# ELIMINATING THURSTON OBSTRUCTIONS AND CONTROLLING DYNAMICS ON CURVES 

MARIO BONK, MIKHAIL HLUSHCHANKA, AND ANNINA ISELI


#### Abstract

Every Thurston map $f: S^{2} \rightarrow S^{2}$ on a 2 -sphere $S^{2}$ induces a pull-back operation on Jordan curves $\alpha \subset S^{2} \backslash P_{f}$, where $P_{f}$ is the postcritical set of $f$. Here the isotopy class [ $f^{-1}(\alpha)$ ] (relative to $P_{f}$ ) only depends on the isotopy class [ $\alpha$ ]. We study this operation for Thurston maps with four postcritical points. In this case a Thurston obstruction for the map $f$ can be seen as a fixed point of the pull-back operation.

We show that if a Thurston map $f$ with a hyperbolic orbifold and four postcritical points has a Thurston obstruction, then one can "blow up" suitable arcs in the underlying 2-sphere and construct a new Thurston map $\widehat{f}$ for which this obstruction is eliminated. We prove that no other obstruction arises and so $\widehat{f}$ is realized by a rational map. In particular, this allows for the combinatorial construction of a large class of rational Thurston maps with four postcritical points.

We also study the dynamics of the pull-back operation under iteration. We exhibit a subclass of our rational Thurston maps with four postcritical points for which we can give positive answer to the global curve attractor problem.


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## 1. Introduction

In this paper, we consider Thurston maps and the dynamics of the induced pull-back operation on Jordan curves on the underlying 2-sphere. By definition, a Thurston map is a branched covering map $f: S^{2} \rightarrow S^{2}$ on a topological 2-sphere $S^{2}$ such that $f$ is not a homeomorphism and every critical point of $f$ (points where $f$ is not a local homeomorphism) has a finite orbit under iteration of $f$. These maps are named after William Thurston who introduced them in his quest for a better understanding of the dynamics of postcriticallyfinite rational maps on the Riemann sphere. We refer to [BM17, Chapter 2] for general background on Thurston maps and related concepts.

For a branched covering map $f: S^{2} \rightarrow S^{2}$ we denote by $C_{f}$ the set of critical points of $f$ and by $f^{n}$ the $n$-th iterate of $f$ for $n \in \mathbb{N}$. Then the postcritical set of $f$ is defined as

$$
P_{f}=\bigcup_{n \in \mathbb{N}}\left\{f^{n}(c): c \in C_{f}\right\} .
$$

For a Thurston map $f$ this set has finite cardinality $2 \leq \# P_{f}<\infty$ (for the first inequality see [BM17, Corollary 2.13]).

A Thurston map $f$ often admits a description in purely combinatorial-topological terms. In this context, it is an interesting question whether $f$ can be realized (in a suitable sense) by a rational map with the same combinatorics. Roughly speaking, this means that $f$ is conjugate to a rational map "up to isotopy" (see Section 3 for the precise definition).

It is not hard to see that each Thurston map with two or three postcritical points is realized. The situation is much more complicated for Thurston maps $f$ with $\# P_{f} \geq 4$. William Thurston found a necessary and sufficient condition when a Thurston map can be realized by a rational map [DH93. Namely, if $f$ has an associated hyperbolic orbifold (this is always true apart from some well-understood exceptional maps), then $f$ is realized if and only if $f$ has no (Thurston) obstruction. Such an obstruction is given by a finite collection of disjoint Jordan curves in $S^{2} \backslash P_{f}$ (up to isotopy) with certain invariance properties (see Section 3.2 for more discussion).

The "if part" of this statement gives a positive criterion for $f$ to be realized, but it is very hard to apply in practice, because, at least in principle, it involves the verification of infinitely many conditions for the map $f$. For this reason, in each individual case a successful
verification for a map, or a class of maps, is difficult and usually constitutes an interesting result in its own right.

We mention two results in this direction. The first one is the "arcs intersecting obstructions" theorem by Kevin Pilgrim and Tan Lei [PL98, Theorem 3.2] that gives control on the position of an obstruction and has many applications in holomorphic dynamics (see, for instance, [PL98, DMRS19]). The other one is the "mating criterion" by Tan Lei, Mary Rees, and Mitsuhiro Shishikura that addresses the question when two postcriticallyfinite quadratic polynomials can be topologically glued together to form a rational map (see [Lei92, Ree92, Shi00]).

The investigation of obstructions of a Thurston map $f: S^{2} \rightarrow S^{2}$ is closely related to the study of the pull-back operation on Jordan curves. It is easy to show that if $\alpha \subset S^{2} \backslash P_{f}$ is a Jordan curve, then the isotopy class $\left[f^{-1}(\alpha)\right]$ (rel. $P_{f}$ ) only depends on the isotopy class $[\alpha]$ (see Lemma 3.4). Intuitively, the number of postcritical points of a Thurston map can be seen as a measure of its combinatorial complexity. In this paper, we focus on the simplest non-trivial case, namely Thurston maps $f$ with $\# P_{f}=4$. In this case the pull-back operation gives rise to a well-defined map, the slope map, on these isotopy classes [ $\alpha$ ] (we will discuss this in more detail below). The search for obstructions of $f$ amounts to understanding the fixed points of the slope map.

There exist various natural constructions that allow one to combine or modify given (rational) Thurston maps to obtain a new dynamical system. The most studied constructions are mating (see [Lei92, SL00]), tuning (see [Ree92]), and capture (see [Hea88, Lei97]). In this paper, we study the operation of blowing up arcs, originally introduced by Kevin Pilgrim and Tan Lei in [PL98]. This operation can be applied to an arbitrary Thurston map $f$ and results in a new Thurston map $\widehat{f}$ that is of higher degree, but combinatorially closely related to the original map $f$. In particular, $f$ and $\widehat{f}$ have the same set of postcritical points and the same dynamics on them. Nevertheless, the dynamical behavior of Jordan curves under the pull-back operation for the original map $f$ and the new map $\widehat{f}$ may differ drastically.

We show that, if a Thurston map $f: S^{2} \rightarrow S^{2}$ with $\# P_{f}=4$ has an obstruction $\alpha$, then one can naturally modify $f$ by blowing up certain arcs to produce a new Thurston map $\widehat{f}$ for which this obstruction $\alpha$ is eliminated. The main result of this paper is the fact that then no new obstructions arise for $\widehat{f}$ and so it is realized by a rational map.
Theorem 1.1. Let $f: S^{2} \rightarrow S^{2}$ be a Thurston map with $\# P_{f}=4$ and a hyperbolic orbifold. Suppose that $f$ has an obstruction represented by a Jordan curve $\alpha \subset S^{2} \backslash P_{f}$, and $E \neq \varnothing$ is a finite set of arcs in $\left(S^{2}, f^{-1}\left(P_{f}\right)\right)$ that satisfy the $\alpha$-restricted blow-up conditions.

Let $\widehat{f}$ be a Thurston map obtained from $f$ by blowing up arcs in $E$ (with some multiplicities) so that $\lambda_{\widehat{f}}(\alpha)<1$. Then $\widehat{f}$ is realized by a rational map.

The technical verbiage and the notation in this formulation will be explained in subsequent sections (see in particular (3.3) for the definition of the "eigenvalue" $\lambda_{\widehat{f}}(\alpha)$ and Definition 6.5 for $\alpha$-restricted blow-up conditions).

Recently, Dylan Thurston provided a positive characterization when a Thurston map is realized, at least in the case when each critical point eventually lands in a critical cycle under iteration. He proved that such a Thurston map $f$ is realized by a rational map if and only if there is an "elastic spine" (that is, a planar embedded graph in $S^{2} \backslash P_{f}$ with a suitable metric on it) that gets "looser" under backwards iteration (see Thu16, Thu20] for more details). In concrete cases, especially for Thurston maps that should be realized


Figure 1. The $(4 \times 4)$-Lattès map.
by rational maps with Julia sets homeomorphic to the Sierpiński carpet, the application of Dylan Thurston's criterion is not so straightforward. Moreover, his criterion is only valid for Thurston maps with periodic critical points. In contrast, for some maps for which Dylan Thurston's criterion is not applicable or hard to apply, Theorem 1.1 can be used to verify that the maps are realized. In particular, many maps obtained by blowing up Lattès maps (see below) are of this type.
1.1. Blowing up Lattès maps. We will now discuss a special case of Theorem 1.1 in detail to give the reader some intuition for the geometric ideas behind this statement and its proof.

Let $\mathbb{P}$ be a pillow obtained from two copies of the unit square $[0,1]^{2} \subset \mathbb{R}^{2} \cong \mathbb{C}$ glued together along their boundaries. We consider the two copies of $[0,1]^{2}$ in $\mathbb{P}$ as the front and back side of $\mathbb{P}$ and call them the tiles of level 0 or simply 0 -tiles. We denote by $A:=(0,0) \in \mathbb{P}$ the lower left corner of $\mathbb{P}$ (see the right part of Figure 1 ). The pillow $\mathbb{P}$ is a topological 2 -sphere. Actually, if we consider $\mathbb{P}$ as an abstract polyhedral surface, then $\mathbb{P}$ carries a conformal structure making $\mathbb{P}$ conformally equivalent to the Riemann sphere $\widehat{\mathbb{C}}$. See Section 2.4 for more discussion.

We now fix $n \in \mathbb{N}$ with $n \geq 2$. We subdivide each of the two 0 -tiles of $\mathbb{P}$ into $n^{2}$ small squares of sidelength $1 / n$, called the 1-tiles. We color these 1-tiles in a checkerboard fashion black and white so that the 1 -tile in the front 0 -tile that contains the vertex $A$ on its boundary is colored white (see the left part of Figure 1). We map this white 1-tile to the front 0-tile of the right-hand pillow by an orientation-preserving Euclidean similarity that fixes the vertex $A$. This similarity scales distances by the factor $n$. We can uniquely extend the similarity by a successive Schwarz reflection process to the whole pillow $\mathbb{P}$ to obtain a continuous map $\mathcal{L}_{n}: \mathbb{P} \rightarrow \mathbb{P}$. Then on each 1-tile $S$ the map $\mathcal{L}_{n}$ is a Euclidean similarity that sends $S$ to the front or back 0 -tile of $\mathbb{P}$ depending on whether $S$ is white or black. We call $\mathcal{L}_{n}$ the $(n \times n)$-Lattès map, because under a suitable conformal equivalence $\mathbb{P} \cong \widehat{\mathbb{C}}$, the map $\mathcal{L}_{n}$ is conjugate to a rational map obtained from $n$-multiplication of a Weierstrass $\wp$-function. See Figure 1 for an illustration of the map $\mathcal{L}_{4}$. Here, the marked points on the left pillow $\mathbb{P}$ (the domain of the map) correspond to the preimage points $\mathcal{L}_{4}^{-1}(A)$. Note that there is exactly one preimage of $A$ in the interior of the back side of the pillow.

It is easy to see that the $(n \times n)$-Lattès map $\mathcal{L}_{n}: \mathbb{P} \rightarrow \mathbb{P}$ is a Thurston map with four postcritical points, namely, the four corners of the pillow $\mathbb{P}$. The map $\mathcal{L}_{n}$ is realized by a rational map, because it is even conjugate to such a map.

We now modify the map $\mathcal{L}_{n}$ by gluing in vertical or horizontal flaps to $\mathbb{P}$. This is a special case of the more general construction of blowing up arcs mentioned above. We will describe


Figure 2. Gluing in a flap.
this in detail in Section 4, but will illustrate the procedure in Figure 2, where we show how to glue in one horizontal flap.

We cut the pillow $\mathbb{P}$ open along a horizontal side $e$ of one of the 1 -tiles. Note that in this process $e$ is "doubled" into two arcs $e^{\prime}$ and $e^{\prime \prime}$ with common endpoints. We then take two disjoint copies of the Euclidean square $[0,1 / n]^{2}$ and identify them along three corresponding sides to obtain a flap F. It has two "free" sides on its boundary. We glue each free side to one of the $\operatorname{arcs} e^{\prime}$ and $e^{\prime \prime}$ of the cut in the obvious way.

In general, one can repeat this construction and glue several flaps at the location given by the arc $e$. We assume that this has been done simultaneously for $n_{h} \geq 0$ flaps along horizontal edges and $n_{v} \geq 0$ flaps along vertical edges. By this procedure we obtain a "flapped" pillow $\widehat{\mathbb{P}}$, which is still a topological 2 -sphere (see the left part of Figure 3). By construction it is tiled by $2 n^{2}+2\left(n_{h}+n_{v}\right)$ squares of sidelength $1 / n$, which we consider as the 1 -tiles of $\widehat{\mathbb{P}}$. The checkerboard coloring of the base surface $\mathbb{P}$ extends in a unique way to the new surface $\widehat{\mathbb{P}}$. The original $(n \times n)$-Lattès map $\mathcal{L}_{n}: \mathbb{P} \rightarrow \mathbb{P}$ can naturally be "extended" to a continuous map $\widehat{\mathcal{L}}: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ so that each 1-tile $S$ of $\widehat{\mathbb{P}}$ is mapped to the front or back 0 -tile of $\mathbb{P}$ (depending on the color of $S$ ) by a Euclidean similarity scaling distances by the factor $n$. See Figure 3 for an illustration of a map $\widehat{\mathcal{L}}$ obtained from the Lattès map $\mathcal{L}_{4}$ by gluing in flaps at a vertical and a horizontal edge. Similarly as in Figure 1, on the left we marked the preimages of $A$ under $\widehat{\mathcal{L}}$.

In order to obtain a Thurston map $f: \mathbb{P} \rightarrow \mathbb{P}$ from this construction, we need to choose a homeomorphism $\phi: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$. Roughly speaking, $\phi$ is a homeomorphism that identifies $\widehat{\mathbb{P}}$ with $\mathbb{P}$ and fixes each corner of the pillow. The precise choice of $\phi$ is somewhat technical and so we refer to Section 4.2 for the details. Then, one easily observes that the postcritical set $P_{f}$ consists of the four corners of the pillow $\mathbb{P}$. The map $f$ is uniquely determined up to Thurston equivalence (see Definition 3.2) independently of the choice of $\phi$ under suitable restrictions. We refer to $f$ as a Thurston map obtained from the ( $n \times n$ )-Lattès map by gluing $n_{h}$ horizontal and $n_{v}$ vertical flaps to $\mathbb{P}$.


Figure 3. An example of a map $\widehat{\mathcal{L}}$ obtained from $\mathcal{L}_{4}$ by gluing in flaps.

Now the following statement is true. As we will explain, it can be seen as a special case of our main result.

Theorem 1.2. Let $n \in \mathbb{N}$ with $n \geq 2$ and $f: \mathbb{P} \rightarrow \mathbb{P}$ be a Thurston map obtained from the $(n \times n)$-Lattès map $\mathcal{L}_{n}$ by gluing $n_{h} \geq 0$ horizontal and $n_{v} \geq 0$ vertical flaps to $\mathbb{P}$, where $n_{h}+n_{v}>0$. Then the map $f$ has a hyperbolic orbifold. It has an obstruction if and only if $n_{h}=0$ or $n_{v}=0$. In particular, if $n_{h}>0$ and $n_{v}>0$, then $f$ is realized by a rational map.

If $n_{h}=n_{v}=0$, then no flaps were glued to $\mathbb{P}$ and the map $f$ coincides with the original $(n \times n)$-Lattès map $\mathcal{L}_{n}$ (strictly speaking, only if we choose the homeomorphism $\phi$ used in the construction above to be the identity on $\mathbb{P}$, as we may). Then $f=\mathcal{L}_{n}$ has a parabolic orbifold. Therefore, Thurston's criterion as formulated in Section 3.2 does not apply.

If $n_{h}=0$ or $n_{v}=0$, but $n_{h}+n_{v}>0$ as in Theorem 1.2, then it is immediate to see that $f$ has an obstruction (see Section 5.1) and therefore $f$ cannot be realized by a rational map. So the interesting part of Theorem 1.2 is the claim that if $n_{h}>0$ and $n_{v}>0$, then $f$ has no obstruction.

Even though Theorem 1.2 follows from our more general statement formulated in Theorem 1.1, we will give a complete proof. We will argue by contradiction and assume that a map $f$ with $n_{h}>0$ and $n_{v}>0$ has an obstruction. In principle, there are infinitely many candidates represented by essential isotopy classes of Jordan curves $\alpha \subset \mathbb{P} \backslash P_{f}$. These isotopy classes in turn are distinguished by different slopes in $\widehat{\mathbb{Q}}=\mathbb{Q} \cup\{\infty\}$ (as will be explained in Section 2.5). For such an isotopy class represented by $\alpha$ to be an obstruction, it has to be $f$-invariant in the sense that $f^{-1}(\alpha)$ should contain a component $\widetilde{\alpha}$ isotopic to $\alpha$ rel. $P_{f}$. It seems to be a very intricate problem to find all slopes in $\widehat{\mathbb{Q}}$ that give an invariant isotopy class for $f$. Since we have been able to decide this question only for very simple maps $f$, we proceed in a more indirect manner.

We assume that the Jordan curve $\alpha \subset \mathbb{P} \backslash P_{f}$ is $f$-invariant and gives an obstruction. We then investigate the mapping degrees of $f$ on components of $f^{-1}(\alpha)$ and consider intersection numbers of some relevant curves together with a careful counting argument. We heavily use the fact that the horizontal and vertical Jordan curves (see (2.4)) are $f$-invariant. Ultimately, we arrive at a contradiction. See Section 5 for the details of this argument.

Our idea to use intersection numbers (as in Lemma 5.5) to control possible locations of obstructions and dynamics on curves is not new (see, for example, PL98, Theorem 3.2], [CPL16, Section 8], and [Par18]). However, the previously available results do not provide sharp enough estimates applicable in our situation.

One can think of Theorem 1.2 in the following way. Suppose that instead of directly passing from the Lattès map $\mathcal{L}_{n}$ to a map, let us now call it $\widehat{f}$, obtained by gluing $n_{h}>0$ horizontal and $n_{v}>0$ vertical flaps to $\mathbb{P}$, we first create an intermediate map $f$ obtained by gluing $n_{h}>0$ horizontal, but no vertical flaps. Then $f$ has a hyperbolic orbifold and an obstruction given by a "horizontal" Jordan curve $\alpha$. In the passage from $f$ to $\widehat{f}$ we kill this obstruction, because we glue additional vertical flaps that serve as obstacles and increase the mapping degree on some pullbacks of $\alpha$. Theorem 1.1 then says that no other obstructions arise for $\widehat{f}$. Therefore, Theorem 1.1 generalizes Theorem 1.2 if we interpret it in the way just described. The proof of Theorem 1.1 is based on a refinement and extension of the ideas that we use to establish Theorem 1.2. The proof of Theorem 1.1 is based on the ideas that we use to establish Theorem 1.2, but substantial refinements and extensions are required.
1.2. The global curve attractor problem. The mapping properties of Jordan curves play an important role in Thurston's characterization of rational maps. The original proof of this statement associates with a given Thurston map $f: S^{2} \rightarrow S^{2}$ with a hyperbolic orbifold a certain Teichmüller space $\mathcal{T}_{f}$ and a real-analytic map $\sigma_{f}: \mathcal{T}_{f} \rightarrow \mathcal{T}_{f}$, called Thurston's pullback map. One can show that the map $f$ is realized by a rational map if and only if $\sigma_{f}$ has a fixed point [DH93]. This reduction to a fixed point problem in a Teichmüller space has also been successfully applied by Thurston in the other contexts such as uniformization problems and the theory of 3 -manifoldsDH (there is a rich literature on the subject; see, for example, [Thu88, Thu98, Ota01, FLP12, Hub16]).

In recent years, the pullback map $\sigma_{f}$ and its dynamical properties have been subject to deeper investigations (see, for example, [Pil12, Sel12, Lod13, KPS16]). In particular, Nikita Selinger showed in [Sel12] that $\sigma_{f}$ extends to the Weil-Petersson boundary of $\mathcal{T}_{f}$. The behavior of $\sigma_{f}$ on this boundary is closely related to the behavior of Jordan curves under pull-back by $f$. This in turn leads to the following difficult open question in holomorphic dynamics, called the global curve attractor problem (see [Lod13, Section 9]).

Conjecture. Let $f: S^{2} \rightarrow S^{2}$ be a Thurston map with a hyperbolic orbifold that is realized by a rational map. Then there exists a finite set $\mathscr{A}(f)$ of Jordan curves in $S^{2} \backslash P_{f}$ such that for every Jordan curve $\gamma \subset S^{2} \backslash P_{f}$, all pullbacks $\widetilde{\gamma}$ of $\gamma$ under $f^{n}$ are contained in $\mathscr{A}(f)$ up to isotopy rel. $P_{f}$ for all sufficiently large $n \in \mathbb{N}$.

A set of Jordan curves $\mathscr{A}(f)$ as in this conjecture is called a global curve attractor of $f$. We will give a solution of this problem for maps as in Theorem 1.2 with $n=2$ and $n_{h}, n_{v} \geq 1$. Unfortunately, our methods only apply for $n=2$ and not for $n \geq 3$.
Theorem 1.3. Let $f: \mathbb{P} \rightarrow \mathbb{P}$ be a Thurston map obtained from the $(2 \times 2)$-Lattès map by gluing $n_{h} \geq 1$ horizontal and $n_{v} \geq 1$ vertical flaps to the pillow $\mathbb{P}$. Then $f$ has a global curve attractor $\mathscr{A}(f)$.

One can show that the Julia set of a rational map as provided by Theorem 1.2 is either a Sierpiński carpet or the whole Riemann sphere depending on whether the map has periodic critical points or not (see Proposition 9.1). Accordingly, Theorem 1.3 provides the first examples of maps with Sierpiński carpet Julia set for which an answer to the global curve attractor problem is known. In fact, we obtain such maps with arbitrary large degrees.

Recently, Belk-Lanier-Margalit-Winarski proved the existence of a finite global curve attractor for all postcritically-finite polynomials [BLMW19]. The conjecture is also known to be true for all critically fixed rational maps (that is, rational maps for which each critical point is fixed) and some nearly Euclidean Thurston maps (that is, Thurston maps with exactly four postcritical points and only simple critical points); see [Hlu19] and [Lod13, FKK+17]. In [KL19] Gregory Kelsey and Russell Lodge verified the conjecture for all quadratic non-Lattès maps with four postcritical points. However, for general postcritcally-finite rational maps the conjecture remains wide open.

Since the maps we consider have four postcritical points, it is convenient to reformulate the global curve attractor problem by introducing the slope map (it is closely related to the Thurston pull-back map $\sigma_{f}$ on the Weil-Petersson boundary of $\mathcal{T}_{f}$ ). To define it in the special case relevant for us, we consider the marked pillow $(\mathbb{P}, V)$, where $V$ is the set consisting of the four corners of $\mathbb{P}$, and assume that $f: \mathbb{P} \rightarrow \mathbb{P}$ is a Thurston map with $P_{f}=V$. Up to topological conjugacy, every Thurston map with four postcritical points can be assumed to have this form.

As we already mentioned, there is a bijective correspondence between isotopy classes $[\alpha]$ of essential Jordan curves $\alpha$ in $(\mathbb{P}, V)$ and slopes $r / s \in \widehat{\mathbb{Q}}$ (see Lemma 2.3). We introduce the additional symbol $\odot$ to represent peripheral Jordan curves in $(\mathbb{P}, V)$. We now define the slope map $\mu_{f}: \widehat{\mathbb{Q}} \cup\{\odot\} \rightarrow \widehat{\mathbb{Q}} \cup\{\odot\}$ associated with $f$ as follows. We set $\mu_{f}(\odot):=\odot$. This corresponds to the fact that each pullback of a peripheral Jordan curve $\alpha$ in ( $\mathbb{P}, V$ ) under $f$ is peripheral (see Corollary 3.5 (i)). If $r / s \in \widehat{\mathbb{Q}}$ is an arbitrary slope, then we choose a Jordan curve $\alpha$ in $(\mathbb{P}, V)$ whose isotopy class $[\alpha]$ is represented by $r / s$. If all pullbacks of $\alpha$ under $f$ are peripheral, we set $\mu_{f}(r / s):=\odot$. Otherwise, there exists an essential pullback $\widetilde{\alpha}$ of $\alpha$ under $f$. Then the isotopy class [ $\widetilde{\alpha}$ ] is independent of the choice of the essential pullback $\widetilde{\alpha}$ (see Corollary $3.5($ ii $)$ ) and so it is represented by a unique slope $r^{\prime} / s^{\prime} \in \widehat{\mathbb{Q}}$. In this case, we set $\mu_{f}(r / s):=r^{\prime} / s^{\prime}$. In this way, $\mu_{f}(x) \in \widehat{\mathbb{Q}} \cup\{\odot\}$ is defined for all $x \in \widehat{\mathbb{Q}} \cup\{\odot\}$. Since the map $\mu_{f}$ has the same source and target, we can iterate it. If $n \in \mathbb{N}_{0}$, then we denote by $\mu_{f}^{n}$ the $n$-th iterate of $\mu_{f}$. We will then prove the following statement.

Theorem 1.4. Let $f: \mathbb{P} \rightarrow \mathbb{P}$ be a Thurston map obtained from the $(2 \times 2)$-Lattès map by gluing $n_{h} \geq 1$ horizontal and $n_{v} \geq 1$ vertical flaps to the pillow $\mathbb{P}$. Then there exists a finite set $S \subset \widehat{\mathbb{Q}} \cup\{\odot\}$ with the following property: for each $x \in \widehat{\mathbb{Q}} \cup\{\odot\}$ there exists $N \in \mathbb{N}_{0}$ such that $\mu_{f}^{n}(x) \in S$ for all $n \geq N$.

Note that $P_{f}=V$ in this case; so our previous considerations apply and the map $\mu_{f}$ is defined. It is clear that the previous theorem leads to the solution of the global curve attractor problem for the maps $f$ considered:

Proof of Theorem 1.3 based on Theorem 1.4. To obtain a finite attractor $\mathscr{A}(f)$, pick a Jordan curve in each isotopy class represented by a slope in $S$ and add five Jordan curves that represent the isotopy classes of peripheral Jordan curves in $(\mathbb{P}, V)$ (one for null-homotopic curves and one for each corner of $\mathbb{P}$ ).

For the proof of Theorem 1.4 we will establish a certain monotonicity property of the slope map $\mu_{f}$ for a map $f$ as in the statement (see Proposition 8.1). Roughly speaking, this monotonicity means that up to isotopy rel. $P_{f}=V$ complicated essential Jordan curves in ( $\mathbb{P}, V$ ) get "simpler" and "less twisted" if we take successive preimages under $f$ and eventually end up in the global curve attractor.

Our methods again rely on the consideration of intersection numbers. The algebraic methods for solving the global curve attractor problem developed in [Pil12] (specifically, [Pil12, Theorem 1.4]) do not apply in general for the maps considered in Theorem 1.3 (see the discussion in Section 9.3).

Some of our ideas can also be used for the study of the global dynamics of the slope map for Thurston maps that are not covered by Theorem 1.4. In particular, we are able to describe the iterative behavior of $\mu_{f}$ for a specific obstructed Thurston map $f$ obtained by blowing up the $(2 \times 2)$-Lattès map (see Section 9.2 for the details). This provides an answer to a question by Kevin Pilgrim (see [Pil21, Question 4.4]).

While it is straightforward to compute $\mu_{f}(x)$ for individual values $x \in \widehat{\mathbb{Q}} \cup\{\odot\}$, we have been unable to give an explicit formula for $\mu_{f}$ for the maps $f$ we consider. In general, these slope maps show very complicated behavior. Currently, very few explicit computations of slope maps are known in the literature. Except for some very special situations (for example, when the slope map is constant, that is, when $\mu_{f}(x)=\odot$ for all $\left.x \in \widehat{\mathbb{Q}} \cup\{\odot\}\right)$, we are only aware
of computations of slope maps for nearly Euclidian Thurston maps in CFPP12, Section 5] and [Lod13, Section 6]. See also [FPP18] for some general properties of the slope map $\mu_{f}$.

An undergraduate student at UCLA, Darragh Glynn, performed some computer experiments to compute $\mu_{f}$ for maps $f$ as in Theorem 1.2 for $n \geq 3$ (and $n_{h}, n_{v} \geq 1$ corresponding to the rational case). His results show that in these cases the map $\mu_{f}$ does not have the monotonicity property as for $n=2$, but indicate that these maps $f$ still have a global curve attractor (see Section 9.2 for more discussion).
1.3. Organization of this paper. Our paper is organized as follows. In the next two sections we review some background. In Section 2, we fix notation and state some basic definitions. We also discuss isotopy classes of Jordan curves in spheres with four marked points, how isotopy classes of such curves correspond to slopes in $\widehat{\mathbb{Q}}$, as well as some relevant facts about intersection numbers. Even though all of this is fairly standard, we give complete proofs in the appendix, because it is hard to track down this material in the literature with a detailed exposition.

In Section 3 we recall some basics about Thurston maps and the relevant concepts for a precise formulation of Thurston's characterization of rational maps for Thurston maps with four postcritical points - the only case relevant for us (see Section 3.2).

We explain the blow-up procedure for arcs in Section 4 and relate this to the procedure of gluing flaps to the pillow $\mathbb{P}$ (see Section 4.2). The proof of Theorem 1.2 is then given in Section 5.

The proof of our main result, Theorem 1.1, requires more preparation. This is the purpose of Section 6. There we introduce the concept of essential circuit length that will allow us to formulate tight estimates for the number of essential pullbacks of a Jordan curve under a Thurston map with four postcritical points. This is formulated in the rather technical Lemma 6.2 which is of crucial importance though. The proof of Theorem 1.1 is then given in Section 7.

Section 8 is devoted to the proof of Theorem 1.4. In Section 9, we discuss some further directions related to this work. As we already mentioned, the appendix is devoted to the discussion of isotopy classes and intersection numbers of Jordan curves in spheres with four marked points.
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## 2. Preliminaries

In this section, we discuss background relevant for the rest of the paper.
2.1. Notation and basic concepts. We denote by $\mathbb{N}=\{1,2, \ldots\}$ the set of natural numbers and by $\mathbb{N}_{0}=\{0,1,2, \ldots\}$ the set of natural numbers including 0 . The sets of integers, real numbers, and complex numbers are denoted by $\mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$, respectively. We write $i$ for the imaginary unit in $\mathbb{C}$, and $\operatorname{Im}(z)$ for the imaginary part of a complex number $z \in \mathbb{C}$.

As usual, $\mathbb{R}^{2}:=\{(x, y): x, y \in \mathbb{R}\}$ is the Euclidean plane and $\widehat{\mathbb{C}}:=\mathbb{C} \cup\{\infty\}$ is the Riemann sphere. Here and elsewhere, we write $A:=B$ for emphasis when an object $A$ is defined to be another object $B$. When we consider two objects $A$ and $B$, and there is a natural identification between them that is clear from the context, we write $A \cong B$. For example,
$\mathbb{R}^{2} \cong \mathbb{C}$ if we identify a point $(x, y) \in \mathbb{R}^{2}$ with $x+i y \in \mathbb{C}$. We will freely switch back and forth between these different viewpoints of $\mathbb{R}^{2} \cong \mathbb{C}$.

We use the notation $\mathbb{I}:=[0,1] \subset \mathbb{R}$ for the closed unit interval, $\mathbb{D}:=\{z \in \mathbb{C}:|z|<1\}$ for the open unit disk in $\mathbb{C}$, and $\mathbb{Z}^{2}:=\{x+i y: x, y \in \mathbb{Z}\}$ for the square lattice in $\mathbb{C}$. If $z, w \in \mathbb{C}$, then we write $[z, w]:=\{z+t(w-z): t \in \mathbb{I}\}$ for the line segment in $\mathbb{C}$ joining $z$ and $w$. We also use the notation $\left[z_{0}, w_{0}\right):=\left[z_{0}, w_{0}\right] \backslash\left\{w_{0}\right\}$ and $\left(z_{0}, w_{0}\right):=\left[z_{0}, w_{0}\right] \backslash\left\{z_{0}, w_{0}\right\}$.

The cardinality of a set $X$ is denoted by $\# X \in \mathbb{N}_{0} \cup\{\infty\}$ and the identity map on $X$ by $\operatorname{id}_{X}$. If $X$ is a topological space and $M \subset X$, then $\operatorname{cl}(M)$ denotes the closure, $\operatorname{int}(M)$ the interior, and $\partial M$ the boundary of $M$ in $X$.

Let $f: X \rightarrow Y$ be a map between sets $X$ and $Y$. If $M \subset X$, then $f \mid M$ stands for the restriction of $f$ to $M$. If $N \subset Y$, then $f^{-1}(N):=\{x \in X: f(x) \in N\}$ is the preimage of $N$ in $X$. Similarly, $f^{-1}(y):=\{x \in X: f(x)=y\}$ is the preimage of a point $y \in Y$.

Let $f: X \rightarrow X$ be a map. For $n \in \mathbb{N}$ we denote by

$$
f^{n}:=\underbrace{f \circ \cdots \circ f}_{n \text { factors }}
$$

the $n$-th iterate of $f$. It is convenient to define $f^{0}:=\operatorname{id}_{X}$. For $n \in \mathbb{N}_{0}$ we denote by $f^{-n}(M):=$ $\left\{x \in X: f^{n}(x) \in M\right\}$ and $f^{-n}(p):=\left\{x \in X: f^{n}(x)=p\right\}$ the preimages of a set $M \subset X$ and a point $p \in X$ under $f^{n}$, respectively.

A surface $S$ is a connected and oriented topological 2-manifold. We denote its Euler characteristic by $\chi(S)$. Note that $\chi(S) \in\{2,1,0,-1, \ldots\} \cup\{-\infty\}$. Throughout this paper, we use the notation $S^{2}$ for a (topological) 2-sphere, that is, $S^{2}$ indicates a surface homeomorphic to the Riemann sphere $\widehat{\mathbb{C}}$. An annulus is a surface homeomorphic to $\{z \in \mathbb{C}: 1<|z|<2\}$.

A Jordan curve $\alpha$ in a surface $S$ is the image $\alpha=\eta(\partial \mathbb{D})$ of a (topological) embedding $\eta: \partial \mathbb{D} \rightarrow S$ of the unit circle $\partial \mathbb{D}=\{z \in \mathbb{C}:|z|=1\}$ into $S$. An arc $e$ in $S$ is the image $e=\iota(\mathbb{I})$ of an embedding $\iota: \mathbb{I} \rightarrow S$. Then $\iota(0)$ and $\iota(1)$ are the endpoints of $e$, and we define $\partial e:=\{\iota(0), \iota(1)\}$. The set $\operatorname{int}(e):=e \backslash \partial e$ is called the interior of $e$. The notions of endpoints and interior of $e$ only depend on $e$ and not on the choice of the embedding $\iota$. Note that the notation $\partial e$ and $\operatorname{int}(e)$ is ambiguous, because it should not be confused with the boundary and interior of $e$ as a subset of $S$. For arcs $e$ in a surface $S$, we will only use $\partial e$ and $\operatorname{int}(e)$ with the meaning just defined.

A subset $U$ of a surface $S$ is called an open or closed Jordan region if there exists a topological embedding $\eta: \operatorname{cl}(\mathbb{D})=\{z \in \mathbb{C}:|z| \leq 1\} \rightarrow S$ such that $U=\eta(\mathbb{D})$ or $U=\eta(\operatorname{cl}(\mathbb{D}))$, respectively. In both cases, $\partial U=\eta(\partial \mathbb{D})$ is a Jordan curve in $S$. A crosscut $e$ in an open or closed Jordan region $U$ is an $\operatorname{arc} e \subset \operatorname{cl}(U)$ such that $\operatorname{int}(e) \subset \operatorname{int}(U)$ and $\partial e \subset \partial U$.

A path $\gamma$ in a surface $S$ is a continuous map $\gamma:[a, b] \rightarrow S$, where $[a, b] \subset \mathbb{R}$ is a compact (non-degenerate) interval. As is common, we will use the same notation $\gamma$ for the image $\gamma([a, b])$ of the path if no confusion can arise. The path $\gamma$ joins two sets $M, N \subset S$ if $\gamma(a) \in M$ and $\gamma(b) \in N$, or vice versa. A loop in $S$ based at $p \in S$ is a path $\gamma:[a, b] \rightarrow S$ such that $\gamma(a)=\gamma(b)=p$. The loop $\gamma$ is called simple if $\gamma$ is injective on $[a, b)$. So essentially a simple loop is a Jordan curve run through with some parametrization.

Let $M, N, K$ be subsets of a surface $S$. We say that $K$ separates $M$ and $N$ if every path in $S$ joining $M$ and $N$ meets $K$. Note that here $K$ is not necessarily disjoint from $M$ or $N$. We say that $K$ separates a point $p \in S$ from a set $M \subset S$ if $K$ separates $\{p\}$ and $M$.

Let $Z \subset S$ be a finite set of points in a surface $S$. Then we refer to the pair $(S, Z)$ as a marked surface, and the points in $Z$ as the marked points in $S$. The most important case for us will be when $S=S^{2}$ is a 2 -sphere and $Z \subset S^{2}$ consists of four points.

A Jordan curve $\alpha$ in a marked surface $(S, Z)$ is a Jordan curve $\alpha \subset S \backslash Z$. An arc $e$ in $(S, Z)$ is an $\operatorname{arc} e \subset S$ with $\partial e \subset Z$ and $\operatorname{int}(e) \subset S \backslash Z$. We say that a Jordan curve $\alpha$ in a marked sphere $\left(S^{2}, Z\right)$ is essential if each of the two connected components of $S^{2} \backslash \alpha$ contains at least two points of $Z$; otherwise, we say that $\alpha$ is peripheral.

Let $\left(S^{2}, Z\right)$ be a marked sphere with $\# Z=4$. A core arc of an essential Jordan curve $\alpha$ in $\left(S^{2}, Z\right)$ is an arc in $\left(S^{2}, Z\right)$ that is contained in one of the two connected components of $S^{2} \backslash \alpha$ and joins the two points in $Z$ that lie in this component.

Let $A$ be an annulus. Then a core curve of $A$ is a Jordan curve $\beta \subset A$ such that under some homeomorphism $\varphi: A \rightarrow A^{\prime}$ the curve $\beta^{\prime}=\varphi(\beta)$ separates the boundary components of $A^{\prime}=\{z \in \mathbb{C}: 1<|z|<2\}$.
2.2. Branched covering maps. Let $X$ and $Y$ be surfaces. Then a continuous map $f: X \rightarrow$ $Y$ is called a branched covering map if for each point $q \in Y$ there exists an open set $V \subset Y$ homeomorphic to $\mathbb{D}$ with $q \in V$ that is evenly covered in the following sense: for some index set $J \neq \varnothing$ we can write $f^{-1}(V)$ as a disjoint union

$$
\begin{equation*}
f^{-1}(V)=\bigcup_{j \in J} U_{j} \tag{2.1}
\end{equation*}
$$

of open sets $U_{j} \subset X$ such that $U_{j}$ contains precisely one point $p_{j} \in f^{-1}(q)$. Moreover, we require that for each $j \in J$ there exists $d_{j} \in \mathbb{N}$ and orientation-preserving homeomorphisms $\varphi_{j}: U_{j} \rightarrow \mathbb{D}$ with $\varphi_{j}\left(p_{j}\right)=0$ and $\psi_{j}: V \rightarrow \mathbb{D}$ with $\psi_{j}(q)=0$ such that

$$
\left(\psi_{j} \circ f \circ \varphi_{j}^{-1}\right)(z)=z^{d_{j}}
$$

for all $z \in \mathbb{D}$ (see [BM17, Section A.6] for more background on branched covering maps). For given $f$, the number $d_{j}$ is uniquely determined by $p=p_{j}$, and called the local degree of $f$ at $p$ and denoted by $\operatorname{deg}(f, p)$. A point $p \in X$ with $\operatorname{deg}(f, p) \geq 2$ is called a critical point of $f$. The set of all critical points of $f$ is a discrete set in $X$ and denoted by $C_{f}$. If $f$ is a branched covering map, then it is a covering map (in the usual sense) from $X \backslash f^{-1}\left(f\left(C_{f}\right)\right)$ onto $Y \backslash f\left(C_{f}\right)$.

In the following, suppose $X$ and $Y$ are compact surfaces, and $f: X \rightarrow Y$ is a branched covering map. Then $C_{f} \subset X$ is a finite set. Moreover, if $\operatorname{deg}(f) \in \mathbb{N}$ denotes the topological degree of $f$, then

$$
\sum_{p \in f^{-1}(q)} \operatorname{deg}(f, p)=\operatorname{deg}(f)
$$

for each $q \in Y$.
If $\gamma:[a, b] \rightarrow Y$ is a path, then we call a path $\widetilde{\gamma}:[a, b] \rightarrow X$ a lift of $\gamma($ under $f$ ) if $f \circ \widetilde{\gamma}=\gamma$. Every path $\gamma$ in $Y$ has a lift $\widetilde{\gamma}$ in $X$ (see [BM17, Lemma A.18]), but in general $\widetilde{\gamma}$ is not unique. If $\gamma([a, b)) \subset Y \backslash f\left(C_{f}\right)$ and $x_{0} \in f^{-1}(\gamma(a))$, then there exists a unique lift $\widetilde{\gamma}:[a, b] \rightarrow X$ of $\gamma$ under $f$ with $\widetilde{\gamma}(a)=x_{0}$. This easily follows from standard existence and uniqueness theorems for lifts under covering maps (see [BM17, Lemma A.6]).

If $e \subset Y$ is an arc, then an $\operatorname{arc} \widetilde{e} \subset X$ is called a lift of $e($ under $f$ ) if $f \mid \widetilde{e}$ is a homeomorphism of $\widetilde{e}$ onto $e$. It easily follows from the existence and uniqueness statements for lifts of paths just discussed that if $e$ is an arc in $\left(Y, f\left(C_{f}\right)\right), y_{0} \in \operatorname{int}(e)$, and $x_{0} \in f^{-1}\left(y_{0}\right)$, then there exists a unique lift $\widetilde{e} \subset X$ of $e$ with $x_{0} \in \widetilde{e}$.

Let $V \subset Y$ be an open and connected set and $U \subset f^{-1}(V)$ be a (connected) component of $f^{-1}(V)$. Then $f \mid U: U \rightarrow V$ is also a branched covering map. Each point $q \in V$ has the same number $d \in \mathbb{N}$ of preimages under $f \mid U$ counting local degrees. We set $\operatorname{deg}(f \mid U):=d$. If the Euler characteristic $\chi(V)$ is finite, then $\chi(U)$ is also finite and we have the Riemann-Hurwitz formula

$$
\begin{equation*}
\chi(U)+\sum_{p \in U \cap C_{f}}(\operatorname{deg}(f, p)-1)=\operatorname{deg}(f \mid U) \cdot \chi(V) \tag{2.2}
\end{equation*}
$$

2.3. Planar embedded graphs. A planar embedded graph in a sphere $S^{2}$ is a pair $G=$ $(V, E)$, where $V$ is a finite set of points in $S^{2}$ and $E$ is a finite set of $\operatorname{arcs}$ in $\left(S^{2}, V\right)$ with pairwise disjoint interiors. The sets $V$ and $E$ are called the vertex and edge sets of $G$, respectively. Note that our notion of a planar embedded graph does not allow loops, that is, edges that connect a vertex to itself, but it does allow multiple edges, that is, distinct edges that join the same pair of vertices. The degree of a vertex $v$ in $G$, denoted $\operatorname{deg}_{G}(v)$, is the number of edges of $G$ incident to $v$. Note that $2 \cdot \# E=\sum_{v \in V} \operatorname{deg}_{G}(v)$.

