L:=(H_{tL})^{1/t}$. We observe that this definition agrees with [BT95a, §2.1].

Proposition 6.10. *For every $\mathbf{y}\in\mathscr{Y}(\mathbb{K})$, we have*

$$
H_L(\pi(\mathbf{y}))=\prod_{\nu\in\Omega_\mathbb{K}}\sup_{\sigma\in\Sigma_{\max}}|\mathbf{y}^{L(\sigma)}|_\nu.
$$

For every $\mathfrak{c}\in\mathcal{C}^r$ and $\mathbf{y}\in\mathscr{Y}^\mathfrak{c}(\mathcal{O}_\mathbb{K})$, we have

$$
H_L(\pi^\mathfrak{c}(\mathbf{y}))=\mathfrak{N}(\mathfrak{c}^{-L})\prod_{\nu\in\Omega_\infty}\sup_{\sigma\in\Sigma_{\max}}|\mathbf{y}^{L(\sigma)}|_\nu.
$$

Proof. By definition of H_L and of π we have, for $\mathbf{y}\in\mathscr{Y}(\mathbb{K})$,

$$
H_L(\pi(\mathbf{y}))=\prod_{\nu\in\Omega_\mathbb{K}}\sup_{m\in P_{tL}\cap M}|\mathbf{y}^{tL+(\chi^m)}|_\nu^{1/t}.
$$

Let $m\in P_{tL}\cap M$. By (6.3) there are $\lambda_\sigma\in\mathbb{R}_{\geq 0}$ for $\sigma\in\Sigma_{\max}$ such that $\sum_{\sigma\in\Sigma_{\max}}\lambda_\sigma=1$ and $m=-\sum_{\sigma\in\Sigma_{\max}}\lambda_\sigma u_{\sigma,tL}$, so that $\mathbf{y}^{tL+(\chi^m)}=\mathbf{y}^{\sum_{\sigma\in\Sigma_{\max}}\lambda_\sigma tL(\sigma)}$. This proves the first statement. For the second statement we argue as in the proof of [Pie16, Proposition 2]. Fix $\mathfrak{c}\in\mathcal{C}^r$ and $\mathbf{y}\in\mathscr{Y}^\mathfrak{c}(\mathcal{O}_\mathbb{K})$. For every prime ideal $\mathfrak{p}$ of $\mathcal{O}_\mathbb{K}$ we write $v_\mathfrak{p}$ for the associated valuation. Then

$$
\min_{\sigma\in\Sigma_{\max}}v_\mathfrak{p}(\mathbf{y}^{L(\sigma)})=\min_{\sigma\in\Sigma_{\max}}v_\mathfrak{p}(\mathbf{y}^{L(\sigma)}\mathfrak{c}^{-L(\sigma)})+v_\mathfrak{p}(\mathfrak{c}^L)=v_\mathfrak{p}(\mathfrak{c}^L),
$$

where the first equality holds as $[L(\sigma)]=[L]$ in $\mathrm{Pic}(X)_\mathbb{R}$, and the second equality follows from (6.1) as $\mathbf{y}^{L(\sigma)}\in\mathfrak{c}^{L(\sigma)}$ for all $\sigma\in\Sigma_{\max}$. $\square$

The following lemma will ensure the Northcott property for H_L.

Lemma 6.11. *If L satisfies Assumption 6.3, then there is $\alpha > 0$ such that for every $\mathfrak{c} \in \mathcal{C}^r$ and $B > 0$, every point $\mathbf{y} \in \bigoplus_{i=1}^s \mathfrak{c}^{D_i}$ with*

$$\prod_{\nu \in \Omega_\infty} \sup_{\sigma \in \Sigma_{\max}} |\mathbf{y}^{L(\sigma)}|_\nu \leq \mathfrak{N}(\mathfrak{c}^L) B \tag{6.7}$$

and $y_1, \dots, y_s \neq 0$ satisfies

$$\prod_{\nu \in \Omega_\infty} |\mathbf{y}_i|_\nu \leq \mathfrak{N}(\mathfrak{c}^{D_i}) B^\alpha$$

for all $i \in \{1, \dots, s\}$.

Proof. Let $\mathbf{y} \in \bigoplus_{i=1}^s \mathfrak{c}^{D_i}$ such that (6.7) holds and $y_1, \dots, y_s \neq 0$. Fix $i \in \{1, \dots, s\}$ and choose $\sigma \in \Sigma_{\max}$ such that $\alpha_{i,\sigma} > 0$. Recall that $\prod_{\nu \in \Omega_\infty} |y_i|_\nu = \mathfrak{N}(y_i \mathcal{O}_{\mathbb{K}})$ by the product formula. Since $y_j \in \mathfrak{c}^{D_j}$ for all $j \in \{1, \dots, s\}$, we have

$$\mathfrak{N}(y_i \mathcal{O}_{\mathbb{K}})^{\alpha_{i,\sigma}} \mathfrak{N}(\mathfrak{c}^{L(\sigma) - \alpha_{i,\sigma} D_i}) \leq \mathfrak{N}(\mathbf{y}^{L(\sigma)} \mathcal{O}_{\mathbb{K}}) \leq \prod_{\nu \in \Omega_\infty} \sup_{\sigma \in \Sigma_{\max}} |\mathbf{y}^{L(\sigma)}|_\nu \leq \mathfrak{N}(\mathfrak{c}^L) B.$$

