Derivations of negative degree on quasihomogeneous isolated complete intersection singularities.

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DERIVATIONS OF NEGATIVE DEGREE ON QUASIHOMOGENEOUS ISOLATED COMPLETE INTERSECTION SINGULARITIES

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ABSTRACT. J. Wahl conjectured that every quasihomogeneous isolated normal singularity admits a positive grading for which there are no derivations of negative weighted degree. We confirm his conjecture for quasihomogeneous isolated complete intersection singularities of either order at least 3 or embedding dimension at most 5. For each embedding dimension larger than 5 (and each dimension larger than 3), we give a counter-example to Wahl's conjecture.

CONTENTS

INTRODUCTION

By a singularity we mean a quotient A of a convergent power series ring over a valued field K of characteristic zero (see §1). We use the acronym *negative derivation* for a derivation of negative weighted degree on a quasihomogeneous singularity. The question of existence of such negative derivations has important consequences in rational homotopy theory (see [Mei82, Thm. A]) and in deformation theory (see [Wah82, Thm. 3.8]).

By a result of Kantor [Kan79], quasihomogeneous curve and hypersurface singularities do not admit any negative derivations. J. Wahl [Wah82, Thm. 2.4, Prop. 2.8] reached the same conclusion in (the much deeper) case of quasihomogeneous normal surface singularities. Motivated by his cohomological characterization of projective space in [Wah83a], he formulates the following conjecture in [Wah83b, Conj. 1.4].

Conjecture (Wahl)**.** *Let R be a normal graded ring, with isolated singularity. Then there is a normal graded $\bar{R}$, with $\hat{R} \cong \hat{\bar{R}}$, so that $\bar{R}$ has no derivations of negative weight.*

In case R is a graded normal locally complete intersection with isolated singularity, $\hat{R}$ becomes a quasihomogeneous normal isolated complete intersection singularity (ICIS) and Wahl's conjecture can be rephrased as follows (see Lemma 5 and Remark 7).

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Conjecture (Wahl, ICIS case). *Any quasihomogeneous normal ICIS has no negative derivations with respect to some positive grading.*

For quasihomogeneous normal ICIS, there is an explicit description of all derivations due to Kersken [Ker84]. Based on this description, we prove our main

Theorem 1. *For any quasihomogeneous normal ICIS of order at least 3 there are no negative derivations with respect to any positive grading.*

Proof. This follows from Corollary 12 and Proposition 16. □

Our investigations lead to a family of counter-examples to Wahl's Conjecture. In order to describe it, we first fix our notation. A quasihomogeneous singularity can be represented as

$$A = P/\mathfrak{a}, \quad \mathfrak{a} = \langle g_1, \dots, g_t \rangle \trianglelefteq K\langle\langle x_1, \dots, x_n \rangle\rangle =: P \tag{0.1}$$

where $g_1, \dots, g_t$ are homogeneous polynomials of degree $p_i := \deg(g_i)$ with respect to weights $w_1, \dots, w_n \in \mathbb{Z}_+$ on the variables $x_1, \dots, x_n$ (see §1). We order these weights and degrees decreasingly as

$$\begin{aligned} w_1 \geq \cdots \geq w_n > 0, \\ p_1 \geq \cdots \geq p_t. \end{aligned} \tag{0.2}$$

Example 2. Let $n \geq 6$ and pick $c_7, \dots, c_n \in K \setminus \{1\}$ pairwise different such that $c_i^9 + 1 \neq 0$ for all i. Assigning weights $8, 8, 5, 2, \dots, 2$ to the variables $x_1, \dots, x_n$, the equations

$$\begin{aligned} g_1 :=& x_1x_4 + x_2x_5 + x_3^2 - x_4^5 + \sum_{i=7}^{n} x_i^5 \\ g_2 :=& x_1x_5 + x_2x_6 + x_3^2 + x_6^5 + \sum_{i=7}^{n} c_i x_i^5 \end{aligned} \tag{0.3}$$

define a quasihomogeneous complete intersection A as in (0.1) with isolated singularity. On A there is a derivation

$$\eta := \begin{vmatrix} \partial_1 & \partial_2 & \partial_3 \\ x_4 & x_5 & 2x_3 \\ x_5 & x_6 & 2x_3 \end{vmatrix} = 2x_3(x_5 - x_6)\partial_1 - 2x_3(x_4 - x_5)\partial_2 + (x_4x_6 - x_5^2)\partial_3 \tag{0.4}$$

of degree -1. We work out the details of this example in §4.

We show that Example 2.8 gives a counter-example to the ICIS case of Wahl's conjecture of minimal embedding dimension $n = 6$.

