

# The $\mathcal{N} = 1$ effective actions of D-branes in Type IIA and IIB orientifolds

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## ABSTRACT

We discuss the four-dimensional  $\mathcal{N} = 1$  effective actions of single space-time filling Dp-branes in general Type IIA and Type IIB Calabi-Yau orientifold compactifications. The effective actions depend on an infinite number of normal deformations and gauge connection modes. For D6-branes the  $\mathcal{N} = 1$  Kähler potential, the gauge-coupling function, the superpotential and the D-terms are determined as functions of these fields. They can be expressed as integrals over chains which end on the D-brane cycle and a reference cycle. The infinite deformation space will reduce to a finite-dimensional moduli space of special Lagrangian submanifolds upon imposing F- and D-term supersymmetry conditions. We show that the Type IIA moduli space geometry is captured by three real functionals encoding the deformations of special Lagrangian submanifolds, holomorphic three-forms and Kähler two-forms of Calabi-Yau manifolds. These elegantly combine in the  $\mathcal{N} = 1$  Kähler potential, which reduces after applying mirror symmetry to the results previously determined for space-time filling D3-, D5- and D7-branes. We also propose general chain integral expressions for the Kähler potentials of Type IIB D-branes.

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# 1 Introduction

In recent years there has been vast progress in understanding the effective supergravity theories arising in Type II string compactifications. From a phenomenological perspective four-dimensional effective theories which are  $\mathcal{N} = 1$  supersymmetric and admit non-trivial gauge groups are of particular interest. A prominent set-up admitting these features are Calabi-Yau orientifold compactifications with space-time filling D-branes [1, 2, 3, 4, 5]. Such compactifications admit in addition to the bulk moduli also a universal class of deformation and Wilson line moduli associated to the D-branes. It will be the task of this work to study the four-dimensional  $\mathcal{N} = 1$  characteristic data encoding the dynamics of the combined open and closed sector moduli. We will first concentrate on Type IIA compactifications with space-time filling D6-branes and later turn to Type IIB compactifications with D3-, D5-, or D7-branes in the discussion of mirror symmetry.

Concentrating on Type IIA Calabi-Yau orientifolds the supersymmetric space-time filling objects are O6-planes and D6-branes wrapped on a supersymmetric three-cycle in the internal Calabi-Yau space. The orientifold planes are supersymmetric since they wrap special Lagrangian cycles which arise as the fix-point locus of an anti-holomorphic and isometric involution of the Calabi-Yau space. On such cycles the Kähler form and the imaginary part of the holomorphic three-form of the Calabi-Yau manifold vanish. Similarly, in a supersymmetric background the D6-branes also have to wrap special Lagrangian cycles which preserve the same supersymmetry as the O6-planes [6, 7, 2]. We will mainly focus on a simple set-up and consider the dynamics of one space-time filling D6-brane and its non-intersecting orientifold image. Global tadpole cancellation conditions impose topological constraints on this configuration and generically imply that there will be additional D6-branes. Their dynamics can be included in the analysis, but will be neglected for simplicity.<sup>3</sup>

In order to determine the  $\mathcal{N} = 1$  effective theory one needs to include the fluctuations of the fields around a given background. Therefore, the scalar fields in the effective theory will include the deformations both of the internal Calabi-Yau geometry as well as the deformations of the D6-branes. As a first step the effective action including only the closed string zero modes in a Calabi-Yau orientifold background can be derived [11, 12]. The reduction considers a finite set of complex and Kähler structure deformations which are compatible with the orientifold involution. This set of real deformations is complexified by the axion-like scalars arising as zero modes of the R-R and NS-NS form fields of Type IIA string theory. It was argued in refs. [11, 13] that the Kähler metric on the  $\mathcal{N} = 1$  field space is captured by two real functionals that have been studied intensively by Hitchin in [14, 15]. Using mirror symmetry at large volume and large complex structure, the Type IIA Kähler potential can be exactly matched with its Type IIB counterparts [11]. This reproduces the expressions found for Type IIB orientifolds with O3/O7 planes and O5 planes [16, 17]. In this work we extend the computation of the  $\mathcal{N} = 1$  characteristic data to the

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<sup>3</sup>For state of the art model building in Type IIA see, for examples, refs. [8, 9, 10], and references therein. Reviews can be found in [1, 2, 5].

open string sector and complete the leading-order mirror identification including space-time filling D-branes.

In the first part of the present paper we will focus on the contribution from a D6-brane and its orientifold image. The degrees of freedom associated to a D6-brane wrapped on a supersymmetric three-cycle  $L_0$  are easiest summarized when considering a fixed background Calabi-Yau geometry. In addition to the  $U(1)$  gauge field the D6-brane can admit brane deformations and non-trivial Wilson lines. A massless deformation preserving the  $\mathcal{N} = 1$  supersymmetry along a normal vector field is associated to a harmonic one-form on the special Lagrangian cycle  $L_0$  [18]. For a fixed background Calabi-Yau geometry there are  $b^1(L_0) = \dim H^1(L_0, \mathbb{R})$  real massless deformations, which combine with the Wilson line scalars into complex fields. We will derive the effective action for these massless modes which keep  $L_0$  special Lagrangian. However, more interestingly, one can also include massive deformations around  $L_0$  by extending the analysis to include non-harmonic one-forms on  $L_0$ . These deformations either violate the Lagrangian condition or the condition that the three-cycle is ‘special’, as we discuss in more detail in the main text. We will show that these deformations induce a scalar potential consisting of an F-term part, rendering non-Lagrangian deformations massive, and a D-term part, rendering non-special deformations massive. Performing a Kaluza-Klein reduction of the D6-brane action we derive the  $\mathcal{N} = 1$  open string Kähler metric and gauge coupling function, and explicitly extract the D6-brane superpotential and D-terms. We argue that these functions take an elegant form when using chain integrals over a four-chain ending on the internal D6-brane three-cycle and a reference cycle  $L_0$ .

In order to determine the Kähler potential it is crucial to include also Calabi-Yau deformations parameterizing the bulk degrees of freedom. In fact, we show that at the classical level the open string scalars only enter in the  $\mathcal{N} = 1$  Kähler potential through a redefinition of the complex coordinates for the Calabi-Yau deformations. This is a generic feature which is already known from Type IIB Calabi-Yau orientifold compactifications with single D3-, D7- or D5-branes [19, 20, 21, 22], as well as Type II orientifolded orbifolds [23, 2]. In the type IIA compactifications we find that the full  $\mathcal{N} = 1$  Kähler potential has an elegant form in terms of the functionals for real two- and three-forms studied by Hitchin [14, 15], and the Kähler potential arising in the study of the deformation space of special Lagrangian submanifolds [24, 25]. We also comment on a generalization of the  $\mathcal{N} = 1$  data to an infinite set of D-brane deformations. As in a fixed background, this generalization will be crucial in the evaluation of the superpotential and D-terms. As in [22] it will be crucial to keep the non-dynamical four-dimensional three-forms in the Kaluza-Klein Ansatz to derive the scalar potential. In addition, we will also be able to extract the kinetic mixing terms of the bulk and brane  $U(1)$  vector fields in the effective action. Such mixing can have profound phenomenological applications [26].

In the last part of this paper we turn to the discussion of mirror symmetry at large volume and large complex structure. We match the known  $\mathcal{N} = 1$  data for single D3-, D5- or D7-branes with

the data found in the D6-brane reduction. This allows us to give chain integral expressions also for the Type IIB reductions which complete the results of [19, 20, 21, 22]. Our strategy to gain a better understanding of the structure of the brane couplings is to use the formulation of mirror symmetry proposed by Strominger, Yau and Zaslow (SYZ) [27]. Neglecting singular fibers it allows to view the compactification Calabi-Yau space as a three-torus fibration over a three-sphere. This allows us to identify different brane wrappings in the Type IIA and Type IIB picture, and yields a matching of the leading order  $\mathcal{N} = 1$  data.

The present work is organized as follows. In section 2 we calculate the IIA orientifold effective action with a space-time filling D6-brane. We first recall the results for the bulk orientifold reduction, and summarize the conditions to include supersymmetric D6-branes in the orientifold background. We then derive the kinetic terms and the scalar potential for an infinite set of normal deformations, Wilson line modes, and  $U(1)$  vector modes on the D6-brane. These computations are performed using a Kaluza-Klein reduction of the Dirac-Born-Infeld and Chern-Simons actions around a supersymmetric configuration. In section 3 we analyze the moduli space of our configuration. We focus on the finite set of massless bulk and brane modes. As first steps we consider the moduli spaces for the bulk sector and the brane sector separately. We introduce the necessary mathematical tools to describe these spaces as Lagrangian embeddings into vector space. In general, the geometry of the open-closed moduli space is more complicated. However, we show that it is possible to encode the complete moduli space in a single elegant Kähler potential as a function of non-trivial local complex coordinates. In section 3 also the kinetic terms for the massless  $U(1)$  gauge field on the D6-brane are discussed, and a kinetic mixing with the massless bulk  $U(1)$ 's is found. In section 4 we discuss the infinite deformation space around a supersymmetric configuration. We give an explicit form of the Kähler potential using chain integrals. We also derive the non-vanishing D-terms and the superpotential for open deformations violating the special Lagrangian condition for supersymmetric D6-branes. Our results should be mirror symmetric to Type IIB orientifold configurations. In section 5 we argue that it is possible to relate the moduli fields obtained in section 3 with the moduli space for IIB orientifold configurations with D3-, D5- and D7-branes. Relations between the homology of the cycles for the mirror configurations can be inferred using the SYZ construction of mirror symmetry. We find elegant expressions for the Type IIB Kähler potentials and  $\mathcal{N} = 1$  complex coordinates.

## 2 The dimensional reduction of the D6-brane action

We start our discussion by fixing the background geometry of our set-up. In the following, we consider the direct product of a compact Calabi-Yau orientifold  $Y/\mathcal{O}$  and flat Minkowski space  $\mathbb{R}^{1,3}$ . We are interested in compactifications with space-time filling D6-branes and O6-planes which preserve  $\mathcal{N} = 1$  supersymmetry in four space-time dimensions. This fixes the orientifold projection

to be of the form [28, 11]

$$\mathcal{O} = (-1)^{F_L} \Omega_p \sigma^* , \quad \sigma^* J = -J , \quad \sigma^* \Omega = e^{2i\theta} \bar{\Omega} , \quad (2.1)$$

where  $\theta$  is some real phase. Here  $\Omega_p$  is the world-sheet parity reversal,  $F_L$  is the space-time fermion number in the left-moving sector, and  $\sigma$  is an anti-holomorphic and isometric involution of the compact Calabi-Yau manifold  $Y$ . The four-dimensional spectrum consists of fields arising as in the zero mode expansion of the ten-dimensional closed string fields into harmonics of the internal space as well as zero modes arising from massless open strings ending on the D6-branes. In the following we will focus on the chiral multiplets in both sectors.

## 2.1 On the four-dimensional Kaluza-Klein spectrum

### Closed string sector

Let us start with a brief discussion of the closed string sector following ref. [11]. The four-dimensional scalars, vectors, two- and three-forms will arise in the expansions of the ten-dimensional fields into harmonic forms of  $Y$  which have to transform in a specified way under the orientifold parity to yield modes which remain in the orientifolded  $\mathcal{N} = 1$  spectrum. More specifically, the ten-dimensional metric and the dilaton are invariant under the action of  $\sigma$  while the NS-NS B-field transforms as  $\sigma^* B_2 = -B_2$ . The R-R fields  $C_1, C_3, C_5, C_7$  remain in the orientifold spectrum if they obey  $\sigma^* C_p = (-1)^{(p+1)/2} C_p$ . Clearly, in type IIA string theory not all the odd-dimensional R-R forms  $C_p$  are independent. Denoting by  $G_{p+1}$  the R-R field strengths are

$$G_2 = dC_1 , \quad G_{p+1} = dC_p - H_3 \wedge C_{p-2} , \quad H_3 = dB_2 . \quad (2.2)$$

When considering all  $G_{p+1}, p = 1 \dots 9$  a the duality constraint

$$G_{p+1} = (-1)^{(p+1)/2} *_10 G_{9-p} , \quad (2.3)$$

has to be imposed to relate the lower and higher-dimensional forms. This can be extended to include Romans mass  $G_0$  which appears in the massive extension of Type IIA supergravity [29]. Using all forms  $G_{p+1}$  one can use a democratic formulation of Type II supergravity [30]. The bosonic kinetic terms of the ten-dimensional action are then given by

$$S_{\text{dem}}^{(10)} = - \int \frac{1}{2} R *_10 1 + \frac{1}{4} H_3 \wedge *_10 H_3 + \sum_{p=1}^9 \frac{1}{8} G_{p+1} \wedge *_10 G_{p+1} . \quad (2.4)$$

This is only an auxiliary action since the duality condition (2.3) has to be imposed by hand in addition to the equations of motion. When coupling the bulk supergravity to a D-brane it turns out to be useful to also introduce another basis  $\mathcal{A}_q$  of  $q$ -forms with a redefined duality relation

$$\mathcal{A} = \sum_q \mathcal{A}_q = e^{-B_2} \wedge \sum_p C_p , \quad d\mathcal{A}_q = (-1)^{(q+1)/2} (*_B d\mathcal{A})_q , \quad (2.5)$$

where the ‘B-twisted’ Hodge star is given by  $*_B = e^{-B_2} *_10 e^{B_2}$ . Clearly, the supergravity action (2.4) can be easily rewritten in terms of the  $\mathcal{A}_q$ .

To perform the Kaluza-Klein expansion of the closed string fields one notes that the anti-holomorphic involution does not preserve the  $(p, q)$  split of the Dolbeault cohomology groups, but rather maps a  $(p, q)$ - to a  $(q, p)$ -form. One thus splits the real de Rham cohomologies into  $\sigma^*$ -eigenspaces  $H_{\pm}^n(Y)$ . It was shown in ref. [11] that the  $\mathcal{N} = 1$  coordinates on the closed string field-space arise by expanding a complex two-form  $J_c$  and three-form  $\Omega_c$  into a basis of  $H_-^2(Y, \mathbb{R})$  and  $H_+^3(Y, \mathbb{R})$ , respectively. More precisely, in accord with (2.1) we expand

$$J_c = B_2 + iJ = (b^a + iv^a) \omega_a = t^a \omega_a , \quad (2.6)$$

where  $a = 1, \dots, h_-^{(1,1)}$  labels a basis  $\omega_a$  of  $H_-^2(Y, \mathbb{R})$ . We thus find the same complex structure as in the underlying  $N = 2$  theory with the dimension of the Kähler moduli space truncated from  $h^{(1,1)}$  to  $h_-^{(1,1)}$ .

The complex three-form  $\Omega_c$  contains the degrees of freedom arising from the complex structure moduli, the dilaton as well as the scalars from the R-R forms. We combine these as

$$\Omega_c = 2 \operatorname{Re}(C\Omega) + iC_3^{\text{sc}} = N'^k \alpha_k - T'_\lambda \beta^\lambda , \quad (2.7)$$

where  $C \propto e^{-\phi+i\theta}$ , as given in (2.10), contains the dilaton, and  $k = 1, \dots, n_-, \lambda = 1, \dots, n_+$  label a basis  $(\alpha_k, \beta^\lambda)$  of  $H_+^3(Y, \mathbb{R})$ . Here  $C_3^{\text{sc}}$  is the part R-R three-form which is also a three-form on the Calabi-Yau manifold  $Y$  and hence descends to scalars in four dimensions. We can use the expansion of  $\Omega$  into the full symplectic basis  $(\alpha_K, \beta^K)$  of  $H^3(Y, \mathbb{R})$  as  $\Omega = X^K \alpha_K - \mathcal{F}_K \beta^K$ . Under  $\sigma^*$  this basis splits into a basis  $(\alpha_k, \beta^\lambda)$  of  $H_+^3(Y, \mathbb{R})$  and a dual basis  $(\alpha_\lambda, \beta^k)$  of  $H_-^3(Y, \mathbb{R})$ . We thus find the explicit expressions

$$N'^k = 2 \operatorname{Re}(CX^k) + i\xi^k , \quad T'_\lambda = 2 \operatorname{Re}(C\mathcal{F}_\lambda) + i\tilde{\xi}_\lambda . \quad (2.8)$$

Note that the split of the  $h^{(2,1)} + 1$  basis elements of  $H_+^3(Y, \mathbb{R})$  into  $n_-$  elements  $\alpha_k$  and  $n_+$  elements  $\beta^\lambda$  does depend on the point in the complex structure moduli space on which one evaluates  $C\Omega$ . In fact, at the large complex structure point the precise split will determine whether this type IIA set-up is dual to an orientifold with O3/O7 planes or O5/O9 planes as we will discuss in detail in section 5. It is important to point out, that the complex coordinates  $(N'^k, T'_\lambda)$  are the correct complex scalars in the  $\mathcal{N} = 1$  chiral multiplets in the absence of D6-branes, but will receive corrections upon introducing dynamical D6-branes.

Before discussing the open string spectrum let us comment further on the complex function  $C$  appearing in (2.7). Since the orientifold projection is an anti-holomorphic involution the complex structure deformations will be real. In fact,  $C$  has a phase factor  $e^{-i\theta}$  and is defined to compensate rescalings of  $\Omega$  such that  $C\Omega$  has a fixed normalization

$$e^{2\phi} C\Omega \wedge \overline{C\Omega} = \frac{1}{6} J \wedge J \wedge J . \quad (2.9)$$

It turns out to be convenient to introduce the four-dimensional dilaton  $D$  by setting  $e^{-2D} = e^{-2\phi}\mathcal{V}$ , where  $\mathcal{V} = \frac{1}{6} \int_Y J \wedge J \wedge J$  is the string-frame volume of the Calabi-Yau space. The compensator field is then given by

$$C = e^{-D-i\theta} e^{K^{\text{cs}}/2} = e^{-\phi-i\theta} \mathcal{V}^{1/2} e^{K^{\text{cs}}/2}, \quad (2.10)$$

where  $K^{\text{cs}} = -\ln[-i \int \Omega \wedge \bar{\Omega}]$ .

Let us note that the R-R three-form in general also leads to  $U(1)$  vectors in four space-time dimensions via the expansion

$$C_3^{\text{vec}} = A^\alpha \wedge \omega_\alpha \quad (2.11)$$

where  $\omega_\alpha$  is a basis of  $H_+^2(Y, \mathbb{R})$ . Their holomorphic gauge coupling function  $f_{\alpha\beta}$  has also been determined in ref. [11]. Denoting by  $\mathcal{K}_{\alpha\beta a} = \int_Y \omega_\alpha \wedge \omega_\beta \wedge \omega_a$ , the intersection form of two elements of  $H_+^2(Y, \mathbb{R})$  with one element of  $H_+^2(Y, \mathbb{R})$  one finds that  $f_{\alpha\beta} = i\mathcal{K}_{\alpha\beta a} t^a$ .

### Open string sector: supersymmetric D6-branes

Let us next discuss the inclusion of space-time filling D6-branes into our set-up. In the background configuration these have to be chosen such that they preserve the same supersymmetry as the O6-planes which arise as the fix-point set of  $\sigma$ . In fact, since  $\sigma$  is an anti-holomorphic involution the O6-planes wrap special Lagrangian cycles satisfying

$$J|_{\text{O6-plane}} = 0, \quad \text{Im}(C\Omega)|_{\text{O6-plane}} = 0. \quad (2.12)$$

Let us consider a single D6-brane wrapped on a three-cycle  $L$  in  $Y$ . We will consider the case where  $L$  is mapped to a three-cycle  $L' = \sigma(L)$  which is in a different cohomology class and does not intersect  $L$ .<sup>4</sup> For this situation the pair of the D6-brane and its image D6-brane is merely an auxiliary description of a single smooth D6-brane wrapping a cycle in the orientifold  $Y/\mathcal{O}$ . Note that the number of D6-branes is restricted by tadpole cancellation. In cohomology one has to satisfy<sup>5</sup>

$$\sum_{\text{D6}} [L + L'] = 4[L_{\text{O6}}], \quad (2.13)$$

where the sum is over all D6-branes present in the compactification and  $L_{\text{O6}}$  is the fix-point set of the involution indicating the location of the O6-plane.

