LOCAL CENTRAL LIMIT THEOREM AND POTENTIAL KERNEL ESTIMATES FOR A CLASS OF SYMMETRIC HEAVY-TAILTED RANDOM VARIABLES

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Abstract. In this article, we study a class of heavy-tailed random variables on $\mathbb{Z}$ in the domain of attraction of an α-stable random variable of index $\alpha \in (0,2)$ satisfying a certain expansion of their characteristic function. Our results include sharp convergence rates for the local (stable) central limit theorem of order $n^{-(1+\frac{1}{\alpha})}$, a detailed expansion of the characteristic function of a long-range random walk with transition probability proportional to $|x|^{-(1+\alpha)}$ and $\alpha \in (0,2)$ and furthermore detailed asymptotic estimates of the discrete potential kernel (Green's function) up to order $\mathcal{O}\left(|x|^{\frac{\alpha-2}{3}+\varepsilon}\right)$ for any $\varepsilon > 0$ small enough, when $\alpha \in [1,2)$.

1. Introduction and overview of the results

Central limit theorems, local central limit theorems (LCLT) and potential kernel estimates are fundamental results in probability theory. They are important to study convergences of sequences of random variables for a variety of contexts in probability and statistical physics. Applications include mixing rates of Lorentz gases [26], asymptotic shapes in Internal Diffusion Limited Aggregation [21], scaling limit of the discrete Gaussian Free Field [9], convergence of discrete Gaussian multiplicative chaos [29] and bounds on size of the largest component for percolation on a box [27].

In this paper, we study a class of i.i.d. heavy-tailed random variables $(X_i)_{i\in\mathbb{N}}$ with support on $\mathbb{Z}$ which are in the domain of attraction of a symmetric α-stable random variable $\bar{X}$ with index $\alpha \in (0,2)$ and satisfy a particular expansion of their characteristic function. We will prove a LCLT result providing sharp convergence rates for $p^n_X(\cdot)$, the law of $S_n := \sum_{i=1}^n X_i$, explicit asymptotic behaviour of its discrete potential kernel and additionally obtain a detailed expansion of the characteristic function for the step size of a long-range random walk in $\mathbb{Z}$.

There exists a vast literature providing different types of LCLT results (or local stable limit theorems) in the stable setting with explicit and implicit convergence rates, e.g. [5, 6, 7, 10, 14, 17, 25, 30, 31]. To our knowledge, the best explicit non-uniform convergence rate for 1d absolutely continuous

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X was proven in [12], where the author showed under some integrability conditions on the characteristic function that for any $\alpha \in (0,2)$:

$$|x|^\alpha \left|p_X^n(x) - p_{\bar{X}}^n(x)\right| \le C n^\gamma, \tag{1.1}$$

where $\bar{X}$ is the stable distribution of index α and $\gamma = 1 - \frac{2}{\alpha}$ if $\alpha \in [1,2)$ and $\gamma = 1 - \frac{1}{\alpha}$ if $\alpha \in (0,1)$. As for uniform bounds in x, one can use classical results of convergence of random variables (such as in [30, 31]) which imply that

$$n^{\frac{1}{\alpha}} \left|p_X^n(x) - p_{\bar{X}}^n(x)\right| = o(1). \tag{1.2}$$

In [7] the author studied LCLT and large deviation estimates for random variables in the Cauchy domain of attraction for mainly asymmetric 1d random walks using renewal theory. Renewal theory was also used in [10] to obtain large deviation results for Lévy walks and in [19, 24] in the dynamical systems setting. A different approach proving LCLT results was taken in a series of papers [18, 20, 23], where the authors use subadditivity of diverse metrics (Kolmogorov, Zoltarev or Mallows distance) to prove LCLT's for continuous heavy-tailed random variables.

Concerning discrete potential kernel or Green's function behaviour there has been some asymptotic estimates obtained in [1, 4, 7, 8, 33] and [32] in the continuum. In [33], the author proves that for $\alpha \in (0,2)$ the discrete potential kernel is asymptotic to $\|x\|^{d-\alpha} L(|x|)$ where $L(\cdot)$ is a slowly varying function, whereas [8] obtains similar asymptotics for processes on $\mathbb{Z}^d$ with index $\alpha = (\alpha_1, ..., \alpha_d)$ and $\alpha \in (0,2]^d$.

The uniform bound given in (1.2) is indeed sharp, as for each $\varepsilon \in (0,2)$ one can use examples from this article to construct sequences in which the term $o(1)$ in (1.2) is of order $\mathcal{O}(n^{-\varepsilon})$. Let us make a brief analogy to the LCLT rates in the classical domain of attraction of a Gaussian distribution. For convenience, we will stay in the symmetric distribution case. Under additional moment conditions, say $\mathbb{E}(|X|^3) < \infty$ or $\mathbb{E}(X^4) < \infty$, the speed of convergence in the LCLT can be improved from $\mathcal{O}(n^{-\frac{1}{2}})$, given in (1.2), to $\mathcal{O}(n^{-1})$ and $\mathcal{O}(n^{-\frac{3}{2}})$ respectively, see [22]. The Edgeworth expansion [11] tells us that these speeds are indeed optimal. In general, one can use cumulants of higher order to get an expansion of the characteristic function and to derive more information about the rate of convergence of such laws. Notice that this is not possible in the context of variables in the domain of attraction of an α-stable distribution, as moments, and therefore cumulants cease to exist. Therefore, we will need to derive the further expansions of the characteristic function analytically.

Let us state the main results from this paper. Assume that the common characteristic function of the random variables X_i's satisfies the following expansion with respect to $\alpha \in (0,2)$ and regularity set $R_\alpha \subset (\alpha, 2+\alpha)$:

$$\phi_X(\theta) = 1 - \kappa_\alpha |\theta|^\alpha + \sum_{\beta \in R_\alpha} \kappa_\beta |\theta|^\beta + \mathcal{O}\Big(|\theta|^{2+\alpha}\Big) \tag{1.3}$$

as $|\theta| \longrightarrow 0$ with constants $\kappa_\alpha > 0, \kappa_\beta \in \mathbb{R}$. This class turns out to have nice properties, it is closed under e.g. addition and convex combinations. The concept of the regularity set R_α is similar to the *index set* A, which appears in the definition of *regularity structures* in [16].

We will show in Proposition 4.1 that the symmetric long-range random walk with transition probability $p(x,y) = c_\alpha |x-y|^{-(1+\alpha)}$ for $\alpha \in (0,2)$ falls into this class with $R_\alpha = \{2\}$ and determine the precise expansion of the characteristic function.

One of the main results, Theorem 3.2, yields sharp convergence rates:

$$\sup_{x\in\mathbb{Z}} \left|p^n_X(x) - p^n_{\bar{X}}(x)\right| \lesssim n^{-\frac{\beta_1+1-\alpha}{\alpha}}$$

where $\beta_1 = \min(J^+_\alpha)$ and $J^+_\alpha \subset (\alpha, 2+\alpha)$ is a set which depends on the regularity set R_α. For a particular case where $R_\alpha \in \{\varnothing, \{2\}\}$ we prove in Theorem 3.1 that given a random variable Z symmetric, with finite support and variance $|\kappa_2|$ and $\bar{Z} \sim \mathcal{N}(0, |\kappa_2|)$. Then if

(1) $\kappa_2 = 0$ we have that $\sup_{x\in\mathbb{Z}} |p^n_X(x) - p^n_{\bar{X}}(x)| \lesssim n^{-(1+\frac{1}{\alpha})}$
(2) $\kappa_2 > 0$ we have that $\sup_{x\in\mathbb{Z}} |p^n_{X+Z}(x) - p^n_{\bar{X}}(x)| \lesssim n^{-(1+\frac{1}{\alpha})}$
(3) $\kappa_2 < 0$ we have that $\sup_{x\in\mathbb{Z}} |p^n_X(x) - p^n_{\bar{X}+\bar{Z}}(x)| \lesssim n^{-(1+\frac{1}{\alpha})}$.

Note that depending on the sign of the constant κ_2 in the expansion we will modify either the original law $p^n_X(\cdot)$ or the limiting law $p^n_{\bar{X}}(\cdot)$ in such a way that the strong convergence rate $n^{-(1+\frac{1}{\alpha})}$ prevails. This modification introduces an error of order $\mathcal{O}(n^{-\frac{1}{\alpha}+(1-\frac{2}{\alpha})})$ which will vanish as $n \to \infty$.

The proofs involve a careful analysis of the laws $p^n_X(\cdot)$ and $p^n_{\bar{X}}(\cdot)$ in terms of their characteristic functions. The modification idea is natural and has shown to be very fruitful for example in [13] where the authors used it to obtain better convergence rates of a truncated Green's function in $\mathbb{Z}^2$. Furthermore we provide explicit potential kernel bounds for $\alpha \in [1,2)$:

In Theorem 3.5 we will prove that there exist explicit constants C_α, C_0, C_δ such that for $|x| \to \infty$ and $\delta := \min(R_\alpha)$ we have

(i) If $\delta < 2\alpha - 1$, then there exists a constant C_δ such that

$$a_X(0,x) = C_\alpha |x|^{\alpha-1} + C_\delta |x|^{2\alpha-\delta-1} + \mathcal{O}(|x|^{2\alpha-\delta-1}),$$

(ii) if $\delta > 2\alpha - 1$, then there exists a constant C_0 such that

$$a_X(0,x) = C_\alpha |x|^{\alpha-1} + C_0 + o(1),$$

(iii) if $\delta = 2\alpha - 1$, then there exists a constant C_δ such that

$$a_X(0,x) = C_\alpha |x|^{\alpha-1} + C_\delta \log|x| + \mathcal{O}(1).$$

as $x \to \infty$.

In particular for $R_\alpha \in \{\varnothing, \{2\}\}$ we prove in Theorems 3.4 and 3.6 that there exist constants $C_0, C_\alpha, ..., C_{m_\alpha}$ such that:

(i) if $\alpha \in (1,2)$ and $\kappa_2 = 0$ then

$$a_X(0,x) = C_\alpha |x|^{\alpha-1} + C_0 + \mathcal{O}(|x|^{\frac{\alpha-2}{3}+\varepsilon}),$$

for any ε small enough

(ii) if $\alpha \in (1,2)$ and $\kappa_2 \neq 0$ then

$$a_X(0,x) = C_\alpha |x|^{\alpha-1} + \sum_{m=1}^{m_\alpha} C_m |x|^{\alpha-1-m(2-\alpha)} + C_0' \log|x| + \mathcal{O}(1)$$

(iii) if $\alpha = 1$ we have

$$a_X(0,x) = C_\alpha \log(|x|) + C_0 + o(1),$$

where the term $o(1)$ can be estimated if $\kappa_2 = 0$.

The proofs of the potential kernel bounds are original and they exploit the asymptotics of the characteristic function together with Hölder continuity instead of using the LCLT as a starting point like in the classical case [22].

The novelty of the paper includes to study the expansion of the characteristic function in terms of regularity sets, sharp convergence bounds in the LCLT and explicit asymptotic expansion with error bounds of the potential kernel and characteristic function of a random walk for a class of heavy-tailed random variables whose characteristic function satisfies (1.3) which did not exist in the literature yet.

Structure of the article. In Section 2, we provide the setting and introduce necessary definitions. In Section 3, we state our main Theorems. The next section 4 deals with determining the expansion of the characteristic function for an explicit example of a long-range random walk and showing that it falls into the class we consider in this article. Section 5 contains all proofs regarding LCLT's and in Section 6 we demonstrate estimates on the discrete potential kernels. In Section 7, we present some final remarks on the possibility and limitations of generalising our techniques to the cases $\alpha < 1$ and/or $d \geq 2$, non-lattice and continuous time random walks. Some technical lemmas are postponed to the Appendix.

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2. Definitions

In this section we will introduce all necessary notation and define the main objects. We will denote by $\mathbb{T} = (-\pi, \pi]$ the one-dimensional torus. Given $z \in \mathbb{R}$ and $r > 0$, we write $B_r(z)$ to denote the interval $(z - r, z + r)$ around z with radius r. For $f, g : \mathbb{R} \to \mathbb{R}$ we write

$$f(x) \lesssim g(x)$$

if there exists a universal constant $C > 0$, which does not depend on x, such that $f(x) \leq Cg(x)$, analogously for $\gtrsim$. If f, g are such that $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$ we will write $f(x) \asymp g(x)$. The functions $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceil functions, respectively. We will write $\mathbb{Z}^+ := \{0\} \cup \mathbb{N}$.