Theorem 3. *Exactly up to embedding dimension* 5*, all quasihomogeneous ICIS have no negative derivations with respect to some positive grading.*

Proof. This follows from Kantor [Kan79], [Wah82, Thm. 2.4, Prop. 2.8], Proposition 18, Example 2 and Corollary 12. □

As a consequence of our arguments we obtain a simple special case of the following conjecture due to S. Halperin.

Conjecture (Halperin). *On any graded zero-dimensional complete intersection there are no negative derivations.*

The following result bounds the degree of negative derivations (see also [Ale91, Prop.]). The bound does not require a complete intersection hypothesis and it is independent of further hypotheses as for instance in [Hau02, Thm. 2].

Proposition 4. *For any quasihomogeneous zero-dimensional singularity A as in (0.1) there are no derivations of degree strictly less than $p_n - p_1$. In particular, Halperin's conjecture holds true if $p_1 = p_n$.*

Proof. As A is assumed to be zero-dimensional, condition $\mathfrak{A}(k)$ on page 4 must hold true for all $k = 1, \dots, n$. Then the claim follows from Remark 14 and Lemma 15. □

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1. Graded analytic algebras

Consider a (local) analytic algebra $A = (A, \mathfrak{m}_A)$ over a (possibly trivially) valued field K of characteristic zero. We assume in addition that A is non-regular and can be represented as a quotient $A = P/\mathfrak{a}$ of a convergent power series ring $P := K\langle\langle x_1, \dots, x_n\rangle\rangle \trianglerighteq \mathfrak{a}$. In the sequel such an A will be referred to as a *singularity*. We choose n minimal such that $n = \operatorname{embdim} A$ and set $d := \dim A$.

A K_+-grading on A is given by a *diagonalizable derivation* $\chi \in \operatorname{Der}_K A =: \Theta_A$ which means that $\mathfrak{m}_A$ is generated by eigenvectors $x_1, \dots, x_n$ (see [SW73, (2.2),(2.3)]). Such a derivation is also called an *Euler derivation.* We refer to $w_1, \dots, w_n$ defined by $w_i := \chi(x_i)/x_i$ as the *eigenvalues of* χ. More generally, we call χ-eigenvectors $f \in A$ *homogeneous* and define their *degree* to be the corresponding eigenvalue denoted by $\deg(f) := \chi(f)/f \in k$. We denote by A_a the K-vector space of all such eigenvector $f \in A$ with $\deg(f) = a$. This defines a K-subalgebra

$$\bar{A} := \bigoplus_{a \in K} A_a \subset A \subset \hat{A}. \tag{1.1}$$

The derivation $\chi \in \Theta_A$ lifts to $\chi \in \Theta_P := \operatorname{Der}_K P$ (see [SW73, (2.1)]). In particular, P is K_+-graded and $\mathfrak{a} \trianglelefteq P$ is a χ-invariant ideal and hence homogeneous (see [SW73, (2.4)]). Pick homogeneous $g_1, \dots, g_t \in \mathfrak{a}$ inducing a K-vector space basis of $\mathfrak{a}/\mathfrak{m}_A\mathfrak{a}$. Then $\mathfrak{a} = \langle g_1, \dots, g_t\rangle$ by Nakayama's Lemma. We set $p_i := \deg(g_i)$ ordered as in (0.2). To summarize, we can write A as in (0.1).

A K_+-grading is called a positive grading if $w_i \in \mathbb{Z}_+$ for all $i = 1, \dots, n$ (see [SW73, §3, Def.]). We call A *quasihomogeneous* if it admits a positive grading. In this case, we shall always normalize χ to make the w_i coprime and order the variables according to (0.2). Positivity of weights enforces $g_i \in \bar{P} = K[x_1, \dots, x_n]$ and that

$$\bar{A} = \bigoplus_{i \geq 0} A_i = \bar{P}/\bar{\mathfrak{a}}, \quad \bar{\mathfrak{a}} = \langle g_1, \dots, g_t\rangle \trianglelefteq K[x_1, \dots, x_n] = \bar{P}, \tag{1.2}$$

is a (positively) graded-local k-algebra with completion

$$\hat{\bar{A}} = \hat{A} \tag{1.3}$$

and graded maximal ideal $\mathfrak{m}_{\bar{A}} = \bar{\mathfrak{m}}_A := \bigoplus_{i>0} A_i$. The preceding discussion enables us to reformulate Wahl's Conjecture in the language of Scheja and Wiebe as follows.