Hence, $\mathfrak{N}(\mathbf{y}_i \mathcal{O}_{\mathbb{K}}) \leq \mathfrak{N}(\mathfrak{c}^{D_\rho}) B^{1/\alpha_{i,\sigma}}$. □

6.4. **Campana points.** From now on we assume that $\mathbb{K} = \mathbb{Q}$. For every $i \in \{1, \dots, s\}$, we fix a positive integer m_i and we denote by $\mathscr{D}_i$ the closure of D_i in $\mathscr{X}$. Let $\mathbf{m} := (m_1, \dots, m_s)$ and

$$\Delta := \sum_{i=1}^s \left(1 - \frac{1}{m_i}\right) \mathscr{D}_i.$$

The support of the restriction of Δ to the fibers over $\operatorname{Spec} \mathbb{Z}$ is a strict normal crossing divisor (see for example [CLT10, §5.1]), hence $(\mathscr{X}, \Delta)$ is a Campana orbifold as in [PSTVA19, Definition 3.1]. We denote by $(\mathscr{X}, \Delta)(\mathbb{Z})$ the set of Campana $\mathbb{Z}$-points as in [PSTVA19, Definition 3.4], and by $\mathscr{Y}(\mathbb{Z})_{\mathbf{m}}$ the preimage of $(\mathscr{X}, \Delta)(\mathbb{Z})$ under $\pi|_{\mathscr{Y}(\mathbb{Z})}$. Then a point of $\mathscr{Y}(\mathbb{Z})$ with coordinates $(y_1, \dots, y_s)$ belongs to $\mathscr{Y}(\mathbb{Z})_{\mathbf{m}}$ if and only if y_i is nonzero and m_i-full for all $i \in \{1, \dots, s\}$.

For every semiample divisor L that satisfies Assumption 6.3 and every $B > 0$, let $N_{\mathbf{m},L}(B)$ be the number of points in $(\mathscr{X}, \Delta)(\mathbb{Z})$ of height H_L at most B. Since $\pi : \mathscr{Y} \to \mathscr{X}$ is a $\mathbb{G}_m^r$-torsor, we have

$$N_{\mathbf{m},L}(B) = \frac{1}{2^r} \sharp \left\{ \mathbf{y} \in \mathscr{Y}(\mathbb{Z})_{\mathbf{m}} : H_L(\pi(\mathbf{y})) \leq B \right\}.$$

6.5. **Heuristics.** In this subsection we give a heuristic argument based on the hyperbola method in support of [PSTVA19, Conjecture 1.1] for split toric varieties.

We assume that $-(K_X + \Delta)$ is ample and that L is a big and semiample $\mathbb{Q}$-divisor, not necessarily equal to $-[K_X + \Delta]$ in $\operatorname{Pic}(X)_{\mathbb{R}}$, that satisfies Assumption 6.3. We recall that [PSTVA19, Conjecture 1.1] for $(\mathscr{X}, \Delta)$ predicts the asymptotic formula

$$N_{\mathbf{m},L}(B) \sim c B^{a(L)} (\log B)^{b(L)-1}, \quad B \to +\infty, \tag{6.8}$$

where c is a positive constant,

$$a(L) := \inf\{t \in \mathbb{R} : t[L] + [K_X + \Delta] \in \operatorname{Eff}(X)\} \tag{6.9}$$

and $b(L)$ is the codimension of the minimal face of $\operatorname{Eff}(X)$ that contains $a(L)[L] + [K_X + \Delta]$. In particular, $b(L)$ is a positive integer, and $a(L)$ is a positive real number, as $-[K_X + \Delta]$ is ample. Since $\operatorname{Eff}(X)$ is closed in the euclidean topology, the infimum in the definition of $a(L)$ is actually a minimum.

6.5.1. *Combinatorial description of $a(L)$.* Recall the notation introduced in Section 6.2. We now give a characterization of $a(L)$ as the solution of certain linear programming problems.