The realization of $G$ is the subset $\mathcal{G}$ of $S^{2}$ given by

$$
\mathcal{G}:=V \cup \bigcup_{e \in E} e .
$$

A face of $G$ is a connected component of $S^{2} \backslash \mathcal{G}$. Usually, we conflate a planar embedded graph $G$ with its realization $\mathcal{G}$. Then it is understood that $\mathcal{G}$ contains a finite set $V \subset \mathcal{G}$ of distinguished points that are the vertices of the graph. Its edges are the closures of the components of $\mathcal{G} \backslash V$.

A subgraph of a planar embedded graph $G=(V, E)$ is a planar embedded graph $G^{\prime}=$ $\left(V^{\prime}, E^{\prime}\right)$ with $V^{\prime} \subset V$ and $E^{\prime} \subset E$. A path of length $n$ between vertices $v$ and $v^{\prime}$ in $G$ is a sequence $v_{0}, e_{0}, v_{1}, e_{1}, \ldots, e_{n-1}, v_{n}$, where $v_{0}=v, v_{n}=v^{\prime}$, and $e_{k}$ is an edge incident to the vertices $v_{k}$ and $v_{k+1}$ for $k=0, \ldots, n-1$. A path that does not repeat vertices is called a simple path.

A path $v_{0}, e_{0}, v_{1}, e_{1}, \ldots, e_{n-1}, v_{n}$ with $v_{0}=v_{n}$ and $n \geq 2$ is called a circuit of length $n$ in $G$ and is denoted by $\left(e_{0}, e_{1}, \ldots, e_{n-1}\right)$. Such a circuit is called a simple cycle if all vertices $v_{k}$, $k=0, \ldots, n-1$, are distinct.

A planar embedded graph $G$ is called connected if any two distinct vertices of $G$ can be joined by a path in $G$. Equivalently, $G$ is connected if its realization $\mathcal{G}$ is connected as a subset of $S^{2}$. Note that if $G$ is connected, then each face of $G$ is simply-connected.

As follows from [Die05, Lemma 4.2.2], the topological boundary $\partial U$ of each face $U$ of $G$ may be viewed as the realization of a subgraph of $G$. Moreover, a walk around any connected component of the boundary $\partial U$ traces a circuit $\left(e_{0}, e_{1}, \ldots, e_{n-1}\right)$ in $G$ such that each edge of $G$ appears zero, one, or two times in the sequence $e_{0}, e_{1}, \ldots, e_{n-1}$. We will say that the circuit $\left(e_{0}, e_{1}, \ldots, e_{n-1}\right)$ traces (a connected component of) the boundary $\partial U$. If $U$ is simply-connected, then $\partial U$ is connected, and the length of the (essentially unique) circuit that bounds $U$ is called the circuit length of $U$ in $G$.

A planar embedded graph $(V, E)$ is called bipartite if we can split $V$ into two disjoint subsets $V_{1}$ and $V_{2}$ such that each edge $e \in E$ has one endpoint in $V_{1}$ and one in $V_{2}$.
2.4. The Euclidean square pillow. As discussed in the introduction, we consider a square pillow $\mathbb{P}$ obtained from gluing two identical copies of the unit square $\mathbb{I}^{2} \subset \mathbb{R}^{2}$ along their boundaries by the identity map. Then $\mathbb{P}$ is a topological 2 -sphere. We equip $\mathbb{P}$ with the induced path metric that agrees with the Euclidean metric on each of the two copies of the


Figure 4. The Euclidean square pillow $\mathbb{P}$.
unit square. We call this metric space $\mathbb{P}$ the Euclidean square pillow. The vertices and edges of the unit square $\mathbb{I}^{2}$ in $\mathbb{P}$ are called the vertices and edges of $\mathbb{P}$. One copy of $\mathbb{I}^{2}$ in $\mathbb{P}$ is called the front and the other copy the back side of $\mathbb{P}$. In a dynamical context, we also refer to these two copies of $\mathbb{I}^{2}$ as the 0 -tiles of $\mathbb{P}$. We color the front side of $\mathbb{P}$ white, and its back side black. Finally, we equip $\mathbb{P}$ with the orientation that agrees with the standard orientation on the front side $\mathbb{I}^{2}$ of $\mathbb{P}$ (represented by the positively-oriented standard flag $\left((0,0), \mathbb{I} \times\{0\}, \mathbb{I}^{2}\right)$; see [BM17, Appendix A.4]).

We label the vertices and edges of $\mathbb{P}$ in counterclockwise order by $A, B, C, D$ and $a, b, c, d$, respectively, so that $A \in \mathbb{P}$ corresponds to the vertex $(0,0) \in \mathbb{I}^{2}$ and the edge $a \subset \mathbb{P}$ corresponds to $[0,1] \times\{0\} \subset \mathbb{I}^{2}$. Then $a$ has the endpoints $A$ and $B$. We can view the boundary $\partial \mathbb{I}^{2}$ of $\mathbb{I}^{2}$ as a planar embedded graph in $\mathbb{P}$ with the vertex set $V:=\{A, B, C, D\}$ and the edge set $E:=\{a, b, c, d\}$. We call $a$ and $c$ the horizontal edges, and $b$ and $d$ the vertical edges of $\mathbb{P}$; see Figure 4.

The pillow $\mathbb{P}$ is an example of a Euclidean polyhedral surface, that is, a surface obtained by gluing Euclidean polygons along boundary edges by using isometries. Note that the metric on $\mathbb{P}$ is locally flat except at its vertices, which are Euclidean conic singularities. So $\mathbb{P}$ is an orbifold (see, for example, Mil06a, Appendix E] and [BM17, Appendix A.9]).

An alternative description for the pillow $\mathbb{P}$ can be given as follows. We consider the unit square $\mathbb{I}^{2} \subset \mathbb{R}^{2} \cong \mathbb{C}$ and map it to the upper half-plane in $\widehat{\mathbb{C}}$ by a conformal map, normalized so that the vertices $0,1,1+i, i$ are mapped to $0,1, \infty,-1$, respectively. By Schwarz reflection, this map can be extended to a meromorphic function $\wp: \mathbb{C} \rightarrow \widehat{\mathbb{C}}$. Then $\wp$ is a Weierstrass $\wp$-function (up to a postcomposition with a Möbius transformation) that is doubly periodic with respect to the lattice $2 \mathbb{Z}^{2}:=\{2 k+2 n i: k, n \in \mathbb{Z}\} \subset \mathbb{C}$. Actually, for $z, w \in \mathbb{C}$ we have

$$
\begin{equation*}
\wp(z)=\wp(w) \text { if and only if } z-w \in 2 \mathbb{Z}^{2} \text { or } z+w \in 2 \mathbb{Z}^{2} . \tag{2.3}
\end{equation*}
$$

We can push forward the Euclidean metric on $\mathbb{C}$ to the Riemann sphere $\widehat{\mathbb{C}}$ by $\wp$. With respect to this metric, called the canonical orbifold metric for $\wp$, the sphere $\widehat{\mathbb{C}}$ is isometric to the Euclidean square pillow $\mathbb{P}$. In the following, we identify the pillow $\mathbb{P}$ with $\widehat{\mathbb{C}}$ by the orientation-preserving isometry that maps the vertices $A, B, C, D$ to $0,1, \infty,-1$, respectively. Then we can consider $\wp: \mathbb{C} \rightarrow \widehat{\mathbb{C}} \cong \mathbb{P}$ as a map onto the pillow $\mathbb{P}$. Actually, $\wp$ is the universal orbifold covering map for $\mathbb{P}$ (see [BM17, Section A.9] for more background). A very intuitive description of this map can be given if we color the squares $[k, k+1] \times[n, n+1], k, n \in \mathbb{Z}$, in checkerboard manner black and white so that $[0,1] \times[0,1]$ is white. Restricted to such a square $S$, the map $\wp$ is an isometry that sends $S$ to the white 0 -tile of $\mathbb{P}$ if $S$ is white, and to the black 0 -tile $\mathbb{P}$ if $S$ is black; see Figure 5 for an illustration. Here, the points in the


Figure 5. The map $\wp: \mathbb{C} \rightarrow \mathbb{P}$.
complex plane $\mathbb{C}$ marked by a black dot (on the left) are mapped to $A$ by $\wp$ and are elements of $\wp^{-1}(A)=2 \mathbb{Z}^{2}$.
2.5. Isotopies and intersection numbers. Let $X$ and $Y$ be topological spaces. Then a continuous map $H: X \times \mathbb{I} \rightarrow Y$ is called a homotopy from $X$ to $Y$. For $t \in \mathbb{I}$ we denote by $H_{t}:=H(\cdot, t): X \rightarrow Y$ the time-t map of the homotopy. The homotopy $H$ is called an isotopy if $H_{t}$ is a homeomorphism from $X$ onto $Y$ for each $t \in \mathbb{I}$. If $Z \subset X$, then a homotopy $H: X \times \mathbb{I} \rightarrow Y$ is said to be a homotopy relative to $Z$ (abbreviated " $H$ is a homotopy rel. $Z$ ") if $H_{t}(p)=H_{0}(p)$ for all $p \in Z$ and $t \in \mathbb{I}$. In other words, the image of each point in $Z$ remains fixed during the homotopy $H$. Isotopies rel. $Z$ are defined in a similar way.

Two homeomorphisms $h_{0}, h_{1}: X \rightarrow Y$ are called isotopic (rel. $Z \subset X$ ) if there exists an isotopy $H: X \times \mathbb{I} \rightarrow Y($ rel. $Z)$ with $H_{0}=h_{0}$ and $H_{1}=h_{1}$. Given $M, N, Z \subset X$, we say that $M$ is isotopic to $N$ rel. $Z$ (or $M$ can be isotoped into $N$ rel. $Z$ ), denoted by $M \sim N$ rel. $Z$, if there exists an isotopy $H: X \times \mathbb{I} \rightarrow X$ rel. $Z$ with $H_{0}=\mathrm{id}_{X}$ and $H_{1}(M)=N$. Recall that $\mathrm{id}_{X}$ is the identity map on $X$.

Let $(S, Z)$ be a marked surface (with a finite, possibly empty set $Z \subset S$ of marked points). If $\alpha$ is a Jordan curve in $(S, Z)$, then its isotopy class $[\alpha]$ (with $(S, Z)$ understood) consists of all Jordan curves $\beta$ in $(S, Z)$ such that $\alpha \sim \beta$ rel. $Z$.

The following statement gives a sufficient condition for two Jordan curves in $(S, Z)$ to be isotopic rel. $Z$.
Lemma 2.1. Let $\alpha$ and $\beta$ be disjoint Jordan curves in a marked surface $(S, Z)$. Suppose there is an annulus $U \subset S \backslash Z$ such that $\partial U=\alpha \cup \beta$. Then $\alpha$ and $\beta$ are isotopic rel. $Z$.

Proof. This is standard and we will only give a sketch of the proof. Since Jordan curves in surfaces are tame, one can slightly enlarge the annulus $U$ to an annulus $U^{\prime} \subset S^{2} \backslash Z$ that contains $\alpha$ and $\beta$. Then $\alpha$ can be isotoped into $\beta$ by an isotopy on $U^{\prime}$ that is the identity near $\partial U^{\prime}$. This isotopy on $U^{\prime}$ can be extended to an isotopy on $S^{2}$ rel. $Z$ that isotopes $\alpha$ into $\beta$.

Let $(S, Z)$ be a marked surface. If $\alpha$ and $\beta$ are arcs or Jordan curves in $(S, Z)$, we define their (unsigned) intersection number as

$$
\mathrm{i}(\alpha, \beta):=\inf \left\{\#\left(\alpha^{\prime} \cap \beta^{\prime}\right): \alpha \sim \alpha^{\prime} \text { rel. } Z \text { and } \beta \sim \beta^{\prime} \text { rel. } Z\right\} .
$$

The relevant marked surface $(S, Z)$ here will be understood from the context, and we suppress it from our notation for intersection numbers. If we want to emphasize it, we will say that we consider intersection numbers in $(S, Z)$. The intersection number is always finite, because
we can always reduce to the case when $\alpha$ and $\beta$ are piecewise geodesic with respect to some Riemannian metric on $S$ (see [Bus10, Lemma A.8]). If $\alpha$ and $\beta$ satisfy $\mathrm{i}(\alpha \cap \beta)=\#(\alpha \cap \beta)$, then we say that $\alpha$ and $\beta$ are in minimal position (in their isotopy classes rel. $Z$ ).

Suppose $\alpha$ and $\beta$ are arcs or Jordan curves in $(S, Z)$. Then we say that $\alpha$ and $\beta$ meet transversely at a point $p \in \alpha \cap \beta \cap(S \backslash Z)$ (or $\alpha$ crosses $\beta$ at $p$ ) if $p$ is an isolated point in $\alpha \cap \beta$ and if the following condition is true for a (small) $\operatorname{arc} \sigma \subset \alpha$ containing $p$ as an interior point such that $\sigma \cap \beta=\{p\}$ : let $\sigma^{L}$ and $\sigma^{R}$ be the two subarcs of $\sigma$ into which $\sigma$ is split by $p$, then with suitable orientation of $\beta$ near $p$ the arc $\sigma^{L}$ lies to the left and $\sigma^{R}$ to the right of $\beta$. We say that $\alpha$ and $\beta$ meet transversely or have transverse intersection if the set $\alpha \cap \beta$ is finite and if $\alpha$ and $\beta$ meet transversely at each point $p \in \alpha \cap \beta \cap(S \backslash Z)$.

Lemma 2.2. Suppose $\alpha$ and $\beta$ are Jordan curves or arcs in a marked surface (S, $Z$ ). If $\alpha$ and $\beta$ are in minimal position, then $\alpha$ and $\beta$ meet transversely.

Proof. This is essentially a standard fact (see, for example, [Bus10, pp. 416-417]), and we will only give an outline of the proof.

Since $\#(\alpha \cap \beta)=\mathrm{i}(\alpha \cap \beta)$, the set $\alpha \cap \beta$ consists of finitely many isolated points. To reach a contradiction, suppose that $\alpha$ and $\beta$ do not meet transversely at some point $p \in \alpha \cap \beta \cap(S \backslash Z)$. Then there exists an arc $\sigma \subset \alpha$ containing $p$ as an interior point such that $\sigma \cap \beta=\{p\}$ and with the following property: if $\sigma_{1}$ and $\sigma_{2}$ denote the two subarcs of $\sigma$ into which $\sigma$ is split by $p$, then $\sigma_{1}$ and $\sigma_{2}$ lie on the same side of $\beta$ (equipped with some orientation locally near $p$ ). In other words, $\alpha$ touches $\beta$ locally near $p$ from one side and does not cross $\beta$ at $p$.

We can then modify the curve $\alpha$ near $p$ by an isotopy that pulls the subarc $\sigma$ away from $\beta$ so that the new curve $\alpha$ does not have the intersection point $p$ with $\beta$ while no new intersection points of $\alpha$ and $\beta$ arise. This contradicts our assumption that for the original curve $\alpha$ we have $\#(\alpha \cap \beta)=\mathrm{i}(\alpha \cap \beta)$.
2.6. Jordan curves in spheres with four marked points. If $\left(S^{2}, Z\right)$ is a marked sphere where $Z \subset S^{2}$ consists of exactly four points, then, up to homeomorphism, we may assume that $S^{2}$ is equal to the pillow $\mathbb{P}$, and $Z=V=\{A, B, C, D\}$ consists of the four vertices of $\mathbb{P}$. We will freely switch back and forth between a general marked sphere ( $S^{2}, Z$ ) with $\# Z=4$ and $(\mathbb{P}, V)$.

We need some statements about isotopy classes of Jordan curves and $\operatorname{arcs}$ in $(\mathbb{P}, V)$ and their intersections numbers. They are "well known", but unfortunately we have been unable to track down a comprehensive account in the literature. Accordingly, we will provide a complete treatment. This may be of independent interest apart from the main objective of the paper. We will give the statements in this section, but will provide the details of the proofs in the appendix.

As we will see, there is a natural way to define a bijection between the set of isotopy classes $[\gamma]$ of essential Jordan curves $\gamma$ in $(\mathbb{P}, V)$ and the set of extended rational numbers $\widehat{\mathbb{Q}}:=\mathbb{Q} \cup\{\infty\}$. Throughout this paper, whenever we write $r / s \in \widehat{\mathbb{Q}}$, we assume that $r \in \mathbb{Z}$ and $s \in \mathbb{N}_{0}$ are two relatively prime integers. We allow $s=0$ here, in which case we assume $r=1$. Then $r / s=1 / 0:=\infty \in \widehat{\mathbb{Q}}$.

We say that a (straight) line $\ell \subset \mathbb{C}$ has slope $r / s \in \widehat{\mathbb{Q}}$ if it is given as

$$
\ell=\left\{z_{0}+(s+i r) t: t \in \mathbb{R}\right\} \subset \mathbb{C}
$$



Figure 6. A line $\ell_{2}$ and the corresponding Jordan curve $\tau_{2}=\wp\left(\ell_{2}\right)$ in $\mathbb{P}$.
for some $z_{0} \in \mathbb{C}$. We use the notation $\ell_{r / s}\left(z_{0}\right)$ for the unique line in $\mathbb{C}$ with slope $r / s$ passing through $z_{0} \in \mathbb{C}$, and the notation $\ell_{r / s}$ (when the point $z_{0}$ is not important) for any line in $\mathbb{C}$ with slope $r / s$.

Let $\ell_{r / s} \subset \mathbb{C}$ be any line with slope $r / s \in \widehat{\mathbb{Q}}$. If $\ell_{r / s}$ does not contain any point in the lattice $\mathbb{Z}^{2}=\wp^{-1}(V)$ and so $\ell_{r / s} \subset \mathbb{C} \backslash \mathbb{Z}^{2}$, then $\tau_{r / s}:=\wp\left(\ell_{r / s}\right)$ is a Jordan curve in $\mathbb{P} \backslash V$. Actually, $\tau_{r / s}$ is a simple closed geodesic in the Euclidean square pillow $\mathbb{P}$ (see Figure 6 for an illustration). If $\ell_{r / s}$ contains a point in $\mathbb{Z}^{2}$, then $\xi_{r / s}:=\wp\left(\ell_{r / s}\right)$ is a geodesic $\operatorname{arc}$ in $(\mathbb{P}, V)$.

It is easy to see that every simple closed geodesic or geodesic arc $\tau$ in $(\mathbb{P}, V)$ has the form $\tau=\wp\left(\ell_{r / s}\right)$ for a line $\ell_{r / s} \subset \mathbb{C}$ with some slope $r / s \in \widehat{\mathbb{Q}}$. In the following, we use the notation $\tau_{r / s}$ for a simple closed geodesic and $\xi_{r / s}$ for a geodesic arc obtained in this way.

It follows from (2.3) that for fixed $r / s \in \widehat{\mathbb{Q}}$ we obtain precisely two distinct arcs $\xi_{r / s}$ and $\xi_{r / s}^{\prime}$ of the form $\wp\left(\ell_{r / s}\left(z_{0}\right)\right)$ depending on $z_{0} \in \mathbb{Z}^{2}$. For each simple closed geodesic $\tau_{r / s}$ the arcs $\xi_{r / s}$ and $\xi_{r / s}^{\prime}$ are core arcs of $\tau_{r / s}$ lying in different components of $\mathbb{P} \backslash \tau_{r / s}$ (see the appendix for more details). In particular, $\tau_{r / s}$ is always an essential Jordan curve in $(\mathbb{P}, V)$.

It turns out that the isotopy classes of essential Jordan curves in $(\mathbb{P}, V)$ are closely related to the simple closed geodesics $\tau_{r / s}$.

Lemma 2.3. Let $\gamma$ be an essential Jordan curve in $(\mathbb{P}, V)$. Then there exists a unique slope $r / s \in \widehat{\mathbb{Q}}$ with the following property. Let $\ell_{r / s}$ be any line in $\mathbb{C}$ with slope $r / s$ and $\ell_{r / s} \subset \mathbb{C} \backslash \mathbb{Z}^{2}$, and set $\tau_{r / s}:=\wp\left(\ell_{r / s}\right)$. Then $\tau_{r / s}$ is an essential Jordan curve in $(\mathbb{P}, V)$ with $\gamma \sim \tau_{r / s}$ rel. $V$. Moreover, the map $[\gamma] \mapsto r / s$ gives a bijection between isotopy classes $[\gamma]$ of essential Jordan curves $\gamma$ in $(\mathbb{P}, V)$ and slopes $r / s \in \widehat{\mathbb{Q}}$.

While this is well known (see, for example, [FM12, Proposition 2.6] or [KS94, Proposition 2.1]), we find the available proofs too sketchy. This is the reason why we provide a detailed proof in the appendix. Implicit in Lemma 2.3 is the fact that the isotopy class [ $\tau_{r / s}$ ] of $\tau_{r / s}=\wp\left(\ell_{r / s}\right)$ only depends on $r / s$ and not on the specific choice of the line $\ell_{r / s}$ with $\ell_{r / s} \subset \mathbb{C} \backslash \mathbb{Z}^{2}$ (see Lemma 10.7 for an explicit statement).

Recall that $a, c$ denote the horizontal, and $b, d$ the vertical edges of $\mathbb{P}$. In the following, we denote by $\alpha^{h}=\tau_{0}$ a horizontal essential Jordan curve in $(\mathbb{P}, V)$ (corresponding to slope 0 and separating the edges $a$ and $c$ of $\mathbb{P})$ and by $\alpha^{v}=\tau_{\infty}$ a vertical essential Jordan curve in $(\mathbb{P}, V)$ (corresponding to slope $\infty$ and separating $b$ from $d$ ). To be specific, we set

$$
\begin{equation*}
\alpha^{h}:=\wp(\mathbb{R} \times\{1 / 2\}) \text { and } \alpha^{v}:=\wp(\{1 / 2\} \times \mathbb{R}) . \tag{2.4}
\end{equation*}
$$



Figure 7. Counting intersections of $\tau_{2}$ with the horizontal curve $\alpha^{h}$ and the horizontal edges $a$ and $c$.

The following lemma summarizes the intersection properties of essential Jordan curves and $\operatorname{arcs}$ in $(\mathbb{P}, V)$.

Lemma 2.4. Let $\alpha$ and $\beta$ be essential Jordan curves in $(\mathbb{P}, V)$ and $r / s, r^{\prime} / s^{\prime} \in \widehat{\mathbb{Q}}$ be the unique slopes such that $\alpha \sim \tau_{r / s}$ and $\beta \sim \tau_{r^{\prime} / s^{\prime}}$ rel. $V$, where $\tau_{r / s}$ and $\tau_{r^{\prime} / s^{\prime}}$ are simple closed geodesics in $(\mathbb{P}, V)$ with slopes $r / s$ and $r^{\prime} / s^{\prime}$, respectively. Let $\xi$ be a core arc of $\beta$, and $\xi_{r^{\prime} / s^{\prime}}$ be a geodesic arc in $(\mathbb{P}, V)$ with slope $r^{\prime} / s^{\prime}$. Then the following statements are true for intersection numbers in $(\mathbb{P}, V)$ :
(i) If $r / s=r^{\prime} / s^{\prime}$, then $\mathrm{i}(\alpha, \beta)=0$, and if $r / s \neq r^{\prime} / s^{\prime}$, then $\mathrm{i}(\alpha, \beta)=\#\left(\tau_{r / s} \cap \tau_{r^{\prime} / s^{\prime}}\right)=$ $2\left|r s^{\prime}-s r^{\prime}\right|>0$.
(ii) $\mathrm{i}(\alpha, \xi)=\#\left(\tau_{r / s} \cap \xi_{r^{\prime} / s^{\prime}}\right)=\frac{1}{2} \mathrm{i}(\alpha, \beta)=\left|r s^{\prime}-s r^{\prime}\right|$.
(iii) $\mathrm{i}(\alpha, a)=\#\left(\tau_{r / s} \cap a\right)=|r|, \mathrm{i}(\alpha, c)=\#\left(\tau_{r / s} \cap c\right)=|r|$.
(iv) $\mathrm{i}(\alpha, b)=\#\left(\tau_{r / s} \cap b\right)=s, \mathrm{i}(\alpha, d)=\#\left(\tau_{r / s} \cap d\right)=s$.
(v) $\mathrm{i}\left(\alpha, \alpha^{h}\right)=2|r|$ and $\mathrm{i}\left(\alpha, \alpha^{v}\right)=2 s$.

We will prove this lemma in the appendix. Note that Figure 7 illustrates (iii) and (v) when $\alpha=\tau_{2}$. It follows from the statement that $\tau_{r / s}$ for $r / s \neq 0, \infty$ is in minimal position with each of the curves $a, b, c, d, \alpha^{h}, \alpha^{v}$.

Let $\gamma$ be a Jordan curve or an arc in a surface $S$, and $M_{1}, M_{2} \subset S$ be two disjoint sets with $0<\#\left(\gamma \cap M_{j}\right)<\infty$ for $j=1,2$. We say that the points in $\gamma \cap M_{1} \neq \varnothing$ and $\gamma \cap M_{2} \neq \varnothing$ alternate on $\gamma$ if any two points in one of the sets are separated by the other, that is, any subarc $\sigma \subset \gamma$ with both endpoints in either of the sets $\gamma \cap M_{1}$ or $\gamma \cap M_{2}$ must contain a point in the other set.

More intuitively, this situation when the points in $\gamma \cap M_{1}$ and $\gamma \cap M_{2}$ alternate on an arc $\gamma$ can be described as follows. Suppose we traverse $\gamma$ in some (injective) parametrization starting from one of its endpoints. Then we will first meet a point in either $M_{1}$ or $M_{2}$, say in $M_{1}$. Then as we continue along $\gamma$, we will meet a point in $M_{2}$, then a point in $M_{1}$, etc. A similar remark applies when $\gamma$ is a Jordan curve. Note that is this case $\#\left(\gamma \cap M_{1}\right)=\#\left(\gamma \cap M_{2}\right)$.

Lemma 2.5. Let $\tau=\wp\left(\ell_{r / s}\right)$ be a simple closed geodesic or a geodesic arc in $(\mathbb{P}, V)$ obtained from a line $\ell_{r / s} \subset \mathbb{C}$ with slope $r / s \in \widehat{\mathbb{Q}}$. If $r / s \neq 0$, then the sets $a \cap \tau$ and $c \cap \tau$ are non-empty and finite, and the points in $a \cap \tau$ and $c \cap \tau$ alternate on $\tau$.

A similar statement is true if $r / s \neq \infty$ and we replace $a, c$ with $b, d$, respectively. Lemma 2.5 is related to a similar statement in a more general setting that is the key to proving Lemma 2.3 (see Lemma 10.3).

Proof. Define $\omega:=s+i r \in \mathbb{Z}^{2}$. Suppose first that $\tau=\wp\left(\ell_{r / s}\right)$ is a simple closed geodesic. Then $\tau=\wp\left(\left[z_{0}, w_{0}\right]\right)$, where $z_{0} \in \ell_{r / s} \subset \mathbb{C} \backslash \mathbb{Z}^{2}$ and $w_{0}=z_{0}+2 \omega$, and the map $u \in[0,1] \mapsto$ $\wp\left(u z_{0}+(1-u) w_{0}\right)$ provides a parametrization of $\tau$ as a simple loop as follows from (2.3).

Now the set $\wp^{-1}(a \cup c)$ consists precisely of the lines $\ell_{0}(n i)=\{z \in \mathbb{C}: \operatorname{Im}(z)=n\}, n \in \mathbb{Z}$. These lines alternate in the sense that $\wp$ maps $\ell_{0}(n i)$ onto $a$ or $c$ depending on whether $n \in \mathbb{Z}$ is even or odd, respectively. Since $r / s \neq 0$, we have $r \in \mathbb{Z} \backslash\{0\}$, and so $\operatorname{Im}\left(w_{0}-z_{0}\right)=\operatorname{Im}(2 \omega)=2 r$ is a non-zero even integer. This implies that the line segment $\left[z_{0}, w_{0}\right] \subset \ell_{r / s}$ has non-empty intersections (consisting of finitely many points) with each of the sets $\wp^{-1}(a)$ and $\wp^{-1}(c)$. Moreover, the points in these intersections alternate on the segment $\left[z_{0}, w_{0}\right]$. From this together with the fact that

$$
\#\left(\wp^{-1}(a) \cap\left[z_{0}, w_{0}\right)\right)=|r|=\#\left(\wp^{-1}(c) \cap\left[z_{0}, w_{0}\right)\right)
$$

the statement follows (the latter fact is needed to argue that the points in $a \cap \tau$ and $c \cap \tau$ alternate on the simple closed geodesic $\tau$ ).

If $\tau$ is a geodesic arc, then there exists $z_{0} \in \mathbb{Z}^{2}$ such that $\tau=\wp\left(\left[z_{0}, w_{0}\right]\right)$, where $w_{0}=$ $z_{0}+\omega$. The map $\wp$ sends $\left[z_{0}, w_{0}\right]$ homeomorphically onto $\tau$. Again, $\left[z_{0}, w_{0}\right]$ has non-empty intersections consisting of finitely many points with each of the sets $\wp^{-1}(a)$ and $\wp^{-1}(c)$. Moreover, the points in the sets $\wp^{-1}(a) \cap\left[z_{0}, w_{0}\right] \neq \varnothing$ and $\wp^{-1}(c) \cap\left[z_{0}, w_{0}\right] \neq \varnothing$ alternate on the segment $\left[z_{0}, w_{0}\right]$. The statement also follows in this case.

We conclude this section with a statement related to the previous considerations formulated for an arbitrary sphere with four marked points.

Lemma 2.6. Let $\left(S^{2}, Z\right)$ be a marked sphere with $\# Z=4$, and $\alpha, \gamma$ be essential Jordan curves in $\left(S^{2}, Z\right)$. Suppose that $a_{\alpha}$ and $c_{\alpha}$ are core arcs of $\alpha$ that lie in different components of $S^{2} \backslash \alpha$. Then the following statements are true:
(i) $\mathrm{i}(\alpha, \gamma)=2 \mathrm{i}\left(a_{\alpha}, \gamma\right)=2 \mathrm{i}\left(c_{\alpha}, \gamma\right)$.
(ii) If $\mathrm{i}(\alpha, \gamma)>0$, then there exists a Jordan curve $\gamma^{\prime}$ in $\left(S^{2}, Z\right)$ with $\gamma^{\prime} \sim \gamma$ rel. $Z$ such that $\gamma^{\prime}$ is in minimal position with $\alpha, a_{\alpha}, c_{\alpha}$ and the points in $a_{\alpha} \cap \gamma^{\prime} \neq \varnothing$ and $c_{\alpha} \cap \gamma^{\prime} \neq \varnothing$ alternate on $\gamma^{\prime}$.

Proof. We may identify the marked sphere $\left(S^{2}, Z\right)$ with the pillow $(\mathbb{P}, V)$ by a homeomorphism that sends $\alpha, a_{\alpha}, c_{\alpha}$ to $\alpha^{h}, a, c$, respectively. Then, by Lemma 2.3, the curve $\gamma$ is isotopic to a simple closed geodesic $\tau_{r / s}$ with slope $r / s \in \widehat{\mathbb{Q}}$. Statement (i) then follows from Lemma 2.4 (iii) and (v).

If $\mathrm{i}(\alpha, \gamma)=\mathrm{i}\left(\alpha^{h}, \tau_{r / s}\right)=2|r|>0$, then we can choose $\gamma^{\prime}=\tau_{r / s}$ in (ii). Indeed, then $\gamma^{\prime}=\tau_{r / s} \sim \gamma$ rel. $V=Z$, and $\gamma^{\prime}=\tau_{r / s}$ is in minimal position with $\alpha^{h}=\tau_{0}, a_{\alpha}=a, c_{\alpha}=c$ as follows from Lemma 2.4 (i) and (iii). Since $r / s \neq 0$ in this case, the statement about alternation follows from Lemma 2.5.

## 3. Thurston maps

Here we provide a very brief summary of some relevant definitions and facts. For more details we refer the reader to [BM17, Chapter 2].

Let $f: S^{2} \rightarrow S^{2}$ be a branched covering map of a topological 2 -sphere $S^{2}$. A point $p \in S^{2}$ is called periodic (for $f$ ) if $f^{n}(p)=p$ for some $n \in \mathbb{N}$. Recall that $C_{f}$ denotes the set of all critical points of $f$. The union

$$
P_{f}=\bigcup_{n \in \mathbb{N}} f^{n}\left(C_{f}\right)
$$

of the orbits of critical points is called the postcritical set of $f$. Note that

$$
f\left(P_{f}\right) \subset P_{f} \subset f^{-1}\left(P_{f}\right) .
$$

The map $f$ is said to be postcritically-finite if its postcritical set $P_{f}$ is finite, in other words, if every critical point of $f$ has a finite orbit under iteration.
Definition 3.1. A Thurston map is a postcritically-finite branched covering map $f: S^{2} \rightarrow S^{2}$ of topological degree $\operatorname{deg}(f) \geq 2$.

Natural examples of Thurston maps are given by rational Thurston maps, that is, post-critically-finite rational maps on the Riemann sphere $\widehat{\mathbb{C}}$.

The ramification function of a Thurston map $f: S^{2} \rightarrow S^{2}$ is a function $\alpha_{f}: S^{2} \rightarrow \mathbb{N} \cup\{\infty\}$ such that $\alpha_{f}(p)$ for $p \in S^{2}$ is the lowest common multiple of all local degrees $\operatorname{deg}\left(f^{n}, q\right)$, where $q \in f^{-n}(p)$ and $n \in \mathbb{N}$ are arbitrary. In particular, $\alpha_{f}(p)=1$ for $p \in S^{2} \backslash P_{f}$ and $\alpha_{f}(p) \geq 2$ for $p \in P_{f}$.
Definition 3.2. Two Thurston maps $f: S^{2} \rightarrow S^{2}$ and $g: \widehat{S}^{2} \rightarrow \widehat{S}^{2}$, where $\widehat{S}^{2}$ is another topological 2-sphere, are called Thurston equivalent if there are homeomorphisms $h_{0}, h_{1}: S^{2} \rightarrow \widehat{S}^{2}$ that are isotopic rel. $P_{f}$ such that $h_{0} \circ f=g \circ h_{1}$.

We say that a Thurston map is realized (by a rational map) if it is Thurston equivalent to a rational map. Otherwise, we say that it is obstructed.

The orbifold $\mathcal{O}_{f}$ associated with a Thurston map $f$ is the pair $\left(S^{2}, \alpha_{f}\right)$. The Euler characteristic of $\mathcal{O}_{f}$ is

$$
\begin{equation*}
\chi\left(\mathcal{O}_{f}\right):=2-\sum_{p \in P_{f}}\left(1-\frac{1}{\alpha_{f}(p)}\right) . \tag{3.1}
\end{equation*}
$$

Here we set $1 / \infty:=0$.
The Euler characteristic of the orbifold $\mathcal{O}_{f}$ satisfies $\chi\left(\mathcal{O}_{f}\right) \leq 0$. We call $\mathcal{O}_{f}$ hyperbolic if $\chi\left(\mathcal{O}_{f}\right)<0$, and parabolic if $\chi\left(\mathcal{O}_{f}\right)=0$.

If $f: S^{2} \rightarrow S^{2}$ is a Thurston map, then $f\left(C_{f} \cup P_{f}\right) \subset P_{f}$, which implies $C_{f} \cup P_{f} \subset f^{-1}\left(P_{f}\right)$. The reverse inclusion is related to the parabolicity of $\mathcal{O}_{f}$.
Lemma 3.3. Let $f: S^{2} \rightarrow S^{2}$ be a Thurston map. If $f$ has a parabolic orbifold, then $f^{-1}\left(P_{f}\right)=$ $C_{f} \cup P_{f}$. Moreover, conversely, if $\# P_{f} \geq 4$ and $f^{-1}\left(P_{f}\right) \subset C_{f} \cup P_{f}$, then $f$ has a parabolic orbifold.

The second part follows from [DH93, Lemma 2], but for the convenience of the reader we will provide the simple proof. Here the assumption $\# P_{f} \geq 4$ cannot be omitted as some examples with $\# P_{f}=3$ show (such as the Thurston map arising from the "barycentric subdivision rule"; see [BM17, Example 12.21]).
Proof. Let $\alpha_{f}$ be the ramification function of $f$. Then $p \in P_{f}$ if and only if $\alpha_{f}(p) \geq 2$.
First suppose that $f$ has a parabolic orbifold. Then $\alpha_{f}(q) \cdot \operatorname{deg}(f, q)=\alpha_{f}(f(q))$ for all $q \in S^{2}$ (see [BM17, Proposition 2.14]). So if $q \in f^{-1}\left(P_{f}\right)$, then $f(q) \in P_{f}$ which implies

$$
\alpha_{f}(q) \cdot \operatorname{deg}(f, q)=\alpha_{f}(f(q)) \geq 2
$$

This is only possible if $\alpha_{f}(q) \geq 2$ in which case $q \in P_{f}$, or if $\operatorname{deg}(f, q) \geq 2$ in which case $q \in C_{f}$. Hence $q \in C_{f} \cup P_{f}$, and so $f^{-1}\left(P_{f}\right) \subset C_{f} \cup P_{f}$. Since the reverse inclusion is true for all Thurston maps, we see that $f^{-1}\left(P_{f}\right)=C_{f} \cup P_{f}$ if $f$ has a parabolic orbifold.

For the converse suppose that $f: S^{2} \rightarrow S^{2}$ is an arbitrary Thurston map with $\# P_{f} \geq 4$ and $f^{-1}\left(P_{f}\right) \subset C_{f} \cup P_{f}$. Let $d:=\operatorname{deg}(f) \geq 2$. Note that $f^{-1}\left(P_{f}\right) \subset\left(C_{f} \backslash P_{f}\right) \cup P_{f}$ by our hypotheses.

Each point $p \in S^{2}$ has precisely $d$ preimages counting multiplicities, that is,

$$
d=\sum_{q \in f^{-1}(p)} \operatorname{deg}(f, q) .
$$

Furthermore, since $C_{f} \subset f^{-1}\left(P_{f}\right)$ and $\operatorname{deg}(f, q) \geq 2$ for $q \in S^{2}$ if and only if $q \in C_{f}$, the Riemann-Hurwitz formula implies

$$
\begin{aligned}
\#\left(C_{f} \backslash P_{f}\right) \leq \# C_{f} & \leq \sum_{c \in C_{f}}(\operatorname{deg}(f, c)-1)=\sum_{q \in f^{-1}\left(P_{f}\right)}(\operatorname{deg}(f, q)-1) \\
& =2 d-2 .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
d \cdot \# P_{f} & =\sum_{q \in f^{-1}\left(P_{f}\right)} \operatorname{deg}(f, q)=\sum_{q \in f^{-1}\left(P_{f}\right)}(\operatorname{deg}(f, q)-1)+\# f^{-1}\left(P_{f}\right) \\
& =(2 d-2)+\# f^{-1}\left(P_{f}\right) \leq(2 d-2)+\#\left(C_{f} \backslash P_{f}\right)+\# P_{f} \\
& \leq 4(d-1)+\# P_{f} .
\end{aligned}
$$

Hence $(d-1) \cdot \# P_{f} \leq 4(d-1)$, and so $4 \leq \# P_{f} \leq 4$. This implies $\# P_{f}=4$ and that all the previous inequalities must be equalities. In particular, $\#\left(C_{f} \backslash P_{f}\right)=2 d-2$, which shows that at all critical points the local degree of $f$ is equal to 2 and no critical points belong to $P_{f}$.

As a consequence, under iteration of $f$ the orbit of any point $p \in S^{2}$ passes through at most one critical point. It follows that we have $\alpha_{f}(p)=1$ for $p \in S^{2} \backslash P_{f}$ and $\alpha_{f}(p)=2$ for $p \in P_{f}$. This implies that the Euler characteristic (see (3.1)) of the orbifold $\mathcal{O}_{f}$ associated with $f$ is equal to

$$
\chi\left(\mathcal{O}_{f}\right)=2-\sum_{p \in P_{f}}\left(1-\frac{1}{\alpha_{f}(p)}\right)=2-(1 / 2+1 / 2+1 / 2+1 / 2)=0 .
$$

We conclude that $f$ has a parabolic orbifold.
3.1. The $(n \times n)$-Lattès map. In general, a Lattès map is a rational Thurston map with parabolic orbifold that does not have periodic critical points. Here we provide the analytic definition for the Lattès maps that we use in this paper and interpret this from a more geometric perspective. See [Mil06b] and [BM17, Chapter 3] for a general discussion of Lattès maps.

Let $\mathbb{P} \cong \widehat{\mathbb{C}}$ be the Euclidean square pillow and $\wp: \mathbb{C} \rightarrow \mathbb{P}$ be its universal orbifold covering map, as discussed in Section 2.4. Fix a natural number $n \geq 2$. It follows from (2.3) that there is a unique (and well-defined) map $\mathcal{L}_{n}: \mathbb{P} \rightarrow \mathbb{P}$ such that

$$
\begin{equation*}
\mathcal{L}_{n}(\wp(z))=\wp(n z) \text { for } z \in \mathbb{C} . \tag{3.2}
\end{equation*}
$$

We call $\mathcal{L}_{n}$ the $(n \times n)$-Lattès map. In fact, $\mathcal{L}_{n}$ is a rational map under the identification $\mathbb{P} \cong \widehat{\mathbb{C}}$ as discussed in Section 2.4 .

Alternatively, we can describe the map $\mathcal{L}_{n}$ in a combinatorial fashion as follows. Recall that the front side of $\mathbb{P}$ is colored white, and the back side black. These are the two 0 -tiles of $\mathbb{P}$, and we subdivide each of them into $n^{2}$ squares of sidelength $1 / n$. We refer to these
small squares as 1-tiles (with $n$ understood), and color them in a checkerboard fashion black and white so that the 1 -tile $S$ in the white side of $\mathbb{P}$ with the vertex $A$ on its boundary is colored white. We map $S$ to the white side of the pillow $\mathbb{P}$ by an orientation-preserving Euclidean similarity (that scales by the factor $n$ ) so that the vertex $A$ is fixed. If we extend this map by reflection to the whole pillow, we get the $(n \times n)$-Lattès map $\mathcal{L}_{n}$ (see Figure 1 for $n=4$ ). The map $\mathcal{L}_{n}$ sends each black or white 1-tile homeomorphically (by a similarity) onto the 0 -tile in $\mathbb{P}$ of the same color.

Based on this combinatorial description, it is easy to see that each critical point of $\mathcal{L}_{n}$ has local degree 2 and that the postcritical set of $\mathcal{L}_{n}$ coincides with the set of vertices of $\mathbb{P}$, that is, $P_{\mathcal{L}_{n}}=\{A, B, C, D\}=V$. One can also check that for the ramification function of $\mathcal{L}_{n}$ we have $\alpha_{\mathcal{L}_{n}}(p)=2$ for each $p \in P_{\mathcal{L}_{n}}$. Substituting this into (3.1), we see that $\chi\left(\mathcal{O}_{\mathcal{L}_{n}}\right)=0$. Thus, $\mathcal{L}_{n}$ has a parabolic orbifold.
3.2. Thurston's characterization of rational maps. Thurston maps can often be described from a combinatorial viewpoint as the Lattès map $\mathcal{L}_{n}$ in Section 3.1 (see, for instance, [CFKP03] and [BM17, Chapter 12]). The question whether a given Thurston map $f$ can be realized by a rational map is usually difficult to answer except in some special cases. William Thurston provided a sharp, purely topological criterion that answers this question. The formulation and proof of this celebrated result can be found in [DH93]. In this section, we introduce the necessary concepts and formulate the result only when $\# P_{f}=4$, which is the relevant case for this paper.