Given finite sets of positive real numbers $A, B \subset \mathbb{R}^+$, we define its sum by

$$A + B := \{a + b : a \in A, b \in B\}.$$

and

$$\operatorname{span}(A) := \left\{ \sum_{a \in A} l_a a : l_a \in \mathbb{Z}^+, a \in A \right\}.$$

Let $C^{k,\gamma}(\mathbb{R})$ denote the space of function in $\mathbb{R}$ with $k \geq 0$ derivatives s.t the k-th derivative is γ- Hölder continuous with $\gamma \in (0, 1]$. We will denote by $C^{k,\gamma}(\mathbb{T})$ the subspace of $C^{k,\gamma}(\mathbb{R})$ composed by 2π-periodic functions. The notation $f \in C^{k,\gamma-}(\mathbb{T})$ will be used for $f \in C^{k,\gamma-\varepsilon}(\mathbb{T})$ for $\varepsilon > 0$ sufficiently small. Similarly we will use the short notation $f(x) = \mathcal{O}(|x|^{\beta\pm})$ for $f(x) = \mathcal{O}(|x|^{\beta\pm\varepsilon})$ for all $\varepsilon > 0$ sufficiently small, where $\mathcal{O}(\cdot)$ is the standard big-O notation.

We will call $\mathcal{F}$ the Fourier transform of f given by

$$\mathcal{F}(f)(\theta) := \int_{\mathbb{R}} f(x) e^{i\theta \cdot x} dx$$

for $\theta \in \mathbb{R}$ resp. $\mathcal{F}_{\mathbb{T}}$ for $k \in \mathbb{N}$

$$\mathcal{F}_{\mathbb{T}}(f)(k) := \int_{\mathbb{T}} f(x) e^{ik \cdot x} dx.$$

Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of i.i.d. random variables defined on some common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $p_X(\cdot)$ the probability distribution of X, with support in $\mathbb{Z}$ and assume that $p_X(-x) = p_X(x)$ for all $x \in \mathbb{Z}$. We write shorthand X instead of X_i when we refer to one single random variable. Call $S_n := \sum_{i=0}^{n} X_i$ its sum and abbreviate by $p_X^n(\cdot)$ the corresponding probability distribution. Denote by

$$\phi_X(\theta) := \mathbb{E}\left[e^{i\theta \cdot X}\right], \;\; \theta \in \mathbb{R}$$

its common characteristic function.

In the following let us define the class of random variables which we will consider in this article.

Definition 2.1. *Let $\alpha \in (0,2]$ and let $R_\alpha \subset (\alpha, 2+\alpha)$ be a finite set. We call the probability distribution $p_X(\cdot)$ of a symmetric random variable X with support in $\mathbb{Z}$ admissible of index α and regularity set R_α (or just admissible) if its corresponding characteristic function $\phi_X(\theta)$ admits the following expansion*

$$\phi_X(\theta) = 1 - \kappa_\alpha |\theta|^\alpha + \sum_{\beta \in R_\alpha} \kappa_\beta |\theta|^\beta + \mathcal{O}\Big(|\theta|^{2+\alpha}\Big) \tag{2.1}$$

as $|\theta| \longrightarrow 0$, for constants $\kappa_\alpha > 0$ and $\kappa_\beta \in \mathbb{R} \setminus \{0\}$, for all $\beta \in R_\alpha$.

It is important to recall that the constants $\kappa_\alpha, \kappa_\beta$, given in the definition above, depend on the law of $p_X(\cdot)$. However, to keep notation short, we omit to explicit this dependence.

Our *regularity set* R_α is a finite collection of powers of $|\theta|$ in the expansion of the characteristic function, up to orders which are strictly smaller than $2+\alpha$.

In order to obtain sharp convergence rates of the LCLT, expansions up to an error term of order $\mathcal{O}(|\theta|^{2\alpha})$ are enough. In fact, for the LCLT this order of the error term is optimal. Regarding the potential kernel estimates choosing an error of order $\mathcal{O}(|\theta|^{2+\alpha})$ improves the expansion compared to choosing $\mathcal{O}(|\theta|^{2\alpha})$. For us, choosing $\mathcal{O}(|\theta|^{2+\alpha})$ is a natural choice since it appears in the expansion of the characteristic function for the distribution of the step size of a long-range random walk, see Section 4.

Furthermore let

$$J_\alpha := \operatorname{span}(R_\alpha^+) \cap (\alpha, 2+\alpha), \tag{2.2}$$

where $R_\alpha^+ := R_\alpha \cup \{\alpha\}$. In a similar way we define $J_\alpha^+ := J_\alpha \cup \{2+\alpha\}$. Remark that if $R_\alpha = \emptyset$ we have that $J_\alpha = \alpha\mathbb{N} \cap (\alpha, 2+\alpha)$ and in particular $\beta_1 = \min(J_\alpha^+) \leq 2\alpha$ for any admissible distribution.

Using the expansion given in (2.1) and the Taylor polynomial of $\log(1+t)$ for $|t| < 1$, setting $t := \phi_X(\theta) - 1$, we get that $\phi_X(\cdot)$ can be written as

$$\phi_X(\theta) = e^{-\kappa_\alpha |\theta|^\alpha + r_X(\theta) + \mathcal{O}(|\theta|^{2+\alpha})}, \quad \text{as } |\theta| \longrightarrow 0, \tag{2.3}$$

where

$$r_X(\theta) = \sum_{j \in J_\alpha} \eta_j |\theta|^j,$$

and the coefficients η_j are combinations of coefficients coming from the expansion of the logarithm and the powers $|\theta|^\alpha$ resp. $|\theta|^\beta$. In particular, for $\alpha \in (1,2)$ and $R_\alpha = \{2\}$, we have $r_X(\theta) = \kappa_2 |\theta|^2 - \frac{(\kappa_\alpha)^2}{2} |\theta|^{2\alpha}$.

The class of admissible probability distributions should be seen as a natural and well-behaved collection of probability distributions in the domain of attraction of an α-stable distribution. Indeed, in the classical central limit theorem case, one usually requires finite moments of order 3 or 4 to study LCLT's. This can be understood as a convenient way of making assumptions about characteristic functions of such variables. Once the term $|\theta|^\alpha$

for $\alpha \in (0,2)$ appears in the expansion of ϕ_X, $\lceil \alpha \rceil$ moments cease to exist. Hence, in the stable case we need to make assumptions directly in the terms of the expansion of the characteristic function instead of their moments.

Remark that symmetric random variables with support in $\mathbb{Z}$ and finite fourth moment have an admissible distribution of index $\alpha = 2$ and $R_\alpha \in \{\varnothing, \{2\}\}$. Both LCLT and potential kernel estimates for such random variables are well understood, see [22]. For this reason, we will concentrate on the case $\alpha \in (0,2)$.

The class of admissible probability distributions is closed under natural operations. Let $p_{X_1}(\cdot)$ and $p_{X_2}(\cdot)$ be admissible distributions of independent X_1 and X_2 of indexes $\alpha_1, \alpha_2 \in (0,2]$, $\alpha_1 \le \alpha_2$ and regularity sets R_{α_1}, R_{α_2} respectively. We have that their convolution equal to

$$p_X(x) := p_{X_1} * p_{X_2}(x)$$

is admissible of index α_1 and regularity set

$$R'_{\alpha_1} \subset (R_{\alpha_1} + R^+_{\alpha_2}) \cap (\alpha_1, 2 + \alpha_1).$$

Moreover convex combinations

$$p_{\tilde{X}}(x) := q \cdot p_{X_1}(x) + (1-q) p_{X_2}(x) \tag{2.4}$$

for $q \in (0,1)$ are admissible of index α_1 and regularity set

$$R^*_\alpha \subset \Big(R_{\alpha_1} \cup R^+_{\alpha_2}\Big) \cap (\alpha_1, 2 + \alpha_1).$$

We can only write the regularity sets as subsets since there might be cancellations due to the convolution or convex combinations.

Note that $\tilde{X} := UX_1 + (1-U)X_2$ where U is a Bernoulli r.v. with parameter q, independent from X_1 and X_2, has distribution $p_{\tilde{X}}(\cdot)$.

Our main example of an admissible distribution of index $\alpha \in (0,2)$ and $R_\alpha = \{2\}$ is given by

$$p_\alpha(x) := \begin{cases} c_\alpha |x|^{-(1+\alpha)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \tag{2.5}$$

where c_α is a normalising constant. We will discuss this example in Section 4. However, using similar ideas, one can show that the distribution given by

$$\tilde{p}_\alpha(x) = \tilde{p}_\alpha(-x) := \frac{1}{2|x|^\alpha} - \frac{1}{2(|x|+1)^\alpha}, \text{ for } x \in \mathbb{Z} \setminus \{0\}$$

is admissible of index α and regularity set $R_\alpha := \{2, 1+\alpha\}$.

An example of a distribution which is *not admissible* is $p_\alpha(\cdot)$, defined in (2.5) with $\alpha = 2$. In fact, in this case the characteristic function has the expansion

$$\phi_X(\theta) = 1 - \kappa_2 |\theta|^2 \log|\theta| + \mathcal{O}(|\theta|^2).$$

Let $p_{\bar{X}}(\cdot)$ denote the density of a symmetric *α-stable random variable* $\bar{X}$ of index $\alpha \in (0,2)$ and scale parameter $c = (\kappa_\alpha)^{1/\alpha}$. Its with characteristic function is given by

$$\phi_{\bar{X}}(\theta) = e^{-\kappa_\alpha |\theta|^\alpha}. \tag{2.6}$$

Its n-th convolution will be abbreviated by $p^n_{\bar{X}}(\cdot)$. Notice that if $p_X(\cdot)$ is admissible, $n^{-\frac{1}{\alpha}} S_n$ converges to $\bar{X}$ in law. We will subdivide the class of admissible distributions in a subclass w.r.t. regularity sets $R_\alpha \in \{\varnothing, \{2\}\}$ and a subclass w.r.t. general R_α. The first subclass will be further subdivided in three classes which will have different asymptotic behaviour as $n \to \infty$.

Definition 2.2. *Let $p_X(\cdot)$ be admissible of index α with regularity set $R_\alpha \in \{\varnothing, \{2\}\}$. Then $p_X(\cdot)$ belongs to one of the following three classes:*

(i) *repaired if $R_\alpha = \emptyset$*
(ii) *locally repairable if $R_\alpha = \{2\}$ and $\kappa_2 > 0$*
(iii) *asymptotically repairable if $R_\alpha = \{2\}$ and $\kappa_2 < 0$.*

A *locally repairable* probability distribution $p_X(\cdot)$ can be *repaired* by convoluting it with a simple discrete random variable with variance $2|\kappa_2|$ which plays the part of a *repairer*. Analogously, we can repair an *asymptotically repairable* probability distribution $p_X(\cdot)$. This repairing is not performed on $p_X(\cdot)$ itself. Instead, we repair its asymptotic distribution $p_{\bar{X}}(\cdot)$ by convoluting $\bar{X}$ with a normal random variable with variance $2|\kappa_2|$. In both cases, the aim is to change either the original random variable X or its stable limit $\bar{X}$ in order to cancel the contribution from κ_2.

Definition 2.3. *Let $p_X(\cdot)$ be admissible of index $\alpha \in (0, 2)$ with regularity set $R_\alpha \in \{\varnothing, \{2\}\}$ and let κ_2 be the constant defined in the expansion of $\phi_X(\cdot)$.*

(i) *If $p_X(\cdot)$ is locally repairable, we call the repairer an independent random variable Z with probability distribution given by*

$$p_Z(x) = \begin{cases} \frac{\kappa_2}{M^2}, & \text{if } |x| = M \\ 1 - \frac{2\kappa_2}{M^2}, & \text{if } x = 0 \\ 0, & \text{otherwise,} \end{cases} \tag{2.7}$$

where $M = \lceil \sqrt{2\kappa_2} \rceil \in \mathbb{N}$.

(ii) *If $p_X(\cdot)$ is asymptotically repairable, we call an asymptotic repairer a random variable $\bar{Z}$ such that $\bar{Z} \sim \mathcal{N}(0, 2|\kappa_2|)$. $\bar{Z}$ and $\bar{X}$ are independent and $\bar{X}$ be a r.v. with characteristic function given by* (2.6).

By construction, the characteristic function of a repairer Z satisfies the expansion

$$\phi_Z(\theta) = 1 - \kappa_2 |\theta|^2 + \mathcal{O}(\theta^4), \qquad \text{as } |\theta| \longrightarrow 0.$$

It is easy to see that $p_{X+Z}(\cdot) = p_X * p_Z(\cdot)$ is in fact repaired. The asymptotic repairer $\bar{Z}$ is such that the characteristic function of $\bar{X} + \bar{Z}$ equal to

$$\phi_{\bar{X}+\bar{Z}}(\theta) = e^{-\kappa_\alpha |\theta|^\alpha - \kappa_2 |\theta|^2}.$$

Note that in both cases we do not change the limiting distribution of $n^{-1/\alpha}S_n$. Indeed, this modification will introduce an error of order $\mathcal{O}(n^{1-\frac{3}{\alpha}})$ which vanishes as $n \to \infty$.