Proposition 6.12. (i) The number $a(L)$ is the minimal value of the function $\sum_{\sigma\in\Sigma_{\max}}\lambda_\sigma$ subject to the conditions

$$\lambda_\sigma \geq 0, \quad \forall \sigma\in\Sigma_{\max}, \tag{6.10}$$

$$\sum_{\sigma\in\Sigma_{\max}} \lambda_\sigma \alpha_{i,\sigma} \geq \frac{1}{m_i}, \quad \forall i\in\{1,\dots,s\}. \tag{6.11}$$

(ii) Assume, additionally, that L is ample. Fix $\sigma\in\Sigma_{\max}$ and define $\gamma_i := \sum_{j\in\mathscr{J}_\sigma}\frac{\beta_{\sigma,i,j}}{m_j}$. Then $a(L)$ is the minimal value of the function λ_0 subject to the conditions

$$\lambda_0, \lambda_j \geq 0, \quad \forall j\in\mathscr{J}_\sigma, \tag{6.12}$$

$$\lambda_0\alpha_{i,\sigma} - \sum_{j\in\mathscr{J}_\sigma}\beta_{\sigma,i,j}\lambda_j \geq \frac{1}{m_i}+\gamma_i, \quad \forall i\in\mathscr{I}_\sigma. \tag{6.13}$$

Proof. To prove part (i) we observe that for $t\in\mathbb{R}$ the condition

$$t[L]+[K_X+\Delta]\in \mathrm{Eff}(X) \tag{6.14}$$

is equivalent to $\varphi^{-1}(t[L]+[K_X+\Delta])\cap C\neq\emptyset$. Now, $\varphi^{-1}(t[L]+[K_X+\Delta]) = tL+K_X+\Delta+M_{\mathbb{R}}$. If $D\in(tL+K_X+\Delta+M_{\mathbb{R}})\cap C$, then $D-(K_X+\Delta)\in C$ as $-(K_X+\Delta)=\sum_{i=1}^s\frac{1}{m_i}D_i\in C$ and C is a cone. Then (6.14) holds if and only if there exists a divisor $D'\in(tL+M_{\mathbb{R}})\cap C$ such that $D'+K_X+\Delta\in C$. By Lemma 6.4 this is equivalent to the existence of $\lambda_\sigma\in\mathbb{R}_{\geq 0}$ for all $\sigma\in\Sigma_{\max}$ such that $\sum_{\sigma\in\Sigma_{\max}}\lambda_\sigma = t$ and $\sum_{\sigma\in\Sigma_{\max}}\lambda_\sigma L(\sigma)+K_X+\Delta\in C$.

Now we prove part (ii). Condition (6.14) is equivalent to the existence of $D\in C$ such that $t[L]+[K_X+\Delta]=\varphi(D)$ in $\mathrm{Pic}(X)_{\mathbb{R}}$. Since X is proper and smooth, the last equality is equivalent to $tL(\sigma)+(K_X+\Delta)(\sigma)=D(\sigma)$. Write $D=\sum_{i=1}^s\lambda_iD_i$. Then $D\in C$ if and only if $\lambda_1,\dots,\lambda_s\geq 0$. We have

$$tL(\sigma)+(K_X+\Delta)(\sigma)-D(\sigma)=\sum_{i\in\mathscr{I}_\sigma}\left(t\alpha_{i,\sigma}-\frac{1}{m_i}-\lambda_i-\gamma_i-\sum_{j\in\mathscr{J}_\sigma}\beta_{\sigma,i,j}\lambda_j\right)D_i.$$

Using the fact that $\lambda_i\geq 0$ for all $i\in\mathscr{I}_\sigma$ if $D\in C$, we see that condition (6.14) is equivalent to the existence of $\lambda_j\in\mathbb{R}_{\geq 0}$ for all $j\in\mathscr{J}_\sigma$ that satisfy the conditions in the statement for $\lambda_0=t$. □

6.5.2. *Heuristic argument for $a(L)$.* Next we give a heuristic argument in support of [PSTVA19, Conjecture 1.1] (and [BM90, §3.3]) regarding the expected exponent $a(L)$ of B in the asymptotic formula (6.8) for split toric varieties over $\mathbb{Q}$.

Up to a positive constant, $N_{\mathbf{m},L}(B)$ is the cardinality S of the set of m_i-full positive integers y_i for $i\in\{1,\dots,s\}$ that satisfy the conditions $\mathbf{y}^{L(\sigma)}\leq B$ for all $\sigma\in\Sigma_{\max}$. We recall that $\mathbf{y}^{L(\sigma)}=\prod_{i=1}^s y_i^{\alpha_{i,\sigma}}$.

One of the ideas of the hyperbola method is to dissect the region of summation for the variables $y_1,\dots,y_s$, into different boxes. Assume that we consider a box where say $y_i\sim B_i$ (here we mean that for example $B_i\leq y_i\leq 2B_i$ for $i\in\{1,\dots,s\}$), and let $B_i=B^{t_i}$. What contribution do such vectors $\mathbf{y}=(y_1,\dots,y_s)$ give to computing the cardinality S? First we note that the contribution from this box is

$$\text{box contribution} = \prod_{i=1}^s B_i^{\frac{1}{m_i}} = B^{\sum_{i=1}^s t_i\frac{1}{m_i}}.$$

In order for this to be a box that we count by S, the parameters $(t_1, \ldots, t_s)$ need to satisfy

$$\sum_{i=1}^{s} t_i \alpha_{i,\sigma} \leq 1, \quad \forall \sigma \in \Sigma_{\max} \tag{6.15}$$

and

$$t_1, \ldots, t_s \geq 0. \tag{6.16}$$

In order to find the size of S we hence have the following linear programming problem $\mathcal{P}$: Maximize the function

$$\sum_{i=1}^{s} t_i \frac{1}{m_i} \tag{6.17}$$

under the conditions (6.15) and (6.16). The conditions (6.15) and (6.16) define a polytope $\widetilde{P}$ in $\mathbb{R}^s$ and by the theory of linear programming we know that the maximum of the function $\sum_{i=1}^{s} t_i \frac{1}{m_i}$ is obtained on at least one of its vertices.