In the following, let $f: S^{2} \rightarrow S^{2}$ be a Thurston map. The map $f$ defines a natural pullback operation on Jordan curves in $\left(S^{2}, P_{f}\right)$ : a pullback of a Jordan curve $\gamma \subset S^{2} \backslash P_{f}$ under $f$ is a connected component $\widetilde{\gamma}$ of $f^{-1}(\gamma)$. Since $f$ is a covering map from $S^{2} \backslash f^{-1}\left(P_{f}\right)$ onto $S^{2} \backslash P_{f}$, each pullback $\widetilde{\gamma}$ of $\gamma$ is a Jordan curve in $\left(S^{2}, P_{f}\right)$. Moreover, $f \mid \widetilde{\gamma}: \widetilde{\gamma} \rightarrow \gamma$ is a covering map. For some $k \in \mathbb{N}$ with $1 \leq k \leq \operatorname{deg}(f)$ each point $p \in \gamma$ has precisely $k$ distinct preimages in $\widetilde{\gamma}$. Here $k$ is the (unsigned) mapping degree of $f \mid \widetilde{\gamma}$ which we denote by $\operatorname{deg}(f: \widetilde{\gamma} \rightarrow \gamma)$.

Recall that a Jordan curve $\gamma \subset S^{2} \backslash P_{f}$ is called essential if each of the two connected components of $S^{2} \backslash \gamma$ contains at least two points from $P_{f}$, and is called peripheral otherwise.

We will need the following standard facts.
Lemma 3.4. Let $f: S^{2} \rightarrow S^{2}$ be a Thurston map and let $\gamma$ and $\gamma^{\prime}$ be Jordan curves in $\left(S^{2}, P_{f}\right)$ with $\gamma^{\prime} \sim \gamma$ rel. $P_{f}$. Then there is a bijection $\widetilde{\gamma} \leftrightarrow \widetilde{\gamma}^{\prime}$ between the pullbacks $\widetilde{\gamma}$ of $\gamma$ and the pullbacks $\widetilde{\gamma}^{\prime}$ of $\gamma^{\prime}$ under $f$ such that for all pullbacks corresponding under this bijection we have $\widetilde{\gamma} \sim \widetilde{\gamma}^{\prime}$ rel. $P_{f}$ and $\operatorname{deg}(f: \widetilde{\gamma} \rightarrow \gamma)=\operatorname{deg}\left(f: \widetilde{\gamma}^{\prime} \rightarrow \gamma^{\prime}\right)$.

For the proof see [BM17, Lemma 6.9]. A consequence of this statement is that the isotopy classes of curves in $f^{-1}(\gamma)$ rel. $P_{f}$ only depend on the isotopy class [ $\gamma$ ] rel. $P_{f}$ and not on the specific choice of $\gamma$.
Corollary 3.5. Let $f: S^{2} \rightarrow S^{2}$ be a Thurston map, and $\gamma$ be Jordan curve in $\left(S^{2}, P_{f}\right)$.
(i) If $\gamma$ is peripheral, then every pullback of $\gamma$ under $f$ is also peripheral.
(ii) Suppose that $\# P_{f}=4$ and let $\widetilde{\gamma}$ be a pullback of $\gamma$ under $f$. If $\gamma$ and $\widetilde{\gamma}$ are essential, then the isotopy class $[\widetilde{\gamma}]$ rel. $P_{f}$ only depends on the isotopy class $[\gamma]$ rel. $P_{f}$ and not on the specific choice of $\gamma$ and its essential pullback $\widetilde{\gamma}$.
Proof. (i) Since $\gamma$ is peripheral, $\gamma$ can be isotoped (rel. $P_{f}$ ) into a Jordan curve $\gamma^{\prime}$ inside a small open Jordan region $V \subset S^{2}$ such that $\#\left(V \cap P_{f}\right) \leq 1$ and $V$ is evenly covered by the branched covering map $f$ as in (2.1).

Then for each component $U_{j}$ of $f^{-1}(V)$, the map $f \mid U_{j}: U_{j} \rightarrow V$ is given by $z \in \mathbb{D} \mapsto z^{d_{j}} \in \mathbb{D}$ for some $d_{j} \in \mathbb{N}$ after orientation-preserving homeomorphic coordinate changes in the source and target. This implies that $\#\left(U_{j} \cap P_{f}\right) \leq 1$ and that each pullback of $\gamma^{\prime}$ in $U_{j}$ is peripheral. Hence all pullbacks of $\gamma^{\prime}$ under $f$ are peripheral and the same is true for the pullbacks of $\gamma$ as follows from Lemma 3.4.
(ii) Suppose $\widetilde{\gamma}$ and $\widetilde{\gamma}^{\prime}$ are two distinct essential pullbacks of $\gamma$ under $f$. Since these are components of $f^{-1}(\gamma)$, the Jordan curves $\widetilde{\gamma}$ and $\widetilde{\gamma}^{\prime}$ are disjoint. Then the set $S^{2} \backslash\left(\widetilde{\gamma} \cup \widetilde{\gamma}^{\prime}\right)$ is a disjoint union $S^{2} \backslash\left(\widetilde{\gamma} \cup \widetilde{\gamma}^{\prime}\right)=V \cup U \cup V^{\prime}$, where $V, V^{\prime} \subset S^{2}$ are Jordan regions and $U \subset S^{2}$ is an annulus with $\partial U=\widetilde{\gamma} \cup \widetilde{\gamma}^{\prime}$. Since $\widetilde{\gamma}$ and $\widetilde{\gamma}^{\prime}$ are essential, both $V$ and $V^{\prime}$ must contain at least two postcritical points. Now $\# P_{f}=4$, and so $U \cap P_{f}=\varnothing$. Lemma 2.1 then implies that $\widetilde{\gamma}$ and $\widetilde{\gamma}^{\prime}$ are isotopic rel. $P_{f}$.

It follows that the isotopy class $[\widetilde{\gamma}]$ rel. $P_{f}$ does not depend on the choice of the essential pullback $\widetilde{\gamma}$ of $\gamma$. At the same time, Lemma 3.4 implies that $[\widetilde{\gamma}]$ only depends on the isotopy class $[\gamma]$, as desired.

For a general Thurston map $f$ the concept of an invariant multicurve is important to decide whether $f$ is realized or obstructed. By definition, a multicurve is a non-empty finite family $\Gamma$ of essential Jordan curves in $S^{2} \backslash P_{f}$ that are pairwise disjoint and pairwise non-isotopic rel. $P_{f}$.

Suppose now that $\# P_{f}=4$. Then any two essential Jordan curves in $S^{2} \backslash P_{f}$ are either isotopic rel. $P_{f}$ or have a non-empty intersection (as follows from the argument in the proof of Corollary 3.5 (ii)). Thus, in this case each multicurve $\Gamma$ consists of a single essential Jordan curve $\gamma$ in $S^{2} \backslash P_{f}$. We say that an essential Jordan curve $\gamma$ in $S^{2} \backslash P_{f}$ is $f$-invariant if each essential pullback of $\gamma$ under $f$ is isotopic to $\gamma$ rel. $P_{f}$.
Definition 3.6. Let $f: S^{2} \rightarrow S^{2}$ be a Thurston map with $\# P_{f}=4$ and let $\gamma \subset S^{2} \backslash P_{f}$ be an essential $f$-invariant Jordan curve. We denote by $\gamma_{1}, \ldots, \gamma_{n}$ for $n \in \mathbb{N}_{0}$ all the essential pullbacks of $\gamma$ under $f$ and define

$$
\begin{equation*}
\lambda_{f}(\gamma):=\sum_{j=1}^{n} \frac{1}{\operatorname{deg}\left(f: \gamma_{j} \rightarrow \gamma\right)} \tag{3.3}
\end{equation*}
$$

Then $\gamma$ is called a (Thurston) obstruction for $f$ if $\lambda_{f}(\gamma) \geq 1$.
Note that if $n=0$, then the sum in (3.3) is the empty sum and so $\lambda_{f}(\gamma)=0$. It immediately follows from Lemma 3.4 that whether or not $\gamma$ is an obstruction for $f$ only depends on the isotopy class $[\gamma]$ rel. $P_{f}$.

The following theorem gives a criterion when a Thurston map $f$ with $\# P_{f}=4$ is realized.
Theorem 3.7 (Thurston's criterion). Let $f: S^{2} \rightarrow S^{2}$ be a Thurston map with $\# P_{f}=4$ and suppose that $f$ has a hyperbolic orbifold. Then $f$ is realized by a rational map if and only if $f$ has no obstruction.

With a suitable definition of an obstruction (as an invariant multicurve that satisfies certain mapping properties) this statement is also true for general Thurston maps with a hyperbolic orbifold; see [DH93] or BM17, Section 2.6].

The example of the $(n \times n)$-Lattès map $\mathcal{L}_{n}: \mathbb{P} \rightarrow \mathbb{P}$ with $n \geq 2$ shows that Theorem 3.7 is false if $f$ has a parabolic orbifold. Indeed, let $\gamma$ be any essential Jordan curve in $\left(\mathbb{P}, \bar{P}_{\mathcal{L}_{n}}\right)$, where $P_{\mathcal{L}_{n}}=V=\{A, B, C, D\}$ consists of the vertices of the pillow $\mathbb{P}$. Since only the isotopy class [ $\gamma$ ] rel. $V$ matters, by Lemma 2.3 we may assume without loss of generality that


Figure 8. The two pullbacks of a curve $\gamma=\tau_{2}$ under $\mathcal{L}_{2}$.
$\gamma=\tau_{r / s}=\wp\left(\ell_{r / s}\left(z_{0}\right)\right)$ with $z_{0} \in \mathbb{C}, r / s \in \widehat{\mathbb{Q}}$, and $\ell_{r / s}\left(z_{0}\right) \subset \mathbb{C} \backslash \mathbb{Z}^{2}$. Here $r$ and $s$ are relatively prime integers and so there exist $p, q \in \mathbb{Z}$ such that $p r+q s=1$. Let $\widetilde{\omega}:=-p+i q$. Using (2.3) and (3.2) one can verify that under $\mathcal{L}_{n}$ the curve $\gamma$ has exactly $n$ distinct pullbacks

$$
\begin{equation*}
\gamma_{j}=\wp\left(\ell_{r / s}\left(\left(z_{0}+2 j \widetilde{\omega}\right) / n\right)\right), \quad j=1, \ldots, n \tag{3.4}
\end{equation*}
$$

Moreover, each curve $\gamma_{j}$ is isotopic to $\gamma$ rel. $P_{\mathcal{L}_{n}}=V$ and $\operatorname{deg}\left(\mathcal{L}_{n}: \gamma_{j} \rightarrow \gamma\right)=n$ for all $j=1, \ldots, n$; see Figure 8 for an illustration. Thus, $\lambda_{\mathcal{L}_{n}}(\gamma)=1$ and $\gamma$ is an obstruction.

## 4. Blowing up arcs

Here, we describe the operation of "blowing up arcs", originally introduced by Kevin Pilgrim and Tan Lei in [PL98, Section 2.5]. This operation allows us to define and modify various Thurston maps and plays a crucial role in this paper. We will first describe the general construction and then illustrate it for Lattès maps. As we will explain, the procedure of "gluing a flap" that we introduced in Section 1.1 can be viewed as a special case of blowing up arcs for Lattès maps.

The construction of blowing up arcs will involve a finite collection $E$ of arcs with pairwise disjoint interiors in a 2-sphere $S^{2}$. We denote by $V$ the set of endpoints of these arcs and consider ( $V, E$ ) as an embedded graph in $S^{2}$. In the construction we will make various topological choices. The following general statement guarantees that we do not run into topological difficulties. In the formulation we equip $S^{2}$ with a "nice" metric $d$ so that $\left(S^{2}, d\right)$ is isometric to $\widehat{\mathbb{C}}$ carrying the spherical metric with length element $d s=2|d z| /\left(1+|z|^{2}\right)$.

Proposition 4.1. Let $G=(V, E)$ be a planar embedded graph in $S^{2}$ and $\mathcal{G} \subset S^{2}$ be its realization. Then there exists a planar embedded graph $G^{\prime}=\left(V, E^{\prime}\right)$ in $S^{2}$ with the same vertex set such that its realization $\mathcal{G}^{\prime}$ is isotopic to $\mathcal{G}$ rel. $V$ and such that each edge of $G^{\prime}$ is a piecewise geodesic arc in $\left(S^{2}, d\right)$.

An outline of the proof is given in [Bol79, Chapter I, §4]; the proposition also follows from [Bus10, Lemma A.8].
4.1. The general construction. Before we provide a formal definition, we give some rough idea of how to "blow up" arcs. In the following, $f: S^{2} \rightarrow S^{2}$ is a fixed Thurston map. Let $e$ be an arc in $S^{2}$ such that the restriction $f \mid e$ is a homeomorphism onto its image. We cut the sphere $S^{2}$ open along $e$ and glue in a closed Jordan region $D$ along the boundary. In this way we obtain a new 2 -sphere on which we can define a branched covering map $\widehat{f}$ as follows: $\widehat{f}$ maps the complement of $\operatorname{int}(D)$ in the same way as the original map $f$ and it maps int $(D)$

Choose regions $W_{e} \supset e \quad$ Open up the arcs $e \quad$ Subdivide each $D_{e}$ into to regions $D_{e} \subset W_{e}$ regions $D_{e}^{1}, \ldots, D_{e}^{m}$


Figure 9. Setup for blowing up the arcs $e_{1}$ and $e_{2}$ (in the sphere on the left) with the multiplicities $m_{e_{1}}=1$ and $m_{e_{2}}=2$.
to the complement of $f(e)$ by a homeomorphism that matches the map $f \mid e$. We say that $\widehat{f}$ is obtained from $f$ by blowing up the arc $e$ with multiplicity 1 .

Now we proceed to give a rigorous definition of the blow-up operation in the general case, where several arcs $e$ are blown up simultaneously with possibly different multiplicitities $m_{e} \geq 1$ resulting in a new Thurston map $\widehat{f}$. To this end, let $E$ be a finite set of arcs in $\left(S^{2}, f^{-1}\left(P_{f}\right)\right)$ with pairwise disjoint interiors such that the restriction $f \mid e: e \rightarrow f(e)$ is a homeomorphism for each $e \in E$. In this case, we say that $E$ satisfies the blow-up conditions.

We assume that each arc $e \in E$ has an assigned multiplicity $m_{e} \in \mathbb{N}$. Since each $e \in E$ is an arc in $\left(S^{2}, f^{-1}\left(P_{f}\right)\right)$, its interior $\operatorname{int}(e)$ is disjoint from $f^{-1}\left(P_{f}\right) \supset P_{f}$ and so $\operatorname{int}(e)$ does not contain any critical or postcritical point of $f$.

For each arc $e \in E$, we choose an open Jordan region $W_{e} \subset S^{2}$ so that the following conditions hold:
(A1) The open Jordan regions $W_{e}, e \in E$, are pairwise disjoint.
(A2) For distinct $\operatorname{arcs} e_{1}, e_{2} \in E$, we have $\operatorname{cl}\left(W_{e_{1}}\right) \cap \operatorname{cl}\left(W_{e_{2}}\right)=\partial e_{1} \cap \partial e_{2}$.
(A3) $\operatorname{int}(e) \subset W_{e}$ and $\partial e \subset \partial W_{e}$ for each $e \in E$.
(A4) $\operatorname{cl}\left(W_{e}\right) \cap f^{-1}\left(P_{f}\right)=e \cap f^{-1}\left(P_{f}\right)=\partial e$ for each $e \in E$.
(A5) $f \mid \operatorname{cl}\left(W_{e}\right)$ is a homeomorphism onto its image for each $e \in E$.
The existence of such a choice (and also of the choices below) can easily be justified based on Propositon 4.1 and we will skip the details.

Let $e \in E$ and $W_{e}$ be chosen as above. Then we choose a closed Jordan region $D_{e}$ so that $e$ is a crosscut in $D_{e}$ and $D_{e} \backslash \partial e \subset W_{e}$. The two endpoints of $e$ lie on the Jordan curve $\partial D_{e}$ and partition it into two arcs, which we denote by $\partial D_{e}^{+}$and $\partial D_{e}^{-}$. One can think of $D_{e}$ as the resulting region if $e$ has been "opened up". This is illustrated in the left and middle parts of Figure 9 .

In order to define the desired Thurston map $\widehat{f}$, we want to collapse $D_{e}$ back to $e$. For this we choose a continuous map $h: S^{2} \times \mathbb{I} \rightarrow S^{2}$ with the following properties:
(B1) $h$ is a pseudo-isotopy, that is, $h_{t}:=h(\cdot, t)$ is a homeomorphism on $S^{2}$ for each $t \in[0,1)$.
(B2) $h_{0}$ is the identity map on $S^{2}$.
(B3) $h_{t}$ is the identity map on $S^{2} \backslash \bigcup_{e \in E} W_{e}$ for each $t \in[0,1]$.
(B4) $h_{1}$ is a homeomorphism of $S^{2} \backslash \bigcup_{e \in E} D_{e}$ onto $S^{2} \backslash \bigcup_{e \in E} e$, and $h_{1}$ maps $\partial D_{e}^{+}$and $\partial D_{e}^{-}$ homeomorphically onto $e$ for each $e \in E$.
It is easy to see that if we equip $S^{2}$ with some metric, then the set $h_{t}\left(D_{e}\right)$ Hausdorff converges to $e$ as $t \rightarrow 1^{-}$. This implies that $h_{1}\left(D_{e}\right)=e$. So intuitively, the deformation process described by $h$ collapses each closed Jordan region $D_{e}$ to $e$ at time 1 so that the points in $S^{2} \backslash \bigcup_{e \in E} W_{e}$ remain fixed.

We now make yet another choice. For a fixed arc $e \in E$, let $m=m_{e}$. We choose $m-1$ crosscuts $e^{1}, \ldots, e^{m-1}$ in $D_{e}$ with the same endpoints as $e$ such that these crosscuts have pairwise disjoint interiors. We set $e^{0}:=\partial D_{e}^{+}$and $e^{m}:=\partial D_{e}^{-}$. The arcs $e^{0}, \ldots, e^{m}$ subdivide the closed Jordan region $D_{e}$ into $m$ closed Jordan regions $D_{e}^{1}, \ldots, D_{e}^{m}$, called components of $D_{e}$. This is illustrated in the right-hand part of Figure 9 .

We may assume that the labeling is such that $\partial D_{e}^{k}=e^{k-1} \cup e^{k}$ for $k=1, \ldots, m$. For each $k=1, \ldots, m$, we now choose a continuous map $\varphi_{k}: D_{e}^{k} \rightarrow S^{2}$ with the following properties:
(C1) $\varphi_{k}$ is an orientation-preserving homeomorphism of $\operatorname{int}\left(D_{e}^{k}\right)$ onto $S^{2} \backslash f(e)$ and maps $e^{k-1}$ and $e^{k}$ homeomorphically onto $f(e)$.
(C2) $\varphi_{1}\left|e^{0}=f \circ h_{1}\right| e^{0}, \varphi_{m}\left|e^{m}=f \circ h_{1}\right| e^{m}$, and $\varphi_{k}\left|e^{k}=\varphi_{k+1}\right| e^{k}$ for $k=1, \ldots, m-1$.
Note that, by the earlier discussion, $h_{1}$ maps $e^{0}=\partial D_{e}^{+}$and $e^{m}=\partial D_{e}^{-}$homeomorphically onto $e$ and $f$ is a homeomorphism of $e$ onto $f(e)$. These choices of the maps $\varphi_{k}$ depend on $e$, but we suppress this in our notation for simplicity.

A map $\widehat{f}: S^{2} \rightarrow S^{2}$ can now be defined as follows:
(D1) If $p \in S^{2} \backslash \bigcup_{e \in E} \operatorname{int}\left(D_{e}\right)$, we set $\widehat{f}(p)=f\left(h_{1}(p)\right)$.


Figure 10. The map $\widehat{f}$ is obtained from $f$ by blowing up the $\operatorname{arcs} e_{1}$ and $e_{2}$ with multiplicities $m_{e_{1}}=1$ and $m_{e_{2}}=2$.
(D2) If $p \in D_{e}$ for some $e \in E$, then $p$ lies in one of the components $D_{e}^{k}$ of $D_{e}$ and we set $\widehat{f}(p)=\varphi_{k}(p)$.
The matching conditions (C2) above immediately imply that $\widehat{f}$ is well-defined and continuous.

Definition 4.2. We say that the map $\widehat{f}: S^{2} \rightarrow S^{2}$ as described above is obtained from the Thurston map $f$ by blowing up each arc $e \in E$ with multiplicity $m_{e}$.

Figure 10 illustrates the construction of $\widehat{f}$. Here, we blow up the arcs $e_{1}$ and $e_{2}$ from Figure 9 with multiplicities $m_{e_{1}}=1$ and $m_{e_{2}}=2$. The arcs $f\left(e_{1}\right)$ and $f\left(e_{2}\right)$ share an endpoint (since $e_{1}$ and $e_{2}$ do), but in general they could have more points in common or even coincide. For simplicity, we chose to draw them with disjoint interiors.

By construction, $\widehat{f}$ acts in a similar way as $f$ outside the closed Jordan regions $D_{e_{1}}=D_{e_{1}}^{1}$ and $D_{e_{2}}=D_{e_{2}}^{1} \cup D_{e_{2}}^{2}$. More precisely, $\widehat{f}=f \circ h_{1}$ on $S^{2} \backslash\left(\operatorname{int}\left(D_{e_{1}}\right) \cup \operatorname{int}\left(D_{e_{2}}\right)\right)$, where $h_{1}$ collapses the closed Jordan regions $D_{e_{1}}$ and $D_{e_{2}}$ onto $e_{1}$ and $e_{2}$, respectively. At the same time, $\widehat{f}$ maps $\operatorname{int}\left(D_{e_{1}}\right)$ homeomorphically onto $S^{2} \backslash f\left(e_{1}\right)$, and each of the regions $\operatorname{int}\left(D_{e_{2}}^{1}\right)$ and $\operatorname{int}\left(D_{e_{2}}^{2}\right)$ homeomorphically onto $S^{2} \backslash f\left(e_{2}\right)$.

In the next section, we want to relate "blowing up arcs" with "gluing flaps" as discussed in the introduction. In order to set this up, we consider the $(2 \times 2)$-Lattès map $f=\mathcal{L}_{2}$. We choose two edges $e_{1}$ and $e_{2}$ of a 1-tile in $\mathbb{P}$ as shown in the pillow on the left in Figure 11 . Note that $f$ sends $e_{1}$ and $e_{2}$ homeomorphically onto the edges $c$ and $b$ of $\mathbb{P}$, respectively. Thus the set $E=\left\{e_{1}, e_{2}\right\}$ satisfies the blow-up conditions. Figure 11 illustrates the setup for blowing up these arcs $e_{1}$ and $e_{2}$ with the multiplicities $m_{e_{1}}=1$ and $m_{e_{2}}=2$. The resulting map $\widehat{f}: \mathbb{P} \rightarrow \mathbb{P}$ is shown in Figure 12 . The points marked by a dot on the left pillow $\mathbb{P}$ (the


Figure 11. Setup for blowing up the arc set $E=\left\{e_{1}, e_{2}\right\}$ (on the left pillow) with $m_{e_{1}}=1$ and $m_{e_{2}}=2$.


Figure 12. The map $\widehat{f}$ obtained from $f=\mathcal{L}_{2}$ by blowing up the arcs in the set $E=\left\{e_{1}, e_{2}\right\}$ illustrated in Figure 11 with $m_{e_{1}}=1$ and $m_{e_{2}}=2$.
domain of the map) correspond to the preimage points $\widehat{f}^{-1}(V)$. The pillow on the left is subdivided into closed Jordan regions alternately colored black and white. The map $\widehat{f}$ sends each of these closed Jordan regions $U$ homeomorphically onto the back side or front side of the pillow $\mathbb{P}$ depending on whether $U$ is black or white.

The following statement summarizes the main properties of maps $\widehat{f}$ as in Definition 4.2 .
Lemma 4.3. Let $f: S^{2} \rightarrow S^{2}$ be a Thurston map and $E$ be a set of arcs in $\left(S^{2}, f^{-1}\left(P_{f}\right)\right)$ satisfying the blow-up conditions. Suppose $\widehat{f:} S^{2} \rightarrow S^{2}$ is the map obtained by blowing up each arc $e \in E$ with multiplicity $m_{e} \in \mathbb{N}$.

Then $\widehat{f}$ is a Thurston map with $P_{\widehat{f}}=P_{f}$. Moreover, the map $\widehat{f}$ is uniquely determined up to Thurston equivalence independently of the choices in the above construction. More precisely, up to Thurston equivalence $\widehat{f}$ depends only on the original map $f$, the isotopy classes of the arcs in $E$ rel. $f^{-1}\left(P_{f}\right)$, and the multiplicities $m_{e}$ for $e \in E$.

Proof. By construction $\widehat{f}$ is an orientation-preserving local homeomorphism near each point $p \in S^{2} \backslash f^{-1}\left(P_{f}\right)$. By considering the number of preimages of a generic point in $S^{2}$ we see that the topological degree of $\widehat{f}$ is equal to $\operatorname{deg}(f)+\sum_{e \in E} m_{e}>0$. The fact that $\widehat{f:} S^{2} \rightarrow S^{2}$ is a branched covering map can now be deduced from [BM17, Corollary A.14].

We have $\operatorname{deg}(\widehat{f}, p)=1$ for $p \in S^{2} \backslash f^{-1}\left(P_{f}\right)$ and

$$
\operatorname{deg}(\widehat{f}, p)=\operatorname{deg}(f, p)+\sum_{\{e \in E: p \in \in\}} m_{e}
$$

for $p \in f^{-1}\left(P_{f}\right)$. This implies $C_{f} \subset C_{\vec{f}} \subset f^{-1}\left(P_{f}\right)$. Since on the set $f^{-1}\left(P_{f}\right)$ the maps $f$ and $\widehat{f}$ agree, it follows that $P_{\widehat{f}}=P_{f}$. So $\widehat{f}$ has a finite postcritical set, and we conclude that $\widehat{f}$ is indeed a Thurston map.

We omit a detailed justification of the second claim that $\widehat{f}$ is uniquely determined up to Thurston equivalence by $f$, the isotopy classes of the $\operatorname{arcs} E$, and their multiplicities. A proof can be given along the lines of [PL98, Proposition 2].

Remark 4.4. If in the previous statement $E \neq \varnothing$ and $P_{f} \geq 3$, then $\widehat{f}$ has a hyperbolic orbifold. To see this, pick an arc $e \in E$. Then $f(e)$ has at most two points in common with $P_{f}$, and so we can find a point $p \in P_{f} \backslash f(e) \subset P_{\widehat{f}}$. Then it follows from the construction of $\widehat{f}$ that there exists a point $q$ in the interior of the region $D_{e}$ associated with $e$ such that $\widehat{f}(q)=p$ and $\operatorname{deg}(\widehat{f}, q)=1$. Then $q \notin C_{\widehat{f}}$, but we also have

$$
q \in \operatorname{int}\left(D_{e}\right) \subset S^{2} \backslash f^{-1}\left(P_{f}\right) \subset S^{2} \backslash P_{f}=S^{2} \backslash P_{\widehat{f}}
$$

This shows that $q \in \widehat{f}^{-1}(p) \subset \widehat{f}^{-1}\left(P_{\widehat{f}}\right)$, but $q \notin C_{\widehat{f}} \cup P_{\widehat{f}}$, and so $\widehat{f}$ must have a hyperbolic orbifold by the first part of Lemma 3.3.
4.2. Blowing up the $(n \times n)$-Lattès map. Let $\mathbb{P}$ be the Euclidean square pillow and $\mathcal{L}_{n}: \mathbb{P} \rightarrow \mathbb{P}$ be the $(n \times n)$-Lattès map for fixed $n \geq 2$. We denote by $\mathcal{C} \subset \mathbb{P}$ the common boundary of the two sides of $\mathbb{P}$. The set $\mathcal{C}$ may be viewed as a planar embedded graph with the vertex set $V=\{A, B, C, D\}$ and the edge set $\{a, b, c, d\}$ in the notation from Section 2.4. Let $\widetilde{\mathcal{C}}:=\mathcal{L}_{n}^{-1}(\mathcal{C}) \subset \mathbb{P}$ be the preimage of $\mathcal{C}$ under $\mathcal{L}_{n}$, viewed as a planar embedded graph with the vertex set $\mathcal{L}_{n}^{-1}(V)$. In the next section, we will study the question whether a Thurston map is realized by a rational map if it is obtained from $\mathcal{L}_{n}$ by blowing up edges of $\widetilde{\mathcal{C}}$. In
order to facilitate this discussion, we will provide a more concrete combinatorial model for these maps.

By the definition of the map $\mathcal{L}_{n}$, the graph $\widetilde{\mathcal{C}}$ subdivides the pillow $\mathbb{P}$ into $2 n^{2} 1$-tiles, which are squares of sidelength $1 / n$. The edges of the embedded graph $\widetilde{\mathcal{C}}$ are precisely the sides of these squares. We call them the 1-edges of $\mathbb{P}$ (for given $n$ ). The map $\mathcal{L}_{n}$ sends each 1-edge $e$ of $\mathbb{P}$ homeomorphically onto one of the edges $a, b, c, d$ of $\mathcal{C}$. We call $e$ horizontal if $\mathcal{L}_{n}$ maps it onto $a$ or $c$, and vertical if $\mathcal{L}_{n}$ maps it onto $b$ or $d$.

We take two disjoint copies of the Euclidean square $[0,1 / n]^{2} \subset \mathbb{R}^{2}$ and identify the points on three of their sides, say the sides $\{0\} \times[0,1 / n],[0,1 / n] \times\{1 / n\}$, and $\{1 / n\} \times[0,1 / n]$. We call the object obtained a flap $F$. Note that it is homeomorphic to the closed unit disk and has two "free" sides corresponding to the two copies of $[0,1 / n] \times\{0\}$ in $F$.

We can cut the pillow $\mathbb{P}$ open along one of the edges of $\widetilde{\mathcal{C}}$ and glue in a flap $F$ to the pillow by identifying each copy of $[0,1 / n] \times\{0\}$ in the flap with one side of the slit by an isometry (see Figure 2). In this way, we get a new polyhedral surface homeomorphic to $S^{2}$. One can also glue multiple copies of the flap to the slit by an isometry and obtain a polyhedral surface $\widehat{\mathbb{P}}$ homeomorphic to $S^{2}$. This can be described more concretely as follows. Let $e$ be an edge in $\widetilde{\mathcal{C}}$ and $F_{1}, \ldots, F_{m}$ be $m \geq 1$ copies of the flap. For each $k=1, \ldots, m$, we denote the two copies of $[0,1 / n] \times\{0\}$ in the flap $F_{k}$ by $e_{k}^{\prime}$ and $e_{k}^{\prime \prime}$. We now construct a new polyhedral surface $\widehat{\mathbb{P}}$ in the following way:
(i) First, we cut the original pillow $\mathbb{P}$ open along the edge $e$.
(ii) Then, for each $k=1, \ldots, m-1$, we identify the edge $e_{k}^{\prime \prime}$ of $F_{k}$ with the edge $e_{k+1}^{\prime}$ of $F_{k+1}$ by an isometry. We get a polyhedral surface $D_{e}$ homeomorphic to a closed disk, whose boundary consists of two edges $e_{1}^{\prime}$ and $e_{m}^{\prime \prime}$.
(iii) Finally, we glue the disk $D_{e}$ to the pillow $\mathbb{P}$ cut open along $e$ by identifying the edges $e_{1}^{\prime}$ and $e_{m}^{\prime \prime}$ in $\partial D_{e}$ with the two sides of the slit by an isometry so that $e_{1}^{\prime}$ and $e_{m}^{\prime \prime}$ are identified with different sides of the slit. We obtain a polyhedral surface $\widehat{\mathbb{P}}$ that is homeomorphic to a 2 -sphere.
More generally, we can cut open $\mathbb{P}$ simultaneously along several edges $e$ of $\widetilde{\mathcal{C}}$ and, by the method described, glue $m_{e} \in \mathbb{N}$ copies of the flap to the slit obtained from each edge $e$. If these edges $e$ of $\widetilde{\mathcal{C}}$ with their multiplicities $m_{e}$ are given, then there is essentially only one way of gluing flaps so that the resulting object is a polyhedral surface homeomorphic to $S^{2}$.

Let $\widehat{\mathbb{P}}$ be the polyhedral surface obtained from $\mathbb{P}$ by gluing a total number of $n_{h} \geq 0$ horizontal flaps (i.e., flaps glued along horizontal edges of $\widetilde{\mathcal{C}}$ ) and a total number of $n_{v} \geq 0$ vertical flaps (i.e., flaps glued along vertical edges of $\widetilde{\mathcal{C}}$ ). We call this surface a flapped pillow. We denote by $E$ the set of all edges in $\widetilde{\mathcal{C}}$ along which flaps were glued and by $m_{e}, e \in E$, the corresponding multiplicities. See the left part of Figure 13 for an example of a flapped pillow $\widehat{\mathbb{P}}$ obtained by gluing one horizontal and two vertical flaps at the edges $e_{1}$ and $e_{2}$ from Figure 11.

The polyhedral surface $\widehat{\mathbb{P}}$ is naturally subdivided into

$$
2\left(n^{2}+n_{h}+n_{v}\right)=2 n^{2}+2 \sum_{e \in E} m_{e}
$$

squares of sidelength $1 / n$, called the 1 -tiles of the flapped pillow $\widehat{\mathbb{P}}$. The vertices and the edges of these squares are called the 1-vertices and 1 -edges of $\widehat{\mathbb{P}}$. There is a natural path metric on $\widehat{\mathbb{P}}$ that agrees with the Euclidean metric on each 1-tile. The surface $\widehat{\mathbb{P}}$ equipped


Figure 13. A branched covering map $\widehat{\mathcal{L}}: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ induced by the flapped pillow $\widehat{\mathbb{P}}$ on the left.
with this metric is locally Euclidean with conic singularities at some of the 1-vertices. Such a conic singularity arises at a 1 -vertex $v \in \widehat{\mathbb{P}}$ if $v$ is contained in $k_{v} \neq 4$ distinct 1-tiles.

We will assume that $\widehat{\mathbb{P}}$ has at least one flap, that is, $n_{h}+n_{v} \geq 1$. Let $F_{j}, j=1, \ldots, n_{h}+n_{v}$, be the collection of flaps glued to $\mathbb{P}$. Each flap $F_{j}$ consists of two 1 -tiles in $\widehat{\mathbb{P}}$. We call the four 1-vertices that belong to $F_{j}$ the vertices of the flap $F_{j}$. The boundary $\partial F_{j}$ is a Jordan curve in $\widehat{\mathbb{P}}$ composed of two 1-edges $e_{j}^{\prime}$ and $e_{j}^{\prime \prime}$, which we call the base edges of $F_{j}$. The 1-edge in $F_{j}$ that is opposite to the base edges is called the top edge of the flap $F_{j}$. Note that $\partial e_{j}^{\prime}=\partial e_{j}^{\prime \prime}$ consists of two vertices of $F_{j}$.

We now define the base $B(\widehat{\mathbb{P}}) \subset \widehat{\mathbb{P}}$ of the flapped pillow as

$$
\begin{equation*}
B(\widehat{\mathbb{P}}):=\widehat{\mathbb{P}} \backslash\left(\bigcup_{j=1}^{n_{h}+n_{v}}\left(F_{j} \backslash \partial e_{j}^{\prime}\right)\right) . \tag{4.1}
\end{equation*}
$$

In other words, $B(\widehat{\mathbb{P}})$ is obtained from $\widehat{\mathbb{P}}$ by removing all flaps $F_{j}$ from $\widehat{\mathbb{P}}$, except that we keep the two vertices in $\partial e_{j}^{\prime} \subset F_{j}$ from each flap. There is a natural identification

$$
\begin{equation*}
B(\widehat{\mathbb{P}}) \cong \mathbb{P} \backslash \bigcup_{e \in E} \operatorname{int}(e) \subset \mathbb{P} . \tag{4.2}
\end{equation*}
$$

This means that we can consider the base $B(\widehat{\mathbb{P}})$ both as a subset of $\widehat{\mathbb{P}}$ and of $\mathbb{P}$. Figure 14 illustrates these two viewpoints. This is slightly imprecise, but this point of view will be extremely convenient in the following.


Figure 14. The base $B(\widehat{\mathbb{P}})$ of the flapped pillow $\widehat{\mathbb{P}}$ from Figure 13 depicted in two different ways: as a subset of $\widehat{\mathbb{P}}$ and as the subset $\mathbb{P} \backslash \bigcup_{e \in E} \operatorname{int}(e)$ of $\mathbb{P}$.

We choose the orientation on $\widehat{\mathbb{P}}$ so that the induced orientation on $B(\widehat{\mathbb{P}})$ considered as a subset of $\widehat{\mathbb{P}}$ coincides with the orientation on $B(\widehat{\mathbb{P}})$ considered as a subset of the oriented sphere $\mathbb{P}$ (if we represent orientations on surfaces by flags as described in [BM17, Section A.4], then we simply pick a positively-oriented flag contained in $B(\widehat{\mathbb{P}}) \subset \mathbb{P}$ and declare it to be positively-oriented in $\widehat{\mathbb{P}} \supset B(\widehat{\mathbb{P}}))$.

The set $B(\widehat{\mathbb{P}}) \cong \mathbb{P} \backslash \bigcup_{e \in E} \operatorname{int}(e)$ contains the vertex set $\mathcal{L}_{n}^{-1}(V) \subset \mathbb{P}$ of the graph $\widetilde{\mathcal{C}}=$ $\mathcal{L}_{n}^{-1}(\mathcal{C}) \subset \mathbb{P}$. This means that we can naturally view each vertex of $\widetilde{\mathcal{C}}$ also as a 1 -vertex in $\widehat{\mathbb{P}}$. Let $\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}$ be the 1 -vertices of $\widehat{\mathbb{P}}$ that correspond to the vertices $A, B, C, D$ of the original pillow, respectively. We set $\widehat{V}:=\{\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}\}$ and call $\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}$ the vertices of $\widehat{\mathbb{P}}$.

Recall from Section 3.1 that the faces of the embedded graph $\widetilde{\mathcal{C}} \subset \mathbb{P}$ are colored black and white in a checkerboard manner. This coloring induces a checkerboard coloring on the 1-tiles of the flapped pillow $\widehat{\mathbb{P}}$. The original map $\mathcal{L}_{n}: \mathbb{P} \rightarrow \mathbb{P}$ can now be naturally extended to a continuous map $\widehat{\mathcal{L}}: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ by reflection so that it preserves the coloring: $\widehat{\mathcal{L}}$ maps each 1 -tile of $\widehat{\mathbb{P}}$ by a Euclidean similarity (scaling distances by the factor $n$ ) onto the 0 -tile of $\mathbb{P}$ with the same color; see Figure 13 for an illustration. On the base $B(\widehat{\mathbb{P}})$ the map $\widehat{\mathcal{L}}$ agrees with the original $(n \times n)$-Lattès map $\mathcal{L}_{n}$ (if we consider $B(\widehat{\mathbb{P}})$ as a subset of $\mathbb{P}$ by the identification (4.2)).

It is clear that $\widehat{\mathcal{L}}: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ is a branched covering map. This map is essentially the Thurston map obtained from $\mathcal{L}_{n}$ by blowing up each arc $e \in E$ with muliplicity $m_{e}$. To make this more precise, we need a suitable identification of the source $\widehat{\mathbb{P}}$ with the target $\mathbb{P}$ of $\widehat{\mathcal{L}}$ so that we obtain a self-map on $\mathbb{P}$. For this we choose a natural homeomorphism $\phi: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$, which we will now define.

We view the set $\widehat{\mathcal{C}}:=\widehat{\mathcal{L}}^{-1}(\mathcal{C}) \subset \widehat{\mathbb{P}}$ as a planar embedded graph, whose vertices and edges are precisely the 1 -vertices and the 1 -edges of the flapped pillow $\widehat{\mathbb{P}}$. Each 1 -edge of $\widehat{\mathbb{P}}$ is homeomorphically mapped by $\widehat{\mathcal{L}}$ onto one of the edges of $\mathbb{P}$. Similarly as before, the 1 -edges of $\widehat{\mathbb{P}}$ that are mapped by $\widehat{\mathcal{L}}$ onto $a$ or $c$ are called horizontal, while the 1 -edges of $\widehat{\mathbb{P}}$ that are mapped by $\widehat{\mathcal{L}}$ onto $b$ or $d$ are called vertical.

There is a simple path of length $n$ in the graph $\widehat{\mathcal{C}}$ that connects the vertices $\widehat{A}$ and $\widehat{B}$. Clearly, any such path consists only of horizontal 1-edges in $\widehat{\mathbb{P}}$. We denote by $\widehat{a}$ the realization of the chosen path in the sphere $\widehat{\mathbb{P}}$, which is an arc in $(\widehat{\mathbb{P}}, \widehat{V})$. The arc $\widehat{a}$ may not be uniquely determined (namely, if flaps have been glued to slits obtained from edges $e \subset a$ ), but any two such arcs are isotopic rel. $\widehat{V}$. We define $\widehat{b}, \widehat{c}, \widehat{d}$ in a similar way and call $\widehat{a}, \widehat{c}$ the horizontal edges, and $\widehat{b}, \widehat{d}$ the vertical edges of $\widehat{\mathbb{P}}$.

We now choose an orientation-preserving homeomorphism $\phi: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ that sends $\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}$ to $A, B, C, D$, and $\widehat{a}, \widehat{b}, \widehat{c}, \widehat{d}$ to $a, b, c, d$, respectively. We define $f:=\widehat{\mathcal{L}} \circ \phi^{-1}$, which is a self-map on $\mathbb{P}$. Clearly, $f$ is a branched covering map on $\mathbb{P}$. To refer to this map, we say that $f: \mathbb{P} \rightarrow \mathbb{P}$ is obtained from the $(n \times n)$-Lattès map $\mathcal{L}_{n}$ by gluing $n_{h}$ horizontal and $n_{v}$ vertical flaps to $\mathbb{P}$. More informally, we call both maps $f$ and $\widehat{\mathcal{L}}$ a blown-up Lattès map.

A point in $\widehat{\mathbb{P}}$ is a critical point for $\widehat{\mathcal{L}}: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ if and only if it is on the boundary of at least four 1-tiles subdividing $\widehat{\mathbb{P}}$. This implies that the set $C_{\widehat{\mathcal{L}}}$ of critical points of $\widehat{\mathcal{L}}$ consists of 1 -vertices of $\widehat{\mathbb{P}}$ and that each critical point of $\mathcal{L}_{n}$ is also a critical point for $\widehat{\mathcal{L}}$ (recall that we can view each point in $\mathcal{L}_{n}^{-1}(V) \supset C_{\mathcal{L}_{n}}$ as a 1 -vertex in $\widehat{\mathbb{P}}$ ). Moreover, if a 1 -vertex of $\widehat{\mathbb{P}}$ is a critical point of $\widehat{\mathcal{L}}$, but not of $\mathcal{L}_{n}$, then it must be one of the points in $\widehat{V}$. For example, $\widehat{A} \in \widehat{V}$ is a critical point of $\widehat{\mathcal{L}}$ if and only if a flap was glued to an edge of $\widetilde{\mathcal{C}}$ incident to $A \cong \widehat{A}$. In any
case, since $\widehat{\mathcal{L}}$ sends the 1 -vertices of $\widehat{\mathbb{P}}$ to the vertices of $\mathbb{P}$, the postcritical set of $f=\widehat{\mathcal{L}} \circ \phi^{-1}$ coincides with the vertex set $V$. Thus, $f$ is a Thurston map.