Lemma 5. *The following supplementary structures on a singularity A are equivalent:*

(1) an Euler derivation χ on A with positive eigenvalues,
(2) a positive grading on A,
(3) a positive grading on $\hat{A}$,
(4) a (positively) graded K-algebra $\bar{A}$ such that $\hat{\bar{A}} = \hat{A}$.

Proof. The equivalences of (1), (2), and (3) are due to Scheja and Wiebe (see [SW73, (2.2),(2.3)] and [SW77, (1.6)]). For the equivalence with (4), note that the obvious Euler derivation on a graded K-algebra $\bar{A}$ lifts to an Euler derivation on the completion $\hat{\bar{A}} = \hat{A}$. The converse follows from from (1.1), (1.2) and (1.3). □

Let us assume now that A is an isolated complete intersection singularity (ICIS). We may then take $g_1, \dots, g_t$ to be a regular sequence and $d + t = n$. The isolated singularity hypothesis can be expressed in terms of the Jacobian ideal

$$J_A := \left\langle \left| \frac{\partial g}{\partial x_\nu} \right| \mid |\nu| = t \right\rangle \trianglelefteq A \tag{1.4}$$

of A as follows.

Proposition 6. *A complete intersection singularity A is isolated if and only if J_A is $\mathfrak{m}_A$-primary. An analogous statement holds for $\bar{A}$.*

Proof. We denote by $\Omega^1_{A/k}$ the universally finite module of differentials of A over k. By the standard sequence

$$\mathfrak{a}/\mathfrak{a}^2 \longrightarrow A \otimes_P \Omega^1_{P/k} \longrightarrow \Omega^1_{A/k} \longrightarrow 0,$$

the Jacobian ideal J_A is the 0th Fitting ideal $F^0_A \Omega^1_{A/k}$. By [SS72, (6.4),(6.9)], reducedness of A is equivalent to $\operatorname{rk} \Omega^1_{A/k} = d$ and $A_\mathfrak{p}$ is regular if and only if $\Omega^1_{A_\mathfrak{p}/k}$ is free. Hence, $A_\mathfrak{p}$ being regular is equivalent to $\mathfrak{p} \not\supset F^0_A \Omega^1_{A/k} = J_A$ by [BH93, Lem. 1.4.9]. In particular, A having an isolated singularity means exactly that A/J_A is supported at $\mathfrak{m}_A$ and hence that J_A is $\mathfrak{m}_A$-primary as claimed. The analogous statement for $\bar{A}$ is proved similarly. □

Remark 7. Let A be a quasihomogeneous singularity. By (1.2),

$$J_{\bar{A}} := \bar{J}_A = \left\langle \left| \frac{\partial g}{\partial x_\nu} \right| \mid |\nu| = t \right\rangle \trianglelefteq \bar{A} \tag{1.5}$$

is the Jacobian ideal of $\bar{A}$ defined analogous to (1.4). By (1.3), A is a complete intersection if and only if $\bar{A}$ is locally a complete intersection (see [BH93, Def. 2.3.1, Ex. 2.3.21.(c)]). By Proposition 6, A is an ICIS if and only if J_A is $\mathfrak{m}_A$-primary. This is equivalent to $J_{\bar{A}}$ being $\mathfrak{m}_{\bar{A}}$-primary. The latter is then equivalent to $\bar{A}$ being locally a complete intersection with isolated singularity by (1.5) and Proposition 6. Complete intersections are Cohen–Macaulay and hence (S_2) so normality is equivalent to (R_1) by Serre's Criterion (see [BH93, §2.3, Thm. 2.2.22]). Since $d = \dim A = \dim \bar{A}$ by (1.3) (see [BH93, Cor. 2.1.8]), normality for both A and $\bar{A}$ reduces to $d \geq 2$.

Scheja and Wiebe [SW77, (3.1)] (see also [Sai71, Satz 1.3]) proved that any K_+-graded ICIS is quasihomogeneous unless $t = 1$ and $g_1 \notin \mathfrak{m}_P^3$. Their starting point (see [SW77, (2.5)] and [Sai71, Lem. 1.5]) is that A being an ICIS implies, by Proposition 6, that for each $k = 1, \dots, n$ one of the following two conditions must holds true.

$\mathfrak{A}(k)$ For some $m \geq 2$ and $1 \leq j \leq t$, the monomial x_k^m occurs in g_j.

$\mathfrak{B}(k)$ For some pairwise different $1 \leq \nu_1, \dots, \nu_t \leq n$, each g_j contains a monomial $x_k^{m_j} x_{\nu_j}$ for some $m_j \geq 1$.