In a supersymmetric orientifold background the D6-brane also has to wrap a calibrated and hence supersymmetric cycle. These calibration conditions have been determined in [6]. They imply that the D6-brane wraps a special Lagrangian submanifold  $L_0 \subset Y$  such that

$$J|_{L_0} = 0, \quad \text{Im}(C\Omega)|_{L_0} = 0, \quad 2 \text{Re}(C\Omega)|_{L_0} = e^{-\phi} \text{vol}_{L_0} \quad (2.14)$$

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<sup>4</sup>Note that this is a non-generic situation for a three-cycle in a six-dimensional manifold. Generically D6-branes on three-cycles will intersect in points. At these intersections matter fields can be localized and have to be included in the reduction.

<sup>5</sup>Note that this condition will be modified in the presence of NS-NS background flux  $H_3$  and the Romans mass parameter  $m^0$  with an additional term proportional to  $m^0 H_3$  (see, e.g., ref. [12]).

where  $\text{vol}_{L_0} = \sqrt{\iota^* g_6} d^3 \xi$  is the induced volume form on  $L$ . Note that the first condition in (2.14) implies that  $L_0$  is Lagrangian, while the second condition makes it special Lagrangian. We fixed the coefficient, in particular the phase of  $C\Omega$ , such that the same supersymmetry is preserved as for the orientifold planes (2.12). Finally, we note that it was also shown in [7] that in a supersymmetric background one has

$$F_{\text{D6}} - B_2|_{L_0} = 0 , \quad (2.15)$$

where  $F_{\text{D6}}$  is the field strength of the  $U(1)$  gauge field  $A$  living on the D6-brane. In the following we will always denote the background special Lagrangian cycle wrapped by a supersymmetric D6-brane by  $L_0$ .

For a *fixed* background complex and Kähler structure we can discuss supersymmetric deformations of the D6-branes. In fact, the deformations of  $L_0$  preserving the special Lagrangian conditions (2.14) were studied by McLean [18]. It was shown that a normal vector field  $\eta$  to a compact special Lagrangian cycle is the deformation vector field to a normal deformation through special Lagrangian submanifolds if and only if the corresponding 1-form  $\theta_\eta = \eta \lrcorner J$  on  $L_0$  is harmonic. This reduces the infinite dimensional space of maps  $\iota$  to a deformation space of dimension  $b^1(L_0) = \dim H^1(L_0, \mathbb{R})$ . Furthermore, there are no obstructions to extending a first order deformation to a finite deformation. The tangent space to such deformations can be identified through the cohomology class of the harmonic form with  $H^1(L_0, \mathbb{R})$ . We can thus write a basis of harmonic one-forms  $\theta_i$  on  $L_0$  as

$$\theta_i = s_i \lrcorner J|_{L_0} , \quad * \theta_i = -2e^\phi s_i \lrcorner \text{Im}(C\Omega)|_{L_0} , \quad i = 1, \dots, b^1(L_0) , \quad (2.16)$$

where  $s_i$  is a basis of the real special Lagrangian normal deformations. Let us recall the derivation of the expression for  $*\theta_i$  [25]. We do this more generally, by determining the Hodge-dual of a one form  $\alpha = (X \lrcorner J)|_{L_0}$  for some  $X \in TY|_{L_0}$ . Note that the vector dual to  $\alpha$  by raising the index with the induced metric is  $IX$  where  $I$  is the complex structure on  $Y$ . Hence one checks

$$*(X \lrcorner J)|_{L_0} = (IX) \lrcorner \text{vol}_{L_0} . \quad (2.17)$$

However, on  $L_0$  the volume form is identical to  $2e^\phi \text{Re}(C\Omega)$  by (2.14). This implies

$$*(X \lrcorner J)|_{L_0} = 2e^\phi (IX \lrcorner \text{Re}(C\Omega))|_{L_0} = -2e^\phi (X \lrcorner \text{Im}(C\Omega))|_{L_0} \quad (2.18)$$

where the minus sign arises from evaluating  $I$  on the  $(3, 0)$ -form  $\Omega$ ,  $(IX) \lrcorner \Omega = iX \lrcorner \Omega$ .

We have just introduced the general supersymmetric deformation encoded by  $b^1(L_0)$  scalars  $\eta^i$  arising in the expansion  $\theta_\eta = \eta^i \theta_i$  of the harmonic form  $\theta_\eta$ . The  $\eta^i(x)$  will be real scalar fields in the four-dimensional effective theory depending on the four space-time coordinates  $x$ . Let us next discuss the degrees of freedom due to  $U(1)$  Wilson lines arising from non-trivial one-cycles on the D6-brane world-volume. Later on we will show that these real scalars will complexify the  $\eta^i$ . The Wilson line scalars arise in the expansion of the  $U(1)$  gauge boson  $A_{\text{D6}}$  on the D6-brane as

$$A_{\text{D6}} = A + a^i \tilde{\alpha}_i , \quad (2.19)$$



where  $A$  is a  $U(1)$  gauge field and the  $a^i(x)$  are  $b^1(L_0)$  real scalars in four dimensions. The forms  $\tilde{\alpha}_i$  provide a basis of  $H^1(L_0, \mathbb{Z})$ . Note that in general the  $U(1)$  field strength  $F_{D6} = dA_{D6}$  can additionally admit a background flux  $\langle F_{D6} \rangle = f_{D6}$  in  $H^2(L_0, \mathbb{Z})$ , which can be trivial or non-trivial in  $H^2(Y, \mathbb{R})$ . Since we will focus on the kinetic terms in the following we will set  $f_{D6} = 0$  for most of the discussion. Note that  $F_{D6}$  naturally combines with the NS-NS B-field into the combination  $F_{D6} - \iota^* B_2$ .

To summarize, one finds as massless variations around a supersymmetric vacuum  $h_-^{(1,1)} + h^{(2,1)} + 1$  chiral multiplets from the bulk and  $b^1(L_0)$  chiral multiplets  $(\eta^i, a^i)$  from the D6-brane. The precise organization of these fields into  $\mathcal{N} = 1$  complex coordinates is postponed to section 3.

## 2.2 General deformations of D6-branes

So far we have discussed the supersymmetric background D6-brane and its supersymmetric deformations. However, in general  $L_0$  admits an infinite set of deformations which will render the D6-brane non-supersymmetric. These deformations will be included in the following and shown to be obstructed by a scalar potential. In order to do that, one recalls that the string-frame world-volume action for the D6-brane takes the form

$$S_{D6}^{\text{SF}} = - \int_{\mathcal{W}_7} d^7 \xi e^{-\phi} \sqrt{-\det(\iota^*(g_{10} + B_2) - F_{D6})} + \int_{\mathcal{W}_7} \sum_{q \text{ odd}} \iota^*(C_q) \wedge e^{F_{D6} - \iota^*(B_2)}. \quad (2.20)$$

In this subsection we will derive the scalar potential arising from the first term in (2.20), the Dirac-Born-Infeld action.

### Exponential map and normal coordinate expansion

A general fluctuation of  $L_0$  to a nearby three-cycle  $L_\eta$  is described by real sections  $\eta$  of the normal bundle  $N_Y L_0$ . Clearly, the space of such sections is infinite dimensional as is the space of all  $L_\eta$ . To make the identification between  $L_\eta$  and  $\eta$  more explicit, one recalls that in a sufficiently small neighborhood of  $L_0$  the exponential map  $\exp_\eta$  is a diffeomorphism of  $L_0$  onto  $L_\eta$ . Roughly speaking, one has to consider geodesics through each point  $p$  on  $L_0$  with tangent  $\eta(p)$  and move this point along the geodesic for a distance of  $\|\eta\|$  to obtain the nearby three-cycle  $L_\eta$ .

It is important to consider how the pull-backs of  $J, \text{Im}(C\Omega)$  as well as other two and three-forms of  $Y$  behave when moving from  $L_0$  to  $L_\eta$ . To examine this change one introduces the pull-back of the exponential map

$$E_\eta(\gamma) = \exp_\eta^*(\gamma|_{L_\eta}), \quad (2.21)$$

where  $\eta \in NL_0$ , and  $\gamma \in \Omega^p(Y)$  are  $p$ -forms on  $Y$ . Hence,  $E_\eta$  pulls back  $\gamma$  from  $L_\eta$  to a  $p$ -form  $E_\eta(\gamma) \in \Omega^p(L_0)$  on  $L_0$ . Of particular interest are the evaluation of  $E_\eta$  on  $J$  and  $\text{Im}(C\Omega)$ . In fact,

one shows that [18]<sup>6</sup>

$$E_\eta(J) = d\hat{\mu}_1, \quad E_\eta(\text{Im}(C\Omega)) = d\hat{\mu}_2, \quad (2.22)$$

which means that  $E_\eta(J)$  and  $E_\eta(C\Omega)$  are exact forms on  $L_0$ . In order to study special Lagrangian deformations as in section 2.1 one thus has to consider the space of deformations  $\eta_{\text{sp}}$  such that  $E_{\eta_{\text{sp}}}(J) = 0$  and  $E_{\eta_{\text{sp}}}(\text{Im}(C\Omega)) = 0$  [18].

In the leading order effective action we will often be interested in first order deformation and the linearizations  $E'_\eta(\gamma) := \partial_t E_{t\eta}(\gamma)|_{t=0}$  of  $E_\eta$  will be of importance. A straightforward computation shows that for  $\gamma$  being a closed form on  $Y$  one has

$$d\gamma = 0: \quad E'_\eta(\gamma) = \mathcal{L}_\eta(\gamma)|_{L_0} = d(\eta \lrcorner \gamma)|_{L_0}. \quad (2.23)$$

Here we have used the standard formula for the Lie derivative on a form  $\mathcal{L}_\eta \gamma = d(\eta \lrcorner \gamma) + \eta \lrcorner d\gamma$ . Note that (2.23) immediately implies that

$$E'_\eta(J) = d\theta_\eta, \quad E'_\eta(\text{Im}(C\Omega)) = -2e^\phi d * \theta_\eta. \quad (2.24)$$

where  $\theta_\eta = \eta \lrcorner J|_{L_0}$  and we have again used the fact that  $*\theta_\eta = 2e^\phi \eta \lrcorner \text{Im} C\Omega|_{L_0}$  as in (2.16). One can proceed with the expansion of the exponential map and determine the full normal coordinate expansion. In particular, for a  $p$ -form one finds the small  $t$  expansion

$$\begin{aligned} E_{t\eta}(C_p) &= \frac{1}{p!} \left[ C_{i_1 \dots i_p} + t \cdot \left( \eta^n \partial_n C_{i_1 \dots i_p} - p \nabla_{i_1} \eta^n C_{ni_2 \dots i_p} \right) \right. \\ &\quad + \frac{1}{2} t^2 \cdot \left( \eta^n \partial_n (\eta^m \partial_m C_{i_1 \dots i_p}) - p \nabla_{i_1} \eta^n \eta^m \partial_m C_{ni_2 \dots i_p} - \frac{p(p-1)}{2} \nabla_{i_1} \eta^n \nabla_{i_2} \eta^m C_{nmi_3 \dots i_p} \right. \\ &\quad \left. \left. + \frac{p-2}{2} R_{ni_1 m}^j \eta^n \eta^m C_{ji_2 \dots i_p} \right) + \mathcal{O}(t^3) \right] d\xi^{i_1} \wedge \dots \wedge d\xi^{i_p}. \end{aligned} \quad (2.25)$$

Such normal coordinate expansions have been used for D-branes of different dimensions, for example, in refs. [19, 21, 22].

## The scalar potential for Lagrangian deformations

Using the exponential map and the corresponding normal coordinate expansion one can study the geometric properties of  $L_\eta$  when examined on  $L_0$ . This in particular includes the variations of the volume functional

$$V(L_\eta) = \int_{L_\eta} d^3 \xi e^{-\phi} \sqrt{\det(\iota^* g)} = \int_{L_\eta} e^{-\phi} \text{vol}_{L_\eta}, \quad (2.26)$$

where  $\iota$  is the pull-back of the Calabi-Yau metric to  $L_\eta$ . Despite the fact that the reference cycle  $L_0$  is special Lagrangian the analysis of  $V(L_\eta)$  is still rather involved for a general deformation vector field  $\eta$  [18]. Note that the volume functional (2.26) is obtained from the Dirac-Born-Infeld action in (2.20) by temporarily ignoring the brane field strength  $F_{D6}$  and the NS-NS B-field  $B_2$ .

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<sup>6</sup>This can be deduced from the fact that  $J$  and  $\text{Im}(C\Omega)$  are closed and one has in cohomology that  $[E_\eta(\gamma)] = [\gamma|_{L_0}]$ .

We will first restrict to the case that  $L_\eta$  is Lagrangian. In this case one can study the deformations of the volume functional employing a rather elegant computation. Later on we include the additional terms obtained in the general linearized analysis of [18]. Given a Lagrangian submanifold  $L$  in  $Y$  one notes that its induced volume form is proportional to the pullback of the holomorphic three-form  $\Omega$  to  $L$ . The proportionality factor is in general a function depending on the coordinates  $\xi$  of  $L$ . Thus, comparing  $\Omega|_L$  with the volume form on  $L$  one has

$$e^{-\phi} \text{vol}_L = 2C_{D6} \Omega|_L, \quad C_{D6}(\xi) = |C| e^{-i\theta_{D6}(\xi)}, \quad (2.27)$$

where  $|C|$  is introduced in (2.9). Recall that  $C$  is constant on  $Y$  since it only contains the constant phase of the O6-planes, in accord with the fact that the O6-planes wrap special Lagrangian cycles (2.12). In contrast,  $\theta_{D6}(\xi)$  is a real map, generally depending on the coordinates  $\xi$  on  $L$ . As  $\theta_{D6}$  appears in (2.27) with a  $2\pi$  periodicity it is a map from  $L$  to the circle, and induces a map  $\theta_{D6*} : \pi(L) \rightarrow \pi(S^1)$ . However, in order to avoid anomalies, one demands that  $\theta_{D6}$  actually admits a lift to a function with values on the full real axis. This implies that  $\theta_{D6*}(L)$  vanishes and translates to the condition that the class  $[d\theta_{D6}]$  vanishes. These Lagrangian submanifolds are known as graded Lagrangians [31], and the lift of  $\theta_{D6}$  to a real valued function is the grading.

Let us next consider a family of Lagrangian submanifolds  $L(t)$  which are obtained by deforming an initial Lagrangian  $L(0)$  for a distance  $t$  into the direction of the normal vector field  $\eta$ . For this to be a Lagrangian deformation  $\theta_\eta = \eta \lrcorner J|_L$  has to be closed. On each  $L(t)$  we can introduce a coordinate dependent phase  $\theta_{D6}(\xi, t)$ . We consider the  $t$ -derivative of  $e^{i\theta_{D6}(t)} \text{vol}_{L(t)}$  by evaluating

$$\frac{d}{dt}(e^{-\phi+i\theta_{D6}} \text{vol}_L) = (\mathcal{L}_\eta |C| \Omega)|_L = -e^{-\phi+i\theta_{D6}} (d\theta_{D6} \wedge \eta \lrcorner \text{vol}_L + i(d^* \theta_\eta) \text{vol}_L), \quad (2.28)$$

where  $d^* = *d*$  with  $*$  being the  $t$ -dependent Hodge-star on  $L(t)$ . Comparing real and imaginary parts one finds that

$$\frac{d}{dt} \theta_{D6} = -d^* \theta_\eta, \quad \frac{d}{dt} \text{vol}_L = -d\theta_{D6} \wedge * \theta_\eta, \quad (2.29)$$

Note that a particularly interesting case is when  $\theta_\eta = d\theta_{D6}$ , since in this case the second equation ensures that the volume of  $L$  is decreasing along  $\eta_{d\theta_{D6}}$ . In fact, this normal vector precisely parameterizes the directions to  $L$  in which its volume is most efficiently decreasing. This vector is known as mean curvature vector. Such Lagrangian mean curvature flows have been discussed intensively in the mathematical literature (see, e.g., refs. [32, 33], and references therein).

We are now in the position to evaluate the  $t$ -derivatives of  $\text{vol}_L$  at the point  $t = 0$ . We return to the case that  $L(0) = L_0$  is the background special Lagrangian. One then shows that

$$\frac{d}{dt} \text{vol}_L |_{t=0} = 0, \quad \frac{d^2}{dt^2} \text{vol}_L |_{t=0} = (dd^* \theta_\eta) \wedge * \theta_\eta. \quad (2.30)$$

In this computation it is crucial to use the fact that at  $t = 0$  one has  $\theta_{D6}(0) = \theta_{O6}$  is constant on  $L_0$ . This immediately implies the vanishing of the first derivative of  $\text{vol}_L$  using (2.29). To evaluate

the second derivative both equations (2.29) have to be applied successively. Finally, we can use (2.30) to evaluate the lowest order scalar potential for a Lagrangian brane on  $L(t)$  as

$$\frac{d^2}{dt^2}V(L_{t\eta})|_{t=0} = e^{-\phi} \int_{L_0} d * \theta_\eta \wedge * d * \theta_\eta = 4e^\phi \int_{L_0} d(\eta \lrcorner \text{Im}C\Omega) \wedge * d(\eta \lrcorner \text{Im}C\Omega) , \quad (2.31)$$

where  $V$  is the volume functional (2.26). As we will show later on, this term provides a scalar potential which corresponds to a D-term in the four-dimensional  $\mathcal{N} = 1$  effective theory for the D6-brane.

### The scalar potential for general deformations

Before turning to the details of the Kaluza-Klein reduction let us recall that one can extend the analysis to deformations  $\eta$  for which  $L(t)$  is no longer Lagrangian. In this case  $d\eta \lrcorner J$  does not necessarily vanish and (2.27) is not generally possible. However, one can still evaluate the second derivative of the volume of  $L(t)$  at the point  $t = 0$  as [18]

$$\frac{d^2}{dt^2}V(L_{t\eta})|_{t=0} = e^{-\phi} \int_{L_0} d(\eta \lrcorner J) \wedge * d(\eta \lrcorner J) + 4e^\phi \int_{L_0} d(\eta \lrcorner \text{Im}C\Omega) \wedge * d(\eta \lrcorner \text{Im}C\Omega) . \quad (2.32)$$

The new term depending on  $d(\eta \lrcorner J)$  is the obstruction for  $L(t)$  to be Lagrangian. In the four-dimensional  $\mathcal{N} = 1$  effective theory for the D6-brane this term can be obtained as one of the F-term contributions from a superpotential which we determine in section 4.

### The scalar potential including the B-field

So far we have discussed the scalar potential without the inclusion of the NS-NS B-field of Type IIA string theory and the brane field strength  $F_{D6}$ . To compute the leading order potential including  $F_{D6}$  we note that only the part  $\tilde{F}$  of  $F_{D6}$  contributes to the potential which has indices on the internal three-cycle wrapped by the brane. We perform a Taylor expansion of the Dirac-Born-Infeld action using

$$\sqrt{\det(\mathfrak{A} + \mathfrak{B})} = \sqrt{\det(\mathfrak{A})} \left[ \mathbf{1} + \frac{1}{2} \text{Tr}(\mathfrak{A}^{-1} \mathfrak{B}) + \frac{1}{8} \left( [\text{Tr}(\mathfrak{A}^{-1} \mathfrak{B})]^2 - 2 \text{Tr}([\mathfrak{A}^{-1} \mathfrak{B}]^2) \right) + \dots \right] \quad (2.33)$$

for small fluctuations  $\mathfrak{B}$  and invertible  $\mathfrak{A}$ . The matrix  $\mathfrak{B}$  we want to identify with the normal coordinate expansion of  $B_2 - \tilde{F}$  in (2.20), while  $\mathfrak{A}$  is the background metric of the Calabi-Yau space restricted to  $L_0$ . Recall that the normal coordinate expansion  $E_{t\eta}(B_2)$  was given in (2.25). One notes that the first term in the expansion (2.33) is canceled by tadpole cancellation of the D6-branes with the O6-planes in the background. Moreover, the second and third term in (2.33) do not contribute to the potential since  $\mathfrak{A}$  is symmetric while  $\mathfrak{B}$  is anti-symmetric. Evaluating the remaining term  $\text{Tr}([\mathfrak{A}^{-1} \mathfrak{B}]^2)$  and adding the result (2.32) one finds

$$V_{\text{DBI}}^{\text{SF}} = e^{-\phi} \int_{L_0} [d * \theta_\eta \wedge * d * \theta_\eta + d\theta_\eta \wedge * d\theta_\eta + (\tilde{F} - B_2 - d\theta_\eta^B) \wedge * (\tilde{F} - B_2 - d\theta_\eta^B)] , \quad (2.34)$$

which is still expressed in the ten-dimensional string frame. Here we have introduced the abbreviation

$$\theta_\eta^B = \eta \lrcorner B_2|_{L_0} , \quad (2.35)$$

which is the B-field analog of  $\theta_\eta = \eta \lrcorner J|_{L_0}$ . This concludes the computation of the scalar potential from the Dirac-Born-Infeld action. In a next step we want to introduce a Kaluza-Klein basis and determine the complete leading order effective action including the kinetic terms.