Since we assumed that $\widehat{\mathbb{P}}$ contains at least one flap (that is, $n_{h}+n_{v}>0$ ), the orbifold of the Thurston map $f$ is hyperbolic. Indeed, each 1 -vertex of $\widehat{\mathbb{P}}$ that is a critical point of $\mathcal{L}_{n}$ is also a critical point of $\widehat{\mathcal{L}}$ with the same or larger local degree. Since we glued at least one flap, there is at least one 1 -vertex $v$ contained in six or more 1 -tiles of $\widehat{\mathbb{P}}$. Then $\operatorname{deg}(\widehat{\mathcal{L}}, v)=\operatorname{deg}\left(f, v^{\prime}\right) \geq 3$, where $v^{\prime}=\phi(v)$. Now

$$
X:=f\left(v^{\prime}\right)=\widehat{\mathcal{L}}(v) \in V=\{A, B, C, D\}=P_{f}
$$

and so for the ramification function $\alpha_{f}$ of $f$ we have $\alpha_{f}(X) \geq 3$. On the other hand, for all other points $Y \in V=P_{f}$ we have $\alpha_{f}(Y) \geq \alpha_{\mathcal{L}_{n}}(Y) \geq 2$. It then follows from (3.1) that $\chi\left(\mathcal{O}_{f}\right)<0$, and so $f$ has indeed a hyperbolic orbifold.

The homeomorphism $\phi$ chosen in the definition of $f$ is not unique, but any two such homeomorphisms are isotopic rel. $\widehat{V}$ (this easily follows from [Bus10, Theorem A.5]). This implies that $f$ is uniquely determined up to Thurston equivalence. This map may be viewed (up to Thurston equivalence) as the result of the blowing up operation introduced in Section 4.1 applied to the edges $e \in E$ with the multiplicities $m_{e}$. In particular, if we run the procedure for the map $\widehat{\mathcal{L}}$ indicated in Figure 13, then we obtain the map $\widehat{f}$ illustrated in Figure 12 (up to Thurston equivalence).

## 5. Realizing blown-up Lattès maps

The goal of this section is to determine when a blown-up Lattès map is realized by a rational map. In particular, we will apply Thurston's criterion to prove Theorem 1.2. The strategies and techniques used in the proof will highlight the main ideas needed for establishing the more general Theorem 1.1.

We fix $n \geq 2, n_{h}, n_{v} \geq 0$, and follow the notation introduced in Section 4.2. In particular, we denote by $\widehat{\mathbb{P}}$ a flapped pillow with $n_{h}$ horizontal and $n_{v}$ vertical flaps, by $\widehat{\mathcal{L}}: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ the respective blown-up $(n \times n)$-Lattès map, and by $\phi: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ the identifying homeomorphism. Then $f: \mathbb{P} \rightarrow \mathbb{P}$ given as $f=\widehat{\mathcal{L}} \circ \phi^{-1}$ is the Thurston map under consideration. We will assume that $n_{h}+n_{v}>0$, and so $\widehat{\mathbb{P}}$ has at least one flap. In this case, $f$ has a hyperbolic orbifold as we have seen, and so we can apply Thurston's criterion. For this we consider essential Jordan curves $\gamma$ in $\left(\mathbb{P}, P_{f}\right)=(\mathbb{P}, V)$ and study their (essential) pullbacks under $f$.

If $\gamma$ is such a curve, then the homeomorphism $\phi$ sends the pullbacks of $\gamma$ under $\widehat{\mathcal{L}}$ to the pullbacks of $\gamma$ under $f$. So in order to understand the isotopy types and mapping properties of the pullbacks under $f$, we will instead look at the pullbacks of $\gamma$ under $\widehat{\mathcal{L}}$. In particular, if $\widehat{\gamma}$ is a pullback of $\gamma$ under $\widehat{\mathcal{L}}$, then $\operatorname{deg}(\widehat{\mathcal{L}}: \widehat{\gamma} \rightarrow \gamma)=\operatorname{deg}(f: \phi(\widehat{\gamma}) \rightarrow \gamma)$ and $\phi(\widehat{\gamma})$ is essential in $\left(\mathbb{P}, P_{f}\right)=(\mathbb{P}, V)$ if and only if $\widehat{\gamma}$ is essential in $(\widehat{\mathbb{P}}, \widehat{V})$, where $\widehat{V}$ denotes the vertex set of the flapped pillow $\widehat{\mathbb{P}}$.

Since the mapping $\widehat{\mathcal{L}}: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ is a similarity map on each 1 -tile of $\widehat{\mathbb{P}}$, the preimage $\widehat{\mathcal{L}}^{-1}(\gamma)$ of a Jordan curve $\gamma$ in $\left(\mathbb{P}, P_{f}\right)$ (or of any subset $\gamma$ of $\mathbb{P}$ ), can be obtained in the following intuitive way: we rescale and copy the part of $\gamma$ that belongs to the white 0 -tile of $\mathbb{P}$ into each white 1 -tile of $\widehat{\mathbb{P}}$ and the part of $\gamma$ that belongs to the black 0 -tile of $\mathbb{P}$ into each black 1-tile.
5.1. The horizontal and vertical curves. Recall that $\alpha^{h}$ and $\alpha^{v}$ (see (2.4)) denote the Jordan curves in $(\mathbb{P}, V)=\left(\mathbb{P}, P_{f}\right)$ that separate the two horizontal and the two vertical edges


Figure 15. Pullbacks of $\alpha^{h}$ for a blown-up (4×4)-Lattès map with $n_{h}=n_{v}=$ 1.
of $\mathbb{P}$, respectively. These two curves are invariant under $f$ and will play a crucial role in the considerations of this section.

Lemma 5.1. Let $f=\widehat{\mathcal{L}} \circ \phi^{-1}: \mathbb{P} \rightarrow \mathbb{P}$ be a Thurston map obtained from the $(n \times n)$-Lattès map, $n \geq 2$, by gluing $n_{h} \geq 0$ horizontal and $n_{v} \geq 0$ vertical flaps to $\mathbb{P}$. Then the following statements are true:
(i) The Jordan curve $\alpha^{h}$ has $n+n_{h}$ pullbacks under $f$. Exactly $n$ of these pullbacks are essential. Each of these essential pullbacks is isotopic to $\alpha^{h}$ rel. $P_{f}$.
(ii) If $\widetilde{\alpha}$ is one of the $n$ essential pullbacks of $\alpha^{h}$, then $\operatorname{deg}\left(f: \widetilde{\alpha} \rightarrow \alpha^{h}\right)=n+n_{\widetilde{\alpha}}$, where $n_{\widetilde{\alpha}} \geq 0$ is the number of distinct vertical flaps in $\widehat{\mathbb{P}}$ that $\phi^{-1}(\widetilde{\alpha})$ meets.
(iii) We have

$$
\lambda_{f}\left(\alpha^{h}\right)=\sum_{\widetilde{\alpha}} \frac{1}{n+n_{\widetilde{\alpha}}},
$$

where the sum is taken over all essential pullbacks $\widetilde{\alpha}$ of $\alpha^{h}$ under $f$.
Analogous statements are true for the curve $\alpha^{v}$.
Proof. Figure 15 illustrates the proof. It is obvious that the Jordan curve $\alpha^{h}$ has exactly $n+n_{h}$ distinct pullbacks under $\widehat{\mathcal{L}}$. Among them, there are $n$ essential pullbacks $\widehat{\alpha}_{1}, \ldots, \widehat{\alpha}_{n}$ that separate the two horizontal edges of $\widehat{\mathbb{P}}$ and thus are isotopic to each other relative to the vertex set $\widehat{V}$ of $\widehat{\mathbb{P}}$. For each $j=1, \ldots, n$, the image $\alpha_{j}:=\phi\left(\widehat{\alpha}_{j}\right)$ is isotopic to $\alpha^{h}$. Moreover, we have $\operatorname{deg}\left(\widehat{\mathcal{L}}: \widehat{\alpha}_{j} \rightarrow \alpha^{h}\right)=n+n_{\alpha_{j}}$. The other $n_{h}$ pullbacks of $\alpha^{h}$ under $\widehat{\mathcal{L}}$ are each contained in one of the horizontal flaps and thus are peripheral in $(\widehat{\mathbb{P}}, \widehat{V})$. Consequently,

$$
\lambda_{f}\left(\alpha^{h}\right)=\sum_{j=1}^{n} \frac{1}{\operatorname{deg}\left(f: \alpha_{j} \rightarrow \alpha^{h}\right)}=\sum_{j=1}^{n} \frac{1}{\operatorname{deg}\left(\widehat{\left.\mathcal{L}: \widehat{\alpha}_{j} \rightarrow \alpha^{h}\right)}=\sum_{j=1}^{n} \frac{1}{n+n_{\alpha_{j}}} . . . . ~\right.}
$$

This completes the proof of the lemma for the curve $\alpha^{h}$. The proof for the curve $\alpha^{v}$ follows from similar considerations.

The following corollary is an immediate consequence of the previous lemma.
Corollary 5.2. Let $f=\widehat{\mathcal{L}} \circ \phi^{-1}: \mathbb{P} \rightarrow \mathbb{P}$ be a Thurston map obtained from the $(n \times n)$-Lattès map, $n \geq 2$, by gluing $n_{h} \geq 0$ horizontal and $n_{v} \geq 0$ vertical flaps to $\mathbb{P}$. Then $\alpha^{h}$ is an obstruction (for $f$ ) if and only if $n_{v}=0$, and $\alpha^{v}$ is an obstruction if and only if $n_{h}=0$.
Proof. Let us first suppose that $n_{v}=0$, that is, $\widehat{\mathbb{P}}$ does not have any vertical flaps. Then by Lemma 5.1, $\alpha^{h}$ has $n$ essential pullbacks under $f$, each of which is mapped onto $\alpha^{h}$ with


Figure 16. Pullbacks of $\alpha^{h}$ for a blown up $(4 \times 4)$-Lattès map with $n_{h}=1$ and $n_{v}=0$.
degree $n$ (this is illustrated in Figure 16 in a special case). Consequently, $\lambda_{f}\left(\alpha^{h}\right)=n \cdot(1 / n)=$ 1 , which means $\alpha^{h}$ is an obstruction for $f$.

If $n_{v}>0$, the flapped pillow $\widehat{\mathbb{P}}$ has at least one vertical flap. Then $n_{\widetilde{\alpha}}>0$ for at least one essential pullback $\widetilde{\alpha}$ of $\alpha^{h}$. Lemma 5.1 implies that $\lambda_{f}\left(\alpha^{h}\right) \leq(n-1) \frac{1}{n}+\frac{1}{n+1}<1$, and so $\alpha^{h}$ is not an obstruction for $f$.

The proof for the vertical curve $\alpha^{v}$ is completely analogous.
The above corollary can be read as follows: the obstruction $\alpha^{h}$ for the ( $n \times n$ )-Lattès map can be eliminated by gluing a vertical flap to $\mathbb{P}$. Similarly, the obstruction $\alpha^{v}$ can be eliminated by gluing a horizontal flap. We will show momentarily that if both of these obstructions are eliminated (that is, if there are both horizontal and vertical flaps) then no other obstructions are present and so the map $f$ is realized.
5.2. Ruling out other obstructions. Now we discuss what happens with the essential curves in ( $\mathbb{P}, P_{f}$ ) that are not isotopic to the horizontal curve $\alpha^{h}$ or the vertical curve $\alpha^{v}$.

Theorem 5.3. Let $f=\widehat{\mathcal{L}} \circ \phi^{-1}: \mathbb{P} \rightarrow \mathbb{P}$ be a Thurston map obtained from the $(n \times n)$-Lattès map, $n \geq 2$, by gluing $n_{h} \geq 0$ horizontal and $n_{v} \geq 0$ vertical flaps to $\mathbb{P}$ and assume that $n_{h}+n_{v}>0$. If $\gamma \subset \mathbb{P} \backslash P_{f}$ is an essential Jordan curve that is not isotopic to either $\alpha^{h}$ or $\alpha^{v}$, then $\gamma$ is not an obstruction for $f$.

Before we turn to the proof of this theorem, we first record how it implies Theorem 1.2 stated in the introduction.

Proof of Theorem 1.2. Let $n, f, n_{h}$, and $n_{v}$ with $n_{h}+n_{v}>0$ be as in the statement. We have seen in Section 4.2 that then $f$ has a hyperbolic orbifold. If $n_{h}=0$ or $n_{v}=0$, then by Corollary 5.2 the curve $\alpha^{v}$ or the curve $\alpha^{h}$ is an obstruction, respectively.

If $n_{h}>0$ and $n_{v}>0$, then $f$ has no obstruction as follows from Corollary 5.2 and Theorem 5.3. Since $f$ has a hyperbolic orbifold, in this case $f$ is realized by a rational map according to Theorem 3.7.

Corollary 5.2 and Theorem 5.3 also imply that if $n_{h}=0$, then $\alpha^{v}$ is the only obstruction for $f$ (up to isotopy rel. $P_{f}$ ). Similarly, $\alpha^{h}$ is the only obstruction if $n_{v}=0$.

Before we go into the details, we will give an outline for the proof of Theorem 5.3. We argue by contradiction and assume that $f$ has an obstruction given by an essential Jordan curve $\gamma$ in $\left(\mathbb{P}, P_{f}\right)$ that is isotopic to neither $\alpha^{h}$ nor $\alpha^{v}$ rel. $P_{f}$. Then $\lambda_{f}(\gamma) \geq 1$. Let $\gamma_{1}, \ldots, \gamma_{k}$ for some $k \in \mathbb{N}$ be all the essential pullbacks of $\gamma$ under $f$, which must be isotopic to $\gamma$ rel. $P_{f}$.

Using facts about intersection numbers and the mapping properties of $f$, one can show that for the number of essential pullbacks of $\gamma$ we have $k \leq n$ and that the corresponding mapping degrees satisfy $\operatorname{deg}\left(f: \gamma_{j} \rightarrow \gamma\right) \geq n$ for all $j=1, \ldots, k$. Since $\lambda_{f}(\gamma) \geq 1$, it follows that there are exactly $k=n$ essential pullbacks and that $\operatorname{deg}\left(f: \gamma_{j} \rightarrow \gamma\right)=n$ for each $j=1, \ldots, n$.

This in turn implies that none of the essential pullbacks $\widehat{\gamma}_{j}:=\phi^{-1}\left(\gamma_{j}\right)$ of $\gamma$ under $\widehat{\mathcal{L}}$ goes over a flap in $\widehat{\mathbb{P}}$. Then all the $n$ pullbacks $\widehat{\gamma}_{1}, \ldots, \widehat{\gamma}_{n}$ belong to the base $B(\widehat{\mathbb{P}})$ of $\widehat{\mathbb{P}}$. This means that each $\widehat{\gamma}_{j}$ can be thought of as a pullback of $\gamma$ under the original $(n \times n)$-Lattès map $\mathcal{L}_{n}$. However, there are only $n$ pullbacks of $\gamma$ under $\mathcal{L}_{n}$, which cross all the edges of the graph $\mathcal{L}_{n}^{-1}(\mathcal{C})$, where $\mathcal{C}$ is the common boundary of the 0 -tiles in $\mathbb{P}$. Consequently, the pullbacks $\widehat{\gamma}_{1}, \ldots, \widehat{\gamma}_{n}$ cross all the 1-edges in the closure of the base $B(\widehat{\mathbb{P}})$. It follows that one of the pullbacks $\widehat{\gamma}_{j}$ must cross one of the base edges of a flap $F$ in $\widehat{\mathbb{P}}$, which would necessarily mean that $\widehat{\gamma}_{j}$ goes over the flap $F$. This gives the desired contradiction and Theorem 5.3 follows.

In the remainder of this section we will fill in the details for this outline. First, we establish several general facts about degrees and intersection numbers.

Let $n \in \mathbb{N}$ and $f: X \rightarrow Y$ be a map between two sets $X$ and $Y$. We say that $f$ is at most $n$-to-1 if $\# f^{-1}(y) \leq n$ for each $y \in Y$. We say that $f$ is $n$-to-1 if $\# f^{-1}(y)=n$ for each $y \in Y$.

Lemma 5.4. Let $f: X \rightarrow Y$ be a map between two sets $X$ and $Y$. Suppose $M \subset X, N \subset Y$, and $f \mid M: M \rightarrow f(M)$ is at most $n$-to- 1 for some $n \in \mathbb{N}$. Then

$$
\#\left(M \cap f^{-1}(N)\right) \leq n \cdot \#(f(M) \cap N)
$$

Proof. The map $f$ sends each point in $M \cap f^{-1}(N)$ to a point in $f(M) \cap N$. Moreover, each point in $f(M) \cap N$ has at most $n$ preimages in $M \cap f^{-1}(N)$ under $f$. The statement follows.

Lemma 5.5. Let $f: S^{2} \rightarrow S^{2}$ be a Thurston map with $\# P_{f}=4$, and $\gamma$ be an essential Jordan curve in $\left(S^{2}, P_{f}\right)$. Suppose that $\widetilde{\alpha}$ is an essential Jordan curve or an arc in $\left(S^{2}, P_{f}\right)$ such that
(i) $f(\widetilde{\alpha})$ and $\widetilde{\alpha}$ are isotopic rel. $P_{f}$,
(ii) the map $f \mid \widetilde{\alpha}: \widetilde{\alpha} \rightarrow f(\widetilde{\alpha})$ is at most $n$-to- 1 , where $n \in \mathbb{N}$,
(iii) $\mathrm{i}(f(\widetilde{\alpha}), \gamma)=\#(f(\widetilde{\alpha}) \cap \gamma)>0$.

Then $k \leq n$, where $k \in \mathbb{N}_{0}$ denotes the number of distinct pullbacks of $\gamma$ under $f$ that are isotopic to $\gamma$ rel. $P_{f}$. Moreover, if $\widetilde{\alpha}$ meets a peripheral pullback of $\gamma$, then $k<n$.

In the formulation and the ensuing proof intersection numbers are with respect to $\left(S^{2}, P_{f}\right)$. Note that since $f(\widetilde{\alpha})$ and $\widetilde{\alpha}$ are isotopic rel. $P_{f}$ by assumption, $f(\widetilde{\alpha})$ is of the same type as $\widetilde{\alpha}$, that is, a Jordan curve or an arc in $\left(S^{2}, P_{f}\right)$.

Proof. Let $\gamma_{1}, \ldots, \gamma_{k}$ be the distinct pullbacks of $\gamma$ under $f$ that are isotopic to $\gamma$ rel. $P_{f}$. Since $f \mid \widetilde{\alpha}: \widetilde{\alpha} \rightarrow f(\widetilde{\alpha})$ is at most $n$-to- 1 , we can apply Lemma 5.4 and conclude that

$$
\#\left(\widetilde{\alpha} \cap f^{-1}(\gamma)\right) \leq n \cdot \#(f(\widetilde{\alpha}) \cap \gamma)=n \cdot \mathrm{i}(f(\widetilde{\alpha}), \gamma)
$$

On the other hand,

$$
\begin{equation*}
n \cdot \mathrm{i}(f(\widetilde{\alpha}), \gamma) \geq \#\left(\widetilde{\alpha} \cap f^{-1}(\gamma)\right) \geq \sum_{j=1}^{k} \#\left(\widetilde{\alpha} \cap \gamma_{j}\right) \geq \sum_{j=1}^{k} \mathrm{i}\left(\widetilde{\alpha}, \gamma_{j}\right)=k \cdot \mathrm{i}(f(\widetilde{\alpha}), \gamma) \tag{5.1}
\end{equation*}
$$



Figure 17. A pullback $\widehat{\gamma}$ going over a horizontal flap in $\widehat{\mathbb{P}}$.
Since $\mathrm{i}(f(\widetilde{\alpha}), \gamma)>0$, we see that $k \leq n$. If $\widetilde{\alpha}$ meets a peripheral pullback of $\gamma$, then the second inequality in (5.1) is strict and we actually have $k<n$.

The next result will lead to the strict inequality from Lemma 5.5 in the proof of Theorem 5.3.
Lemma 5.6. As before, let $\widehat{\mathcal{L}}: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ be the blown-up $(n \times n)$-Lattès map, and suppose $\gamma=\tau_{r / s}$ is a simple closed geodesic in $\mathbb{P}$ with slope $r / s \in \widehat{\mathbb{Q}} \backslash\{0, \infty\}$. Let $\widehat{\gamma}$ be a pullback of $\gamma$ under $\widehat{\mathcal{L}}$. If $\widehat{\gamma}$ intersects the interior of a base edge of a flap $F$ in $\widehat{\mathbb{P}}$, then $\widehat{\gamma}$ also intersects the top edge of $F$.

Proof. We have $\gamma=\wp\left(\ell_{r / s}\right)$, where $\ell_{r / s} \subset \mathbb{C} \backslash \mathbb{Z}^{2}$ is a straight line with slope $r / s \neq 0, \infty$. Then $\gamma$ is an essential Jordan curve in $(\mathbb{P}, V)$. Let $\widehat{\gamma} \subset \widehat{\mathbb{P}}$ be as in the statement. As in Section 2.4, we denote by $a, b, c, d$ the edges of the pillow $\mathbb{P}$. Let $e^{\prime} \subset \widehat{\mathbb{P}}$ be a base edge of a flap $F$ in $\widehat{\mathbb{P}}$ such that $\widehat{\gamma} \cap \operatorname{int}\left(e^{\prime}\right) \neq \varnothing$. We will assume that $F$ is a horizontal flap. Then $\widehat{\mathcal{L}}\left(e^{\prime}\right)=a$ or $\widehat{\mathcal{L}}\left(e^{\prime}\right)=c$. We will make the further assumption that $\widehat{\mathcal{L}}\left(e^{\prime}\right)=a$. The other cases, when $\widehat{\mathcal{L}}\left(e^{\prime}\right)=c$ or when $F$ is a vertical flap, can be treated in a way that is completely analogous to the ensuing argument.

Let $e^{\prime \prime} \subset F$ be the base edge of $F$ different from $e^{\prime}$, and $\widetilde{e}$ be the top edge of $F$. Then $\widehat{\mathcal{L}}\left(e^{\prime \prime}\right)=a$ and $\widehat{\mathcal{L}}(\widetilde{e})=c$. Moreover,

$$
\begin{equation*}
F \cap \widehat{\mathcal{L}}^{-1}(a)=e^{\prime} \cup e^{\prime \prime}=\partial F \quad \text { and } \quad F \cap \widehat{\mathcal{L}}^{-1}(c)=\widetilde{e} . \tag{5.2}
\end{equation*}
$$

We have $\gamma=\wp\left(\ell_{r / s}\right)$ with $r / s \neq 0$, and so by Lemma 2.5 the sets $a \cap \gamma$ and $c \cap \gamma$ are finite and non-empty and the points in these sets alternate on $\gamma$. Since $\widehat{\mathcal{L}}$ is a covering map from $\widehat{\gamma}$ onto $\gamma$, we conclude that the sets $\widehat{\gamma} \cap \widehat{\mathcal{L}}^{-1}(a)$ and $\widehat{\gamma} \cap \widehat{\mathcal{L}}^{-1}(c)$ are also finite and non-empty and the points in these sets alternate on $\widehat{\gamma}$.

We choose a point $p \in \widehat{\gamma} \cap \operatorname{int}\left(e^{\prime}\right)$. Since $\gamma=\wp\left(\ell_{r / s}\right)$, the curve $\gamma$ has transverse intersections with $\operatorname{int}(a)$. It follows that the pullback $\widehat{\gamma}$ has a transverse intersection with $\operatorname{int}\left(e^{\prime}\right)$ at $p$, and so $\widehat{\gamma}$ crosses into the interior of the flap $F$ at $p$. Therefore, if we travel along $\widehat{\gamma}$ starting at $p \in \widehat{\gamma} \cap \widehat{\mathcal{L}}^{-1}(a)$ and traverse into the interior of the flap $F$, we must meet $\widehat{\mathcal{L}}^{-1}(c)$ before we possibly exit $F$ through its boundary $\partial F=e^{\prime} \cup e^{\prime \prime}=F \cap \widehat{\mathcal{L}}^{-1}(a)$ (see (5.2)). Now $F \cap \widehat{\mathcal{L}}^{-1}(c)=\widetilde{e}$ and so this implies that the pullback $\widehat{\gamma}$ meets the top edge $\widetilde{e}$ (see Figure 17 for an illustration). The statement follows.

We are now ready to prove the main result of this section.
Proof of Theorem 5.3. Let $f: \mathbb{P} \rightarrow \mathbb{P}$ be a Thurston map as in the statement, obtained from the $(n \times n)$-Lattès map $\mathcal{L}_{n}, n \geq 2$, by gluing $n_{h} \geq 0$ horizontal and $n_{v} \geq 0$ vertical flaps, where we assume $n_{h}+n_{v}>0$. As described in the beginning of this section, then $f=\widehat{\mathcal{L}} \circ \phi^{-1}$, where $\widehat{\mathcal{L}}: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ is a branched covering map on the associated flapped pillow $\widehat{\mathbb{P}}$ and $\phi: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ is an
identifying homeomorphism as discussed in Section 4.2. Note that $\widehat{\mathbb{P}}$ has at least one flap, since $n_{h}+n_{v}>0$. Following the notation from Section 4.2 , we denote by $\widehat{a}, \widehat{b}, \widehat{c}, \widehat{d}$ the arcs in $(\widehat{\mathbb{P}}, \widehat{V})$ that under $\phi$ correspond to the edges $a, b, c, d$ of $\mathbb{P}$, respectively.

We now argue by contradiction and assume that there exists an essential Jordan curve $\gamma$ in $\left(\mathbb{P}, P_{f}\right)=(\mathbb{P}, V)$ that is not isotopic to $\alpha^{h}$ or $\alpha^{v}$ rel. $P_{f}=V$, but is an obstruction for $f$, that is, $\lambda_{f}(\gamma) \geq 1$. Since we can replace $\gamma$ with any curve in the same isotopy class, by Lemma 2.3 we may assume that $\gamma=\wp\left(\ell_{r / s}\right)$ for a straight line $\ell_{r / s} \subset \mathbb{C} \backslash \mathbb{Z}^{2}$ with slope $r / s \in \widehat{\mathbb{Q}}$. Since $\gamma$ is not isotopic to $\alpha^{h}$ or $\alpha^{v}$ rel. $V$, we have $r / s \neq 0, \infty$, and so $r, s \neq 0$. Then it follows from Lemma 2.4 (iii) and (iv) that

$$
\begin{array}{r}
\#(a \cap \gamma)=\mathrm{i}(a, \gamma)=|r|=\mathrm{i}(c, \gamma)=\#(c \cap \gamma)>0  \tag{5.3}\\
\#(b \cap \gamma)=\mathrm{i}(b, \gamma)=s=\mathrm{i}(d, \gamma)=\#(d \cap \gamma)>0
\end{array}
$$

In particular, $\gamma \subset \mathbb{P} \backslash V$ intersects the interiors of all four edges of $\mathbb{P}$.
Let $\gamma_{1}, \ldots, \gamma_{k}$ for some $k \in \mathbb{N}$ denote all the pullbacks of $\gamma$ under $f$ that are isotopic to $\gamma$ rel. $P_{f}$. By construction of the blown-up map, the arc $\widehat{a}=\phi^{-1}(a)$ consists of $n$ 1-edges in $\widehat{\mathbb{P}}$, each of which is homeomorphically mapped onto $a$ by $\widehat{\mathcal{L}}$. This implies that the map $f \mid a: a \rightarrow a$ is at most $n$-to-1. By (5.3) we can apply Lemma 5.5 to $a$ and $\gamma$ and conclude that $k \leq n$.

By Lemma 5.1, the horizontal curve $\alpha^{h}$ has $n$ distinct pullbacks under $f$ that are isotopic to $\alpha^{h}$ rel. $P_{f}$. Since

$$
\begin{equation*}
\mathrm{i}\left(\gamma, \alpha^{h}\right)=\#\left(\gamma \cap \alpha^{h}\right)=2|r|>0 \tag{5.4}
\end{equation*}
$$

by Lemma 2.4 (i), we can apply Lemma 5.5 again, this time to $\gamma_{j}$ and $\alpha^{h}$ (in the roles of $\widetilde{\alpha}$ and $\gamma$, respectively), and conclude that $n \leq \operatorname{deg}\left(f: \gamma_{j} \rightarrow \gamma\right)$ for all $j=1, \ldots, k$.

Then

$$
1 \leq \lambda_{f}(\gamma)=\sum_{j=1}^{k} \frac{1}{\operatorname{deg}\left(f: \gamma_{j} \rightarrow \gamma\right)} \leq k / n \leq 1
$$

Therefore, $k=n$ and $\operatorname{deg}\left(f: \gamma_{j} \rightarrow \gamma\right)=n$ for each $j=1, \ldots, n$.
This shows that the curve $\gamma$ has exactly $n$ essential pullbacks under $\widehat{\mathcal{L}}$ given by $\widehat{\gamma}_{1}:=$ $\phi^{-1}\left(\gamma_{1}\right), \ldots, \widehat{\gamma}_{n}:=\phi^{-1}\left(\gamma_{n}\right)$. Here, the isotopy classes are considered with respect to the vertex set $\widehat{V}$ of $\widehat{\mathbb{P}}$. We will now use the second part of Lemma 5.5 to show that none of these pullbacks goes over a flap in $\widehat{\mathbb{P}}$.
Claim. For each flap $F$ in $\widehat{\mathbb{P}}$ and each pullback $\widehat{\gamma}_{j}, j=1, \ldots, n$, we have $F \cap \widehat{\gamma}_{j}=\varnothing$.
To see that the Claim is true, suppose some pullback $\widehat{\gamma}_{j}$ meets a flap $F$ in $\widehat{\mathbb{P}}$. We may assume that $F$ is a horizontal flap; the other case, when $F$ is vertical, can be treated by similar considerations. Then $F$ contains a peripheral pullback $\widehat{\alpha}^{h}$ of $\alpha^{h}$ under $\widehat{\mathcal{L}}$, which separates the union $\partial F$ of the two base edges of $F$ from the top edge of $F$. We will first show that $\widehat{\gamma}_{j}$ intersects $\widehat{\alpha}^{h}$.

Note that $\partial F$ is a Jordan curve and that $\operatorname{int}(F)$ does not contain any point from the vertex set $\widehat{V}$ of $\widehat{\mathbb{P}}$. It follows that the curve $\widehat{\gamma}_{j}$ must intersect $\partial F$, because $\widehat{\gamma}_{j}$ is essential in $(\widehat{\mathbb{P}}, \widehat{V})$. Since the curve $\gamma$ does not pass through $P_{f}=V$, its pullback $\widehat{\gamma}_{j}$ under $\widehat{\mathcal{L}}$ does not pass through any 1-vertex in $\widehat{\mathbb{P}}$. Consequently, $\widehat{\gamma}_{j}$ must meet the interior of one of the two base edges of $F$, which compose the boundary $\partial F$. Lemma 5.6 now implies that $\widehat{\gamma}_{j}$ also meets the top edge of $F$. Therefore, $\widehat{\gamma}_{j}$ meets the peripheral pullback $\widehat{\alpha}^{h}$ in $F$.

It follows that $\gamma_{j}=\phi\left(\widehat{\gamma}_{j}\right)$ meets the peripheral pullback $\phi\left(\widehat{\alpha}^{h}\right)$ of $\alpha^{h}$ under $f$. Lemma 5.5 now implies that $n<\operatorname{deg}\left(f: \gamma_{j} \rightarrow \gamma\right)=n$, which is a contradiction. This finishes the proof of the Claim.

The Claim implies that each essential pullback $\widehat{\gamma}_{j}, j=1, \ldots, n$, belongs to the base $B(\widehat{\mathbb{P}})$ of $\widehat{\mathbb{P}}$. By $(4.2)$ we can identify $B(\widehat{\mathbb{P}})$ with the subset $\mathbb{P} \backslash \bigcup_{e \in E} \operatorname{int}(e)$ of the original pillow $\mathbb{P}$, where $E$ denotes the non-empty subset of all 1-edges of $\mathbb{P}$ along which flaps were glued in the construction of $\widehat{\mathbb{P}}$.

Under this identification, the map $\widehat{\mathcal{L}}$ on $B(\widehat{\mathbb{P}})$ coincides with the $(n \times n)$-Lattès map $\mathcal{L}_{n}$. Thus we may view $\widehat{\gamma}_{1}, \ldots, \widehat{\gamma}_{n}$ as pullbacks of $\gamma$ under $\mathcal{L}_{n}$ on the original pillow $\mathbb{P}$. Now $\gamma$ has exactly $n$ pullbacks under $\mathcal{L}_{n}$ (see (3.4)). This implies that

$$
\mathcal{L}_{n}^{-1}(\gamma)=\widehat{\gamma}_{1} \cup \cdots \cup \widehat{\gamma}_{n}
$$

Since $\gamma$ meets the interior of every edge of $\mathbb{P}$, the set $\mathcal{L}_{n}^{-1}(\gamma)=\widehat{\gamma}_{1} \cup \cdots \cup \widehat{\gamma}_{n}$ meets the interior of every 1 -edge of $\mathbb{P}$. This is impossible, because

$$
\widehat{\gamma}_{1} \cup \cdots \cup \widehat{\gamma}_{n} \subset B(\widehat{\mathbb{P}})=\mathbb{P} \backslash \bigcup_{e \in E} \operatorname{int}(e)
$$

does not meet the interior of any 1-edge in $E \neq \varnothing$. This is a contradiction and the statement follows.

## 6. Essential circuit length

In order to prove our main result, Theorem 1.1, we need some preparation, in particular a refined version of Lemma 5.5. We will also address the question how blowing up arcs modifies the pullbacks of a curve $\alpha$ under natural restrictions on the blown-up arcs. First, we introduce some terminology and establish some auxiliary facts.

Let $U \subset S^{2}$ be an open and connected set, and $\sigma \subset S^{2}$ be an arc. We say that $\sigma$ is an arc in $U$ ending in $\partial U$ if there exists an endpoint $p$ of $\sigma$ such that $\sigma \backslash\{p\} \subset U$ and $p \in \partial U$.

Let $\mathcal{G}$ be a connected planar embedded graph in $S^{2}$ and $U$ be one of its faces. Then $U$ is simply connected, and so we can find a homeomorphism $\varphi: \mathbb{D} \rightarrow U$. Since we want some additional properties of $\varphi$ here, it is easiest to equip $S^{2}$ with a complex structure and choose a conformal map $\varphi: \mathbb{D} \rightarrow U$.

Since $\partial U$ is a finite union of edges of $\mathcal{G}$, this set is locally connected and so the conformal map $\varphi$ extends to a surjective continuous map $\varphi: \operatorname{cl}(\mathbb{D}) \rightarrow \operatorname{cl}(U)$ [Pom92, Theorem 2.1]. This extension has the following property: if $\sigma$ is an arc in $U$ ending in $\partial U$, then $\varphi^{-1}(\sigma)$ is an arc in $\mathbb{D}$ ending in $\partial \mathbb{D}$ (see [Pom92, Proposition 2.14]). For given $\mathcal{G}$ and $U$, we fix, once and for all, such a $\operatorname{map} \varphi=\varphi_{\mathcal{G}, U}$ from $\operatorname{cl}(\mathbb{D})$ onto $\operatorname{cl}(U)$.

Let $\left(e_{1}, e_{2}, \ldots, e_{n}\right)$ be a circuit in $\mathcal{G}$ that traces the boundary $\partial U$. Recall from Section 2.3 that the number $n$ is called the circuit length of $U$ in $\mathcal{G}$, and each edge $e \in \partial U$ appears exactly once or twice in the sequence $e_{1}, e_{2}, \ldots, e_{n}$ depending on whether the face $U$ lies on one or both sides of $e$, respectively. Then there is a corresponding decomposition $\partial \mathbb{D}=\sigma_{1} \cup \cdots \cup \sigma_{n}$ of the unit circle $\partial \mathbb{D}$ into non-overlapping subarcs $\sigma_{1}, \ldots, \sigma_{n}$ of $\partial \mathbb{D}$ such that $\varphi=\varphi_{\mathcal{G}, U}$ is a homeomorphism of $\sigma_{m}$ onto $e_{m}$ for each $m=1, \ldots, n$.

Let $0<\epsilon<1$. We say that a Jordan curve $\beta \subset U$ is an $\epsilon$-boundary of $U$ with respect to (wrt.) $\mathcal{G}$ if $\beta^{\prime}:=\varphi^{-1}(\beta) \subset A_{\epsilon}:=\{z \in \mathbb{C}: 1-\epsilon<|z|<1\}$, and $\beta^{\prime}$ separates 0 from $\partial \mathbb{D}$. Then $\beta^{\prime}$ is a core curve of the annulus $A_{\epsilon}$.

For the remainder of this section, $f: S^{2} \rightarrow S^{2}$ is a Thurston map. All isotopies on $S^{2}$ are relative to $P_{f}$, and we consider intersection numbers in $\left(S^{2}, P_{f}\right)$.

Let $e$ be an arc in $\left(S^{2}, P_{f}\right)$. Then we can naturally view the set $\mathcal{G}:=f^{-1}(e)$ as a planar embedded graph with the vertex set $f^{-1}(\partial e)$ and the edges given by the lifts of $e$ under $f$. Note that $\mathcal{G}$ is bipartite.

Lemma 6.1. Let $f: S^{2} \rightarrow S^{2}$ be a Thurston map, e be an arc and $\gamma$ be a Jordan curve in $\left(S^{2}, P_{f}\right)$ with $\#(e \cap \gamma)=\mathrm{i}(e, \gamma)$. Suppose that $\mathcal{H}$ is a connected subgraph of $\mathcal{G}=f^{-1}(e)$ and $U$ is a face of $\mathcal{H}$. Let $2 n$ with $n \in \mathbb{N}$ be the circuit length of $U$ in $\mathcal{H}$. Then for each $0<\epsilon<1$ there exists an $\epsilon$-boundary $\beta$ of $U$ wrt. $\mathcal{H}$ such that $\#\left(\beta \cap f^{-1}(\gamma)\right)=2 n \cdot \mathrm{i}(e, \gamma)$.

Note that the circuit length of $U$ in $\mathcal{H}$ is even since $\mathcal{H}$ is a bipartite graph.
Proof. Let $s:=\mathrm{i}(e, \gamma) \in \mathbb{N}_{0}$. Then $e$ and $\gamma$ have exactly $s$ distinct points in common, say $p_{1}, \ldots, p_{s} \in e \cap \gamma$. Each point $p_{j}$ lies in int $(e)$, because $\partial e \subset P_{f}$ and $\gamma \subset S^{2} \backslash P_{f}$. Since $e$ and $\gamma$ are in minimal position, they meet transversely (see Lemma 2.2 ), that is, if we travel along $\gamma$ towards one of the points $p_{j}$ (according to some orientation of $\gamma$ ), then near $p_{j}$ we stay on one side of $e$, but cross over to the other side of $e$ if we pass $p_{j}$.

This implies that we can find disjoint subarcs $\sigma_{1}, \ldots, \sigma_{s}$ of $\gamma$ such that each arc $\sigma_{j}$ contains $p_{j}$ in its interior, but contains no other point in $e \cap \gamma$. Moreover, $p_{j}$ splits $\sigma_{j}$ into two non-overlapping subarcs $\sigma_{j}^{L}$ and $\sigma_{j}^{R}$ with the common endpoint $p_{j}$ so that with some fixed orientation of $e$ the $\operatorname{arc} \sigma_{j}^{L}$ lies to the left and $\sigma_{j}^{R}$ lies to the right of $e$. Note that if $\gamma^{\prime}:=\gamma \backslash\left(\sigma_{1} \cup \cdots \cup \sigma_{s}\right)$, then $e \cap \gamma^{\prime}=\varnothing$.

For the given face $U$ of $\mathcal{H}$ we fix a $\operatorname{map} \varphi=\varphi_{\mathcal{H}, U}: \operatorname{cl}(\mathbb{D}) \rightarrow \operatorname{cl}(U)$ as discussed in the beginning of this section. Let $\left(e_{1}, \ldots, e_{2 n}\right)$ be a circuit in $\mathcal{H}$ that traces the boundary $\partial U$. As we have already pointed out, the number of edges in the circuit is even, because $\mathcal{H}$ is a bipartite graph. With suitable orientation of each arc $e_{m}$, the face $U$ lies on the left of $e_{m}$. If an arc appears twice in the list $e_{1}, \ldots, e_{2 n}$, then it will carry opposite orientations in its two occurrences.

We want to investigate the set $f^{-1}(\gamma) \cap \operatorname{cl}(U)$ near $\partial U$. Note that $f$ maps each arc $e_{m}$ homeomorphically onto $e$. This implies that $f$ is a homeomorphism on a suitable Jordan region that contains $e_{m}$ as a crosscut. It follows that we can pull back the local picture near points in $e \cap \gamma$ to a similar local picture for points in $e_{m} \cap f^{-1}(\gamma)$. So if we choose the $\operatorname{arcs} \sigma_{j}$ small enough, as we may assume, and pull them back by $f$, then it is clear that $f^{-1}(\gamma) \cap \operatorname{cl}(U)$ can be represented in the form

$$
f^{-1}(\gamma) \cap \operatorname{cl}(U)=K \cup \bigcup_{m=1}^{2 n} \bigcup_{j=1}^{s} \sigma_{m, j}
$$

where $K$ has positive distance to $\partial U=e_{1} \cup \cdots \cup e_{2 n}$ (with respect to some base metric on $S^{2}$ ). Moreover, each $\sigma_{m, j}$ is an arc in $U$ ending in $e_{m} \subset \partial U$ such that $f$ is a homeomorphism from $\sigma_{m, j}$ onto $\sigma_{j}^{L}$ or $\sigma_{j}^{R}$ depending on whether $f \mid e_{m}: e_{m} \rightarrow e$ is orientation-preserving or orientation-reversing. If we remove from each arc $\sigma_{m, j}$ its endpoint in $e_{m}$, then the half-open arcs obtained are all disjoint. Two arcs $\sigma_{m, j}$ and $\sigma_{m^{\prime}, j^{\prime}}$ share an endpoint precisely when $j=j^{\prime}$ and they arise from edges $e_{m}$ and $e_{m^{\prime}}$ with the same underlying set, but with opposite orientations. In this case, $f$ sends one of these arcs to $\sigma_{j}^{L}$, and the other one to $\sigma_{j}^{R}$.

Since $K$ has positive distance to $\partial U$, it is clear that if $\beta$ is an $\epsilon$-boundary of $U$ wrt. $\mathcal{H}$ for $\epsilon>0$ small enough (as we may assume), then $\beta \cap K=\varnothing$. So in order to control $\#\left(\beta \cap f^{-1}(\gamma)\right)$, we have to worry only about the intersections of $\beta$ with the $\operatorname{arcs} \sigma_{m, j}$.

Note that there are exactly $2 n \cdot s=2 n \cdot \mathrm{i}(e, \gamma)$ of these arcs. If we pull them back by the map $\varphi$, then we obtain pairwise disjoint arcs in $\mathbb{D}$ ending in $\partial \mathbb{D}$. The statement now follows
from the following fact, whose precise justification we leave to the reader: if $\alpha_{1}, \ldots, \alpha_{M}$ with $M \in \mathbb{N}_{0}$ are pairwise disjoint arcs in $\mathbb{D}$ ending in $\partial \mathbb{D}$, then for each $0<\epsilon<1$ there exists a core curve $\beta^{\prime}$ of the annulus $A_{\epsilon}=\{z \in \mathbb{C}: 1-\epsilon<|z|<1\}$ such that $\#\left(\beta^{\prime} \cap\left(\alpha_{1} \cup \cdots \cup \alpha_{M}\right)\right)=M$.