The dual linear programming problem $\mathcal{D}$ is given by the following problem: Minimize the function

$$\sum_{\sigma \in \Sigma_{\max}} \lambda_\sigma$$

under the conditions (6.10) and (6.11). By the strong duality property in linear programming, both problems have a finite optimal solution and these values are equal. Since $a(L)$ is positive, by Proposition 6.12(i) it is the solution of the dual linear programming problem $\mathcal{D}$ and also of $\mathcal{P}$.

6.5.3. *Heuristic argument for $b(L)$.* Now we give a heuristic argument in support of [PSTVA19, Conjecture 1.1] (and [BM90, §3.3]) regarding the expected exponent $a(L)$ of B in the asymptotic formula (6.8) for split toric varieties over $\mathbb{Q}$. We keep the setting introduced above.

If we cover the region of summation $\mathbf{y}^{L(\sigma)} \leq B$ for all $\sigma \in \Sigma_{\max}$ by dyadic boxes, then the maximal value of the count is attained on boxes, that are located at the maximal face $\widetilde{F}$ of the polytope $\widetilde{P}$ where the function in (6.17) is maximized. Working with a dyadic dissection this suggests that the leading term should be of order $B^{a(L)}(\log B)^k$, where k is equal to the dimension of the face $\widetilde{F}$. The next proposition shows that $k = b(L) - 1$. Hence, the heuristic expectation we obtained from the hyperbola method matches the prediction in [PSTVA19, Conjecture 1.1].

We recall from Lemma 6.7 that $\widetilde{F} \subseteq \widetilde{H}$, where $\widetilde{H}$ is the space of linear functions on $\mathrm{Pic}(X)_{\mathbb{R}}$. With this identification, the cone generated by $\widetilde{P} \cap \widetilde{H}$ is the space of linear functions on $\mathrm{Pic}(X)_{\mathbb{R}}$ that are nonnegative on $\mathrm{Eff}(X)$ (i.e., the cone in $\widetilde{H}$ dual to $\mathrm{Eff}(X)$) by [CLS11, Proposition 1.2.8].

Proposition 6.13. *The cone generated by $\widetilde{F}$ is dual to the minimal face of* $\mathrm{Eff}(X)$ *that contains $a(L)[L] + [K_X + \Delta]$. In particular, $b(L) = \dim \widetilde{F} + 1$.*

Proof. Fix $\sigma \in \Sigma_{\max}$. Then $\widetilde{P} \cap \widetilde{H} = \widetilde{P}_\sigma \cap \widetilde{H}$. Let $\mathbf{t} = (t_1, \ldots, t_s) \in \mathrm{cone}(\widetilde{P} \cap \widetilde{H})$. We recall that $[L] = \sum_{i \in \mathscr{I}_\sigma} \alpha_{i,\sigma}[D_i]$ and $[K_X + \Delta] = -\sum_{i \in \mathscr{I}_\sigma} \left(\frac{1}{m_i} + \gamma_i\right)[D_i]$, so that

$$\mathbf{t}(a(L)[L] + [K_X + \Delta]) = a(L) \sum_{i \in \mathscr{I}_\sigma} \alpha_{i,\sigma} t_i - \sum_{i \in \mathscr{I}_\sigma} \left(\frac{1}{m_i} + \gamma_i\right) t_i.$$

If $\mathbf{t} \in \widetilde{F}$, then $\mathbf{t}(a(L)[L] + [K_X + \Delta]) = 0$. Conversely, if $\mathbf{t} \neq 0$ and $t(a(L)[L] + [K_X + \Delta]) = 0$, then $\alpha := \sum_{i \in \mathscr{I}_\sigma} \alpha_{i,\sigma} t_i > 0$ as $\frac{1}{m_i} + \gamma_i > 0$ and $t_i \geq 0$ for all

$i \in \mathscr{I}_\sigma$. So $(\alpha^{-1}t_1, \dots, \alpha^{-1}t_s) \in \widetilde{F}$, and $\mathbf{t} \in \operatorname{cone}(\widetilde{F})$. Thus, $\operatorname{cone}(\widetilde{F})$ is the face of $\operatorname{cone}(\widetilde{P} \cap \widetilde{H})$ defined by

$$\mathbf{t}(a(L)[L] + [K_X + \Delta]) = 0.$$

By [CLS11, Definition 1.2.5, Proposition 1.2.10] the face of $\operatorname{Eff}(X)$ dual to $\operatorname{cone}(\widetilde{F})$ is the smallest face of $\operatorname{Eff}(X)$ that contains $a(L)[L] + [K_X + \Delta]$, and $b(L) = \dim \operatorname{cone}(\widetilde{F}) = \dim \widetilde{F} + 1$. □

6.6. **The technical assumption.** Here we spell out a technical assumption for the application of Theorem 1.1 to toric varieties.

