

Five-Brane Superpotentials and Heterotic/F-theory Duality

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ABSTRACT

Under heterotic/F-theory duality it was argued that a wide class of heterotic five-branes is mapped into the geometry of an F-theory compactification manifold. In four-dimensional compactifications this identifies a five-brane wrapped on a curve in the base of an elliptically fibered Calabi-Yau threefold with a specific F-theory Calabi-Yau fourfold containing the blow-up of the five-brane curve. We argue that this duality can be reformulated by first constructing a non-Calabi-Yau heterotic threefold by blowing up the curve of the five-brane into a divisor with five-brane flux. Employing heterotic/F-theory duality this leads us to the construction of a Calabi-Yau fourfold and four-form flux. Moreover, we obtain an explicit map between the five-brane superpotential and an F-theory flux superpotential. The map of the open-closed deformation problem of a five-brane in a compact Calabi-Yau threefold into a deformation problem of complex structures on a dual Calabi-Yau fourfold with four-form flux provides a powerful tool to explicitly compute the five-brane superpotential.

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1 Introduction

The study of string compactifications leading to $\mathcal{N} = 1$ supersymmetric four-dimensional low-energy effective theories is of conceptual as well as of phenomenological interest. Two prominent approaches to obtain such effective theories are either to consider heterotic $E_8 \times E_8$ string theory on a Calabi-Yau manifold with non-trivial vector bundles, or to study F-theory compactifications on singular Calabi-Yau fourfolds. At first, these two approaches appear to be very different in nature, since the data determining the effective dynamics are encoded by seemingly different objects. However, at least if one focuses on certain compact geometries, the heterotic and F-theory picture are believed to be dual descriptions of the same physics [1, 2]. The dictionary of this duality not only contains the map for the vector bundles of the heterotic string, but also includes heterotic five-branes wrapped on curves in the Calabi-Yau threefold [3, 4, 5]. These are often necessary in a consistent heterotic compactification to ensure anomaly cancellation. Using this duality either of the two descriptions can be used to answer specific questions about the four-dimensional physics. In this work we will focus on parts of the effective action which are efficiently calculable in F-theory, but admit a natural physical interpretation in the heterotic theory.

An important question in the study of the four-dimensional $\mathcal{N} = 1$ low-energy effective action is the explicit computation of the superpotential and gauge-coupling functions which depend holomorphically on the chiral multiplets. In the following we will mainly focus on the study of the superpotential of a heterotic five-brane wrapped on a curve \mathcal{C} in a Calabi-Yau threefold Z_3 . It was shown in ref. [6] that this superpotential depends on the deformation modes of the curve \mathcal{C} and the complex structure moduli of Z_3 via the chain integral $\int_{\Gamma} \Omega$, where Ω is the holomorphic three-form on Z_3 , and Γ is a three-chain which admits \mathcal{C} as a boundary component. We will argue by using heterotic/F-theory duality that this chain integral is mapped to the flux superpotential of an F-theory compactification upon constructing an appropriate Calabi-Yau fourfold \hat{X}_4 encoding the five-brane dynamics, and the associated four-form flux G_4 . The F-theory flux superpotential can then be computed explicitly by solving Picard-Fuchs differential equations determining the closed period integrals of the holomorphic four-form on \hat{X}_4 , and using mirror symmetry to identify the superpotential solution [7]. Earlier discussions and computations of the periods of the holomorphic four-form can be found in refs. [8, 9, 10].

The computation of brane superpotentials given by chain integrals has been of significant interest in the D-brane literature. Starting with [11] the superpotential for D5-branes has been studied intensively for non-compact Calabi-Yau threefolds [12, 13]. More recently, there has been various attempts to extend this to compact threefolds [14, 15, 16, 17, 18, 19, 20, 21].

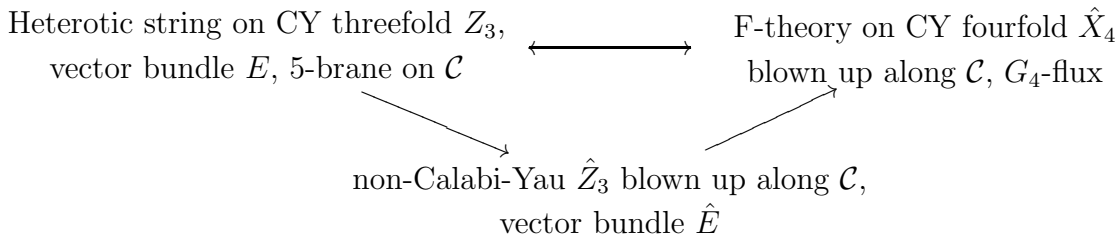
In particular, it was proposed in refs. [13] to use the variation of mixed Hodge structure for an auxiliary divisor capturing the variations of the curve \mathcal{C} . While initially studied in non-compact geometries, extensions to compact Calabi-Yau threefolds with D5-branes have appeared in refs. [16, 19, 20]. In this proposal the deformations of an appropriately chosen divisor are effectively identified with the deformations of the curve. In contrast, it was argued in [18] that the deformation problem of the curve \mathcal{C} in Z_3 admits a natural map to a geometric setup in which the curve is blown up into a divisor. In this case, the divisor is rigid² and the deformations of the curve appear as new complex structure deformations of the blown-up threefold \hat{Z}_3 . In this work we will use the blow-up construction to study the duality of an heterotic five-brane to an F-theory compactification fourfold. Let us note that in refs. [22, 19, 20, 23] it was proposed to use non-compact Calabi-Yau fourfolds to compute the D5-brane superpotential and a connection with F-theory was indicated. Let us stress that the approach we are using here is different in nature, and rather completes the approach initiated in our works [18, 7].

The map of heterotic string theory on a Calabi-Yau threefold Z_3 with five-branes to an F-theory compactification is best studied for elliptically fibered Z_3 . It was shown in ref. [24] that there exist elegant constructions of heterotic vector bundles on these threefolds. Furthermore, a five-brane wrapped on a curve \mathcal{C} in the base B_2 of this elliptic fibration was argued to map entirely into the geometry of an F-theory compactification. Using the adiabatic argument of [25] the heterotic string on Z_3 is equivalent to F-theory on an elliptic K3- fibered Calabi-Yau fourfold X_4 with base B_2 . This implies, in particular, that the three-dimensional base B_3 of the elliptically fibered F-theory fourfold X_4 is a holomorphic \mathbb{P}^1 -fibration over B_2 . It was then argued in refs. [4, 5, 26] that in the presence of a heterotic five-brane one has to blow up the curve \mathcal{C} into a rigid divisor in B_3 . The deformations of the curve \mathcal{C} then map to complex structure deformations of the blown-up Calabi-Yau fourfold \hat{X}_4 , and hence can be constrained by a calculable flux superpotential. Note that certain five-branes can also be interpreted as special gauge bundle configurations of the heterotic string, the so-called small instantons. In the small instanton/five-brane transition the deformation moduli of the curve \mathcal{C} are identified with heterotic bundle moduli. This yields yet another identification of superpotentials, since the five-brane superpotential arises as a localization of the Chern-Simons superpotential for the bundle moduli [11]. Both types of superpotentials are efficiently calculable on the F-theory side using the geometric tools for Calabi-Yau fourfolds.

To study the duality map between the heterotic and F-theory setup, one can alternatively start by blowing up the heterotic threefold Z_3 along the five-brane curve \mathcal{C} into \hat{Z}_3 [18]. This can be made explicit by realizing \hat{Z}_3 as a complete intersection. The non-Calabi-Yau

²More accurately, such a divisor is described as an isolated divisor.

threefold \hat{Z}_3 contains the five-brane moduli as a subsector of its complex structure moduli. The heterotic superpotential crucially depends on the pull-back $\hat{\Omega}$ of the holomorphic three-form Ω to \hat{Z}_3 . Since $\hat{\Omega}$ vanishes along the blow-up divisor D , the heterotic flux, specifying the five-brane, localizes on elements in $H^3(\hat{Z}_3 - D, \mathbb{Z})$. This is equivalent to considering relative three-forms in $H^3(\hat{Z}_3, D, \mathbb{Z})$.³ Identifying the elements of this relative group with elements in the dual fourfold cohomology, one finds an explicit map between the heterotic five-brane and F-theory fluxes. We propose, and explicitly demonstrate for examples, that the F-theory geometry \hat{X}_4 can in turn be entirely constructed from \hat{Z}_3 . In particular, this identification becomes apparent when also realizing \hat{X}_4 as a complete intersection. In this way the complex structure moduli of \hat{Z}_3 naturally form a subsector of the complex structure moduli of \hat{X}_4 . In summary, the general idea of this discussion is to reformulate and slightly extend the heterotic/F-theory duality map schematically as:



where the horizontal arrow indicates the action of heterotic/F-theory duality.

Following this general strategy the paper is organized as follows. In section 2 we first recall the connection between small instantons and heterotic five-branes. This allows us to introduce the respective heterotic superpotentials. Moreover, we discuss the general blow-up procedure of the heterotic Calabi-Yau threefold, and comment on the representation and properties of the holomorphic three-form on the blown-up geometry \hat{Z}_3 . In section 3, we first review the heterotic/F-theory duality, highlighting the map of five-branes into a Calabi-Yau fourfold geometry. We then discuss the F-theory flux superpotential and describe how it is matched with its heterotic counterpart. In the last section we study two classes of examples. Firstly, we discuss the geometrical construction of the heterotic blow-up threefold and its associated Calabi-Yau fourfold constructed as complete intersections. Secondly, we investigate an example for which we explicitly compute the superpotential and confirm the map between five-brane deformations and fourfold complex structure moduli.

³This should be compared with the use of relative cohomology for the auxiliary non-rigid divisor in the constructions of refs. [13, 16, 19, 20].

2 Heterotic Five-Branes and Superpotentials

In this section we review the construction of $\mathcal{N} = 1$ vacua by compactification of the heterotic string on a Calabi-Yau threefold Z_3 with vector bundle E and a number of space-time filling five-branes. We discuss the relation of the bundle moduli and five-brane deformations via a small instanton transition in section 2.1. The heterotic superpotential for these moduli fields will be introduced in section 2.2. It will be argued that the motion of the five-brane inside Z_3 is constraint by a superpotential given by the integral of the holomorphic three-form Ω over a chain ending on the five-brane. In section 2.3 we discuss the general blow-up procedure and some properties of the resulting geometry.

2.1 Transition between Heterotic Vector Bundles and Five-Branes

Let us begin by reviewing the heterotic compactifications. Besides the choice of a Calabi-Yau threefold Z_3 , a consistent heterotic vacuum requires a choice of stable holomorphic vector-bundles $E = E_1 \oplus E_2$ over Z_3 which determine the gauge group preserved in the perturbative $E_8 \times E_8$ of the heterotic theory. In general we can additionally have five-branes wrapping holomorphic curves \mathcal{C} in the threefold Z_3 . This setup is further constrained by the general heterotic anomaly cancellation condition

$$\lambda(E_1) + \lambda(E_2) + [\mathcal{C}] = c_2(Z_3) , \quad (2.1)$$

where $\lambda(E)$ is the fundamental characteristic class of the vector bundle E , which, for example, is $c_2(E)$ for $SU(N)$ bundles and $c_2(E)/60$ for E_8 bundles. This condition dictates consistent choices of the cohomology class $[\mathcal{C}]$ of the curve \mathcal{C} in the presence of non-trivial vector bundles to match the curvature of the threefold Z_3 as measured by the second Chern class $c_2(Z_3)$. In particular, it implies that \mathcal{C} corresponds to an effective class in $H_2(Z_3, \mathbb{Z})$ [27].

The analysis of the moduli space of the heterotic string on Z_3 requires the study of three a priori very different pieces. Firstly, we have the geometric moduli spaces of the threefold Z_3 consisting of the complex structure as well as Kähler moduli space. Secondly, there are the moduli of the bundles E_1 and E_2 which parameterize different gauge-field backgrounds on Z_3 . Finally, if the five-brane is wrapped on a non-rigid curves \mathcal{C} , the deformations of \mathcal{C} within Z_3 of the various five-branes have to be taken into account. The entire moduli space is in general very complicated and difficult to analyze. This problem, however, becomes more tractable if one focuses on elliptically fibered Calabi-Yau threefolds Z_3 . It was shown in ref. [24] that there exist elegant constructions of the vector bundle E on these threefolds. Moreover, the moduli space of five-branes on elliptically fibered Z_3 has been discussed in

great detail in ref. [28]. In general, it admits several different branches corresponding to the number and type of five-branes present. However, there are distinguished points in the moduli space corresponding to enhanced gauge symmetry [29, 30] of the heterotic string that allow for a clear physical interpretation and that we now discuss in more detail. It will turn out that at these points an interesting transition is possible where a five-brane completely dissolves into a finite size instanton of the bundle E and vice versa.

Let us start with a threefold Z_3 with $c_2(Z_3) \neq 0$ and no five-branes. Thus, the anomaly condition (2.1) forces us to turn on a background bundle E with non-trivial second Chern class $c_2(E)$ in order to cancel $c_2(Z_3)$. Then the bundle is topologically non-trivial and carries bundle instantons characterized by the topological second Chern number [31]

$$[c_2] = - \int_{Z_3} J \wedge \mathcal{F} \wedge \mathcal{F}, \quad (2.2)$$

where J denotes the Kähler form on Z_3 and \mathcal{F} the field strength of the background bundle. The heterotic gauge group G_1 in four dimensions is generically broken and given by the commutant of the holonomy group of the bundle E in E_8 . Varying the moduli of E_1 one can ask whether it is possible to restore parts or all of the broken gauge symmetry by flattening out the bundle as much as possible [32]. To show how this can be achieved, one first decomposes $c_2(E)$ into its components each of which being dual to an irreducible curve \mathcal{C}_i in Z_3 . Since the invariant $[c_2]$ has to be kept fixed, the best we can do is to consecutively split off the components of $c_2(E)$ and to localize the curvature of E on the corresponding curves \mathcal{C}_i . This should be contrasted with the generic situation, where the curvature is smeared out all over Z_3 . In the localization limit the holonomy of the bundle around each individual curve \mathcal{C}_i becomes trivial and the gauge group G enhances accordingly. Having reached this so-called small instanton configuration at the boundary of the moduli space of the bundle, the dynamics of (this part of) the gauge bundle can be completely described by a five-brane on \mathcal{C}_i [29].