Now we are ready to provide a refined version of Lemma 5.5.
Lemma 6.2. Let $f: S^{2} \rightarrow S^{2}$ be a Thurston map with $\# P_{f}=4, \alpha$ and $\gamma$ be essential Jordan curves in $\left(S^{2}, P_{f}\right)$, c be a core arc of $\alpha$, and assume that $\#(c \cap \gamma)=\mathrm{i}(c, \gamma)>0$.

Let $\mathcal{H}$ be a connected subgraph of $\mathcal{G}:=f^{-1}(c)$, and $U$ be a face of $\mathcal{H}$ such that for small enough $\epsilon>0$ each $\epsilon$-boundary $\beta$ of $U$ wrt. $\mathcal{H}$ is isotopic to $\alpha$ rel. $P_{f}$. Let $2 n$ with $n \in \mathbb{N}$ be the circuit length of $U$ in $\mathcal{H}$.

Then $k \leq n$, where $k \in N_{0}$ denotes the number of pullbacks of $\gamma$ under $f$ that are isotopic to $\gamma$ rel. $P_{f}$. Moreover, if $\partial U \subset \mathcal{H}$ meets a peripheral pullback of $\gamma$ under $f$, then $k<n$.

Proof. Let $\gamma_{1}, \ldots, \gamma_{k}$ be the pullbacks of $\gamma$ under $f$ that are isotopic to $\gamma$ rel. $P_{f}$. Then by Lemma 6.1, for sufficiently small $\epsilon>0$, we can find an $\epsilon$-boundary $\beta$ of $U$ wrt. $\mathcal{H}$ such that $\beta \sim \alpha$ rel. $P_{f}$ and

$$
\#\left(\beta \cap f^{-1}(\gamma)\right)=2 n \cdot \mathrm{i}(c, \gamma)=n \cdot \mathrm{i}(\alpha, \gamma),
$$

where the last equality follows from Lemma 2.6. Hence, we have

$$
\begin{equation*}
n \cdot \mathrm{i}(\alpha, \gamma)=\#\left(\beta \cap f^{-1}(\gamma)\right) \geq \sum_{j=1}^{k} \#\left(\beta \cap \gamma_{j}\right) \geq \sum_{j=1}^{k} \mathrm{i}\left(\beta, \gamma_{j}\right)=k \cdot \mathrm{i}(\alpha, \gamma) . \tag{6.1}
\end{equation*}
$$

Since $\mathrm{i}(\alpha, \gamma)=2 \cdot \mathrm{i}(c, \gamma)>0$, we conclude that $k \leq n$, as desired.
To see the second statement, we have to revisit the proof of Lemma 6.1. There we identified $2 n \cdot \mathrm{i}(c, \gamma)=n \cdot \mathrm{i}(\alpha, \gamma)$ distinct $\operatorname{arcs} \sigma$ in $U$ ending in $\partial U$ (they were called $\sigma_{m, j}$ in the proof). These arcs were subarcs of $f^{-1}(\gamma)$ and accounted for all possible intersections of $\beta$ with $f^{-1}(\gamma)$ for sufficiently small $\epsilon>0$; with a suitable choice of $\beta$ each of these arcs $\sigma$ gave precisely one such intersection point. Now if a peripheral pullback $\widetilde{\gamma} \subset f^{-1}(\gamma)$ of $\gamma$ under $f$ meets $\partial U$, then one of these $\operatorname{arcs} \sigma$ is a subarc of $\widetilde{\gamma}$. It follows that the first inequality in (6.1) must be strict and so $k<n$.

For the rest of this section, we fix a Thurston map $f: S^{2} \rightarrow S^{2}$ with $\# P_{f}=4$, an essential Jordan curve $\alpha$ in $\left(S^{2}, P_{f}\right)$, and core arcs $a$ and $c$ of $\alpha$ that lie in different components of $S^{2} \backslash \alpha$. We can view the set $\mathcal{G}:=f^{-1}(a \cup c)$ as a planar embedded graph in $S^{2}$ with the set of vertices $f^{-1}\left(P_{f}\right)$ and the edge set consisting of the lifts of $a$ and $c$ under $f$. Then $\mathcal{G}$ is a bipartite graph.

Let $U$ be the unique connected component of $S^{2} \backslash(a \cup c)$. Then $U$ is an annulus and $\alpha$ is its core curve. The connected components $\widetilde{U}$ of $f^{-1}(U)$ are precisely the complementary components of $\mathcal{G}=f^{-1}(a \cup c)$ in $S^{2}$. It easily follows from the Riemann-Hurwitz formula (see (2.2)) that each $\widetilde{U}$ is an annulus, and that $f: \widetilde{U} \rightarrow U$ is a covering map. Moreover, each such annulus contains precisely one pullback $\widetilde{\alpha}$ of $\alpha$ under $f$.

This setup is illustrated in Figure 18. The points marked in black indicate the four postcritical points of $f$. The sphere on the right contains two core arcs $a$ and $c$ of a Jordan curve $\alpha$ in $\left(S^{2}, P_{f}\right)$. On the left the lifts of $a$ and $c$ under $f$ are shown in blue and magenta colors, respectively, and the pullbacks of $\alpha$ in green.

We call a connected component $\widetilde{U}$ of $f^{-1}(U)=S^{2} \backslash \mathcal{G}$ essential or peripheral, depending on whether the unique pullback $\widetilde{\alpha}$ of $\alpha$ contained in $\widetilde{U}$ is essential or peripheral in $\left(S^{2}, P_{f}\right)$, respectively. Each boundary $\partial \widetilde{U}$ has exactly two connected components. One of them is


Figure 18. A Thurston map $f$. The sphere on the right shows a Jordan curve $\alpha$ in $\left(S^{2}, P_{f}\right)$ and two core arcs $a$ and $c$. The sphere on the left shows the pullbacks of $\alpha$ under $f$ and the planar embedded graph $\mathcal{G}=f^{-1}(a \cup c)$.
mapped by $f$ to $a$ and the other one to $c$; accordingly, we denote them by $\partial_{a} \widetilde{U}$ and $\partial_{c} \widetilde{U}$, respectively. Then we have

$$
\partial \widetilde{U}=\partial_{a} \widetilde{U} \cup \partial_{c} \widetilde{U}, \quad \partial \widetilde{U}_{a}=f^{-1}(a) \cap \partial \widetilde{U}, \quad \text { and } \quad \partial \widetilde{U}_{c}=f^{-1}(c) \cap \partial \widetilde{U} .
$$

The sets $\partial_{a} \widetilde{U}$ and $\partial_{c} \widetilde{U}$ are subgraphs of $\mathcal{G}$. Since $\widetilde{U}$ is a connected subset of $S^{2} \backslash \mathcal{G} \subset S^{2} \backslash \partial_{a} \widetilde{U}$, there exists a unique face $V_{a}$ of $\partial_{a} \widetilde{U}$ (considered as a subgraph of $\mathcal{G}$ ) such that $\widetilde{U} \subset V_{a}$. Similarly, there exists a unique face $V_{c}$ of $\partial_{c} \widetilde{U}$ with $\widetilde{U} \subset V_{c}$. By definition, the circuit length of $\partial_{a} \widetilde{U}$ or of $\partial_{c} \widetilde{U}$ is the circuit length of $V_{a}$ in $\partial_{a} \widetilde{U}$ or of $V_{c}$ in $\partial_{c} \widetilde{U}$, respectively.

Then the following statement is true.
Lemma 6.3. The circuit lengths of $\partial_{a} \widetilde{U}$ and $\partial_{c} \widetilde{U}$ are both equal to $2 \cdot \operatorname{deg}(f: \widetilde{U} \rightarrow U)$.
We call the identical circuit lengths of $\partial_{a} \widetilde{U}$ and $\partial_{c} \widetilde{U}$ the circuit length of $\widetilde{U}$ (for fixed $f$, $\alpha, a$, and $c$ ).
Proof. It is clear that the subgraph $\partial_{a} \widetilde{U}$ of $\mathcal{G}$ is bipartite, and so $\partial_{a} \widetilde{U}$ has even circuit length $2 n$ with $n \in \mathbb{N}$. Let $d:=\operatorname{deg}(f: \widetilde{U} \rightarrow U)$. It is enough to show that $2 n=2 d$, because the roles of $\partial_{a} \widetilde{U}$ and $\partial_{c} \widetilde{U}$ are symmetric, and so the same identity will then also be true for the circuit length of $\partial_{c} \widetilde{U}$.

To see that $2 n=2 d$, we use a similar idea as in the proof of Lemma 6.1. We choose a point $p \in \operatorname{int}(a)$ and an $\operatorname{arc} \sigma \subset S^{2} \backslash c$ with $p \in \operatorname{int}(\sigma)$ that meets $a$ transversely in $p$, but has no other point with $a$ in common. Then $p$ splits $\sigma$ into two non-overlapping subarcs $\sigma^{L}$ and $\sigma^{R}$ with the common endpoint $p$ so that with some fixed orientation of $a$ the $\operatorname{arc} \sigma^{L}$ lies to the left and $\sigma^{R}$ lies to the right of $a$.

Let $\left(e_{1}, \ldots, e_{2 n}\right)$ be a circuit in $\partial_{a} \widetilde{U}$ that traces the boundary $\partial V_{a}=\partial_{a} \widetilde{U}$, where $V_{a}$ is the unique face of $\partial_{a} \widetilde{U}$ with $\widetilde{U} \subset V_{a}$. With suitable orientation of each arc $e_{m}$, the face $\widetilde{U}$ lies on the left of $e_{m}$. We now consider the set $f^{-1}(\sigma) \cap \operatorname{cl}(\widetilde{U})$ near $\partial_{a} \widetilde{U}$. If we choose $\sigma$ small enough (as we may), then as in the proof of Lemma 6.1 we see that

$$
\begin{equation*}
f^{-1}(\sigma) \cap \operatorname{cl}(\widetilde{U})=\bigcup_{m=1}^{2 n} \sigma_{m}, \tag{6.2}
\end{equation*}
$$

where each $\sigma_{m}$ is an arc in $\widetilde{U}$ ending in $e_{m} \subset \partial_{a} \widetilde{U}$ such that $f$ is a homeomorphism from $\sigma_{m}$ to $\sigma^{L}$ or $\sigma^{R}$ depending on whether $f \mid e_{m}: e_{m} \rightarrow a$ is orientation-preserving or orientationreversing. If we remove from each $\operatorname{arc} \sigma_{m}$ its endpoint in $e_{m}$, then the half-open arcs obtained
are all disjoint. On the other hand, since $f: \widetilde{U} \rightarrow U$ is a $d$-to- 1 covering map, there are precisely $d$ distinct lifts of $\sigma^{L} \backslash\{p\}$ and $d$ distinct lifts of $\sigma^{R} \backslash\{p\}$ under $f$ contained in $\widetilde{U}$. These must be precisely the half-open arcs obtained from $\sigma_{m}, m=1, \ldots, 2 n$. It follows that $2 n=2 d$, as desired.

Suppose $\widetilde{U}$ is an essential component of $f^{-1}(U)$. We consider a circuit in $\partial \widetilde{U} \subset \mathcal{G}$ and denote by $\mathcal{H} \subset \mathcal{G}$ the underlying graph of the circuit. In the following, we will often conflate the circuit with its underlying graph $\mathcal{H}$, where we think of $\mathcal{H}$ as traversed as a circuit in some way. Since $\widetilde{U}$ is a connected set in $S^{2} \backslash \mathcal{G} \subset S^{2} \backslash \mathcal{H}$, there exists a unique face $V$ of $\mathcal{H}$ such that $\widetilde{U} \subset V$. By definition, for $0<\epsilon<1$, an $\epsilon$-boundary $\beta$ of $\widetilde{U}$ wrt. $\mathcal{H}$ is an $\epsilon$-boundary of $V$ wrt. $\mathcal{H}$. This is an abuse of terminology, because even for small $\epsilon>0$ such an $\epsilon$-boundary $\beta$ may not lie in $\widetilde{U}$, but it is convenient in the following. Note that for small enough $\epsilon>0$ such $\epsilon$-boundaries for fixed $\mathcal{H}$ have the same isotopy type rel. $P_{f}$.

By definition the essential circuit length of $\widetilde{U}$ is the minimal length of all circuits $\mathcal{H}$ in $\partial \widetilde{U}$ such that for all small enough $\epsilon>0$ each $\epsilon$-boundary of $\widetilde{U}$ wrt. $\mathcal{H}$ is isotopic to a core curve of $\widetilde{U}$ rel. $P_{f}$. As we will see momentarily, if we run through $\partial_{a} \widetilde{U}$ and $\partial_{c} \widetilde{U}$ as circuits, then they have this property and so the essential circuit length of $\widetilde{U}$ is well-defined. We call a circuit $\mathcal{H}$ in $\partial \widetilde{U}$ that realizes the essential circuit length an essential circuit for $\widetilde{U}$.

Lemma 6.4. We have the inequality

$$
\text { circuit length of } \widetilde{U} \geq \text { essential circuit length of } \widetilde{U} \text {. }
$$

For example, consider the annulus $\widetilde{U}$ containing the pullback $\widetilde{\alpha}$ in Figure 18. Then the circuit length of $\widetilde{U}$ equals 6 , while the essential circuit length of $\widetilde{U}$ equals 4 .
Proof. Consider $\partial_{a} \widetilde{U}$ as a circuit in $\partial \widetilde{U}$. Let $V_{a}$ be the unique face of $\partial_{a} \widetilde{U}$ that contains $\widetilde{U}$. Then, for sufficiently small $\epsilon>0$, each $\epsilon$-boundary $\beta$ of $V_{a}$ wrt. $\partial_{a} \widetilde{U}$ necessarily separates $\partial_{a} \widetilde{U}$ and $\partial_{c} \widetilde{U}$, and is thus a core curve of $\widetilde{U}$. The statement follows.

Let $\widehat{f}$ be a Thurston map obtained from $f$ by blowing up some set of $\operatorname{arcs} E$ in $S^{2} \backslash f^{-1}\left(P_{f}\right)$. Under certain natural assumptions on the arcs in $E$, we want to describe the components of $\widehat{f}^{-1}(U)$ and their properties in terms of the components of $f^{-1}(U)$. We first formulate suitable conditions that allow such a comparison.

Definition 6.5 ( $\alpha$-restricted blow-up conditions). Let $f: S^{2} \rightarrow S^{2}$ be a Thurston map with $\# P_{f}=4, \alpha$ be an essential Jordan curve in $\left(S^{2}, P_{f}\right)$, and $a$ and $c$ be core arcs of $\alpha$ that lie in different components of $S^{2} \backslash \alpha$. Suppose $E \neq \varnothing$ is a finite set of $\operatorname{arcs}$ in $\left(S^{2}, f^{-1}\left(P_{f}\right)\right)$ satisfying the blow-up conditions, that is, the interiors of the arcs in $E$ are disjoint and $f: e \rightarrow f(e)$ is a homeomorphism for each $e \in E$.

We say that $E$ satisfies the $\alpha$-restricted blow-up conditions if

$$
\begin{equation*}
\mathrm{i}(f(e), \alpha)=\#(f(e) \cap \alpha)=1 \quad \text { and } \quad f(\operatorname{int}(e)) \cap a=\varnothing=f(\operatorname{int}(e)) \cap c \tag{6.3}
\end{equation*}
$$

for each $e \in E$.
In other words, for each $e \in E$ the arc $f(e)$ is in minimal position with respect to $\alpha$ and intersects $\alpha$ only once, and $f(\operatorname{int}(e))=\operatorname{int}(f(e))$ belongs to the annulus $U=S^{2} \backslash(a \cup c)$. Note that the endpoints of $f(e)$ lie in $P_{f} \subset a \cup c=\partial U$; see Figure 19 for an illustration.

Condition (6.3) is somewhat artificial, because it depends not only on $\alpha$, but also on the choices of $a$ and $c$. One can show that up to isotopy it can be replaced by the more natural


Figure 19. The Thurston map $f$ from Figure 18 and a set $E=\left\{e_{1}, e_{2}\right\}$ of $\operatorname{arcs}$ in $\left(S^{2}, f^{-1}\left(P_{f}\right)\right)$ satisfying the $\alpha$-restricted blow-up conditions.
condition $\mathrm{i}(f(e), \alpha)=1$ for all $e \in E$. Since the justification of this claim involves some topological machinery that is beyond the scope of the paper, we prefer to work with $(\overline{6.3})$.

Now the following statement is true.
Lemma 6.6. Let $f: S^{2} \rightarrow S^{2}$ be a Thurston map with $\# P_{f}=4, \alpha$ be an essential Jordan curve in $\left(S^{2}, P_{f}\right)$, a and c be core arcs of $\alpha$ that lie in different components of $S^{2} \backslash \alpha$. Suppose a set $E$ of arcs in $\left(S^{2}, f^{-1}\left(P_{f}\right)\right)$ satisfies the $\alpha$-restricted blow-up conditions and we are given multiplicities $m_{e} \in \mathbb{N}$ for $e \in E$.

Then the Thurston map $\widehat{f}$ obtained from $f$ by blowing up each arc $e \in E$ with multiplicity $m_{e}$ can be constructed so that it satisfies the following conditions:
(i) $\mathcal{G}=f^{-1}(a \cup c)$ is a subgraph of $\widehat{\mathcal{G}}:=\widehat{f}^{-1}(a \cup c)$.
(ii) Each complementary annulus $\widehat{U}$ of $\widehat{\mathcal{G}}$ is contained in a unique complementary annulus $\widetilde{U}$ of $\mathcal{G}$. Moreover, the assignment $\widehat{U} \mapsto \widetilde{U}$ is a bijection between the complementary annuli of $\widehat{\mathcal{G}}$ and of $\mathcal{G}$.
Let $\widehat{U}$ and $\widetilde{U}$ be corresponding annuli as in (ii). Then the following statements are true:
(iii) The core curves of $\widehat{U}$ and of $\widetilde{U}$ are isotopic rel. $P_{f}=P_{\widehat{f}}$. In particular, $\widehat{U}$ is essential if and only if $\widetilde{U}$ is essential.
(iv) If $\widehat{U}$ (and hence also $\widetilde{U}$ ) is essential, then the essential circuit lengths of $\widehat{U}$ and $\widetilde{U}$ are the same. Moreover, if $\mathcal{H}$ is an essential circuit for $\widehat{U}$, then $\mathcal{H}$ is an essential circuit for $\widetilde{U}$. In particular, $\mathcal{H} \subset \partial \widetilde{U} \subset \partial \widehat{U}$.


Figure 20. A Thurston map $\widehat{f}$ obtained from the Thurston map $f$ in Figures 18 and 19 by blowing up the $\operatorname{arcs} e_{1}$ and $e_{2}$ with multiplicities $m_{e_{1}}=2$ and $m_{e_{2}}=1$, respectively.

In (i) it is understood that the planar embedded graph $\mathcal{G}=f^{-1}(a \cup c) \subset S^{2}$ has the vertex set $f^{-1}\left(P_{f}\right)$ and that $\widehat{\mathcal{G}}=\widehat{f}^{-1}(a \cup c)$ has the vertex set $\widehat{f}^{-1}\left(P_{\widehat{f}}\right) \supset f^{-1}\left(P_{f}\right)$.

To illustrate the lemma, we consider the Thurston map $\widehat{f}$ that is in indicated in Figure 20 and obtained from the Thurston map $f$ in Figure 18 by blowing up the arcs $e_{1}$ and $e_{2}$ in Figure 19 with multilicities $m_{e_{1}}=2$ and $m_{e_{2}}=1$. Here, the lifts of $a$ and $c$ under the blown-up map $\widehat{f}$ are shown in blue and magenta colors on the left sphere, respectively. Comparing these figures, we immediately see that in this case the statements of the lemma are true.

Proof. Let $e \in E$ be arbitrary. Since $E$ satisfies the conditions in Definition 6.5, the set $f(e)$ is an arc in $\left(S^{2}, P_{f}\right)$, and so $f(e)$ has its endpoints in $P_{f}$. By (6.3) the arc $f(e)$ meets $\alpha$ precisely once and is in minimal position with respect to $\alpha$. In particular, $f(e)$ meets $\alpha$ transversely by Lemma 2.2 . This implies that the endpoints of $f(e)$ lie in different core arcs of $\alpha$, and so one endpoint of $f(e)$ lies in $a$ and the other one in $c$. It follows that $e$ has one endpoint in $f^{-1}(a)$ and the other one in $f^{-1}(c)$.

The set int $(e)$ belongs to a unique annulus $\widetilde{U}$ obtained as a complementary component of $\mathcal{G}=f^{-1}(a \cup c)$. Then one endpoint of $e$ is in $\partial_{a} \widetilde{U}=f^{-1}(a) \cap \partial \widetilde{U}$ and the other one in $\partial_{c} \widetilde{U}=f^{-1}(c) \cap \partial \widetilde{U}$. In the blow-up construction described in Section 4.1, we can choose the open Jordan region $W_{e}$ so that $W_{e} \subset \widetilde{U}$ for each $e \in E$ (of course, here the annulus $\widetilde{U}$ depends on $e$ ). Now we make choices of the subsequent ingredients in the blow-up construction as discussed in Section 4.1. That is, for each fixed $\operatorname{arc} e \in E$, we choose a closed Jordan region $D_{e}$ inside $W_{e}$. It is is subdivided into $m=m_{e}$ components $D_{e}^{1}, \ldots, D_{e}^{m}$. In addition, we also choose a pseudo-isotopy $h: S^{2} \times \mathbb{I} \rightarrow S^{2}$ satisfying conditions (B1) (B4), as well as maps $\varphi_{k}: D_{e}^{k} \rightarrow S^{2}, k=1, \ldots, m=m_{e}$, satisfying conditions (C1) and (C2), Let $\widehat{f}$ be the Thurston map obtained by blowing up each arc $e \in E$ with multiplicity $m_{e}$ according to these choices. We claim that $\widehat{f}$ satisfies all the conditions in the statement.

It immediately follows from (B3) and the definition of $\widehat{f}$ that $\mathcal{G}=f^{-1}(a \cup c)$ is a subgraph of $\widehat{\mathcal{G}}:=\widehat{f}^{-1}(a \cup c)$, and so statement (i) is true.

We have $\widehat{\mathcal{G}} \backslash \mathcal{G} \subset \bigcup_{e \in E} D_{e}$. Condition (C1) now implies that for each $e \in E$ and each $k=$ $1, \ldots, m=m_{e}$, the set $\widehat{\mathcal{G}} \cap D_{e}^{k}$ consists of two disjoint edges, one of which is homeomorphically mapped onto $a$ and the other one onto $c$ by $\widehat{f}$. We will call these edges $a$-sticks and $c$ sticks, respectively. Each closed Jordan region $D_{e}$ contains exactly $m_{e} a$-sticks, which have a common endpoint in $\partial e \cap f^{-1}(a)$, and exactly $m_{e} c$-sticks with a common endpoint in $\partial e \cap f^{-1}(c)$. The edge set of $\widehat{\mathcal{G}}$ consists of all the edges of $\mathcal{G}$ together with all the $a$-sticks and $c$-sticks.

Each complementary component $\widehat{U}$ of $\widehat{\mathcal{G}}$ is equal to a unique complementary component $\widetilde{U}$ of $\mathcal{G}$ with all the $a$ - and $c$-sticks removed that are contained in $\operatorname{cl}(\widetilde{U})$. Statement (ii) follows. Furthermore, since $P_{f} \cap \widetilde{U}=\varnothing$, statement (iii) follows as well.

To prove (iv), let $\widetilde{U}$ and $\widehat{U}$ be corresponding essential annuli as in (ii). Viewing $\partial \widetilde{U}$ as a subgraph of $\mathcal{G}$ and $\partial \widehat{U}$ as a subgraph of $\widehat{\mathcal{G}}$, we see that $\partial \widetilde{U}$ is a subgraph of $\partial \widehat{U}$. The additional edges of $\partial \widehat{U}$ are exactly the $a$ - and $c$-sticks contained in $\operatorname{cl}(\widetilde{U})$. It follows from the definition that the essential circuit length of $\widehat{U}$ is greater than or equal to the essential circuit length of $\widetilde{U}$.

Let $\mathcal{H}$ be an essential circuit for $\widehat{U}$ and suppose $\mathcal{H}$ contains an $a$ - or $c$-stick $\sigma$. Then one of the endpoints of $\sigma$ has degree 1 in $\partial \widehat{U}$, and so $\sigma$ must appear in two consecutive positions in the circuit $\mathcal{H}$. Omitting these two occurrences of $\sigma$ from $\mathcal{H}$ we get a shorter circuit $\mathcal{H}^{\prime}$ in
$\partial \widehat{U}$ such that for all small enough $\epsilon>0$ each $\epsilon$-boundary of $\widehat{U}$ wrt. $\mathcal{H}^{\prime}$ is isotopic to a core curve of $\widehat{U}$ rel. $P_{\widehat{f}}=P_{f}$. This contradicts the choice of $\mathcal{H}$, and it follows that $\mathcal{H}$ does not contain any $a$ - or $c$-sticks. Consequently, $\mathcal{H} \subset \partial \widetilde{U} \subset \partial \widehat{U}$, and the definition of the essential circuit length together with (iii) imply that $\mathcal{H}$ is an essential circuit for $\widetilde{U}$. Statement (iv) follows.

## 7. ELiminating obstructions By BLOWIng UP ARCS

The goal of this section is to show that the blow-up surgery can be applied to an obstructed Thurston map $f$ with four postcritical points in such a way that the resulting map $\widehat{f}$ is realized by a rational map. The precise formulation is given in Theorem 1.1 (see also Remark 7.2). We will prove this statement by contradiction. For this we assume that $\widehat{f}$ has an obstruction, and will carefully analyze some related mapping degrees. This leads to a very tight situation, where in some inequalities we actually have equality. From this we want to conclude that $f$ has a parabolic orbifold, in contradiction to our hypotheses in Theorem 1.1. We first formulate a related criterion for parabolicity.

Lemma 7.1. Let $f: S^{2} \rightarrow S^{2}$ be a Thurston with $\# P_{f}=4$. Suppose there exists a Jordan curve $\alpha$ in $\left(S^{2}, P_{f}\right)$ such that the following conditions are true:
(i) $\alpha$ is an obstruction for $f$.
(ii) $\alpha$ has no peripheral pullbacks under $f$.
(iii) If we choose core arcs a and cof $\alpha$ in different components of $S^{2} \backslash \alpha$ and consider the graph $\mathcal{G}=f^{-1}(a \cup c)$, then $\mathcal{G}$ has precisely $n \in \mathbb{N}$ essential complementary components $U_{1}, \ldots, U_{n}$ with core curves isotopic to $\alpha$ rel. $P_{f}$. Moreover, we assume the essential circuit length of $U_{j}$ is equal to $2 n$ for each $j=1, \ldots, n$.
Then $f$ has a parabolic orbifold.
Proof. Each $U_{j}$ contains precisely one pullback $\alpha_{j}$ of $\alpha$ under $f$. The curves $\alpha_{1}, \ldots, \alpha_{n}$ are all the pullbacks of $\alpha$ under $f$. Then it follows from our assumptions that

$$
\begin{aligned}
2 \operatorname{deg}\left(f: \alpha_{j} \rightarrow \alpha\right) & =2 \operatorname{deg}\left(f: U_{j} \rightarrow U\right) \\
& =\text { circuit length of } U_{j} \quad \text { (by Lemma 6.3) } \\
& \geq \text { essential circuit length of } U_{j} \quad \text { (by Lemma 6.4) } \\
& =2 n,
\end{aligned}
$$

and so $\operatorname{deg}\left(f: \alpha_{j} \rightarrow \alpha\right) \geq n$ for $j=1, \ldots, n$. On the other hand, $\alpha$ is an obstruction for $f$, and so

$$
1 \leq \lambda_{f}(\alpha)=\sum_{j=1}^{n} \frac{1}{\operatorname{deg}\left(f: \alpha_{j} \rightarrow \alpha\right)} \leq n / n=1 .
$$

It follows that we have equality in all previous inequalities. In particular,

$$
\text { circuit length of } U_{j}=\text { essential circuit length of } U_{j}=2 n
$$

for $j=1, \ldots, n$.
We want to apply the second part of Lemma 3.3, that is, we want to show that $f^{-1}\left(P_{f}\right) \subset$ $C_{f} \cup P_{f}$. To see this, let $v \in f^{-1}\left(P_{f}\right)$ be arbitrary. Then $v$ is a vertex of $\mathcal{G}=f^{-1}(a \cup c)$. If $v$ is incident with two or more edges in $\mathcal{G}$, then $v \in C_{f}$.

Otherwise, $v$ is the endpoint of precisely one edge $e$ in $\mathcal{G}$, and so $\operatorname{deg}_{\mathcal{G}}(v)=1$. We claim that then $v \in P_{f}$; to see this, we argue by contradiction and assume that $v \notin P_{f}$. Since $\alpha$ has no peripheral pullbacks, we have

$$
e \subset \mathcal{G}=\bigcup_{j=1}^{n} \partial U_{j},
$$

and so $e \subset \partial U_{j}$ for some $U_{j}$. Then $e$ is contained in a circuit $\left(e_{1}, \ldots, e_{2 n}\right)$ of length $2 n$ that traces one of the components of $\partial U_{j}$. Since $\operatorname{deg}_{\mathcal{G}}(v)=1$, the circuit must traverse $e$ twice with opposite orientations, that is, the edge $e$ appears precisely in two consecutive entries in the cycle $\left(e_{1}, \ldots, e_{2 n}\right)$. Erasing these two occurrences from the cycle, we obtain a new circuit in $\partial U_{j} \subset \mathcal{G}$ with length $2 n-2$. Let $\mathcal{H}$ denote the underlying subgraph of $\mathcal{G}$ corresponding to this shortened circuit and let $U$ be the face of $\mathcal{H}$ that contains $U_{j}$. Since the endpoint $v$ of $e$ does not belong to $P_{f}$, for every small enough $\epsilon>0$ the $\epsilon$-boundary of $\mathcal{H}$ wrt. $U$ is isotopic to the curve $\alpha_{j}$ rel. $P_{f}$. Then the essential circuit length of $U_{j}$ is $\leq 2 n-2$, contradicting the fact that $2 n$ is the essential circuit length of $U_{j}$ by our hypotheses. So we must have $v \in P_{f}$.

It follows that $f^{-1}\left(P_{f}\right) \subset C_{f} \cup P_{f}$, and so $f$ has a parabolic orbifold by Lemma 3.3.
We are now ready to prove our main result.
Proof of Theorem 1.1. As in the statement, suppose $f: S^{2} \rightarrow S^{2}$ is a Thurston map with $\# P_{f}=4$ and a hyperbolic orbifold. We assume that $f$ has an obstruction given by a Jordan curve $\alpha$ in $\left(S^{2}, P_{f}\right)$. We choose core $\operatorname{arcs} a$ and $c$ for $\alpha$ that lie in different components of $S^{2} \backslash \alpha$, and assume that $E \neq \varnothing$ is a finite set of $\operatorname{arcs}$ in $\left(S^{2}, f^{-1}\left(P_{f}\right)\right)$ that satisfies the $\alpha$-restricted blow-up conditions as in Definition 6.5.

We assume that we obtained a Thurston map $\bar{f}: S^{2} \rightarrow S^{2}$ by blowing up arcs in $E$ (with some multiplicities) so that $\lambda_{\widehat{f}}(\alpha)<1$. Then $P_{\widehat{f}}=P_{f}$ and $\widehat{f}$ has a hyperbolic orbifold (see Lemma 4.3 and Remark 4.4). Up to replacing $\widehat{f}$ with a Thurston equivalent map, we may also assume that the statements in Lemma 6.6 are true for the map $\widehat{f}$. We now argue by contradiction and assume that $\widehat{f}$ is not realized by a rational map. Then by Theorem 3.7 the map $\widehat{f}$ has an obstruction given by a Jordan curve $\gamma$ in $\left(S^{2}, P_{f}\right)$.

We set $U=S^{2} \backslash(a \cup c)$. Since $E$ satisfies the $\alpha$-restricted blow-up conditions, we have $\#(f(e) \cap \alpha)=1$ and $f(\operatorname{int}(e)) \cap a=f(\operatorname{int}(e)) \cap c=\varnothing$ for each $e \in E$. In other words, $f(e)$ intersects $\alpha$ only once and $\operatorname{int}(f(e))$ belongs to $U$. Then each arc in $E$ intersects only one pullback of $\alpha$ and only once.

Since $\gamma$ is an obstruction for $\widehat{f}$, but $\alpha$ is not, the curves $\alpha$ and $\gamma$ are not isotopic rel. $P_{\widehat{f}}=P_{f}$. So we have $\mathrm{i}(\alpha, \gamma)>0$ for intersection numbers in $\left(S^{2}, P_{f}\right)$ as follows from Lemma 2.1. By Lemma 2.6 (i) we have

$$
\begin{equation*}
\mathrm{i}(a, \gamma)=\mathrm{i}(c, \gamma)=\frac{1}{2} \mathrm{i}(\alpha, \gamma)>0 \tag{7.1}
\end{equation*}
$$

As follows from Lemma $2.6 \mid$ (ii), by replacing $\gamma$ with an isotopic curve rel. $P_{\widehat{f}}=P_{f}$ if necessary, we may also assume that

$$
\begin{equation*}
\#(\alpha \cap \gamma)=\mathrm{i}(\alpha, \gamma), \#(a \cap \gamma)=\mathrm{i}(a, \gamma), \#(c \cap \gamma)=\mathrm{i}(c, \gamma) \tag{7.2}
\end{equation*}
$$

and that the points in the non-empty and finite sets $a \cap \gamma$ and $c \cap \gamma$ alternate on $\gamma$.
We denote by $\alpha_{1}, \ldots, \alpha_{n}$ with $n \in \mathbb{N}$ the pullbacks of $\alpha$ under $f$ that are isotopic to $\alpha$ rel. $P_{f}$. Now we consider the graphs $\mathcal{G}=f^{-1}(a \cup c)$ and $\widehat{\mathcal{G}}=\widehat{f}^{-1}(a \cup c)$ as in Section 6. By Lemma 6.6, $\mathcal{G}$ is a subgraph of $\widehat{\mathcal{G}}$. Moreover, the following facts are true for their complementary components. Each $\alpha_{j}$ is a core curve in an essential annulus $U_{j}$ that is
a component of $S^{2} \backslash \mathcal{G}$. Each $U_{j}$ contains precisely one component $\widehat{U}_{j}$ of $S^{2} \backslash \widehat{\mathcal{G}}$. This component is essential and contains precisely one essential pullback $\widehat{\alpha}_{j}$ of $\alpha$ under $\widehat{f}$. The essential circuit length of $U_{j}$ is the same as the essential circuit length of $\widehat{U}_{j}$. The curves $\widehat{\alpha}_{1}, \ldots, \widehat{\alpha}_{n}$ are precisely all the distinct essential pullbacks of $\alpha$ under $\widehat{f}$. They are isotopic to $\alpha$ rel. $P_{f}$.

Let $\gamma_{1}, \ldots, \gamma_{k}$ with $k \in \mathbb{N}$ be the pullbacks of $\gamma$ under $\widehat{f}$ that are isotopic to $\gamma$ rel. $P_{\widehat{f}}=P_{f}$. Applying Lemma 6.3 and Lemma 6.2 (for the latter $(7.1)$ and $(7.2)$ are important) to an essential circuit for $\widehat{U}_{j}$, we see that

$$
\begin{align*}
\operatorname{deg}\left(f: \alpha_{j} \rightarrow \alpha\right) & =\operatorname{deg}\left(f: U_{j} \rightarrow U\right)  \tag{7.3}\\
& =\frac{1}{2} \cdot \text { circuit length of } U_{j} \\
& \geq \frac{1}{2} \cdot \text { essential circuit length of } U_{j} \\
& =\frac{1}{2} \cdot \text { essential circuit length of } \widehat{U}_{j} \\
& \geq k
\end{align*}
$$

for $j=1, \ldots, n$.
On the other hand, $\alpha$ has $n$ distinct essential pullbacks $\widehat{\alpha}_{1}, \ldots, \widehat{\alpha}_{n}$ under $\widehat{f}$, and so Lemma 5.5 implies that $\operatorname{deg}\left(\widehat{f}: \gamma_{m} \rightarrow \gamma\right) \geq n$ for $m=1, \ldots, k$ (again (7.1) and (7.2) are used here). Since $\alpha$ and $\gamma$ are obstructions for $f$ and $\widehat{f}$, respectively, we conclude that

$$
1 \leq \lambda_{f}(\alpha)=\sum_{j=1}^{n} \frac{1}{\operatorname{deg}\left(f: \alpha_{j} \rightarrow \alpha\right)} \leq n / k
$$

and

$$
1 \leq \lambda_{\widehat{f}}(\gamma)=\sum_{m=1}^{k} \frac{1}{\operatorname{deg}\left(\widehat{f:} \gamma_{m} \rightarrow \gamma\right)} \leq k / n
$$

It follows that $k=n$, which forces $\operatorname{deg}\left(f: \alpha_{j} \rightarrow \alpha\right)=\operatorname{deg}\left(\widehat{f}: \gamma_{j} \rightarrow \gamma\right)=n$ for $j=1, \ldots, n$. If we combine this with (7.3), then we also see that

$$
\begin{equation*}
\text { circuit length of } U_{j}=\text { essential circuit length of } U_{j} \tag{7.4}
\end{equation*}
$$

$$
=\text { essential circuit length of } \widehat{U}_{j}=2 n
$$

for $j=1, \ldots, n$.
We now want to apply Lemma 7.1 to our map $f$ and the obstruction $\alpha$. In order to verify the hypotheses of Lemma 7.1, it remains to show that $\alpha$ has no peripheral pullbacks under $f$, or equivalently, no peripheral pullbacks under $\widehat{f}$ (see Lemma 6.6 (iii)).

We argue by contradiction and assume that $\alpha$ has some peripheral pullbacks under $\widehat{f}$. Then there exists at least one peripheral annulus in the complement of $\widehat{\mathcal{G}}$. Such an annulus is disjoint from each annulus $\widehat{U}_{j}$. We can then travel from a point $p$ of such a peripheral annulus to a point in the set $\widehat{U}_{1} \cup \cdots \cup \widehat{U}_{n}$ along an arc $\sigma$ in $S^{2} \backslash \widehat{f}^{-1}\left(P_{\widehat{f}}\right)$ that crosses each edge in the graph $\widehat{\mathcal{G}}$ transversely. Then there is a first point $q$ on $\sigma$ where we enter the closure $M$ of $\widehat{U}_{1} \cup \cdots \cup \widehat{U}_{n}$. The point $q$ is necessarily an interior point of an edge $e$ of $\widehat{\mathcal{G}}$ contained in the boundary $\partial \widehat{U}_{j}$ for some $j \in\{1, \ldots, n\}$. Interior points of the subarc of $\sigma$ between $p$ and $q$ that are close to $q$ do not lie in $M \cup \widehat{\mathcal{G}}$. Hence such points must belong to a peripheral component $\widehat{U}$ of $\widehat{\mathcal{G}}$. Then necessarily $e \subset \partial \widehat{U}$.

In other words, there exists an edge $e$ in the graph $\widehat{\mathcal{G}}$ that belongs to the boundary of an essential annulus $\widehat{U}_{j}$ and a peripheral annulus $\widehat{U}$. Clearly, $\widehat{f}(e)=a$ or $\widehat{f}(e)=c$. In the following, we will assume that $\widehat{f}(e)=c$, that is, $e \subset \partial_{c} \widehat{U}_{j} \cap \partial_{c} \widehat{U}$; the other case, $\widehat{f}(e)=a$, is completely analogous.

Since $\mathrm{i}(c, \gamma)=\#(c \cap \gamma)>0$, there exists a pullback $\widehat{\gamma}$ of $\gamma$ under $\widehat{f}$ that meets $e$ transversely. Consequently, this pullback $\widehat{\gamma}$ meets both $\widehat{U}_{j}$ and $\widehat{U}$. Since $\widehat{f}: \widehat{\gamma} \rightarrow \gamma$ is a covering map and the points in $a \cap \gamma \neq \varnothing$ and $c \cap \gamma \neq \varnothing$ alternate on $\gamma$, the points in $\widehat{f}^{-1}(a) \cap \widehat{\gamma} \neq \varnothing$ and $\widehat{f}^{-1}(c) \cap \widehat{\gamma} \neq \varnothing$ alternate on $\widehat{\gamma}$. This implies that the curve $\widehat{\gamma}$ also meets the sets $\partial_{a} \widehat{U}_{j}$ and $\partial_{a} \widehat{U}$, and hence both components of the boundary of $\widehat{U}_{j}$ and of $\widehat{U}$. We conclude that $\widehat{\gamma}$ meets the core curve $\widehat{\alpha}_{j}$ of $\widehat{U}_{j}$ and the core curve $\widehat{\alpha}$ of $\widehat{U}$. Note that $\widehat{\alpha}$ is a peripheral pullback of $\alpha$ under $\widehat{f}$. In order to show that this is impossible, we consider two cases.

Case 1: $\widehat{\gamma}$ is an essential pullback of $\gamma$ under $\widehat{f}$, say $\widehat{\gamma}=\gamma_{m}$ for some $m \in\{1, \ldots, n\}$. Then Lemma 5.5 (for the map $\widehat{f}$, and $\alpha, \widehat{\gamma}$ in the roles of $\gamma, \widetilde{\alpha}$, respectively) shows $n<\operatorname{deg}(\widehat{f}: \widehat{\gamma} \rightarrow \gamma)$, because $\alpha$ has $n$ essential pullbacks and $\widehat{\gamma}$ meets a peripheral pullback of $\alpha$. On the other hand, we know that $\operatorname{deg}(\widehat{f}: \widehat{\gamma} \rightarrow \gamma)=\operatorname{deg}\left(\widehat{f}: \gamma_{m} \rightarrow \gamma\right)=n$. This is a contradiction.

Case 2: $\widehat{\gamma}$ is a peripheral pullback of $\gamma$ under $\widehat{f}$. Let $\mathcal{H}:=\partial_{c} U_{j}$. Then it follows from Lemma 6.6 that $\mathcal{H} \subset \partial_{c} \widehat{U}_{j}$. Moreover, (7.4) implies that $\mathcal{H}$ (considered as a circuit) realizes the essential circuit lengths of $U_{j}$ and $\widehat{U}_{j}$, which are both equal to $2 n$.

Now $e \subset \partial_{c} U_{j}=\mathcal{H}$, and so $\mathcal{H}$ meets the peripheral pullback $\widehat{\gamma}$ of $\gamma$ under $\widehat{f}$. The second part of Lemma 6.2 applied to $\mathcal{H}$ implies that the number $k$ of essential pullbacks of $\gamma$ under $\widehat{f}$ is less than $n$, contradicting $k=n$.

To summarize, these contradictions show that $\alpha$ has no peripheral pullbacks under $\widehat{f}$, and hence no peripheral pullbacks under $f$ by Lemma 6.6 (iii). So we can apply Lemma 7.1 and conclude that $f$ has a parabolic orbifold. This is yet another contradiction, because $f$ has a hyperbolic orbifold by our hypotheses. This shows that our initial assumption that $\widehat{f}$ has an obstruction is false. Hence $\hat{f}$ is realized by a rational map. This completes the proof of Theorem 1.1.