Assumption 6.14. Let $\varepsilon > 0$. For every nonempty subset $J \subsetneq \{1, \dots, s\}$ and every $\tau = (\tau_i)_{i \in \{1,\dots,s\} \smallsetminus J} \in [0, \varepsilon]^{\{1,\dots,s\} \smallsetminus J}$, let

$$\widetilde{P}^J_\tau := \widetilde{P} \cap \{t_i = \tau_i \ \forall i \in \{1, \dots, s\} \smallsetminus J\},$$

where $\widetilde{P} \subseteq \mathbb{R}^s$ is the polytope defined by (6.5) for $L = \sum_{i=1}^s m_i^{-1} D_i$. Let a be the maximal value of $\sum_{i=1}^s m_i^{-1} t_i$ on $\widetilde{P}$. Let k be the dimension of the face $\widetilde{F}$ of $\widetilde{P}$ where $\sum_{i=1}^s m_i^{-1} t_i = a$. Let a^J_τ be the maximal value of $\sum_{i \in J} m_i^{-1} t_i$ on $\widetilde{P}^J_\tau$. Let k^J_τ be the dimension of the face $\widetilde{F}^J_\tau$ of $\widetilde{P}^J_\tau$ where $\sum_{i \in J} m_i^{-1} t_i = a^J_\tau$. Assume that there exists $\eta > 0$ such that for sufficiently small $\varepsilon > 0$, if $a^J_\tau \geq a - \eta$ then $k^J_\tau \leq k - 1$.

If L is ample, by Lemma 6.7(ii) we know that $a = 1$ and $k = r - 1$ in the setting of Assumption 6.14, and that the face $\widetilde{F}$ is not contained in any coordinate hyperplane. As a consequence of Lemma 4.17(v) we see that if $r \geq n + 1$, then Assumption 6.14 is satisfied for all ample divisors $L = \sum_{i=1}^s m_i^{-1} D_i$. Since toric blow-ups are birational transformations that increase the Picard rank, by Remark 6.5 there are infinitely many examples of toric Campana orbifolds to which Theorem 1.2 applies.

Among toric varieties of smaller Picard rank, we observe that whenever $\widetilde{F}$ does not intersect any coordinate hyperplane of $\mathbb{R}^s$ then Assumption 6.14 is satisfied by Lemma 4.17(iv). This is the case of projective spaces for all choices of $L = \sum_{i=1}^s m_i^{-1} D_i$.

The blow-up of the projective plane in one point and arbitrary finite products of projective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ are examples of toric varieties for which Assumption 6.14 can be verified by computations, but are not covered by Lemma 4.17.

While at present we cannot show that Assumption 6.14 always holds under the remaining assumptions of Theorem 1.2, we are not aware of any examples of toric Campana orbifolds that don't satisfy Assumption 6.14.

6.7. **Proof of Theorem 1.2.** From now on we work in the setting of Theorem 1.2. In particular, $L = -(K_X + \Delta) = \sum_{i=1}^s m_i^{-1} D_i$, and

$$N(B) = \frac{1}{2^r} \sharp \{\mathbf{y} \in \mathscr{Y}(\mathbb{Z})_{\mathbf{m}} : H(\mathbf{y}) \leq B\},$$

where $H(\mathbf{y}) := \sup_{\sigma \in \Sigma_{\max}} |\mathbf{y}^{L(\sigma)}|$, by Section 6.4 and Proposition 6.10.

6.7.1. *Möbius inversion.* For $B > 0$ and $\mathbf{d} \in (\mathbb{Z}_{>0})^s$, let $A(B, \mathbf{d})$ be the set of points $\mathbf{y} = (y_1, \dots, y_s) \in \mathbb{Z}^s$ such that $H(\mathbf{y}) \leq B$, y_i is nonzero and m_i-full and $d_i \mid y_i$ for all $i \in \{1, \dots, s\}$. We observe that $A(B, \mathbf{d})$ is a finite set by Lemma 6.11. Then

$$N(B) = \frac{1}{2^r} \sum_{\mathbf{d} \in (\mathbb{Z}_{>0})^s} \mu(d) \sharp A(B, \mathbf{d}), \tag{6.18}$$

where μ is the function introduced in [Sal98, Definition and proposition 11.9].

6.7.2. *The estimate.* Under the assumptions of Theorem 1.2, we can apply Theorem 1.1 to obtain an estimate for the cardinality of the sets $A(B,\mathbf{d})$ as follows.