The following result gives numerical constraints for A to be a quasihomogeneous ICIS.

Lemma 8. *If A is a quasihomogeneous ICIS then*

$$p_1 + \cdots + p_j \geq w_1 + \cdots + w_j + j \tag{1.6}$$

for all $j = 1, \dots, t$.

Proof. We proceed by induction on j. Assume that $p_1+\cdots+p_{j-1} \geq w_1+\cdots+w_{j-1}+j-1$ but $p_1+\cdots+p_j \leq w_1+\cdots+w_j+j-1$. Then $p_j \leq w_j$ and hence $g_i = g_i(x_{j+1},\ldots,x_n)$ for all $i = j,\ldots,n$. Then J_A maps to zero in

$$A/\langle x_{j+1},\ldots,x_n\rangle = K\langle\langle x_1,\ldots,x_j\rangle\rangle/\langle g_1,\ldots,g_{j-1}\rangle$$

and hence J_A cannot be $\mathfrak{m}_A$-primary as required by Proposition 6. □

2. Negative derivations

Let A be a quasihomogeneous singularity as in §1. The target of our investigations is the positively graded A-module $\Theta_A = \operatorname{Der}_K A$ of K-linear derivations on A. More precisely, we are concerned with the question whether its negative part

$$\Theta_{A,<0} = \Theta_{\bar{A},<0} = \bigoplus_{i<0} \Theta_{A,i}$$

is trivial. A priori this condition depends on the choice of a grading. In Proposition 9 below, we shall prove the independence of this choice for a general singularity under a strong hypothesis satisfied in the ICIS case (see Corollary 12). To this end, we write (see [SW73, (2.1)])

$$\Theta_A = \Theta_{\mathfrak{a}\subset P}/\mathfrak{a}\Theta_P \tag{2.1}$$

as a quotient of a (k,P)-Lie algebra

$$\Theta_{\mathfrak{a}\subset P} := \{\delta \in \Theta_P \mid \delta\mathfrak{a} \subset \mathfrak{a}\} \trianglerighteq \mathfrak{a}\Theta_P$$

of *logarithmic derivations* along $\mathfrak{a}$ by the (k,P)-Lie ideal $\mathfrak{a}\Theta_P$.

Proposition 9. *Let A be a quasihomogeneous singularity with positive grading given by χ and assume that*

$$\Theta_{\mathfrak{a}\subset P} = P\chi + \Theta'_P + \mathfrak{a}\Theta_P, \tag{2.2}$$

$$\textit{for some} \quad \Theta'_P \subset \mathfrak{m}_P^2\Theta_P. \tag{2.3}$$

Then the condition $\Theta_{A,<0} = 0$ and the $p_1,\ldots,p_t$ in (0.2) are independent of the chosen positive grading.

Proof. Consider a second positive grading with corresponding Euler derivation χ' (see Lemma 5). By (2.1) and (2.2), any $\delta \in \Theta_A$ lifts to an element of $\Theta_{\mathfrak{a}\subset P}$ of the form

$$\delta = c\chi + \delta_+, \quad \delta_+ = a\chi + \eta, \quad c \in K, \quad a \in \mathfrak{m}_P, \quad \eta \in \Theta'_P, \tag{2.4}$$

denoted by the same symbol. By (2.3) and the Leibniz rule,

$$\chi\mathfrak{m}_P^k \subset \mathfrak{m}_P^k, \quad \delta_+\mathfrak{m}_P^k \subset \mathfrak{m}_P^{k+1} \tag{2.5}$$

for all $k \geq 1$. Specializing to $\delta = \chi$, this implies that $\chi_+ = 0$ and $\chi' = c\chi$ on $\mathfrak{m}_A/\mathfrak{m}_A^2 = \mathfrak{m}_P/\mathfrak{m}_P^2$ and hence $c = 1$ by the definition of a positive grading and our normalization of weights.

Using (2.1), we equip Θ_A with the decreasing $\mathfrak{m}_P$-adic filtration $F^\bullet$ induced from Θ_P which is defined as follows

$$F^k\Theta_A = (\Theta_{\mathfrak{a}\subset P} \cap \mathfrak{m}_P^k\Theta_P)/(\mathfrak{a}\Theta_P \cap \mathfrak{m}_P^k\Theta_P).$$