Remark 7.2. Let $f: S^{2} \rightarrow S^{2}$ be a Thurston map with $\# P_{f}=4$ and a hyperbolic orbifold, and suppose $f$ has an obstruction represented by a Jordan curve $\alpha$ in $\left(S^{2}, P_{f}\right)$. Then there always exist a set of $\operatorname{arcs} E \neq \varnothing$ in $\left(S^{2}, f^{-1}\left(P_{f}\right)\right)$ satisfying the $\alpha$-restricted blow-up conditions and multiplicities $m_{e}, e \in E$, such that the corresponding blown-up map $\widehat{f}$ satisfies $\lambda_{\mathscr{f}}(\alpha)<1$. We are then exactly in the setup of Theorem 1.1.

To see this, we first fix some core $\operatorname{arcs} a$ and $c$ of $\alpha$ lying in different components of $S^{2} \backslash \alpha$. We now choose an arc $e_{0}$ in $\left(S^{2}, P_{f}\right)$ with $\mathrm{i}\left(e_{0}, \alpha\right)=\#\left(e_{0} \cap \alpha\right)=1$ and $\operatorname{int}\left(e_{0}\right) \subset S^{2} \backslash(a \cup c)$. Let $E$ be the (non-empty) set of all lifts of $e_{0}$ under $f$. Then $E$ is a set of $\operatorname{arcs}$ in $\left(S^{2}, f^{-1}\left(P_{f}\right)\right)$ and it is clear that $E$ satisfies the $\alpha$-restricted blow-up conditions. If $\widetilde{\alpha}$ is any pullback of $\alpha$ under $f$, then there exists at least one arc $e \in E$ that meets $\widetilde{\alpha}$ (necessarily in an interior point of $e$ ). Blowing up the arc $e$ with some multiplicity $m_{e} \in \mathbb{N}$ increases the mapping degree for the corresponding pullback $\widehat{\alpha}$ under $\widehat{f}$ by $m_{e}$ and does not change the isotopy class of this pullback, that is, $[\widehat{\alpha}]=[\widetilde{\alpha}]$ rel. $P_{\widehat{f}}=P_{f}$ (this easily follows from Lemma 6.6 and its proof). Note that each pullback $\widehat{\alpha}$ of $\alpha$ under $\widehat{f}$ corresponds to a pullback $\widetilde{\alpha}$ of $\alpha$ under $f$ (this is essentially Lemma 6.6 (ii)). It follows that if we choose the multiplicities $m_{e}, e \in E$, large enough, then for the Thurston map $\widehat{f}$ we will have $\lambda_{\widehat{f}}(\alpha)<1$ and so $\alpha$ is not an obstruction
for $\widehat{f}$. By Theorem 1.1 the map $\widehat{f}$ is actually realized by a rational map. So by a suitable blow-up operation an obstructed Thurston map $f$ (with $P_{f}=4$ and a hyperbolic orbifold) can be turned into a Thurston map $\widehat{f}$ that is realized.

## 8. Global curve attractors

In this section we will prove Theorem 1.4. We consider the pillow $\mathbb{P}$ with its vertex set $V=\{A, B, C, D\}$. For the remainder of this section, $f: \mathbb{P} \rightarrow \mathbb{P}$ is a Thurston map obtained from the ( $2 \times 2$ )-Lattès map by gluing $n_{h} \geq 1$ horizontal and $n_{v} \geq 1$ vertical flaps to $\mathbb{P}$. Then $f$ is Thurston equivalent to a rational map by Theorem 1.2. In the following, all isotopies on $\mathbb{P}$ are considered relative to $P_{f}=V$.

In order to prove Theorem 1.4 , we want to show that Jordan curves in $(\mathbb{P}, V)$ are getting "less twisted" under taking preimages under $f$. To formalize this, we define the complexity $\|x\|$ of $x \in \widehat{\mathbb{Q}} \cup\{\odot\}$ as $\|x\|:=0$ for $x=\odot$ and $\|x\|:=|r|+s$ for $x=r / s \in \widehat{\mathbb{Q}}$. Recall that $\odot$ represents the isotopy classes of all peripheral curves, and that for a slope $r / s \in \widehat{\mathbb{Q}}$ we use the convention that the numbers $r \in \mathbb{Z}$ and $s \in \mathbb{N}_{0}$ are relatively prime and that $r=1$ if $s=0$. Note that $\|x\|=0$ for $x \in \widehat{\mathbb{Q}} \cup\{\odot\}$ if and only if $x=\odot$.

The complexity admits a natural interpretation in terms of intersection numbers. To see this, recall that $\alpha^{h}$ and $\alpha^{v}($ see $(\overline{2.4}))$ represent simple closed geodesics in $(\mathbb{P}, V)$ that separate the two horizontal and the two vertical edges of $\mathbb{P}$, respectively. Suppose the slope $r / s \in \widehat{\mathbb{Q}}$ corresponds to the isotopy class $[\gamma]$ of a (necessarily essential) Jordan curve $\gamma$ in $(\mathbb{P}, V)$. Then by Lemma 2.4 (v),

$$
\|r / s\|=|r|+s=\frac{1}{2} \mathrm{i}\left(\gamma, \alpha^{h}\right)+\frac{1}{2} \mathrm{i}\left(\gamma, \alpha^{v}\right)
$$

Moreover, if $\gamma$ is peripheral, then $\mathrm{i}\left(\gamma, \alpha^{h}\right)+\mathrm{i}\left(\gamma, \alpha^{v}\right)=0$, which agrees with the fact that $\|\odot\|=0$.

As we will see, under the slope map $\mu_{f}$ (as defined in Section 1.2) complexities do not increase, and actually strictly decrease unless the slope belongs to a certain finite set. More precisely, we will show the following statement.
Proposition 8.1. Let $f: \mathbb{P} \rightarrow \mathbb{P}$ be a Thurston map obtained from the $(2 \times 2)$-Lattès map by gluing $n_{h} \geq 1$ horizontal and $n_{v} \geq 1$ vertical flaps to the pillow $\mathbb{P}$. Then the following statements are true:
(i) $\left\|\mu_{f}(x)\right\| \leq\|x\|$ for all $x \in \widehat{\mathbb{Q}} \cup\{\odot\}$.
(ii) $\left\|\mu_{f}(x)\right\|<\|x\|$ for all $x \in \widehat{\mathbb{Q}} \cup\{\odot\}$ with $\|x\|>8$.

Since the set $\{x \in \widehat{\mathbb{Q}} \cup\{\odot\}:\|x\| \leq 8\}$ is finite, we actually have the strict inequality in (i) with at most finitely many exceptions. The proof of the proposition will show that $\left\|\mu_{f}(x)\right\|=\|x\|$ if and only if $\mu_{f}(x)=x$ (see Remark 8.6). As we will see below, Theorem 1.4 easily follows from Proposition 8.1.

Before we proceed with the proof of this proposition, we will establish several auxiliary results. As in Section 2.4, $a, b, c, d$ are the edges of the pillow $\mathbb{P}$, and $\wp: \mathbb{C} \rightarrow \mathbb{P}$ denotes the Weierstrass function that is doubly periodic with respect to the lattice $2 \mathbb{Z}^{2}$.

We are interested in simple closed geodesics and geodesic arcs $\tau$ in $(\mathbb{P}, V)$. Recall that every such geodesic has the form $\tau=\wp\left(\ell_{r / s}\right)$ for a line $\ell_{r / s} \subset \mathbb{C}$ with slope $r / s \in \widehat{\mathbb{Q}}$. If $\ell_{r / s} \subset \mathbb{C} \backslash \mathbb{Z}^{2}$, then $\tau=\wp\left(\ell_{r / s}\right)$ is a simple closed geodesic in $(\mathbb{P}, V)$, that is, $\tau \subset \mathbb{P} \backslash V$. If $\ell_{r / s}$ contains a point in $\mathbb{Z}^{2}$, then $\tau=\wp\left(\ell_{r / s}\right)$ is a geodesic arc in $(\mathbb{P}, V)$, that is, its interior lies in $\mathbb{P} \backslash V$ and its endpoints are in $V$.

Lemma 8.2. Let $\tau$ be a simple closed geodesic or a geodesic arc in $(\mathbb{P}, V)$ with slope $r / s \in \widehat{\mathbb{Q}}$. We consider the 1 -edges of $\mathbb{P}$ with respect to the $(n \times n)$-Lattès map $\mathcal{L}_{n}, n \geq 2$, that is, the lifts of the edges $a, b, c, d$ of $\mathbb{P}$ under $\mathcal{L}_{n}$. Then the following statements are true:
(i) If $|r|>2 n$, then $\tau$ intersects the interior of every horizontal 1-edge of $\mathbb{P}$.
(ii) If $s>2 n$, then $\tau$ intersects the interior of every vertical 1 -edge of $\mathbb{P}$.

Proof. We will only show the first part of the statement. The proof of the second part is completely analogous.

Let $\tau$ be a simple closed geodesic or a geodesic arc in $(\mathbb{P}, V)$ with slope $r / s \in \widehat{\mathbb{Q}}$ where $|r|>2 n$. Suppose that $e$ is a horizontal 1-edge and $\widetilde{e}$ is a lift of $e$ under $\wp$. Then the $\operatorname{arc} \widetilde{e}$ is a line segment of length $1 / n$ contained in a line $\ell_{0} \subset \mathbb{C}$ parallel to the real axis. In order to show that $\tau$ meets int $(e)$, it suffices to represent the given geodesic $\tau$ in the form $\tau=\wp\left(\ell_{r / s}\right)$ for a line $\ell_{r / s} \subset \mathbb{C}$ with $\ell_{r / s} \cap \operatorname{int}(\widetilde{e}) \neq \varnothing$.

For this we choose $p, q \in \mathbb{Z}$ such that $p r+q s=1$ and define $\omega:=2(s+i r)$ and $\widetilde{\omega}:=2(-p+i q)$. The numbers $\omega$ and $\widetilde{\omega}$ form a basis of the period lattice $2 \mathbb{Z}^{2}$ of $\wp$. In particular, if $\tau=$ $\wp\left(\ell_{r / s}\left(z_{0}\right)\right)$ for some $z_{0} \in \mathbb{C}$, then $\tau=\wp\left(\ell_{r / s}\left(z_{0}+j \widetilde{\omega}\right)\right)$ for all $j \in \mathbb{Z}$. The lines $\ell_{r / s}\left(z_{0}+j \widetilde{\omega}\right)$, $j \in \mathbb{Z}$, are parallel and equally spaced. Actually, two consecutive lines in this family differ by a translation by $\widetilde{\omega}$. Since $r \neq 0$, these lines are not parallel to the real axis and so they will cut out subsegments of equal length on the line $\ell_{0}$ that contains $\widetilde{e}$. To determine the length of these segments, we write $\widetilde{\omega}$ in the form

$$
\begin{equation*}
\widetilde{\omega}=u+v \omega \tag{8.1}
\end{equation*}
$$

with $u, v \in \mathbb{R}$. It is easy to see that (8.1) implies that $u=-2 / r$ (multiply (8.1) by the complex conjugate of $\omega$ and take imaginary parts), and so the lines in our family cut $\ell_{0}$ into subsegments of length $|u|=2 /|r|$. Since $|u|=2 /|r|<1 / n$ by our hypotheses, one of these lines meets $\operatorname{int}(\widetilde{e})$. This implies that $\tau \cap \operatorname{int}(e) \neq \varnothing$, as desired.

We now want to see what happens to a geodesic $\operatorname{arc} \xi$ in $(\mathbb{P}, V)$ if we take preimages under a map $f$ as in Proposition 8.1. Unless $\xi$ has slope in a finite exceptional set, suitable sets $\mathcal{H}$ in the preimage $f^{-1}(\xi)$ will meet the interior of a flap glued to the pillow $\mathbb{P}$, and consequently a peripheral pullback of the horizontal curve $\alpha^{h} \subset \mathbb{P}$ or of the vertical curve $\alpha^{v} \subset \mathbb{P}$. We will formulate some relevant statements in a slightly more general situation. We first introduce some terminology.

Suppose $Z \subset S^{2}$ consists of four distinct points. We say that $K \subset S^{2}$ essentially separates $Z$ if we can split $Z$ into two pairs (that is, into disjoint subsets $Z_{1}, Z_{2} \subset Z$ consisting of two points each) such that $K$ separates $Z_{1}$ and $Z_{2}$. Note that $K$ trivially has this property if $K \cap Z$ consists of two or more points.

Now let $n \in \mathbb{N}, n \geq 2$, and consider the $(n \times n)$-Lattès map $\mathcal{L}_{n}: \mathbb{P} \rightarrow \mathbb{P}$. If $\xi$ is a geodesic arc in $(\mathbb{P}, V)$, then the preimage $\mathcal{L}_{n}^{-1}(\xi)$ is a disjoint union of simple closed geodesics and geodesic arcs in $(\mathbb{P}, V)$. Note that each connected component of $\mathcal{L}_{n}^{-1}(\xi)$ essentially separates $V$, but no proper subset of such a component does. It follows that if $K \subset \mathcal{L}_{n}^{-1}(\xi)$ is a connected set, then it essentially separates $V$ if and only if $K$ is a simple closed geodesic or a geodesic arc in $(\mathbb{P}, V)$.

Let $\widehat{\mathcal{L}}: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ be a branched covering map obtained from the $(n \times n)$-Lattès map by gluing flaps to $\mathbb{P}$. As in Section 4.2 , we denote by $\widehat{V}$ the vertex set and by $B(\widehat{\mathbb{P}})$ the base of the flapped pillow $\widehat{\mathbb{P}}$. By construction, $\widehat{\mathcal{L}}$ maps each 1 -tile of $\widehat{\mathbb{P}}$ by a Euclidean similarity (with scaling factor $n$ ) onto a 0 -tile of $\mathbb{P}$. We also recall that we can naturally view the base $B(\widehat{\mathbb{P}})$
as a subset of $\mathbb{P}($ see $(4.2))$ and, with such an identification, the map $\widehat{\mathcal{L}}$ coincides with $\mathcal{L}_{n}$ on $B(\widehat{\mathbb{P}})$.

Suppose that a geodesic arc $\xi$ in $(\mathbb{P}, V)$ joins two distinct points $X, Y \in V$. We consider $\widehat{\mathcal{G}}:=\widehat{\mathcal{L}}^{-1}(\xi)$ as a planar embedded graph in $\widehat{\mathbb{P}}$ with the set of vertices $\widehat{\mathcal{L}}^{-1}(\{X, Y\})$ and the edges given by the lifts of $\xi$ under $\widehat{\mathcal{L}}$.
Lemma 8.3. Let $\widehat{\mathcal{L}}: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ be a branched covering map obtained from the $(n \times n)$-Lattès map with $n \geq 2$ by gluing $n_{h} \geq 1$ horizontal and $n_{v} \geq 1$ vertical flaps to $\mathbb{P}$. Suppose $\xi$ is a geodesic arc in $(\mathbb{P}, V)$ with slope $r / s \in \widehat{\mathbb{Q}} \backslash\{0, \infty\}$ and $\widehat{\xi}$ is a lift of $\xi$ under $\widehat{\mathcal{L}}$.

Let $F$ be a flap in $\widehat{\mathbb{P}}$ with the base edges $e^{\prime}$ and $e^{\prime \prime}$. If

$$
\widehat{\xi} \cap\left(\operatorname{int}(F) \cup \operatorname{int}\left(e^{\prime}\right) \cup \operatorname{int}\left(e^{\prime \prime}\right)\right) \neq \varnothing,
$$

then $\widehat{\xi}$ meets a base edge and the top edge of the flap $F$.
Proof. The proof is similar to the proof of Lemma 5.6. Recall that $a, b, c, d$ denote the edges of the pillow $\mathbb{P}$. Suppose $\xi \subset \mathbb{P}, \widehat{\xi} \subset \widehat{\mathbb{P}}$, and $e^{\prime}, e^{\prime \prime} \subset F$ are as in the statement of the lemma. Let $\widetilde{e} \subset F$ be the top edge of $F$.

Without loss of generality we will assume that $F$ is a horizontal flap. Then $\widehat{\mathcal{L}}\left(e^{\prime}\right)=a$ or $\widehat{\mathcal{L}}\left(e^{\prime}\right)=c$. We will make the further assumption that $\widehat{\mathcal{L}}\left(e^{\prime}\right)=a$. The other cases, when $\widehat{\mathcal{L}}\left(e^{\prime}\right)=c$ or when $F$ is a vertical flap, can be treated in a way that is completely analogous to the ensuing argument. Then $\widehat{\mathcal{L}}\left(e^{\prime \prime}\right)=a$ and $\widehat{\mathcal{L}}(\widetilde{e})=c$. Moreover,

$$
\begin{equation*}
\widehat{\mathcal{L}}^{-1}(a \cup c) \cap F=e^{\prime} \cup e^{\prime \prime} \cup \widetilde{e} . \tag{8.2}
\end{equation*}
$$

Since $\xi$ is a geodesic arc in $(\mathbb{P}, V)$ with slope $r / s \neq 0$, by Lemma 2.5 the sets $a \cap \xi$ and $c \cap \xi$ are non-empty and finite, and the points in these sets alternate on $\xi$. We claim that there is a point $p \in \widehat{\xi} \cap \operatorname{int}(F)$. By our hypotheses this can only fail if $\widehat{\xi}$ meets either $\operatorname{int}\left(e^{\prime}\right)$ or $\operatorname{int}\left(e^{\prime \prime}\right)$ in a point $q$. Since the $\operatorname{arc} \xi$ has a transverse intersection with $\operatorname{int}(a)$ at $\widehat{\mathcal{L}}(q)$, the $\operatorname{arc} \widehat{\xi}$ has a transverse intersection with $\operatorname{int}\left(e^{\prime}\right)$ or $\operatorname{int}\left(e^{\prime \prime}\right)$ at $q$. Then $\widehat{\xi}$ meets $\operatorname{int}(F)$ in a point $p$ in any case.

Since the points in $a \cap \xi \neq \varnothing$ and $c \cap \xi \neq \varnothing$ alternate on $\xi$, the points in $\widehat{\xi} \cap \widehat{\mathcal{L}}^{-1}(a) \neq \varnothing$ and $\widehat{\xi} \cap \widehat{\mathcal{L}}^{-1}(c) \neq \varnothing$ alternate on $\widehat{\xi}$. Note that $F \cap \widehat{\mathcal{L}}^{-1}(c)=\widetilde{e}$ and $F \cap \widehat{\mathcal{L}}^{-1}(a)=\partial F=e^{\prime} \cup e^{\prime \prime}$. So, if we trace the arc $\widehat{\xi}$ starting from $p$ in two different directions, we must meet a base edge of $F$ in one direction and the top edge of $F$ in the other direction. The statement follows.

Now the following fact is true.
Lemma 8.4. Let $\widehat{\mathcal{L}}: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ be a branched covering map obtained from the ( $n \times n$ )-Lattès map with $n \geq 2$ by gluing $n_{h} \geq 1$ horizontal and $n_{v} \geq 1$ vertical flaps to $\mathbb{P}$. Suppose that $\xi$ is a geodesic arc in $(\mathbb{P}, V)$ with slope $r / s \in \widehat{\mathbb{Q}}$ and $\mathcal{H}$ is any connected subgraph of $\widehat{\mathcal{G}}=\widehat{\mathcal{L}}^{-1}(\xi)$ that essentially separates $\widehat{V} \subset \widehat{\mathbb{P}}$. If $|r|+s>4 n$, then $\mathcal{H}$ meets a base edge and the top edge of a flap in $\widehat{\mathbb{P}}$.
Proof. Suppose $\xi$ is a geodesic arc in $(\mathbb{P}, V)$ with slope $r / s \in \widehat{\mathbb{Q}}$, where $|r|+s>4 n$, and $\mathcal{H}$ is a connected subgraph of $\widehat{\mathcal{G}}=\widehat{\mathcal{L}}^{-1}(\xi)$ that essentially separates the vertex set $\widehat{V}$ of $\widehat{\mathbb{P}}$. Note that then $r / s \neq 0, \infty$, which will allow us to apply Lemma 8.3. Each edge of the graph $\widehat{\mathcal{G}}=\widehat{\mathcal{L}}^{-1}(\xi)$ is a lift $\widehat{\xi}$ of $\xi$ as in this lemma.

We now argue by contradiction and suppose that there is no flap $F$ in $\widehat{\mathbb{P}}$ such that $\mathcal{H}$ meets both a base edge and the top edge of $F$. By the definition of $B(\widehat{\mathbb{P}})($ see $(4.1))$ and

Lemma 8.3, this means that each edge of $\mathcal{H}$, and thus the graph $\mathcal{H}$ itself, is contained in $B(\widehat{\mathbb{P}})$. Consequently, we can consider $\mathcal{H}$ as a connected subset of $\mathbb{P} \supset B(\widehat{\mathbb{P}})$. On $B(\widehat{\mathbb{P}})$ the maps $\widehat{\mathcal{L}}$ and $\mathcal{L}_{n}$ are identical. Therefore, we can also regard $\mathcal{H}$ as a connected subset of $\mathcal{L}_{n}^{-1}(\xi)$.

The set $\mathcal{H}$, now considered as a subset of $\mathbb{P}$, essentially separates $V \subset \mathbb{P}$. To see this, let $\widehat{V}_{1}, \widehat{V}_{2} \subset \widehat{V} \subset \widehat{\mathbb{P}}$ be two pairs of vertices separated by $\mathcal{H}$ in $\widehat{\mathbb{P}}$. We can identify $\widehat{V}_{1}$ with a pair $V_{1}$ and $\widehat{V}_{2}$ with a pair $V_{2}$ of vertices of $\mathbb{P}$. We claim that $V_{1}$ and $V_{2}$ are separated by $\mathcal{H}$ in $\mathbb{P}$. Indeed, if this was not the case, then we could find a path $\beta$ in $\mathbb{P}$ that joins $V_{1}$ and $V_{2}$ without meeting $\mathcal{H}$. This path can be modified as follows to a path $\widehat{\beta}$ in $\widehat{\mathbb{P}}$ that joins $\widehat{V}_{1}$ and $\widehat{V}_{2}$ and does not meet $\mathcal{H} \subset \widehat{\mathbb{P}}$ : if $\beta$ meets some 1-edge $e$ of $\mathbb{P}$ to which one or several flaps are glued, then on $\beta$ there is a first point $p \in e$ and a last point $q \in e$. We now replace the part of $\beta$ between $p$ and $q$ by a path that joins points corresponding to $p$ and $q$ in $\widehat{\mathbb{P}}$, travels on these flaps, and does not meet $\mathcal{H}$. If we make such replacements for all these edges $e$ consecutively, then we obtain a path $\widehat{\beta}$ that joins $\widehat{V}_{1}$ and $\widehat{V}_{2}$, but is disjoint from $\mathcal{H}$. But such a path $\widehat{\beta}$ cannot exist, because $\mathcal{H}$ separates $\widehat{V}_{1}$ and $\widehat{V}_{2}$ in $\widehat{\mathbb{P}}$.

We see that $\mathcal{H} \subset \mathcal{L}_{n}^{-1}(\xi)$ indeed essentially separates $V$. Since $\mathcal{H}$ is connected, the discussion above (after the definition of essential separation) implies that $\mathcal{H}$ is a simple closed geodesic or a geodesic arc in $(\mathbb{P}, V)$ with slope $r / s$. Since $|r|+s>4 n$, either $|r|>2 n$ or $s>2 n$. Thus, by Lemma 8.2, the geodesic $\mathcal{H}$ meets each horizontal 1-edge of $\mathbb{P}$ in the first case or each vertical 1-edge of $\mathbb{P}$ in the second case. Since $n_{h} \geq 1$ and $n_{v} \geq 1$, in either case, $\mathcal{H}$ must meet the interior of a 1-edge along which a flap is glued and hence cannot be a subset of $B(\widehat{\mathbb{P}})$. This is a contradiction and the lemma follows.

Remark 8.5. Suppose we are in the setup of Lemma 8.4. Then the connected set $\mathcal{H}$ meets a base edge and the top edge of a flap, say a horizontal flap $F$. Then there exists a peripheral pullback $\widehat{\alpha}$ of the horizontal curve $\alpha^{h}$ under the map $\widehat{\mathcal{L}}$ that is contained in $F$. Let $e^{\prime}$ and $e^{\prime \prime}$ be the base edges of $F$, and $\widetilde{e}$ be the top edge of $F$. Then the curve $\widehat{\alpha}$ separates $\partial F=e^{\prime} \cup e^{\prime \prime}$ from $\widetilde{e} \subset F$. Since $\mathcal{H}$ is connected and meets both $\widetilde{e}$ and $e^{\prime} \cup e^{\prime \prime}$, we conclude that $\mathcal{H} \cap \widehat{\alpha} \neq \varnothing$. If $\beta$ is a connected set that traces $\mathcal{H}$ closely, then it will also have points close to $\widetilde{e}$ and close to $e^{\prime} \cup e^{\prime \prime}$. Again this will imply that $\beta \cap \widehat{\alpha} \neq \varnothing$. This remark will become important in the proof of Proposition 8.1.

A completely analogous statement to Lemma 8.4 is true (with a very similar proof) if we assume that $\xi$ is a simple closed geodesic in $(\mathbb{P}, V)$ and $\mathcal{H}$ is an essential pullback of $\xi$ under $\widehat{\mathcal{L}}$.

We now turn to the proof of Proposition 8.1.
Proof of Proposition 8.1. Let $f: \mathbb{P} \rightarrow \mathbb{P}$ be a Thurston map obtained from the $(2 \times 2)$-Lattès map by gluing $n_{h} \geq 1$ horizontal and $n_{v} \geq 1$ vertical flaps to $\mathbb{P}$. Then $P_{f}=V$, where $V=\{A, B, C, D\}$ is the set of vertices of $\mathbb{P}$, and $A$ is the unique point in $P_{f}=V$ that is fixed by $f$.

To prove the first statement, let $x \in \widehat{\mathbb{Q}} \cup\{\odot\}$ be arbitrary. If $x=\odot$, then $\mu_{f}(\odot)=\odot$ and $\left\|\mu_{f}(\odot)\right\|=\|\odot\|=0$. So in the following we will assume that $x=r / s \in \widehat{\mathbb{Q}}$. Let $\gamma \subset \mathbb{P} \backslash V$ be a simple closed geodesic with slope $r / s \in \widehat{\mathbb{Q}}$. Then $\gamma$ is an essential Jordan curve and so each of the two complementary components of $\gamma$ in $\mathbb{P}$ contains precisely two postcritical points of $f$. Let $\xi$ and $\xi^{\prime}$ be core arcs of $\gamma$ belonging to different components of $\mathbb{P} \backslash \gamma$. Here we may assume that $\xi$ and $\xi^{\prime}$ are geodesic arcs in $(\mathbb{P}, V)$ with slope $r / s$.

As before, we denote by $\alpha^{h}$ and $\alpha^{v}$ simple closed geodesics in $(\mathbb{P}, V)$ that separate the two horizontal and the two vertical edges of $\mathbb{P}$, respectively. Then, by Lemma 2.4 , we have:

$$
\begin{aligned}
\mathrm{i}\left(\gamma, \alpha^{h}\right) & =2 \mathrm{i}\left(\xi, \alpha^{h}\right)=\#\left(\gamma \cap \alpha^{h}\right)=2|r|, \\
\mathrm{i}\left(\gamma, \alpha^{v}\right) & =2 \mathrm{i}\left(\xi, \alpha^{v}\right)=\#\left(\gamma \cap \alpha^{v}\right)=2 s, \\
\|x\| & =|r|+s=\frac{1}{2} \mathrm{i}\left(\gamma, \alpha^{h}\right)+\frac{1}{2} \mathrm{i}\left(\gamma, \alpha^{v}\right) .
\end{aligned}
$$

We call a point $p \in \mathbb{P}$ a 1-vertex if $f(p) \in P_{f}=\{A, B, C, D\}$. We say that a 1-vertex is of type $A, B, C$, or $D$ if it is a preimage of $A, B, C$, or $D$ under $f$, respectively.

Without loss of generality, we may assume that the core arc $\xi$ connects the point $A$ with a point $X \in\{B, C, D\}$. Then $\xi^{\prime}$ joins the two points in $\{B, C, D\} \backslash X$. Let $\mathcal{G}=f^{-1}\left(\xi \cup \xi^{\prime}\right)$, which we view as a planar embedded graph with the set of vertices $f^{-1}(V)$. Note that the degree of a vertex $p$ in $\mathcal{G}$ is equal to the local degree of the map $f$ at $p$. In addition, the graph $\mathcal{G}$ has the following properties:
(P1) $\mathcal{G}$ is a bipartite graph. In particular, 1-vertices of type $A$ are connected only to 1-vertices of type $X$ and vice versa.
(P2) Each postcritical point of $f$ is a 1-vertex of type $A$. If a 1-vertex of type $A$ has degree $\geq 2$ in $\mathcal{G}$, then it must be a postcritical point.
The analog of (P1) is valid for arbitrary Thurston maps with four postcritical points. To see that $(\mathrm{P} 2)$ is true, note that the $(2 \times 2)$-Lattès map sends each of the four vertices of $\mathbb{P}$ to $A$. This remains true if we glue any number of flaps to $\mathbb{P}$. Moreover, gluing additional flaps can only create additional preimages of $A$ of degree 1 in $\mathcal{G}$.

If every pullback of $\gamma$ under $f$ is peripheral, then $\mu_{f}(x)=\odot$, and so

$$
\begin{equation*}
\left\|\mu_{f}(x)\right\|=\|\odot\|=0<|r|+s=\|x\| . \tag{8.3}
\end{equation*}
$$

Suppose $\gamma$ has an essential pullback $\widetilde{\gamma}$ under $f$. Then $\mu_{f}(x) \in \widehat{\mathbb{Q}}$ is the slope corresponding to the isotopy class of $\widetilde{\gamma}$. By the discussion in Section 6 , the pullback $\widetilde{\gamma}$ belongs to a unique component $\widetilde{U}$ of $\mathbb{P} \backslash \mathcal{G}$. We use the notation $\partial_{\xi} \widetilde{U}:=f^{-1}(\xi) \cap \partial \widetilde{U}$. Then $\partial_{\xi} \widetilde{U}$ is a subgraph of $\mathcal{G}$ that only contains 1 -vertices of type $A$ and $X$. Since $\widetilde{\gamma}$ is essential, $\partial_{\xi} \widetilde{U}$ satisfies:
(P3) $\#\left(\partial_{\xi} \widetilde{U} \cap P_{f}\right) \leq 2$.
Our goal now is to simplify the pullback $\widetilde{\gamma}$ using an isotopy depending on the combinatorics of $\partial_{\xi} \widetilde{U}$. More precisely, we will construct a curve $\beta$ that is isotopic to $\widetilde{\gamma}$, but has fewer intersections with $\alpha^{h}$ and $\alpha^{v}$. In order to obtain a suitable curve $\beta$, we now distinguish several cases that exhaust all possibilities.

Case 1: $\partial_{\xi} \widetilde{U}$ does not contain any simple cycle. Then $\partial_{\xi} \widetilde{U}$ is a tree and, since $\widetilde{\gamma}$ is essential, there are exactly two postcritical points in $\partial_{\xi} \widetilde{U}$. These are 1 -vertices of type $A$ by (P2). Let $\mathcal{H} \subset \partial_{\xi} \widetilde{U}$ be the unique simple path that joins these two postcritical points in $\partial_{\xi} \widetilde{U}$. By (P1) the path $\mathcal{H}$ must have length $\geq 2$, because the endpoints of $\mathcal{H}$ have type $A$ and the vertices of types $A$ and $X$ alternate on $\mathcal{H}$.

If the length of $\mathcal{H}$ was $\geq 3$, then $\mathcal{H}$ would contain at least one additional point $p$ of type $A$ apart from its endpoints. Then $\operatorname{deg}_{\mathcal{H}}(p)=2$, ${\operatorname{so~} \operatorname{deg}_{\mathcal{G}}(p) \geq 2 \text {, which means } p \text { must be }}^{\text {a }}$ a postcritical point by (P2). But then $\mathcal{H} \subset \partial_{\xi} \widetilde{U}$ contains at least three postcritical points, which contradicts (P3). We conclude that $\mathcal{H}$ has length 2; see Figure 21 (Case 1).

Case 1


Case 2a


Case 2b


Figure 21. Different combinatorial types of the graph $\partial_{\xi} \widetilde{U}$. The subgraph in magenta corresponds to the appropriate choice of $\mathcal{H}$ in each case. The vertices in black indicate the postcritical points.

Let $\widehat{U}=S^{2} \backslash \mathcal{H}$. Then the annulus between $\widetilde{\gamma}$ and $\mathcal{H}$ contains no postcritical points of $f$, and hence for sufficiently small $\epsilon$ each $\epsilon$-boundary $\beta$ of $\widehat{U}$ wrt. $\mathcal{H}$ is isotopic to $\widetilde{\gamma}$ as follows from Lemma 2.1.

Case 2: $\partial_{\xi} \widetilde{U}$ contains a simple cycle. Then by (P1) one of the vertices of such a cycle must be of type $A$. Since this vertex has degree equal to 2 in the cycle, and hence degree $\geq 2$ in $\mathcal{G}$, it must be a postcritical point by (P2). It follows that $\#\left(\partial_{\xi} \widetilde{U} \cap P_{f}\right) \geq 1$. So by (P3), either $\#\left(\partial_{\xi} \widetilde{U} \cap P_{f}\right)=1$ or $\#\left(\partial_{\xi} \widetilde{U} \cap P_{f}\right)=2$.

Case 2a: $\#\left(\partial_{\xi} \widetilde{U} \cap P_{f}\right)=1$. Since $\widetilde{\gamma}$ is essential, there are exactly two postcritical points in the component of $S^{2} \backslash \widetilde{\gamma}$ that contains $\partial_{\xi} \widetilde{U}$. One of them belongs to $\partial_{\xi} \widetilde{U}$, while the other one belongs to a face of $\partial_{\xi} \widetilde{U}$ disjoint from $\widetilde{U}$. This postcritical point then necessarily belongs to a face of a simple cycle $\mathcal{H}$ in $\partial_{\xi} \widetilde{U}$. This simple cycle $\mathcal{H}$ then necessarily contains the unique postcritical point in $\partial_{\xi} \widetilde{U}$ as we have seen above. Moreover, $\mathcal{H}$ must have length 2 , because otherwise $\mathcal{H}$ has an even length $\geq 4$ by (P1). But then $\mathcal{H}$ contains another 1-vertex of type $A$ with degree $\geq 2$, which is necessarily a postcritical point by (P2). Then $\mathcal{H} \subset \partial_{\xi} \widetilde{U}$ contains at least two postcritical points, which contradicts our assumption for this case. So $\mathcal{H}$ has indeed length 2; see Figure 21 (Case 2a).

Let $\widehat{U}$ denote the face of $\mathcal{H}$ that contains $\widetilde{U}$. Then again the annulus between $\widetilde{\gamma}$ and $\mathcal{H}$ contains no postcritical points of $f$, and hence each $\epsilon$-boundary $\beta$ of $\widehat{U}$ wrt. $\mathcal{H}$ is isotopic to $\widetilde{\gamma}$ for sufficiently small $\epsilon$.

Case 2b: \# $\left.\partial_{\xi} \widetilde{U} \cap P_{f}\right)=2$. Let $\mathcal{H}$ be a simple path in $\partial_{\xi} \widetilde{U}$ that joins the two postcritical points in $\partial_{\xi} \widetilde{U}$. By the same reasoning as in Case 1, $\mathcal{H}$ has length 2; see Figure 21 (Case 2b). Let $\widehat{U}=S^{2} \backslash \mathcal{H}$. Since $\widetilde{\gamma}$ is essential, there are no postcritical points in the annulus between $\widetilde{\gamma}$ and $\mathcal{H}$. Thus, each $\epsilon$-boundary $\beta$ of $\widehat{U}$ wrt. $\mathcal{H}$ is isotopic to $\widetilde{\gamma}$ for sufficiently small $\epsilon$.

Note that in all cases $\mathcal{H}$ essentially separates $V=P_{f}$, because in all cases $\mathcal{H}$ separates the pairs of points in $V$ contained in different complementary components of $\widetilde{\gamma}$. Moreover, by our choice the circuit length of $\widehat{U}$ is equal to 4 in Cases 1 and 2 b , and equal to 2 in Case 2a. So in each case it is $\leq 4$. Since $\xi$ and $\alpha^{h}$ are in minimal position as follows from Lemma 2.4, we can apply Lemma 6.1 to the face $\widehat{U}$ of $\mathcal{H}$. Hence for each sufficiently small $\epsilon>0$ we can always find an $\epsilon$-boundary $\beta$ of $\widehat{U}$ wrt. $\mathcal{H}$ that is isotopic to $\widetilde{\gamma}$ and satisfies $\#\left(\beta \cap f^{-1}\left(\alpha^{h}\right)\right) \leq 4 \mathrm{i}\left(\xi, \alpha^{h}\right)$.

Let $\widetilde{\alpha}_{1}$ and $\widetilde{\alpha}_{2}$ be the two pullbacks of $\alpha^{h}$ under $f$ that are isotopic to $\alpha^{h}$ (there are exactly two such pullbacks by Lemma 5.1). Then in all cases we have

$$
\begin{align*}
2 \mathrm{i}\left(\widetilde{\gamma}, \alpha^{h}\right) & =2 \mathrm{i}\left(\beta, \alpha^{h}\right)=\mathrm{i}\left(\beta, \widetilde{\alpha}_{1}\right)+\mathrm{i}\left(\beta, \widetilde{\alpha}_{2}\right)  \tag{8.4}\\
& \leq \#\left(\beta \cap \widetilde{\alpha}_{1}\right)+\#\left(\beta \cap \widetilde{\alpha}_{2}\right) \\
& \leq \#\left(\beta \cap f^{-1}\left(\alpha^{h}\right)\right) \\
& \leq 4 \mathrm{i}\left(\xi, \alpha^{h}\right) \\
& =2 \mathrm{i}\left(\gamma, \alpha^{h}\right) .
\end{align*}
$$

Thus, $\mathrm{i}\left(\widetilde{\gamma}, \alpha^{h}\right) \leq \mathrm{i}\left(\gamma, \alpha^{h}\right)$. The same reasoning (with a possibly different choice of $\beta$ ) also shows $\mathrm{i}\left(\widetilde{\gamma}, \alpha^{v}\right) \leq \mathrm{i}\left(\gamma, \alpha^{v}\right)$. Combining these inequalities, we conclude:

$$
\begin{equation*}
\left\|\mu_{f}(x)\right\|=\frac{1}{2} \mathrm{i}\left(\widetilde{\gamma}, \alpha^{h}\right)+\frac{1}{2} \mathrm{i}\left(\widetilde{\gamma}, \alpha^{v}\right) \leq \frac{1}{2} \mathrm{i}\left(\gamma, \alpha^{h}\right)+\frac{1}{2} \mathrm{i}\left(\gamma, \alpha^{v}\right)=\|x\| . \tag{8.5}
\end{equation*}
$$

This completes the proof of the first part of the statement.
Note that the second inequality in (8.4) is strict if $\beta$ intersects a peripheral pullback of $\alpha^{h}$. A similar statement is also true for the analogous inequality for the curve $\alpha^{v}$. We now assume that $x=r / s \in \widehat{\mathbb{Q}}$ satisfies $\|x\|>8$. We will argue that then either inequality (8.4) or the analogous inequality for $\alpha^{v}$ is strict. This will lead to $\left\|\mu_{f}(x)\right\|<\|x\|$.

To see this, first note that $f=\widehat{\mathcal{L}} \circ \phi^{-1}$, where $\widehat{\mathcal{L}}: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ is the associated branched covering map obtained by blowing up the ( $2 \times 2$ )-Lattès map and $\phi: \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ is a suitable homeomorphism (see Section 4.2 for the details). We proceed as in the first part of the proof and again represent $x$ by a simple closed geodesic $\gamma$ in $(\mathbb{P}, V)$ with slope $x=r / s$. We may assume that $\gamma$ has an essential pullback $\widetilde{\gamma}$ under $f$, because otherwise we have the desired strict inequality by (8.3).

We choose a geodesic core arc $\xi$ in $(\mathbb{P}, V)$ and a connected set $\mathcal{H} \subset f^{-1}(\xi)$ as before and define $\widehat{\mathcal{H}}:=\phi^{-1}(\mathcal{H})$. Then $\widehat{\mathcal{H}}$ is a connected subset of $\widehat{\mathcal{L}}^{-1}(\xi)$ that essentially separates $\widehat{V}$, where $\widehat{V}=\phi^{-1}(V)$ is the set of vertices of the flapped pillow $\widehat{\mathbb{P}}$. Since $\|x\|=|r|+s>8$, we can apply Lemma 8.4 (with $n=2$ ) and conclude that the set $\widehat{\mathcal{H}}$ will meet a base edge and the top edge of some flap $F$ in $\widehat{\mathbb{P}}$. We will assume that $F$ is a horizontal flap, the case of a vertical flap being completely analogous.

If $\epsilon$ is small enough, then the $\epsilon$-boundary $\beta$ constructed above traces $\mathcal{H}$ very closely in the sense that for each point in $\mathcal{H}$, there is a nearby point in $\beta$. The same is true for $\widehat{\beta}:=\phi^{-1}(\beta)$ and $\widehat{\mathcal{H}}$. Using Remark 8.5, this implies that if $\epsilon$ is sufficiently small (as we may assume), then $\widehat{\beta}$ will meet the peripheral pullback $\widehat{\alpha}$ of $\alpha^{h}$ under $\widehat{\mathcal{L}}$ that is contained in the horizontal flap $F$. Consequently, $\beta$ meets the peripheral pullback $\phi(\widehat{\alpha})$ of $\alpha^{h}$ under $f$. As we already pointed out, this leads to a strict inequality in (8.4) and thus also in (8.5). The statement follows.

The proof of Theorem 1.4 is now easy.


Figure 22. A flapped pillow.
Proof of Theorem 1.4. Let $f: \mathbb{P} \rightarrow \mathbb{P}$ be a Thurston map as in the statement. Then Proposition 8.1 implies that if $x \in \widehat{\mathbb{Q}} \cup\{\odot\}$ is arbitrary, then the complexities of the elements $x, \mu_{f}(x), \mu_{f}^{2}(x), \ldots$ of the orbit of $x$ under iteration of $\mu_{f}$ strictly decrease until this orbit eventually reaches the finite set $S:=\{u \in \widehat{\mathbb{Q}} \cup\{\odot\}:\|u\| \leq 8\}$. From this point on, the orbit of $x$ stays in $S$. The statement follows.

Remark 8.6. The proofs of Theorems 1.3 and 1.4 show that a global curve attractor $\mathcal{A}(f)$ for $f$ can be obtained from Jordan curves corresponding to slopes in the finite set $S=$ $\{x \in \widehat{\mathbb{Q}} \cup\{\odot\}:\|x\| \leq 8\}$. Actually, (8.4) and (8.5) imply that $\left\|\mu_{f}(x)\right\|=\|x\|$ if and only if $\mu_{f}(x)=x$. Therefore, the minimal global curve attractor $\mathcal{A}(f)$ corresponds to the set $\left\{x \in \widehat{\mathbb{Q}} \cup\{\odot\}: \mu_{f}(x)=x\right\} \subset S$. In other words, the minimal $\mathcal{A}(f)$ consists of peripheral curves and essential curves that are invariant under $f$ (up to isotopy).