Proposition 6.15. *There is $B_0 > 0$ such that for every s-tuple $\mathbf{d} = (d_1,\dots,d_s)$ of squarefree positive integers, $B > B_0$ and $\epsilon > 0$, we have*

$$\sharp A(B,\mathbf{d}) = 2^s\alpha(L)\left(\sum_{\sigma\in\Sigma_{\max}}\prod_{i\in\mathscr{I}_\sigma} m_i^{-1}\right)\left(\prod_{i=1}^s c_{m_i,d_i}\right)B(\log B)^{r-1} + O_\varepsilon\left(\left(\prod_{i=1}^s d_i\right)^{-\frac{2}{3}+\varepsilon} B(\log B)^{r-2}(\log\log B)^{s-1}\right),$$

where $\alpha(L)$ is defined in Lemma 6.8 and c_{m_i,d_i} is defined in (5.11).

Proof. We write $\sharp A(B,\mathbf{d}) = 2^s\sharp A'(B,\mathbf{d})$, where $A'(B,\mathbf{d})$ is the set of positive integers $y_1,\dots,y_s$ that satisfy the conditions

$$d_i \mid y_i, \quad y_i \text{ is } m_i\text{-full} \quad \forall i\in\{1,\dots,s\}$$

and the inequalities

$$\prod_{i=1}^s y_i^{\alpha_{i,\sigma}} \le B, \quad \forall\sigma\in\Sigma_{\max}. \tag{6.19}$$

We observe that

$$\sharp A'(B,\mathbf{d}) = \sum_{y_1,\dots,y_s\in\mathbb{Z}_{>0}:(6.19)} f_{\mathbf{m},\mathbf{d}}(y_1,\dots,y_s),$$

where $f_{\mathbf{m},\mathbf{d}}$ is the function defined in Subsection 5.1. The function $f_{\mathbf{m},\mathbf{d}}$ satisfies *Property I* and *Property II* in Section 4 and Condition (4.1) by Lemma 5.6 and (5.14), see the remarks after Lemma 5.6. Assumption 4.3 is satisfied as (6.5) defines an s-dimensional polytope. Assumption 4.5 is satisfied by Lemma 6.7(ii). For B large enough, the first part of Assumption 4.14 is a consequence of Assumption 6.14. Hence, we can combine Theorem 1.1 and Lemma 5.6 to compute

$$\sharp A'(B,\mathbf{d}) = (s-1-k)!\left(\prod_{i=1}^s c_{m_i,d_i}\right)cB^a(\log B)^k + O\left(\left(\prod_{i=1}^s d_i\right)^{-\frac{2}{3}+\varepsilon} B^a(\log B)^{k-1}(\log\log B)^{s-1}\right), \tag{6.20}$$

where $a = 1$ and $k = r-1$ by Lemma 6.7(ii), and

$$c = \frac{\alpha(L)}{(s-r)!}\sum_{\sigma\in\Sigma_{\max}}\prod_{i\in\mathscr{I}_\sigma} m_i^{-1}$$

by Lemma 6.9. □

We combine the proposition above with the Möbius inversion to obtain an estimate for $N(B)$.

Proposition 6.16. *For sufficiently large $B > 0$, we have*

$$N(B) = cB(\log B)^{r-1} + O(B(\log B)^{r-2}(\log\log B)^{s-1}),$$

where

$$c = 2^{s-r}\alpha(L)\left(\sum_{\sigma\in\Sigma_{\max}}\prod_{i\in\mathscr{I}_\sigma} m_i^{-1}\right)\sum_{\mathbf{d}\in(\mathbb{Z}_{>0})^s}\mu(d)\left(\prod_{i=1}^s c_{m_i,d_i}\right). \tag{6.21}$$

Proof. By (6.18) and Proposition 6.15 we have

$$\left|N(B) - cB(\log B)^{r-1}\right| / (B(\log B)^{r-2}(\log\log B)^{s-1}) \ll_\varepsilon \sum_{\mathbf{d}\in(\mathbb{Z}_{>0})^s} \frac{|\mu(d)|}{\prod_{i=1}^s d_i^{\frac{2}{3}-\varepsilon}}.$$

For $\varepsilon = \frac{1}{12}$, the right hand side converges by [Sal98, Lemma 11.15]. □

Theorem 1.2 follows from Proposition 6.16 and the interpretation of the leading constant that we carry out in the following section.

6.7.3. *The leading constant.* Here we prove that the leading constant (6.21) can be written as

$$\alpha(X,\Delta)\tau(X,\Delta)\prod_{i=1}^s \frac{1}{m_i}, \tag{6.22}$$

where $\alpha(X,\Delta) := \alpha(-(K_X+\Delta))$ has been introduced in Lemma 6.8, and

$$\tau(X,\Delta) = \int_{\overline{\mathscr{X}(\mathbb{Z})}} H_\Delta(x)\delta_\Delta(x)\tau_X,$$

where τ_X is the Tamagawa measure on the set of adelic points $X(\mathbb{A}_\mathbb{Q})$ defined in [CLT10, Definition 2.8], H_Δ is the height function defined by the divisor Δ as in [BT95a, §2.1], $\overline{\mathscr{X}(\mathbb{Z})}$ is the closure of the set of rational points $\mathscr{X}(\mathbb{Z})$ inside the space of adelic points on X, and $\delta_\Delta = \prod_{p\in\Omega_f}\delta_{\Delta,p}$, where $\delta_{\Delta,p}$ is the characteristic function of the set of Campana points in $\mathscr{X}(\mathbb{Z}_p)$ for each finite place p.