### 2.3 A Kaluza-Klein basis

In performing a Kaluza-Klein reduction of the D6-brane action to four space-time dimensions we like to include all massive modes corresponding to arbitrary deformations of  $L_0$  to  $L_\eta$ . This means that we include sections  $s_I$  of  $NL_0$  which yield one-forms in the contraction with  $J$

$$\theta_I = s_I \lrcorner J|_{L_0} \in \Omega^1(L_0) . \quad (2.36)$$

For a compact  $L_0$  it is possible to label these one-forms by indices  $I = 1, \dots, \infty$  by considering the Kaluza-Klein eigenmodes of the Laplacian  $\Delta_{L_0}$ . In this case the zero modes  $\Delta_{L_0}\theta_i = 0$  are precisely the harmonic forms  $\theta_i$  introduced in (2.16). However, the basis adopted to  $\Delta_{L_0}$  is not always useful, since it explicitly depends on the metric inherited from the ambient Calabi-Yau manifold. In the following we will therefore work with a general countable basis of  $\Omega^1(L_0)$ , and later use the induced metric to interpret the final expressions after performing the reduction. In general we will always demand that the one-forms  $\theta_I$  are finite in the  $L^2$ -metric

$$\mathcal{G}(\tilde{\alpha}, \tilde{\beta}) = \int_{L_0} \tilde{\alpha} \wedge * \tilde{\beta} , \quad (2.37)$$

where  $\tilde{\alpha}, \tilde{\beta} \in \Omega^1(L_0)$ .

Let us now turn to the discussion of the  $U(1)$  gauge field on the D6-brane. It admits the general expansion

$$A_{\text{D6}} = A^J h_J + a^I \hat{\alpha}_I , \quad (2.38)$$

where  $h_J \in C^\infty(L_0)$  is a basis of functions on  $L_0$  and  $\hat{\alpha}_J \in \Omega^1(L_0)$  is a basis of one-forms on  $L_0$ . Here again a countable basis can be chosen due to the compactness of  $L_0$ . Note that the field-strength of  $A_{\text{D6}}$  is given by

$$F_{\text{D6}} = F^J h_J - A^J \wedge dh_J + da^I \wedge \hat{\alpha}_I + \tilde{F} , \quad \tilde{F} = a^I d\hat{\alpha}_I + f_{\text{D6}} , \quad (2.39)$$

where  $f_{\text{D6}} \in H^2(L_0, \mathbb{R})$  is a background flux of  $F_{\text{D6}}$  on  $L_0$ . The terms  $dh_J$  and  $d\hat{\alpha}_I$  arise due to the fact that the functions  $h_J$  need not to be constant on  $L_0$  and the one-forms  $\hat{\alpha}_I$  need not to be closed.

We thus find that an infinite tower of scalars  $a^I$  which are coefficients of *exact* forms are actually gauged by the gauge fields  $A^J$  for which  $dh_J \neq 0$ . Moreover, scalars  $a^I$  arising in the expansion in

*non-closed* forms appear without four-dimensional derivative in the expansion (2.39). To see this, we introduce a special basis adopted to the metric induced on  $L_0$ . More precisely, via the Hodge decomposition each one-form  $\hat{\alpha}_I$  can be uniquely decomposed into a harmonic form, an exact form  $d\hat{h}_I$  and an co-exact form  $d^*\hat{\gamma}_I$  on  $L_0$  as

$$\hat{\alpha}_I = \mu_I^i \tilde{\alpha}_i + d\hat{h}_I + d^*\hat{\gamma}_I , \quad (2.40)$$

where  $\tilde{\alpha}_i$  are the  $b^1(L_0)$  harmonic forms introduced in (2.19). We thus pick a basis of the space of exact forms  $\Omega_{\text{ex}}^1(L_0)$  denoted by  $dh_I$  and a basis  $d^*\gamma_I$  of the space  $\Omega_{\text{co-ex}}^1(L_0)$  which are exact with respect to  $d^*$ . By appropriate redefinition we can introduce scalars  $\hat{a}^J$  parameterizing the expansion in  $dh_J$ . Denoting the coefficients of the non-closed forms  $d^*\gamma_I$  by  $\tilde{a}^I$ , and the coefficients of the harmonic forms by  $a^j$  the expansion (2.39) reads

$$\begin{aligned} F_{\text{D6}} &= F^I h_I + da^j \wedge \tilde{\alpha}_j + \mathcal{D}\hat{a}^I \wedge dh_I + d\tilde{a}^J \wedge d^*\gamma_I + \tilde{F} , \\ \mathcal{D}\hat{a}^I &= d\hat{a}^I - A^I , \quad \tilde{F} = \tilde{a}^I dd^*\gamma_I + f_{\text{D6}} . \end{aligned} \quad (2.41)$$

From this we conclude that precisely the scalars  $\hat{a}^I$  are gauged by  $A^I$ . Since the four-dimensional effective theory is an  $\mathcal{N} = 1$  supersymmetric theory one infers that there will be D-terms induced due to these gaugings  $\mathcal{D}\hat{a}^I$ , while F-terms are induced due to  $\tilde{F}$ . We will determine the D-term in section 4, and check that it matches the moment map analysis of ref. [34].

## 2.4 The four-dimensional effective action

We can now determine the kinetic terms for the chiral multiplets of the D6-brane coupled to the bulk supergravity. Since the bulk action has been Kaluza-Klein reduced on the orientifold background in ref. [11] we will focus on the reduction of the D6-brane action (2.20). The contributions entirely due to bulk fields are later included in the determination of the  $\mathcal{N} = 1$  characteristic data.

### Dirac-Born-Infeld action

Let us start by considering the Kaluza-Klein reduction of the first term in (2.20), i.e. the Dirac-Born-Infeld action. We expand the determinant in (2.20) to quadratic order in the fluctuations around the supersymmetric background. These are precisely the fluctuations of the embedding  $\iota$  of  $L$  parameterized by the fields  $\eta^i$  of (2.16) and the Wilson line scalars  $a^i$  introduced in (2.19). The normal coordinate expansions of the ten-dimensional metric on the D6-brane world-volume is given to leading order by

$$\iota^* g_{10} = (e^{2D} \eta_{\mu\nu} + g(\partial_\mu \eta, \partial_\nu \eta)) dx^\mu \cdot dx^\nu + (\iota^* g + \delta(\iota^* g))_{mn} d\xi^m \cdot d\xi^n , \quad (2.42)$$

where  $g_{mn}$  is the induced metric on  $L$ , and  $\delta(\iota^* g)_{mn}$  is the metric variation induced by the variation of the background Kähler and complex structure. Note that the four-dimensional metric  $\eta_{\mu\nu}$  is

rescaled to the four-dimensional Einstein frame.<sup>7</sup> One first performs the Taylor expansion of the determinant while using (2.42). Inserting the result together with  $F_{D6}$  given in (2.41) into the first part of (2.20) we obtain the four-dimensional action

$$S_{\text{DBI}}^{(4)} = - \int \frac{1}{2} \text{Re} f_{rIJ} F^I \wedge *F^J + e^{2D} \mathcal{G}_{ij} da^i \wedge *da^j + e^{2D} \tilde{\mathcal{G}}_{IJ} d\tilde{a}^I \wedge *d\tilde{a}^J \\ + e^{2D} \mathcal{G}_{IJ} \mathcal{D}\hat{a}^I \wedge *\mathcal{D}\hat{a}^J + e^{2D} \hat{\mathcal{G}}_{IJ} d\eta^I \wedge *d\eta^J + V_{\text{DBI}} *1 , \quad (2.43)$$

in the four-dimensional Einstein frame. The covariant derivative  $\mathcal{D}\hat{a}^I$  was introduced in (2.41) and indicates the gauging of the infinite tower of scalars  $\hat{a}^I$ . The potential term  $V_{\text{DBI}}$  depends on the deformations  $\delta(\iota^*g)_{mn}$  of the calibration conditions (2.14) induced by the variation of the induced metric on  $L_\eta$  which we computed in (2.32). Moreover, one obtains an additional term depending on the modes violating the background condition  $F_{D6} - B_2|_{L_0} = 0$  as in (2.34). Explicitly we find

$$V_{\text{DBI}} = \frac{e^{3\phi}}{\mathcal{V}^2} \int_{L_0} d^*\theta_\eta \wedge *d^*\theta_\eta + \frac{e^{3\phi}}{\mathcal{V}^2} \int_{L_0} \left( d\theta_\eta \wedge *d\theta_\eta + (\tilde{F} - B_2 - d\theta_\eta^B) \wedge *(\tilde{F} - B_2 - d\theta_\eta^B) \right) , \quad (2.44)$$

where  $\tilde{F}$  is defined in (2.41). In the following we will discuss the metric functions appearing in the kinetic terms of (2.43).

The first term in (2.43) is the kinetic term for the  $U(1)$  gauge bosons  $A^I$ . The gauge coupling function is thus given to leading order by

$$f_{rIJ} = \int_{L_0} (2 \text{Re}(C\Omega) + iC_3) h_I h_J , \quad (2.45)$$

where the volume of  $L_0$  has been replaced using (2.14). Note that  $\text{Re} f_{rIJ}$  admits a simple geometrical interpretation as  $L^2$ -metric on the space of functions on  $L_0$ . More generally, without introducing a specific basis and restricting to a special Lagrangian one writes for two functions  $h, \tilde{h}$  on  $L$

$$\text{Re} f_r(h, \tilde{h})|_L = e^{-\phi} \int_L h \wedge *\tilde{h} , \quad (2.46)$$

which readily reduces to (2.45) on  $L = L_0$  using  $*1 = \text{vol}_L$  and (2.14).

The second, third and fourth term in (2.43) are the kinetic terms for the Wilson line moduli  $a^i, \tilde{a}^I, \hat{a}^I$ , where the later appear with the covariant derivative  $\mathcal{D}\hat{a}^I = d\hat{a}^I + A^I$  as introduced in (2.41). The appearing metrics take the form

$$\mathcal{G}_{ij} = \frac{1}{2} e^{-\phi} \mathcal{G}(\tilde{\alpha}_i, \tilde{\alpha}_j) , \quad \tilde{\mathcal{G}}_{IJ} = \frac{1}{2} e^{-\phi} \mathcal{G}(d^*\gamma_I, d^*\gamma_J) , \quad \mathcal{G}_{IJ} = \frac{1}{2} e^{-\phi} \mathcal{G}(dh_I, dh_J) , \quad (2.47)$$

where  $\mathcal{G}$  is the  $L^2$ -metric defined in (2.37), and  $\tilde{\alpha}_i, dh^I$  and  $d^*\gamma_I$  are the one-form basis introduced in (2.41). The fifth term in (2.43) contains the field space metric for the deformations  $\eta^I$  and is of the form

$$\hat{\mathcal{G}}_{IJ} = \int_{L_0} g(s_I, s_J) \text{Re}(C\Omega) = \frac{1}{2} e^{-\phi} \mathcal{G}(\theta_I, \theta_J) . \quad (2.48)$$

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<sup>7</sup>Recall that the four-dimensional metric in the Einstein frame  $\eta$  is related to the string frame metric  $\eta^{\text{SF}}$  via  $\eta = e^{-2D} \eta^{\text{SF}}$ .

where  $\theta_I$  are the one-forms on  $L_0$  introduced in (2.36). Let us comment on the derivation of the second identity in (2.48). Here we first have to use the fact that  $g(s_i, s_j) = J(s_i, I s_j) = (I s_j) \lrcorner \theta_i$ , where  $J$  is the Kähler form and  $I$  is the complex structure on  $Y$ . Next we deduce from  $J \wedge \text{Re}(C\Omega) = 0$  that we can move the  $I s_j$  to obtain  $\theta_i \wedge (I s_j) \lrcorner \text{Re} C\Omega$ . However, since  $C\Omega$  is a  $(3, 0)$ -form one deduces using

$$2(I s_j) \lrcorner \text{Re} C\Omega = -2s_j \lrcorner \text{Im}(C\Omega) = e^{-\phi} * \theta_j , \quad (2.49)$$

and the identity (2.16) the second equality in (2.48).

This completes our reduction of the Dirac-Born-Infeld action. Let us stress that the reduction so far only included the leading order terms. In order to fully extract the  $\mathcal{N} = 1$  characteristic data, however, we will need to match also higher order terms. It turns out that an efficient strategy to proceed is to include these by using supersymmetry and a careful study of the the Chern-Simons action. We will turn to the Kaluza-Klein reduction of this part of the D-brane action in the following.

### Chern-Simons action

Let us now turn to the dimensional reduction of the Chern-Simons part of the D6-brane action. In the reduction one can again perform a normal coordinate expansion of the form-fields appearing in the action. However, we will take here a somewhat different route and parameterize the normal variations by introducing a four-chain  $\mathcal{C}_4$  which contains the three-cycle  $L_\eta$  in its boundary

$$\partial \mathcal{C}_4 = L_\eta - L_0 , \quad (2.50)$$

where  $L_0$  is the reference three-cycle, the supersymmetric background cycle.

We consider the Chern-Simons action containing the R-R forms  $C_3, C_5$  and  $C_7$  given by

$$S_{\text{CS}} = \int_{\mathcal{W}_7^{(0)}} e^{F-B_2} \wedge (C_3 + C_5 + C_7) + S_{\text{CS}}^{\mathcal{C}_4}. \quad (2.51)$$

Here  $\mathcal{W}_7^{(0)} = \mathcal{M}^{3,1} \times L_0$ ,

$$S_{\text{CS}}^{\mathcal{C}_4} = \int_{\mathcal{W}_8} d[e^{F-B_2} \wedge (C_3 + C_5 + C_7)] , \quad (2.52)$$

and  $\mathcal{W}_8 = \mathcal{M}^{3,1} \times \mathcal{C}_4$  such that  $\mathcal{W}_7 \subset \partial \mathcal{W}_8$ . This is in a similar spirit as the constructions in [35]. To perform the Kaluza-Klein reduction of (2.52) we consider the expansion of  $\mathcal{A}$ , the wedge product between the R-R forms and the B-field introduced in (2.5), as

$$\begin{aligned} \sum_{p=3,5,7,9} e^{-B_2} \wedge C_p &= (\xi^k \alpha_k - \tilde{\xi}_\lambda \beta^\lambda) + (A^\alpha \wedge \omega_\alpha + A_\alpha \wedge \tilde{\omega}^\alpha) \\ &+ (C_2^\lambda \wedge \alpha_\lambda - \tilde{C}_k^2 \wedge \beta^k) + (C_3^0 + C_3^a \wedge \omega_a + C_a^3 \wedge \tilde{\omega}^a + C_0^3 \wedge \text{vol}_Y) . \end{aligned} \quad (2.53)$$



In (2.53),  $(\alpha_\lambda, \beta^k)$  is a basis of  $H_-^3(Y, \mathbb{R})$ ,  $\omega_a, \omega_\alpha$  are basis of  $H_-^2(Y, \mathbb{R}), H_+^2(Y, \mathbb{R})$ , and  $\tilde{\omega}^a, \tilde{\omega}^\alpha$  are a basis of  $H_+^4(Y, \mathbb{R}), H_-^4(Y, \mathbb{R})$ . Here we introduced the four-dimensional two-forms  $(C_2^\lambda, \tilde{C}_k^2)$  which are dual to the scalars  $(\xi^k, \tilde{\xi}_\lambda)$ , already introduced in (2.8). The vectors  $A^\alpha$  have been already introduced in (2.11), and  $A_\alpha$  are their four-dimensional duals. Moreover, the Kaluza-Klein expansion (2.53) also contains the four-dimensional three-forms  $(C_3^0, C_3^a, C_a^3, C_0^3)$  which are non-dynamical, but will crucially contribute to the scalar potential as in ref. [22].

Note also that the fields defined in (2.53) are not the expansions from the R-R forms alone, but in general combine with the NS-NS two-form  $B_2$ . Denoting by a hat  $\hat{\phantom{x}}$  the fields which arise in the expansion of the R-R forms alone, one finds, for example, that

$$B_2\text{-corrected: } \begin{cases} \text{vectors:} & A^\alpha = \hat{A}^\alpha, & A_\alpha = \hat{A}_\alpha - \hat{A}^\beta b^a \mathcal{K}_{\beta a \alpha}, \\ \text{3-forms:} & C_3^0 = \hat{C}_3^0, & C_3^a = \hat{C}_3^a + \hat{C}_3^0 b^a, & \text{etc.} \end{cases} \quad (2.54)$$

where  $\hat{A}^\alpha, \hat{C}_3^0$  and  $\hat{A}_\alpha, \hat{C}_3^a$  denote the space-time vector bosons and three-forms coming from the expansion of  $C_3$  and  $C_5$ , respectively. In contrast, the scalars and two-forms in (2.53) have no mixing with the B-field such that

$$\text{no } B_2\text{-correction:} \quad \text{scalars: } (\xi^k, \tilde{\xi}_\lambda) \quad \text{2-forms: } (C_2^\lambda, \tilde{C}_k^2). \quad (2.55)$$

As discussed in more detail in section 5 the situation is precisely reversed under mirror symmetry. In fact, using the results on the side without  $B_2$  corrections mirror symmetry can be used to compute the corrected couplings.

The Chern Simons action is dimensionally reduced by inserting (2.53) into (2.52). Focusing on the couplings of  $A^\alpha$  and  $(C_2^\lambda, \tilde{C}_k^2)$  in favor over their duals, one finds <sup>8</sup>

$$S_{\text{CS}}^{(4)} = \int \frac{1}{2} \text{Im} f_{\text{r}}{}_{IJ} F^I \wedge F^J - (\delta_{I\lambda} dC_2^\lambda - \delta_I^k d\tilde{C}_k^2) \wedge A^I - (\mathcal{I}_{I\lambda} dC_2^\lambda - \mathcal{I}_I^k d\tilde{C}_k^2) \wedge da^I + \mathcal{L}_{\text{mix}} + \mathcal{L}_3. \quad (2.56)$$

Here  $\mathcal{L}_{\text{mix}}$  corresponds to the mixing of the brane and bulk gauge bosons

$$\mathcal{L}_{\text{mix}} = (a^J \Delta_{(I)J\alpha} + \Gamma_{(I)\alpha}) dA^\alpha \wedge F^I + \tilde{\mathcal{J}}_{(I)}^\alpha dA_\alpha \wedge F^I, \quad (2.57)$$

and  $\mathcal{L}_3$  is the term which depends on the three-form field strengths as

$$\mathcal{L}_3 = dC_3^0 \left( \frac{1}{2} a^I a^J \Delta_{IJ} + a^J \tilde{\Gamma}_J \right) + dC_3^a (a^J \Delta_{Ja} + \Gamma_a) + dC_a^3 \tilde{\mathcal{J}}^a. \quad (2.58)$$

In order to display the couplings appearing in this action we first define the integral  $\mathcal{I}(\tilde{\alpha}, \alpha)$  between a one-form  $\tilde{\alpha}$  on  $L_\eta$  and a three-form  $\alpha$  on  $Y$ , as well as the integral  $\mathcal{J}(\tilde{\beta}, \omega)$  between a

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<sup>8</sup>One could also include the couplings to  $A_\alpha$  and  $(\xi^k, \tilde{\xi}_\lambda)$ . In this case one has to analyze also the bulk action keeping all forms and their duals as in ref. [30].

two-form  $\tilde{\beta}$  on  $L_\eta$  and a two-form  $\omega$  on  $Y$ . To do that we again extend the forms defined on  $L_0$  to the chain  $\mathcal{C}_4$  such that they are constant along the normal directions of  $L_\eta$  in  $Y$ . We define

$$\mathcal{I}(\tilde{\alpha}, \alpha) = \int_{\mathcal{C}_4} \tilde{\alpha} \wedge \alpha, \quad \mathcal{J}(\tilde{\beta}, \omega) = \int_{\mathcal{C}_4} \tilde{\beta} \wedge \omega. \quad (2.59)$$

Furthermore, we will also need a pairing  $\delta$  between a function  $h$  on  $L_0$  and three-form  $\alpha$  on  $Y$ , as well as a pairing  $\Delta$  between a one-form  $\gamma$  on  $L_0$  and a two-form on  $Y$ . Hence, we set

$$\delta(h, \alpha) = \int_{L_0} h \alpha + \mathcal{I}(dh, \alpha), \quad \Delta(\gamma, \beta) = \int_{L_0} \gamma \wedge \beta + \mathcal{J}(d\gamma, \beta). \quad (2.60)$$

Note that these latter definitions include terms supported on  $L_0$  which are non-vanishing even in the limit of vanishing normal displacement  $\eta$ . This redefinition is necessary since  $\mathcal{I}$  and  $\mathcal{J}$  vanish for a vanishing normal displacement. In fact, we can expand (2.59) to first order in  $\eta$  for small normal displacement in  $\mathcal{C}_4 = L_\eta - L_0$  and obtain

$$\mathcal{I}(\tilde{\alpha}, \alpha) = \int_{L_0} \tilde{\alpha} \wedge \eta \lrcorner \alpha + \dots, \quad \mathcal{J}(\tilde{\beta}, \omega) = \int_{L_0} \tilde{\beta} \wedge \eta \lrcorner \omega + \dots, \quad (2.61)$$

which has a leading term linear in  $\eta$ .