Let us remark that alternatively one could *repair* by taking a convex combination as in [13]. Different repairing methods might be more convenient depending on the context.

Finally let us define the potential kernel for a random walk, whose transition probability $p_X(\cdot) := p_X(\cdot,\cdot)$ is admissible of index $\alpha \in [1,2)$ and regularity set R_α.

Definition 2.4. *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of symmetric, i.i.d. random variables such that $p_X(\cdot)$ is admissible. Call $S_n = \sum_{i=1}^n X_i$ and $p_X^n(\cdot,\cdot)$ its transition probability. Then the potential kernel is defined by*

$$a_X(0,x) = \sum_{n=0}^{\infty}(p_X^n(0,x) - p_X^n(0,0)), \quad \textit{for } x \in \mathbb{Z}.$$

3. Results

3.1. **Local central limit theorem.** In this section we state our results regarding LCLT's for heavy-tailed i.i.d. random variables with admissible probability distribution. First for the subclass $R_\alpha \in \{\varnothing, \{2\}\}$ and then for general R_α.

Theorem 3.1. *Let $\alpha \in (0,2)$ and $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with admissible law $p_X(\cdot)$ and $R_\alpha \in \{\varnothing, \{2\}\}$. Let furthermore $p_{\bar{X}}(\cdot)$ denote the law of the symmetric α-stable random variable with scale parameter $(\kappa_\alpha)^{1/\alpha}$, $p_Z(\cdot)$ the law of the repairer and $p_{\bar{Z}}(\cdot)$ the law of the asymptotic repairer. Then we have that,*

(i) *if $p_X(\cdot)$ is repaired,*
$$\sup_{x\in\mathbb{Z}} |p_X^n(x) - p_{\bar{X}}^n(x)| \lesssim n^{-(1+\frac{1}{\alpha})}.$$
(ii) *if $p_X(\cdot)$ is locally repairable,*
$$\sup_{x\in\mathbb{Z}} |p_{X+Z}^n(x) - p_{\bar{X}}^n(x)| \lesssim n^{-(1+\frac{1}{\alpha})}.$$
(iii) *if $p_X(\cdot)$ is asymptotically repairable,*
$$\sup_{x\in\mathbb{Z}} |p_X^n(x) - p_{\bar{X}+\bar{Z}}^n(x)| \lesssim n^{-(1+\frac{1}{\alpha})}.$$

The next Theorem gives LCLT convergence rates for admissible distributions w.r.t. general R_α.

Theorem 3.2. *Let $\alpha \in (0,2)$ and $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with common admissible law $p_X(\cdot)$. Let furthermore $p_{\bar{X}}(\cdot)$ denote the law of the symmetric α-stable random variable with scale parameter*

$(\kappa_\alpha)^{1/\alpha}$. *Then, there exists a collection of constants* $\{C_j, j\in J_\alpha\}$ *s.t. for all* $x\in\mathbb{Z}$,

$$\left| p_X^n(x) - p_{\tilde{X}}^n(x) - \sum_{j\in J_\alpha} C_j \frac{u_j\left(\frac{x}{n^{1/\alpha}}\right)}{n^{(1+j-\alpha)/\alpha}} \right| \lesssim n^{-\frac{3}{\alpha}}, \tag{3.1}$$

where

$$u_j(x) := \frac{1}{2\pi}\int_{\mathbb{R}} |\theta|^j e^{-\kappa_\alpha|\theta|^\alpha}\cos(\theta x)d\theta. \tag{3.2}$$

A careful analysis of the function $u_j(x)$ shows that

$$|u_j(x)| \lesssim \frac{1}{|x|^{\alpha+j+1}}. \tag{3.3}$$

Indeed, this bound is significantly weaker than its equivalent Theorem 2.3.7 in [22]. There, the integrands in (3.2) are given by $g_j(\theta) := \kappa_j\theta^j e^{-c|\theta|^2}$, and therefore $g_j(\cdot)$ are in Schwartz functions with rapidly decaying derivatives.

A simple triangular inequality leads us to the following corollary.

Corollary 3.3. *Under the conditions of Theorem 3.2, calling* $\beta_1 := \min(J_\alpha^+)$ *and* $\beta_2 := \min(J_\alpha^+\setminus\{\beta_1\})$, *we have that*

$$\left|p_X^n(x) - p_{\tilde{X}}^n(x)\right| = o\left(\sum_{j\in J_\alpha} C_j \frac{u_j\left(\frac{x}{n^{1/\alpha}}\right)}{n^{(1+j-\alpha)/\alpha}}\right).$$

In particular we have that

$$\sup_{x\in\mathbb{Z}}\left|p_X^n(x) - p_{\tilde{X}}^n(x)\right| \lesssim n^{-\frac{(\beta_1+1-\alpha)}{\alpha}}$$

and

$$\left|p_X^n(x) - p_{\tilde{X}}^n(x)\right| \lesssim \left(n^{-\frac{(\beta_2+1-\alpha)}{\alpha}}\right) \vee \left(n^{\frac{2}{\alpha}}|x|^{-(\alpha+\beta_1+1)}\right).$$

Note that from Corollary 3.3 we can deduce that the rate of convergence is sharp. More precisely we have seen that the speed is of order $\mathcal{O}(n^{-\gamma})$ where $\gamma = \frac{\beta_1+1-\alpha}{\alpha}$. If $p_X(\cdot)$ is repaired, then $\beta_1 = \min\{2\alpha, 2+\alpha\} = 2\alpha \geq 2$ which leads to $\gamma = \frac{\alpha+1}{\alpha}$. For $\alpha \geq 1$ and $p_X(\cdot)$ is locally repairable we have that $\beta_1 = \min\{2, 2\alpha, 2+\alpha\} = 2$. Without repairing, the best uniform bound we can get is

$$\left|p_X^n(x) - p_{\tilde{X}}^n(x)\right| \lesssim n^{1-3/\alpha},$$

which is much weaker than the bound in Theorem 3.1, especially for α close to 2. Theorem 3.1 states that repairing a probability distribution preserves the convergence rates. Note that for $\alpha < 1$, we have that $\beta_1 < 2$ so repairing will not provide better convergence bounds beyond the once in Corollary 3.3.

In Section 7, we discuss how one could potentially repair a distribution using heavy-tailed random variables instead of random variables with finite variance.

3.2. **Potential kernel estimates for long-range random walks.** The next Theorem 3.4 presents potential kernel estimates for long-range random walks with admissible law $p_X(\cdot)$. It exemplifies that repairing distributions provides good potential kernel expansions. This will be proven in Section 6. Note that the results in this Section hold for $\alpha \in [1,2)$. For further considerations on $\alpha < 1$, we refer to Section 7.

We will first treat the case $\alpha \in (1,2)$ and $\alpha = 1$ for the subclass described by $R_\alpha \in \{\varnothing, \{2\}\}$ separately. First we give bounds for repaired distributions when $\alpha \in (1,2)$, where we have an expansion up to some vanishing error as $|x| \to \infty$. After that we compute all terms of the expansion for locally and asymptotically repairable distributions up to order $\mathcal{O}(1)$. Finally, we present the general admissible case, in which we obtain the first and second terms of the expansion which will depend on $\delta := \min(R_\alpha)$.

Theorem 3.4. *Let $\alpha \in (1,2)$ and $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with common admissible distribution $p_X(\cdot)$ of index α and regularity set $R_\alpha \in \{\varnothing, \{2\}\}$.*

(i) *Assume that $p_X(\cdot)$ is repaired, then there exist constants $C_0, C_\alpha \in \mathbb{R}$ such that*

$$a_X(0,x) = C_\alpha |x|^{\alpha-1} + C_0 + \mathcal{O}(|x|^{\frac{\alpha-2}{3}+})$$

as $x \to \infty$, where

$$C_\alpha = \frac{1}{\pi\kappa_\alpha}\int_0^\infty \frac{\cos(\theta)-1}{\theta^\alpha}d\theta$$

and

$$C_0 = -\frac{\pi^{1-\alpha}}{2\pi\kappa_\alpha(\alpha-1)} + \frac{1}{\pi}\int_0^\pi \frac{\phi_X(\theta)-(1-\kappa_\alpha\theta^\alpha)}{\kappa_\alpha\theta^\alpha(1-\phi_X(\theta))}d\theta.$$

(ii) *Assume that $p_X(\cdot)$ is locally or asymptotically repairable. Let $m_\alpha := \lceil \frac{\alpha-1}{2-\alpha}\rceil - 1$, then there exist constants $C_0', C_1, \dots, C_{m_\alpha+1}$ such that*

$$a_X(0,x) = C_\alpha|x|^{\alpha-1} + \sum_{m=1}^{m_\alpha} C_m|x|^{(\alpha-1)-m(2-\alpha)} + C_0'\log|x| + \mathcal{O}(1)$$

as $|x| \to \infty$, where for $1 \le m \le m_\alpha + 1$

$$C_m := \frac{\kappa_2^m}{\pi\kappa_\alpha^{m+1}}\int_0^\infty \theta^{m(2-\alpha)-\alpha}(\cos(\theta)-1)d\theta,$$

and the sum is zero if $m_\alpha = 0$. Moreover,

$$C_0' := \begin{cases} 0, & \text{if } \frac{2}{2-\alpha} \notin \mathbb{N} \\ C_{m_\alpha+1}, & \text{if } \frac{2}{2-\alpha} \in \mathbb{N}. \end{cases}$$

Note that $m_\alpha \to \infty$ as $\alpha \to 2$, therefore, performing a repair (whenever possible) becomes more relevant for larger values of α. The following theorem treats the general admissible case.

Theorem 3.5. *Let $\alpha \in (1,2)$ and $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with common admissible distribution $p_X(\cdot)$ of index α and regularity set R_α. Let $\delta := \min(R_\alpha)$ and*

$$C_\alpha = \frac{1}{\pi\kappa_\alpha}\int_0^\infty \frac{\cos(\theta)-1}{\theta^\alpha}d\theta.$$

(i) *If $\delta < 2\alpha - 1$, then there exists a constant C_δ such that*

$$a_X(0,x) = C_\alpha|x|^{\alpha-1} + C_\delta|x|^{2\alpha-\delta-1} + \mathcal{O}(|x|^{2\alpha-\delta-1})$$

as $|x| \to \infty$, where

$$C_\delta = \frac{\kappa_\delta}{\pi\kappa_\alpha}\int_0^\infty \theta^{\delta-2\alpha}(\cos(\theta)-1)d\theta.$$

(ii) *If $\delta > 2\alpha - 1$, then there exists a constant C_0 such that*

$$a_X(0,x) = C_\alpha|x|^{\alpha-1} + C_0 + o(1).$$

as $x \to \infty$, where

$$C_0 = -\frac{\pi^{1-\alpha}}{2\pi\kappa_\alpha(\alpha-1)} + \frac{1}{\pi}\int_0^\pi \frac{\phi_X(\theta)-(1-\kappa_\alpha\theta^\alpha)}{\kappa_\alpha\theta^\alpha(1-\phi_X(\theta))}d\theta.$$

(iii) *If $\delta = 2\alpha - 1$, then there exists a constant C_δ such that*

$$a_X(0,x) = C_\alpha|x|^{\alpha-1} + C_\delta \log|x| + \mathcal{O}(1).$$

as $x \to \infty$, where

$$C_\delta := \frac{\kappa_\delta}{\pi\kappa_\alpha}\int_0^\pi \frac{\cos(\theta)-1}{\theta}d\theta$$

Finally, we include the result for the potential kernel for $\alpha = 1$, when $R_\alpha \in \{\varnothing, \{2\}\}$.

Theorem 3.6. *Let $\alpha = 1$ and $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with common admissible law $p_X(\cdot)$ and $R_\alpha \in \{\varnothing, \{2\}\}$. Then*

$$a_X(0,x) = -\frac{1}{\pi\kappa_1}\log|x| + C_0 + o\left(1\right).$$

where

$$C_0 := \frac{\gamma + \log\pi}{\pi\kappa_1},$$

and γ is the Euler-Mascheroni constant. Additionally, if $p_X(\cdot)$ is repaired, we have that the term $o(1)$ is in fact $\mathcal{O}\left(|x|^{-\frac{1}{3}+}\right)$.

4. Example: 1d long-range random walk

In this section we will discuss a typical example of an admissible probability distribution with index $\alpha \in (0,2)$ and regularity set $R_\alpha = \{2\}$.

Let $p_\alpha : \mathbb{Z} \times \mathbb{Z} \to [0,1]$ denote the transition probability of a long-range random walk in $\mathbb{Z}$, defined by $p_\alpha(x,y) = p_\alpha(x-y) = c_\alpha |x-y|^{-(1+\alpha)}$ where $x, y \in \mathbb{Z}$, $\alpha \in (0,2)$ and $c_\alpha > 0$ a normalising constant.