Small instanton configurations thus allow for transitions between branches of the moduli space with different numbers of five-branes, that consequently map bundle moduli to five-brane moduli and vice versa [33]. This is precisely what we need for our later F-theory analysis. Note that this transition is completely consistent with (2.1) since we have just shifted irreducible components between the two summands $c_2(E)$ and $[\mathcal{C}]$. Thus, we are in the following allowed to think about the small instanton configuration as the presence of a five-brane. In particular, doing this transition for all components of $c_2(E)$ the full perturbative heterotic gauge group $E_8 \times E_8$ can be restored. Turning this argument around, a heterotic string with full $E_8 \times E_8$ gauge symmetry on a threefold Z_3 with non-trivial $c_2(Z_3)$ has to contain five-branes to cancel the anomaly according to (2.1). In our concrete example

of section 4 we will precisely encounter this situation guiding us to the interpretation of the F-theory flux superpotential in terms of a superpotential for a particular class of five-branes.

To precisely specify the five-branes we will consider later, we note that on an elliptically fibered Calabi-Yau threefold the five-brane class $[\mathcal{C}]$ can be decomposed as

$$\mathcal{C} = n_f F + \mathcal{C}_B, \quad (2.3)$$

where \mathcal{C}_B denotes a curve in the base B_2 of the elliptic fibration, F denotes the elliptic fiber, and n_f is a positive integer. This is a split into five-branes vertical to the projection $\pi : Z_3 \rightarrow B_2$, where the integer n_f counts the number of five-branes wrapping the elliptic fiber, and into horizontal five-branes on \mathcal{C}_B in the base B_2 . Both cases are covered by (2.1), but lead to different effects in the F-theory dual theory. Vertical five-branes correspond to spacetime filling three-branes at a point in the base B_3 of the F-theory fourfold X_4 [3, 24]. Conversely, horizontal five-branes on the curve \mathcal{C}_B map completely to the geometry of the F-theory side. They map to seven-branes supported on a divisor in the fourfold base B_3 which projects onto the curve \mathcal{C} in B_2 [2, 34, 35] that has to be blown-up in B_3 into a divisor D [4, 5, 26]. Of course, there can be mixed types of five-branes as well. It will be precisely the horizontal five-branes corresponding to blow-ups into exceptional divisors D for which our analysis and calculation of the superpotential will be performed.

2.2 The Heterotic Superpotential

The small instanton transition implies a transition between bundle and five-brane moduli [33]. Since both types of moduli are generally obstructed by a superpotential also the superpotentials for bundle and five-brane have to be connected by the transition. As was argued in [6] in the context of M-theory on a Calabi-Yau threefold, a spacetime-filling M5-brane supported on a curve \mathcal{C} in general induces a superpotential

$$W_{M5} = \int_{\Gamma} \Omega, \quad (2.4)$$

where Γ denotes a three-chain bounded by \mathcal{C} and an unimportant reference curve \mathcal{C}_0 in the homology class of \mathcal{C} . It depends on both the moduli of the five-brane on \mathcal{C} as well as the complex structure moduli of Z_3 in the holomorphic three-form Ω . On the other hand, the perturbative superpotential for the heterotic bundle moduli is given by the holomorphic Chern-Simons functional [31]

$$W_{CS} = \int_{Z_3} \Omega \wedge (A \bar{\partial} A + \frac{2}{3} A \wedge A \wedge A), \quad (2.5)$$

where A denotes the gauge connection that depends on the bundle moduli. The dependence on the complex structure moduli of Z_3 is implicit through Ω .

To see how the two superpotentials (2.4) and (2.5) are mapped onto each other in the transition, let us assume a single instanton solution \mathcal{F} with $\mathcal{F} \wedge \mathcal{F}$ dual to an irreducible curve \mathcal{C} . Displaying the explicit moduli dependence of the configuration \mathcal{F} [36], in the small instanton limit $\mathcal{F} \wedge \mathcal{F}$ reduces to the delta function $\delta_{\mathcal{C}_i}$ of four real scalar parameters. They describe the position moduli of the instanton normal to the curve in the class $[\mathcal{C}_i]$ on which it is localized. Inserting the gauge configuration \mathcal{F} into W_{CS} , the holomorphic Chern-Simons functional is effectively dimensionally reduced to the curve \mathcal{C} [11]. In the vicinity of \mathcal{C} we may write the holomorphic three-form as $\Omega = d\omega$ which we insert into (2.5) in the background $\mathcal{F} \wedge \mathcal{F}$ to obtain

$$W_{\text{CS}} = \int_{\mathcal{C}} \omega \quad (2.6)$$

after a partial integration. Adding a constant given by the integral of ω over the reference curve \mathcal{C}_0 this precisely matches the chain integral (2.4). Applying the above discussion, we can think about the M5-brane moduli in W_{M5} as the bundle moduli describing the position of the instanton configuration \mathcal{F} , that in the small instanton limit precisely map to sections $H^0(\mathcal{C}_i, N_{Z_3}\mathcal{C}_i)$ of the normal bundle to \mathcal{C}_i .

We will verify this matching of moduli explicitly from the perspective of the F-theory dual setup later on. There we will on the one hand identify some of the fourfold complex structure moduli with the heterotic bundle moduli, on the other hand, however, show that part of the F-theory flux superpotential depending on the same complex structure moduli really calculates the superpotential of a five-brane on a curve. This way, employing heterotic/F-theory duality, we show in the case of an example the equivalence of the small instanton/five-brane picture.

To complete the discussion of perturbative heterotic superpotentials, let us also comment on the flux superpotential due to bulk fluxes. In general, the heterotic B -field can have a non-trivial background field strength H_3^{flux} that has to be in $H^3(Z_3, \mathbb{Z})$ due to the flux quantization condition. The induced superpotential will be intimately linked to (2.4) and (2.5) due to the anomaly cancellation condition

$$dH_3 = \text{Tr}(\mathcal{R} \wedge \mathcal{R}) - \text{Tr}(\mathcal{F} \wedge \mathcal{F}) - \sum_i \delta_{\mathcal{C}_i} , \quad (2.7)$$

which yields, with an appropriate definition of the traces, the condition (2.1) if one restricts to cohomology classes. The superpotential in terms of this H_3 reads [37, 38]

$$W_{\text{het}} = \int_{Z_3} \Omega \wedge H_3 = W_{\text{flux}} + W_{\text{CS}} + W_{\text{M5}} , \quad (2.8)$$

where the different terms can be associated to the various contributions in H_3 in (2.7). In order to discuss the flux part, we expand $H_3^{\text{flux}} = N^i \alpha_i - M_i \beta^i$ in the integral basis α_i, β^i of $H^3(Z_3, \mathbb{Z})$ with integer flux numbers N^i, M_i . Then one can write the flux superpotential as

$$W_{\text{flux}} = \int_{Z_3} \Omega \wedge H_3^{\text{flux}} = M_i X^i - N^i F_i, \quad (2.9)$$

where we introduced the period expansion $\Omega = X^i \alpha_i - F_i \beta^i$. In general, the periods (X^i, F_i) admit a complicated dependence on the complex structure deformations of Z_3 . It is the great success of algebraic geometry that this superpotential can be calculated explicitly for a wide range of examples, see [39] and [40, 41] for reviews. This is due to the fact that the periods X^i, F_i obey differential equations, the so-called Picard-Fuchs equations⁴, that can be solved explicitly and thus allow to determine the complete moduli dependence of W_{flux} . To end our discussion of the flux superpotential, let us stress that strictly speaking there is a back-reaction of H_3^{flux} which renders Z_3 to be non-Kähler [43]. Since our main focus will be on the five-brane superpotential, we will not be concerned with this back-reaction in the following.

2.3 The Blow-Up of the Heterotic Calabi-Yau Threefold

The form of the superpotential W_{M5} of (2.4) is rather universal. It occurs, for example, also for D5-branes on curves in Type IIB orientifold compactifications.⁵ In the following we will apply the blow-up procedure suggested in ref. [18] for the study of the chain integral for D5-branes to the heterotic setup. The idea is to find a purely geometric description that puts the dynamics of the five-brane and the geometry of Z_3 on an equal footing. To achieve this, we blow up the curve \mathcal{C} into a rigid divisor D in a non-Calabi-Yau threefold \hat{Z}_3 . This embeds the deformation modes of \mathcal{C} in Z_3 as well as the complex structure deformations of Z_3 into the deformation problem of only complex structures of \hat{Z}_3 . We will see explicitly later that this alternative view on the heterotic string with five-branes allows for a direct geometric interpretation of the fate of the five-brane dynamics in heterotic/F-theory duality. Here we provide the geometrical tools to describe the blow-up of Z_3 along a curve \mathcal{C} which we will later use in the construction of explicit examples in section 4.

For concreteness, let us consider a Calabi-Yau threefold Z_3 described as the hypersurface $\{P = 0\}$ in a projective or toric ambient space V_4 . Consider then a curve \mathcal{C} specified by two additional constraints $\{h_1 = h_2 = 0\}$ in the ambient space intersecting transversally Z_3 . For

⁴See for example [42] for a review.

⁵Similar expressions arise for higher dimensional branes with world-volume flux inducing D5-charge supported on the same curves.

the example of horizontal curves \mathcal{C} supporting horizontal five-branes in an elliptic Z_3 , the constraints take the form

$$h_1 \equiv \tilde{z} = 0, \quad h_2 \equiv g_5 = 0, \quad (2.10)$$

where $\{\tilde{z} = 0\}$ restricts to the base B_2 and g_5 specifies \mathcal{C} within B_2 . In general, the constraints h_1, h_2 describe divisors in the ambient space that descend to divisors⁶ in Z_3 as well upon intersecting with $\{P = 0\}$, called D_1 and D_2 . Locally, (h_1, h_2) can be considered as normal coordinates to the curve \mathcal{C} in Z_3 . Thus, the normal bundle $N_{Z_3}\mathcal{C}$ of the curve takes the form $N_{Z_3}\mathcal{C} = \mathcal{O}_{Z_3}(D_1) \oplus \mathcal{O}_{Z_3}(D_2)$, where $\mathcal{O}_{Z_3}(D_i)$ denotes the line bundle of D_i as read off from the scalings of the section h_i . As the divisors D_i , also their line bundles $\mathcal{O}_{Z_3}(D_i)$ are induced from the bundles $\mathcal{O}(D_i)$ on the ambient space V_4 .

To describe the blown-up threefold \hat{Z}_3 , we introduce the total space of the projective bundle $\mathbb{P}(\mathcal{O}(D_1) \oplus \mathcal{O}(D_2))$. This total space describes a \mathbb{P}^1 -fibration over the ambient space V_4 on which we introduce the \mathbb{P}^1 -coordinates $(l_1, l_2) \sim \lambda(l_1, l_2)$. Then, the blow-up \hat{Z}_3 is given by the complete intersection [44]

$$P = 0, \quad Q = l_1 h_2 - l_2 h_1 = 0, \quad (2.11)$$

in the projective bundle. This is easily checked to describe \hat{Z}_3 . The first constraint depending only on the coordinates of the base V_4 of the projective bundle restricts to the threefold Z_3 . The second constraint then fibers the \mathbb{P}^1 non-trivially over Z_3 to describe the blow-up along \mathcal{C} . Away from $h_1 \neq 0$ or $h_2 \neq 0$ we can solve (2.11) for l_1 or l_2 respectively. Thus, (2.11) describes a point in the \mathbb{P}^1 -fiber for every point in Z_3 away from the curve. However, if $h_1 = h_2 = 0$ the coordinates (l_1, l_2) are unconstrained and parameterize the full \mathbb{P}^1 , which is fibered over \mathcal{C} as its normal bundle $N_{Z_3}\mathcal{C}$. Thus, we have replaced the curve by the exceptional divisor D that is given by the projectivization of its normal bundle in Z_3 , i.e. the ruled surface $D = \mathbb{P}(N_{Z_3}\mathcal{C})$ over \mathcal{C} . We denote the blow-down map by

$$\pi : \hat{Z}_3 \longrightarrow Z_3. \quad (2.12)$$

Having described the construction of the blow-up, one can also determine details on the cohomology of D and \hat{Z}_3 [44]. For a single smooth curve \mathcal{C} the non-vanishing Hodge numbers of D are determined to be

$$h^{0,0} = h^{2,2} = 1, \quad h^{1,0} = g, \quad h^{1,1} = 2 \quad (2.13)$$

as usual for a ruled surface D over a genus g curve \mathcal{C} . One element, which we denote by $\eta|_D$, of $H^{1,1}(D)$ is induced from the ambient space \hat{Z}_3 and given by $\eta = c_1(N_{\hat{Z}_3}D)$. The

⁶The Lefschetz-Hyperplane theorem tells us that indeed any divisor and line bundle in Z_3 is induced from the ambient space [44].

second element spanning $H^{1,1}(D)$ is given by the Poincaré dual $[\mathcal{C}]_D$ of the curve \mathcal{C} in D , $[\mathcal{C}]_D = c_1(N_D\mathcal{C})$. It is related to the first Chern class $c_1(\mathcal{C})$ and thus to the genus as

$$c_1(N_D\mathcal{C}) = -c_1(\mathcal{C}) - 2\eta, \quad (2.14)$$

by using the adjunction formula in \hat{Z}_3 . Note that as a blow-up divisor D is rigid in \hat{Z}_3 . The first and second Chern class of \hat{Z}_3 are affected by the blow-up as

$$c_1(\hat{Z}_3) = \pi^*(c_1(Z_3)) - c_1(N_{\hat{Z}_3}D), \quad (2.15)$$

$$c_2(\hat{Z}_3) = \pi^*(c_2(Z_3) + [\mathcal{C}]) - \pi^*(c_1(Z_3))D. \quad (2.16)$$

Clearly, if Z_3 is a Calabi-Yau manifold one can use $c_1(Z_3) = 0$ to find

$$c_1(\hat{Z}_3) = -\eta, \quad c_2(\hat{Z}_3) = \pi^*(c_2(Z_3) + [\mathcal{C}]), \quad (2.17)$$

in particular that \hat{Z}_3 is no more Calabi-Yau.