Due to (2.3), (2.4) and (2.5) this is a filtration by (k,P)-Lie ideals and

$$\delta_+F^k\Theta_A \subset F^{k+1}\Theta_A$$

for the adjoint action of δ_+. Therefore, for any $k \geq 1$, the adjoint action of $\chi' = \chi + \chi_+$ on the truncation

$$F^{\leq k}\Theta_A := \Theta_A / F^{k+1}\Theta_A$$

is triangularizable with semisimple part equal to that of χ. Thus, χ' and χ have the same eigenvalues on $F^{\leq k}\Theta_A$ for any $k \geq 1$. The first claim then follows by choosing k sufficiently large. A similar argument yields the second claim. □

For a Gorenstein singularity A, there is a natural way to produce elements of Θ_A. The A-submodule $\Theta'_A \subset \Theta_A$ of *trivial derivations* is by definition the image of the inclusion

$$\Omega^{d-1}_{A/K} \hookrightarrow \omega^{d-1}_{A/K} = \mathrm{Hom}_A(\Omega^1_{A/K}, \omega^d_{A/K}) = \Theta_A \otimes_A \omega^d_{A/K} \cong \Theta_A. \tag{2.6}$$

We return to the case of an ICIS singularity A. For $1 \leq \nu_0 < \cdots < \nu_t \leq n$ with complementary indices $1 \leq \mu_1 < \cdots < \mu_{d-1} \leq n$, the lift to P of the image of $dx_{\mu_1} \wedge \cdots \wedge dx_{\mu_{d-1}}$ can be written (up to sign) explicitly as

$$\delta_\nu := \begin{vmatrix} \partial_{\nu_0} & \cdots & \partial_{\nu_t} \\ \partial_{\nu_0} g_1 & \cdots & \partial_{\nu_t} g_1 \\ \vdots & & \vdots \\ \partial_{\nu_0} g_t & \cdots & \partial_{\nu_t} g_t \end{vmatrix}. \tag{2.7}$$

Note that

$$\deg \delta_\nu = p_1 + \cdots + p_t - w_{\nu_0} - \cdots - w_{\nu_t}, \tag{2.8}$$

$$\delta_\nu g_j = 0 \tag{2.9}$$

for all $j = 1, \ldots, t$ and ν. Consider the P-module

$$\Theta'_P := \langle \delta_\nu \mid 1 \leq \nu_0 < \cdots < \nu_t \leq n \rangle_P \subset \Theta_P. \tag{2.10}$$

The key to our investigations is the following result due to Kersken [Ker84, (5.2)]. From now on we assume in addition that A is quasihomogeneous and normal, that is, $\dim A \geq 2$.

Theorem 10 (Kersken). *Let A be a quasihomogeneous normal ICIS. Then the module Θ_A of K-linear derivations on A is generated by the Euler derivation χ and the trivial derivations Θ'_A.*

Although Kersken only states that Θ'_A is minimally generated by the δ_ν in (2.7), his arguments show that together with χ they form a minimal set of generators of Θ_A. We denote by $\mu(-)$ the minimal number of generators.

Corollary 11. *Let A be quasihomogeneous normal ICIS. Then Θ_A is minimally generated by the Euler derivation χ and the trivial derivations δ_ν in (2.7). In particular,*

$$\mu(\Theta_A) = \binom{n}{t+1} + 1.$$

Proof. Since the case $d = 2$ is covered by [Wah87, Prop. 1.12], we may assume that $d \geq 3$. In this case, the inclusion (2.6) fits into the following commutative diagramm with exact rows and columns (see [Ker84, Proof of (4.8)] or [Wah87, Prop. 1.7]).

(2.11)
$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & H^1_{\mathfrak{m}_A}(\Omega^d_{A/K}) & \underset{\cong}{\xrightarrow{\chi}} & H^1_{\mathfrak{m}_A}(\Omega^{d-1}_{A/K}) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \omega^d_{A/K} & \xrightarrow{\chi} & \omega^{d-1}_{A/K} & \xrightarrow{\chi} & \omega^{d-2}_{A/K} \\
 & & \uparrow & & \uparrow & & \uparrow\cong \\
0 & \longrightarrow & \Omega^d_{A/K} & \xrightarrow{\chi} & \Omega^{d-1}_{A/K} & \xrightarrow{\chi} & \Omega^{d-2}_{A/K} \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & & 0
\end{array}
$$

It follows that

$$\chi(\omega^{d-1}_{A/K}) \cong \chi(\Omega^{d-1}_{A/K}) \cong \Omega^{d-1}_{A/K}/\chi(\Omega^d_{A/K})$$

where $\chi(\Omega^d_{A/K}) \subset \mathfrak{m}_A\Omega^{d-1}_{A/K}$ and hence

$$\mu(\chi(\Omega^{d-1}_{A/K})) = \mu(\Omega^{d-1}_{A/K}) = \mu(\Theta'_A).$$

Now the middle row of (2.11) yields an exact sequence

$$0 \longrightarrow A \xrightarrow{\chi} \Theta_A \longrightarrow \chi(\omega^{d-1}_{A/K}) \otimes (\omega^d_{A/K})^{-1} \longrightarrow 0$$