The characteristic function is equal to

$$\phi_\alpha(\theta) = c_\alpha \sum_{x \in \mathbb{Z}\setminus\{0\}} \frac{e^{ix\theta}}{|x|^{1+\alpha}}. \tag{4.1}$$

Proposition 4.1. *Let X denote the step size of the long-range random variable with probability distribution given by $p_\alpha(\cdot)$ defined above, $\alpha \in (0,2)$. The distribution $p_\alpha(\cdot)$ is admissible of index α and locally repairable, i.e. for $\alpha \neq 1$:*

$$\phi_\alpha(\theta) = 1 - \kappa_\alpha |\theta|^\alpha + \kappa_2 |\theta|^2 + \mathcal{O}(|\theta|^{2+\alpha}) \text{ as } |\theta| \to 0$$

with coefficients κ_α, κ_2 given by

$$\kappa_\alpha = -2c_\alpha \cos\left(\frac{\pi\alpha}{2}\right)\Gamma(-\alpha)$$

and

$$\kappa_2 = 2c_\alpha \left(\frac{1}{2(2-\alpha)} - \frac{1}{4} - K_2 \right)$$

where

$$K_2 = \frac{1-\alpha}{2}\left(\left(\frac{2^{2-\alpha}-1}{2-\alpha} - \frac{3(2^{1-\alpha}-1)}{2(1-\alpha)} \right) + \frac{1}{2\Gamma(\alpha)} \sum_{m=1}^{\infty} (-1)^m (\zeta(m+\alpha)-1) \frac{m\Gamma(m+\alpha)}{\Gamma(m+2)(m+2)} \right),$$

with $\zeta(\cdot)$ denoting the zeta function and $\Gamma(\cdot)$ the Gamma function. In the case $\alpha = 1$ we have that

$$\phi_1(\theta) = 1 - \frac{3}{\pi}|\theta| + \frac{3}{2\pi^2}|\theta|^2 + \mathcal{O}(|\theta|^3) \text{ as } |\theta| \to 0.$$

Proof. To prove this statement, we will use the Euler-Maclaurin formula [2], which states that for a given smooth function $f \in C^\infty(\mathbb{R})$, we have that

$$\sum_{x=1}^{M} f(x) - \int_1^M f(x)dx = \frac{f(1)+f(M)}{2} + R_\alpha^M, \tag{4.2}$$

where the remainder term R_α^M can be computed explicitly by

$$R_\alpha^M = \int_1^M f'(x) P_1(x) dx,$$

and $P_1(x) = B_1(x - \lfloor x \rfloor)$ with $B_1(\cdot)$ being the first periodized Bernoulli function, that is: $P_1(x) = (x - \lfloor x \rfloor) - \frac{1}{2}$. We will apply this formula to the function $f(x) = \frac{1-\cos(\theta x)}{|x|^{1+\alpha}}$. Without loss of generality, we assume that $\theta > 0$. The left-hand side of (4.2) becomes

$$\sum_{x=1}^{M} \frac{1-\cos(\theta x)}{x^{1+\alpha}} - \int_1^M \frac{1-\cos(\theta x)}{x^{1+\alpha}} dx.$$

Notice that, as we let M go to infinity, we get that the expression above converges to

$$\frac{1-\phi_\alpha(\theta)}{2c_\alpha} - \int_1^\infty \frac{1-\cos(\theta x)}{x^{1+\alpha}} dx, \tag{4.3}$$

where c_α was the normalising constant used in the definition of $p_\alpha(\cdot)$. By a change of variables $z = x\theta$ in the above integral, we get

$$\frac{1-\phi_\alpha(\theta)}{2c_\alpha} - \theta^\alpha \int_\theta^\infty \frac{1-\cos(z)}{z^{1+\alpha}} dz.$$

For $\alpha \in (0,2) \setminus \{1\}$, we can write

$$\int_0^\infty \frac{1-\cos(z)}{z^{1+\alpha}} dz = -\cos\Big(\frac{\pi\alpha}{2}\Big)\Gamma(-\alpha) > 0,$$

so, by writing

$$\begin{aligned}\theta^\alpha \int_\theta^\infty \frac{1-\cos(z)}{z^{1+\alpha}} dz &= \theta^\alpha \int_0^\infty \frac{1-\cos(z)}{z^{1+\alpha}} dz - \theta^\alpha \int_0^\theta \frac{1-\cos(z)}{z^{1+\alpha}} dz \\ &= \theta^\alpha\Big(-\cos\Big(\frac{\pi\alpha}{2}\Big)\Gamma(-\alpha)\Big) - \frac{1}{2(2-\alpha)}\theta^2 + \mathcal{O}(\theta^4),\end{aligned}$$

where in the last line we used a simple Taylor expansion of $\cos(\cdot)$.

Now we turn to the right-hand side of (4.2). Note that $f(M) \to 0$ as $M \to \infty$. Hence

$$\begin{aligned}\lim_{M\to\infty} \frac{f(1)+f(M)}{2} + R_\alpha^M &= \frac{1}{2}(1-\cos(\theta)) + R_\alpha^\infty \\ &= \frac{1}{4}\theta^2 + \mathcal{O}(\theta^4) + R_\alpha^\infty,\end{aligned}$$

where

$$R_\alpha^\infty = \theta^{1+\alpha} \int_\theta^\infty \Big(\frac{z\sin z - (1+\alpha)(1-\cos(z))}{z^{2+\alpha}}\Big) P_1\Big(\frac{z}{\theta}\Big) dz. \tag{4.4}$$

We explore this integral in more detail in Lemma A.1, in which we prove that

$$R_\alpha^\infty = K_2\theta^2 + \mathcal{O}(\theta^{2+\alpha}),$$

where K_2 is a constant depending on α which is defined in (A.2). We will first focus on $\alpha > 1$ and express κ_2 as

$$\kappa_2 = 2c_\alpha \left(\frac{1}{2(2-\alpha)} - \frac{1}{4} - K_2 \right).$$

To complete the proof that $\kappa_2 > 0$, we need to examine K_2. As $\alpha > 1$, for $m \geq 1$, we have $m + \alpha > 2$ and therefore

$$\begin{aligned}
\zeta(m+\alpha) - 1 &= \frac{1}{2^{m+\alpha}} + \sum_{k \geq 3} \frac{3^{m+\alpha}}{3^{m+\alpha}} \frac{1}{k^{m+\alpha}} \\
&\leq \frac{1}{2^{m+\alpha}} + \frac{1}{3^{m+\alpha}} \sum_{k \geq 3} \left(\frac{3}{k}\right)^2 \\
&\leq \frac{1}{2^{m+\alpha}} \left(1 + 9\Big(\zeta(2) - \frac{5}{4}\Big) \right) \leq \frac{5}{2^{m+\alpha}},
\end{aligned}$$

where $\zeta(z)$ is the zeta-function. Moreover, using Gautschi's inequality for the ratio of two Gamma functions, see e.g. [28], we can write

$$(m+2)^{\alpha-2} < \frac{\Gamma(m+\alpha)}{\Gamma(m+2)} < (m+1)^{\alpha-2} < m^{\alpha-2}.$$

The upper bound on K_2 will follow from the lower bound on $\frac{K_2}{1-\alpha}$. We remove all even summands m in the definition of K_2 and bound further

$$\begin{aligned}
\frac{2K_2}{1-\alpha} \geq &\Bigg(\left(\frac{2^{2-\alpha}-1}{2-\alpha} - \frac{3(2^{1-\alpha}-1)}{2(1-\alpha)} \right) \\
&\quad - \frac{1}{2\Gamma(\alpha)} \sum_{m=0}^{\infty} (\zeta(2m+1+\alpha) - 1) \frac{(2m+1)\Gamma(2m+1+\alpha)}{\Gamma(2m+3)(2m+3)} \Bigg) \\
\geq &\left(\left(\frac{2^{2-\alpha}-1}{2-\alpha} - \frac{3(2^{1-\alpha}-1)}{2(1-\alpha)} \right) - \frac{5}{2\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{(2m+2)^{\alpha-2}}{2^{2m+1+\alpha}} \right) \\
\geq &\left(\Big(\frac{2^{2-\alpha}-1}{2-\alpha} - \frac{3(2^{1-\alpha}-1)}{2(1-\alpha)} \Big) - \frac{5}{12\Gamma(\alpha)} \right).
\end{aligned}$$

Call $u : (0,2) \to \mathbb{R}$ the map

$$t \mapsto \frac{1-t}{2} \left(\Big(\frac{2^{2-t}-1}{2-t} - \frac{3(2^{1-t}-1)}{2(1-t)} \Big) - \frac{5}{12\Gamma(t)} \right) \tag{4.5}$$

which is increasing for $t > 1$ and simple analysis shows that $u(t)$ is bounded from above by $\frac{1}{4}$. Now we collect all previous contributions to the constant κ_2 and show that the sum above cannot flip the sign. This concludes that

$$\kappa_2 = 2c_\alpha\left(\frac{1}{2(2-\alpha)} - \frac{1}{4} - K_2\right) > \frac{(\alpha-1)c_\alpha}{2-\alpha}$$

is positive for $\alpha > 1$.

For $\alpha < 1$, the strategy is similar, only this time, we proceed to get a function $u'(\cdot)$ similar to (4.5) but bounding $\frac{K_2}{2(1-\alpha)}$ from below (as $1-\alpha$ is now positive).

For the case $\alpha = 1$ the analysis becomes much simpler. This is because the first order term in (A.1) vanishes. Since $\alpha = 1$, the terms $\theta^{1+\alpha}$ and θ^2 collapse to the same term. The normalization constant is equal to $c_1 = \frac{1}{2\zeta(2)} = \frac{3}{\pi^2}$.

Again, using Euler-Maclaurin we get that, for $\theta > 0$

$$\frac{1-\phi_1(\theta)}{2c_1} - \int_1^\infty \frac{1-\cos(\theta x)}{x^2}dx = \frac{1-\cos(\theta)}{2} + R_1^\infty, \tag{4.6}$$

where the remainder term will be of order

$$R_1^\infty = \int_1^\infty \left(\frac{1-\cos(\theta\cdot)}{(\cdot)^2}\right)'(x)P_p(x)dx = \mathcal{O}(\theta^3).$$

Since

$$\int_0^\infty \frac{1-\cos(z)}{z^2}dz = \frac{\pi}{2},$$

we can write

$$\begin{aligned}\theta\int_\theta^\infty \frac{1-\cos(z)}{z^2}dz &= \theta\int_0^\infty \frac{1-\cos(z)}{z^2}dz - \theta\int_0^\theta \frac{1-\cos(z)}{z^2}dz\\ &= \frac{\pi}{2}\theta - \frac{1}{2}\theta^2 + \mathcal{O}(\theta^4)\end{aligned}$$

where in the last line we used a simple Taylor expansion. Collecting all coefficients corresponding to the powers of θ we obtain the result.

□

5. Proofs of Local Central Limit Theorems

In this section we will prove Theorems 3.1 and 3.2.

Proof of Theorem 3.1. We will prove cases (i) and (iii) since case (ii) is a corollary of case (i).

Case (i): $p_X(\cdot)$ repaired

Consider $(X_i)_{i\in\mathbb{N}}$ resp. a sequence of symmetric i.i.d. α-stable random variables $(\bar{X}_i)_{i\in\mathbb{N}}$ with scale parameter $(\kappa_\alpha)^{1/\alpha}$ and laws $p_X(\cdot)$ resp. $p_{\bar{X}}(\cdot)$. Let $S_n = \sum_{i=1}^n X_i$ resp. $\bar{S}_n = \sum_{i=1}^n \bar{X}_i$ with probability distributions denoted by $p_X^n(\cdot)$ resp. $p_{\bar{X}}^n(\cdot)$. We want to compare the probability distributions $p_X^n(\cdot)$

and $p^n_{\tilde{X}}(\cdot)$ using their representation in terms of inverse Fourier transforms. More precisely we have that

$$p^n_X(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi^n_X(\theta)e^{-ix\theta}d\theta$$

resp.

$$p^n_{\tilde{X}}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-n\kappa_\alpha|\theta|^\alpha}e^{-i\theta\cdot x}d\theta.$$

Using a change of variable formula, we get

$$p^n_X(x) = \frac{1}{2\pi n^{1/\alpha}}\int_{-\pi n^{1/\alpha}}^{\pi n^{1/\alpha}} \phi^n_X\left(\frac{\theta}{n^{1/\alpha}}\right)e^{-ix\frac{\theta}{n^{1/\alpha}}}d\theta.$$

Given $\varepsilon > 0$, notice that $\sup_{\theta\in\mathbb{T}\setminus[-\varepsilon,\varepsilon]}|\phi_X(\theta)| < 1$, as X is supported in $\mathbb{Z}$, see [22, Lemma 2.3.2]. To get

$$p^n_X(x) = \frac{1}{2\pi n^{1/\alpha}}\int_{-\varepsilon n^{1/\alpha}}^{\varepsilon n^{1/\alpha}} \phi^n_X\left(\frac{\theta}{n^{1/\alpha}}\right)e^{-ix\frac{\theta}{n^{1/\alpha}}}d\theta + \mathcal{O}(e^{-cn})$$

for some positive constant $c > 0$. Analogously, we have that

$$\begin{aligned}
p^n_{\tilde{X}}(x) &= \frac{1}{2\pi n^{1/\alpha}}\int_{-\infty}^{\infty} e^{-\kappa_\alpha|\theta|^\alpha}e^{-\frac{ix\theta}{n^{1/\alpha}}}d\theta \\
&= \frac{1}{2\pi n^{1/\alpha}}\int_{-\varepsilon n^{1/\alpha}}^{\varepsilon n^{1/\alpha}} e^{-\kappa_\alpha|\theta|^\alpha}e^{-ix\frac{\theta}{n^{1/\alpha}}}d\theta \\
&\quad + \frac{1}{2\pi n^{1/\alpha}}\int_{|\theta|>\varepsilon n^{1/\alpha}} e^{-\kappa_\alpha|\theta|^\alpha}e^{-ix\frac{\theta}{n^{1/\alpha}}}d\theta.
\end{aligned}$$

One can easily check that,

$$\int_{|\theta|>\varepsilon n^{1/\alpha}} e^{-\kappa_\alpha|\theta|^\alpha}e^{-\frac{ix\theta}{n^{1/\alpha}}}d\theta = \mathcal{O}(e^{-c'n}),$$

for some constant $c' > 0$. Write $\phi^n_X\left(\frac{\theta}{n^{1/\alpha}}\right) = [1 + F_n(\theta)]e^{-\kappa_\alpha|\theta|^\alpha}$.