It was argued in [18] that the complex structure moduli space of \hat{Z}_3 contains the complex structure moduli of Z_3 as well as the deformation of \mathcal{C} within Z_3 . The basic reason for this is roughly that the complex structure deformations of the rigid divisor D contain the deformation moduli of the curve \mathcal{C} and thus embed them into the complex structure of \hat{Z}_3 . This way the deformations of the pair (Z_3, \mathcal{C}) form a subsector of the geometrical deformations of \hat{Z}_3 . This allows for the study of the combined superpotential of five-brane (2.4) and flux (2.9) as well. First we use the formal unification of the two superpotentials in terms of the relative homology group $H_3(Z_3, \mathcal{C}, \mathbb{Z})$ consisting of three-cycles $H_3(Z_3, \mathbb{Z})$ and three-chains $\Gamma^{\mathcal{C}}$ ending on the curve \mathcal{C} . Then the superpotential can be written as [13]

$$W_{\text{flux}} + W_{\text{M5}} = \sum_i \tilde{N}^i \int_{\Gamma_{\mathcal{C}}^i} \Omega \quad (2.18)$$

with respect to an integral basis $\Gamma_{\mathcal{C}}^i$ of the relative group $H_3(Z_3, \mathcal{C}, \mathbb{Z})$. Here the integers \tilde{N}^i correspond to the three-form flux quanta (M_i, N^i) in (2.9) and the five-brane windings. In particular Ω has to be interpreted as a relative form.

It has been argued in ref. [18] that in the blow-up $\pi : \hat{Z}_3 \rightarrow Z_3$ the superpotential (2.18) is lifted to \hat{Z}_3 as follows. First we have to replace Ω by its equivalent on \hat{Z}_3 , the pullback form

$$\hat{\Omega} = \pi^*(\Omega), \quad \hat{\Omega}|_D = 0 \quad (2.19)$$

that can be shown to vanish on D , see [18] for details and references. Consequently we can write the heterotic superpotentials as

$$W_{\text{flux}} + W_{\text{M5}} = \int_{\hat{Z}_3} H_3 \wedge \hat{\Omega} = \int_{\hat{Z}_3 - D} H_3 \wedge \hat{\Omega} = \int_{\Gamma_{H_3}} \hat{\Omega} \quad (2.20)$$

such that it only depends on the topology of the open manifold $Z_3 - \mathcal{C} = \hat{Z}_3 - D$. Here, we naturally obtain Γ_{H_3} as the Poincaré dual of the flux H_3 in the group $H_3(\hat{Z}_3 - D, \mathbb{Z})$.

These replacements can also be understood in the language of relative (co)homology. On the one hand we can treat $\hat{\Omega}$ as a relative form exploiting the fact that any element in the relative group $H^3(\hat{Z}_3, D, \mathbb{Z})$ can be represented by a form vanishing on D . On the other hand the element Γ_{H_3} maps to the relative homology since Lefschetz and Poincaré duality relate the de Rham homology of the open manifold to the relative homology as

$$H_3(\hat{Z}_3 - D, \mathbb{Z}) = H_3(\hat{Z}_3, D, \mathbb{Z}) \quad (2.21)$$

This identification of (co-)homology groups gets completed by the equivalence $H_3(Z_3, \mathcal{C}, \mathbb{Z}) = H_3(\hat{Z}_3, D, \mathbb{Z})$ telling us that we have consistently replaced all relevant topological quantities on Z_3 by those on the blow-up \hat{Z}_3 . Finally, we expand the element Γ_{H_3} in a basis Γ_D^i of $H_3(\hat{Z}_3 - D, \mathbb{Z}) = H_3(\hat{Z}_3, D, \mathbb{Z})$ to obtain an expansion of the superpotential by relative periods of $\hat{\Omega}$ as

$$W_{\text{flux}} + W_{\text{M5}} = \sum_i \tilde{N}^i \int_{\Gamma_D^i} \hat{\Omega} = \sum_i \tilde{N}^i \int_{\hat{Z}_3} \hat{\Omega} \wedge \gamma_i^D. \quad (2.22)$$

Here γ_i^D are the Poincaré duals in $H^3(\hat{Z}_3, D, \mathbb{Z})$.

Similar to the Calabi-Yau threefold case where every element in $H^3(Z_3, \mathbb{Z})$ can be obtained upon differentiating Ω with respect to the complex structure, it is possible to obtain a basis of $H^3(\hat{Z}_3, D, \mathbb{Z})$ the same way. More precisely we can write the basis elements γ_i^D as differentials of $\hat{\Omega}$ evaluated at the large complex structure point,

$$\gamma_i^D = \mathcal{R}_i \hat{\Omega}|_{z=0}. \quad (2.23)$$

The operators \mathcal{R}_i are polynomials in the differentials $\theta_a = z_a \frac{d}{dz_a}$. Such a representation can be made explicit by noting that $\hat{\Omega}$ can be written as a residue integral [45]

$$\hat{\Omega} = \int_{\epsilon_1} \int_{\epsilon_2} \frac{\Delta}{PQ}, \quad (2.24)$$

where P, Q are the two constraints (2.11) which define \hat{Z}_3 . The form Δ denotes a top-form on the five-dimensional ambient space $\mathbb{P}(\mathcal{O}(D_1) \oplus \mathcal{O}(D_2))$ that is invariant under its torus actions and the ϵ_i are loops around $\{P = 0\}$, $\{Q = 0\}$. For the type of ambient space we consider, the measure Δ takes the schematic form [46]

$$\Delta = \Delta_V \wedge (l_1 dl_2 - l_2 dl_1), \quad (2.25)$$

where Δ_V denotes the invariant form on the toric base V_4 and (l_1, l_2) the coordinates of the \mathbb{P}^1 -fiber. This makes it possible to study some of the afore-mentioned properties of $\hat{\Omega}$ explicitly.

The crucial achievement of the blow-up to \hat{Z}_3 is the fact that all moduli dependence of the superpotential is now contained in the complex structure dependence of $\hat{\Omega}$. Thus it is possible, analogous the Calabi-Yau case, to derive Picard-Fuchs type differential equations for $\hat{\Omega}$ by studying its complex structure dependence explicitly. Upon the algebraic representation of \hat{Z}_3 by the complete intersection (2.11) it is now possible to find an explicit residue representation of $\hat{\Omega}$ such that Griffiths-Dwork reduction can be used to derive the desired differential equations for $\hat{\Omega}$, among whose solutions we find the superpotential W .

So far the discussion of the blow-up procedure and the determination of the brane and flux superpotential was entirely in the heterotic theory Z_3 . However, we will shed more light on the connection between the brane geometry of (Z_3, \mathcal{C}) and the classical complex geometry of the blow-up \hat{Z}_3 in the context of heterotic/F-theory duality. More precisely, we argue that the five-brane superpotential is mapped to a flux superpotential for F-theory compactified on a dual Calabi-Yau fourfold \hat{X}_4 . Starting with \hat{Z}_3 , the fourfold \hat{X}_4 can be represented as a complete intersection generalizing (2.11). However, in contrast to \hat{Z}_3 the fourfold \hat{X}_4 can also be represented as a hypersurface. This fact allows us to directly compute the flux superpotential. Such a computation has been performed in ref. [7] for a set of examples, and confirmed that the five-brane superpotential is naturally contained in the F-theory flux superpotential. In the next section we will discuss this duality in detail and outline the construction of \hat{X}_4 and the F-theory flux G_4 .

3 F-Theory Blow-Ups and the Superpotential

Here we turn to the discussion of F-theory compactifications on elliptic Calabi-Yau fourfolds X_4 yielding $\mathcal{N} = 1$ effective theories in four dimensions. We will discuss the basic geometric ingredients encoding the seven-brane content as well as the three-brane tadpole in its most general form including G_4 -flux in section 3.1. There, we will readily restrict to F-theory fourfolds X_4 with a heterotic dual on an elliptic threefold Z_3 . The F-theory dual to an $E_8 \times E_8$ heterotic string with small instantons/five-branes is discussed in section 3.2 requiring a blow-up in the F-theory base B_3 along curves \mathcal{C} in B_2 . We will argue that the five-brane moduli and superpotential are mapped to complex structure moduli of X_4 and the flux superpotential. Finally in section 3.4, we will construct the appropriate G_4 -flux inducing the flux superpotential dual to the heterotic brane superpotential.

3.1 F-Theory and Heterotic/F-Theory Duality

We prepare for our further discussion by briefly reviewing the necessary aspects of F-theory and heterotic/F-theory duality.

An F-theory compactification to four dimensions is in general defined by an elliptically fibered Calabi-Yau fourfold X_4 with a section. This section can be used to express the fourfold X_4 as an analytic hypersurface in the projective bundle $\mathbb{P}(\mathcal{O}_{B_3} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$ with coordinates (z, x, y) for which the constraint equation can be brought to the Weierstrass form

$$y^2 = x^3 + fxz^4 + gz^6. \quad (3.1)$$

The Calabi-Yau condition on X_4 implies $\mathcal{L} = K_{B_3}^{-1}$ and f, g have to be sections of \mathcal{L}^4 and \mathcal{L}^6 for the constraint (3.1) to transform as a section of \mathcal{L}^6 . F-theory defined on X_4 automatically takes care of a consistent inclusion of spacetime-filling seven-branes. These are supported on the in general reducible divisors Δ in the base B_3 determined by the degeneration loci of (3.1) given by the discriminant

$$\Delta = \{\Delta = 27g^2 + 4f^3 = 0\}. \quad (3.2)$$

The degeneration type of the fibration specified by the order of vanishing of f, g and Δ along the irreducible components Δ_i of the discriminant have an ADE-type classification that physically specifies the four-dimensional gauge group G [34].

There are further building blocks necessary to specify a consistent F-theory setup. This is due to the fact that a four-dimensional compactification generically has a three-brane tadpole of the form [47, 48, 49]

$$\frac{\chi(X_4)}{24} = n_3 + \frac{1}{2} \int_{X_4} G_4 \wedge G_4. \quad (3.3)$$

In the case that the Euler characteristic $\chi(X_4)$ of X_4 is non-zero a given number n_3 of spacetime-filling three-branes on points in B_3 and a specific amount of quantized four-form flux G_4 have to be added in order to fulfill (3.3).

For a generic setup with three-branes and flux, the four-dimensional gauge symmetry as determined by the seven-branes is not affected. However, if the three-brane happens to collide with a seven-brane, it can dissolve, by the same transition as discussed in section 2.1, into a finite-size instanton on the seven-brane worldvolume that breaks the four-dimensional gauge group G . During this transition the number n_3 of three-branes jumps and a flux G_4 is generated describing the gauge instanton on the seven-brane worldvolume [41]. In particular, in case of a heterotic dual theory the three-branes on the F-theory side precisely

correspond to vertical five-branes on the heterotic threefold [24]. Thus, under duality the three-brane/instanton transition is precisely the F-theory dual of the transition of a vertical five-brane into a finite size instanton breaking the gauge group on the heterotic side accordingly. However, we will not encounter this any further since we restrict our discussion to the case that the gauge bundle on those seven branes dual to the perturbative heterotic gauge group is trivial and no three-branes sit on top of their worldvolumes.

Let us now come to a more systematic discussion of heterotic/F-theory duality. The fundamental duality that underlies it in any dimensions is the eight-dimensional equivalence of the heterotic string compactified on T^2 and F-theory on elliptic K3 [1]. The eight-dimensional gauge symmetry G is determined in the heterotic string as the commutant of an $E_8 \times E_8$ -bundle on T^2 with structure group H . This precisely matches the singularity type G of the elliptic fibration of K3 in the F-theory formulation. Using the adiabatic argument [25] it is possible to consider a family of dual eight-dimensional theories parameterized by a base manifold B_n to obtain dualities between the heterotic string and F-theory in lower dimensions.

This way a four-dimensional heterotic string on the elliptic threefold Z_3 is equivalent to F-theory on the elliptic K3-fibered Calabi-Yau fourfold X_4 . Consequently, the three-dimensional base B_3 of the elliptic fibration of X_4 has to be ruled over the base B_2 of the heterotic threefold Z_3 , i.e. B_3 is a holomorphic \mathbb{P}^1 -fibration over B_2 . It turns out that precisely this fibration data of B_3 is crucial for the construction of the dual heterotic theory, in particular the stable vector bundle E on Z_3 that determines the four-dimensional gauge group G . To analyze this issue in a more refined way it is necessary to use the methods developed in [24], in particular the spectral cover. However, instead of delving into the technical details, we will focus on the results essential for our further discussion.

The basic strategy of the spectral cover is to obtain the stable holomorphic bundle E on the elliptic threefold Z_3 roughly speaking by fibering the stable bundles on the fiber torus so that they globally fit into a stable bundle on the threefold Z_3 [24]. This way, the topological data of the bundle E can be determined in terms of the cohomology of the two-dimensional base B_2 . For example, for our case of interest, $H = SU(n)$ and E_8 ,⁷ the second Chern class $c_2(E)$ of the bundle E schematically reads

$$\lambda(E) = \eta\sigma + \pi^*(\omega), \tag{3.4}$$

where η and ω are up to now arbitrary classes in $H^2(B_2, \mathbb{Z})$ and $\sigma = c_1(\mathcal{O}(\sigma))$ is Poincaré dual to the section σ of $\pi : Z_3 \rightarrow B_2$. The class η is essential in the general construction of

⁷Strictly speaking, there is no spectral cover description of E_8 bundles. However, upon application of the method of parabolics very similar results to the $SU(n)$ case can be obtained [24].

the spectral cover. However, its meaning is further clarified in heterotic/F-theory duality, where it can be constructed from the base B_3 of the dual F-theory.

Consider the heterotic string with an $E_8 \times E_8$ -bundle on Z_3 . Besides the required singularities of the elliptic fibration of X_4 to match the heterotic gauge group G only the part B_2 of the F-theory geometry is fixed by duality. The threefold B_3 can be freely specified by choosing the \mathbb{P}^1 -fibration over B_2 as follows. Fixing a line bundle L over B_2 the threefold B_3 is described as the total space of the projective bundle $\mathbb{P}(\mathcal{O} \oplus L)$. There are two distinguished classes in $H^2(B_3, \mathbb{Z})$, namely the B_2 -independent class of the hyperplane of the \mathbb{P}^1 -fiber denoted by $r = c_1(\mathcal{O}(1))$ and the line bundle L with $c_1(L) = t$. Then, the heterotic bundle $E = E_1 \times E_2$ is specified by [24]

$$\eta(E_1) = 6c_1 + t, \quad \eta(E_2) = 6c_1 - t, \quad (3.5)$$

meaning that the choice of \mathbb{P}^1 -fibration uniquely determines the η -classes of the two bundles. In particular, we note that the heterotic anomaly (2.1) is trivially fulfilled without the inclusion of any horizontal five-branes.