Since $\chi \notin \mathfrak{m}_A\Theta_A$, the claim follows. $\square$

Note that Θ'_P in (2.10) satisfies (2.3) due to (2.7) unless $t = 1$ and $g_1 \notin \mathfrak{m}_P^3$. As a consequence of Proposition 9 and Theorem 10 we therefore obtain the following result. It is crucial for Example 2 to be a counter-example to Wahl's Conjecture.

Corollary 12. *Let A be a quasihomogeneous normal ICIS. Unless $t = 1$ and $g_1 \notin \mathfrak{m}_P^3$, the condition $\Theta_{A,<0} = 0$ and the $p_1, \dots, p_t$ in (0.2) are independent of the choice of a positive grading.* $\square$

We shall now derive numerical constraints for minimal negative trivial derivations. To this end, suppose that $0 \neq \eta \in \Theta_{A,<0}$. For reasons of degree (see (0.2)), η can be written as

$$\eta = q_1\partial_1 + \cdots q_n\partial_n, \quad q_i = q_i(x_{i+1}, \dots, x_n) \tag{2.12}$$

By Theorem 10, we may assume that $\eta = \delta_\nu \neq 0$ is a trivial derivation as in (2.7). By (0.2) and (2.8), we may further assume that $\nu_i = i+1$ for $i = 0, \dots, t$. Explicitly, we may write

$$q_i = (-1)^{i-1} \begin{vmatrix} \partial_1 g_1 & \cdots & \widehat{\partial_i g_1} & \cdots & \partial_{t+1} g_1 \\ \vdots & & \vdots & & \vdots \\ \partial_1 g_t & \cdots & \widehat{\partial_i g_t} & \cdots & \partial_{t+1} g_t \end{vmatrix}, \quad q_{t+2} = \cdots = q_n = 0. \tag{2.13}$$

Now (2.8) and (2.9) specialize to the following simple

Lemma 13. *For η as in (2.12) with (2.13), we have*

$$\eta g_j = 0 \tag{2.14}$$

for all $j = 1, \ldots, t$. If $\Theta_{A,<0} \neq 0$ for a quasihomogeneous normal ICIS then

$$p_1 + \cdots + p_t < w_1 + \cdots + w_{t+1}. \tag{2.15}$$

□

Remark 14. For degree reasons (see (0.2)), the identity (2.14) holds true for any $\eta \in \Theta_{A,<p_t - p_1}$ and any quasihomogeneous singularity A as in (0.1).

We now link the conditions $\mathfrak{A}(k)$ and $\mathfrak{B}(k)$ from page 4 to the existence of a negative derivation as in (2.12).

Lemma 15. *Assume that the identity (2.14) holds true for all $j = 1, \ldots, t$. Then $\mathfrak{A}(k)$ implies $q_k = 0$ in (2.12) for a suitable choice of coordinates.*

Proof. Pick $k \in \{1, \ldots, t+1\}$ such that $\mathfrak{A}(k)$ holds. Then some g_j contains x_k^m, $m > 1$, and all other monomials in g_j contain only strictly lower powers of x_k by homogeneity. Let $t_{k,j} = t_{k,j}(x_1, \ldots, \widehat{x_k}, \ldots, x_n)$ denote the coefficient of x_k^{m-1} in g_j, and assume, without loss of generality, that the coefficient of x_k^m is $\frac{1}{m}$. Note that $t_{k,j}$ is independent of variables of weight larger than w_k. Expanding (2.14) with respect to the variable x_k and taking the terms involving x_k^{m-1} gives

$$q_k x_k^{m-1} = q_k \partial_k \left(\frac{1}{m} x_k^m \right) = - \sum_{i \neq k} q_i \partial_i (t_{k,j} x_k^{m-1}) = - \sum_{i \neq k} q_i \partial_i (t_{k,j}) x_k^{m-1}$$

and hence

$$\eta = \sum_{i \neq k} q_i \cdot (\partial_i - \partial_i(t_{k,j}) \partial_k). \tag{2.16}$$

The χ-homogeneous coordinate change

$$x_i' = \begin{cases} x_k + t_{k,j}, & \text{if } i = k, \\ x_i, & \text{else.} \end{cases}$$

replaces $\partial_i - \partial_i(t_{k,j})\partial_k$ in (2.16) by ∂_i', and thus q_k in (2.12) by 0. Iterating this process yields the claim. □