Hence, we can concentrate our efforts into bounding

$$\int_{-\varepsilon n^{1/\alpha}}^{\varepsilon n^{1/\alpha}} F_n(\theta)e^{-\kappa_\alpha|\theta|^\alpha}e^{-\frac{ix\theta}{n^{1/\alpha}}}d\theta. \tag{5.1}$$

Now we write $F_n(\theta) = e^{g_n(\theta)} - 1$ which is formally equal to

$$F_n(\theta) = \sum_{k=1}^{\infty}\frac{1}{k!}\left(\mathcal{O}\left(\frac{|\theta|^{2+\alpha}}{n^{2/\alpha}}\right)\right)^k$$

and use that for $|\theta| < \varepsilon n^{1/\alpha}$ (possibly for smaller value of ε), we have for every $k \geq 1$

$$\left(\mathcal{O}\left(\frac{|\theta|^{2+\alpha}}{n^{2/\alpha}}\right)\right)^k \lesssim \frac{|\theta|^{2\alpha}}{n}.$$

Note that the error term does not depend on k. With this, we get

$$\begin{aligned}\left|p_X^n(x)-p_{\bar X}^n(x)\right| &= \Big|\frac{1}{2\pi n^{1/\alpha}}\int_{|\theta|<\varepsilon n^{1/\alpha}} e^{-\frac{ix\cdot\theta}{n^{1/\alpha}}} e^{-\kappa_\alpha|\theta|^\alpha}F_n(\theta)d\theta\Big| \\ &\lesssim \frac{1}{n^{1+1/\alpha}}\underbrace{\int_{|\theta|<\varepsilon n^{1/\alpha}} e^{-\kappa_\alpha|\theta|^\alpha}|\theta|^{2\alpha}d\theta}_{\mathcal{O}(1)}\end{aligned}$$

and that the integral on the r.h.s. is bounded as $n \longrightarrow \infty$.

Case (iii): $p_X(\cdot)$ **asymptotically repairable**

We will prove the statement in a similar manner, so we will only highlight the main differences. Write

$$\begin{aligned}p_{\bar X+\bar Z}^n(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-n\kappa_\alpha|\theta|^\alpha-n\kappa_2|\theta|^2}e^{-ix\theta}d\theta \\ &= \frac{1}{2\pi n^{1/\alpha}}\int_{-\infty}^{\infty} e^{-\kappa_\alpha|\theta|^\alpha-n^{(1-2/\alpha)}\kappa_2|\theta|^2}e^{-\frac{ix\theta}{n^{1/\alpha}}}d\theta\end{aligned}$$

and write $\phi_X^n\left(\frac{\theta}{n^{1/\alpha}}\right) = [1+F_n(\theta)]\exp\left(-\kappa_\alpha|\theta|^\alpha - n^{(1-2/\alpha)}\kappa_2|\theta|^2\right)$. Notice that $1-\frac{2}{\alpha}<0$.

One can easily check that,

$$\int_{|\theta|>\varepsilon n^{1/\alpha}} e^{-\kappa_\alpha|\theta|^\alpha-n^{1-\frac{2}{\alpha}}\kappa_2|\theta|^2}e^{-ix\frac{\theta}{n^{1/\alpha}}}d\theta = \mathcal{O}(e^{-cn}),$$

for some constant $c>0$. The statement will follow once we bound

$$\int_{-\varepsilon n^{1/\alpha}}^{\varepsilon n^{1/\alpha}} F_n(\theta)e^{-\kappa_\alpha|\theta|^\alpha-n^{1-\frac{2}{\alpha}}\kappa_2|\theta|^2}e^{-ix\frac{\theta}{n^{1/\alpha}}}d\theta \lesssim n^{-1/\alpha}.$$

Analogously to before write $F_n(\theta)=e^{g_n(\theta)}-1$ and note that for $|\theta|\le\varepsilon n^{1/\alpha}$, we have

$$|F_n(\theta)| \lesssim \frac{|\theta|^{2\alpha}}{n}.$$

This concludes the claim.

□

We proceed with the proof of Theorem 3.2.

Proof of Theorem 3.2. Using similar ideas as before in the proof of Theorem 3.1, assume again that $\theta>0$, we write

$$p_X^n(x) = \frac{1}{2\pi n^{1/\alpha}}\int_{-\varepsilon n^{1/\alpha}}^{\varepsilon n^{1/\alpha}}[1+F_n(\theta)]e^{-\kappa_\alpha|\theta|^\alpha}e^{-ix\theta n^{-\frac{1}{\alpha}}}d\theta + \mathcal{O}(e^{-cn^{1/\alpha}})$$

for some positive constant $c>0$. We have that

$$F_n(\theta) = \sum_{j\in J_\alpha} C_j\frac{n}{n^{j/\alpha}}|\theta|^j + \mathcal{O}\left(\frac{|\theta|^{2+\alpha}}{n^{2/\alpha}}\right), \tag{5.2}$$

where we used the Taylor polynomial of

$$t \mapsto e^{\sum_{\beta \in R_\alpha} n^{1-\beta/\alpha} \kappa_\beta t^\beta}$$

truncated at level $\mathcal{O}(\frac{t^{2+\alpha}}{n^{2/\alpha}})$.

Define

$$u_j(x) := \frac{1}{2\pi} \int_{\mathbb{R}} |\theta|^j e^{-\kappa_\alpha |\theta|^\alpha} \cos(\theta x) d\theta,$$

hence we have that for $|\theta| < \varepsilon n^{1/\alpha}$

$$\begin{aligned} &\left| p_X^n(x) - p_{\bar{X}}^n(x) - \sum_{j \in J_\alpha} C_j \frac{u_j\left(\frac{x}{n^{1/\alpha}}\right)}{n^{(1+j-\alpha)/\alpha}} \right| \\ &\lesssim \int_{-\varepsilon n^{1/\alpha}}^{\varepsilon n^{1/\alpha}} \frac{|\theta|^{2+\alpha} e^{-\kappa_\alpha |\theta|^\alpha}}{n^{3/\alpha}} d\theta + \sum_{j \in J_\alpha} C_j \int_{\mathbb{R} \setminus [-\varepsilon n^{1/\alpha}, \varepsilon n^{1/\alpha}]} \frac{|\theta|^j e^{-\kappa_\alpha |\theta|^\alpha}}{n^{(1+j-\alpha)/\alpha}} d\theta, \\ &\lesssim n^{-3/\alpha} + \mathcal{O}\left(e^{-cn}\right) \end{aligned}$$

for some $c > 0$ and n large enough.

□

6. Proofs for Potential Kernel expansion

In this section we will develop potential kernel estimates stated in Theorems 3.4 and 3.6. The strategy will be to use detailed knowledge of the expansion $\phi_X(\cdot)$ and not the LCLT theorem as was done for the equivalent problem in the classical case in [22].

Proof of Theorem 3.4. **Case (i) $p_X(\cdot)$ repaired**
Let us evaluate the expression

$$a_X(0, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1 - \phi_X(\theta)} (\cos(\theta x) - 1) d\theta.$$

The idea is to compare $a_X(0, x)$ with the potential kernel $a_{\bar{X}}(\cdot, \cdot)$ of a symmetric stable process $(\bar{X}_t)_{t \geq 0}$ with multiplicative constant κ_α whose characteristic function is given by $\phi_{\bar{X}_t}(\theta) = e^{-\kappa_\alpha t |\theta|^\alpha}$. This is more convenient since it can be explicitly computed. Using that $(t, \theta) \mapsto e^{-\kappa_\alpha t |\theta|^\alpha} (\cos(\theta x) - 1)$ is in $L^1(\mathbb{R}_+ \times \mathbb{R})$, we can use Fubini to get

$$\begin{aligned} a_{\bar{X}}(0, x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^\infty e^{-t\kappa_\alpha |\theta|^\alpha} dt (\cos(\theta x) - 1) d\theta \\ &= \left(\frac{1}{2\pi \kappa_\alpha} \int_{\mathbb{R}} \frac{1}{|\theta|^\alpha} (\cos(\theta) - 1) d\theta \right) |x|^{\alpha - 1} \end{aligned}$$

which gives the expression for the constant C_α. We write

$$a_X(0, x) = a_{\bar{X}}(0, x) + \underbrace{\left(a_X(0, x) - a'_{\bar{X}}(0, x)\right)}_{A} + \underbrace{\left(a_{\bar{X}}(0, x) - a'_{\bar{X}}(0, x)\right)}_{B},$$

where

$$a'_{\bar{X}}(0,x) := \frac{1}{2\pi\kappa_\alpha}\int_{-\pi}^{\pi}\frac{1}{|\theta|^\alpha}(\cos(\theta x)-1)d\theta.$$

The reminder of the proof is divided into two parts: estimating the term in A by using Hölder continuity and then the term in B by using an interplay of Fourier transform in the torus $\mathbb{T}$ and in $\mathbb{R}$ plus a trick involving dyadic partitions of the unity.

We start by analysing the term

$$\begin{aligned} a_X(0,x) - a'_{\bar{X}}(0,x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{1-\phi_X(\theta)} - \frac{1}{\kappa_\alpha|\theta|^\alpha}\right)(\cos(\theta x)-1)d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_X(\theta)}{\kappa_\alpha|\theta|^\alpha(1-\phi_X(\theta))}(\cos(\theta x)-1)d\theta. \end{aligned}$$

Remember that h_X was defined as

$$h_X(\theta) := \phi_X(\theta) - (1-\kappa_\alpha|\theta|^\alpha) = \mathcal{O}(|\theta|^{2+\alpha}).$$

since $p_X(\cdot)$ is repaired.

It is important to notice that $h_X(\theta)$ is in $C^{1,\alpha-1-}(\mathbb{T})$ due to Lemma B.2 and the continuity of $\theta \mapsto 1-\kappa_\alpha|\theta|^\alpha$. Denote by $\tilde{h}_X(\theta) := \frac{h_X(\theta)}{\kappa_\alpha|\theta|^\alpha(1-\phi_X(\theta))}$ which is in $L^1(\mathbb{T})$, as $(1-\phi_X(\theta)) \neq 0$ for all $\theta \in \mathbb{T}\setminus\{0\}$ again due to the fact that X is supported in $\mathbb{Z}$.

Hence, we write for A

$$a_X(0,x) - a'_{\bar{X}}(0,x) = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{h}_X(\theta)d\theta + \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{h}_X(\theta)\cos(\theta x)d\theta}_{I(x)}.$$

The first integral in the r.h.s. is finite and does not depend on x. We will show that the second integral on the r.h.s. above is of order $\mathcal{O}(|x|^{\frac{\alpha-2}{3+\varepsilon}})$.

This estimate is based on the fact that such integrals are Fourier coefficients of a function in $C^{0,\frac{2-\alpha}{3+\varepsilon}}(\mathbb{T})$ for some $\varepsilon > 0$ small enough.