So far, the above discussion is not the most general setup possible since it does not allow for the presence of horizontal five-branes. It turns out that the F-theory dual to the $E_8 \times E_8$ heterotic string has to be analyzed more thoroughly in order to naturally include horizontal five-branes to the setup.

3.2 The Five-Brane Dual: Blowing Up in F-Theory

In this section we will discuss the F-theory dual of horizontal five-branes [5, 26] as will be essential for our understanding of the five-brane superpotential.

Thus, we now restrict our considerations completely to F-theory compactifications with a heterotic dual. Then B_3 is the total space of the projective bundle $\mathbb{P}(\mathcal{O}_{B_2} \oplus L)$ where we now assume $L = \mathcal{O}_{B_2}(-\Gamma)$ for an effective divisor Γ in B_2 . This fibration $p : B_3 \rightarrow B_2$ has two holomorphic sections denoted C_0, C_∞ with

$$C_\infty = C_0 + p^*\Gamma. \quad (3.6)$$

Then, the perturbative gauge group $G = G_1 \times G_2$, where we denote the group factors from the first E_8 as G_1 and from the second E_8 as G_2 , is realized by seven-branes over C_0 and C_∞ with singularity type G_1 and G_2 , respectively [2, 35]. On the other hand, components of the discriminant on which Δ vanishes of order greater than one that project onto curves \mathcal{C}_i in B_2 correspond to heterotic five-branes on the same curves in Z_3 [2, 34, 35]. Consequently,

the corresponding seven-branes induce a gauge symmetry that is a non-perturbative effect due to five-branes on the heterotic side.

Since the understanding of horizontal five-branes is the central point of our discussion let us analyze the consequences of these vertical components of the discriminant for the F-theory geometry more thoroughly. Guided by our example of section 4.3, we will consider the enhanced symmetry point with $G = E_8 \times E_8$ due to small instantons/five-branes such that the heterotic bundle is trivial. In general, an analysis of the local F-theory geometry near the five-brane curve \mathcal{C} is possible [26] applying the method of stable degeneration [50, 24]. However, since the essential point in the analysis is the trivial heterotic gauge bundle, the results of [26] carry over to our situation immediately.

As follows in general using (3.6) the canonical bundle of the ruled base B_3 reads

$$K_{B_3} = -2C_0 + p^*(K_{B_2} - \Gamma) = -C_0 - C_\infty + p^*(K_{B_2}). \quad (3.7)$$

From this we obtain the classes F, G and Δ of the divisors defined by f, g and Δ as sections of $K_{B_3}^{-4}, K_{B_3}^{-6}$ and $K_{B_3}^{-12}$, respectively. To match the heterotic gauge symmetry $G = E_8 \times E_8$, there have to be II^* fibers over the divisors C_0, C_∞ in B_3 . Since II^* fibers require that f, g and Δ vanish to order 4, 5 and 10 over C_0 and C_∞ , their divisor classes split accordingly with remaining parts

$$\begin{aligned} F' &= F - 4(C_0 + C_\infty) = -4p^*(K_{B_2}), \\ G' &= G - 5(C_0 + C_\infty) = C_0 + C_\infty - 6p^*(K_{B_2}), \\ \Delta' &= \Delta - 10(C_0 + C_\infty) = 2C_0 + 2C_\infty - 12p^*(K_{B_2}). \end{aligned} \quad (3.8)$$

This generic splitting implies that the component Δ' can locally be described as a quadratic constraint in a local normal coordinate k to C_0 or C_∞ , respectively. Thus, Δ' can be understood locally as a double cover over C_0 respectively C_∞ branching over each irreducible curve \mathcal{C}_i of $\Delta' \cdot C_0$ and $\Delta' \cdot C_\infty$. In fact, near one irreducible curve \mathcal{C}_i intersecting say C_0 the splitting (3.9) implies that the sections f, g take the form

$$f = k^4 f', \quad g = k^5(g_5 + kg_6) \equiv k^5 g' \quad (3.9)$$

with f' denoting a section of KB_3^{-4} and g_5, g_6 sections of $KB_3^{-6} \otimes L, KB_3^{-6}$, respectively. The discriminant then takes the form $\Delta = k^{10} \Delta'$ where Δ' is calculated from f' and g' . Thus, the intersection curve is given by $g_5 = 0$ and the degree of the discriminant Δ rises by two over \mathcal{C}_i with f' and g' vanishing of order zero and one. Precisely the singular curves \mathcal{C}_i in X_4 that appear in g as above are the locations of the small instantons/horizontal five-branes in Z_3 [5, 26] on the heterotic side. In the fourfold X_4 the collision of a II^* and a I_1 singularity over \mathcal{C}_i induces a singularity of X_4 exceeding Kodaira's classification of singularities. Thus, it

requires a blow-up $\pi : \tilde{B}_3 \rightarrow B_3$ in the three-dimensional base of the curves \mathcal{C}_i into divisors D_i . This blow-up can be performed without violating the Calabi-Yau condition since the shift in the canonical class of the base, $K_{\tilde{B}_3} = \pi^* K_{B_3} + D_i$, can be absorbed into a redefinition of the line bundle $\mathcal{L}' = \pi^* \mathcal{L} - D_i$ entering (3.1) such that $K_{X_4} = p^*(K_{B_3} + \mathcal{L}) = p^*(K_{\tilde{B}_3} + \mathcal{L}') = 0$.

To describe this blow-up explicitly let us restrict to the local neighborhood of one irreducible curve \mathcal{C}_i of the intersection of Δ and C_0 . We note that the curve \mathcal{C}_i in B_2 is given by the two constraints

$$h'_1 \equiv k = 0, \quad h'_2 \equiv g_5 = 0, \quad (3.10)$$

for k and g_5 being sections of the normal bundle $N_{B_3} C_0$ and of $K B_3^{-6} \otimes L$, respectively. Then if X_4 is given as a hypersurface $P' = 0$ we obtain the blow-up as the complete intersection

$$P' = 0, \quad Q' = l_1 h'_2 - l_2 h'_1 = 0, \quad (3.11)$$

where, as in (2.11), we have introduced coordinates (l_1, l_2) parameterizing the \mathbb{P}^1 -fiber.

However, at least in a local description, we can introduce a local normal coordinate t to \mathcal{C}_i in B_2 such that $g_5 = t g'_5$ for a section g'_5 which is non-vanishing at $t = 0$. Then by choosing a local coordinate k_1 of the \mathbb{P}^1 -fiber we can solve the blow-up relation Q' of (3.11) to obtain $k = k_1 t$. This coordinate transformation can be inserted into the constraint $P' = 0$ of X_4 to obtain the blown-up fourfold \hat{X}_4 as a hypersurface. The f', g' of this hypersurface are given by

$$f' = k_1^4 f, \quad g' = k_1^5 (g_5 + k_1 t g_6 + \dots) \quad (3.12)$$

In particular, calculating the discriminant Δ' of \hat{X}_4 it can be demonstrated that the I_1 singularity no longer hits the II^* singularity over C_0 [26]. This way we have one description of \hat{X}_4 as the complete intersection (3.11) and another as a hypersurface. Both will be of importance for the explicit examples discussed in sections 4.1, 4.3 and in particular section 4.2.

To draw our conclusions of this blow-up, we summarize what we just discussed. The F-theory counterpart of a heterotic string with full perturbative gauge group is given by a fourfold with II^* fibers over the sections C_0, C_∞ in B_3 . The component Δ' of the discriminant enhances the degree of Δ on each intersection curve \mathcal{C}_i such that a blow-up in B_3 becomes necessary. On the other hand, each blow-up corresponds to a small instanton in the heterotic bundle [2, 32], that we previously described in section 2.1 as a horizontal five-brane on the curve \mathcal{C}_i in the heterotic threefold Z_3 . Indeed, this can be viewed as a consequence of the observation mentioned above that a vertical component of the discriminant with degree greater than one corresponds to a horizontal five-brane [35] as the degree of Δ' on C_0 and C_∞ is two.

We finish this discussion by a brief look at the moduli map between F-theory and its heterotic dual, where we focus on the fate of the five-brane moduli in the just mentioned blow-up process. The first step in the moduli analysis is to relate the dimensions of the various moduli spaces in both theories and to point to possible mismatches where moduli of some ingredients are missing. In particular, this happens in the presence of heterotic five-branes. Indeed it was argued in [5] that the relation of the fourfold Hodge numbers $h^{3,1}(X_4)$ and $h^{1,1}(X_4)$ counting complex structure and Kähler deformations, respectively, to $h^{2,1}(Z_3)$, $h^{1,1}(Z_3)$ and the bundle moduli and characteristic data has to be modified in the presence of five-branes. The extra contribution is due to deformation moduli of the curve \mathcal{C}_i supporting the five-brane counted by $h^0(\mathcal{C}_i, N_{Z_3}\mathcal{C}_i)$ as well as the blow-ups in B_3 increasing $h^{1,1}(B_3)$ such that we obtain [5]

$$\begin{aligned} h^{3,1}(X_4) &= h^{2,1}(Z_3) + I(E_1) + I(E_2) + h^{2,1}(X_4) + 1 + \sum_i h^0(\mathcal{C}_i, N_{Z_3}\mathcal{C}_i), \\ h^{1,1}(X_4) &= 1 + h^{1,1}(B_3) + \text{rk}(G). \end{aligned} \quad (3.13)$$

Here the sum index i runs over all irreducible curves \mathcal{C}_i and we denote the rank of the four-dimensional gauge group by $\text{rk}(G)$. The index $I(E_{1,2})$ counts a topological invariant of the bundle moduli and is given by [24, 51]

$$I(E_i) = \text{rk}(E_i) + \int_{B_2} (4(\eta_i\sigma - \lambda_i) + \eta_i c_1(B_2)) . \quad (3.14)$$

The map for $h^{3,1}(X_4)$ reflects the fact that the four-dimensional gauge symmetry G is on the heterotic side determined by the gauge bundle E whereas on the F-theory side G is due to the seven-brane content defined by the discriminant Δ that is sensitive to a change of complex structure. For an explicit demonstration of this map exploiting the techniques of [4] we refer to our work [7].

Let us now discuss how (3.13) changes during the blow-up procedure. To actually perform the blow-up along the curve \mathcal{C}_i it is necessary to first degenerate the constraint of X_4 such that X_4 develops the singularity over \mathcal{C}_i described above. This requires a tuning of the coefficients entering the fourfold constraint thus restricting the complex structure of X_4 accordingly which means $h^{3,1}(X_4)$ is lowered. Then, we perform the actual blow-up by introducing the new Kähler class associated to the complexified volume of the exceptional divisor D_i . Thus, we end up with a new fourfold with decreased $h^{3,1}$ and $h^{1,1}(\tilde{B}_3)$ increased by one. This is also clear from the general argument [35] that, enforcing a given gauge group G in four dimensions, the complex structure moduli have to respect the form of Δ dictated by the singularity type G . Since the blow-up which is dual to the heterotic small instanton/five-brane transition enhances the gauge symmetry G , the form of the discriminant becomes

more restrictive, thus fixing more complex structures. In this picture the blow-down can be understood as switching on moduli decreasing the singularity type of the elliptic fibration.

Similarly, we can understand (3.13) from the heterotic side. For each transition between small instanton and five-brane, the bundle loses parts of its moduli since the small instanton is on the boundary of the bundle moduli space. Consequently, the index I reduces accordingly. In the same process, the five-brane in general gains position moduli counted by $h^0(\mathcal{C}_i, N_{Z_3\mathcal{C}_i})$, that have to be added to (3.13).

We close the discussion of moduli by making a more refined and illustrative statement about the heterotic meaning of the Kähler modulus of the exceptional divisors D_i . To do so we have to consider heterotic M-theory on $Z_3 \times S^1/\mathbb{Z}_2$. In this picture the instanton/five-brane transition can be understood [52] as a spacetime-filling five-brane wrapping \mathcal{C}_i and moving on S^1/\mathbb{Z}_2 onto the end-of-the-world brane where one perturbative E_8 gauge group is located. There, it dissolves into a finite size instanton of the heterotic bundle E . With this in mind the distance of the five-brane on the interval S^1/\mathbb{Z}_2 away from the end-of-world brane precisely maps [26] to the Kähler modulus of the divisor D_i resolving \mathcal{C}_i in B_3 .

3.3 The F-Theory Flux Superpotential

In this section we discuss the F-theory flux superpotential and recall how mirror symmetry for Calabi-Yau fourfolds allows to compute its explicit form [8, 9, 10, 7]. Recall, that the F-theory superpotential is induced by four-form flux G_4 and given by [53]

$$W_{G_4}(\underline{t}) = \int_{X_4} G_4 \wedge \Omega_4(\underline{t}) = N^a \Pi^b(\underline{t}) \eta_{ab}, \quad a, b = 1, \dots, b^4(X_4), \quad (3.15)$$

where \underline{t} collectively denote the $h^{3,1}(X_4)$ complex structure deformations of X_4 . Note that in order to compute W_{G_4} it is necessary to expand in a basis γ^a of the integral homology group $H_4(X_4, \mathbb{Z})$. The $N^a = \int_{\gamma^a} G_4 \in \mathbb{Z}/2$ are the flux quantum numbers in this basis, while $\Pi^a(\underline{t}) = \int_{\gamma^a} \Omega_4(\underline{t})$ are the periods of the holomorphic $(4,0)$ -form Ω_4 . The constant intersection matrix

$$\eta_{ab} = \int_{X_4} \hat{\gamma}_a \wedge \hat{\gamma}_b, \quad \int_{\gamma^a} \hat{\gamma}_b = \delta_b^a, \quad (3.16)$$

is defined for the integral basis $\hat{\gamma}_a$ of the cohomology group $H^4(X_4, \mathbb{Z})$ which is dual to γ_a . Note that in contrast to $H^3(Z_3, \mathbb{Z})$ of Calabi-Yau threefolds the fourth cohomology group of X_4 does not carry a symplectic structure which necessitates the introduction of η_{ab} . The last expression in formula (3.15) is therefore obtained by expanding $G_4 = N^a \hat{\gamma}_a$ and $\Omega_4(\underline{t}) = \Pi^a(\underline{t}) \hat{\gamma}_a$ in the cohomology basis.