Having introduced the pairings we can display the couplings in (2.56), (2.57) and (2.58). Let us start with the couplings in (2.56) obtained as

$$\mathcal{I}_{I\lambda} = \mathcal{I}(\hat{\alpha}_I, \alpha_\lambda), \quad \mathcal{I}_I^k = \mathcal{I}(\hat{\alpha}_I, \beta^k), \quad \delta_{I\lambda} = \delta(h_I, \alpha_\lambda), \quad \delta_I^k = \delta(h_I, \beta^k). \quad (2.62)$$

Furthermore, in the mixed term  $\mathcal{L}_{\text{mix}}$ , given in (2.57), for the gauge bosons one finds

$$\Delta_{(I)J\alpha} = \Delta(h_I \hat{\alpha}_J, \omega_\alpha), \quad \Gamma_{(I)\alpha} = \mathcal{J}(h_I f_{\text{D6}}, \omega_\alpha), \quad \tilde{\mathcal{J}}_{(I)}^\alpha = \int_{\mathcal{C}_4} h_I \tilde{\omega}^\alpha. \quad (2.63)$$

Finally, we introduce the coefficients in (2.58) as

$$\Delta_{Ja} = \Delta(\hat{\alpha}_J, \omega_a), \quad \Gamma_a = \mathcal{J}(f_{\text{D6}}, \omega_a), \quad (2.64)$$

for couplings between the ambient space two-forms  $\omega_a$  and forms  $\hat{\alpha}_J$  and  $f_{\text{D6}}$  on the D6-brane. The remaining couplings are

$$\Delta_{IJ} = \int_{L_0} \hat{\alpha}_I \wedge d\hat{\alpha}_J, \quad \tilde{\Gamma}_J = \int_{L_0} \hat{\alpha}_J \wedge f_{\text{D6}}, \quad \tilde{\mathcal{J}}^a = \int_{\mathcal{C}_4} \tilde{\omega}^a. \quad (2.65)$$

It is not hard to interpret the different terms appearing in the action (2.56). The first term corresponds to the theta-angle term of the gauge theory on the D6-brane and thus contains the imaginary part of the gauge kinetic function. The second is a Green-Schwarz term which indicates that the scalar fields  $(\xi^k, \tilde{\xi}_\lambda)$  dual to the two-forms  $(C_k^2, \tilde{C}_2^\lambda)$  are gauged by the D6-brane vector fields  $A^I$ . In fact, upon elimination of  $(C_k^2, \tilde{C}_2^\lambda)$  one finds the covariant derivative

$$D\xi^k = d\xi^k + \delta_I^k A^I, \quad D\tilde{\xi}_\lambda = d\tilde{\xi}_\lambda + \delta_{I\lambda} A^I, \quad (2.66)$$

We will show that the corresponding D-term appears in  $V_{\text{DBI}}$  as expected from supersymmetry in section 4. The third term in (2.56) will be of importance for the derivation of the Kähler potential and complex coordinates on the  $\mathcal{N} = 1$  field space. Upon elimination of  $(C_k^2, \tilde{C}_2^\lambda)$  it induces a mixing of the kinetic terms of  $a^I = (a^i, \hat{a}^I)$  and  $(\xi^k, \tilde{\xi}_\lambda)$ . More precisely, one finds the modified four-dimensional kinetic terms

$$\mathcal{L}_{C_3}^{\text{kin}} = G_{kl} \nabla \xi^k \wedge * \nabla \xi^l + G^{\lambda\kappa} \nabla \tilde{\xi}_\lambda \wedge * \nabla \tilde{\xi}_\kappa + 2G_k^\lambda \nabla \xi^k \wedge * \nabla \tilde{\xi}_\lambda \quad (2.67)$$

where the modified derivatives  $\nabla$  are defined by

$$\nabla \xi^k \equiv D \xi^k + \mathcal{I}_I^k da^I \quad \text{and} \quad \nabla \tilde{\xi}_\lambda \equiv D \tilde{\xi}_\lambda + \mathcal{I}_{I\lambda} da^I, \quad (2.68)$$

with the metric  $G$  given as in the closed string case,

$$G_{kl} = \frac{1}{2} e^{2D} \int_Y \alpha_k \wedge * \alpha_l, \quad G^{\lambda\kappa} = \frac{1}{2} e^{2D} \int_Y \beta^\lambda \wedge * \beta^\kappa, \quad G_k^\lambda = -\frac{1}{2} e^{2D} \int_Y \alpha_l \wedge * \beta^\lambda. \quad (2.69)$$

Note that the form of the metric  $G$  for  $\nabla \xi^k$  and  $\nabla \tilde{\xi}_\lambda$  closely resembles the form of the metric  $\mathcal{G}_{ij}$  for the scalars  $a^i$  as seen from (2.43) and (2.47). We will exploit this observation in the detailed study of the moduli space geometry later on. This similarity only occurs in the  $\mathcal{N} = 1$  orientifold for which the field space metric is Kähler. In the underlying  $\mathcal{N} = 2$  set-ups the moduli space containing the R-R scalars is a quaternionic manifold.

The  $\mathcal{L}_{\text{mix}}$  is a kinetic mixing term between the  $U(1)$  from the brane with the vector field from the  $C_3$  expansion. This term will be important in the derivation of the gauge coupling function in section 3.

The term  $\mathcal{L}_3$  given by (2.58) contains the four-dimensional three-forms which arise in the expansion of  $C_3, C_5, C_7$ . Very similar to the analysis in ref. [22] they will be crucial to complete the scalar potential contributions in  $V_{\text{DBI}}$  to supersymmetric F-terms which can be obtained from a superpotential. To find the scalar potential from the three-form potential one has to eliminate the forms  $dC_3^0, dC_a^3$  and  $dC_3^a$  from the complete four-dimensional effective action. In particular, in addition to  $\mathcal{L}_3$  one also has to include the reduction of the ten-dimensional kinetic term in (2.4). The resulting action for the three-forms will be given in terms of the matrix  $\mathcal{N}_{\hat{A}\hat{B}}$  defined as

$$\mathcal{N}_{\hat{A}\hat{B}} = \begin{pmatrix} -\frac{1}{3} \mathcal{K}_{abc} b^a b^b b^c & \frac{1}{2} \mathcal{K}_{Bab} b^a b^b \\ \frac{1}{2} \mathcal{K}_{Aab} b^a b^b & -\mathcal{K}_{ABa} b^a \end{pmatrix} - i\mathcal{V} \begin{pmatrix} 1 + 4G_{ab} b^a b^b & -4G_{Ba} b^a \\ -4G_{Aa} b^a & 4G_{AB} \end{pmatrix}, \quad (2.70)$$

where  $\hat{A} = \{0, a, \alpha\}$ , and one has to use  $\mathcal{K}_{ab\alpha} = \mathcal{K}_{\alpha\beta\gamma} = 0$ . Using these definitions we find after rescaling to the Einstein that

$$S_{3\text{-form}} = \int \frac{1}{4} e^{-4D} (\text{Im} \mathcal{N})^{-1 \hat{a}\hat{b}} (dC_{\hat{a}}^3 - \mathcal{N}_{\hat{a}\hat{c}} dC_{\hat{c}}^3) \wedge * (dC_{\hat{b}}^3 - \bar{\mathcal{N}}_{\hat{b}\hat{d}} dC_{\hat{d}}^3) + \mathcal{L}_3, \quad (2.71)$$

where  $C_{\hat{a}}^3 = (C_3^0, C_3^a)$  and  $C_{\hat{a}}^3 = (C_0^3, C_a^3)$ , and  $\mathcal{L}_3$  is the D-brane coupling defined in (2.58). As in ref. [22] we next dualize  $dC_3^0, dC_3^a$  and  $dC_a^3, dC_0^3$  into flux scalars  $e_0, e_a, m^a, m^0$ . In ref. [42] the

interpretation of these scalars as quantized fluxes has been provided. They also arise as background values of the field strengths  $F_2 = m^a \omega_a$ ,  $F_4 = e_a \tilde{\omega}^a$  and  $F_6 = e_0 \text{vol}_Y$  as their expansions into harmonic forms on  $Y$ . In addition there is Romans mass parameter  $F_0 = G_0 = m^0$ . After dualization of the three-forms one finds the scalar potential

$$V_{\text{flux+CS}} = \frac{1}{4} e^{-4D} (\text{Im}\mathcal{N})^{-1 \hat{a}\hat{b}} (\tilde{e}_{\hat{a}} - \mathcal{N}_{\hat{a}\hat{c}} \tilde{m}^{\hat{c}}) \wedge *(\tilde{e}_{\hat{b}} - \bar{\mathcal{N}}_{\hat{b}\hat{d}} \tilde{m}^{\hat{d}}), \quad (2.72)$$

where

$$\begin{aligned} \tilde{e}_0 &= e_0 + \frac{1}{2} \int_{\mathcal{C}_4} \tilde{F} \wedge \tilde{F} + \frac{1}{2} \int_{L_0} \tilde{F} \wedge a^I \hat{\alpha}_I, \\ \tilde{e}_a &= e_a + \int_{\mathcal{C}_4} \tilde{F} \wedge \omega_a + \int_{L_0} a^I \hat{\alpha}_I \wedge \omega_a, \\ \tilde{m}^a &= m^a + \int_{\mathcal{C}_4} \tilde{\omega}^a, \quad \tilde{m}^0 = m^0. \end{aligned} \quad (2.73)$$

The additional terms in the definitions (2.73) arise precisely because of the term  $\mathcal{L}_3$  from the D6-brane. Luckily, apart from these shifts, the closed string moduli dependence of the potential (2.72) agrees with the analog expression found in ref. [11], and we will thus be able to integrate it into a superpotential without much effort.

### Restriction of the brane action to harmonic modes

To conclude our reduction of the D6-brane action let us also give the result which is obtained by restricting to harmonic forms. This corresponds to a truncation of the Kaluza-Klein tower of the brane fields to include only the lightest states. The resulting action will be useful in the next section when analyzing the moduli space. The Kaluza-Klein Ansatz for the D6-brane field strength, eqn. (2.41), simplifies to

$$F_{\text{D6}} = F + da^i \wedge \tilde{\alpha}_i + f_{\text{D6}}. \quad (2.74)$$

This implies that the DBI action reduces to

$$S_{\text{DBI}}^{(4)} = - \int \frac{1}{2} \text{Re} f_{\text{r}} F \wedge *F + e^{2D} \mathcal{G}_{ij} da^i \wedge *da^j + e^{2D} \hat{\mathcal{G}}_{ij} d\eta^i \wedge *d\eta^j, \quad (2.75)$$

with the metric  $\mathcal{G}_{ij}$  being the same as in (2.47), and  $\hat{\mathcal{G}}_{ij}$  the restriction of (2.48) to supersymmetric deformations (i.e., harmonic one-forms  $\theta_i$ ). The gauge coupling function (2.45) simplifies to

$$\text{Re} f_{\text{r}} = \int_{L_0} 2 \text{Re}(C\Omega), \quad (2.76)$$

as we restrict  $h_I$  to the only harmonic function, the constant function which we normalized to 1. We did not include the scalar potential  $V_{\text{DBI}}$  since it vanishes when restricting to the harmonic subset of forms, as we will show in section 4.

The truncation of the Chern-Simons action to the harmonic modes is

$$S_{\text{CS}}^{(4)} = \int \frac{1}{2} \text{Im} f_r F \wedge F - (\delta_\lambda dC_2^\lambda - \delta^k d\tilde{C}_k^2) \wedge A - (\mathcal{I}_{i\lambda} dC_2^\lambda - \mathcal{I}_i^k d\tilde{C}_k^2) \wedge da^i \quad (2.77)$$

$$+ (a^j \Delta_{j\alpha} + \Gamma_\alpha) dA^\alpha \wedge F + \tilde{\mathcal{J}}^\alpha dA_\alpha \wedge F + dC_3^a (a^j \Delta_{ja} + \Gamma_a) + dC_a^3 \tilde{\mathcal{J}}^a + dC_3^0 (a^j \tilde{\Gamma}_j) ,$$

with couplings

$$\delta_\lambda = \int_{L_0} \alpha_\lambda, \quad \delta^k = \int_{L_0} \beta^k, \quad \mathcal{I}_{i\lambda} = \int_{\mathcal{C}_4} \tilde{\alpha}_i \wedge \alpha_\lambda, \quad \mathcal{I}_i^k = \int_{\mathcal{C}_4} \tilde{\alpha}_i \wedge \beta^k, \quad (2.78)$$

$$\Delta_{iA} = \int_{L_0} \tilde{\alpha}_i \wedge \omega_A, \quad \Gamma_A = \int_{\mathcal{C}_4} f_{\text{D6}} \wedge \omega_A, \quad A = \{a, \alpha\}, \quad \tilde{\Gamma}_i = \int_{L_0} \tilde{\alpha}_i \wedge f_{\text{D6}},$$

and  $\tilde{\mathcal{J}}^A = \int_{\mathcal{C}_4} \tilde{\omega}^A$  as defined in (2.65). One realizes that the couplings  $(\delta_\lambda, \delta^k)$  and  $\Delta_{iA}, \tilde{\Gamma}_i$  are constants, while the couplings  $(\mathcal{I}_{i\lambda}, \mathcal{I}_i^k)$  and  $\Gamma_A$  depend on the brane deformations through the chain  $\mathcal{C}_4$ .

Let us take a closer look at the three-form couplings  $\mathcal{L}_3$ . We can expand the  $\mathcal{C}_4$  chain around the  $L_0$  cycle to see the explicit dependence on the brane deformations. Just like (2.61), we obtain, up to first order in the open fields,

$$\mathcal{L}_3 = dC_3^a \int_{L_0} (a^j \tilde{\alpha}_j \wedge \omega_a + \eta^j s_j \lrcorner \omega_a \wedge f_{\text{D6}}) + dC_a^3 \int_{L_0} \eta^j s_j \lrcorner \tilde{\omega}^a + dC_3^0 \int_{L_0} a^j \tilde{\alpha}_j \wedge f_{\text{D6}}. \quad (2.79)$$

Note that this implies that  $\mathcal{L}_3$  is non-vanishing also in the case we restrict to harmonic forms only. However, note that (2.79) describes a coupling between the open and closed sector. In fact, the scalar potential (2.72) arising from (2.79) is obtained as an F-term potential when varying the superpotential with respect to the closed string fields  $t^a$ .

### 3 The open-closed moduli space and the Hitchin functionals

In this section we discuss the geometry of the moduli space of the bulk sector and brane sector in more detail. In the first part, section 3.1, we assume that the open moduli are frozen and discuss the geometry of the moduli space  $\mathcal{M}^Q$  of the dilaton and the real complex structure deformations following [11]. In section 3.2 we discuss the moduli space of special Lagrangian deformations  $\eta^i$  following the work of Hitchin [24, 25]. This description will be slightly extended by including the NS-NS B-field. The open moduli space has finite dimension  $b^1(L_0)$  and can be encoded by the variation of harmonic one- or two-forms on  $L_0$ .

In the complete set-up, with varying open and closed modes, the definition of being special Lagrangian crucially depends on both the Kähler as well as the complex structure moduli of  $Y$ . In fact, the normal vectors  $s_i$  used in order to define the one-forms  $\theta_i = s_i \lrcorner J$  need to be chosen such that  $\theta_i$  is harmonic. This notion changes when varying the complex and Kähler structure of  $Y$ . Nevertheless, if such a change does not alter the topology of  $Y$  and  $L_0$ , one expects to find a new

embedding map  $\iota'$  which makes  $L_\eta$  supersymmetric in  $Y$  and posses also  $b^1(L_0)$  special Lagrangian deformations. This suggest to view the full moduli space as fibration of the open string moduli space  $\mathcal{M}_o^{\mathbb{C}}$  over the closed string moduli space  $\mathcal{M}_{\mathbb{C}}^K \times \mathcal{M}_{\mathbb{C}}^Q$ , where  $\mathcal{M}_{\mathbb{C}}^K$  is the space spanned by the complexified Kähler deformations. In section 3.3 we will explore the local geometry of this full moduli space in more detail. Note that we are still dealing with only a finite set of deformations. In the absence of background fluxes these remain massless due to the vanishing of the scalar potential.

In section 3.4 we also analyze the gauge coupling function and the kinetic mixing for the brane and bulk  $U(1)$  gauge fields. In particular, we comment on its holomorphicity properties.

### 3.1 The orientifold moduli space

Let us first discuss the moduli space  $\mathcal{M}^Q$  for the closed string modes  $e^D$  and the  $h^{(2,1)}$  real complex structure deformations denoted by  $q^K$ . Its metric takes the form

$$\frac{1}{2}G = dD \cdot dD + K_{KL}^{\text{cs}} dq^K \cdot dq^L, \quad (3.1)$$

where  $K_{KL}^{\text{cs}}$  is the Weil-Petersson metric restricted to the slice of real complex structure deformations preserving the orientifold constraint (2.1). As suggested already in (2.7) and (2.8) one describes the geometry of this space by considering the the three-form  $2 \text{Re}(C\Omega) \in H_+^3(Y, \mathbb{R})$  with periods  $U^k = 2 \text{Re}(CX^k)$  and  $U_\lambda = 2 \text{Re}(C\mathcal{F}_\lambda)$ . In these new coordinates  $U^K = (U^k, U_\lambda)$  the metric  $G$  in (3.1) is obtained as a second derivative of the real function [11]

$$K^Q(V) = -2 \ln \left[ i \int_Y C\Omega \wedge \overline{C\Omega} \right] = -2 \log [e^{-2D}]. \quad (3.2)$$

Then the first derivatives of  $K_c$  are given by

$$\frac{1}{2} \frac{\partial K^Q}{\partial U^k} = 2 e^{2D} \text{Im}(C\mathcal{F}_k) \equiv V_k, \quad \frac{1}{2} \frac{\partial K^Q}{\partial U_\lambda} = -2 e^{2D} \text{Im}(CX^\lambda) \equiv V^\lambda. \quad (3.3)$$

The second derivatives of  $K^Q$  can be evaluated explicitly as well

$$G = \frac{\partial^2 K^Q}{\partial U^K \partial U^L} dU^K \cdot dU^L = G_{kl} dU^k \cdot dU^l + G^{\lambda\kappa} dU_\lambda \cdot dU_\kappa + 2G_k^\lambda dU^k \cdot dU_\lambda. \quad (3.4)$$

Here one checks that the components of  $G$  in these coordinates are precisely as defined in (2.69) by either using a truncated version of  $\mathcal{N} = 2$  special geometry as in ref. [11] or by applying the techniques developed by Hitchin in ref. [14] as done in [13].<sup>9</sup> The fact that (2.69) is the metric for the scalars  $(\xi^k, \tilde{\xi}_\lambda)$  allows us to identify local complex coordinates  $N'^k = U^k + i\xi^k$  and  $T'_\kappa = U_\lambda + i\tilde{\xi}_\lambda$  on a Kähler manifold  $\mathcal{M}_{\mathbb{C}}^Q$  which is locally of the form  $\mathcal{M}^Q \times H_+^3(Y, \mathbb{R}/\mathbb{Z})$ . The Kähler potential for the metric  $G$  on  $\mathcal{M}_{\mathbb{C}}^Q$  is precisely  $K^Q(N + \bar{N}, T + \bar{T})$  given in (3.2).