We write

$$f_1(\theta) := \frac{h_X(\theta)}{|\theta|^{2\alpha}}$$

and

$$f_2(\theta) := \frac{|\theta|^\alpha\left(\kappa_\alpha|\theta|^\alpha - h_X(\theta)\right)}{|\theta|^{2\alpha}} = \kappa_\alpha - \frac{h_X(\theta)}{|\theta|^\alpha}.$$

Now, we use Lemma B.1 to determine the degree of Hölder continuity of $f_1(\cdot)$ and $f_2(\cdot)$. For $f_1(\cdot)$ we can choose $\beta = \alpha-1-\varepsilon$ for any $\varepsilon \in (0,\alpha-1)$, $\beta_0 = 2+\alpha$ and $\beta_1 = 2\alpha$ to obtain that $f_1(\cdot)$ is Hölder continuous with $\alpha_1 = \frac{2-\alpha}{3+\varepsilon}$ for $\alpha > 1$. For $f_2(\cdot)$, we can choose $\beta = \alpha-1-\varepsilon$, $\beta_0 = 2+\alpha$ and $\beta_1 = \alpha$ which yields to an order $\alpha_2 = \frac{2}{3+\varepsilon}$. Since $f_2(\cdot)$ is bounded away from 0 we have that the reciprocal $1/f_2(\cdot)$ is Hölder continuous of order α_2 as well. Therefore the product $f_1(\cdot)\cdot\frac{1}{f_2(\cdot)}$ is Hölder continuous of order $\alpha_1 \wedge \alpha_2 = \alpha_1$. This implies that $I(x) = \mathcal{O}(|x|^{-\alpha_1})$, see [15, Theorem 3.3.9].

For the second part of the proof, we estimate the term $B = a_{\bar{X}}(0,x) - a'_{\bar{X}}(0,x)$. To do so, let $\varphi \in C^\infty(\mathbb{R})$ be a symmetric cutoff function such that $\varphi \equiv 1$ in $\mathbb{R} \setminus [-\pi+\eta, \pi-\eta]$ for some arbitrarily small $\eta > 0$ and such that $\varphi \equiv 0$ in $[-\pi+2\eta, \pi-2\eta]$, we now have

$$2\pi\kappa_\alpha \left[a_{\bar{X}}(0,x) - a'_{\bar{X}}(0,x)\right] = \int_{\mathbb{R}\setminus\mathbb{T}} \frac{1}{|\theta|^\alpha}(\cos(\theta x) - 1)d\theta$$
$$= -\underbrace{\int_{\mathbb{R}\setminus[-\pi,\pi]} \frac{1}{|\theta|^\alpha} d\theta}_{\frac{\pi^{1-\alpha}}{\alpha-1}} + \underbrace{\int_{\mathbb{R}} \varphi(\theta)\frac{1}{|\theta|^\alpha}\cos(\theta x)d\theta}_{J_1(x)}$$
$$+ \underbrace{\int_{\mathbb{R}} \left[1_{[|\theta|>\pi]} - \varphi(\theta)\right]\frac{1}{|\theta|^\alpha}\cos(\theta x)d\theta}_{J_2(x)}.$$

The constant $-\frac{\pi^{1-\alpha}}{2\pi\kappa_\alpha(\alpha-1)}$ gives the second contribution to C_0. We write $J_1(x) = \mathcal{F}\left(\frac{\varphi(\cdot)}{|\cdot|^\alpha}\right)(x)$,

In order to analyse $J_1(x)$ we need to use a dyadic partition of the unity to show that this term decays faster than any polynomial. Let ψ_{-1}, ψ_0 be two radial functions such that $\psi_{-1} \in C_c^\infty(B_\pi(0))$ and $\psi_0 \in C_c^\infty(B_{2\pi}(0) \setminus B_\pi(0))$. It satisfies

$$1 \equiv \psi_{-1}(\theta) + \sum_{j=0}^{\infty} \underbrace{\psi_0(2^{-j}\theta)}_{=:\psi_j(\theta)}. \tag{6.1}$$

Such functions exist by Proposition 2.10 in [3], it is an application of Littlewood-Payley theory. Define

$$\mu(\theta) := \frac{\varphi(\theta)}{|\theta|^\alpha}\psi_{-1}(\theta) \qquad \text{and} \qquad \nu(\theta) := \frac{\varphi(\theta)}{|\theta|^\alpha}\psi_0(\theta) \equiv \frac{1}{|\theta|^\alpha}\psi_0(\theta),$$

where, in the identity, we used that $\varphi \equiv 1$ in the $\mathrm{supp}(\psi_0)$. We have that both $\mu, \nu \in C_c^\infty(\mathbb{R})$ and therefore their Fourier transforms decay faster than any polynomial, that is, for any $N > 1$, we have that

$$\mathcal{F}(\nu)(x), \mathcal{F}(\mu)(x) = \mathcal{O}(|x|^{-N}). \tag{6.2}$$

The fact that we can exchange the infinite sum with the Fourier transform is a result of the dominated convergence theorem.

Multiply both sides of (6.1) by $\varphi(\theta)/|\theta|^\alpha$, compute $\mathcal{F}$ and use the scaling property of the Fourier transform to get

$$J_1(x) = \mathcal{F}(\mu)(x) + \sum_{j=0}^{\infty} 2^{(1-\alpha)j}\mathcal{F}(\nu)(2^j x). \tag{6.3}$$

By using (6.2) and (6.3), we get that $J_1(x) = \mathcal{O}(|x|^{-N})$ for all $N \geq 1$. Finally we estimate $J_2(x)$

$$\begin{aligned} J_2(x) &= \int_{-\pi}^{\pi} \left[1_{[|\theta|>\pi]} - \varphi(\theta)\right] \frac{1}{|\theta|^\alpha} \cos(\theta x) d\theta \\ &= -\int_{-\pi}^{\pi} \varphi(\theta) \frac{1}{|\theta|^\alpha} \cos(\theta x) d\theta \end{aligned}$$

where we used that $\varphi \equiv 1$ for $|x| > \pi$. We can write $J_2(x) = \mathcal{F}_{\mathbb{T}}(g)(x)$. Notice that g is $C^{0,1}(\mathbb{T})$, and therefore $J_2(x)$ decays as $\mathcal{O}(|x|^{-1})$ which is faster than $\mathcal{O}(|x|^{\frac{\alpha-2}{3+\varepsilon}})$ because $\alpha \in (1,2)$. This concludes the proof of the second part. Note that alternatively we could have interpreted the integral $a_{\bar{X}}(\cdot,\cdot) - a'_X(\cdot,\cdot)$ as a generalized hypergeometric function and study its series expansion which is more implicit. We preferred this more explicit way as it seems more feasible to generalise to higher dimensions.

Case (ii) $p_X(\cdot)$ **locally or asymptotically repairable**
Here we follow a similar idea as in case (ii). Write again

$$a_X(0,x) = \left(a_X(0,x) - a'_{\bar{X}}(0,x)\right) + \left(a_{\bar{X}}(0,x) - a'_{\bar{X}}(0,x)\right) + a'_{\bar{X}}(0,x).$$

The last two terms are exactly the same as in the proof of (i). However, the first term behaves differently due the presence of $\kappa_2|\theta|^2$. We have that

$$\frac{1}{(1-\phi_X(\theta))} - \frac{1}{\kappa_\alpha|\theta|^\alpha} = \frac{h_X(\theta)}{\kappa_\alpha|\theta|^\alpha(1-\phi_X(\theta))} = \mathcal{O}\left(|\theta|^{2-2\alpha}\right) \tag{6.4}$$

as $|\theta| \to 0$, which blows up slower than $\mathcal{O}(|\theta|^{-\alpha})$ for any $\alpha < 2$. The main idea is to perform a telescopic sum together with expression (6.4) until we get a function in $L^1(\mathbb{T})$, which will require exactly m_α iterations.

Note that, in this proof we are only interested in characterising the potential kernel up to a constant order, therefore, we will not need to use information on the degree of continuity of a remainder term as in previous proofs. Instead, we will compute the first m_α terms by hand an use that the remainder is in $L^1(\mathbb{T})$, for which an application of the Riemann-Lebesgue Lemma [15, Proposition 3.3.1] will be enough.

Let

$$a_X(0,x) - a'_{\bar{X}}(0,x) = \frac{1}{2\kappa_\alpha\pi} \int_{-\pi}^{\pi} \frac{h_X(\theta)}{|\theta|^\alpha(1-\phi_X(\theta))} \left(\cos(\theta x) - 1\right) d\theta.$$

For $\alpha < 3/2$ we have that $m_\alpha = 0$ and $\tilde{h}_X(\cdot) := \frac{h_X(\cdot)}{|\cdot|^\alpha(1-\phi_X(\cdot))}$ is in $L^1(\mathbb{T})$. Indeed,

$$a_X(0,x) - a'_{\bar{X}}(0,x) = \int_{-\pi}^{\pi} \tilde{h}_X(\theta) \cos(\theta x)\, d\theta - \int_{-\pi}^{\pi} \tilde{h}_X(\theta) d\theta.$$

The second term on the r.h.s. is a constant, whereas the first vanishes as $|x| \to \infty$ as before.

For the case $\alpha \in (\frac{3}{2}, \frac{5}{3})$ the proof is analogous to the proof of (i): we compare the integral to its counterpart with $1-\phi_X(\theta)$ substituted by $\kappa_\alpha|\theta|^\alpha$ in the denominator. Here we have:

$$\begin{aligned}
a_X(0,x) - a'_{\bar{X}}(0,x) := & \underbrace{\frac{\kappa_2}{2(\kappa_\alpha)^2\pi}\int_{-\pi}^{\pi} \frac{h_X(\theta)}{|\theta|^{2\alpha}} (\cos(\theta x) - 1)\, d\theta}_{I(x)} \\
& + \underbrace{\frac{1}{2\kappa_\alpha\pi}\int_{-\pi}^{\pi} \left(\frac{h_X(\theta)}{|\theta|^\alpha(1-\phi_X(\theta))} - \frac{\kappa_2 h_X(\theta)}{\kappa_\alpha|\theta|^{2\alpha})}\right) (\cos(\theta x) - 1)\, d\theta}_{R_0(x)} .
\end{aligned}$$

The last remainder term $R_0(x)$ is of order $\mathcal{O}(1)$ as $|x| \longrightarrow \infty$ for any $\alpha < 2$, again due to the fact that we can interpret it as the Fourier transform of a $L^1(\mathbb{T})$ function.

Since we assumed $\alpha > \frac{3}{2}$, $\theta \mapsto |\theta|^{2-2\alpha}(\cos(\theta x) - 1)$ is in $L^1(\mathbb{R})$ and therefore

$$\begin{aligned}
I(x) = & |x|^{2\alpha-3}\frac{\kappa_2}{2(\kappa_\alpha)^2\pi}\int_{-\pi x}^{\pi x} |\theta|^{2-2\alpha}(\cos(\theta) - 1)\, d\theta \\
& + \frac{\kappa_2}{2\kappa_\alpha\pi}\int_{-\pi}^{\pi} \frac{h_X(\theta) - |\theta|^2}{|\theta|^{2\alpha}} (\cos(\theta x) - 1)\, d\theta \\
= & \underbrace{|x|^{2\alpha-3}\frac{\kappa_2}{2(\kappa_\alpha)^2\pi}\int_{-\infty}^{\infty} |\theta|^{2-2\alpha}(\cos(\theta) - 1)\, d\theta}_{I_1(x)} \\
& - \underbrace{|x|^{2\alpha-3}\frac{\kappa_2}{2(\kappa_\alpha)^2\pi}\int_{\mathbb{R}\setminus[-\pi x,\pi x]} |\theta|^{2-2\alpha}(\cos(\theta) - 1)\, d\theta}_{R_{1,1}(x)} \\
& + \underbrace{\frac{\kappa_2}{2\kappa_\alpha\pi}\int_{-\pi}^{\pi} \frac{h_X(\theta) - |\theta|^2}{|\theta|^{2\alpha}} (\cos(\theta x) - 1)\, d\theta}_{R_{1,2}(x)} .
\end{aligned}$$

Both terms $R_{1,1}, R_{1,2} = \mathcal{O}(1)$ as $|x| \longrightarrow \infty$, since

$$|x|^{2\alpha-3}\left|\int_{\mathbb{R}\setminus[-\pi x,\pi x]} |\theta|^{2-2\alpha}(\cos(\theta) - 1)\, d\theta\right| = \mathcal{O}(1),$$

for any $\alpha < 2$. More generaly, let $\alpha \in (1,2)$ and $2/(2-\alpha) \notin \mathbb{N}$, we write

(6.5)
$$\begin{aligned}
&\int_{-\pi}^{\pi} \frac{h_X(\theta)}{|\theta|^\alpha(1-\phi_X(\theta))} (\cos(\theta x) - 1)\, d\theta \\
&\qquad = \sum_{m=1}^{m_\alpha} \underbrace{\int_{-\pi}^{\pi} \frac{\kappa_2^m}{\kappa_\alpha^m} \frac{(h_X(\theta))^m}{|\theta|^{(m+1)\alpha}} (\cos(\theta x) - 1)\, d\theta}_{I_m(x)} \\
&\qquad\quad + \underbrace{\int_{-\pi}^{\pi} \frac{\kappa_2^{m_\alpha+1}}{\kappa_\alpha^{m_\alpha+1}} \frac{(h_X(\theta))^{m_\alpha+1}}{|\theta|^{(m_\alpha+1)\cdot\alpha}(1-\phi_X(\theta))} (\cos(\theta x) - 1)\, d\theta}_{R(x)} \\
&\qquad = \sum_{m=1}^{m_\alpha} I_m(x) + R(x).
\end{aligned}$$

We chose $m_\alpha = \lceil \frac{\alpha-1}{2-\alpha} \rceil - 1$ as the minimal value of m such that

$$\frac{(h_X(\theta))^{m_\alpha+1}}{(1-\phi_X(\theta))|\theta|^{m_\alpha+1}} \in L^1(\mathbb{T}).$$

Analogously as before we argue that $R(x) = \mathcal{O}(1)$ as $|x| \longrightarrow \infty$.