On the F-theory side one has the following consistency condition on the flux. The first constraint comes from the quantization condition for G_4 , which depends on the second Chern class of X_4 in the following way [54]

$$G_4 + \frac{c_2(X_4)}{2} \in H^4(X_4, \mathbb{Z}) . \quad (3.17)$$

More restrictive is the condition that G_4 has to be primitive, i.e. orthogonal to the Kähler form of X_4 . In the F-theory limit of vanishing elliptic fiber this yields the constraints

$$\int_{X_4} G_4 \wedge J_i \wedge J_j = 0 . \quad (3.18)$$

for every generator J_i , $i = 1, \dots, h^{1,1}(X_4)$ of the Kähler cone. To discuss the two conditions further it is useful to remind us of the fact that the (co)homology of a Calabi-Yau splits into a horizontal and a vertical subspace

$$H_H^4(X_4, \mathbb{Z}) = \bigoplus_{k=0}^4 H_H^{4-k,k}(X_4, \mathbb{Z}) , \quad H_V^4(X_4, \mathbb{Z}) = \bigoplus_{k=0}^4 H_V^{k,k}(X_4, \mathbb{Z}) . \quad (3.19)$$

Since we have an even number of complex dimensions the group $H^{2,2}(X_4, \mathbb{C})$ contains both parts and splits accordingly into the vertical and the horizontal subspace as [8]

$$H^{2,2}(X_4) = H_V^{2,2}(X_4) \oplus H_H^{2,2}(X_4) . \quad (3.20)$$

Analogous to the two-dimensional case of K3 and in contrast to the Calabi-Yau threefold case, the derivatives of Ω_4 with respect to the complex structure modulo the differential ideal given by the Picard-Fuchs operators generate only the horizontal subspace. The remaining part is the vertical subspace which is the natural ring of polynomials in the Kähler cone generators J_i modulo the ideal defining the intersection ring. Mirror symmetry exchanges the vertical and the horizontal subspace. A corollary of these statements is that the allowed fluxes in the superpotential (3.15) are in the horizontal subspace. On the other hand Chern classes are in the vertical subspace, so that half integral flux quantum numbers are not allowed if condition (3.18) is met. Now, the most important task on the fourfold side is to find the periods which correspond to the integrals over an integral basis of $H_4(X_4, \mathbb{Z})$.

The first step to determine the periods is to determine the Picard-Fuchs equations $\mathcal{L}_\kappa \Pi^a(\underline{t}) = 0$ satisfied by the periods. The Picard-Fuchs operators \mathcal{L}_κ are differential operators in the complex structure moduli \underline{t} . In general, the \mathcal{L}_κ can be determined by applying Griffiths-Dwork reduction [45]. One identifies the \mathcal{L}_κ which yield exact forms when applied to Ω_4 , i.e.

$$\mathcal{L}_\kappa \Omega_4 = dw_\kappa , \quad (3.21)$$

where w_κ are three-forms on X_4 . To derive the Picard-Fuchs operators \mathcal{L}_κ one uses an explicit expression for the holomorphic four-form Ω_4 via the Griffiths residuum expression [45]. For Calabi-Yau fourfold hypersurfaces and complete intersections $\{P_1 = \dots = P_s = 0\}$ with $dP_1 \wedge \dots \wedge dP_s \neq 0$ in toric varieties \mathbb{P}_Δ of dimensions $s + 4$ the four-form Ω_4 can be expressed as

$$\Omega_4 = \int_{\epsilon_1} \dots \int_{\epsilon_s} \prod_{k=1}^s \frac{a_0^{(k)}}{P_k} \Delta . \quad (3.22)$$

Here ϵ_i are paths in \mathbb{P}_Δ , which encircle $P_i = 0$ and Δ is an measure invariant under the torus action. The parameter $a_0^{(k)}$ denotes a distinguished coefficient in the defining constraint P_k as introduced below. This method is general but tedious. However, the operators \mathcal{L}_κ can also be determined by the toric data. They are related to the scaling relations of the dual toric variety $\mathbb{P}_{\tilde{\Delta}}$ that happens to be the ambient space of the mirror fourfold \tilde{X}_4 of X_4 . This nicely connects to the framework of toric mirror symmetry [55, 42, 56] where the charge vectors $\ell^{(a)}$, defining the Kähler cone of the mirror \tilde{X}_4 , determine a canonical set of differential operators, the GKZ-system, from which the Picard-Fuchs system for the complex structure of X_4 is obtained. From these operators \mathcal{L}_κ one can evaluate a finite set of solutions $\Pi^a(\underline{t})$.

In a second step, one has to identify the solutions corresponding to the integral basis of $H_4(X_4, \mathbb{Z})$. A strategy to do this was outlined in [7] (see also refs. [9, 10]) and made concrete in simple examples. The key idea is to use the structure of the solution near conifold divisors in the moduli space, where a four-cycle ν and therefore the corresponding period $\int_\nu \Omega_4$ vanishes. The vanishing cycle ν can often be identified directly with generators of $H_4(X_4, \mathbb{Z})$. Associated to each vanishing cycle, there will be a monodromy action on the period vector that is generated by encircling the divisor in the moduli space and is patching the, in general redundant, generators of these monodromies globally together.

Most information comes from the large complex structure, i.e. the point of maximal unipotent monodromy whose location is the origin in the Mori cone coordinate system $z_a = (-1)^{\ell_0^{(a)}} \prod_{j=0}^m a_j^{\ell_j^{(a)}}$ for a toric hypersurface X_4 . For every entry $\ell_j^{(a)}$ of the Mori vectors $\ell^{(a)}$ there are parameters a_j that are just the coefficients of the constraint $P_1 = 0$ defining X_4 . At the point $\underline{z} = 0$ several cycles γ_a vanish and we have one analytic solution $X^0(z) = \int_{\gamma_0} \Omega_4$ and $h^{3,1}(X_4)$ logarithmic periods $X^a(z) = \int_{\gamma_a} \Omega_4 = X^0(z) \log(z_a) + \Sigma_a(z)$. Then the mirror map is given by

$$t^a = \frac{X^a}{X^0} . \quad (3.23)$$

Noting that $t^a \sim \log(z_a)$ at this point we can use these flat coordinates to write the leading logarithmic structure of the period vector as

$$\Pi^T = \left(\int_{\gamma_0} \Omega_4, \dots, \int_{\gamma_{b_H^4}} \Omega_4 \right) = X^0 \left(1, t^a, \frac{1}{2} C_{ab}^\delta t^a t^b, \frac{1}{3!} C_{bcd}^a t^b t^c t^d, \frac{1}{4!} C_{abcd} t^a t^b t^c t^d \right) . \quad (3.24)$$

In particular, the grading $(\{k\}) = (0, 1, 2, 3, 4)$ in powers of t^a corresponds to a grading of $\gamma_a \in H_4(X_4)$. In the complex structure given by the point z the dual cohomology group has the natural grading $H_H^4(X_4, \mathbb{Z})$. Mirror symmetry maps this group to the vertical cohomology $H_V(\tilde{X}_4, \mathbb{Z})$. Thus, the Greek indices in (3.24) run from 1 to $h_H^{2,2}(X_4) = h_V^{2,2}(\tilde{X}_4)$, the Latin indices from 1 to $h^{3,1}(X_4) = h^{1,1}(\tilde{X}_4)$. Note that we have introduced the constant coefficients $C_{ab}^\delta = \eta^{(2)\delta\gamma} C_{ab\gamma}^0$, $C_{abc}^e = \eta^{(1)ed} C_{abcd}^0$ that are related to the classical intersection numbers $C_{ab\gamma}^0$ and C_{abcd}^0 . These are calculated in the classical geometry of \tilde{X}_4 as follows. Let us denote a basis of $H_V(\tilde{X}_4, \mathbb{Z})$ by

$$A_{p_k}^{(k)} = a_{p_k}^{i_1, \dots, i_k} \tilde{J}_{i_1} \wedge \dots \wedge \tilde{J}_{i_k}, \quad (3.25)$$

where the \tilde{J}_{i_n} are the generators of the Kähler cone of the mirror \tilde{X}_4 . Then one has

$$C_{abcd}^0 = \int_{\tilde{X}_4} A_a^{(1)} \wedge A_b^{(1)} \wedge A_c^{(1)} \wedge A_d^{(1)}, \quad C_{ab\gamma}^0 = \int_{\tilde{X}_4} A_a^{(1)} \wedge A_b^{(1)} \wedge A_\gamma^{(2)} \quad (3.26)$$

and $\eta_{ab}^{(1)} = \int_{\tilde{X}_4} A_a^{(1)} \wedge A_b^{(3)}$ as well as $\eta_{\gamma\delta}^{(2)} = \int_{\tilde{X}_4} A_\gamma^{(2)} \wedge A_\delta^{(2)}$ denote subblocks of η_{ab} at grade $k = 1$ and $k = 2$ respectively whose inverses are indicated by upper indices. By formally replacing the \tilde{J}_i with $\theta_i = z_i \frac{d}{dz_i}$, we get a map

$$\mu : H_V(\tilde{X}_4, \mathbb{Z}) \longrightarrow H_H^4(X_4, \mathbb{Z}) \quad (3.27)$$

given by

$$\mu : A_{p_k}^{(k)} \longmapsto \mathcal{R}_{p_k}^{(k)} \Omega_4 \Big|_{z=0} := a_{p_k}^{i_1, \dots, i_k} \theta_{i_1} \dots \theta_{i_k} \Omega_4 \Big|_{z=0}, \quad (3.28)$$

which preserves the grading. This implies that one can think of the integral basis $\hat{\gamma}_a$ in terms of their corresponding differential operators $\mathcal{R}_{p_k}^{(k)}$ acting on Ω_4 .

The representation of the integral basis as differential operators will be particularly useful in the identification of the heterotic and F-theory superpotential. In particular, this formalism allows us to express the flux G_4 in an integral basis in the form

$$G_4 = \sum_{k=0}^4 \sum_{p_k} N^{p_k(k)} \mathcal{R}_{p_k}^{(k)} \Omega_4 \Big|_{z=0}. \quad (3.29)$$

In the next section we will argue that the heterotic/F-theory duality map is obtained by a matching of the operators $\mathcal{R}_{p_k}^{(k)}$ with their analogs in the heterotic blow-up.

3.4 Duality of the Heterotic and F-Theory Superpotentials

Let us finally turn to the matching of the heterotic and F-theory superpotentials. Recall, that the heterotic superpotential (2.8), is formally given by

$$W_{\text{het}}(\underline{t}^c, \underline{t}^g, \underline{t}^o) = W_{\text{flux}}(\underline{t}^c) + W_{\text{CS}}(\underline{t}^c, \underline{t}^g) + W_{\text{M5}}(\underline{t}^c, \underline{t}^o), \quad (3.30)$$

where t^c, t^g and t^o denote the complex structure, bundle and five-brane moduli respectively. The last two terms are not inequivalent, since tuning the \underline{t}^g or \underline{t}^o moduli one can condense or evaporate five-branes and explore different branches of the heterotic moduli space. Clearly the moduli spaces parametrized by \underline{t}^c and \underline{t}^g do not factorize globally in complex structure and bundle moduli since the notion of a holomorphic gauge bundle on Z_3 depends on the complex structure of Z_3 . Similarly, \underline{t}^c and \underline{t}^o do not factorize as the notion of a holomorphic curve in Z_3 does depend on the complex structure of Z_3 . This is also reflected in the fact that flux and brane superpotential can be unified into one superpotential (2.18) for which the splitting into W_{M5} and W_{flux} is just a matter of basis choice of $H_3(Z_3, \mathcal{C}, \mathbb{Z})$.

The key point of the construction is of course that we can map the heterotic moduli $(\underline{t}^c, \underline{t}^g, \underline{t}^o)$ to the complex structure moduli \underline{t} of X_4 which are encoded in the fourfold period integrals. To make the equivalence

$$W_{\text{het}}(\underline{t}^c, \underline{t}^g, \underline{t}^o) = W_{G_4}(\underline{t}) , \quad (3.31)$$

precise, we need to establish a dictionary between the topological data on the heterotic side, which consist of the heterotic flux quanta, the topological classes of gauge bundles and the class of the curves \mathcal{C} , and the F-theory flux quanta.

In order to study the duality map, we will restrict our considerations to the map between five-brane moduli and complex structure deformations of Z_3 to complex structure deformations of \hat{X}_4 . This can be achieved by restricting the heterotic gauge bundle E to be of trivial $SU(1) \times SU(1)$ type. In this case one needs to include heterotic five-branes to satisfy the anomaly cancellation condition (2.1). In accord with the discussion of section 3.2 the dual fourfold \hat{X}_4 can be realized as a complete intersection blown up along the five-brane curves. As above, we will restrict the discussion to a single five-brane. We want to match this description with the heterotic theory on \hat{Z}_3 . One can now identify the blow-up constraints

$$Q = l_1 g_5(\underline{u}) - l_2 \tilde{z} \longmapsto Q' = l_1 g_5(\underline{u}) - l_2 k , \quad \tilde{z} \longmapsto k , \quad (3.32)$$

where \underline{u} denote coordinates on the base B_2 , $\{\tilde{z} = 0\}$ defines the base B_2 in Z_3 , and $\{z = 0\} \cap \{k = 0\}$ defines the base B_2 in X_4 .⁸ The map (3.32) is possible since both Z_3 and X_4 share the twofold base B_2 with the curve \mathcal{C} . The identification of \tilde{z} with k corresponds to the fact that in heterotic/F-theory duality the elliptic fibration of Z_3 is mapped to the \mathbb{P}^1 -fibration of B_3 . Clearly, the map (3.32) identifies the deformations of \mathcal{C} realized as coefficients in the constraint $\{Q = 0\}$ of \hat{Z}_3 with the complex structure deformations of \hat{X}_4 realized as coefficient in $\{Q' = 0\}$. We also have to match the remaining constraints $\{P = 0\}$ and

⁸Note that the \mathbb{P}^1 -fibration $B_3 \rightarrow B_2$ has actually two sections. As in section 3.2, $k = 0$ is one of the two sections, say, the zero section.

$\{P' = 0\}$ of \hat{Z}_3 and \hat{X}_4 , respectively. Clearly, there will not be a general match. However, as was argued in ref. [4] for Calabi-Yau fourfold hypersurfaces, one can split $P' = 0$ as $P + V_E$ yielding a map

$$P + V_E \longmapsto P' \quad , \quad (3.33)$$

where V_E is describing the spectral cover of the dual heterotic bundles $E = E_1 \oplus E_2$. Again, this requires an identification of \tilde{z} and k . For $SU(1)$ bundles this map was given in (3.32), but can be generalized for non-trivial bundles. Note that the maps (3.32) and (3.33) can also be formulated in terms of the GKZ systems of the complete intersections \hat{Z}_3 and \hat{X}_4 . It implies that the $\ell_i^{(a)}$ of \hat{X}_4 contain the GKZ system of Z_3 and the five-brane ℓ -vectors, similar to the situation encountered in refs. [19, 20, 7, 23, 21].