In principle, a global curve attractor for a map $f$ as in Theorem 1.4 depends on the locations of the flaps. By Remark 8.6 for each concrete case one can easily determine the exact attractor by checking if a slope $x \in \widehat{\mathbb{Q}}$ with $\|x\| \leq 8$ is invariant. For example, by using a computer program written by Darragh Glynn, we verified that for the map $f$ corresponding to the flapped pillow in Figure 22 (with one horizontal flap and one vertical flap glued at the two 1 -edges of $\mathbb{P}$ incident to the vertex $B$ ) the invariant slopes are $0, \infty, 1,-1$.

## 9. Further discussion

In this section, we briefly discuss some additional topics related to the investigations in this paper.
9.1. Julia sets of blown-up Lattès maps. An obvious question is what we can say about the Julia sets of the rational maps provided by Theorem 1.2 (for the definitions of the Julia and Fatou sets of rational maps and other basic notions in complex dynamics, see [Mil06a].

Proposition 9.1. Let $g: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a rational map that is Thurston equivalent to a map $f: \mathbb{P} \rightarrow \mathbb{P}$ obtained from the $(n \times n)$-Lattès map $\mathcal{L}_{n}$ with $n \geq 2$ by gluing $n_{h} \geq 1$ horizontal and $n_{v} \geq 1$ vertical flaps to the pillow $\mathbb{P}$. Then the following statements are true:
(i) The Julia set of $g$ is equal to $\widehat{\mathbb{C}}$ if $n$ is even and the vertex $A$ is not contained in a flap, or if $n$ is odd and none of the points in $V$ is contained in a flap.
(ii) The Julia set of $g$ is equal to a Sierpinski carpet in $\widehat{\mathbb{C}}$ if $n$ is even and $A$ is contained in a flap, or if $n$ is odd and at least one of the points in $V$ is contained in a flap.

Obviously, these cases cover all possibilities and so the Julia set of $g$ is either the whole Riemann sphere $\widehat{\mathbb{C}}$ or a Sierpiński carpet, i.e., a subset of $\widehat{\mathbb{C}}$ homeomorphic to the standard $1 / 3$-Sierpiński carpet fractal. As we will see, in the first case the map $g$ has no periodic critical points, while it has periodic critical points (namely critical fixed points) in the second case.
Proof. Let $g$ be a rational map as in the statement. To see what the Julia set of $g$ is, we will check whether $g$ has periodic critical points or not, and verify in the former case that $g$ has no Levy arcs (see below for the definition). These conditions are invariant under Thurston equivalence and therefore it is enough to consider the map $f$. Then $P_{f}=V$, where $V=\{A, B, C, D\}$ is the set of vertices of $\mathbb{P}$. By definition of the $(n \times n)$-Lattès map $\mathcal{L}_{n}$, for each $X \in V$ we have $\mathcal{L}_{n}(X)=A$ if $n$ is even and $\mathcal{L}_{n}(X)=X$ if $n$ is odd. Since $f \mid V$ agrees with $\mathcal{L}_{n} \mid V$, this implies that for each $X \in V$ we also have $f(X)=A$ if $n$ is even and $f(X)=X$ if $n$ is odd.

Since the orbit of each critical point under iteration of $f$ passes through the set $P_{f}=V$, this shows that any periodic critical point of $f$ must be equal to the point $A$ if $n$ is even or must belong to $V$ if $n$ is odd. Now for $X \in V$ we have $\operatorname{deg}_{\mathcal{L}_{n}}(X)=1$ and so

$$
\operatorname{deg}_{f}(X)=n_{X}+\operatorname{deg}_{\mathcal{L}_{n}}(X)=n_{X}+1
$$

where $n_{X} \in \mathbb{N}_{0}$ is the number of flaps that contain $X$. These considerations show that $f$, and hence also $g$, has no periodic critical points in case (i). Hence the Julia set of $g$ is the whole Riemann sphere $\widehat{\mathbb{C}}$ in this case (see [Mil06a, Corollary 19.8]).

In case (ii), the map $f$, and hence also $g$, has a critical fixed point, and so the Fatou set of $g$ is non-empty. To show that its Julia set is a Sierpiński carpet, we use the following criterion that follows from [BD18, Lemma 4.16]: the Julia set of $g$ is a Sierpiński carpet if and only if $g$, or equivalently $f$, has no Levy arcs. Here a Levy arc of $f$ is a path $\alpha$ in $\mathbb{P}$ satisfying the following conditions:
(L1) $\alpha$ is an arc in $(\mathbb{P}, V)$, or $\alpha$ is a simple loop based at a point $X \in V$ such that $\alpha \backslash\{X\} \subset \mathbb{P} \backslash V$ and each component of $\mathbb{P} \backslash \alpha$ contains at least one point in $V$.
(L2) There exist $k \in \mathbb{N}$ and a lift $\widetilde{\alpha}$ of $\alpha$ under $f^{k}$ such that $\alpha$ and $\widetilde{\alpha}$ are isotopic rel. $V$.
Now suppose that $f$ has a Levy arc $\alpha$ with $\widetilde{\alpha}$ and $k \in \mathbb{N}$ as in (L2). Then $f^{k} \mid \widetilde{\alpha}$ is a 1-to- 1 map and either $\mathrm{i}\left(\alpha, \alpha^{h}\right)>0$ or $\mathrm{i}\left(\alpha, \alpha^{v}\right)>0$. Without loss of generality, we may assume that $\mathrm{i}\left(\alpha, \alpha^{h}\right)=\#\left(\alpha \cap \alpha^{h}\right)>0$. If $\alpha$ is an arc in $(\mathbb{P}, V)$, then we can apply Lemma 5.5 with $\gamma:=\alpha^{h}$, $f:=f^{k}$, and $\widetilde{\alpha}:=\widetilde{\alpha}$ and conclude that the number of distinct pullbacks of $\alpha^{h}$ under $f^{k}$ that are isotopic to $\alpha^{h}$ is at most 1. This is also true if $\alpha$ is a simple loop as in (L1) by the argument in the proof of Lemma 5.5.

We reach a contradiction, because it follows from Lemma 5.1 that $\alpha^{h}$ has $n^{k}>1$ such pullbacks. Consequently, $f$ and $g$ do not have any Levy arcs and so the Julia set of $g$ is a Sierpiński carpet.
9.2. The global curve attractor problem. We were able to prove the existence of a finite global curve attractor only for blown-up $(n \times n)$-Lattès maps with $n=2$. The proof of Theorem 1.4 crucially relies on Proposition 8.1, which says that the (naturally defined) complexity of curves does not increase under the pullback operation. The latter statement is false in general for blown-up $(n \times n)$-Lattès maps with $n \geq 3$.

Numerical computations by Darragh Glynn suggest that for some blown-up (3×3)-Lattès map $f$ one can have infinitely many slopes $x \in \widehat{\mathbb{Q}}$ such that $\left\|\mu_{f}(x)\right\|>\|x\|$. For example, consider the map $f$ obtained from the $(3 \times 3)$-Lattès map by blowing up once the horizontal
and vertical edges incident to the vertex $B$ of $\mathbb{P}$. Then one can prove the following general relation for the slope map $\mu_{f}$ :

$$
\mu_{f}(r / s)=r^{\prime} / s^{\prime} \Rightarrow \mu_{f}(r /(s+24 r))=r^{\prime} /\left(s^{\prime}+22 r^{\prime}\right) .
$$

Based on this one can show that $\left\|\mu_{f}(x)\right\|>\|x\|$ for all

$$
x \in\left\{1 /(m+24 k): m \in\{7,8,9,15,16,17\}, k \in \mathbb{N}_{0}\right\} .
$$

Actually, it seems that in this case the slope map $\mu_{f}$ has orbits with arbitrarily many strict increases of complexity. For instance, we have two jumps of complexity for the orbit of slope $1 / 9$ under $\mu_{f}$ :

$$
1 / 9 \rightarrow 3 / 25 \rightarrow 3 / 23 \rightarrow 1 / 7 \rightarrow 3 / 19 \rightarrow 3 / 17 \rightarrow 1 / 5 \rightarrow 1 / 5 .
$$

Note that this orbit stabilizes at the fixed point $1 / 5$ of $\mu_{f}$. The numerical computations by Darragh Glynn also show that there are examples of blown-up $(n \times n)$-Lattès maps with $n \geq 5$ for which the slope map has periodic cycles of length $\geq 2$.

It is natural to ask what one can say about the behavior of the slope map $\mu_{f}$ for an obstructed Thurston map $f$ (with $\# P_{f}=4$ ). It was already observed in [KPS16] that for a blown-up $(2 \times 2)$-Lattès map $f$ with only vertical flaps glued to the pillow $\mathbb{P}$, there are infinitely many (non-isotopic) invariant essential Jordan curves. Indeed, for such a map $f$ the curve $\alpha^{v}$ is $f$-invariant and satisfies $\lambda_{f}\left(\alpha^{v}\right)=1$. One can use this to show that $f$ commutes with $T^{2}$ (up to isotopy rel. $P_{f}$ ), where $T$ is a Dehn twist about $\alpha^{v}$. This implies that each curve $T^{2 n}\left(\alpha^{h}\right)$ is $f$-invariant. In fact, it is easy to verify directly that each essential Jordan curve with slope $x \in \mathbb{Z} \cup\{\infty\}$ is $f$-invariant, or equivalently, that $\mu_{f}(x)=x$ for $x \in \mathbb{Z} \cup\{\infty\}$.

However, for such a blown-up ( $2 \times 2$ )-Lattès map $f$ with only vertical flaps glued to the pillow $\mathbb{P}$, the general behavior of the slope map $\mu_{f}$ under iteration has not been analyzed before. The considerations in the proof of the first part of Proposition 8.1 also apply in this situation. In particular, (8.4) and (8.5) are still true and show that the orbit of an arbitrary $x \in \widehat{\mathbb{Q}} \cup\{\infty\}$ under $\mu_{f}$ eventually lands in a fixed point of $\mu_{f}$. Moreover, results in Section 8 provide a method to determine all fixed slopes for $\mu_{f}$.

The easiest case is the map $f$ obtained from the $(2 \times 2)$-Lattès map $\mathcal{L}_{2}$ by gluing at least one vertical flap to each of the four vertical 1-edges in the "middle" of the pillow $\mathbb{P}$. If $\xi$ is a geodesic arc in ( $\mathbb{P}, V$ ) with an endpoint in $A \in V$ and slope $x \in \mathbb{Q} \backslash \mathbb{Z}$, then each component of $\mathcal{L}_{2}^{-1}(\xi)$ must pass through the interior of one of the four vertical 1-edges in the middle of $\mathbb{P}$. Consequently, we can apply the considerations in the proof of Lemma 8.4 and in the second part of the proof of Proposition 8.1 and conclude that $\left\|\mu_{f}(x)\right\|<\|x\|$ for $x \in \mathbb{Q} \backslash \mathbb{Z}$. Thus, the orbit of each $x \in \widehat{\mathbb{Q}} \cup\{\odot\}$ under $\mu_{f}$ eventually lands in $\mathbb{Z} \cup\{\infty, \odot\}$ (that is, in a fixed point of $\mu_{f}$ ). Since the map $f$ is easily seen to be expanding (see BM17, Definition 2.2 and Theorem 14.1]), this provides an answer to a question raised by Kevin Pilgrim.
9.3. Twisting problems. To this date, many natural problems related to Thurston equivalence remain rather mysterious and are often very difficult to solve. Twisting problems are examples of this nature.

To explain this, suppose we are given a rational Thurston map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$. Let $\phi: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be an orientation-preserving homeomorphism that fixes the postcritical set $P_{f}$ pointwise. We now consider the branched covering map $g:=\phi \circ f$ on $\widehat{\mathbb{C}}$, called the $\phi$-twist of $f$. Then $C_{g}=C_{f}$ and $g$ has the same dynamics on $C_{f}$ as $f$. In particular, $P_{g}=P_{f}$; so $g$ has a finite postcritical set and is a Thurston map.

This leads us to the natural questions: Is $g$ realized? And if yes, to which rational map is $g$ equivalent depending on the isotopy type of $\phi$ ? In fact, there are only finitely many rational maps $g$ (up to Möbius conjugation) that can arise in this way from a fixed map $f$. A famous instance of this question, called the "twisted rabbit problem", was solved by Laurent Bartholdi and Volodymyr Nekrashevych in [BN06] (see also [Lod13, BLMW19]).

In our context we can ask which twists of maps as in Theorem 1.2 are realized. We do not have an answer to this question, but it seems that this leads to non-trivial and difficult problems. For example, consider the blown-up $(2 \times 2)$-Lattès map $f: \mathbb{P} \rightarrow \mathbb{P}$ corresponding to the flapped pillow $\widehat{\mathbb{P}}$ as in Figure 22 . Then $\widehat{\mathbb{P}}$ has one horizontal and one vertical flap, and so $f$ is realized by Theorem 1.2. One can check that the Jordan curve $\gamma:=\wp\left(\ell_{3 / 13}\right)$ has exactly two essential pullbacks $\gamma_{1}, \gamma_{2} \sim \wp\left(\ell_{1 / 3}\right)$ under $f$ with $\operatorname{deg}\left(f: \gamma_{1} \rightarrow \gamma\right)=1$ and $\operatorname{deg}\left(f: \gamma_{2} \rightarrow \gamma\right)=2$. We now choose an orientation-preserving homeomorphism $\phi: \mathbb{P} \rightarrow \mathbb{P}$ that maps $\gamma$ onto $\gamma_{1}$, while fixing each point in $V=P_{f}$. Then the curve $\gamma_{1}$ is an obstruction for the twisted map $g:=\phi \circ f$ with $\lambda_{g}\left(\gamma_{1}\right)=3 / 2$. An analogous construction applies to some other essential Jordan curves, for instance, with slopes $3 / 23$ and $3 / 49$, and gives twists of $f$ with an obstruction.

It follows from this discussion that the mapping class biset associated with the map $f$ above is not contracting (see BD17, BD18] for the definitions). Thus, the algebraic methods for solving the global curve attractor problem developed in Pil12 (see, specifically, Pil12, Theorem 1.4]) do not apply in general for the maps considered in Theorem 1.2 .
9.4. Thurston maps with more than four postcritical points. While in this paper we only discuss the case of Thurston maps $f: S^{2} \rightarrow S^{2}$ with $\# P_{f}=4$, it is natural to ask if one can adapt Theorem 1.1 to the case when $\# P_{f}>4$. The main difficulty is that an obstruction in this case is in general not given by a unique essential Jordan curve in $\left(S^{2}, P_{f}\right)$, but by a multicurve. Of course, this fact complicates the analysis of pullback properties of curves and their intersection numbers. However, we expect that one can naturally generalize our result for an arbitrary Thurston map: given an obstructed Thurston map $f$ one can eliminate all possible multicurve obstructions by successively applying the blow-up operation and obtain a Thurston map that is realized.
9.5. Other combinatorial constructions of rational maps. The dynamical behavior of curves under the pullback operation is an important topic in holomorphic dynamics. While in this paper we only study the realization and the global curve attractor problems, one is led to similar considerations, for example, in the study of iterated monodromy groups (see [HM18]). For these investigations it is important to have explicit classes of rational maps at hand that are constructed in combinatorial fashion and against which conjectures can be tested or which lead to the discovery of general phenomena. The maps provided by Theorem 1.1 may be useful in this respect. Another interesting class of maps worthy of further investigation are Thurston maps constructed from tilings of the Euclidean or hyperbolic plane as in [BM17, Example 12.25].

## 10. Appendix: Isotopy classes of Jordan curves in spheres with four MARKED POINTS

In this appendix, we will provide proofs for Lemmas 2.3 and 2.4. Our presentation is rather detailed. We need some additional auxiliary facts that we will discuss first. Throughout, we will rely on the notation and terminology established in Section 2 .

In the following, we will consider a marked sphere $\left(S^{2}, Z\right)$, where $Z \subset S^{2}$ consists of precisely four points. If $M \subset S^{2}$ and $\alpha$ is a Jordan curve in $\left(S^{2}, Z\right)$, then we say that $\alpha$ is in minimal position with the set $M$ if $\#(\alpha \cap M) \leq \#\left(\alpha^{\prime} \cap M\right)$ for all Jordan curves $\alpha^{\prime}$ in $\left(S^{2}, Z\right)$ with $\alpha \sim \alpha^{\prime}$ rel. $Z$.

Let $\alpha$ and $\beta$ be Jordan curves or $\operatorname{arcs}$ in $\left(S^{2}, Z\right)$. We say that subarcs $\alpha^{\prime} \subset \alpha$ and $\beta^{\prime} \subset \beta$ form a bigon $U$ in $\left(S^{2}, Z\right)$ if $\alpha^{\prime}$ and $\beta^{\prime}$ have the same endpoints, but disjoint interiors and if $U \subset S^{2}$ is an open Jordan region with $\partial U=\alpha^{\prime} \cup \beta^{\prime}$ and $U \subset S^{2} \backslash Z$.

Lemma 10.1. Let $\gamma$ be a Jordan curve in a marked sphere $\left(S^{2}, Z\right)$ with $\# Z=4$, and let a and $c$ be disjoint arcs in $\left(S^{2}, Z\right)$. Suppose $\gamma$ is in minimal position with the set $a \cup c$. Then $\gamma$ meets each of the arcs a and c transversely and no subarcs of $\gamma$ and of a or c form a bigon $U$ in $\left(S^{2}, Z\right)$ with $U \cap(\gamma \cup a \cup c)=\varnothing$.
Proof. These facts are fairly standard in contexts like this (see, for example, FM12, Section 1.2.4]), and so we will only give an outline of the proof.

Our assumptions imply that $\gamma$ meets each arc $a$ and $c$ transversally and has only finitely many intersections with $a \cup c$ (Lemma 2.1 and its proof apply mutatis mutandis to our situation). We now argue by contradiction and assume that a subarc $\gamma^{\prime} \subset \gamma$ and a subarc $\sigma$ of $a$ or $c$, form a bigon $U$ in $\left(S^{2}, Z\right)$ with $U \cap(\gamma \cup a \cup c)=\varnothing$. Note that then $\operatorname{cl}(U) \subset S^{2} \backslash Z$. Hence we can modify the curve $\gamma$ near $U$ by an isotopy in $S^{2} \backslash Z$ that pulls the subarc $\gamma^{\prime}$ of $\gamma$ through $U$ and away from $\sigma$ so that the new Jordan curve $\gamma$ does not intersect $\sigma \subset a \cup c$ and no new intersection points with $a \cup c$ arise. This leads to a contradiction, because the original curve $\gamma$ was in minimal position with $a \cup c$.

A topological space $D$ is called a closed topological disk if there exists a homeomorphism $\eta: \operatorname{cl}(\mathbb{D}) \rightarrow D$ of the closed unit disk $\operatorname{cl}(\mathbb{D}) \subset \mathbb{C}$ onto $D$. This is an abstract version of the notion of a closed Jordan region contained in a surface. The set $\partial D:=\eta(\partial \mathbb{D})$ is a Jordan curve independent of $\eta$ and called the boundary of $D$. The interior of $D$ is defined as $\operatorname{int}(D):=\eta(\mathbb{D})=D \backslash \partial D$.

Similarly as for closed Jordan regions, an arc $\alpha$ contained in a closed topological disk $D$ is called a crosscut (in $D$ ) if $\partial \alpha \subset \partial D$ and $\operatorname{int}(\alpha) \subset \operatorname{int}(D)$. A crosscut $\alpha$ splits $D$ into two compact and connected sets $S$ and $S^{\prime}$ called the sides of $\alpha$ (in $D$ ) such that $D=S \cup S^{\prime}$ and $S \cap S^{\prime}=\alpha$. With suitable orientations of $\alpha$ and $D$, one side of $\alpha$ lies on the left and the other side on the right of $\alpha$. Each non-empty connected set $c \subset D$ that does not meet $\alpha$ is contained in precisely one side of $\alpha$.

If $\alpha_{1}, \ldots, \alpha_{n}$ for $n \in \mathbb{N}$ are pairwise disjoint crosscuts in a closed topological disk $D$, then we can define an abstract graph $G=(V, E)$ in the following way: we consider each component $U$ of $D \backslash\left(\alpha_{1} \cup \cdots \cup \alpha_{n}\right)$ as a vertex of $G$. We join two distinct vertices represented by components $U$ and $U^{\prime}$ by an edge, if one of the crosscuts $\alpha_{j}$ is contained in the boundary of both $U$ and $U^{\prime}$. Accordingly, the edges of $G$ are in bijective correspondence with the crosscuts $\alpha_{1}, \ldots, \alpha_{n}$.

Lemma 10.2. Let $n \in \mathbb{N}$ and $\alpha_{1}, \ldots, \alpha_{n}$ be pairwise disjoint crosscuts in a closed topological disk $D$, and let $G$ be the graph obtained from the components of the set $D \backslash\left(\alpha_{1} \cup \cdots \cup \alpha_{n}\right)$ as described. The the following statements are true:
(i) The graph $G$ is a finite tree with at least two vertices.
(ii) Let $c \subset D$ be a connected set and suppose that $c \cap \alpha_{k}=\varnothing$ for some $k \in\{1, \ldots, n\}$. Then there exists $m \in\{1, \ldots, n\}$ and a side $S$ of $\alpha_{m}$ such that $S \backslash \alpha_{m}$ is disjoint from all the sets $c, \alpha_{1}, \ldots, \alpha_{n}$.

If $\alpha=\alpha_{m}$ and $S$ are as in (ii), then there exists a subarc $\beta$ of $\partial D$ with the same endpoints as $\partial \alpha \subset D$ such that $\partial S=\alpha \cup \beta$. Then $U:=\operatorname{int}(S)$ is an open Jordan region bounded by the union of the arcs $\alpha$ and $\beta$ whose only common points are their endpoints. This region $U$ does not meet $c$ nor any of the $\operatorname{arcs} \alpha_{1}, \ldots, \alpha_{n}$. In the proof of Lemma 10.3 we will use such a region $U$ to obtain a bigon in an appropriate context.

Proof. (i) This is intuitively clear, and we leave the details to the reader. By induction on the number $n$ of crosscuts one can show that $G$ is a finite connected graph with at least two vertices. Since a crosscut splits $D$ into two sides, it easily follows that the removal of any edge from $G$ disconnects it. Hence $G$ cannot contain any simple cycle and must be a tree.
(ii) The graph $G$ is a tree; so if we remove the edge corresponding to the crosscut $\alpha_{k}$ from $G$, then we obtain two disjoint non-empty subgraphs $G_{1}$ and $G_{2}$ of $G$. The connected components of $D \backslash\left(\alpha_{1} \cup \cdots \cup \alpha_{n}\right)$ corresponding to the vertices of $G_{1}$ are contained in one side $S^{\prime}$ of $\alpha_{k}$, while the other connected components of $D \backslash\left(\alpha_{1} \cup \cdots \cup \alpha_{n}\right)$ corresponding to the vertices of $G_{2}$ lie in the other side $S^{\prime \prime}$ of $\alpha_{k}$. Since $c$ is connected and does not meet $\alpha_{k}$, it must be contained in one of the sides of $\alpha_{k}$, say $c \subset S^{\prime \prime}$. Then $c$ is disjoint from $S^{\prime}$ and hence from all the sets that correspond to vertices in $G_{1}$.

The tree $G$ has a leaf $v$ in $G_{1} \neq \varnothing$, i.e., there exists a vertex $v$ of $G_{1}$ such that $v$ is connected to the rest of $G$ by precisely one edge. Then the connected component of $D \backslash\left(\alpha_{1} \cup \cdots \cup \alpha_{n}\right)$ corresponding to $v$ has exactly one of the crosscuts, say $\alpha_{m}$ with $m \in\{1, \ldots, n\}$, on its boundary. Then this component has the form $S \backslash \alpha_{m}$, where $S$ is the unique side of $\alpha_{m}$ contained in $S^{\prime}$. Then $S \backslash \alpha_{m}$ is disjoint from $c \subset S^{\prime \prime}$ and from all the crosscuts $\alpha_{1}, \ldots, \alpha_{n}$.

We can now prove a statement that is the key to the understanding of isotopy classes of Jordan curves in a sphere with four marked points.

Lemma 10.3. Let $\gamma$ be a Jordan curve in a marked sphere $\left(S^{2}, Z\right)$ with $\# Z=4$, and let a and $c$ be disjoint arcs in $\left(S^{2}, Z\right)$. Suppose $\gamma$ is in minimal position with the set $a \cup c$. Then the sets $a \cap \gamma$ and $c \cap \gamma$ are non-empty and finite, and the points in these sets alternate on $\gamma$ unless $\gamma$ is peripheral or $\gamma \cap(a \cup c)=\varnothing$.

Proof. In the given setup, each of the disjoints sets $\partial a$ and $\partial c$ contains two points in $Z$. Since $\# Z=4$, it follows that $a$ connects two of the points in $Z$, while $c$ connects the other two points.

We may assume that $\gamma$ is essential and that at least one of the arcs $a$ or $c$ meets $\gamma$, say $a \cap \gamma \neq \varnothing$, because otherwise we are in an exceptional situation as in the statement.

If none of the arcs $a$ and $c$ meets $\gamma$ in more than two points, then $\#(a \cap \gamma)=1$ and $\#(c \cap \gamma) \leq 1$. Now $a$ and $\gamma$ meet transversely by Lemma 10.1. This implies that the endpoints of $a$ lie in different components of $S^{2} \backslash \gamma$. Since $\gamma$ is essential, each of these components contains precisely two points of $Z=\partial a \cup \partial c$. Hence the endpoints of $c$ also lie in different components of $S^{2} \backslash \gamma$. This implies that $c \cap \gamma \neq \varnothing$ and so $\#(c \cap \gamma)=1$ in the case under consideration. So both $a$ and $c$ meet $\gamma$ in exactly one point. It follows that the statement is true in this case.

We are reduced to the situation where at least one of the arcs $a$ or $c$ meets $\gamma$ in at least two (but necessarily finitely many) points, say $n:=\#(a \cap \gamma) \geq 2$. We now endow $\gamma$ and $a$ with some orientations. With the given orientation we denote the initial point of $a$ by $x_{0}$ and its terminal point by $x_{1}$. Let $y_{1}, \ldots, y_{n}, y_{n+1}=y_{1}$ denote the $n \geq 2$ intersection points of $\gamma$ with $a$ that we encounter while traversing $\gamma$ once starting from some point in $\gamma \backslash a$. The
same $n$ points also appear on $a$. We denote them by $p_{1}, \ldots, p_{n}$ in the order they appear if we traverse $a$ starting from $x_{0}$. For $k=1, \ldots, n$, we denote by $\gamma\left[y_{k}, y_{k+1}\right]$ the subarc of $\gamma$ obtained from traversing $\gamma$ with the given orientation from $y_{k}$ to $y_{k+1}$.

Claim. $c \cap \gamma\left[y_{k}, y_{k+1}\right] \neq \varnothing$ for each $k=1, \ldots, n$.
To see this, we argue by contradiction and assume that $c \cap \gamma\left[y_{k}, y_{k+1}\right]=\varnothing$ for some $k \in$ $\{1, \ldots, n\}$. Our goal now is to show that some subarcs of $a$ and $\gamma$ form a bigon $U$ in $\left(S^{2}, Z\right)$ with $U \cap(\gamma \cup a \cup c)=\varnothing$. This is a contradiction with Lemma 10.1, because $\gamma$ and $a \cup c$ are in minimal position.

To produce such bigon $U$, we want to apply Lemma 10.2. In order to do this, we slit the sphere $S^{2}$ open along the arc $a$. This results in a closed topological disk $D$ whose boundary $\partial D$ consists of two copies $a^{+}$and $a^{-}$of the arc $a$. The set $S^{2} \backslash \operatorname{int}(a)$ can be identified with $\operatorname{int}(D)$, while each point $\operatorname{in} \operatorname{int}(a)$ is doubled into one corresponding point in $a^{+}$and one in $a^{-}$.

The arcs $a^{+}$and $a^{-}$have their endpoints $x_{0}$ and $x_{1}$ in common. We identify $a^{+}$with the original arc $a$ with the same orientation. Then we can think of the intersection points $p_{1}, \ldots, p_{n}$ of $\gamma$ with $a$ as lying on $a^{+}=a$, while each of the points $p_{j}$ has a corresponding point $q_{j}$ on $a^{-}$.

Each arc $\gamma\left[y_{j}, y_{j+1}\right]$ corresponds to a crosscut $\gamma_{j}$ in $D$ for $j=1, \ldots, n$. These crosscuts have their endpoints in the set $P \cup Q$, where $P:=\left\{p_{1}, \ldots, p_{n}\right\}$ and $Q:=\left\{q_{1}, \ldots, q_{n}\right\}$. Moreover, the crosscuts $\gamma_{1}, \ldots, \gamma_{n}$ are pairwise disjoint. Indeed, the only possible common intersection point of two of these arcs could be a common endpoint of two consecutive arcs $\gamma_{j}$ and $\gamma_{j+1}$ (where $\gamma_{n+1}:=\gamma_{1}$ ) corresponding to $y_{j+1} \in a$; but in the process of creating $D$, the point $y_{j+1}$ is doubled into the points $p_{\ell}$ and $q_{\ell}$ for some $\ell \in\{1, \ldots, n\}$. Since $\gamma$ meets $a$ transversely, one of these points will be the terminal point of $\gamma_{j}$, while the other one will be the initial point of $\gamma_{j+1}$, and so actually $\gamma_{j} \cap \gamma_{j+1}=\varnothing$. It follows that the hypotheses of Lemma 10.2 are satisfied.

It is clear that the arc $c$, now considered as a subset of $D$, does not meet the crosscut $\gamma_{k}$ corresponding to $\gamma\left[y_{k}, y_{k+1}\right]$. Hence by Lemma 10.2 , there exists $m \in\{1, \ldots, n\}$ and a side $S$ of $\gamma_{m}$ in $D$ such that $S \backslash \gamma_{m}$ is disjoint from $c$ and all the $\operatorname{arcs} \gamma_{1}, \ldots, \gamma_{n}$. Then there exists an $\operatorname{arc} \beta \subset \partial D=a^{+} \cup a^{-}$with the same endpoints as $\gamma_{m}$ such that $\partial S=\gamma_{m} \cup \beta$. The set $S \backslash \gamma_{m}$ is disjoint from $\gamma_{1} \cup \cdots \cup \gamma_{n} \supset P \cup Q$, and so the arc $\beta$ has its endpoints in the set $P \cup Q$, but no other points in common with $P \cup Q$.

This implies that neither $x_{0}$ nor $x_{1}$ are contained in $\beta$; indeed, suppose $x_{0} \in \beta$, for example. Then the endpoints of $\beta$ and hence of $\gamma_{m}$ are necessarily the points $p_{1}$ and $q_{1}$. Collapsing $D$ back to $S^{2}$, we see that the endpoints $y_{m}$ and $y_{m+1}$ of $\gamma\left[y_{m}, y_{m+1}\right]$ are the same. This is a contradiction (here the assumption $n \geq 2$ is crucial). We arrive at a similar contradiction (using the points $p_{n}$ and $q_{n}$ ) if we assume $x_{1} \in \beta$. It follows that $\beta \subset \operatorname{int}\left(a^{+}\right)$or $\beta \subset \operatorname{int}\left(a^{-}\right)$.

These considerations imply that if we pass back to $S^{2}$ by identifying corresponding points in $a^{+}$and $a^{-}$, then from $\operatorname{int}(S)$ we obtain a bigon $U \subset S^{2}$ bounded by the subarc $\gamma\left[y_{m}, y_{m+1}\right]$ of $\gamma$ and a subarc $\widetilde{\beta}$ of $a$, where $U$ is disjoint from $\gamma \cup a \cup c$. This is impossible by Lemma 10.1 since $\gamma$ is in minimal position with $a \cup c$. This contradiction shows that the Claim is indeed true.

The Claim implies that $c$ has at least $n \geq 2$ intersection points with $\gamma$. Hence we can reverse the roles of $a$ and $c$ and get a similar statement as the Claim also for the arc $c$. This implies that the (finitely many) points in $a \cap \gamma \neq \varnothing$ and $c \cap \gamma \neq \varnothing$ alternate on $\gamma$.

As in Section 2.4, we now consider the pillow $\mathbb{P}$ with its set $V$ of vertices as the marked points, and the Weierstrass function $\wp: \mathbb{C} \rightarrow \mathbb{P}$ that is doubly periodic with respect to the lattice $2 \mathbb{Z}^{2}$. We will now revert to the notation $a$ and $c$ for the horizontal edges of $\mathbb{P}$. Recall that $\mathbb{I}=[0,1]$.

Lemma 10.4. Let $\alpha: \mathbb{I} \rightarrow \mathbb{C}$ be a simple loop or a homeomorphic parametrization of an arc. Suppose that the endpoints $z_{0}=\alpha(0)$ and $w_{0}=\alpha(1)$ of $\alpha$ lie in $\mathbb{C} \backslash \wp^{-1}(a \cup c)$ and that $\wp\left(z_{0}\right)=\wp\left(w_{0}\right)$. Suppose further that either $\alpha \cap \wp^{-1}(a \cup c)=\varnothing$, or all of the following conditions are true: $\alpha$ meets each of the lines in $\wp^{-1}(a \cup c)$ transversely, we have

$$
0<\#\left(\alpha \cap \wp^{-1}(a)\right)=\#\left(\alpha \cap \wp^{-1}(c)\right)<\infty,
$$

and the points in $\alpha \cap \wp^{-1}(a)$ and $\alpha \cap \wp^{-1}(c)$ alternate on $\alpha$.
Then $w_{0}-z_{0} \in 2 \mathbb{Z}^{2}$. Here $w_{0}-z_{0} \neq 0$ unless $\alpha \cap \wp^{-1}(a \cup c)=\varnothing$.
Proof. Note that the set $\wp^{-1}(a \cup c)$ consists precisely of the lines $L_{n}:=\{z \in \mathbb{C}: \operatorname{Im}(z)=n\}$, $n \in \mathbb{Z}$. Moreover, such a line $L_{n}$ is mapped to $a$ or $c$ depending on whether $n$ is even or odd, respectively.

We consider the second case first when $\alpha \cap \wp^{-1}(a \cup c) \neq \varnothing$. We denote by $0<u_{1}<\cdots<u_{k}<1$, $k \in \mathbb{N}$, all the (finitely many) $u$-parameter values with $\alpha\left(u_{j}\right) \in \wp^{-1}(a \cup c)$ for $j=1, \ldots, k$. Since $\alpha$ meets each line $L_{n}$ transversely and the points in $\alpha \cap \wp^{-1}(a) \neq \varnothing$ and $\alpha \cap \wp^{-1}(c) \neq \varnothing$ alternate on $\alpha$, it is clear that the values $\operatorname{Im}\left(\alpha\left(u_{j}\right)\right)$ either strictly increase by 1 in each step $j=1, \ldots, k$, or strictly decrease by 1 in each step. Here we have strict increase if $\operatorname{Im}\left(\alpha\left(u_{1}\right)-\alpha(0)\right)>0$, and strict decrease if $\operatorname{Im}\left(\alpha\left(u_{1}\right)-\alpha(0)\right)<0$. This implies that $k>0$ is precisely the number of lines $L_{n}, n \in \mathbb{Z}$, that separate $z_{0}$ and $w_{0}$. So $z_{0}$ and $w_{0}$ lie in different components of $\mathbb{C} \backslash \wp^{-1}(a \cup c)$ which shows that $z_{0} \neq w_{0}$.

By our hypotheses, $\#\left(\alpha \cap \wp^{-1}(a)\right)=\#\left(\alpha \cap \wp^{-1}(c)\right)$, which implies that the number $k$ of intersection points of $\alpha$ with $\wp^{-1}(a \cup c)$ is even. Since $\wp\left(z_{0}\right)=\wp\left(w_{0}\right)$, by (2.3) we have $w_{0}= \pm z_{0}+v_{0}$ with $v_{0} \in 2 \mathbb{Z}^{2}$. We have to rule out the minus sign here.

We argue by contradiction and assume that $w_{0}=-z_{0}+v_{0}$. Then $v_{1}:=\frac{1}{2}\left(z_{0}+w_{0}\right)=\frac{1}{2} v_{0} \in \mathbb{Z}^{2}$, and so the endpoints $z_{0}$ and $w_{0}$ of $\alpha$ are in symmetric position to the point $v_{1} \in \mathbb{Z}^{2}$. This implies that the number $k$ of lines $L_{n}, n \in \mathbb{Z}$, separating $z_{0}$ and $w_{0}$ is odd, contradicting what we have just seen. We conclude $w_{0}=z_{0}+v_{0}$ with $v_{0} \in 2 \mathbb{Z}^{2}$, and the statement follows in this case. Note that the exact same argument leading to $w_{0}-z_{0} \in 2 \mathbb{Z}^{2}$ also applies if $\alpha \cap \wp^{-1}(a \cup c)=\varnothing$.

We call a Jordan curve $\gamma$ in $(\mathbb{P}, V)$ null-homotopic in $(\mathbb{P}, V)$ if $\gamma$ can be homotoped in $\mathbb{P} \backslash V$ to a point, i.e., if there exists a homotopy $H: \partial \mathbb{D} \times \mathbb{I} \rightarrow \mathbb{P} \backslash V$ such that $H_{0}$ is a homeomorphism of $\partial \mathbb{D}$ onto $\gamma$ and $H_{1}$ is a constant map.

Lemma 10.5. Let $\gamma$ be an essential Jordan curve in $(\mathbb{P}, V)$. Then $\gamma$ is not null-homotopic in $(\mathbb{P}, V)$.

This statement sounds somewhat tautological, because in topology "essential" is often defined as "not null-homotopic". Recall though that in our context $\gamma$ is called essential if each of the two components of $\mathbb{P} \backslash \gamma$ contains precisely two of the points in $V$. In the proof, we will use some standard facts about winding numbers; see [Bur79, Chapter 4] for the basic definitions and background.

Proof. On an intuitive level, every homotopy contracting $\gamma$ to a point must slide over all the points in one of the complementary components of $\gamma$. Hence it cannot stay in $\mathbb{P} \backslash V$, and so $\gamma$ is not null-homotopic in $(\mathbb{P}, V)$.

To make this more rigorous, we argue by contradiction. By the Schönflies theorem we may identify $\mathbb{P}$ with $\widehat{\mathbb{C}}$ and $\gamma$ with $\partial \mathbb{D}$ and assume that 0 and $\infty$ belong to the set $Z \subset \widehat{\mathbb{C}}$ of marked points corresponding to the points in $V$. We now argue by contradiction and assume that there exists a homotopy $H: \partial \mathbb{D} \times \mathbb{I} \rightarrow \widehat{\mathbb{C}} \backslash Z \subset \mathbb{C} \backslash\{0\}$ such that $H_{0}$ is a homeomorphism on $\partial \mathbb{D}$ and $H_{1}$ is a constant map. Then for each $t \in \mathbb{I}$, the map $u \in \mathbb{I} \mapsto \alpha_{t}(u):=H_{t}\left(e^{2 \pi i u}\right)$ is a loop in $\mathbb{C} \backslash\{0\}$. Each loop $\alpha_{t}, t \in \mathbb{I}$, has the same winding number $\operatorname{ind}_{\alpha_{t}}(0)$ around 0 , because this winding number is invariant under homotopies in $\mathbb{C} \backslash\{0\}$ (see Bur79, Theorem 4.12]). On the other hand, $\operatorname{ind}_{\alpha_{0}}(0)= \pm 1$, because $\alpha_{0}$ is a simple loop (see Bur79, Theorem 4.42]), while $\operatorname{ind}_{\alpha_{1}}(0)=0$, because $\alpha_{1}$ is a constant loop. This is a contradiction.

An element $x$ of a rank-2 lattice $\Gamma$ (such as $2 \mathbb{Z}^{2}$ ) in $\mathbb{C}$ is called primitive if it cannot be represented in the form $x=n y$ with $y \in \Gamma$ and $n \in \mathbb{N}, n \geq 2$. Note that then $x \neq 0$.

Lemma 10.6. Let $\gamma$ be a Jordan curve in $(\mathbb{P}, V)$ parametrized as a simple loop $\beta: \mathbb{I} \rightarrow \mathbb{P}$, and let $\alpha: \mathbb{I} \rightarrow \mathbb{C}$ be a lift of $\beta$ under $\wp$.
(i) If $\alpha(0)=\alpha(1)$ and $\alpha$ is a path in a convex subset of $\mathbb{C} \backslash \mathbb{Z}^{2}$, then $\gamma$ is null-homotopic in $(\mathbb{P}, V)$.
(ii) If $\gamma$ is essential, then $\alpha(1)-\alpha(0)$ is a primitive element of $2 \mathbb{Z}^{2}$. It is uniquely determined up to sign by the isotopy class $[\gamma]$ of $\gamma$ rel. $V$.

Here we call $\alpha$ a lift of $\beta$ under $\wp$ if $\beta=\wp \circ \alpha$. In this lemma and its proof, we will carefully distinguish between a path and its image set (unlike elsewhere in the paper).
Proof. Note that since $\beta(\mathbb{I})=\gamma \subset \mathbb{P} \backslash V$ and $\wp$ is a covering map over $\mathbb{P} \backslash V$, a lift $\alpha$ of $\beta$ under $\wp$ exists. Moreover, for each choice of $z_{0} \in \wp^{-1}(\beta(0))$, there exists a unique lift $\alpha$ of $\beta$ such that $\alpha(0)=z_{0}$ (for these standard facts see [Hat02, Section 1.3, Proposition 1.30]). We will use this uniqueness property of lifts repeatedly in the following.
(i) The idea for the first part is very simple. We use a "straight-line homotopy" between $\alpha$ and the constant path $u \in \mathbb{I} \mapsto \alpha(0)$ and push it to $\mathbb{P} \backslash V$ by applying $\wp$.

More precisely, we define

$$
H(u, t):=\wp((1-t) \alpha(u)+t \alpha(0))
$$

for $u, t \in \mathbb{I}$. Since the path $\alpha$ lies in a convex set $K \subset \mathbb{C} \backslash \mathbb{Z}^{2}$, we have

$$
\alpha_{t}(u):=(1-t) \alpha(u)+t \alpha(0) \in K
$$

for all $u, t \in \mathbb{I}$, and so $H(\mathbb{I} \times \mathbb{I}) \subset \wp\left(\mathbb{C} \backslash \mathbb{Z}^{2}\right)=\mathbb{P} \backslash V$. Hence $H$ is a homotopy in $\mathbb{P} \backslash V$. Moreover, $H_{t}(u)=\left(\wp \circ \alpha_{t}\right)(u)$ for all $u \in \mathbb{I}$, and so $H_{t}=\wp \circ \alpha_{t}$ for all $t \in \mathbb{I}$. In particular, $H_{0}=\wp \circ \alpha_{0}=\wp \circ \alpha=\beta$. Moreover, $H_{1}(u)=\alpha(0)$ for $u \in \mathbb{I}$, and so $H_{1}$ is a constant path.

For all $t \in \mathbb{I}$ we have

$$
\alpha_{t}(1)-\alpha_{t}(0)=(1-t)(\alpha(1)-\alpha(0))=0 .
$$

Hence $\wp\left(\alpha_{t}(1)\right)=\wp\left(\alpha_{t}(0)\right)$ for $t \in \mathbb{I}$, and so $H_{t}=\wp \circ \alpha_{t}$ is a loop in $\mathbb{P} \backslash V$ for all $t \in \mathbb{I}$. By identifying the points $(0, t)$ and $(1, t)$ for each $t \in \mathbb{I}$, we obtain a homotopy $\bar{H}: \partial \mathbb{D} \times \mathbb{I} \rightarrow \mathbb{P} \backslash V$ such that

$$
\bar{H}\left(e^{2 \pi i u}, t\right)=H(u, t)
$$

for all $u, t \in \mathbb{I}$. Since $\bar{H}_{0}\left(e^{2 \pi i u}\right)=H_{0}(u)=\beta(u)$ for $u \in \mathbb{I}$, we see that $\bar{H}_{0}$ is a homeomorphism of $\partial \mathbb{D}$ onto $\beta(\mathbb{I})=\gamma$. On the other hand, $\bar{H}_{1}$ is a constant map. Hence $\gamma$ is null-homotopic in $(\mathbb{P}, V)$.
(ii) The proof is somewhat tedious as we have to worry about different choices of the curve in $[\gamma]$, its different parametrizations as a simple loop, and the different lifts of these parametrizations under $\wp$.