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<sup>9</sup> $H(\text{Re}(C\Omega)) = i \int_Y C\Omega \wedge \overline{C\Omega}$  is also known as entropy or Hitchin functional of the real three-form  $\text{Re}(C\Omega)$ .

Note that originally  $\mathcal{M}_{\mathbb{C}}^Q$  was found as the  $\mathcal{N} = 1$  field-space obtained by truncating the underlying quaternionic geometry spanned by the  $\mathcal{N} = 2$  hypermultiplets. Each hypermultiplet has been truncated to a single  $\mathcal{N} = 1$  chiral multiplet such that  $\mathcal{M}^Q$  has half the real dimension of the quaternionic space. However, in order to prepare for the discussion of the moduli space of special Lagrangian submanifolds, one notes that  $\mathcal{M}^Q$  can also be viewed as a Lagrangian submanifold of a vector space. The map embedding  $\mathcal{M}^Q$  in this special way is of the form

$$F^Q : \mathcal{M}^Q \hookrightarrow V \times V^* , \quad V = H_+^3(Y, \mathbb{R}) , \quad (3.5)$$

where  $V^* \cong H_-^3(Y, \mathbb{R})$  is the dual vector space of  $V$ .  $\mathcal{M}^Q$  is Lagrangian with respect to the natural symplectic structure  $\mathfrak{w}$  on  $V \times V^*$ , i.e.  $F^{Q*}\mathfrak{w} = 0$ , and its metric is induced by the natural metric  $\mathfrak{g}$  on  $V \times V^*$ , i.e.  $F^{Q*}\mathfrak{g} = G$ . Explicitly,  $\mathfrak{w}$  and  $\mathfrak{g}$  are given by

$$\mathfrak{w}((a, a'), (b, b')) = a'(b) - b'(a) , \quad \mathfrak{g}((a, a'), (b, b')) = a'(b) + b'(a) , \quad (3.6)$$

where  $a'(a)$  is the application of  $a' \in V^*$  to  $a \in V$ . In the case at hand,  $\mathfrak{w}$  and  $\mathfrak{g}$  can be evaluated using the wedge product  $a'(b) = \int_Y a' \wedge b$ , and the map  $F^Q$  is given by

$$F^Q : (D, q^K) \mapsto (U^k \alpha_k - U_\lambda \beta^\lambda, V^\lambda \alpha_\lambda + V_k \beta^k) = (2\text{Re}(C\Omega), -2e^{2D}\text{Im}(C\Omega)) . \quad (3.7)$$

Since  $\mathcal{M}^Q$  is a Lagrangian subspace of  $V \times V^*$  with induced metric  $G$ , it can be obtained as a graph of a function  $K^Q$  which is the potential introduced in (3.2). Note that the embedding of  $\mathcal{M}^Q$  satisfies another special property, since

$$\mathfrak{g}(F^Q(p), F^Q(p)) = 2(V_k U^k + V^\lambda U_\lambda) = 4 , \quad (3.8)$$

for every point  $p$  on  $\mathcal{M}^Q$ . This additional condition corresponds to the fact that, upon complexification with the R-R scalars, the Kähler metric satisfies the no-scale type condition

$$K_{M'K}^{-Q} K^Q M'^K \bar{M}'^L K_{M'L}^Q = 4 , \quad (3.9)$$

where  $M'^K = (N'^k, T'_\lambda)$  are the complex coordinates introduced in (2.8).

It worthwhile to mention that a similar logic can also be applied to the Kähler sector of the orientifold theory. In this case the functional  $K^K$  is simply given by the logarithm of the Calabi-Yau volume. One finds that for the coefficients  $v^a$  of  $J = v^a \omega_a$  that

$$\frac{\partial^2 K^K}{\partial v^a \partial v^b} = \frac{1}{4\mathcal{V}} \int_Y \omega_a \wedge * \omega_b , \quad K^K = -\ln \left[ \frac{4}{3} \int_Y J \wedge J \wedge J \right] , \quad (3.10)$$

which is the analog of (3.2) and (3.4). Here also one finds a natural Lagrangian embedding  $F^K$  of the moduli space  $\mathcal{M}^K$  into a vector space, which is now of the form  $V \times V^* = H_-^2(Y, \mathbb{R}) \times H_+^4(Y, \mathbb{R})$ . Here the special non-scale property of  $K^K$  translates to  $\mathfrak{g}(F^{\text{ks}}(p), F^K(p)) = 3$  for each  $p$  in  $\mathcal{M}^K$ . The complexification of  $\mathcal{M}^K$  is via the  $B_2$  scalars as in (2.6) and we locally have  $\mathcal{M}_{\mathbb{C}}^K = \mathcal{M}^K \times H_-^2(Y, \mathbb{R}/\mathbb{Z})$ .

Let us conclude the discussion of the moduli space  $\mathcal{M}^Q \times \mathcal{M}^K$  by presenting yet another way to motivate its geometrical structures. In an orientifold compactification it is well-known that the orientifold planes, located on the fix-points of the involution  $\sigma$ , are not dynamical and hence do not possess moduli at weak string coupling. Hence, all deformations in  $\mathcal{M}^Q \times \mathcal{M}^K$  need to preserve the embedding of the fix-planes and thus the conditions (2.12). Clearly, this is indeed the case for the scaling of  $e^D$ . Also the real complex structure and Kähler structure deformations chosen such that  $\text{Im}(C\Omega)$  and  $J$  remain elements of  $H^3_-(Y, \mathbb{R})$  and  $H^2_-(Y, \mathbb{R})$  ensure that these forms vanish on the fix-point locus of  $\sigma$ . In the discussion of the D6-brane moduli space we will turn the story around and consider the variations of the D-brane embedding maps  $\iota$  which preserve the conditions (2.14) for fixed closed string fields.

### 3.2 The moduli space of D6-branes on special Lagrangian submanifolds

In the following we will discuss the moduli space of a supersymmetric D6-brane wrapped on a special Lagrangian cycle on a Calabi-Yau manifold  $Y$  with fixed complex and Kähler structure following [24, 25]. At the end of this subsection we propose a simple modification to include the B-field.

#### The geometry of the moduli space of special Lagrangian submanifolds

To begin with, recall that the space of maps from a three-dimensional manifold  $L$  into  $Y$  is infinite dimensional if no further restrictions are imposed. However, we have derived the potential (2.44) for modes violating the supersymmetry constrains (2.14) rendering these fields massive. Reducing the general deformation problem to embeddings  $\iota$  which preserve (2.14) reduces the problem to the study of a finite dimensional deformation space  $\mathcal{M}_o$ . In section 2.1 we already stated McLeans result that this moduli space  $\mathcal{M}_o$  is  $b^1(L_0)$ -dimensional. At linear order its geometry can be studied by considering the variations of harmonic one-forms  $\eta^i \theta_i = \eta^i s_i \lrcorner J$  on  $L_0$ . Here  $s_i$  is a normal vector parameterizing a deformation through special Lagrangian submanifolds, and  $J$  is a fixed background Kähler form which vanishes on  $L_0$ . The Hodge dual to  $\theta_i$  on  $L_0$  can be obtained as contraction of  $\text{Im}(C\Omega)$  with  $s_i$  as given in (2.49). The variations of the  $\theta_i$  and  $*\theta_i$  are analyzed by expanding these forms in an integral basis  $\tilde{\alpha}_i$  of  $H^1(L_0, \mathbb{Z})$  and  $\tilde{\beta}^i$  of  $H^2(L_0, \mathbb{Z})$  respectively,

$$\theta_i = \lambda_i^j \tilde{\alpha}_j, \quad \frac{1}{2} e^{-\phi} * \theta_i = \mu_{ji} \tilde{\beta}^j, \quad (3.11)$$

where  $\lambda_i^j(\eta)$  and  $\mu_{ij}(\eta)$  define the periods of  $\theta_i$  and  $e^{-\phi} * \theta_i$ . Explicitly they are given by

$$\lambda_i^j = \int_{L_0} s_i \lrcorner J \wedge \tilde{\beta}^j, \quad \mu_{ij} = - \int_{L_0} s_j \lrcorner \text{Im}(C\Omega) \wedge \tilde{\alpha}_i. \quad (3.12)$$

Note that we have introduced an additional factor of the dilaton, which is constant for a fixed background, but will later allow us to make contact to the metrics found in section 2. By using the



closedness of  $J$  and  $\text{Im}(C\Omega)$  one shows that there exist functions  $(u^i, v_i)$  such that [24]

$$\frac{\partial u^i}{\partial \eta^j} = \lambda_j^i, \quad \frac{\partial v_i}{\partial \eta^j} = \mu_{ij}. \quad (3.13)$$

In fact,  $(u^i, v_i)$  are the analogs of  $(U^K, V_K)$  for the orientifold moduli space (3.3).

Let us point out that the harmonic one-forms  $\theta_i^\eta$  can be constructed on each  $L_\eta$  obtained by a supersymmetric deformation of  $L_0$  [24]. Generalizing (3.11) we can pull back  $\theta_i^\eta$  from  $L_\eta$  to  $L_0$  using the exponential map  $E$  introduced in section 2.2. Following the strategy of section 2.4 we can then use the chain  $\mathcal{C}_4$  to write

$$\lambda_i^j = \partial_{\eta^i} \int_{\mathcal{C}_4} J \wedge \tilde{\beta}^j, \quad \mu_{ji} = -\partial_{\eta^i} \int_{\mathcal{C}_4} \text{Im}C\Omega \wedge \tilde{\alpha}_j. \quad (3.14)$$

which at linear order reproduces (3.11) on  $L_0$ . Inserting (3.14) into (3.13) this provides us with a chain integral expression for the coordinates  $(u^i, v_i)$ .

To obtain the differential geometrical structure on  $\mathcal{M}_o$  one follows a similar logic as in (3.5) and (3.7), and defines the map

$$\begin{aligned} F_o: \mathcal{M}_o &\hookrightarrow V \times V^* = H^1(L, \mathbb{R}) \times H^2(L, \mathbb{R}), \\ \eta^i &\mapsto (u^i \tilde{\alpha}_i, v_i \tilde{\beta}^i). \end{aligned} \quad (3.15)$$

Using this map  $\mathcal{M}_o$  is embedded as a Lagrangian submanifold with respect to the natural symplectic form  $\mathfrak{w}$  in (3.6) on  $V \times V^*$ , where now  $a'(b) = \int_L a' \wedge b$  [24]. Moreover, the induced metric obtained from  $\mathfrak{g}$ , defined in (3.6), is evaluated to be

$$F_o^* \mathfrak{g} = \mathcal{G}_{ij} du^i \cdot dv^j = \widehat{\mathcal{G}}_{ij} d\eta^i \cdot d\eta^j, \quad (3.16)$$

where  $\mathcal{G}_{ij}$  is explicitly given in (2.47) and  $\widehat{\mathcal{G}}_{ij}$  can be found in (2.48). It is straightforward to evaluate the metrics in terms of the periods  $\lambda_j^i$  and  $\mu_{ij}$  using (3.11) and (3.13) as

$$\widehat{\mathcal{G}}_{ij} = \mu_{ki} \lambda_j^k, \quad \mathcal{G}_{ij} = \mu_{ik} (\lambda^{-1})_j^k. \quad (3.17)$$

From the fact that  $\mathcal{M}_o$  is a Lagrangian submanifold one finds that it can be locally represented by a single function  $K_o$  with  $v_i = \partial K_o / \partial u^i$ . This is the direct analog of (3.3). Moreover, using the fact that  $F_o^* \mathfrak{g} = du_i \cdot dv^i$  the metric on  $\mathcal{M}_o$  is the Hessian of  $K_o$  with respect to  $u^i$ , i.e.  $\mathcal{G}_{ij} = \partial^2 K_o / \partial u^i \partial u^j$ .

As in the case of the orientifold moduli space, we next have to define a complexification of  $\mathcal{M}_o$  to obtain the space  $\mathcal{M}_o^{\mathbb{C}}$ . Let us first consider the case of vanishing B-field. Since the metric  $\mathcal{G}_{ij}$  in the coordinates  $u^i$  agrees with the metric for the Wilson line moduli  $a^i$ , found in (2.43), one defines complex coordinates  $\zeta^i$  on  $\mathcal{M}_o^{\mathbb{C}}$  as

$$\text{no B-field:} \quad \zeta^i = u^i + ia^i, \quad (3.18)$$

and identifies  $K_o(\zeta + \bar{\zeta})$  as a Kähler potential such that

$$\mathcal{G}_{ij} = \frac{\partial^2 K_o}{\partial u^i \partial u^j} = 4 \frac{\partial^2 K_o}{\partial \zeta^i \partial \bar{\zeta}^j} . \quad (3.19)$$

The metric  $\mathcal{G}_{ij}$  on  $\mathcal{M}_o^{\mathbb{C}}$  satisfies an important additional property. In fact, it turns out that  $\mathcal{M}_o^{\mathbb{C}}$  is actually a non-compact Calabi-Yau manifold with non-vanishing holomorphic  $b^1(L_0)$ -form  $\widehat{\Omega} = d\zeta^1 \wedge \dots \wedge d\zeta^{b^1}$  with constant length with respect to the Kähler form on  $\mathcal{M}_o^{\mathbb{C}}$  [24]. However, it is important to note that  $K_o$  cannot be simply extended to a Kähler potential on a compact Calabi-Yau manifold due to its apparent shift symmetry  $\zeta \rightarrow \zeta + ic$ , for constants  $c$ . As well-known these shift symmetries will however be broken by non-perturbative effects coupling with instanton factors  $e^{-\zeta^i}$ . There has been much progress in understanding such corrections for in the holomorphic superpotential by explicitly computing the Type IIB chain integrals. Recent works in this direction include [36, 37, 38, 39], and references therein. The study of corrections to the Kähler potential is significantly more involved, since it is not protected by holomorphicity.

### Open coordinates with B-field

So far we have analyzed in this subsection the open moduli space for vanishing  $B_2$  and  $f_{D6}$ . We want to generalize this in the following. To include the B-field we note from (3.14) and (3.13) that  $u^i$  can be written by using the four-chain in (2.50) as

$$u^i = \int_{\mathcal{C}_4} J \wedge \tilde{\beta}^i = \int_{L_0} \eta \lrcorner J \wedge \tilde{\beta}^i + \dots , \quad (3.20)$$

where we have also given the  $\eta$  expansion for small fluctuations around  $L_0$ . One can now replace  $J$  in (3.20) by  $-iJ_c = J - iB_2$  as used for the closed coordinates in (2.6). This leads us to modify (3.18) as

$$\zeta^i = u_c^i + ia^i , \quad u_c^i = -i \int_{\mathcal{C}_4} J_c \wedge \tilde{\beta}^i . \quad (3.21)$$

Note that  $u_c^i$  is the complexification of  $u^i$  with a B-field correction which can be absorbed by a shift of  $a^i$ . This implies that (3.19) remains to be valid.

In the definition (3.21) we have used the chain  $\mathcal{C}_4$  with boundaries  $L_0$  and  $L_\eta$ . It is desirable to introduce a similar extension which allows to include the gauge field. To do that we introduce an extension  $\mathcal{F}_{D6} = d\mathcal{A}_{D6}$  of the gauge connection  $A_{D6}$  to the chain  $\mathcal{C}_4$  such that

$$\mathcal{A}_{D6}|_{L_0} = A_{D6}^0 , \quad \mathcal{A}_{D6}|_{L_\eta} = A_{D6}^0 - a^I \hat{\alpha}_I , \quad (3.22)$$

where  $\hat{\alpha}_I$  and  $A_{D6}^0$  have been transported trivially from  $L_0$  to  $L_\eta$  along the geodesic given by  $\eta$ . Here  $A_{D6}^0$  is a background gauge bundle on  $L_0$  which for fixed  $B_2$  allows to satisfy the supersymmetry conditions on  $L_0$ . In other words, for a constant  $B_2$  along the chain,  $\mathcal{F}_{D6}$  might satisfy the supersymmetry conditions on  $L_0$  but violate the supersymmetry conditions on  $L_\eta$  due to non-trivial

Wilson line scalars  $a^I$ . Importantly this prescription can also be used for  $\eta \rightarrow 0$ . In this case, one does not deform  $L_0$  but changes the gauge connection by non-trivial scalars  $a^I$ . The imaginary part of the  $\mathcal{N} = 1$  coordinates arising from the gauge connection  $A_{D6}$  can now be also written as a chain integral  $\int_{\mathcal{C}_4} \mathcal{F}_{D6} \wedge \tilde{\beta}^i$ . Thus, we find that the  $\zeta^i$  are given by the elegant expression

$$\zeta^i = -i \int_{\mathcal{C}_4} (J_c - \mathcal{F}_{D6}) \wedge \tilde{\beta}^i . \quad (3.23)$$

At leading order in the  $\eta$ -expansion the complex coordinates  $\zeta^i$  are encoded by a one-form  $\mathcal{A}_c$  on  $L_0$  with expansion

$$\mathcal{A}_c = -i\eta \lrcorner J_c + iA_{D6} = \zeta^i \tilde{\alpha}_i , \quad (3.24)$$

into a basis  $\tilde{\alpha}_i$  of  $H^1(L_0, \mathbb{Z})$ . Let us close by noting that (3.23) naturally includes a possible D6-brane flux. It would be interesting to evaluate all expressions found below including this flux. However, we will keep  $f_{D6} = 0$  in most of the computations.

### 3.3 The open-closed Kähler potential and $\mathcal{N} = 1$ coordinates

In the following we determine the  $\mathcal{N} = 1$  data for the kinetic terms of the four-dimensional effective action by specifying the  $\mathcal{N} = 1$  complex coordinates, the Kähler potential and the gauge coupling function for the U(1) gauge theory on the D6-brane. We will do this by only including a finite set of deformations specified in the last two subsections. Note that these deformations will be obstructed by a scalar potential, since one always needs to impose the supersymmetry conditions (2.14) for the deformed D6-brane which depend on both the open as well as closed moduli. One thus expects that only a space of complex dimension smaller than  $\frac{1}{2}b^3(Y) + h_-^{1,1}(Y) + b^1(L_0)$  can be studied as a true open-closed moduli space which is classically un-obstructed by a scalar potential in the absence of background fluxes. This can be also understood by noting that Type IIA compactifications with D6-branes will admit an M-theory embedding as a compactification on a  $G_2$ -manifold [40, 41, 42]. The finite number of massless deformations of this manifold will incorporate the subset of the closed and open deformations of section 3.1 and 3.2 which are flat directions of the supersymmetry conditions (2.14).