Finally, for $m \le m_\alpha$ we have

$$\frac{h_X^m(\theta)}{\kappa_\alpha^m |\theta|^{m\alpha}(1-\phi_X(\theta))} = \frac{\kappa_2^m}{\kappa_\alpha^{m+1}} |\theta|^{m(2-\alpha)-\alpha} + \mathcal{O}\left(|\theta|^{m(2-\alpha)-1}\right),$$

and as $\alpha < 2$, we have that $m(2-\alpha) - 1 > -1$, using a change of variable we get

$$\begin{aligned}
I_m(x) &= \frac{\kappa_2^m}{\kappa_\alpha^m} \int_{-\pi}^{\pi} |\theta|^{m(2-\alpha)-\alpha} (\cos(\theta x) - 1)\, d\theta + \mathcal{O}(1) \\
&= |x|^{(\alpha-1)-m(2-\alpha)} \frac{\kappa_2^m}{\kappa_\alpha^m} \int_{-\infty}^{\infty} |\theta|^{m(2-\alpha)-\alpha} (\cos(\theta) - 1)\, d\theta \\
&\quad - \frac{\kappa_2^m}{\kappa_\alpha^m} \int_{\mathbb{R}\setminus[-\pi|x|,\pi|x|]} |\theta|^{m(2-\alpha)-\alpha} (\cos(\theta x) - 1)\, d\theta + \mathcal{O}(1).
\end{aligned}$$

Where the first integral in the second line is finite because $m < m_\alpha$. Again, notice that the last integral is of order $\mathcal{O}(1)$ as $|x| \longrightarrow \infty$.

Finally, if $2/(2-\alpha) \in \mathbb{N}$, we have that

$$\frac{(h_X(\theta))^{m_\alpha+1}}{(1-\phi_X(\theta))|\theta|^{m_\alpha+1}} - \frac{\kappa_2^{m_\alpha+1}}{\kappa_\alpha^{m_\alpha+2}|\theta|} \in L^1(\mathbb{T}).$$

Now, we proceed like before, but also taking into account the contribution of the integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos(x\theta) - 1}{|\theta|} d\theta = \frac{1}{\pi} \int_0^{\pi|x|} \frac{\cos(\theta) - 1}{\theta} d\theta$$

and using Lemma A.2. This concludes the proof. □

Proof of Theorem 3.6. We will only prove the repaired case, as the other case is just an adaptation of the arguments of Theorem 3.4 case *(ii)* together with the considerations we will present here.

Instead of comparing $a_X(\cdot,\cdot)$ and $a_{\bar{X}}(\cdot,\cdot)$ and $a'_{\bar{X}}(\cdot,\cdot)$, we we will only compare $a_X(\cdot,\cdot)$ and $a'_{\bar{X}}(\cdot,\cdot)$. That is, we have

$$a_X(0,x) := \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{1-\phi_X(\theta)}(\cos(\theta x)-1)d\theta$$

and we define

$$a'_{\bar{X}}(0,x) := \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{\kappa_1|\theta|}(\cos(\theta x)-1)d\theta.$$

Write now

$$a_X(0,x) := a'_{\bar{X}}(0,x) + \left(a_X(0,x) - a'_{\bar{X}}(0,x)\right).$$

To evaluate the second term, we use a very similar approach to the one in the proof of Theorem 3.4. Using the second part of the statement of Lemma B.1, we get $\theta \mapsto \frac{1}{\kappa_1|\theta|} - \frac{1}{1-\phi_X(\theta)}$ is in $C^{0,1/3-}(\mathbb{T})$.

It remains to evaluate $a'_X(0,x)$. Note that

$$a'_{\bar{X}}(0,x) = \frac{1}{\pi\kappa_1}\int_0^{\pi|x|}\frac{\cos(\theta)-1}{\theta}d\theta.$$

Again, using Lemma A.2, we conclude the result. □

7. Final remarks and generalisations

In this section we quickly discuss possible generalisations and limitations of our results and techniques.

LCLT for higher dimensions and the asymmetric case. The notion of an admissible distribution in higher dimensions is straightforward. Let $X_i \in \mathbb{Z}^d$, we do expect that the transition probability of a long-range random walk $p_\alpha(x) = c_{d,\alpha}\|x\|^{-(d+\alpha)}$ is admissible for any norm $\|\cdot\|$ in $\mathbb{Z}^d$.

Provided that such examples exist, we can generalise our LCLT results in Theorem 3.1, Theorem 3.2 and Corollary 3.3 immediately to d-dimensional walks. Let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. random variables on $\mathbb{Z}^d$ with admissible law of index $\alpha \in (0,2)$. Then

$$\sup_{x\in\mathbb{Z}^d}|p_X^n(x) - p_{\bar{X}}^n(x)| \lesssim n^{-\frac{\beta_1+d-\alpha}{\alpha}}.$$

and assuming that $p_X(\cdot)$ is repaired we obtain convergence rates of order $\mathcal{O}(n^{-(d+\alpha)/\alpha})$. The notions of repairer and asymptotic repairer can be trivially generalised to d-dimensions. We believe that an appropriate shift extends the results to the asymmetric case as well.

Continuous-time random walks. All the results presented here, could be easily extended to the continuous version of a random walk with admissible law. For the continuous time random walk, at each point, the walker waits a Poisson clock of rate 1 then makes a single step according to a admissible distribution. Both LCLT and potential kernel statements and proofs are essentially the same.

Further repairers. In this article we only studied repairers for probability distributions $p_X(\cdot)$ which are α-admissible with a regularity set $R_\alpha = \{2\}$. However, suppose that $p_X(\cdot)$ is an admissible distribution, let $\delta := \min(R_\alpha)$ and $\kappa_\delta > 0$ so we are in the locally repairable case. We could define a repairer Z as an admissible distribution $p_Z(\cdot)$ with index δ such that $\kappa_\delta = -\kappa'_\delta$, i.e. the coefficient corresponding to $|\theta|^\delta$ in the expansion of the characteristic function of X is equal to the negative value of the coefficient κ'_δ multiplying $|\theta|^\delta$ in the expansion of the characteristic function of Z. Then, $\min(R'_\alpha) > \delta$, where R'_α is the regularity set of $X + Z$. Hence repairing would allow to obtain precise estimates on its potential kernel beyond the constant order of the error. A similar idea could be used to improve the rates of convergence in the LCLT for distributions such that $\min(R_\alpha) < 2\alpha$, by performing multiple repairs to cancel each of the terms in $r_X(\theta)$.

Non-lattice walks/ Random variables in $\mathbb{R}$. We believe that a combination of the ideas of the present paper and [31] would be enough to prove our results in the context of non-lattice walks and absolutely continuous random variables, possibly depending on a further integrability assumption over the characteristic function. However, we cannot say the same about potential kernel estimates. Here we are relying on the fact that smoothness implies decay of the Fourier coefficients on the torus. This relation is not simple in the full $\mathbb{R}^d$.

Improvement of the error bounds in the Potential Kernel. Notice that the decay of the error term in the potential kernel remainder is equivalent to the degree of continuity of the function $\tilde{h}_X(\cdot)$ (defined in the proof of Theorem 3.4). This function contains the contribution of the regularity set and error term. In general, $h_X \in C^{1,\alpha-1-}(\mathbb{T})$ but we do not necessarily have $h_X \in C^{1,\alpha-1}(\mathbb{T})$. Under these assumptions, it falls upon the sharpness of Lemma B.1 (which we believe is close to optimal) to decide the maximum degree of continuity of $\tilde{h}_X(\cdot)$. There are two ways that one could try to obtain better bounds for a specific distribution. The first is by showing $\tilde{h}_X(\cdot)$ has a higher degree of continuity by hand for the specific examples. The second is by expanding the characteristic function up to a smaller error, which is computationally demanding.

Potential kernel estimates in higher dimensions and/or $\alpha < 1$, asymmetric case. Unfortunately, our results do not generalise for Green function estimates for $d \geq 2$ and $\alpha \in (0, 2)$ immediately without further assumptions on the degree of continuity of the remainder of the function $\tilde{h}_X(\cdot)$. We would need to guarantee that the remainder would decay faster than $\|x\|^{\alpha-d}$, which is the first order term.

The same argument applies to $\alpha < 1$ and $d = 1$, the degree of continuity of $\phi_X(\cdot)$ becomes too low to guarantee that its Fourier transform will decay faster than $|x|^{\alpha-1}$.

One could try to expand ideas from the proof of Theorem 1.4 in [13] to tackle the $d \geq 2$ and/or $\alpha < 1$ case. There the authors demonstrate a detailed expansion for the Green's function in $d = 2$, $\alpha \in (0, 2)$ for a trucated long-range random walk.

Regarding adding asymmetry in the random walk, we expect that the potential kernel in this case can be written in terms of its continuous counterpart and an error term of order $\mathcal{O}(|x|^{-\alpha/3})$ for the repaired case and $\alpha > 1$.

Appendix A. Evaluation of some special integrals

Lemma A.1. *For $\alpha \in (0, 2)\backslash\{1\}$, we have that R^∞_α defined in (4.4) satisfies*

$$R^\infty_\alpha(\theta) = K_2|\theta|^2 + \mathcal{O}(|\theta|^{2+\alpha}) \tag{A.1}$$

where

$$K_2 = \frac{1-\alpha}{2}\Bigg(\Big(\frac{2^{2-\alpha}-1}{2-\alpha} - \frac{3(2^{1-\alpha}-1)}{2(1-\alpha)}\Big) + \frac{1}{2\Gamma(\alpha)}\sum_{m=1}^{\infty}(-1)^m(\zeta(m+\alpha)-1)\frac{m\Gamma(m+\alpha)}{\Gamma(m+2)(m+2)}\Bigg). \tag{A.2}$$

Proof. Recall that $\theta > 0$ and

$$R^\infty_\alpha = \theta^{1+\alpha}\int_\theta^\infty \Big(\frac{z\sin(z) - (1+\alpha)(1-\cos(z))}{z^{2+\alpha}}\Big)P_1\Big(\frac{z}{\theta}\Big)dz$$

and $P_1(x) = (x - \lfloor x \rfloor) - \frac{1}{2}$. Note that this integral is finite. Indeed, one can prove this by observing that $|P(z)| \leq \frac{1}{2}$. We shall now divide the integral in R^∞_α in two parts, one going from θ to 1 and the other 1 to ∞, as we will use different techniques to bound them.

$$R^\infty_\alpha = \underbrace{\theta^{1+\alpha}\int_\theta^1 \frac{z\sin(z) - (1+\alpha)(1-\cos(z))}{z^{2+\alpha}}P_1\Big(\frac{z}{\theta}\Big)dz}_{I_1} + \underbrace{\theta^{1+\alpha}\int_1^\infty \frac{z\sin(z) - (1+\alpha)(1-\cos(z))}{z^{2+\alpha}}P_1\Big(\frac{z}{\theta}\Big)dz}_{I_2}.$$

We start by analysing I_2 and proving that $I_2 = \mathcal{O}(|\theta|^{2+\alpha})$,

$$I_2 = \theta^{1+\alpha} \int_1^\infty \frac{z\sin(z) - (1+\alpha)(1-\cos(z))}{z^{2+\alpha}} P_1\Big(\frac{z}{\theta}\Big) dz.$$

For convenience, we assume that $\theta^{-1} \in \mathbb{N}$. To treat the general case we need to compare the expressions between for θ^{-1} and $\lfloor \theta^{-1} \rfloor$.