To prove the statement, we first consider a special case, namely we choose a Jordan curve $\gamma_{0}$ in $(\mathbb{P}, V)$ that lies in the same isotopy class rel. $V$ as $\gamma$ with the additional property that $\gamma_{0}$ is in minimal position with the set $a \cup c$. Since $\gamma_{0}$ cannot be a subset of $a \cup c$, we can parametrize $\gamma_{0}$ as a simple loop $\beta_{0}: \mathbb{I} \rightarrow \mathbb{P}$ such that $\beta_{0}(0)=\beta_{0}(1) \notin a \cup c$. We now consider a lift $\alpha_{0}: \mathbb{I} \rightarrow \mathbb{C}$ of $\beta_{0}$ under $\wp$. Note that $\alpha_{0}$ is a simple loop or a homeomorphic parametrization on an arc in $\mathbb{C}$.

Since $\gamma$ is essential, the Jordan curve $\gamma_{0}$ is also essential. Indeed, under an isotopy rel. $V$ that deforms $\gamma$ into $\gamma_{0}$, the complementary components of $\gamma$ in $\mathbb{P}$ are deformed into the complementary components of $\gamma_{0}$ while the points in $V$ stay fixed. Therefore, each of the two components of $\mathbb{P} \backslash \gamma_{0}$ contains precisely two points of $V$.

It follows from Lemma 10.1 that $\gamma_{0}$ meets $a$ and $c$ transversely. Moreover, by Lemma 10.3 either $\gamma_{0} \cap(a \cup c)=\varnothing$, or the sets $a \cap \gamma_{0}$ and $c \cap \gamma_{0}$ are non-empty and finite, and the points in these sets alternate on $\gamma_{0}$. We now define $\bar{\alpha}_{0}:=\alpha_{0}(\mathbb{I})$. Then $\bar{\alpha}_{0}$ is a Jordan curve or an arc in $\mathbb{C}$. Moreover, we either have $\bar{\alpha}_{0} \cap \wp^{-1}(a \cup c)=\varnothing$, or $\bar{\alpha}_{0}$ meets each of the lines in $\wp^{-1}(a \cup c)$ transversely,

$$
0<\#\left(\bar{\alpha}_{0} \cap \wp^{-1}(a)\right)=\#\left(\bar{\alpha}_{0} \cap \wp^{-1}(c)\right)<\infty,
$$

and the points in $\bar{\alpha}_{0} \cap \wp^{-1}(a)$ and $\bar{\alpha}_{0} \cap \wp^{-1}(c)$ alternate on $\bar{\alpha}_{0}$. Let $z_{0}:=\alpha_{0}(0)$ and $w_{0}:=\alpha_{0}(1)$. Then

$$
\wp\left(z_{0}\right)=\wp\left(\alpha_{0}(0)\right)=\beta_{0}(0)=\beta_{0}(1)=\wp\left(\alpha_{0}(1)\right)=\wp\left(w_{0}\right),
$$

and by the choice of $\beta_{0}$ we have $z_{0}, w_{0} \notin \wp^{-1}(a \cup c)$. Therefore, we are exactly in the situation of Lemma 10.4.

It follows that $v_{0}:=w_{0}-z_{0} \in 2 \mathbb{Z}^{2}$. Here $v_{0} \neq 0$. Indeed, otherwise $z_{0}=w_{0}$. Then the second part of Lemma 10.4 implies that the arc $\bar{\alpha}_{0}$ does not meet $\wp^{-1}(a \cup c)$ and so it lies in a connected component of $\mathbb{C} \wp^{-1}(a \cup c)$. This component is an infinite strip, and hence a convex set, contained in $\mathbb{C} \backslash \mathbb{Z}^{2}$. Now (i) implies that $\gamma_{0}$ is null-homotopic in $(\mathbb{P}, V)$. By Lemma 10.5 this contradicts the fact that $\gamma_{0}$ is essential. We conclude that indeed $v_{0}=w_{0}-z_{0} \neq 0$.

This shows that $\alpha_{0}(1)-\alpha_{0}(0)=w_{0}-z_{0}$ is a non-zero element of the lattice $2 \mathbb{Z}^{2}$. We claim that $w_{0}-z_{0}$ is actually a primitive element of $2 \mathbb{Z}^{2}$. To see this, we argue by contradiction and assume that $w_{0}-z_{0}=n y_{0}$ with $y_{0} \in 2 \mathbb{Z}^{2} \backslash\{0\}$ and $n \in \mathbb{N}, n \geq 2$.

We now consider the path $\sigma: \mathbb{I} \rightarrow \mathbb{C} \backslash\{0\}$ given as $\sigma(u)=\exp \left(\frac{2 \pi i}{y_{0}} \alpha_{0}(u)\right)$ for $u \in \mathbb{I}$. This is a loop with winding number

$$
\operatorname{ind}_{\sigma}(0)=\frac{1}{y_{0}}\left(\alpha_{0}(1)-\alpha_{0}(0)\right)=n
$$

around 0 . Now a simple loop in $\mathbb{C} \backslash\{0\}$ has winding number 0 or $\pm 1$ around 0 (see Bur79, Theorem 4.42]), and so $\sigma$ cannot be simple. This implies that there are numbers $0 \leq u<u^{\prime}<1$ such that $\sigma(u)=\sigma\left(u^{\prime}\right)$. This in turn means that $\alpha_{0}\left(u^{\prime}\right)-\alpha_{0}(u)=k y_{0}$ for some $k \in \mathbb{Z}$. Since $y_{0} \in 2 \mathbb{Z}^{2}$, it follows that

$$
\beta_{0}\left(u^{\prime}\right)=\wp\left(\alpha_{0}\left(u^{\prime}\right)\right)=\wp\left(\alpha_{0}(u)\right)=\beta_{0}(u) .
$$

This is impossible, since $\beta_{0}$ is injective on $[0,1)$.
We have shown the first part of the statement for a particular Jordan curve $\gamma_{0}$ in [ $\gamma$ ] with a special parametrization $\beta_{0}$, and a choice of a lift $\alpha_{0}$ of $\beta_{0}$ under $\wp$. We now have to show that the number $v_{0}=w_{0}-z_{0}=\alpha_{0}(1)-\alpha_{0}(0)$ obtained in this way only depends on $[\gamma]$ up to sign. For this we pick an arbitrary Jordan curve in $[\gamma]$ which we will simply call $\gamma$.

Since $\gamma_{0} \sim \gamma$ rel. $V$, there exists an isotopy $H: \mathbb{P} \times \mathbb{I} \rightarrow \mathbb{P}$ rel. $V$ with $H_{0}=\operatorname{id} \mathbb{P}$ and $H_{1}\left(\gamma_{0}\right)=\gamma$. We now define a homotopy $\bar{H}: \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{P} \backslash V$ by setting

$$
\bar{H}(u, t):=H\left(\beta_{0}(u), t\right)
$$

for $u, t \in \mathbb{I}$. Note that $\bar{H}$ maps into $\mathbb{P} \backslash V$ as follows from the facts that $\gamma_{0} \subset \mathbb{P} \backslash V$ and $H$ is an isotopy rel. $V$.

The time- 0 map $\bar{H}(\cdot, 0)=\beta_{0}$ of the homotopy $\bar{H}$ is the parametrization of the loop $\gamma_{0}$, while the time-1 map $\beta:=\bar{H}(\cdot, 1)=H_{1} \circ \beta_{0}$ gives some parametrization of $\gamma=H_{1}\left(\gamma_{0}\right)$ as a simple loop.

By the homotopy lifting theorem (see [Hat02, Proposition 1.30]), there exists a homotopy $\widetilde{H}: \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{C} \backslash \mathbb{Z}^{2}$ such that $\bar{H}=\wp \circ \widetilde{H}$ and $\widetilde{H}_{0}=\widetilde{H}(\cdot, 0)=\alpha_{0}$. Then $\alpha:=\widetilde{H}(\cdot, 1)$ is a lift of $\beta=\bar{H}(\cdot, 1)$ under $\wp$. We want to show that $\alpha(1)-\alpha(0)=v_{0}=w_{0}-z_{0}$.

To see this, we consider the paths $\sigma, \tau: \mathbb{I} \rightarrow \mathbb{C} \backslash \mathbb{Z}^{2}$ defined as $\sigma(t)=\widetilde{H}(0, t)$ and $\tau(t)=$ $\widetilde{H}(1, t)$ for $t \in \mathbb{I}$. Then

$$
\begin{align*}
\bar{\sigma}(t) & :=\wp(\sigma(t))=\wp(\widetilde{H}(0, t))=\bar{H}(0, t)  \tag{10.1}\\
& =H_{t}(\beta(0))=H_{t}(\beta(1))=\bar{H}(1, t)=\wp(\widetilde{H}(1, t))=\wp(\tau(t))
\end{align*}
$$

for $t \in \mathbb{I}$. Note also that

$$
\tau(0)=\widetilde{H}(1,0)=\alpha_{0}(1)=\alpha_{0}(0)+v_{0}=\widetilde{H}(0,0)+v_{0}=\sigma(0)+v_{0}
$$

This implies that the paths $\tau$ and $t \in \mathbb{I} \mapsto \sigma(t)+v_{0}$ have the same initial points. Since $v_{0} \in 2 \mathbb{Z}^{2}$, it follows from (2.3) and (10.1) that the map $\wp$ sends them both to $\bar{\sigma}=\wp \circ \sigma=\wp \circ \tau$, which is a path in $\mathbb{P} \backslash V$. It follows from the uniqueness of lifts under $\wp$ that $\tau(t)=\sigma(t)+v_{0}$ for all $t \in \mathbb{I}$.

This implies that

$$
\alpha(1)-\alpha(0)=\widetilde{H}(1,1)-\widetilde{H}(0,1)=\tau(1)-\sigma(1)=v_{0}
$$

as desired.
Note that for a given parametrization $\beta$ of $\gamma$, the difference $\alpha(1)-\alpha(0)$ is independent up to sign of the choice of the lift $\alpha$ of $\beta$. Indeed, suppose $\alpha^{\prime}$ is another lift of $\beta$ under $\wp$. Then

$$
\wp(\alpha(0))=\beta(0)=\wp\left(\alpha^{\prime}(0)\right),
$$

and so by (2.3) we have

$$
\alpha^{\prime}(0)= \pm \alpha(0)+m_{0}
$$

for some (fixed) choice of the sign $\pm$ and $m_{0} \in 2 \mathbb{Z}^{2}$. Then $t \in \mathbb{I} \mapsto \pm \alpha(t)+m_{0}$ is a lift of $\beta$ with the same initial point as $\alpha^{\prime}$ and so we see that

$$
\alpha^{\prime}(t)= \pm \alpha(t)+m_{0}
$$

for all $t \in \mathbb{I}$. This implies that

$$
\begin{equation*}
\alpha^{\prime}(1)-\alpha^{\prime}(0)= \pm(\alpha(1)-\alpha(0)) \tag{10.2}
\end{equation*}
$$

as desired.

It remains to show that up to $\operatorname{sign} \alpha(1)-\alpha(0)$ is independent of the choice of the parametrization $\beta$ of $\gamma$. For this we consider another parametrization $\beta^{\prime}$ of $\gamma$ as a simple loop. We first assume that $\beta^{\prime}(0)=\beta(0)$. Then there exists a homeomorphism $h: \mathbb{I} \rightarrow \mathbb{I}$ such that $\beta^{\prime}=\beta \circ h$. Here $h$ fixes the endpoints 0 and 1 of $\mathbb{I}$ or interchanges them depending on whether $\beta^{\prime}$ parametrizes $\gamma$ with the same or opposite orientation as $\beta$, respectively. In any case, $\alpha^{\prime}=\alpha \circ h$ is a lift of $\beta^{\prime}$ under $\wp$. It follows that

$$
\alpha^{\prime}(1)-\alpha^{\prime}(0)=\alpha(h(1))-\alpha(h(0))= \pm(\alpha(1)-\alpha(0)),
$$

as desired. As we now know, this relation is independent of the specific choice of the lift $\alpha^{\prime}$ of $\beta^{\prime}$.

Finally, we have to consider the case where $\beta^{\prime}$ has a possibly different initial point than $\beta$, say $p_{0}:=\beta^{\prime}(0) \in \gamma$. By what we have seen, in order to establish $(10.2)$, we can choose any parametrization $\beta^{\prime}$ of $\gamma$ as a simple loop with $\beta^{\prime}(0)=p_{0}$ and any lift $\alpha^{\prime}$ of $\beta^{\prime}$.

We can extend our parametrization $\beta$ on $\mathbb{I}$ periodically to a continuous map $\beta: \mathbb{R} \rightarrow \gamma$ such that $\beta(u+1)=\beta(u)$ for all $u \in \mathbb{R}$. Then $\beta$ lifts under $\wp$ to a continuous map $\alpha: \mathbb{R} \rightarrow \mathbb{C}$ which agrees with the original lift $\alpha$ on $\mathbb{I}$. Then $u \in \mathbb{R} \mapsto \alpha(u+1)-v_{0}$ is also a lift of $\beta$ under $\wp$. The initial point of this lift corresponding to $u=0$ is equal to $\alpha(0)$. The uniqueness of lifts implies that this lift and the original lift $\alpha$ are the same paths and so

$$
\begin{equation*}
\alpha(u+1)=\alpha(u)+v_{0} \tag{10.3}
\end{equation*}
$$

for all $u \in \mathbb{R}$.
We can find $u_{0} \in[0,1)$ such that $\beta\left(u_{0}\right)=p_{0}$. Then $\beta^{\prime}: \mathbb{I} \rightarrow \gamma$ defined as $\beta^{\prime}(u)=\beta\left(u_{0}+u\right)$ for $u \in \mathbb{I}$ is a parametrization of $\gamma$ as a simple loop with the initial point $p_{0}$. Under $\wp$ this path $\beta^{\prime}$ has the lift $\alpha^{\prime}: \mathbb{I} \rightarrow \mathbb{C}$ given by $\alpha^{\prime}(u)=\alpha\left(u+u_{0}\right)$ for $u \in \mathbb{I}$. Then (10.3) implies that

$$
\alpha^{\prime}(1)-\alpha^{\prime}(0)=\alpha\left(u_{0}+1\right)-\alpha\left(u_{0}\right)=v_{0}
$$

as desired. The proof is complete.
We are now almost ready to prove Lemma 2.3. Before we get to this, it is useful to discuss an alternative way to view our pillow $\mathbb{P}$.

We consider a slope $r / s \in \widehat{\mathbb{Q}}$. Then we can choose $p, q \in \mathbb{Z}$ such $p r+q s=1$ and define $\omega:=s+i r$ and $\widetilde{\omega}:=-p+i q$. The numbers $\omega$ and $\widetilde{\omega}$ form a basis of $\mathbb{C} \cong \mathbb{R}^{2}$ over $\mathbb{R}$, and so every point $z \in \mathbb{C}$ can be uniquely written in the form $z=u \widetilde{\omega}+v \omega$ with $u, v \in \mathbb{R}$. Accordingly, the map $z=u \widetilde{\omega}+v \omega \mapsto R(z):=u \widetilde{\omega}-v \omega \mapsto$ for $u, v \in \mathbb{R}$ is a well-defined "skew-reflection" $R$ on $\mathbb{C}$. Note also that $\mathbb{Z}^{2}=\{n \widetilde{\omega}+k \omega: n, k \in \mathbb{Z}\}$.

We consider the parallelogram $Q:=\{u \widetilde{\omega}+v \omega: u \in[0,1], v \in[-1,1]\} \subset \mathbb{C}$. Then it follows from (2.3) that $\wp(Q)=\mathbb{P}$. Moreover, for $z, w \in Q, z \neq w$, we have $\wp(z)=\wp(w)$ if and only if $z, w \in \partial Q$ and $w=R(z)$. Intuitively, this means that the pillow $\mathbb{P}$ can be obtained from $Q$ by "folding" $Q$ in its middle segment $[0, \widetilde{\omega}] \subset Q$ and identifying the points on $\partial Q$ that correspond to each other under the skew-reflection $R$. The map $\wp$ sends the set $\{0, \widetilde{\omega}, \widetilde{\omega}+\omega, \omega\}$ bijectively onto the set $\{A, B, C, D\}$ of vertices of $\mathbb{P}$ (but not necessarily in that order).

From this geometric picture it is clear that for each $t \in(0,1)$ the set

$$
\tau_{r / s}:=\wp([t \widetilde{\omega}-\omega, t \widetilde{\omega}+\omega])=\wp\left(\ell_{r / s}(t \widetilde{\omega})\right)
$$

is a simple closed geodesic in $(\mathbb{P}, V)$. Moreover, the sets

$$
\xi_{r / s}:=\wp([-\omega,+\omega])=\wp\left(\ell_{r / s}(0)\right) \text { and } \xi_{r / s}^{\prime}:=\wp([\widetilde{\omega}-\omega, \widetilde{\omega}+\omega])=\wp\left(\ell_{r / s}(\widetilde{\omega})\right)
$$

are geodesic core arcs of $\tau_{r / s}$ lying in different components of $\mathbb{P} \backslash \tau_{r / s}$. In particular, $\tau_{r / s}$ is an essential Jordan curve in $(\mathbb{P}, V)$.

It follows from $(\overline{2.3})$ that $\wp\left(\ell_{r / s}(t \widetilde{\omega})\right) \cap \wp\left(\ell_{r / s}\left(t^{\prime} \widetilde{\omega}\right)\right) \neq \varnothing$ for $t, t^{\prime} \in \mathbb{R}$ if and only if $t^{\prime}-t \in 2 \mathbb{Z}$ or $t^{\prime}+t \in 2 \mathbb{Z}$. In this case, we have $\wp\left(\ell_{r / s}(t \widetilde{\omega})\right)=\wp\left(\ell_{r / s}\left(t^{\prime} \widetilde{\omega}\right)\right)$. Moreover, $\tau=\wp\left(\ell_{r / s}(t \widetilde{\omega})\right)$ is a simple closed geodesic $\tau_{r / s}$ in $(\mathbb{P}, V)$ if $t \in \mathbb{R} \backslash \mathbb{Z}$, it is equal to the geodesic arc $\xi_{r / s}$ if $t$ is an even integer, and is equal to the geodesic arc $\xi_{r / s}^{\prime}$ if $t$ is an odd integer. Note that $\ell_{r / s}(t \widetilde{\omega})$ for $t \in \mathbb{R}$ contains a point in $\mathbb{Z}^{2}=\{n \widetilde{\omega}+k \omega: n, k \in \mathbb{Z}\}$ if and only if $t \in \mathbb{Z}$.

Lemma 10.7. Let $\tau_{r / s}$ and $\tau_{r / s}^{\prime}$ be two distinct simple closed geodesics in $(\mathbb{P}, V)$ with slope $r / s \in \widehat{\mathbb{Q}}$. Then $\tau_{r / s}$ and $\tau_{r / s}^{\prime}$ are isotopic rel. $V$.
Proof. The previous considerations imply that we may assume that the geodesics are represented in the form $\tau_{r / s}=\wp\left(\ell_{r / s}(t \widetilde{\omega})\right)$ and $\tau_{r / s}^{\prime}=\wp\left(\ell_{r / s}\left(t^{\prime} \widetilde{\omega}\right)\right)$ with $t, t^{\prime} \in(0,1), t \neq t^{\prime}$. We may assume $t<t^{\prime}$. Then

$$
U:=\wp\left(\left\{u \widetilde{\omega}+v \omega: u \in\left(t, t^{\prime}\right), v \in[-1,1]\right\}\right)
$$

is an annulus contained in $\mathbb{P} \backslash V$ with $\partial U=\tau_{r / s} \cup \tau_{r / s}^{\prime}$. It follows from Lemma 2.1 that $\tau_{r / s}$ and $\tau_{r / s}^{\prime}$ are isotopic rel. $V$.

Proof of Lemma 2.3. Let $\gamma$ be an essential Jordan curve in $(\mathbb{P}, V)$. If we parametrize $\gamma$ as a simple loop $\beta: \mathbb{I} \rightarrow \mathbb{P}$ and lift $\beta$ to a path $\alpha: \mathbb{I} \rightarrow \mathbb{C}$ under $\wp$, then by Lemma 10.6 we know that $\alpha(1)-\alpha(0)$ is a primitive element of $2 \mathbb{Z}^{2}$ uniquely determined by $[\gamma]$ up to sign. Hence we can find relatively prime integers $r, s \in \mathbb{Z}$ such that

$$
\begin{equation*}
\alpha(1)-\alpha(0)=2(s+i r) \tag{10.4}
\end{equation*}
$$

By switching signs here, which corresponds to parametrizing $\gamma$ with opposite orientation, we may assume that $r \in \mathbb{Z}, s \in \mathbb{N}_{0}$, and that $r=1$ if $s=0$. Note that with these restrictions on $r$ and $s$, the primitive element $2(s+i r)$ of $2 \mathbb{Z}^{2}$ corresponds to the unique slope $r / s \in \widehat{\mathbb{Q}}$, and every slope in $\widehat{\mathbb{Q}}$ arises from a unique primitive element of $2 \mathbb{Z}^{2}$ in this form.

As before, define $\omega:=s+i r$ and $\widetilde{\omega}:=-p+i q$, where $p, q \in \mathbb{Z}$ are chosen so that $p r+q s=1$. We know that $\xi:=\wp\left(\ell_{r / s}(0)\right)$ and $\xi^{\prime}:=\wp\left(\ell_{r / s}(\widetilde{\omega})\right)$ are disjoint geodesic arcs in ( $\mathbb{P}, V$ ). The sets $\wp^{-1}(\xi)$ and $\wp^{-1}\left(\xi^{\prime}\right)$ consist of parallel lines with slope $r / s$, and these lines alternate in the following sense: each component of $\mathbb{C} \backslash \wp^{-1}\left(\xi \cup \xi^{\prime}\right)$ is an infinite strip whose boundary contains one line in $\wp^{-1}(\xi)$ and one line in $\wp^{-1}\left(\xi^{\prime}\right)$.

We may assume that $\gamma$ is in minimal position with $\xi \cup \xi^{\prime}$. Moreover, we may assume that the parametrization $\beta$ of $\gamma$ as a simple loop was chosen so that $\beta(0)=\beta(1) \notin \xi \cup \xi^{\prime}$.

Then by Lemmas 10.1 and 10.3 we know that either $\xi \cap \gamma=\varnothing=\xi^{\prime} \cap \gamma$, or the following conditions are true: $\gamma$ meets $\xi$ and $\xi^{\prime}$ transversely, the sets $\xi \cap \gamma$ and $\xi^{\prime} \cap \gamma$ are non-empty and finite, and the points in these sets alternate on $\gamma$. We claim that the latter is not possible.

Otherwise, we choose a lift $\alpha$ of $\beta$ under $\wp$. Then $\alpha$ intersects the lines in $\wp^{-1}\left(\xi \cup \xi^{\prime}\right)$ transversely, we have

$$
k:=\#\left(\alpha \cap \wp^{-1}(\xi)\right)=\#\left(\alpha \cap \wp^{-1}\left(\xi^{\prime}\right)\right) \in \mathbb{N},
$$

and the points in $\alpha \cap \wp^{-1}(\xi) \neq \varnothing$ and $\alpha \cap \wp^{-1}\left(\xi^{\prime}\right) \neq \varnothing$ alternate on $\alpha$. An argument very similar to the proof of Lemma 10.4 then shows that $2 k>0$ is the number of lines in the set $\wp^{-1}\left(\xi \cup \xi^{\prime}\right)$ that separate $\alpha(0)$ from $\alpha(1)$. On the other hand, we know that $\alpha(1)-\alpha(0)=2(s+i r)$ and so $\alpha(1)$ and $\alpha(0)$ lie on a line with slope $r / s$ and are not separated by any line in $\wp^{-1}\left(\xi \cup \xi^{\prime}\right)$. This is a contradiction.

This shows that $\xi \cap \gamma=\varnothing=\xi^{\prime} \cap \gamma$. Now consider a simple closed geodesic $\tau_{r / s}=\wp\left(\ell_{r / s}(t \widetilde{\omega})\right)$ with slope $r / s$ and $0<t<1$. Since $\xi \cap \gamma=\varnothing$, we can choose $t$ very close to 0 so that $\tau_{r / s} \cap \gamma=\varnothing$. Now we can apply considerations very similar to the proof of Corollary 3.5 (ii). The complement of $\tau_{r / s} \cup \gamma$ in $\mathbb{P}$ is a disjoint union $\mathbb{P} \backslash\left(\tau_{r / s} \cup \gamma\right)=W \cup U \cup W^{\prime}$, where $W, W^{\prime} \subset \mathbb{P}$ are open Jordan regions and $U \subset \mathbb{P}$ is an annulus with $\partial U=\tau_{r / s} \cup \gamma$. Since $\tau_{r / s}$ and $\gamma$ are essential, both $W$ and $W^{\prime}$ must contain at least two points in $V$. Since $\# V=4$, we have $U \cap V=\varnothing$. Lemma 2.1 now implies that $\tau_{r / s}$ and $\gamma$ are isotopic rel. $V$. By Lemma 10.7 the curve $\gamma$ is actually isotopic to each closed geodesic $\tau_{r / s}$ with slope $r / s$.

The map $[\gamma] \mapsto r / s$ that sends each isotopy class $[\gamma]$ to a slope $r / s \in \widehat{\mathbb{Q}}$ obtained from a primitive element in $2 \mathbb{Z}^{2}$ associated with $[\gamma]$ according to Lemma 10.6 is well-defined. It is clear that it is surjective, because the isotopy class $\left[\tau_{r / s}\right]$ of a geodesic $\tau_{r / s}$ with slope $r / s \in \widehat{\mathbb{Q}}$ is sent to $r / s$. To see that it is injective, suppose two isotopy classes $[\gamma]$ and $\left[\gamma^{\prime}\right]$ are sent to the same slope $r / s \in \widehat{\mathbb{Q}}$ by this map. If $\tau_{r / s}$ is a closed geodesic with slope $r / s$, then by our previous discussion we have $\gamma \sim \tau_{r / s} \sim \gamma^{\prime}$ rel. $V$. Hence $[\gamma]=\left[\gamma^{\prime}\right]$. It follows that the map $[\gamma] \mapsto r / s$ is indeed a bijection.

Remark 10.8. Suppose $\gamma$ is an essential Jordan curve in $(\mathbb{P}, V)$ and its isotopy class [ $\gamma$ ] rel. $V$ corresponds to slope $r / s \in \widehat{\mathbb{Q}}$ according to Lemma 2.3. Let $\beta: \mathbb{I} \rightarrow \gamma$ be a parametrization of $\gamma$ as a simple loop, and $\alpha: \mathbb{I} \rightarrow \mathbb{C}$ be a lift of $\beta$ under $\wp$. Then (10.4) in the proof of Lemma 10.6 shows that we always have $\alpha(1)-\alpha(0)= \pm 2(s+i r)$.

A similar statement is true for $\operatorname{arcs}$ in $(\mathbb{P}, V)$.
Corollary 10.9. Let $\xi$ be an arc in $(\mathbb{P}, V)$. Then $\xi$ is isotopic rel. $V$ to a geodesic arc $\xi_{r / s}$ for some slope $r / s \in \widehat{\mathbb{Q}}$. Moreover, if $\beta: \mathbb{I} \rightarrow \xi$ is a homeomorphic parametrization of $\xi$ and $\alpha$ is any lift of $\beta$ under $\wp$, then $\alpha(1)-\alpha(0)= \pm(s+i r)$.
Proof. The arc $\xi$ joins two of the points in $V$, while the other two points in $V$ do not lie on $\xi$. Hence we may "surround" $\xi$ by an essential Jordan curve $\gamma$ in $(\mathbb{P}, V)$ such that $\xi$ is a core arc of $\gamma$. By Lemma 2.3 we know that $\gamma$ is isotopic rel. $V$ to a closed geodesic $\tau_{r / s}$ in $\mathbb{P}$ with some slope $r / s \in \widehat{\mathbb{Q}}$. Hence $\xi$ is isotopic rel. $V$ to a core arc $\xi^{\prime}$ of $\tau_{r / s}$.

Now any two arcs in the interior of a closed topological disk $D$ with the same endpoints are isotopic by an isotopy that fixes the endpoints of these arcs and the points in $\partial D$ (see [Bus10, Theorem A. 6 (ii)]). This implies that any two core arcs of an essential Jordan curve in $(\mathbb{P}, V)$ are isotopic rel. $V$ if the core arcs have the same endpoints. We know that the closed geodesic $\tau_{r / s}$ has precisely two geodesic arcs with slope $r / s$ as core arcs in different components of $\mathbb{P} \backslash \tau_{r / s}$. Therefore, $\xi^{\prime}$, and hence also $\xi$, is isotopic rel. $V$ to a geodesic arc $\xi_{r / s}$ with slope $r / s$. This proves the first part of the statement.

Let $\beta: \mathbb{I} \rightarrow \xi$ be a homeomorphic parametrization of $\xi$ and $\alpha: \mathbb{I} \rightarrow \mathbb{C}$ be a lift of $\beta$ under $\wp$. By what we have seen, we can choose an isotopy $H: \mathbb{P} \times \mathbb{I} \rightarrow \mathbb{P}$ rel. $V$ with $H_{0}=\mathrm{id}_{\mathbb{P}}$ and $H_{1}(\xi)=\xi_{r / s}$. We use this to define a homotopy $\bar{H}: \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{P}$ by setting

$$
\bar{H}(u, t):=H(\beta(u), t)
$$

for $u, t \in \mathbb{I}$. Note that $\bar{H}_{t}$ for $t \in \mathbb{I}$ gives a homeomorphic parametrization of an arc in $(\mathbb{P}, V)$. These arcs have all the same endpoints. In particular $u \in \mathbb{I} \mapsto \beta^{\prime}(u):=\bar{H}_{1}(u)=H_{1}(\beta(u))$ gives a homeomorphic parametrization of $H_{1}(\xi)=\xi_{r / s}$.

We can lift $\bar{H}_{t}$ under $\wp$ to find a homotopy $\widetilde{H}: \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{C}$ such that $\widetilde{H}_{0}=\alpha$ and $\wp \circ \widetilde{H}_{t}=\bar{H}_{t}$ for all $t \in \mathbb{I}$. To see this, one first applies the standard homotopy lifting theorem (see

Hat02, Proposition 1.30]) to the homotopy $\bar{H}$ restricted $(0,1) \times \mathbb{I}$ and the covering map $\wp: \mathbb{C} \backslash \mathbb{Z}^{2} \rightarrow \mathbb{P} \backslash V \supset \bar{H}((0,1) \times \mathbb{I})$ to obtain a unique homotopy $\widetilde{H}:(0,1) \times \mathbb{I} \rightarrow \mathbb{C}$ with $\widetilde{H}_{0}=\alpha \mid(0,1)$. Now as $u_{0} \rightarrow 0^{+}$, the set $\bar{H}\left(\left(0, u_{0}\right] \times \mathbb{I}\right)$ shrinks to the point $\beta(0) \in V$, and so the connected set $\widetilde{H}\left(\left(0, u_{0}\right] \times \mathbb{I}\right)$ shrinks to a unique point in $\wp^{-1}(V)=\mathbb{Z}^{2}$. This point can only be $\alpha(0)$. Hence $\bar{H}(u, t) \rightarrow \alpha(0)$ uniformly for $t \in \mathbb{I}$ as $u \rightarrow 0^{+}$, and similarly $\bar{H}(u, t) \rightarrow \alpha(1)$ uniformly for $t \in \mathbb{I}$ as $u \rightarrow 1^{-}$. This implies that we can continuously extend $\widetilde{H}$ to a homotopy on $\mathbb{I} \times \mathbb{I}$ with the desired properties by setting $\widetilde{H}(0, t)=\alpha(0)$ and $\widetilde{H}(1, t)=\alpha(1)$ for $t \in \mathbb{I}$.

Then $\alpha^{\prime}:=\widetilde{H}_{1}$ is a lift of $\beta^{\prime}$, because $\wp \circ \alpha^{\prime}=\wp \circ \widetilde{H}_{1}=\bar{H}_{1}=\beta^{\prime}$. Since $\beta^{\prime}$ is a homeomorphic parametrization of the geodesic arc $\xi_{r / s}$, the path $\alpha^{\prime}$ sends $\mathbb{I}$ homeomorphically onto a subsegment of a line $\ell_{r / s} \subset \mathbb{C}$. Since $\alpha^{\prime}$ has its endpoints in $\mathbb{Z}^{2}$ and $\alpha^{\prime}((0,1))$ is disjoint from $\mathbb{Z}^{2}$, this implies $\alpha^{\prime}(1)-\alpha^{\prime}(0)= \pm(s+i r)$. Since $\alpha$ and $\alpha^{\prime}$ have the same endpoints, the statement follows.

Proof of Lemma 2.4. In the proof all isotopies, isotopy classes, intersection numbers, etc. are for isotopies on $\mathbb{P}$ rel. $V$. We will use the facts about the geodesics on $(\mathbb{P}, V)$ discussed before Lemma 10.7 without further reference.
(i) Let $\alpha$ and $\beta$ be essential Jordan curves in $(\mathbb{P}, V)$ as in the statement. As before, we define $\omega=s+i r$ and $\widetilde{\omega}=-p+i q$, where $p, q \in \mathbb{Z}$ and $p r+q s=1$.

First suppose that $r / s=r^{\prime} / s^{\prime}$. Then in the isotopy class $[\alpha]=[\beta]$ we can find simple closed geodesics with slope $r / s$ that are disjoint, for example the curves $\tau_{r / s}=\wp\left(\ell_{r / s}(\widetilde{\omega} / 3)\right)$ and $\tau_{r / s}^{\prime}=\wp\left(\ell_{r / s}(2 \widetilde{\omega} / 3)\right)$. It follows that $\mathrm{i}(\alpha, \beta)=0$ in this case.

We now assume that $r / s \neq r^{\prime} / s^{\prime}$. In order to determine $\mathrm{i}(\alpha, \beta)$, we have to find the minimum of all numbers $\#(\alpha \cap \beta)$, where $\alpha$ and $\beta$ range over the given isotopy classes. By applying a suitable isotopy, we can reduce to the case where $\alpha$ is a fixed curve in its isotopy class and we only have to take variations over $\beta$. So by Lemma 2.3 we may assume that $\alpha=\tau_{r / s}=\wp\left(\ell_{r / s}\right)$ is a simple closed geodesic as in the statement. Then $\tau_{r / s}=\wp\left(\ell_{r / s}\left(t_{0} \widetilde{\omega}\right)\right)$ for some $t_{0} \in(0,1)$. The preimage $\wp^{-1}\left(\tau_{r / s}\right)$ of $\tau_{r / s}$ under $\wp$ consists of the two disjoint families

$$
\begin{equation*}
\mathcal{F}_{1}:=\left\{\ell_{r / s}\left(\left(t_{0}+2 j\right) \widetilde{\omega}\right): j \in \mathbb{Z}\right\} \text { and } \mathcal{F}_{2}:=\left\{\ell_{r / s}\left(\left(-t_{0}+2 j\right) \widetilde{\omega}\right): j \in \mathbb{Z}\right\} \tag{10.5}
\end{equation*}
$$

of distinct lines with slope $r / s$.
Now let $\widetilde{\beta}: \mathbb{I} \rightarrow \mathbb{C}$ be a lift of $\beta$ under $\wp$, where we think of $\beta$ as a simple closed loop in a parametrization with suitable orientation. Then if $z_{0}:=\widetilde{\beta}(0)$ and $w_{0}:=\widetilde{\beta}(1)$, we have $w_{0}-z_{0}=2\left(s^{\prime}+i r^{\prime}\right)$ as follows from Remark 10.8. By changing the basepoint of $\beta$ if neccesary, we may assume that $\widetilde{\beta}(0), \widetilde{\beta}(1) \notin \wp^{-1}\left(\tau_{r / s}\right)$. If $\omega^{\prime}:=s^{\prime}+i r^{\prime}$, then we can write $\omega^{\prime}$ uniquely in the form

$$
\begin{equation*}
\omega^{\prime}=k \omega+n \widetilde{\omega}, \tag{10.6}
\end{equation*}
$$

where $k, n \in \mathbb{Z}$. Note that then $|n|=N:=\left|r s^{\prime}-s r^{\prime}\right|>0$ (to see this, multiply (10.6) by the complex conjugate of $\omega$ and take imaginary parts).

Now each family $\mathcal{F}_{j}, j=1,2$, consists of equally spaced parallel lines with slope $r / s$ such that consecutive lines in each family differ by a translation by $2 \widetilde{\omega}$. This implies that the points $z_{0}$ and $w_{0}=z_{0}+2 \omega^{\prime}$ are separated by precisely $N$ lines from each of the families $\mathcal{F}_{j}$, $j=1,2$. So $\widetilde{\beta}$ must have at least $2 N$ points in common with $\wp^{-1}\left(\tau_{r / s}\right)$. Since $\wp \circ \widetilde{\beta}$ maps $[0,1)$ injectively onto $\beta$, we conclude that $\beta=\wp(\widetilde{\beta})$ has at least $2 N$ points in common with $\tau_{r / s}$. If $\beta=\tau_{r^{\prime} / s^{\prime}}$, then $\widetilde{\beta}$ is a parametrization of the line segment $\left[z_{0}, w_{0}\right]$, and so $\widetilde{\beta}$ meets
$\wp^{-1}\left(\tau_{r / s}\right)$ in precisely $2 N$ points. This means that $\tau_{r / s}$ and $\tau_{r^{\prime} / s^{\prime}}$ have exactly $2 N$ points in common.

It follows that for all $\beta$ we have

$$
2 N \leq \#\left(\wp^{-1}\left(\tau_{r / s}\right) \cap \widetilde{\beta}\right)=\#\left(\tau_{r / s} \cap \beta\right),
$$

and so $2 N \leq \mathrm{i}(\alpha, \beta) \leq \#\left(\tau_{r / s} \cap \tau_{r^{\prime} / s^{\prime}}\right)=2 N$. Thus we have equality here and the statement follows.
(ii) This is a variant of the argument in (i) and we use the same notation.

Since $\beta \sim \tau_{r^{\prime} / s^{\prime}}$, the core arc $\xi$ of $\beta$ is isotopic to a core arc of $\tau_{r^{\prime} / s^{\prime}}$. Now two core arcs of a given essential Jordan curve in $(\mathbb{P}, V)$ are isotopic rel. $V$ if they have the same endpoints (this was pointed out in the proof of Corollary 10.9). It follows that $\xi \sim \xi_{r^{\prime} / s^{\prime}}^{\prime}$, where $\xi_{r^{\prime} / s^{\prime}}^{\prime}$ is one of the two geodesic core arcs of $\tau_{r^{\prime} / s^{\prime}}$. In particular, $\xi_{r^{\prime} / s^{\prime}}^{\prime}=\wp\left(\ell_{r^{\prime} / s^{\prime}}\right)$ for a line $\ell_{r^{\prime} / s^{\prime}} \subset \mathbb{C}$ that contains a point in $\mathbb{Z}^{2}$. Note that $\xi_{r^{\prime} / s^{\prime}}^{\prime}$ possibly differs from the geodesic arc $\xi_{r^{\prime} / s^{\prime}}$ as in the statement (if $\xi_{r^{\prime} / s^{\prime}}^{\prime}$ and $\xi_{r^{\prime} / s^{\prime}}$ lie in different components of $\mathbb{P} \backslash \tau_{r^{\prime} / s^{\prime}}$ ).

If $\widetilde{\xi}: \mathbb{I} \rightarrow \mathbb{C}$ is a lift of $\xi$ under $\wp$ in suitable orientation, and $z_{1}:=\widetilde{\xi}(0) \in \mathbb{Z}^{2}, w_{1}:=\widetilde{\xi}(1) \in \mathbb{Z}^{2}$, then it follows from Corollary 10.9 that $w_{1}-z_{1}=\omega^{\prime}=s^{\prime}+i r^{\prime}$. Now (10.6) implies that there are exactly $N=|n|=\left|r s^{\prime}-s r^{\prime}\right|$ lines in $\mathcal{F}_{1} \cup \mathcal{F}_{2}$ that separate $z_{1}$ and $w_{1}$ (essentially, this follows from the fact that the set $[0, n] \cap\left\{2 j \pm t_{0}: j \in \mathbb{Z}\right\}$, where $t_{0} \in(0,1)$, contains precisely $N=|n|$ points).

Let $\widetilde{\omega}^{\prime}:=-p^{\prime}+i q^{\prime}$, where $p^{\prime}, q^{\prime} \in \mathbb{Z}$ and $p^{\prime} r^{\prime}+q^{\prime} s^{\prime}=1$. Then, by the discussion before Lemma 10.7, the map $\wp$ sends one of the segments [ $z_{1}, w_{1}$ ] and [ $\left.z_{1}+\widetilde{\omega}^{\prime}, w_{1}+\widetilde{\omega}^{\prime}\right]$ homeomorphically onto $\xi_{r^{\prime} / s^{\prime}}$ depending on whether $\xi_{r^{\prime} / s^{\prime}}^{\prime}=\xi_{r^{\prime} / s^{\prime}}$ or $\xi_{r^{\prime} / s^{\prime}}^{\prime} \neq \xi_{r^{\prime} / s^{\prime}}$, respectively. In either case, each segment meets exactly $N$ lines in $\mathcal{F}_{1} \cup \mathcal{F}_{2}$.

Arguing as before, we see that

$$
\#\left(\tau_{r / s} \cap \xi_{r^{\prime} / s^{\prime}}\right)=N \leq \#\left(\wp^{-1}\left(\tau_{r / s}\right) \cap \widetilde{\xi}\right)=\#\left(\tau_{r / s} \cap \xi\right) .
$$

This leads to $N \leq \mathrm{i}(\alpha, \xi) \leq \#\left(\tau_{r / s} \cap \xi_{r^{\prime} / s^{\prime}}\right)=N$. So we have equality here and the statement follows.
(iii) (v) These are special cases of (i) and (ii). For example, $a=\wp(\mathbb{R} \times\{0\})=\wp\left(\ell_{0}(0)\right)$ is a core arc of $\alpha^{h}$ corresponding to slope $r^{\prime} / s^{\prime}=0 / 1=0$. Hence by (ii) we have

$$
\mathrm{i}(\alpha, a)=|r \cdot 1-s \cdot 0|=|r|=\#\left(\tau_{r / s} \cap \xi_{0}\right)=\#\left(\tau_{r / s} \cap a\right) .
$$

The other statements follow from similar considerations.

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Department of Mathematics, University of California, Los Angeles, CA 90095, USA
Email address: mbonk@math.ucla.edu
Aix-Marseille Université, Institut de Mathéatiques de Marseille, 13009 Marseille, France
Mathematisch Instituut, Universiteit Utrecht, 3508 TA Utrecht, The Netherlands
Email address: m.hlushchanka@uu.nl
Department of Mathematics, University of Fribourg, 1700 Fribourg, Switzerland
Department of Mathematics, University of California, Los Angeles, CA 90095, USA
Email address: annina.iseli@math.ucla.edu


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