Let us start by noting that the D6-brane degrees of freedom are still encoded by the complex coordinates  $\zeta^i$  which have been introduced in (3.18) and (3.21). From the closed string sector we find the complexified Kähler structure deformations  $t^a$  introduced in (2.6). As we will check later on, the definition of the remaining closed string complex coordinates is corrected by a functional depending on the open coordinates  $\zeta^i$ . More precisely, they arrange very elegantly as

$$N^k = U^k - 2 \partial_{V_k}(e^{2D} K_o) + i\xi^k , \quad T_\lambda = U_\lambda - 2 \partial_{V^\lambda}(e^{2D} K_o) + i\tilde{\xi}_\lambda , \quad (3.25)$$

where the real scalars  $(\xi^k, \tilde{\xi}_\lambda)$  arise in the expansion (2.7), and we recall that  $U^k = 2\text{Re}(CX^k)$ ,  $U_\lambda = 2\text{Re}(C\mathcal{F}_\lambda)$  as well as  $V_k = 2e^{2D}\text{Im}(C\mathcal{F}_k)$ ,  $V^\lambda = -2e^{2D}\text{Im}(CX^\lambda)$  are periods of  $C\Omega$ . In

summary, we can simply write

$$\zeta^i = u_c^i + ia^i, \quad M^K = U^K - 2\partial_{V_K}(e^{2D}K_o) + i\xi^K, \quad (3.26)$$

where  $\xi^K = (\xi^k, \tilde{\xi}^\lambda)$  and the abbreviations  $U^K = (U^k, U_\lambda)$  and  $V_K = (V_k, V^\lambda)$  are as in (3.3). The real function  $K_o$  is now dependent on both  $u^i$  as well as  $U^K$  (or rather  $V_K$ ). To see this, note that  $e^\phi * \theta_i = 2s_i \text{Im}(C\Omega)$  as introduced in (2.16), clearly depends on  $\text{Im}(C\Omega)$ . Performing the  $\eta$ -expansion of  $K_o$  around  $\eta = 0$  one finds

$$\begin{aligned} -2\partial_{V_k}(e^{2D}K_o) &= -\partial_{V_k}(e^{2D}\mathcal{G}_{ij})|_{\eta=0}u^i u^j + \dots, \\ &= -\frac{1}{2}\int_{L_0}\tilde{\alpha}_i \wedge s_{l\lrcorner}\beta^k \left( \int_{L_0}\tilde{\beta}^j \wedge s_{l\lrcorner}J \right)^{-1} u^i u^j + \dots, \end{aligned} \quad (3.27)$$

as we derive in detail in appendix A. Together with a similar expression for  $\partial_{V^\lambda}(e^{2D}K_o)$ , replacing  $\beta^k \rightarrow \alpha_\lambda$ , one can use (3.27) to derive the leading order effective action. In order to do that, we also need to specify the Kähler potential, to which we will turn next. Realize that as a trivial check of (3.26) one recovers the bulk  $\mathcal{N} = 1$  coordinates  $(N^k, T'_k)$  given in (2.8) if  $K_o = 0$ .

To encode the leading order D6-brane effective action found in (2.43) and (2.56), we finally need to specify the Kähler potential. It is given by

$$K = K^{\text{ks}} + K^{\text{Q}} = -\ln \left[ \frac{4}{3} \int_Y J \wedge J \wedge J \right] - 2 \ln \left[ i \int_Y C\Omega \wedge \overline{C\Omega} \right], \quad e^K = \frac{1}{8} e^{4D} \mathcal{V}^{-1}. \quad (3.28)$$

Note that  $K$  has to be evaluated in terms of the  $\mathcal{N} = 1$  coordinates (3.25) and thus only depends on  $\zeta^i + \bar{\zeta}^i$ ,  $M^K + \bar{M}^K$  and  $t^a - \bar{t}^a$ . This can be done explicitly for the first term  $K^{\text{ks}}$  since

$$K^{\text{ks}}(t, \bar{t}) = -\ln \left[ \frac{i}{6} \mathcal{K}_{abc} (t - \bar{t})^a (t - \bar{t})^b (t - \bar{t})^c \right], \quad (3.29)$$

where  $\mathcal{K}_{abc} = \int_Y \omega_a \wedge \omega_b \wedge \omega_c$  are the triple intersection numbers. It corresponds to the volume of the Calabi-Yau manifold  $Y$  and will be corrected by perturbative and non-perturbative string worldsheet contributions. For the second term  $K^{\text{Q}}$  it is in general hard to find an explicit expression in terms of the  $\mathcal{N} = 1$  coordinates. However, we are nevertheless able to check that the general kinetic terms determined by the derivatives of  $K^{\text{Q}}$  match the leading order terms found by dimensional reduction.

Let us summarize the derivatives of the Kähler potential  $K^{\text{Q}}$ . We note that the derivatives with respect to the closed string moduli  $N^k, T_\lambda$  take the same form as in (3.3),  $\partial_{N^k} K = V_k$ ,  $\partial_{T_\lambda} K = V^\lambda$ . However,  $(V_k, V^\lambda)$  now depend implicitly on the open string coordinates  $\zeta^i$  through the evaluation of the closed string expressions in terms of the  $\mathcal{N} = 1$  coordinates (3.25), i.e. one has to view  $V_K(u^i, U^K)$ . The derivatives with respect to  $\zeta^i$  will be postponed to section 4. In summary one finds that

$$K_i = e^{2D} v_i, \quad K_k = 2e^{2D} \text{Im}(C\mathcal{F}_k), \quad K^\lambda = -2e^{2D} \text{Im}(CX^\lambda). \quad (3.30)$$

where  $K_i = \partial K / \partial \zeta^i$ ,  $K_k = \partial K / \partial N^k$  and  $K_\lambda = \partial K / \partial T_\lambda$ . Also the Kähler metric can be evaluated explicitly. One finds for the derivatives with respect to  $(N^k, T_\lambda, \zeta^i)$  that

$$\begin{aligned} K_{k\bar{l}} &= G_{kl}, & K_{\lambda\bar{\kappa}} &= G^{\lambda\kappa}, & K_{k\bar{\lambda}} &= G_k^\lambda, \\ K_{i\bar{j}} &= e^{2D} \mathcal{G}_{ij} + \mathcal{I}_i^K G_{KL} \mathcal{I}_j^L, & K_{i\bar{k}} &= \mathcal{I}_i^L G_{Lk}, & K_{i\bar{\lambda}} &= \mathcal{I}_i^L G_L^\lambda, \end{aligned} \quad (3.31)$$

where  $G_{KL} = (G_{kl}, G^{\lambda\kappa}, G_k^\lambda)$  was given in (2.69), and  $\mathcal{I}_i^K = (\mathcal{I}_i^k, \mathcal{I}_{i\lambda})$  are the derivatives

$$\mathcal{I}_i^k = \frac{\partial^2 K_o}{\partial V_k \partial \zeta^i}, \quad \mathcal{I}_{i\lambda} = \frac{\partial^2 K_o}{\partial V^\lambda \partial \zeta^i}. \quad (3.32)$$

In appendix A we will check these expressions by an explicit computation, and match these data with the leading order effective action obtained in section 2.

Let us comment on the special form of the Kähler metric (3.31). It can be directly inferred by making use of the invariance of the kinetic terms under the shift symmetries

$$N^k \rightarrow N^k + i\Lambda^k, \quad T_\lambda \rightarrow T_\lambda + i\Lambda_\lambda, \quad (3.33)$$

for arbitrary constants  $(\Lambda^k, \Lambda_\lambda)$ . If such shift symmetries exist in the full four-dimensional effective action one can replace the chiral multiplets  $N^k$  and  $T_\lambda$  by linear multiplets  $(V_k, C_k^2)$  and  $(V^\lambda, C_2^\lambda)$ , as described in more details in appendix B. Here  $V_K = (V_k, V^\lambda)$  are the scalars dual to  $(\text{Re}N^k, \text{Re}T_\lambda)$  given in (3.30) and  $(C_k^2, C_2^\lambda)$  are two-forms dual to the scalars from  $C_3$ . The chiral multiplets and linear multiplets are connected by a Legendre transform, and the new real function encoding the kinetic terms of the multiplets is given by

$$\begin{aligned} \tilde{K}(V, \zeta + \bar{\zeta}) &= K(V) - V_k(N^k + \bar{N}^k) - V^\lambda(T_\lambda + \bar{T}_\lambda) \\ &= K(V) + 4 \frac{\partial(e^{2D} K_o)}{\partial V_K} V_K - 4, \end{aligned} \quad (3.34)$$

where we have inserted (3.26) and used (3.8) to obtain the constant term  $-4$ . The key point to notice is that in this dual picture all quantities are functions of  $V_K, \zeta^i$ . In particular, this implies that now  $K(V) = -2 \ln(e^{-2D}) = -2 \ln(i \int C \Omega \wedge \overline{C \Omega})$  is independent of  $\zeta^i$ , and all equalities found for the moduli space of special Lagrangian cycles of section 3.2 can be directly applied. Since the linear multiplet picture is just an equivalent dual description one can equally express the kinetic terms in the chiral multiplet picture in terms of the derivatives of  $\tilde{K}$ . Let us denote by  $\tilde{K}^{KL} = \partial_{V_K} \partial_{V_L} \tilde{K}$ , and by  $\tilde{K}_{KL}$  its inverse. Similarly, we denote by  $\tilde{K}_{\zeta^i}^K$  and  $\tilde{K}_{\zeta^i \zeta^j}$  the remaining second derivatives with respect to  $\zeta^i$  and  $V_K$ . The expression for the kinetic terms then has the form

$$\begin{aligned} \mathcal{L}^{\text{kin}} &= -(\tilde{K}_{\zeta^i \bar{\zeta}^j} + \tilde{K}_{\zeta^i}^K \tilde{K}_{KL} \tilde{K}_{\bar{\zeta}^j}^L) d\zeta^i \wedge *d\bar{\zeta}^j + \tilde{K}_{KL} (d\text{Re}M^I \wedge *d\text{Re}M^J + d\xi^K \wedge *d\xi^J) \\ &\quad - 2 \tilde{K}_{KL} \tilde{K}_{\zeta^i}^L (d\text{Re}M^I \wedge *dw^j + d\xi^I \wedge *da^j) \end{aligned} \quad (3.35)$$

This is precisely the form of the Kähler metric (3.31) and it remains to check that indeed  $\tilde{K}_{KL} = G_{KL}$ ,  $\tilde{K}_{\zeta^i \bar{\zeta}^j} = e^{2D} \mathcal{G}_{ij}$  and  $\tilde{K}_{\zeta^i}^K = \mathcal{I}_i^K$ . For the leading order actions found in section 2 this is done

in appendix A. Note that the form of the metric (3.35) is also inherited if only a potential term breaks the shift-symmetries (3.33).

Let us make a brief comment on the appearance of the term  $d\text{Re}M^I \wedge *du^j$ . This term corresponds to a kinetic mixing between complex structure and brane deformations, and would be expected to appear in higher order expansions of the Dirac-Born-Infeld action. In this section however it was obtained by simply analyzing the  $\mathcal{N} = 1$  characteristic data and the moduli space.

### 3.4 Gauge coupling functions and kinetic mixing for finite deformations

Having discussed the kinetic terms for the scalars in the  $\mathcal{N} = 1$  effective theory we will now turn to an analysis of the kinetic terms for the  $U(1)$  vectors fields. We have shown in section 2 in the case one focuses on harmonic modes in the reduction that the spectrum contains a D6-brane  $U(1)$  vector  $A$  as well as  $h_+^{(1,1)}$  bulk  $U(1)$  vectors  $A^\alpha$ . The leading gauge coupling function for the brane  $U(1)$  was derived in section 2.4 and given by

$$f_r = \int_{L_0} (2\text{Re}(C\Omega) + iC_3) = \delta_k N'^k - \delta^\lambda T'_\lambda, \quad (3.36)$$

where  $\delta_k = \int_{L_0} \alpha_k$  and  $\delta^\lambda = \int_{L_0} \beta^\lambda$ . However, as we have discussed in section 3.3, the inclusion of the open moduli forces us to introduce the modified complex coordinates  $N^k, T_\lambda$  given in (3.25). In order to obtain a holomorphic gauge coupling function it is expected that (3.36) is modified to

$$f = \delta_k N^k - \delta^\lambda T_\lambda. \quad (3.37)$$

The modifications in (3.37) did not appear in our leading order dimensional reduction, but are expected to arise a higher order in the brane deformations. As we will see shortly open moduli corrections to  $f_r$  are also obtained after a careful treatment of the two dual bulk gauge fields  $A^\alpha, A_\alpha$  introduced in (2.54). Recall that the gauge coupling function for the bulk R-R  $U(1)$  vectors  $A^\alpha$  is simply given by [11]

$$f_{\alpha\beta} = i \int_Y \omega_\alpha \wedge \omega_\beta \wedge \omega_a t^a = i\mathcal{K}_{\alpha\beta a} t^a = -i\bar{\mathcal{N}}_{\alpha\beta}. \quad (3.38)$$

where  $\mathcal{N}_{\alpha\beta}$  is the complex matrix already introduced in (2.70). Clearly,  $f_{\alpha\beta}$  is holomorphic in the complex fields  $t^a$ . Since the  $t^a$  are not corrected by the open moduli one expects the result (3.38) to remain valid also in the leading order reduction with a D6-brane. We will show in the following that this is indeed the case. More interestingly, we find that there are further corrections depending on the open moduli and D6-brane fluxes which induce a kinetic mixing of the brane and bulk  $U(1)$  gauge fields.

Let us now turn to a more careful analysis of the gauge coupling functions including the brane moduli. In order to do that we summarize the action for all vector fields including the dual  $A_\lambda$  introduced in (2.54). The mixing terms proportional to  $dA^\alpha \wedge F$  and  $dA_\alpha \wedge F$  have appeared

in in the reduction of the Chern-Simons action in (2.77). The brane couplings have to be taken into account when eliminating  $A_\alpha$  in favor of  $A^\alpha$  by using vector-vector duality in four dimensions as enforced by (2.3). A detailed calculation can be found in appendix C which uses a procedure similar to the one of ref. [21]. Here we just present the results. The action obtained after a careful elimination of  $A_\lambda$  is

$$S_{\text{vec}}^{(4)} = - \int \frac{1}{2} \text{Re} f_\alpha dA^\alpha \wedge *F + \frac{1}{2} \text{Im} f_\alpha dA^\alpha \wedge F + \frac{1}{2} \text{Im} \mathcal{N}_{\alpha\beta} dA^\alpha \wedge *dA^\beta + \frac{1}{2} \text{Re} \mathcal{N}_{\alpha\beta} dA^\alpha \wedge dA^\beta + \frac{1}{2} \text{Re} f_{\text{cor}} F \wedge *F + \frac{1}{2} \text{Im} f_{\text{cor}} F \wedge F$$

where the gauge coupling function  $f_\alpha$  encoding the kinetic mixing between bulk and brane  $U(1)$ 's is given by

$$f_\alpha = -4(i\bar{\mathcal{N}}_{\alpha\beta} \tilde{\mathcal{J}}^\beta + ia^j \Delta_{j\alpha} + i\Gamma_\alpha) , \quad (3.39)$$

and the corrected gauge coupling function  $f_{\text{cor}}$  for the brane  $U(1)$  is

$$f_{\text{cor}} = f_{\text{r}} + 4(i\bar{\mathcal{N}}_{\alpha\beta} \tilde{\mathcal{J}}^\alpha + ia^j \Delta_{j\beta} + i\Gamma_\alpha) \tilde{\mathcal{J}}^\beta . \quad (3.40)$$

The coefficient functions are given by  $\tilde{\mathcal{J}}^\alpha = \int_{\mathcal{C}_4} \tilde{\omega}^\alpha$ ,  $\Delta_{j\alpha} = \int_{L_0} \tilde{\alpha}_j \wedge \omega_\alpha$  and  $\Gamma_\alpha = \int_{\mathcal{C}_4} \omega_\alpha \wedge f_{\text{D6}}$  as introduced in section 2. Recall that  $\Delta_{j\alpha}$  is independent of the moduli, while  $\tilde{\mathcal{J}}^\beta, \Gamma_\alpha$  depend on the brane deformations through the chain  $\mathcal{C}_4$ .

To study the holomorphicity properties of the gauge couplings we discuss  $f_\alpha$  and  $f_{\text{cor}}$  in turn. One notes that the first term in (3.39) can be rewritten as

$$i\bar{\mathcal{N}}_{\alpha\beta} \tilde{\mathcal{J}}^\alpha = \int_{\mathcal{C}_4} i\bar{\mathcal{N}}_{\alpha\beta} \tilde{\omega}^\alpha = \int_{\mathcal{C}_4} (J - iB) \wedge \omega_\beta = u_c^j \Delta_{j\beta} , \quad (3.41)$$

where we have used (3.21) to obtain the factor  $u_c^j$ . Using this expression it is straightforward to rewrite the gauge coupling  $f_\alpha$  in the absence of brane fluxes as

$$f_\alpha = -4\zeta^j \Delta_{j\beta} , \quad (3.42)$$

which is clearly holomorphic on the open moduli  $\zeta^i = u_c^i + ia^i$ . It would be interesting to extend these arguments to include the D6-brane flux  $f_{\text{D6}}$ .

Let us now turn to the analysis of the corrected gauge coupling function  $f_{\text{cor}}$  of the brane  $U(1)$ . Using (3.40) and (3.39) one sees that it can be written as

$$f_{\text{cor}} = f_{\text{r}} - f_\alpha \tilde{\mathcal{J}}^\alpha , \quad (3.43)$$

the additional term is at least of second order in the open moduli. One notes that the real part of  $f_{\text{cor}}$  is given by

$$\text{Re} f_{\text{cor}} = \text{Re} f_{\text{r}} + 4\text{Im} \mathcal{N}_{\alpha\beta} \tilde{\mathcal{J}}^\alpha \tilde{\mathcal{J}}^\beta = \text{Re} f_{\text{r}} + \text{Re} f_\alpha \text{Re} f^{\alpha\beta} \text{Re} f_\beta , \quad (3.44)$$

which can be inferred from (3.39) and (3.43). This result generalizes to the space of infinite deformations by replacing  $f_r$  with  $f_{rIJ}$ , and  $f_\alpha$  with  $f_{\alpha I}$ . The expressions for these are straightforward generalizations of (3.39)-(3.44) with the abbreviations introduced in section 2.4. Hence, the real part of the gauge coupling function takes the form

$$\text{Re } \mathbf{f} = \begin{pmatrix} \text{Re } f_{rIJ} + \text{Re } f_{\gamma I} \text{Re } f^{\gamma\delta} \text{Re } f_{\delta J} & \text{Re } f_{I\alpha} \\ \text{Re } f_{J\beta} & \text{Re } f_{\alpha\beta} \end{pmatrix}, \quad (3.45)$$

and can be easily inverted. This result will be important in section 4, when we compute the scalar potential coming from D-terms since it involved the inverse  $(\text{Re } \mathbf{f})^{-1}$ .

Let us close this section by making some general remarks about the holomorphicity of the gauge coupling function  $f_{\text{cor}}$  in (3.43). In order to do that, one has express it in terms of the  $\mathcal{N} = 1$  coordinates  $N^k, T_\lambda, t^a$  and  $\zeta^i$ . However, recall from (3.25) that also the  $N^k$  and  $T_\lambda$  receive corrections by the open deformations. In fact, we  $\eta$ -expand

$$\text{Re}(N^k - N'^k)\delta_k - \text{Re}(T_\lambda - T'_\lambda)\delta^\lambda = u^i \left( -\frac{1}{2} \int_{L_0} \tilde{\alpha}_i \wedge \eta \lrcorner \beta^k \int_{L_0} \alpha_k + \frac{1}{2} \int_{L_0} \tilde{\alpha}_i \wedge \eta \lrcorner \alpha_\lambda \int_{L_0} \beta^\lambda \right) + \dots, \quad (3.46)$$

where we have used (3.20) and (3.27). To compare this result, we also  $\eta$ -expand (3.43) to find

$$\text{Re } f_{\text{cor}} - \text{Re } f_r = 4u^i \int_{L_0} \tilde{\alpha}_i \wedge \omega_\alpha \int_{L_0} \eta \lrcorner \tilde{\omega}^\alpha + \dots. \quad (3.47)$$

This indicates that the result for  $f_{\text{cor}}$  cannot be complete. In particular, it is conceivable that a contribution from the two-forms  $\omega_a$  is missing which arises at higher order in the Kaluza-Klein reduction. This is similar to what was found in [21, 22] for D7- and D5-branes on the type IIB side. It would be interesting to complete this computation to higher order and determine the fully corrected gauge coupling function. For example, one loop corrections for the gauge-coupling function were calculated for orbifold models in [43].

## 4 General deformations and the D- and F-term potential

In the previous section we considered D6-branes with a finite number of deformations arising from the expansion into harmonic forms on the brane world-volume. Using harmonic modes one infers that the scalar potential (2.44) vanishes. A non-vanishing potential precisely arises for deformations which violate the supersymmetry conditions that the three-cycle is special Lagrangian. In this section we include such deformations into the discussion and analyze the  $\mathcal{N} = 1$  encoding the geometry on the infinite field space. We discuss the Kähler potential and show that the scalar potential (2.44) indeed arises from a D-term, induced by a gauging, and a holomorphic superpotential. In order to do that we will keep the background geometry fixed and only consider the variations of the brane degrees of freedom.