In this case, we can write the integral above as

$$I_2 = \theta^{1+\alpha} \sum_{k=1/\theta}^{\infty} \int_{k\theta}^{(k+1)\theta} g(z) \left(\frac{z}{\theta} - k - \frac{1}{2} \right) dz,$$

where $g(z) := \frac{z\sin(z)-(1+\alpha)(1-\cos(z))}{z^{2+\alpha}}$. Now, we will use that $\int_{k\theta}^{(k+1)\theta} P_1\left(\frac{z}{\theta}\right) dz = 0$ and sum and subtract the term $g(k\theta)$ in each term of the summands. Hence

$$\begin{aligned}
|I_2| &= |\theta|^{1+\alpha} \left| \sum_{k=1/\theta}^{\infty} \int_{k\theta}^{(k+1)\theta} (g(z) - g(k\theta)) \left(\frac{z}{\theta} - k - \frac{1}{2} \right) dz, \right| \\
&\le |\theta|^{1+\alpha} \sum_{k=1/\theta}^{\infty} \sup_{y \in [k\theta,(k+1)\theta]} |g'(y)| \int_{k\theta}^{(k+1)\theta} |z - k\theta| \left| \frac{z}{\theta} - k - \frac{1}{2} \right| dz, \\
&\le \frac{1}{4} |\theta|^{3+\alpha} \sum_{k=1/\theta}^{\infty} \sup_{y \in [k\theta,(k+1)\theta]} |g'(y)|,
\end{aligned}$$

where we used in the second inequality both a change of variables and that $|z - k\theta| \le \theta$.

For $z > 0$, we have

$$g'(z) = \frac{\cos(z)}{z^{1+\alpha}} - 2(1+\alpha)\frac{\sin(z)}{z^{2+\alpha}} + (1+\alpha)(2+\alpha)\frac{1-\cos(z)}{z^{3+\alpha}}$$

and therefore there is a constant C_α that only depends on α, such

$$|g'(z)| \le \frac{C_\alpha}{z^{1+\alpha}}$$

which implies

$$\sup_{[k\theta,(k+1)\theta]} |g'(z)| \le \frac{C_\alpha}{(\theta k)^{1+\alpha}}.$$

We can now use this in the estimate of $|I_2|$ to get

$$|I_2| \le C\theta^2 \sum_{k=1/\theta}^{\infty} \frac{1}{k^{1+\alpha}} \lesssim |\theta|^{2+\alpha}$$

and $I_2 = \mathcal{O}(|\theta|^{2+\alpha})$

Now, for I_1, we use Taylor expansion of the function $h(z) = z\sin z - (1+\alpha)(1-\cos z)$ to get

$$h(z) = \frac{1-\alpha}{2} z^2 - \frac{3-\alpha}{24} z^4 + r(z),$$

where $r(z) = \mathcal{O}(z^6)$. We get

$$
\begin{aligned}
I_1 &= \theta^{1+\alpha}\frac{1-\alpha}{2}\int_\theta^1 \frac{1}{z^\alpha}P_1\Big(\frac{z}{\theta}\Big)dz - \theta^{1+\alpha}\frac{3-\alpha}{24}\int_\theta^1 z^{2-\alpha}P_1\Big(\frac{z}{\theta}\Big)dz \\
&\quad + \theta^{1+\alpha}\int_\theta^1 \frac{r(z)}{z^{2+\alpha}}P_1\Big(\frac{z}{\theta}\Big)dz \\
&= \frac{1-\alpha}{2}I_{1,1} - \frac{3-\alpha}{24}I_{1,2} + I_{1,3}
\end{aligned}
$$

Again we examine each of the terms separately. We start with the last one. For this, notice that $r(\cdot)$ is a $C^\infty([-1,1])$ function, as it is the difference of two such functions. Moreover, we know that $\tilde{r}(z) := \left|\frac{r(z)}{z^{2+\alpha}}\right|$ and therefore, applying Lemma B.1 we have that $\tilde{r}(\cdot)$ is in $C^{0,\frac{4-\alpha}{6}-}([-1,1])$. Now we can proceed like we did for I_2 to get that $I_{1,3}$ is of order $\mathcal{O}(\theta^{2+\alpha})$.

The first integral $I_{1,1}$ can be written as, again assuming that $\theta^{-1} \in \mathbb{N}$,

$$
\begin{aligned}
I_{1,1} &= \theta^{1+\alpha}\sum_{k=1}^{\lfloor\frac{1}{\theta}\rfloor-1}\int_{k\theta}^{(k+1)\theta}\frac{1}{z^\alpha}\Big(\frac{z}{\theta}-k-\frac{1}{2}\Big)dz \\
&= \theta^2\sum_{k=1}^{\lfloor\frac{1}{\theta}\rfloor-1}k^{2-\alpha}\left[\frac{\left(1+\frac{1}{k}\right)^{2-\alpha}-1}{2-\alpha} - \Big(1+\frac{1}{2k}\Big)\frac{\left(1+\frac{1}{k}\right)^{1-\alpha}-1}{1-\alpha}\right].
\end{aligned}
$$

We now split between $k=1$ and $k>1$.

$$
\begin{aligned}
I_{1,1} &= \theta^2\Big(\frac{2^{2-\alpha}-1}{2-\alpha} - \frac{3(2^{1-\alpha}-1)}{2(1-\alpha)}\Big) \\
&\quad + \theta^2\sum_{k=2}^{\lfloor\frac{1}{\theta}\rfloor-1}k^{2-\alpha}\left[\frac{\left(1+\frac{1}{k}\right)^{2-\alpha}-1}{2-\alpha} - \Big(1+\frac{1}{2k}\Big)\frac{\left(1+\frac{1}{k}\right)^{1-\alpha}-1}{1-\alpha}\right]
\end{aligned}
$$

Use now the full Taylor series of both $(1+x)^{2-\alpha}$ and $(1+x)^{1-\alpha}$ where we are taking $x = \frac{1}{k} \in (0,1)$ to explore the cancellations. We then get that the last sum is equal to

$$
\theta^2\sum_{k=2}^{\lfloor\frac{1}{\theta}\rfloor-1}\sum_{j=3}^{\infty}k^{2-\alpha-j}\frac{(j-2)\Gamma(1-\alpha)}{2j!\Gamma(-\alpha-j+3)}. \tag{A.3}
$$

Therefore using the reflection formula for the Gamma function and a change of variables $m = j-2$, we get

$$
\text{(A.3)} = \frac{\theta^2}{2\Gamma(\alpha)}\sum_{k=2}^{\lfloor\frac{1}{\theta}\rfloor-1}\sum_{m=1}^{\infty}(-1)^m k^{-\alpha-m}\frac{m\Gamma(m+\alpha)}{(m+2)!}.
$$

Now, using Euler-Maclaurin again, one can easily prove that for $\alpha \in (0,2)$ and $m \geq 1$

$$\left| \sum_{k=2}^{\lfloor \frac{1}{\theta} \rfloor - 1} k^{-\alpha-m} - (\zeta(m+\alpha)-1) + \frac{\theta^{m+\alpha-1}}{m+\alpha-1} \right| \leq C\theta^{m+\alpha} \tag{A.4}$$

where the constant C does not depend on m or α. Therefore there exist an explicit constant K_2

$$I_{1,1} = K_2|\theta|^2 + K_{1+\alpha}|\theta|^{1+\alpha} + \mathcal{O}(|\theta|^{2+\alpha}).$$

Finally, we can show in an analogous way that $I_{1,2} = \mathcal{O}(|\theta|^4)$. For the case $\alpha = 1$ we proceed in a similar way. We need to evaluate

$$R_1^{\infty} = \theta^2 \int_{\theta}^{\infty} \Big(\frac{z\sin(z) - 2(1-\cos(z))}{z^3} \Big) P_1\Big(\frac{z}{\theta}\Big) dz,$$

Using similar ideas as before and the fact that $z\sin(z) - 2(1-\cos(z)) = \mathcal{O}(z^4)$ when $|z| \to 0$ instead of the order $\mathcal{O}(z^2)$ that we got for the case $\alpha \in (1,2)$ we conclude the proof. □

Lemma A.2. *Let $z \in [1,\infty)$ define*

$$\operatorname{Cin}(z) := \int_0^z \frac{1-\cos(t)}{t} dt.$$

We have that

$$\operatorname{Cin}(z) = \log z + \gamma + \mathcal{O}(z^{-1}).$$

as $z \longrightarrow \infty$ where γ is the Euler-Mascheroni constant

Proof. By defining

$$\operatorname{Ci}(z) := -\int_z^{\infty} \frac{\cos(t)}{t} dt$$

the linearity of the integral implies that

$$\operatorname{Cin}(z) = \log z - \operatorname{Ci}(z) + \int_1^{\infty} \frac{\cos t}{t} dt + \int_0^1 \frac{1-\cos t}{t} dt.$$

The exact value of the sum of the two integrals is not relevant for us, but it is known to be γ. Therefore,

$$\operatorname{Cin}(z) = -\operatorname{Ci}(z) + \log z + \gamma.$$

finally, it is easy to show that $\operatorname{Ci} = \mathcal{O}(z^{-1})$ as $z \to \infty$. □

Appendix B. Continuity estimates

Lemma B.1. *Let $f \in C^{1,\beta}(I)$ for a closed interval I containing the origin. Additionally, suppose that*

$$f(x) = \mathcal{O}\left(|x|^{\beta_0}\right) \ \ \textit{as } |x| \longrightarrow 0$$

for some $\beta_0 \geq 1+\beta$. Let $1<\beta_1<\beta_0$ and define the function

$$h(x) := \frac{f(x)}{|x|^{\beta_1}}.$$

Then we have that the function h is in $C^{0,\bar\beta}(I)$ where $\bar\beta = \frac{\beta_0-\beta_1}{\beta_0-\beta}$. If instead, we have that $f \in C^{0,\beta}(I)$ for some $\beta \in (0,1)$, and $1/2 \leq \beta_1 < \beta_0 = 1$, we get that $h \in C^{0,\bar\beta}(I)$ with $\bar\beta := \beta(1-\beta_1)$.

Proof. We will prove the first claim, the second can be proved analogously. Let $x, y \in I$ and assume, without loss of generality, that $|x| < |y|$,

$$\begin{aligned}\left|\frac{f(x)}{|x|^{\beta_1}} - \frac{f(y)}{|y|^{\beta_1}} \pm \frac{f(x)}{|y|^{\beta_1}}\right| &= \left|\frac{f(x)}{|x|^{\beta_1}}\left(\frac{|y|^{\beta_1}-|x|^{\beta_1}}{|y|^{\beta_1}}\right) + \frac{f(x)-f(y)}{|y|^{\beta_1}}\right| \\ &\lesssim |x|^{\beta_0-\beta_1}\frac{\left||y|^{\beta_1}-|x|^{\beta_1}\right|}{|y|^{\beta_1}} + \frac{|f(x)-f(y)|}{|y|^{\beta_1}}\end{aligned}$$

Now, we use that for $A, B > C > 0$ real numbers and $\delta \in [0,1]$, we have $C \leq A^\delta B^{1-\delta}$. Regarding the first term on the right hand side, notice that

$$\left||y|^{\beta_1} - |x|^{\beta_1}\right| \lesssim \min\{|y|^{\beta_1}, |y|^{\beta_1-1}|x-y|\}$$

so choosing $A = |y|^{\beta_1}, B = |y|^{\beta_1-1}|x-y|$ and $\delta = \beta_0 - \beta_1$ we can easily see that

$$|x|^{\beta_0-\beta_1}\frac{\left||y|^{\beta_1}-|x|^{\beta_1}\right|}{|y|^{\beta_1}} \lesssim |x-y|^\delta \leq |x-y|^{\bar\beta}.$$

To bound the second term, remark that $|f'(z)| \leq C|y|^\beta$ for all $|z| \leq |y|$ since $f' \in C^{0,\beta}(I)$ and $f'(0)=0$, so

$$|f(x)-f(y)| \lesssim \min\{|y|^{\beta_0}, |y|^\beta|x-y|\}$$

and again choosing $A = |y|^{\beta_0}, B = |y|^\beta|x-y|$ and $\delta = \bar\beta$ the claim follows. □

Lemma B.2. *If $p_X(\cdot)$ is admissible of index $\alpha \in (1,2)$, then $\phi_X(\cdot)$ is in $C^{1,\alpha-1-}(\mathbb{T})$. If $p_X(\cdot)$ is admissible of index 1, then ϕ_X is $C^{0,1-}(\mathbb{T})$.*

Proof. Notice that $p_X(\cdot)$ being admissible implies that it is in the basin of attraction of a α-stable distribution. Therefore given $\beta \geq 0$ we have $\mathbb{E}_X[|X|^\beta] < \infty$ for $\beta \in (0,\alpha)$ and $p_X(x) \lesssim |x|^{-\alpha+}$. Now, we just write that $p_X(\cdot)$ as the inverse Fourier transform of

$$\mathcal{F}_{\mathbb{T}}(\phi_X)(-x) = p_X(x).$$

Then use the classic relations between continuity and decay of Fourier coefficients, see [15, Proposition 3.3.12]. □

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