We observe that by [BT95a, Proposition 2.4.4], the product (6.22) agrees with the expectation formulated in [PSTVA19, §3.3] provided that the domains of integration in the definitions of $\tau(X,\Delta)$ (i.e., $\{x\in\overline{\mathscr{X}(\mathbb{Z})} : \delta_\Delta(x)=1\}$) coincide.

Proposition 6.17. *We have*

$$\tau(X,\Delta) = \left(2^{s-r}\sum_{\sigma\in\Sigma_{\max}}\prod_{j\in\mathscr{J}_\sigma} m_j\right)\sum_{\mathbf{d}\in(\mathbb{Z}_{>0})^s}\left(\mu(\mathbf{d})\prod_{i=1}^s c_{m_i,d_i}\right) > 0.$$

Proof. For every prime number p, we denote by Fr_p the geometric Frobenius acting on $\mathrm{Pic}(\mathscr{X}_{\overline{\mathbb{F}}_p})\otimes_\mathbb{Z}\mathbb{Q}$, and for $t\in\mathbb{C}$ we define $L_p(t,\mathrm{Pic}(X_{\overline{\mathbb{Q}}})) := \det(1-p^{-t}\,\mathrm{Fr}_p)^{-1}$. Since X is split, and hence $\mathscr{X}_{\overline{\mathbb{F}}_p}$ is split, we have $L_p(t,\mathrm{Pic}(X_{\overline{\mathbb{Q}}})) = (1-p^{-t})^{-r}$ for every prime p. Let $\lambda_p := L_p(1,\mathrm{Pic}(X_{\overline{\mathbb{Q}}})) = (1-p^{-1})^{-r}$, and define $\lambda := \lim_{t\to1}(t-1)^r\prod_{p\in\Omega_f}L_p(t,\mathrm{Pic}(X_{\overline{\mathbb{Q}}}))$. Then $\lambda = \lim_{t\to1}(t-1)^r\zeta(t)^r = 1$ by properties of the residue at 1 of the Riemann zeta function. Let $\lambda_\infty := 1$. By [CLT10, Remark 2.9(b)], we have $\tau_X = \lambda\prod_{v\in\Omega_\mathbb{Q}}\lambda_v^{-1}\tau_{X,p}$, where $\tau_{X,v}$ denotes the local measure on $X(\mathbb{Q}_v)$ defined in [CLT10, §2.1.7].

Since split toric varieties satisfy weak approximation (e.g., [Har04, §2]) and $\mathscr{X}$ is projective, we get

$$\tau(X,\Delta) = \int_{X(\mathbb{R})} H_{\Delta,\infty}(x)\tau_{X,\infty}\prod_{p\in\Omega_f}\lambda_p^{-1}\int_{\mathscr{X}(\mathbb{Z}_p)} H_{\Delta,p}(x)\delta_{\Delta,p}(x)\tau_{X,p},$$

where $H_\Delta = \prod_{v\in\Omega_\mathbb{Q}} H_{\Delta,v}$ as in [BT95a, §2.1]. We recall that the global metrization used to define H_δ in [BT95a, §2.1] is equivalent to the one determined by [Sal98, Proposition and definition 9.2].

Since $\tau_{X,\infty}$ is the measure used in [Sal98, Proposition 9.16] for the real place, we have

$$\int_{X(\mathbb{R})} H_{\Delta,\infty}(x)\tau_{X,\infty} = \sum_{\sigma\in\Sigma_{\max}} \left(2^{s-r} \prod_{j\in\mathscr{J}_\sigma} \int_{0\le x_j\le 1} x_j^{\frac{1}{m_j}-1} \mathrm{d}x_j \right) = 2^{s-r} \sum_{\sigma\in\Sigma_{\max}} \prod_{j\in\mathscr{J}_\sigma} m_j.$$