To match the superpotentials as in (3.31) one finally has to identify the integral basis of $H^3(\hat{Z}_3, D, \mathbb{Z})$ with elements of $H^4(\hat{X}_4, \mathbb{Z})$ and show that the relative periods of $\hat{\Omega}_3$ can be identified with a subset of the periods of Ω_4 . In order to do that, one compares the residue integrals (2.24) and (3.22) for $\hat{\Omega}_3$ and Ω_4 represented as complete intersections. Using the maps (3.32) and (3.33) one then shows that each Picard-Fuchs operator annihilating $\hat{\Omega}_3$ is also annihilating Ω_4 . Hence, also a subset of the solutions to the Picard-Fuchs equations can be matched accordingly. As a minimal check, one finds that the periods of Ω_3 on Z_3 before the blow-up arise as a subset of the periods of Ω_4 in specific directions [9, 7]. The map between the cohomologies $H^3(\hat{Z}_3, D, \mathbb{Z}) \hookrightarrow H^4(\hat{X}_4, \mathbb{Z})$ is also best formulated in terms of operators $\mathcal{R}_p^{(i)}$ applied to the forms $\hat{\Omega}$ and Ω_4 ,

$$\mathcal{R}_p^{(i)} \tilde{\Omega}_3(\underline{z}^c, \underline{z}^o) \Big|_{\underline{z}^c = \underline{z}^o = 0} \longmapsto \mathcal{R}_p^{(i)} \Omega_4(\underline{z}) \Big|_{\underline{z} = 0} \quad . \quad (3.34)$$

Note that the preimage of this map will in general contain derivatives with respect to the variables z^o and hence is an element in relative cohomology. It was shown in refs. [16] that one can find differential operators $\mathcal{R}_p^{(i)}$ which span the full space $H^3(\hat{Z}_3, D, \mathbb{Z})$. One now finds that by identifying the heterotic and F-theory moduli at the large complex structure point $\underline{z} = 0$, one obtains an embedding map of the integral basis.

One immediate application of this formalism is that if we know the classical quadratic terms in W_{het} we can fix the dual G_4 -flux and use the periods of the fourfold to determine the instanton parts. In particular, for the five-brane superpotential $W_{\hat{M}5}(t^o, t^c)$ one finds that the dual flux $G_4^{\text{M}5}$ can be expressed as

$$G_4^{\text{M}5} = \sum_p N^{p(2)} \mathcal{R}_p^{(2)} \Omega_4 \Big|_{\underline{z} = 0} \quad (3.35)$$

Note that for G_4 fluxes generated by operators $\mathcal{R}^{(2)}$ the superpotential yields an integral structure of the fourfold symplectic invariants at large volume of the mirror $\text{Mir}(\hat{X}_4)$ of \hat{X}_4

as [8, 9, 57]

$$W_{G_4}^{\text{inst}} = \sum_{\beta \in H_2(\text{Mir}(\hat{X}_4), \mathbb{Z})} n_{\beta}^0(\gamma_{G_4}) \text{Li}_2(\underline{q}^{\beta}), \quad n_{\beta}^0 \in \mathbb{Z}, \quad (3.36)$$

where γ_{G_4} is co-dimension two cycle specified by the flux [7], and $\underline{q}^{\beta} = \exp(\int_{\beta} \tilde{J})$ is the exponential of the mirror Kähler form \tilde{J} integrated over classes β . This integrality structure is inherited to the heterotic superpotentials in geometric phases of their parameter spaces. For superpotential from five-branes wrapped on a curve \mathcal{C} this matches naturally the disk multi-covering formula of [58], since this part is mapped by mirror symmetry to disk instantons ending on special Lagrangians L mirror dual to \mathcal{C} . It would be interesting to explore a generalization of this integral structure to the gauge sector of the heterotic theory.

Finally, there is geometric way to identify the flux which corresponds to a chain integral $\int_{\Gamma} \Omega_3$. The three-chain Γ can be mapped to a three-chain Γ in B_3 whose boundary two-cycles lie in the worldvolume of a seven-brane over which the cycles of the F-theory elliptic fiber degenerates. By fibering the one-cycle of the elliptic fiber which vanishes at the seven-brane locus over Γ , one gets a transcendental cycle in $H_4(X_4, \mathbb{Z})$. Its dual form lies then in the horizontal part $H_H^4(X_4, \mathbb{Z})$ and therefore yields the flux (see ref. [41] for a review on such constructions). For a recent very explicit construction of these cycles in F-theory compactifications on elliptic K3 surfaces and Calabi-Yau threefolds see refs. [59].

4 Examples of Heterotic/F-Theory Dual Pairs

In this section we study concrete examples to demonstrate the concepts discussed in the earlier sections. We will examine two geometries in detail. The first F-theory Calabi-Yau fourfold geometry, discussed in section 4.1 and 4.2, will have few Kähler moduli and many complex structure moduli. In this case we can use toric geometry to compute explicitly the intersection numbers, evaluate both sides of the expression (3.13) yielding the number of deformation moduli of the five-brane curve, and check the anomaly formula (2.1). We also show that the Calabi-Yau fourfold can be explicitly constructed from the heterotic non-Calabi-Yau threefold obtained by blowing up the five-brane curve in section 4.2. The second Calabi-Yau fourfold example, introduced in section 4.3, will admit few complex structure moduli and many Kähler moduli. This allows us to identify the bundle moduli and five-brane moduli under duality by studying the Weierstrass constraint. The F-theory flux superpotential for this configuration was already evaluated in ref. [7], and we will discuss its heterotic dual in section 4.3.

4.1 Example 1: Five-Branes in the Elliptic Fibration over \mathbb{P}^2

We begin the discussion of our first example of heterotic/F-theory dual theories by defining the setup on the heterotic side. Following section 2.1 the heterotic theory is specified by an elliptic Calabi-Yau threefold Z_3 with a stable holomorphic vector bundle $E = E_1 \oplus E_2$ obeying the heterotic anomaly constraint (2.1).

We choose the threefold Z_3 as the elliptic fibration over the base $B_2 = \mathbb{P}^2$ with generic torus fiber $\mathbb{P}_{1,2,3}[6]$. It is given as a hypersurface $P = 0$ in the toric ambient space

$$\Delta(Z_3) = \left(\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 3B + 9H \\ 0 & -1 & 0 & 0 & 2B + 6H \\ 3 & 2 & 0 & 0 & B \\ 3 & 2 & 1 & 1 & H \\ 3 & 2 & -1 & 0 & H \\ 3 & 2 & 0 & -1 & H \end{array} \right) \quad (4.1)$$

with the class of the hypersurface Z_3 given by

$$[Z_3] = \sum D_i = 6B + 18H. \quad (4.2)$$

Here we denoted the two independent toric divisors D_i by H and B , the pullback of the hyperplane class of the \mathbb{P}^2 base respectively the class of the base itself. From the toric data the basic topological numbers of Z_3 are obtained as

$$\chi(Z_3) = 540, \quad h^{1,1}(Z_3) = 2, \quad h^{2,1}(Z_3) = 272. \quad (4.3)$$

The second Chern-class of Z_3 is in general given in terms of the Chern classes $c_1(B_2)$, $c_2(B_2)$ and the section $\sigma : B_2 \rightarrow Z_3$ of the elliptic fibration by $c_2(Z_3) = 12c_1(B_2)\sigma + 11c_1(B_2)^2 + c_2(B_2)$. Here we have $\sigma = B$ and thus obtain

$$c_2(Z_3) = 36H \cdot B + 102H^2. \quad (4.4)$$

To satisfy the heterotic anomaly formula (2.1), we have to construct the heterotic vector bundle $E_1 \oplus E_2$ and compute the characteristic classes $\lambda(E_i)$. Since Z_3 is elliptically fibered the classes $\lambda(E_i)$ can be constructed using the basic methods of [24] that were briefly mentioned in section 3.1. According to (3.4), we first need to specify the classes $\eta_1, \eta_2 \in H^2(B_2, \mathbb{Z})$ essential in the spectral cover construction. We furthermore restrict $E_1 \oplus E_2$ to be an $E_8 \times E_8$ bundle over Z_3 and choose both classes as $\eta_1 = \eta_2 = 6c_1(B_2)$. Then, we use the formula for the second Chern class of E_8 -bundles

$$\lambda(E_i) = \frac{c_2(E_i)}{60} = \eta_i \sigma - 15\sigma^2 + 135\eta_i c_1(B_2) - 310c_1(B_2)^2 \quad (4.5)$$

to obtain $\lambda(E_1) = \lambda(E_2) = 18H \cdot B - 360H^2$. The anomaly condition (2.1) then leads to conditions on the coefficients of the independent classes in $H^4(Z_3)$. For the class $H \cdot B$ contributed by the base via $\sigma \cdot H^2(\mathbb{P}^2, \mathbb{Z})$ this is trivially satisfied by the choice of $\lambda(E_i)$. This implies that no horizontal five-branes are present. For the class of the fiber F the anomaly forces the inclusion of vertical five-branes in the class $\mathcal{C} = c_2(B_2) + 91c_1(B_2)^2 = 822H^2 \equiv n_f F$. Since F is dual to the base B_2 the number n_f of vertical branes is determined by integrating \mathcal{C} over \mathbb{P}^2 ,

$$n_f = \int_{\mathbb{P}^2} \mathcal{C} = 822. \quad (4.6)$$

To conclude the heterotic side we compute the index $I(E_i)$ since it appears in the identification of moduli (3.13) and thus is crucial for the analysis of heterotic/F-theory duality. For Z_3 we use the formula (3.14) to obtain that $I(E_1) = I(E_2) = 8 + 4 \cdot 360 + 18 \cdot 3 = 1502$.

Next we include horizontal five-branes to the setup by shifting the classes η_i appropriately. We achieve this by putting $\eta_2 = 6c_1(B) - H$. The class of the five-brane \mathcal{C} can then be determined analogous to the above discussion by evaluating (4.5) and imposing the anomaly (2.1). It takes the form

$$\mathcal{C} = 91c_1(B_2)^2 + c_2(B_2) - 45c_1(B_2) \cdot H + 15H^2 + H \cdot B = 702H^2 + H \cdot B, \quad (4.7)$$

which means that we have to include five-branes in the base on a curve \mathcal{C} in the class H of the hyperplane of \mathbb{P}^2 . Additionally the number of five-branes on the fiber F is altered to $n_f = 702$. Accordingly, the shifting of η_2 changes the second index to $I_2 = 1019$, whereas $I_1 = 1502$ remains unchanged.

Let us now turn to the dual F-theory description. We first construct the fourfold X_4 dual to the heterotic setup with no five-branes. In this case the base B_3 of the elliptically fibered fourfold is $B_3 = \mathbb{P}^1 \times \mathbb{P}^2$. This can be seen from the relation (3.5) of the classes η_i and the fibration structure of B_3 for E_8 -bundles. Since both classes equal $6c_1$ we have $t = 0$ and thus the bundle $L = \mathcal{O}_{\mathbb{P}^2}$ is trivial as well as the projective bundle $B_3 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2})$. Then the fourfold X_4 is constructed as the elliptic fibration over B_3 with generic fiber given by $\mathbb{P}_{1,2,3}[6]$. Again X_4 is described as a hypersurface in a five-dimensional toric ambient space V_5 as described by the toric data in (4.11) if one drops the point $(3, 2, -1, 0, 1)$ and sets the divisor D to zero. The class of X_4 is then given by

$$[X_4] = \sum_i D_i = 6B + 18H + 12K, \quad (4.8)$$

where the independent divisors are the base B_3 denoted by B , the pullback of the hyperplane H in \mathbb{P}^2 and of the hyperplane K in \mathbb{P}^1 . Then, the basic topological data reads

$$\chi(X_4) = 19728 \quad , \quad h^{1,1}(X_4) = 3 \quad , \quad h^{3,1}(X_4) = 3277 \quad , \quad h^{2,1}(X_4) = 0. \quad (4.9)$$

Now we have everything at hand to discuss heterotic/F-theory duality along the lines of section 3.2, in particular the map of moduli (3.13). As discussed there, the complex structure moduli of the F-theory fourfold are expected to contain the complex structure moduli of Z_3 on the heterotic side as well as the bundle and brane moduli of possible horizontal five-branes. Indeed we obtain a complete matching by adding up all contributions in (3.13),

$$h^{3,1}(X_4) = 3277 = 272 + 1502 + 1502 + 1, \quad (4.10)$$

where it is crucial that no horizontal five-branes with possible brane moduli are present.

To obtain the F-theory dual of the heterotic theory with horizontal five-branes, we have to apply the recipe discussed in section 3.2. We have to perform the described geometric transition of first tuning the complex structure moduli of the fourfold X_4 such that it becomes singular over the curve \mathcal{C} which we then blow up into a divisor D . This way we obtain a new smooth Calabi-Yau fourfold denoted by \hat{X}_4 . The toric data of this fourfold are given by

$$\Delta(\hat{X}_4) = \left(\begin{array}{ccccc|ccc} -1 & 0 & 0 & 0 & 0 & 3D + 3B + 9H + 6K & D_1 \\ 0 & -1 & 0 & 0 & 0 & 2D + 2B + 6H + 4K & D_2 \\ 3 & 2 & 0 & 0 & 0 & B & D_3 \\ 3 & 2 & 1 & 1 & 0 & H & D_4 \\ 3 & 2 & -1 & 0 & 0 & H - D & D_5 \\ 3 & 2 & 0 & -1 & 0 & H & D_5 \\ 3 & 2 & 0 & 0 & 1 & K & D_7 \\ 3 & 2 & 0 & 0 & -1 & K + D & D_8 \\ 3 & 2 & -1 & 0 & 1 & D & D_9 \end{array} \right). \quad (4.11)$$

where we included the last point $(3, 2, -1, 0, 1)$ and a corresponding divisor $D_9 = D$ to perform the blow-up along the curve \mathcal{C} as follows.