Our main technical result is the following

Proposition 16. *Let A be a quasihomogeneous normal ICIS such that $\Theta_{A,<0} \neq 0$. Then $\mathfrak{B}(k)$ holds for at least two indices $k \leq t+1$. Each such k satisfies $k \geq t - d + 2$ and $g_k, \ldots, g_t \notin \mathfrak{m}_P^3$.*

Proof. By hypothesis and Lemma 15, $\mathfrak{B}(k)$ holds for some $k \leq t+1$ with $q_k \neq 0$. Assuming that k is unique, (2.9) reads $q_k \partial_k g_j = 0$ which would imply that g_j is independent of x_k for all $j = 1, \ldots, t$. By the isolated singularity hypothesis, this is impossible.

Combining (1.6) and (2.15), we obtain

$$p_j + \cdots + p_t + j \leq w_j + \cdots + w_{t+1} \tag{2.17}$$

for all $j = 1, \ldots, t$. Using (0.2), $\mathfrak{B}(k)$ and (2.17) for $j = k$, we compute

$$\begin{aligned} m_k w_k + \cdots + m_t w_t &\leq (m_k + \cdots + m_t) w_k \\ &= \deg(\partial_{\nu_k} g_k \cdots \partial_{\nu_t} g_t) \\ &= p_k + \cdots + p_t - w_{\nu_k} - \cdots - w_{\nu_t} \\ &\leq w_k + \cdots + w_{t+1} - k - w_{\nu_k} - \cdots - w_{\nu_t}. \end{aligned}$$

and hence

$$(m_k - 1)w_k + \cdots + (m_t - 1)w_t \leq w_{t+1} - k - w_{\nu_k} - \cdots - w_{\nu_t}.$$

By (0.2), this forces

$$\begin{aligned} m_k = \cdots = m_t = 1, \\ w_{t+1} \geq w_{\nu_k} + \cdots + w_{\nu_t} + k. \end{aligned} \tag{2.18}$$

In particular,

$$\nu_k, \dots, \nu_t \geq t + 2 \tag{2.19}$$

and hence $k \geq t - d + 2$. □

3. ICIS of embedding dimension 5

Lemma 17. *Let A be a quasihomogeneous normal ICIS such that $\Theta_{A,<0} \neq 0$. Then $\mathfrak{A}(k_1)$ and $\mathfrak{B}(k_2)$ for $\{k_1, k_2\} = \{1, 2\}$ is impossible.*

Proof. Assuming the contrary, one of the g_j has a monomial divisible by $x_{k_1}^2$ by $\mathfrak{A}(k_1)$ and each of the g_j has a monomial divisible by x_{k_2} by $\mathfrak{B}(k_2)$. In particular,

$$p_1 + \cdots + p_t \geq 2w_{k_1} + (t-1)w_{k_2} \geq w_1 + \cdots + w_{t+1}$$

contradicting (2.15). □

Proposition 18. *For any quasihomogeneous ICIS A as in (0.1) with $n = 5$ and $t = 2$, we have $\Theta_{A,<0} = 0$.*

Proof. Assume that $\Theta_{A,<0} \neq 0$. By Proposition 16 and Lemma 17, we must have $\mathfrak{B}(1)$ and $\mathfrak{B}(2)$. Using (0.2), (2.18), and (2.19), we may write

$$\begin{aligned} g_1 &= x_1x_4 + c_1x_2^jx_{k_1} + \cdots \\ g_2 &= x_1x_5 + c_2x_2x_{k_2} + \cdots \end{aligned}$$

with $\{k_1, k_2\} = \{4, 5\}$ and $c_1, c_2 \in K^*$. As in the proof of Lemma 17, the inequality (2.15) can only hold true if $j = 1$. In this case,

$$A/(J_A + \langle x_3, \dots, x_n \rangle) = K\langle\langle x_1, x_2 \rangle\rangle / \left\langle \left| \frac{\partial g}{\partial(x_4, x_5)} \right| \right\rangle.$$

for degree reasons (see (0.2)), and hence J_A is not $\mathfrak{m}_A$-primary. This contradicts to the isolated singularity hypothesis. □

4. Counter-examples

Proof of Example 2. The sequence g is clearly regular and defines a complete intersection as in (0.1). Note that η in (0.4) agrees with $\eta = \delta_{1,2,3}$ in (2.12). Since $\deg(g_1) = 10 = \deg(g_2)$, (2.9) shows that η has negative degree $\deg \eta = -1$.