Moreover, if we use the $\mathbb{G}_m^r$-torsor structure on $\pi : \mathscr{Y}(\mathbb{Z}_p) \to \mathscr{X}(\mathbb{Z}_p)$ together with [Sal98, Corollary 2.23, Propositions 9.7, 9.13 and 9.14] we get

$$\int_{\mathscr{X}(\mathbb{Z}_p)} H_{\Delta,p}(x)\delta_{\Delta,p}(x)\tau_{X,p} = \left(\int_{\mathbb{G}_m^r(\mathbb{Z}_p)} \mathrm{d}z\right)^{-1} \int_{\mathscr{Y}(\mathbb{Z}_p)} H_{\Delta,p}(\pi(y))\delta_{\Delta,p}(\pi(y)) \prod_{i=1}^s \mathrm{d}y_i,$$

where dz and $\prod_{i=1}^s \mathrm{d}y_i$ are the Haar measures on $\mathbb{Q}_p^r$ and $\mathbb{Q}_p^s$, respectively, induced by the Haar measure on $\mathbb{Q}_p$ normalized such that $\mathbb{Z}_p$ has volume 1, and $\delta_{\Delta,p}\circ\pi$ is the characteristic function of the property

$$v_p(y_i) > 0 \quad \Rightarrow \quad v_p(y_i) \ge m_i, \qquad \forall i \in \{1,\dots,s\}.$$

We have $\int_{\mathbb{G}_m^r(\mathbb{Z}_p)} \mathrm{d}z = (1-p^{-1})^r$. Let χ be the characteristic function of $\mathscr{Y}$. By [Sal98, Lemma 11.15] we have

$$\begin{aligned}
\int_{\mathscr{Y}(\mathbb{Z}_p)} H_{\Delta,p}(\pi(\mathbf{y}))\delta_{\Delta,p}(\pi(\mathbf{y})) \prod_{i=1}^s \mathrm{d}y_i &= \int_{\mathbb{Z}_p^N} \chi(\mathbf{y})\delta_{\Delta,p}(\pi(\mathbf{y})) \prod_{i=1}^s |y_i|_p^{\frac{1}{m_i}-1} \prod_{i=1}^s \mathrm{d}y_i \\
&= \sum_{(\mathbf{e})\in\{0,1\}^s} \mu((p^{e_1},\dots,p^{e_s})) \int_{p^{e_i}|y_i, 1\le i\le s} \delta_{\Delta,p}(\pi(\mathbf{y})) \prod_{i=1}^s |y_i|_p^{\frac{1}{m_i}-1} \prod_{i=1}^s \mathrm{d}y_i \\
&= \sum_{(\mathbf{e})\in\{0,1\}^s} \mu((p^{e_1},\dots,p^{e_s})) \prod_{i=1}^s \left((1-e_i)\int_{\mathbb{Z}_p^\times} \mathrm{d}y_i + \sum_{j=m_i}^\infty \int_{p^j\mathbb{Z}_p^\times} |y_i|_p^{\frac{1}{m_i}-1} \mathrm{d}y_i \right) \\
&= \sum_{(\mathbf{e})\in\{0,1\}^s} \mu((p^{e_1},\dots,p^{e_s})) \prod_{i=1}^s \left(\left(1-p^{-1}\right) \left(1-e_i + \sum_{j=m_i}^\infty p^{-\frac{j}{m_i}} \right)\right).
\end{aligned}$$

We observe that the product $\left(1-p^{-1}\right)\left(1-e_i+\sum_{j=m_i}^\infty p^{-\frac{j}{m_i}}\right)$ equals

$$\begin{cases} 1+\sum_{j=m_i+1}^{2m_i-1} p^{-\frac{j}{m_i}} & \text{if } e_i = 0 \\ \left(1+\sum_{j=m_i+1}^{2m_i-1} p^{-\frac{j}{m_i}}\right)\left(1+p-p^{\frac{m_i-1}{m_i}}\right)^{-1} & \text{if } e_i = 1. \end{cases}$$

So, remembering (5.2) and (5.10), we get

$$\begin{aligned}
&\tau(X,\Delta)\left(2^{s-r}\sum_{\sigma\in\Sigma_{\max}}\prod_{j\in\mathscr{J}_\sigma} m_j\right)^{-1} \\
&\quad = \left(\prod_{i=1}^s C_{m_i}\right)\left(\prod_{p\in\Omega_f}\sum_{\mathbf{e}\in\{0,1\}^s} \mu((p^{e_1},\dots,p^{e_s})) \prod_{i=1}^s \left(p^{-1}+G_{m_i}\left(p^{-\frac{1}{m_i}}\right)\right)^{e_i}\right).
\end{aligned}$$

We conclude by [Sal98, Lemma 11.15(e)] as $\prod_{p|d_i}(p^{-1}+G_{m_i}(p^{-\frac{1}{m_i}})) \ll_\epsilon d_i^{-1+\epsilon}$ by (5.14) for all $i \in \{1,\dots,s\}$.

To show that $\tau(X,\Delta)$ is positive it suffices to observe that

$$\int_{\mathscr{Y}(\mathbb{Z}_p)} H_{\Delta,p}(\pi(y))\delta_{\Delta,p}(\pi(y))\prod_{i=1}^{s}\mathrm{d}y_i \geq \int_{(\mathbb{Z}_p^{\times})^s}\prod_{i=1}^{s}\mathrm{d}y_i = (1-p^{-1})^s > 0,$$

as the integral on the right is the restriction of the integral on the left to the subset $(\mathbb{Z}_p^{\times})^s \subseteq \mathscr{Y}(\mathbb{Z}_p)$. □

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