Since the curve \mathcal{C} on the heterotic theory is in the class H we have to blow-up over the hyperplane class of \mathbb{P}^2 in B_3 . First we project the polyhedron $\Delta(\hat{X}_4)$ to the base B_3 which is done just by omitting the first and second column in (4.11). Then the last point maps to the point $(-1, 0, 1)$ that subdivides the two-dimensional cone spanned by $(-1, 0, 0)$ and $(0, 0, 1)$ in the polyhedron of B_3 . Thus, upon adding this point the curve $\mathcal{C} = H$ in B_2 corresponding to this cone is removed from B_3 and replaced by the divisor D corresponding to the new point. Thus we see that the toric data (4.11) contain this blown-up base B_3 in the last three columns.

The fourfold is then realized as a generic constraint $P = 0$ in the class

$$[\hat{X}_4] = 6B + 18H + 12K + 6D. \quad (4.12)$$

Note that this fourfold has now three different triangulations which correspond to the various five-brane phases on the dual heterotic side. The topological data for the new fourfold \hat{X}_4 are given by

$$\chi(\hat{X}_4) = 16848 \quad , \quad h^{1,1}(\hat{X}_4) = 4 \quad , \quad h^{3,1}(\hat{X}_4) = 2796 \quad , \quad h^{2,1}(\hat{X}_4) = 0 \quad , \quad (4.13)$$

where the number of complex structure moduli has reduced in the transition as expected.

If we now analyze the map of moduli (3.13) in heterotic/F-theory duality we observe that we have to put $h^0(\mathcal{C}, N_{Z_3\mathcal{C}}) = 2$ in order to obtain a matching. This implies, from the point of view of heterotic/F-theory duality, that the horizontal five-brane wrapped on \mathcal{C} has to have two deformation moduli. Indeed, this precisely matches the fact that the hyperplane class of \mathbb{P}^2 has two deformations since a general hyperplane is given by the linear constraint $a_1x_1 + a_2x_2 + a_3x_3 = 0$ in the three homogeneous coordinates x_i of \mathbb{P}^2 . Upon the overall scaling it thus has two moduli parameterized by the \mathbb{P}^2 with homogeneous coordinates a_i . This way we have found an explicit construction of an F-theory fourfold with complex structure moduli encoding the dynamics of heterotic five-branes.

In section 4.2 we provide further evidence for this identification by showing that one can also construct \hat{X}_4 as a complete intersection starting with a heterotic non-Calabi-Yau threefold. Unfortunately, it will be very hard to compute the complete superpotential for the fourfold \hat{X}_4 since it admits such a large number of complex structure deformations. It would be interesting, however, to extract the superpotential for a subsector of the moduli including the two brane deformations.⁹ Later on we will take a different route and consider examples with only a few complex structure moduli which are constructed by using mirror symmetry.

4.2 Calabi-Yau Fourfolds from Heterotic Non-Calabi-Yau Threefolds

In this section we discuss the example of section 4.1 employing the blow-up proposal of ref. [18] as discussed in section 2.3. More precisely, we will explicitly construct a non-Calabi-Yau threefold \hat{Z}_3 which is obtained by blowing up the horizontal five-brane curve into a divisor. This translates the deformations of \mathcal{C} into new complex structure deformations of \hat{Z}_3 . The F-theory Calabi-Yau fourfold \hat{X}_4 is then naturally obtained from the base of \hat{Z}_3 by an additional \mathbb{P}^1 fibration. \hat{X}_4 is identical to the fourfold considered in section 4.1, despite the fact that it is now realized as a complete intersection.

⁹If one considers exactly the mirror of \hat{X}_4 , as we will in fact do in section 4.3, it might be possible to embed this reduced deformation problem into the complicated deformation problem of \hat{X}_4 constructed in this section.

As in section 4.1 the starting point is the elliptic fibration Z_3 over $B_2 = \mathbb{P}^2$ with a five-brane wrapping the hyperplane class of the base. Let us describe the explicit construction of \hat{Z}_3 . The blow-up geometry \hat{Z}_3 is given by $\mathbb{P}(N_{Z_3}\mathcal{C})$. Z_3 is a hypersurface $\{P = 0\}$ in a toric variety V_4 and the curve \mathcal{C} is given as a complete intersection of two hypersurfaces in Z_3 , i.e. $\mathcal{C} = \{h_1 = 0\} \cap \{h_2 = 0\} \subset Z_3$. The charge vectors of V_4 are given by $\{\ell^{(i)}\}$ with $i = 1, \dots, k$. We are aiming to construct a five-dimensional toric variety which is given by $\hat{V}_5 = \mathbb{P}(N_{V_4}\mathcal{C})$ and use the blow-up equation described in section 2.3. Let us denote the divisor classes defined by h_i by H_i and the charges of h_i by $\mu_i = (\mu_i^{(1)}, \dots, \mu_i^{(k)})$. Then, the coordinates l_i of $N_{V_4}H_i$ transform with charge $\mu_i^{(m)}$ under the k scaling relations. The normal bundle $N_{V_4}\mathcal{C}$ is given by $N_{V_4}H_1 \oplus N_{V_4}H_2$. Since we have to projectivize $N_{V_4}\mathcal{C}$, we have to include another \mathbb{C}^* -action with charge vector $\ell_{\hat{V}_5}^{(k+1)}$ acting non-trivially only on the new coordinates l_i . The new charge vectors of \hat{V}_5 are thus given by the following table

	coordinates of V_4	l_1	l_2
$\ell_{\hat{V}_5}^{(1)}$	$\ell^{(1)}$	$\mu_1^{(1)}$	$\mu_2^{(1)}$
\vdots	\vdots	\vdots	\vdots
$\ell_{\hat{V}_5}^{(k)}$	$\ell^{(k)}$	$\mu_1^{(k)}$	$\mu_2^{(k)}$
$\ell_{\hat{V}_5}^{(k+1)}$	0	1	1

As in (2.11), the blown-up geometry \hat{Z}_3 is now given as a complete intersection

$$P = 0 \quad , \quad l_1 h_2 - l_2 h_1 = 0 . \quad (4.14)$$

To apply this to the elliptic fibration over \mathbb{P}^2 with the polyhedron (4.1), one picks the curve \mathcal{C} given by $\{\tilde{z} = 0\}$ and $\{x_1 = 0\}$. \mathcal{C} is a genus zero curve and we will find that the exceptional divisor D will be the first del Pezzo surface dP_1 in accord with the discussion of section 2.3. We construct the five-dimensional ambient manifold as explained above,

$$\Delta(\hat{Z}_3) = \left(\begin{array}{ccccc|c} -1 & 0 & 0 & 0 & 0 & 3B + 3D + 9H \\ 0 & -1 & 0 & 0 & 0 & 2B + 2D + 6H \\ 3 & 2 & 0 & 0 & 0 & B \\ 3 & 2 & 1 & 1 & 1 & H \\ 3 & 2 & -1 & 0 & 0 & H \\ 3 & 2 & 0 & -1 & 0 & H \\ 3 & 2 & 0 & 0 & -1 & D \\ 0 & 0 & 0 & 0 & -1 & H - D \end{array} \right) . \quad (4.15)$$

Note that one has to include the inner point $(3, 2, 0, 0, 0)$ which corresponds to the base of the elliptic fibration \hat{Z}_3 . Furthermore, one shows that the point $(0, 0, 0, 0, 1)$, required for the

above scalings, can be omitted since the associated divisor does not intersect the complete intersection \hat{Z}_3 . Explicitly the complete intersection \hat{Z}_3 is given by a generic constraint in the class

$$\hat{Z}_3 : (6B + 6D + 18H) \cap H , \quad (4.16)$$

where H, B, D are the divisor classes of the ambient space (4.15). The first divisor in (4.16) is the sum of the first seven divisors in (4.15) and corresponds to the original Calabi-Yau constraint $P = 0$ in (4.14). The second divisor in (4.16) is the sum of the last two divisors and is the class of the second equation of (4.14). This complete intersection threefold has $\chi(\hat{Z}_3) = -538 = \chi(Z_3) - \chi(\mathbb{P}^1) + \chi(dP_1)$, and one checks that the exceptional divisor D has the characteristic data of a del Pezzo 1 surface. This means that we have replaced the hyperplane isomorphic to \mathbb{P}^1 in the base with the exceptional divisor which is dP_1 . It can be readily checked that the first Chern class of \hat{Z}_3 is non-vanishing and equals $-D$.

Having described the blow-up geometry, we now turn to the construction of the fourfold \hat{X}_4 for F-theory. This fourfold will also be constructed as complete intersection, but it will be the same manifold as the fourfold described in section 4.1, equation (4.11). We fiber an additional \mathbb{P}^1 over $\mathbb{P}(\Delta(\hat{Z}_3))$ which is only non-trivially fibered along the exceptional divisor. This is analogous to the construction of the dual fourfold in the heterotic/F-theory duality where one also fibers \mathbb{P}^1 over the base twofold of the Calabi-Yau threefold to obtain the F-theory fourfold. Here we proceed in a similar fashion but construct a \mathbb{P}^1 -fibration over the base of the non-Calabi-Yau manifold \hat{Z}_3 . This base is a complete intersection and thus leads to a realization of \hat{X}_4 as a complete intersection. Concretely, we have the following polyhedron

$$\Delta(\hat{X}_4) = \left(\begin{array}{cccccc|ccc} -1 & 0 & 0 & 0 & 0 & 0 & 3D + 3B + 9H + 6K & D_1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2D + 2B + 6H + 4K & D_2 \\ 3 & 2 & 0 & 0 & 0 & 0 & B & D_3 \\ 3 & 2 & 1 & 1 & 1 & 0 & H & D_4 \\ 3 & 2 & -1 & 0 & 0 & 0 & H & D_5 \\ 3 & 2 & 0 & -1 & 0 & 0 & H & D_6 \\ 3 & 2 & 0 & 0 & 0 & 1 & K & D_7 \\ 3 & 2 & 0 & 0 & 0 & -1 & K + D & D_8 \\ \hline 0 & 0 & 0 & 0 & -1 & 1 & D & D_9 \\ 0 & 0 & 0 & 0 & -1 & 0 & H - D & D_{10} \end{array} \right) . \quad (4.17)$$

The fourfold \hat{X}_4 is given as the following complete intersection

$$\hat{X}_4 : (6B + 6D + 18H + 12K) \cap H . \quad (4.18)$$

Note that this fourfold is indeed Calabi-Yau as can be checked explicitly by analyzing the toric data (4.17). For complete intersections the Calabi-Yau constraint is realized via the two

partitions, so-called nef partitions, in (4.17) as in refs. [62]. The first nef partition yields the sum of the first eight divisors $\sum_{i=1}^8 D_i$ in (4.17) and gives the first constraint in (4.18). The second nef partition yields the sum of the last two divisors $D_9 + D_{10}$ in (4.17) and yield the second constraint in (4.18). The divisors D_7 and D_8 correspond to the \mathbb{P}^1 fiber in the base of \hat{X}_4 obtained by dropping the first two columns in (4.17). This fibration is only non-trivial over the exceptional divisors $D_9 = D$ in the second nef partition of (4.17). Note that if one simply drops K from the expression (4.18) one formally recovers the constraint (4.16) of \hat{Z}_3 . To check that the complete intersection \hat{X}_4 is precisely the fourfold constructed in section 4.1, one has to compute the intersection ring and Chern classes. In particular, it is not hard to show that also (4.17) has three triangulations matching the result of section 4.1.

In summary, we have found that there is a natural construction of \hat{X}_4 as complete intersection with the base obtained from the heterotic non-Calabi-Yau threefold \hat{Z}_3 . Let us stress that this construction will straightforwardly generalize to dual heterotic/F-theory setups with other toric base spaces B_2 and different types of bundles. For example, to study the bundle configurations on Z_3 of section 4.1 with $\eta_{1,2} = 6c_1(B_2) \pm kH$, $k = 0, 1, 2$ one has to replace

$$D_4 \rightarrow (3, 2, 1, 1, k), \quad D_4 \rightarrow (3, 2, 1, 1, 1, k), \quad (4.19)$$

in the polyhedra (4.11) and (4.17), respectively. Moreover, also bundles which are not of the type $E_8 \times E_8$ can be included by generalizing the form of the \mathbb{P}^1 fibration just as in the standard construction of dual F-theory fourfolds.

4.3 Example 2: Five-Brane Superpotential in Heterotic/F-Theory Duality

Let us now discuss a second example for which the F-theory flux superpotential can be computed explicitly since the F-theory fourfold admits only few complex structure moduli. Clearly, using mirror symmetry such fourfolds can be obtained as mirror manifolds of examples with few Kähler moduli. To start with, let us consider heterotic string theory on the *mirror* of the Calabi-Yau threefold which is an elliptic fibration over \mathbb{P}^2 . This mirror is the heterotic manifold Z_3 . One shows by using the methods of ref. [60], that this Z_3 is also elliptically fibered, such that, at least in principle, one can construct the bundles explicitly. The polyhedron of Z_3 is the dual polyhedron to (4.1) and the Weierstrass form of Z_3 is as follows

$$\mu_3 = x^3 + y^2 + xy\tilde{z}a_0u_1u_2u_3 + \tilde{z}^6(a_1u_1^{18} + a_2u_2^{18} + a_3u_3^{18} + a_4u_1^6u_2^6u_3^6). \quad (4.20)$$

The coordinates $\{u_i\}$ are the homogeneous coordinates of the twofold base B_2 . Note that one finds that the elliptic fibration of this Z_3 is highly degenerate over B_2 . The threefold is nevertheless non-singular since the singularities are blown up by many divisors in the

toric ambient space of Z_3 . In writing (4.20) many of the coordinates parameterizing these additional divisors have been set to one.¹⁰ Turning to the perturbative gauge bundle $E_1 \oplus E_2$ we will restrict in the following to the simplest bundle $SU(1) \times SU(1)$ which thus preserves the full perturbative $E_8 \times E_8$ gauge symmetry in four dimensions. To nevertheless satisfy the anomaly condition (2.1) one also has to include five-branes. In particular, we consider a five-brane in Z_3 given by the equations

$$h_1 = b_1 u_1^{18} + b_2 u_1^6 u_2^6 u_3^6 = 0, \quad h_2 = \tilde{z} = 0. \quad (4.21)$$

The curve \mathcal{C} wrapped by the five-brane is thus in the base B_2 of Z_3 . Unfortunately, it is hard to check (2.1) explicitly as in the example of section 4.1 since there are too many Kähler classes in Z_3 . However, one can proceed to construct the associated Calabi-Yau fourfold \hat{X}_4 which encodes a consistent completion of the setup.