It remains to check that A has an isolated singularity, that is, the Jacobian ideal J_A from (1.4) is $\mathfrak{m}_A$-primary. To this end, we may assume that $K = \bar{K}$ which enables us to argue geometrically on the variety

$$\bar{X} := \operatorname{Spec} \bar{A} \subset \mathbb{A}_K^n$$

with $\bar{A}$ as in (1.2) using the Nullstellensatz.

The ideal J_A is the image in A of the Jacobian ideal $\bar{J}_g \trianglelefteq \bar{P}$ of g generated by the 2×2-minors

$$M_{i,j} := \left| \frac{\partial g}{\partial(x_i x_j)} \right|$$

of the Jacobian matrix of g which reads

$$\frac{\partial g}{\partial x} = \begin{pmatrix} x_4 & x_5 & 2x_3 & x_1 - 5x_4^4 & x_2 & 0 & 5x_7^4 & \cdots & 5x_n^4 \\ x_5 & x_6 & 2x_3 & 0 & x_1 & x_2 + 5x_6^4 & 5c_7x_7^4 & \cdots & 5c_nx_n^4 \end{pmatrix}.$$

With this notation we have to show that

$$\operatorname{Sing}\bar{X} = V(g, \bar{J}_g) = \{0\}.$$

Due to those 2×2-minors of $\frac{\partial g}{\partial x}$ which involve only the columns $3, 7, 8, 9, \ldots, n$, only one of components $x_3, x_7, x_8, x_9, \ldots, x_n$ of any $x \in \operatorname{Sing}\bar{X}$ can be non-zero. We may therefore reduce to the case $n \leq 7$.

Because of the 3rd column of $\frac{\partial g}{\partial x}$, we have $\bar{J}_g \cap K[x_1, \ldots, x_6] \supseteq x_3 I$ where

$$I := \langle x_4 - x_5, x_5 - x_6, x_1 - x_2, x_1 - 5x_4^4, x_2 + 5x_6^4 \rangle.$$

Note that $V(I)$ is the x_3-axis which is not contained in $V(g)$. It follows that $\operatorname{Sing}\bar{X} \cap V(x_7)$ is contained in the hyperplane $V(x_3)$. Similarly because of the 7th column of $\frac{\partial g}{\partial x}$ and setting $c := c_7$, we have $\bar{J}_g \cap K[x_1, \ldots, \widehat{x_3}, \ldots, x_7] \supseteq x_7 I'$ where

$$I' := \langle cx_4 - x_5, cx_5 - x_6, cx_2 - x_1, x_1 - 5x_4^4, x_2 + 5x_6^4 \rangle.$$

Using $c^9 + 1 \neq 0$, we find that $V(I')$ is the x_7-axis and conclude $\operatorname{Sing}\bar{X} \cap V(x_3) \subset V(x_7)$ as before. Summarizing the two cases, $\operatorname{Sing}\bar{X}$ is in fact contained in $V(x_3, x_7)$.

Fix a point $(x_1, x_2, 0, x_4, x_5, x_6, 0) \in \operatorname{Sing}\bar{X}$. Successively using the the equations

$$\begin{aligned} M_{1,2} &= x_4x_6 - x_5^2 = 0, \\ M_{2,5} &= x_1x_5 - x_2x_6 = 0, \\ g_2 &= x_1x_5 + x_2x_6 + x_6^5 = 0, \\ M_{4,5} &= x_1(x_1 - 5x_4^4) = 0, \\ M_{5,6} &= x_2(x_2 + 5x_6^4) = 0, \end{aligned}$$

we derive

$$x_4 = 0 \Rightarrow x_5 = 0 \Rightarrow x_2x_6 = 0 \Rightarrow x_6 = 0 \Rightarrow x_1 = x_2 = 0.$$

Similarly $x_6 = 0$ leaves no possibility except $x = 0$ and $x_5 = 0$ reduces to one of these two cases by $M_{1,2} = 0$.

Assume now that x_4, x_5, x_6 are all non zero. Then the minors $M_{1,5}$, $M_{2,4}$, $M_{2,5}$, $M_{2,6}$ give equations

$$x_1x_4 = x_2x_5, \quad x_1 = 5x_4^4, \quad x_1x_5 = x_2x_6, \quad x_2 = -5x_6^4.$$

Substituting into g, we obtain

$$g_1 = 2x_1x_4 - x_4^5 = 9x_4^5, \quad g_2 = 2x_2x_6 + x_6^5 = -9x_6^5$$

and hence $x_4 = x_6 = 0$ contradicting our assumption. □

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