## 4.1 A local Kähler metric for general deformations of $L_0$

In the general reduction performed in section 2.2 we already included a whole tower of normal deformations of  $L_0$  as well as the whole tower of Kaluza-Klein modes in  $F_{D6}$  parameterizing variations around a background connection  $A_0$ . Together, these modes parameterize a neighborhood around  $(L_0, A_0)$  in an infinite dimensional field-space  $\mathcal{V}_o$ . We will focus on the neighborhood around a supersymmetric  $L_0$  and mainly be concerned with the local geometrical structure of  $\mathcal{V}_o$ . In order to do that we study the tangent space to  $\mathcal{V}_o$  at the special Lagrangian  $L_0$  with connection  $A_0$ . This tangent space is identified with

$$T_{(L_0, A_0)}\mathcal{V}_o \cong TY|_{L_0} \cong NL_0 \oplus TL_0 . \quad (4.1)$$

In this we can identify the  $s_I$  introduced in (2.36) as basis of sections of  $NL_0$  and the  $\tilde{s}_I^m = g^{mn}|_{L_0}(\hat{\alpha}_I)_n$  as sections of  $TL_0$ . Note that in defining the tangent vector  $\tilde{s}_I$  we have simply raised the tangent index  $m$  of the one-form  $\hat{\alpha}_I$  introduced in (2.38) by the inverse of the induced metric  $g_{mn}|_{L_0}$ . This also means that we can identify

$$T_{(L_0, A_0)}\mathcal{V}_o \cong \Omega^1(L_0) \oplus \Omega^1(L_0) , \quad (4.2)$$

which is naturally parameterized by the basis vectors  $\theta_I$  and  $\hat{\alpha}_I$  introduced in (2.36) and (2.38).

Using the first identification in (4.1) the tangent space  $T_{(L_0, A_0)}\mathcal{V}_o$  admits a natural symplectic form

$$\varphi(X, Y) = \frac{1}{2}e^{-\phi} \int_{L_0} J(X, Y)|_{L_0} \text{vol}_{L_0} . \quad (4.3)$$

for  $X, Y \in TY|_{L_0}$ . It was shown in [25] that the two-form  $\varphi$  on  $\mathcal{V}_o$  is actually closed. The tangent space (4.1) also admits a natural complex structure  $I$ , which is the induced complex structure from the Calabi-Yau manifold  $Y$ . At  $L_0$  the complex structure  $I$  identifies  $TL_0$  with  $NL_0$  such that complex tangent vectors in  $T_{(L_0, A_0)}\mathcal{V}_o$  are given by

$$\partial_{z^I} = \frac{1}{2}(s_I - iIs_I) , \quad \partial_{\bar{z}^I} = \frac{1}{2}(s_I + iIs_I) . \quad (4.4)$$

Since this complex structure is formally integrable, the manifold  $\mathcal{V}_o$  is Kähler, with Kähler form

$$\varphi(\partial_{z^I}, \partial_{\bar{z}^J}) = \frac{i}{2}e^{-\phi} \int_{L_0} g(s_I, s_J) \text{vol}_{L_0} = i\hat{\mathcal{G}}_{IJ} , \quad \varphi(\partial_{z^I}, \partial_{z^J}) = \varphi(\partial_{\bar{z}^I}, \partial_{\bar{z}^J}) = 0 . \quad (4.5)$$

Here we have used that  $J(Is_I, s_J) = -g(s_I, s_J)$  and the fact that  $L_0$  is Lagrangian such that  $J(s_I, s_J) = -J(Is_I, Is_J) = 0$  for normal vectors  $s_I$  to  $L_0$ . This implies that  $\hat{\mathcal{G}}_{IJ}$  is a Kähler metric, which is locally the second derivative of a Kähler potential  $K_o = K_o(z^I, \bar{z}^I)$ . Explicitly this means that

$$\hat{\mathcal{G}}_{IJ} = \partial_{z^I} \partial_{\bar{z}^J} K_o = \frac{1}{2}e^{-\phi} \int_{L_0} \theta_I \wedge * \theta_J , \quad (4.6)$$

with the forms  $\theta_I$  as introduced in (2.36). Note that the real part of the complex coordinates  $z^I$  are the normal vectors  $\eta^I$ . This should be contrasted to the complex coordinates  $\zeta^i$  which were the

complexifications of the  $u^i$  as discussed in section 3.2. In the appendix D we further analyze the symmetries of the symplectic form (4.3). We argue that the first derivatives of the Kähler potential  $K_o$  are encoded by moment maps of these symmetries.

It is interesting to note that there is a natural generalization of the finite-dimensional analysis of section 3.2 to the infinite dimensional deformation space. The key will be the use of the four-chain  $\mathcal{C}_4$  which interpolates between  $L_0$  and  $L_\eta$ . Clearly, the natural generalization of the complex coordinates in (3.23) is

$$\zeta^I = -i \int_{\mathcal{C}_4} (J_c - \mathcal{F}_{D6}) \wedge \hat{\beta}^I, \quad (4.7)$$

where  $\hat{\beta}^I$  is the infinite basis of two-forms on  $L_0$  which has been trivially extended to the chain  $\mathcal{C}_4$ . We have also included the field strength  $\mathcal{F}_{D6}$  on  $\mathcal{C}_4$  which is obtained from the gauge connection  $\mathcal{A}_{D6}$  introduced in (3.22). A natural proposal for the Kähler potential  $K_o$  is given by

$$K_o(\zeta + \bar{\zeta}) = -\frac{1}{2} \int_{\mathcal{C}_4} J \wedge \hat{\beta}^I \int_{\mathcal{C}_4} \text{Im}(C\Omega) \wedge \hat{\alpha}_I. \quad (4.8)$$

This can be checked by performing an  $\eta$ -expansion around the supersymmetric cycle  $L_0$ . This yields the leading term

$$\begin{aligned} K_o(\zeta + \bar{\zeta}) &= -\frac{1}{2} \int_{L_0} s_{L \lrcorner} J \wedge \hat{\beta}^I \int_{L_0} s_{K \lrcorner} \text{Im}(C\Omega) \wedge \hat{\alpha}_I \eta^L \eta^K + \dots \\ &= \frac{1}{4} e^{-\phi} \int_{L_0} \theta_L \wedge \hat{\beta}^I \int_{L_0} * \theta_K \wedge \hat{\alpha}_I \eta^L \eta^K + \dots \\ &= \frac{1}{2} \widehat{\mathcal{G}}_{LK} \eta^L \eta^K + \dots \\ &= \frac{1}{8} \mathcal{G}_{LK} (\zeta + \bar{\zeta})^L (\zeta + \bar{\zeta})^K + \dots \end{aligned} \quad (4.9)$$

where here we mean by  $\widehat{\mathcal{G}}_{LK}, \mathcal{G}_{LK}$  the leading order metrics independent of  $\eta$ . Here we have used (2.18) on  $L_0$  to rewrite the contraction  $s_{K \lrcorner} \text{Im}(C\Omega)$  into the Hodge-star on  $L_0$ . Using (4.9) one sees that (4.6) is satisfied. Let us stress that in general the evaluation of  $K_o$  as a function of  $\zeta^I + \bar{\zeta}^I$  is non-trivial due to the appearance of the chain  $\mathcal{C}_4$  in both integrals of (4.8). It would be very interesting to compute  $K_o$  explicitly for specific orientifold examples, generalizing the superpotential computations of [36, 37, 38, 39].

## 4.2 The superpotential and D-terms

Having discussed the Kähler potential determining the kinetic terms, we will now examine the scalar potential in more detail. More precisely, we will work in a fixed background geometry by fixing Kähler and complex structure deformations and focus on the leading scalar potential  $V_{\text{DBI}}$  given in (2.44). We will show that  $V_{\text{DBI}}$  splits into an F-term and a D-term piece as

$$V_{\text{DBI}} = V_F + V_D, \quad (4.10)$$

with

$$V_D = \frac{e^{3\phi}}{\mathcal{V}^2} \int_{L_0} d^* \theta_\eta \wedge * d^* \theta_\eta \quad (4.11)$$

and

$$V_F = \frac{e^{3\phi}}{\mathcal{V}^2} \int_{L_0} d\theta_\eta \wedge * d\theta_\eta + (\tilde{F} - B_2 - d\theta_\eta^B) \wedge * (\tilde{F} - B_2 - d\theta_\eta^B) . \quad (4.12)$$

We will show momentarily that  $V_F = e^K \mathcal{G}^{IJ} \partial_{\zeta^I} W \overline{\partial_{\zeta^J} W}$  can be obtained from a superpotential  $W$  and the metric determined from  $K_o$  using only the open string degrees of freedom.

To specify  $W$  we aim to define a functional which picks out deformations  $\eta$  such that  $L_\eta$  is a Lagrangian submanifold  $J|_{L_\eta} = 0$ . In section 2.4 we defined a chain  $\mathcal{C}_4$  with boundaries  $L_\eta$  and  $L_0$ . Recall also that we extended the gauge field  $A_{D6}$  from  $L_0$  to  $\mathcal{C}_4$  as in (3.22), such that the extension  $\mathcal{F}_{D6} = d\mathcal{A}_{D6}$  satisfies

$$\mathcal{F}_{D6}|_{L_0} = f_{D6} , \quad \mathcal{F}_{D6}|_{L_\eta} = f_{D6} + a^I d\hat{\alpha}_I . \quad (4.13)$$

In the following we will again set again the D-brane flux  $f_{D6}$  to zero. One next identifies the superpotential functional

$$W = \int_{\mathcal{C}_4} (J_c - \mathcal{F}_{D6}) \wedge (J_c - \mathcal{F}_{D6}) \quad (4.14)$$

depending on the open string data as well as the complexified Kähler form (2.6). This is an extension of the functional introduced in ref. [34], since we have included the B-field through the complex two-form  $J_c$ . Note that a superpotential of this form has been already discussed in [44, 45].

Let us briefly study the holomorphicity properties of  $W$ . Clearly,  $W$  is holomorphic with respect to variations of the complexified Kähler form  $J_c$  parameterized by the scalars  $t^a$  in (2.6). However, note that one first has to express  $W$  as a function of the open fields  $\zeta^I = u_c^I + ia^I$  introduced in (4.7). To check that  $W$  it is a holomorphic section in the  $\zeta^I$  we show that  $\partial_{\zeta^I} W = (\partial_{u_c^I} + i\partial_{a^I})W = 0$ . The derivative with respect to Wilson lines is

$$\partial_{a^I} W = 2 \int_{L_\eta} (J_c - \mathcal{F}_{D6}) \wedge \hat{\alpha}_I = 2 \int_{L_0} (J_c - \mathcal{F}_{D6}) \wedge \hat{\alpha}_I + 2 \int_{L_0} d(\eta \lrcorner J_c - a^J \hat{\alpha}_J) \wedge \hat{\alpha}_I + \dots \quad (4.15)$$

To evaluate the derivative with respect to  $u_c^I$  we expand the chain integral around the special Lagrangian cycle  $L_0$  in terms of the deformations

$$\begin{aligned} W &= 2 \int_{L_0} \eta \lrcorner J_c \wedge (J_c - \tilde{F}) + \int_{L_0} \eta \lrcorner J_c \wedge \mathcal{L}_\eta J_c + \dots \\ &= 2 \int_{L_0} (\eta \lrcorner J_c) \wedge (J_c - \mathcal{F}_{D6}) + \int_{L_0} \eta \lrcorner J_c \wedge d(\eta \lrcorner J_c + 2a^I \hat{\alpha}_I) + \dots \end{aligned} \quad (4.16)$$

Recalling  $\eta \lrcorner J_c = \theta_\eta^B + i\theta_\eta = iu_c^I \hat{\alpha}_I + \dots$  one sees that by comparing (4.15) with  $\partial_{u_c^I} W$  obtained from (4.16) that the superpotential is holomorphic in  $\zeta^I = u_c^I + ia^I$ .

It is now straightforward to determine the F-term potential using the expression (4.16). The real part of the derivative of (4.16) is given by

$$\text{Re } \partial_{\zeta^I} W = 2 \int_{L_0} d\theta_\eta \wedge \hat{\alpha}_I . \quad (4.17)$$

Note that  $d\theta_\eta$  is a 2-form in  $L_0$  and therefore can be expanded in the infinite basis  $*\hat{\alpha}_I$  as  $d\theta_\eta = c^I *\hat{\alpha}_I$ . The coefficients  $c^I$  can be obtained by taking on both sides the wedge product with  $\alpha_J$  and integrate on  $L_0$ . Inverting this relation for  $c^I$  and taking the Hodge star one finds

$$*d\theta_\eta = \frac{1}{2} e^{-\phi} \hat{\alpha}_I \mathcal{G}^{IJ} \int_{L_0} \hat{\alpha}_J \wedge d\theta_\eta . \quad (4.18)$$

We proceed analogously with the imaginary part  $\text{Im } \partial_{\zeta^I} W$  obtained from (4.16) and expand the two-form  $(B - \tilde{F} + d\theta_\eta^B)$  in the  $*\hat{\alpha}_I$  basis. The F-term potential is thus given by

$$\begin{aligned} V_F &= e^K \mathcal{G}^{IJ} \partial_{\zeta^I} W \overline{\partial_{\zeta^J} W} \\ &= \frac{e^{2D}}{2\mathcal{V}} \int_{L_0} d\theta_\eta \wedge \hat{\alpha}_I \mathcal{G}^{IJ} \int_{L_0} \hat{\alpha}_J \wedge d\theta_\eta \\ &\quad + \frac{e^{2D}}{2\mathcal{V}} \int_{L_0} (B - \tilde{F} + d\theta_\eta^B) \wedge \hat{\alpha}_I \mathcal{G}^{IJ} \int_{L_0} \hat{\alpha}_J \wedge (B - \tilde{F} + d\theta_\eta^B) \\ &= \frac{1}{\mathcal{V}^2} e^{3\phi} \int_{L_0} d\theta_\eta \wedge *d\theta_\eta + (B - \tilde{F} + d\theta_\eta^B) \wedge *(B - \tilde{F} + d\theta_\eta^B) \end{aligned} \quad (4.19)$$

which agrees with the result (4.12) obtained from dimensional reduction, and reduces to the result of McLean [18] in the limit of vanishing B field. As expected, the condition for vanishing of the potential and therefore to preserve supersymmetry is the closedness of  $\theta_\eta$  and  $\theta_\eta^B$ , as well as the condition  $(B - \tilde{F})|_{L_0} = 0$ .

Finally, we also compute the D-term potential in (4.11) induced by the gaugings of the scalars  $\hat{a}^I$  in (2.41) and  $(\xi^k, \tilde{\xi}_\lambda)$  in (2.66). More precisely, these scalars are charged under the gauge transformations  $A^I \rightarrow A^I + d\Lambda^I$  of the  $U(1)$  vectors  $A^I$  as

$$\hat{a}^I \rightarrow \hat{a}^I - \Lambda^I , \quad (\xi^k, \tilde{\xi}_\lambda) \rightarrow (\xi^k - \delta_I^k \Lambda^I, \tilde{\xi}_\lambda - \delta_{\lambda I} \Lambda^I) \quad (4.20)$$

The potential arising from D-terms can be calculated by

$$V_D = \frac{1}{2} \text{Re} f^{AB} D_A D_B , \quad \partial_A D_I = K_{A\bar{B}} X_I^{\bar{B}} , \quad (4.21)$$

where  $X_I^{\bar{B}}$  are the Killing symmetries appearing in the covariant derivative  $D\xi^k = d\xi^k + X_I^k A^I$ . Explicitly they take the form

$$X_I^k = \int_{L_0} h_I \beta^k + \int_{\mathcal{C}_4} dh_I \wedge \beta^k , \quad X_{I\lambda} = \int_{L_0} h_I \alpha_\lambda + \int_{\mathcal{C}_4} dh_I \wedge \alpha_\lambda . \quad (4.22)$$

The leading inverse gauge coupling function is simply

$$(\text{Ref}_r^{-1})^{IJ} = \left( \int_{L_0} h_I h_J 2\text{Re}(C\Omega) \right)^{-1}. \quad (4.23)$$

Integrating (4.21) we obtain the D-terms

$$D_I = -2e^{2D} \left( \int_{L_0} h_I \text{Im}C\Omega + \int_{\mathcal{C}_4} dh_I \wedge \text{Im}C\Omega \right). \quad (4.24)$$

We can expand the chain along an infinite set of brane deformations and obtain

$$D_I = -2e^{2D} \int_{L_0} h_I \text{Im}C\Omega - 2e^{2D} \int_{L_0} h_I d(\eta_{\perp} \text{Im}C\Omega) + \dots, \quad (4.25)$$

where we have used that the functions  $h_I$  are translated constantly along the chain. Now we repeat a similar calculation as for the F-term, by expanding the three forms into  $*h_I$  and noticing that on the  $L_0$  cycle  $\int h_J * h_I = e^{\phi} \int h_J h_I 2\text{Re}(C\Omega) = e^{\phi} \text{Ref}_{rIJ}$ . The potential is then,

$$V_D = \frac{e^{3\phi}}{\mathcal{V}^2} \int_{L_0} 4 \text{Im}C\Omega \wedge * \text{Im}C\Omega + 4 \text{Im}C\Omega \wedge *d*\theta + d*\theta \wedge *d*\theta. \quad (4.26)$$

From the condition  $\text{Im}C\Omega|_{L_0} = 0$  only the last term survives, yielding the remaining term obtained from dimensional reduction. The vanishing of the D-term potential, which is necessary in a supersymmetric vacuum, happens when the two-form  $*\theta_{\eta}$  is closed.

## 5 Mirror Symmetry with D-branes

In this final section we relate the Type IIA  $\mathcal{N} = 1$  characteristic data found in the previous sections with the data for Type IIB orientifolds with space-time filling D3-, D5- and D7-branes. In order to do that, we first review some basics of Type IIB orientifolds following [11]. To define the orientifold set-up starting with Type IIB string theory compactified on a Calabi-Yau manifold  $\tilde{Y}$ , one acts with a discrete involutive symmetry  $\mathcal{O}$  containing worldsheet parity  $\Omega_p$ . In Type IIB one still is left with two options of constructing such an involution. These correspond to the situations with O3/O7 or O5/O9 orientifold planes:

$$\begin{aligned} \mathcal{O}_1 &= \Omega_p \sigma_B (-)^{F_L}, & \sigma_B^* \Omega &= -\Omega, & \text{O3/O7}, \\ \mathcal{O}_2 &= \Omega_p \sigma_B, & \sigma_B^* \Omega &= \Omega, & \text{O5/O9}. \end{aligned} \quad (5.1)$$

Here  $\sigma_B$  is a holomorphic (instead of antiholomorphic, as in the Type IIA case) involutive symmetry  $\sigma_B^2 = 1$  of the Calabi-Yau target space, and  $F_L$  is the space-time fermion number in the left-moving sector. The subspace of fields which are invariant under the orientifold projection has to satisfy

$$\begin{array}{ccc} & \underline{\text{O3/O7}} & \underline{\text{O5/O9}} \\ \sigma_B^* \phi & = \phi, & \sigma_B^* C_0 = -C_0, \\ \sigma_B^* g & = g, & \sigma_B^* C_2 = C_2, \\ \sigma_B^* B_2 & = -B_2, & \sigma_B^* C_4 = -C_4, \end{array} \quad (5.2)$$

where the first column is identical for both involutions  $\sigma_B$  in (5.1). The involution allows us to separate the cohomologies into even and odd eigenspaces  $H^{(p,q)} = H_+^{(p,q)} \oplus H_-^{(p,q)}$ .