The associated fourfold \hat{X}_4 cannot be constructed as it was done in section 4.1. However, one can employ mirror symmetry to first obtain the mirror fourfold $\text{Mir}(\hat{X}_4)$ of \hat{X}_4 as Calabi-Yau fibration

$$\begin{array}{ccc} \text{Mir}(Z_3) & \longrightarrow & \text{Mir}(\hat{X}_4) \\ & & \downarrow \\ & & \mathbb{P}^1 \end{array} \quad (4.22)$$

where $\text{Mir}(Z_3)$ is the mirror of the heterotic threefold Z_3 [4]. This naturally leads us to identify \hat{X}_4 as the mirror to the fourfold (4.11) from section 4.1. This fourfold is also the main example discussed in detail in ref. [7]. In the following we will check that this is indeed the correct identification by using the formalism of refs. [4, 26]. The Weierstrass form of \hat{X}_4 can be computed using the dual polyhedron of (4.11) yielding

$$\mu_4 = y^2 + x^3 + m_1(u_i, w_j, k_m)xyz + m_6(u_i, w_j, k_m)z^6 = 0, \quad (4.23)$$

where

$$\begin{aligned} m_1(w_j, u_i) &= a_0 u_1 u_2 u_3 w_1 w_2 w_3 w_4 w_5 w_6 k_1 k_2, \\ m_6(w_j, u_i) &= a_1 (k_1 k_2)^6 u_1^{18} w_1^{18} w_2^{18} w_5^6 w_6^6 + a_2 (k_1 k_2)^6 u_2^{18} w_3^{18} w_5^{12} \\ &\quad + a_3 (k_1 k_2)^6 u_3^{18} w_4^{18} w_6^{12} + a_4 (k_1 k_2)^6 (u_1 u_2 u_3 w_1 w_2 w_3 w_4 w_5 w_6)^6 \\ &\quad + b_1 k_2^{12} u_1^{18} w_1^{24} w_2^{12} w_3^6 w_4^6 + b_2 k_2^{12} (u_1 u_2 u_3)^6 (w_1 w_3 w_4)^{12} \\ &\quad + c_1 k_1^{12} (u_1 u_2 u_3)^6 (w_2 w_5 w_6)^{12}. \end{aligned} \quad (4.24)$$

The coordinates u_i are the coordinates of the base twofold B_2 as before and w_i, k_1, k_2 are the additional coordinates of the base threefold B_3 . Again, note that we have set many

¹⁰Note that the blow-down of these divisors induces a large non-perturbative gauge group in the heterotic compactification.

coordinates to one to display μ_4 . The chosen coordinates correspond to divisors which include the vertices of $\Delta(X_4)$ and hence determine the polyhedron fully. In particular, one finds that k_1, k_2 are the coordinates of the fiber \mathbb{P}^1 over B_2 . The coefficients a_i, b_1, b_2, c_1 denote coefficients encoding the complex structure deformations of \hat{X}_4 . However, since $h^{3,1}(\hat{X}_4) = 4$, there are only four complex structure parameters rendering six of the a_i redundant. As the first check that \hat{X}_4 is indeed the correct geometry, we use the stable degeneration limit [24] and write μ_4 in a local patch with appropriate coordinate redefinition as follows [4]

$$\mu_4 = p_0 + p_+ + p_- , \quad (4.25)$$

where

$$\begin{aligned} p_0 &= x^3 + y^2 + xy\tilde{z}a_0u_1u_2u_3 + \tilde{z}^6 (a_1u_1^{18} + a_2u_2^{18} + a_3u_3^{18} + a_4u_1^6u_2^6u_3^6) , \\ p_+ &= v\tilde{z}^6 (b_1u_1^{18} + b_2u_1^6u_2^6u_3^6) , \\ p_- &= v^{-1}\tilde{z}^6c_1u_1^6u_2^6u_3^6 . \end{aligned} \quad (4.26)$$

The coordinate v is the affine coordinate of the fiber \mathbb{P}^1 . In the stable degeneration limit $\{p_0 = 0\}$ describes the Calabi-Yau threefold of the heterotic string. In this case p_0 coincides with μ_3 which means that the heterotic Calabi-Yau threefold of \hat{X}_4 is precisely Z_3 . This shows that the geometric moduli of Z_3 are correctly embedded in \hat{X}_4 . The polynomials p_{\pm} encode the perturbative bundles, and the explicit form (4.26) shows that one has a trivial $SU(1) \times SU(1)$ bundle. This fact can also be directly checked by analyzing the polyhedron of \hat{X}_4 using the methods of [34, 61]. One shows explicitly that over each divisor $k_i = 0$ in B_3 a full E_8 gauge group is realized. Since the full $E_8 \times E_8$ gauge symmetry is preserved we are precisely in the situation of section 3.2, where we recalled from ref. [26] that a smooth \hat{X}_4 contains a blow-up corresponding to a heterotic five-brane. We will now check that this allows us to identify the brane moduli in the duality.

Let us now make contact to section 3.2. To make the perturbative $E_8 \times E_8$ gauge group visible in the Weierstrass equation (4.23), we have to include new coordinates $(\tilde{k}_1, \tilde{k}_2)$ replacing (k_1, k_2) . This can be again understood by analyzing the toric data using the methods of [61, 34]. We denote by $(3, 2, \vec{\mu})$ the toric coordinates of the divisor corresponding to \tilde{k}_1 in the Weierstrass model. Then the resolved E_8 singularity corresponds to the points¹¹

$$\begin{aligned} &(3, 2, n\vec{\mu}), \quad n = 1, \dots, 6 \quad , \quad (2, 1, n\vec{\mu}), \quad n = 1, \dots, 4 \quad , \\ &(1, 1, n\vec{\mu}), \quad n = 1, 2, 3 \quad , \quad (1, 0, n\vec{\mu}), \quad n = 1, 2 \quad , \quad (0, 0, \vec{\mu}) \end{aligned} \quad (4.27)$$

¹¹Note that we have chosen the vertices in the $\mathbb{P}_{1,2,3}[6]$ to be $(-1, 0), (0, -1), (3, 2)$ to match the discussion in refs. [34, 61]. However, if one explicitly analyses the polyhedron of \hat{X}_4 one finds that one has to apply a $Gl(2, \mathbb{Z})$ transformation to find a perfect match. This is due to the fact that \hat{X}_4 , in comparison to its mirror $\text{Mir}(\hat{X}_4)$, actually contains the dual torus as elliptic fiber.

While $(3, 2, 6\vec{\mu})$ corresponding to k_1 is a vertex of the polyhedron, $(3, 2, \vec{\mu})$ corresponding to \tilde{k}_1 is an inner point. Using the inner point for \tilde{k}_1 , the Weierstrass form μ_4 changes slightly, while the polynomials p_0, p_+ and p_- can still be identified in the stable degeneration limit. To determine g_5 in (3.9), we compute g of the Weierstrass form in a local patch where $\tilde{k}_2 = 1$

$$g = \tilde{k}_1^5 \left(b_1 u_1^{18} + b_2 u_1^6 u_2^6 u_3^6 + \tilde{k}_1 (a_1 u_1^{18} + a_2 u_2^{18} + \dots) \right) . \quad (4.28)$$

The dots contain only terms of order zero or higher in \tilde{k}_1 . Comparing this with (3.12), g_5 is given by

$$g_5 = b_1 u_1^{18} + b_2 u_1^6 u_2^6 u_3^6. \quad (4.29)$$

This identifies $\{g_5 = 0\}$ with the curve of the five-brane in the base B_2 of Z_3 and is in accord with (4.21). One concludes that \hat{X}_4 is indeed a correct fourfold associated to Z_3 with the given five-brane. As we can see from (4.29), the five-brane has one modulus. If we compare g_5 with p_+ , we see that $p_+ = v\tilde{z}^6 g_5$. This nicely fits with the bundle description. In our configuration, p_+ and p_- should describe $SU(1)$ -bundle since we have the full unbroken perturbative $E_8 \times E_8$ -bundle as described above. The $SU(1)$ -bundles do not have any moduli, such that the moduli space corresponds to just one point [24]. In the explicit discussion of the Weierstrass form in our setting, p_+ has one modulus which corresponds to the modulus of the five-brane. Note that the Calabi-Yau fourfold \hat{X}_4 is already blown up along the curve $\tilde{k}_1 = g_5 = 0$ in the base of \hat{X}_4 . This blow-up can be equivalently described as a complete intersection as we discussed in the previous sections. A simple example of such a construction was presented in section 4.2.

Finally, we consider the computation of the flux superpotential. Here, we do not need to recall all the details, since the superpotential for this configuration was already studied in ref. [7]. The different triangulations of $\text{Mir}(\hat{X}_4)$ correspond to different five-brane configurations. The four-form flux, for one five-brane configurations, was shown to be given in the basis elements

$$\hat{\gamma}_1^{(2)} = \frac{1}{2}\theta_4(\theta_1 + \theta_3)\Omega_4|_{\underline{z}=0} , \quad \hat{\gamma}_1^{(2)} = \frac{1}{7}\theta_2(\theta_2 - 2\theta_1 + 6\theta_4 - \theta_3)\Omega_4|_{\underline{z}=0} , \quad (4.30)$$

where the $\theta_i = z_i \frac{d}{dz_i}$ are the logarithmic derivatives as introduced in (3.28). The moduli z_1, z_2 can be identified as the deformations of the complex structure of the heterotic threefold Z_3 , while z_3 corresponds to the deformation of the heterotic five-brane.¹² A non-trivial check of this identification was already provided in [7], where it was shown that the F-theory flux superpotential in the directions (4.30) matches with the superpotential for a five-brane configuration in a local Calabi-Yau threefold obtained by decompactifying Z_3 . This non-compact five-brane can be described by a point on a Riemann surface in the base B_2 of

¹²The deformation z_4 describes the change in p_- .

Z_3 . Using heterotic F-theory duality as in section 3, one can now argue that the flux (4.30) actually describes a compact heterotic five-brane setup.

5 Conclusion

In this work we have studied the dynamics of heterotic five-branes using the duality between the heterotic string and F-theory compactifications. In particular, we have exploited the fact that five-branes wrapped on the base of an elliptically fibered Calabi-Yau threefold Z_3 map under duality into the geometry of the F-theory Calabi-Yau fourfold X_4 . This implies that the heterotic five-brane superpotential has to be identified with a F-theory flux superpotential. On the heterotic side the five-brane superpotential is given by a chain integral $\int_{\Gamma} \Omega$ over the holomorphic three-form of Z_3 . Upon identifying the F-theory four-form flux which corresponds to this three-chain Γ , the determination of the superpotential becomes a tractable task [7]. This is due to the fact that the deformation moduli of the five-brane are mapped to complex structure deformations of the dual Calabi-Yau fourfold X_4 . Their dynamics is then captured by the periods of the holomorphic four-form on X_4 .

The construction of the F-theory fourfold dual to a five-brane has been argued to involve a blow-up of the five-brane curve [5, 4]. We have provided further evidence for this proposal by noting that this blow-up can also be performed in the heterotic Calabi-Yau threefold. Following our discussion in ref. [18], the deformation moduli of the five-brane curve become new complex structure deformation of the blown-up Kähler threefold. This space is no longer Calabi-Yau and the vanishing of $\hat{\Omega}$ implies, that the heterotic flux naturally maps to the relative cohomology of \hat{Z}_3 . This allows for an equal treatment of the different parts of the superpotential by expressing the complete heterotic flux supporting both the five-brane and flux superpotential as derivatives of $\hat{\Omega}$ with respect to the complex structure of \hat{Z}_3 . Finally, we were able to explicitly show that there exists a natural map of this non-Calabi-Yau threefold to the F-theory Calabi-Yau fourfold. In an upcoming publication [63] such maps from a more general class of non-Calabi-Yau threefolds to Calabi-Yau fourfolds is constructed and verified by explicit computations on both geometries.

By the identification of the fourfold variables (3.23) with the heterotic variables in the superpotentials $W_{G_4}(\underline{t}) = W_{\text{het}}(\underline{t}^c, \underline{t}^g, \underline{t}^o)$, the integral structure (3.36) of the fourfold symplectic invariants at large volume [8, 57] is now inherited to the heterotic superpotentials in geometric phases of their parameter spaces. For the superpotential from five-branes wrapped on a curve \mathcal{C} this matches naturally the disk multi-covering formula of [58], since this part is mapped by mirror symmetry to disk instantons ending on special Lagrangians L mirror

dual to \mathcal{C} . Similar for the heterotic flux superpotential it matches the expectations from the rational curve counting on threefolds as encoded in the period of the threefold. For the Chern-Simons part of the potential we obtain by our construction integer geometric invariants for the gauge bundles on Calabi-Yau threefolds whose precise relation to Donaldson-Thomas invariants is an interesting subject of research. Finally the Picard-Fuchs system of the fourfold allows to analytically continue the superpotential away from the geometric phases of the open/gauge and closed moduli space into the interior of this moduli space and to find the correct open and closed flat coordinates in these regions, see e.g. for the orbifold points [64].

Let us point out some applications of our results. Firstly, the computation of the superpotential is crucial in the study of moduli stabilization. The F-theory fourfold setup provides powerful tools to determine heterotic vacua in which five-brane and bundle moduli are stabilized. As the F-theory flux superpotential can be determined at an arbitrary point in the moduli space, one is able to study a landscape of heterotic vacua with five-branes and gauge-bundle configurations far inside the moduli space. Since the F-theory Kähler potential for the complex structure moduli of the Calabi-Yau fourfold is computable as a function of the periods of the holomorphic four-form, one also expects to determine the kinetic terms of the five-brane and bundle moduli at different points in the moduli space. It will be an interesting task to explicitly determine the heterotic Kähler potential close to singular configurations and to search for interesting supersymmetric and non-supersymmetric vacua in analogy to the Type IIB analysis [39, 40, 41].

A second application will be the study of the heterotic compactifications which are dual to phenomenologically appealing F-theory vacua. Recently, in refs. [65], a promising class of Calabi-Yau fourfolds for GUT model building was constructed by blowing up singular curves in the base of an elliptic fourfold. The geometries were explicitly realized as complete intersections in a toric ambient space. Remarkably, these manifolds share various properties with the geometries constructed in this work. To explore this relation and the use of heterotic/F-theory duality in more detail will be an interesting and important task [66].

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