Let us focus on the closed string sector for the moment. Locally the truncated moduli space of Type IIB orientifolds can then be written as a direct product

$$\mathcal{M}_B^K \times \mathcal{M}_B^Q. \quad (5.3)$$

Here  $\mathcal{M}_B^Q$  is a Kähler manifold and spanned by the dilaton, the Kähler structure deformations, the NS-NS B-field and the R-R scalars.  $\mathcal{M}_B^K$  is a special Kähler manifold spanned by the complex structure deformations of  $\tilde{Y}$  respecting the constraints (5.1). In contrast, recall that in Type IIA  $\mathcal{M}_A^Q$  is spanned by the dilaton, the complex structure deformations and the R-R scalars, while  $\mathcal{M}_A^K$  is spanned by the Kähler deformations and the NS-NS B-field. The Type IIB effective theory also contains  $h_+^{(2,1)}(h_-^{(2,1)})$  vector multiplets for orientifolds with  $O3/O7(O5/O9)$  planes, whereas in Type IIA one as  $h_+^{(1,1)}$  vector multiplets. The number of multiplets from the closed string sector is shown in Table 5.1.

multiplets	IIA $_Y$ O6	IIB $_{\tilde{Y}}$ O3/O7	IIB $_{\tilde{Y}}$ O5/O9
vector multiplets	$h_+^{(1,1)}$	$h_+^{(2,1)}$	$h_-^{(2,1)}$
chiral multiplets in $\mathcal{M}^K$	$h_-^{(1,1)}$	$h_-^{(2,1)}$	$h_+^{(2,1)}$
chiral multiplets in $\mathcal{M}^Q$	$h^{(2,1)} + 1$	$h^{(1,1)} + 1$	$h^{(1,1)} + 1$

Table 5.1: Number of  $N = 1$  multiplets of orientifold compactifications.

Applying mirror symmetry to this  $\mathcal{N} = 1$  set-up one expects that the  $\mathcal{M}_B^Q$  space of type IIB should be identified with the  $\mathcal{M}_A^Q$  moduli space of the mirror IIA, and similarly  $\mathcal{M}_B^K$  with  $\mathcal{M}_A^K$ . Requiring  $\tilde{Y}$  to be the mirror manifold of  $Y$ , the mirror map between the moduli spaces implies that for the different orientifold setups

$$\begin{aligned} O3/O7 & : \quad h_-^{(1,1)}(Y) = h_-^{(2,1)}(\tilde{Y}) , & h_+^{(1,1)}(Y) = h_+^{(2,1)}(\tilde{Y}) , \\ O5/O9 & : \quad h_-^{(1,1)}(Y) = h_+^{(2,1)}(\tilde{Y}) , & h_+^{(1,1)}(Y) = h_-^{(2,1)}(\tilde{Y}) , \end{aligned} \quad (5.4)$$

as well as  $h^{(2,1)}(Y) = h^{(1,1)}(\tilde{Y})$  for both set-ups. The mirror mapping for closed moduli is discussed in more detail in [11], and will be briefly recalled below.

In the following we want to extend the mirror identification to include the leading corrections due to the space-time filling D-branes. As we have seen, at leading order the moduli space  $\mathcal{M}_A^K$  remains unchanged after the inclusion of open string moduli. This is also true for  $\mathcal{M}_B^K$  on the

Type IIB side. In section 3 we have shown that the open string moduli space of the D6-branes is fibered over the closed string moduli space  $\mathcal{M}_A^Q$ . The mirror equivalent of this statement has been established in [19, 21, 22] for  $\mathcal{M}_B^Q$  and the moduli space of D3-, D5- or D7-branes. In the remainder of this section we will therefore focus on the discussion of the  $\mathcal{M}^Q$  and establish the mirror map including the open degrees of freedom.

## 5.1 Mirror of O3/O7 orientifolds

The moduli space  $\mathcal{M}^Q$  is obtained from the four-dimensional scalar parts of the fields  $J, B_2, C_2, C_4$ . To make this more precise, we expand

$$\begin{aligned} B_2 &= b^k \omega_k, & C_2 &= c^k \omega_k, & k &= 1, \dots, h_-^{(1,1)}(\tilde{Y}), \\ J &= v^\lambda \omega_\lambda, & C_4 &= \rho_\lambda \tilde{\omega}^\lambda, & \lambda &= 1, \dots, h_+^{(1,1)}(\tilde{Y}). \end{aligned} \quad (5.5)$$

The complex coordinates and the Kähler potential which encode the local geometry of  $\mathcal{M}_B^Q$  are [17]

$$\begin{aligned} \tau &= C_0 + i e^{-\phi_B}, & G^k &= c^k - \tau b^k, \\ T_\lambda^{\prime B} &= e^{-\phi_B} \frac{1}{2} \mathcal{K}_{\lambda\rho\sigma} v^\rho v^\sigma + i \rho_\lambda - i \frac{1}{2} \mathcal{K}_{\lambda kl} b^k G^l, \end{aligned} \quad (5.6)$$

and

$$K(\tau, G^k, T_\lambda^{\prime B}) = -2 \ln \left[ e^{-2\phi_B} \int_{\tilde{Y}} J \wedge J \wedge J \right] = \ln(e^{4D_B}). \quad (5.7)$$

Here  $D_B$  is the redefined four-dimensional dilaton. The Kähler potential has to be evaluated as a function of the moduli  $\tau, G^k, T_\lambda^{\prime B}$  by solving (5.6) for  $v^a, \phi_B$  and inserting the result into (5.7). The coefficients  $\mathcal{K}_{\lambda bc}$  are the intersection numbers of the basis  $\omega_\lambda$  of  $H_+^{1,1}(\tilde{Y})$  and  $\omega_a$  of  $H_-^{1,1}(\tilde{Y})$ ,  $\mathcal{K}_{\lambda bc} = \int \omega_\lambda \wedge \omega_b \wedge \omega_c$ . Note that the above scalar fields can be also obtained from the expansion

$$- \text{Re} \Phi^{\text{ev}} + i \sum_n e^{-B} \wedge C_{2n} = i\tau + iG^k \omega_k + T_\lambda^{\prime B} \tilde{\omega}^\lambda, \quad (5.8)$$

which has to be evaluated by matching the parts of different form degrees on both sides. Here we have introduced the even form

$$\Phi^{\text{ev}} = e^{-\phi_B} e^{-B_2 + iJ} \quad (5.9)$$

following the notation of [13].

Let us now recall the mirror map to the Type IIA coordinates without inclusion of the open string degrees of freedom. The  $\mathcal{N} = 1$  coordinates  $(N^k, T'_\lambda)$  have been introduced in (2.8). Note that on a Calabi-Yau manifold we can use the rescaling invariance of  $\Omega$  to fix one of the  $X^I$  to be constant. At large complex structure there is a special real symplectic basis of  $H^3(Y)$  which is distinguished by the logarithmic behavior of the solutions in the complex structure moduli of  $Y$ . In particular, this fixes a pair  $(\alpha_0, \beta^0)$ , by demanding that  $X^0$ , the fundamental period, has no logarithmic singularity. One can use the rescaling of  $\Omega$  to set the  $\alpha_0$  period to a constant. Note that

in the orientifold background  $H^3(Y)$  splits into  $H_-^3$  and  $H_+^3$ . The component chosen to eliminate the rescaling property of  $\Omega$  can be either in the positive or negative eigenspace of the orientifold projection. We will see momentarily these choices will correspond to different orientifold set-ups on the Type IIB side.

For the O3/O7 case we fix the component  $X^0\alpha_0$  in  $H_+^3(Y)$ . We define then the special coordinates  $q$  and the scaling parameter  $g_A$  as

$$q^k = \frac{\text{Re}CX^k}{\text{Re}CX^0}, \quad q^\lambda = \frac{\text{Im}CX^\lambda}{\text{Re}CX^0}, \quad g_A^{-1} = 2\text{Re}CX^0. \quad (5.10)$$

Recall that in the underlying  $\mathcal{N} = 2$  theory, the periods of  $\Omega$  are determined by a holomorphic pre-potential  $\mathcal{F}(X)$ . Due to the homogeneity property of  $\mathcal{F}$  we can define a rescaled function  $f$  as

$$\mathcal{F}(2CX) = i(2\text{Re}CX^0)^2 f(q^k, q^\lambda) \quad (5.11)$$

such that  $C\Omega$  can be written as

$$2C\Omega = g_A^{-1} \left[ 1\alpha_0 + q^k\alpha_k + iq^\lambda\alpha_\lambda - f_\lambda\beta^\lambda - i(2f - q^k f_k - q^\lambda f_\lambda)\beta^0 - i f_k\beta^k \right], \quad (5.12)$$

where  $(f_\lambda, f_k)$  are the derivatives of  $f$  with respect to  $(q^\lambda, q^k)$ . The coordinates  $(N'^k, T'_\lambda)$  become in terms of these special coordinates

$$N'^0 = g_A^{-1} + i\xi^0 \quad N'^k = g_A^{-1}q^k + i\xi^k \quad T'_\lambda{}^A = g_A^{-1}f_\lambda + i\tilde{\xi}_\lambda. \quad (5.13)$$

In order to provide complete match with the Type IIB side we need an explicit expression for  $f_\lambda$  at the large complex structure limit of the Calabi-Yau manifold  $Y$ . The results will then be identified with the large volume results of Type IIB. In this limit the  $\mathcal{N} = 2$  pre-potential is given by

$$\mathcal{F}(X) = \frac{1}{6}\kappa_{IJK} \frac{X^I X^J X^K}{X^0}. \quad (5.14)$$

Therefore, inserting the orientifold constraints and switching to special coordinates we find

$$f(q) = -\frac{1}{6}\kappa_{\lambda\mu\rho}q^\lambda q^\mu q^\rho + \frac{1}{2}\kappa_{\lambda kl}q^\lambda q^k q^l, \quad (5.15)$$

such that one can readily evaluate the  $T'_\lambda{}^A$  using (5.13). Now it is straightforward to relate the Type IIA coordinates with the ones from the Type IIB side

$$(-i\tau, -iG^k) \leftrightarrow (N'^0, N'^k) \quad \text{and} \quad -T'_\lambda{}^B \leftrightarrow T'_\lambda{}^A, \quad (5.16)$$

with the matching of the cohomologies for the pair of mirror Calabi-Yau manifolds given in Table 5.2. In terms of the string moduli, the above relations translate into

$$\begin{aligned} g_A^{-1} &= e^{-\phi_B}, & q^k &= -b^k, & q^\lambda &= v^\lambda, \\ \xi_0 &= -C_0, & \xi^k &= -c^k + C_0 b^k, & \tilde{\xi}_\lambda &= -\rho_\lambda + \frac{1}{2}\mathcal{K}_{\lambda kl}c^k b^l - \frac{1}{2}C_0\mathcal{K}_{\lambda kl}b^k b^l. \end{aligned} \quad (5.17)$$



$H^3(Y)$	$H^{\text{even}}(\tilde{Y})$
$\alpha_0 \in H_+^3(Y)$	1
$\alpha_k \in H_+^3(Y)$	$\omega_k \in H_-^2(\tilde{Y})$
$\alpha_\lambda \in H_-^3(Y)$	$\omega_\lambda \in H_+^2(\tilde{Y})$
$\beta^k \in H_-^3(Y)$	$\tilde{\omega}^k \in H_-^4(\tilde{Y})$
$\beta^\lambda \in H_+^3(Y)$	$\tilde{\omega}^\lambda \in H_+^4(\tilde{Y})$
$\beta^0 \in H_-^3(Y)$	$\mathcal{V}^{-1}\text{vol}_{\tilde{Y}}$

Table 5.2: The mirror mapping from the basis of  $H^3(Y)$  to the basis of even cohomologies of the mirror Calabi-Yau  $\tilde{Y}$  in  $O3/O7$  orientifold setups.

### Inclusion of D3 brane moduli

In the discussion of mirror symmetry with D-branes we first consider the setup with spacetime filling D3 branes. The  $\mathcal{N} = 1$  characteristic data were analyzed in [19]. The brane is a point in the internal space  $\tilde{Y}$ , such that the brane deformations  $\eta$  are described by six scalar fields  $\phi^I$  corresponding to the possible movements in  $\tilde{Y}$ . These fields naturally combine into complex fields  $\phi^i, \phi^{\bar{j}}$  with  $i, \bar{j} = 1, 2, 3$  if one uses the inherited complex structure of the Calabi-Yau manifold. Clearly, there are no Wilson line moduli for D3-branes since there is no internal one-cycle on the brane. It turns out that, up to second order in the fields, only the coordinates  $T_\lambda^B$  are corrected by the open moduli [19]

$$\text{Re } T_\lambda^B = \text{Re } T_\lambda^{\prime B} + i(\omega_\lambda)_{i\bar{j}}(\phi_0) \phi^i \phi^{\bar{j}}, \quad (5.18)$$

where the two-form  $(\omega_\lambda)_{i\bar{j}}$  has to be evaluated at the point  $\phi_0$  around which the D3-brane fluctuates. More generally, it was argued in ref. [46] that the D3-brane correction to  $T_\alpha$  can be expressed through the Kähler potential  $K_{\tilde{Y}}$  for the Calabi-Yau metric as

$$\text{Re } T_\lambda^B = \text{Re } T_\lambda^{\prime B} - \partial_{v^\lambda} K_{\tilde{Y}}(\phi_0 + \phi), \quad (5.19)$$

where  $v^\lambda$  are the Kähler moduli introduced in (5.5). To obtain (5.18) one expands  $K_{\tilde{Y}}$  around the point  $\phi_0$  as

$$K_{\tilde{Y}}(\phi_0 + \phi) = K_{\tilde{Y}}^0 + 2\text{Re}[(K_{\tilde{Y}}^0)_i \phi^i] + \text{Re}[(K_{\tilde{Y}}^0)_{ij} \phi^i \phi^j] + (K_{\tilde{Y}}^0)_{i\bar{j}} \phi^i \phi^{\bar{j}} + \dots, \quad (5.20)$$

where  $K_{\tilde{Y}}^0$ , and  $(K_{\tilde{Y}}^0)_i, (K_{\tilde{Y}}^0)_{ij}, (K_{\tilde{Y}}^0)_{i\bar{j}}$  are the Kähler potential and its  $\phi^i$ -derivatives evaluated at  $\phi_0$ . Since the coefficients are constant, the first three terms in (5.20) can be absorbed by a holomorphic redefinition into a new  $T_\lambda^B$ . Clearly, this does not change the complex structure on the  $\mathcal{N} = 1$  moduli space. Using  $(K_{\tilde{Y}}^0)_{i\bar{j}} = -iJ_{i\bar{j}}^0 = -iv^\lambda(\omega_\lambda)_{i\bar{j}}(\phi_0)$  one then recovers (5.18).

Let us now turn to the discussion of mirror symmetry. We aim to match the corrected coordinates  $T_\lambda^B$  as well as the un-corrected  $G^k$  and  $\tau$  with the Type IIA side. This implies that we must

have up to quadratic order in the brane moduli that

$$\begin{aligned} -2\partial_{V^\lambda}(e^{2D_A}K_o) &= \partial_{v^\lambda}K_{\tilde{Y}}(\phi_0 + \phi) \cong -i(\omega_\lambda)_{i\bar{j}}\phi^i\bar{\phi}^{\bar{j}} \\ \partial_{V_0}(e^{2D_A}K_o) &= \partial_{V_k}(e^{2D_A}K_o) = 0, \end{aligned} \quad (5.21)$$

where the  $\cong$  indicates that one has to apply the transformation which identifies (5.19) and (5.18). Using the fact that  $V^\lambda = -e^{2D_B}e^{-\phi_B}v^\lambda$ , as inferred from (5.12), the identification (5.21) implies

$$K_o(\phi, \bar{\phi}) = \frac{1}{2}e^{-\phi_B}K_{\tilde{Y}}. \quad (5.22)$$

The number of open moduli must coincide, so the number of brane deformations on the Type IIB must equal the number of brane and Wilson line moduli on the Type IIA side. Since this number is given by the number of non-trivial one-cycles in  $L_0$ , we must have  $b^1(L_0) = 3$ . However, recall that the open moduli space in Type IIA has shift symmetries,  $\text{Im}\zeta^i \rightarrow \text{Im}\zeta^i + c^i$ , for constants  $c^i$ . These are not manifested in the Type IIB side for a general  $K_{\tilde{Y}}$ , since the Calabi-Yau metric has no continuous symmetries. As we recall below, this can be attributed to the fact that instanton contributions break these symmetries and are not included in this leading order identification.

Before commenting on the corrections to the mirror construction let us make contact to the chain integral form of the Kähler potential as given in (4.8). For a D3-brane we simply have to introduce a one-chain  $\mathcal{C}_1$  which starts at  $\phi_0$  and ends at the point in  $\tilde{Y}$  to which the D3-brane has moved. We also introduce a basis of complex normal vectors  $s_i$  to the point  $\phi_0$  and dual (1,0)-forms  $s_{(1)}^j$  such that

$$s_i \lrcorner s_{(1)}^j = \delta_i^j. \quad (5.23)$$

Note that the index  $i, j$  are counting here the number of such normal vectors. In case we only include the massless modes, one has  $i, j = 1, \dots, 3$ . The complex structure of  $s_i$  and  $s_{(1)}^i$  is induced by the complex structure of  $\tilde{Y}$ , and hence depends on the complex structure moduli. In fact one can use the nowhere vanishing (3,0)-form  $\Omega$  on  $\tilde{Y}$  and introduce a bi-vector  $s^j$  such that  $s_{(1)}^j = \bar{s}^j \lrcorner \Omega$ . To propose a form for  $K_o$  one trivially extended  $s_i, \bar{s}^i$  to the chain  $\mathcal{C}_1$  and writes

$$K_o = \frac{i}{4}e^{-\phi_B} \int_{\mathcal{C}_1} s_i \lrcorner J \int_{\mathcal{C}_1} \bar{s}^i \lrcorner \Omega + c.c. . \quad (5.24)$$

This form of  $K_o$  is very suggestive and yields upon expanding the chain integral the desired leading order expression (5.18). Moreover, we will see in the following that a generalization of this  $K_o$  also arises for D7-brane, and one can generally write in O3/O7 orientifolds for the deformations of a  $D(p+3)$ -brane

$$K_o^{\text{def}} = \frac{i}{4} \int_{\mathcal{C}_{p+1}} s_I \lrcorner \text{Im}\Phi^{\text{ev}} \int_{\mathcal{C}_{p+1}} \bar{s}^I \cdot \Omega + c.c. . \quad (5.25)$$

where  $\Phi^{\text{ev}}$  has been introduced in (5.9), and  $\mathcal{C}_{p+1}$  is a  $(p+1)$ -chain which ends on the internal parts of the D-branes and its reference cycle. Moreover,  $s_I$  is an appropriate basis of complex normal vectors and  $s^J$  are their duals as we discuss below.

Before giving a more careful treatment of the other D-brane configurations let us first comment on a more intuitive understanding of mirror symmetry which we will apply below. It was argued by Strominger, Yau and Zaslow [27] that the Calabi-Yau manifold  $\tilde{Y}$  can be viewed as a three-torus fibration with singular fibers. This manifold can be endowed with a semi-flat metric. In a local patch avoiding possible singular points the metric of the Calabi-Yau manifold can be written as

$$ds^2 = g_{ab}(\tilde{u})d\tilde{u}^a d\tilde{u}^b + 2g_{ia}(\tilde{u})d\tilde{a}^i d\tilde{u}^a + g_{ij}(\tilde{u})d\tilde{a}^i d\tilde{a}^j , \quad i, a = 1, 2, 3 , \quad (5.26)$$

where  $\tilde{a}^i$  are the coordinates on the  $T^3$  fiber and  $\tilde{u}^a$  of the base. Since the coefficient functions in (5.26) are independent of  $\tilde{a}^i$  the shift symmetry is now manifest. In fact, introducing complex coordinates as in the Type IIA setting a Kähler metric in (5.26) can be obtained from a Kähler potential  $K_{\tilde{Y}}(\tilde{u})$  which is independent of  $\tilde{a}^i$ . The argument for the existence of such a  $T^3$ -fibration with a metric of the form (5.26) away from singularities proceeds precisely via mirror symmetry of a pointlike D-brane on  $\tilde{Y}$  which is mapped to a D-brane which wraps a three-torus [27]. Having found a  $T^3$ -fibration in the Type IIB set-up one can equally use T-duality along all  $T^3$ -directions to analyze the setting. Since T-duality exchanges Neumann and Dirichlet boundary conditions, it exchanges the dimensionality of the brane for each wrapped cycle that is T-dualized. Starting with a D3-brane on such a fibered Calabi-Yau manifold, T-duality on the fiber will turn the brane into a D6-brane wrapping the  $T^3$ -fiber. The D6-brane then has  $b^1(L_0) = 3$  deformation moduli in the direction of the base, and there are also  $b^1(L_0) = 3$  Wilson line moduli will be along the torus.

In the following it will be more important that we can use the SYZ-picture also for D7- and D5-branes present in a Type IIB reduction. Clearly, both types of branes will map to D6-branes under mirror symmetry. Away from the singular fibers one can obtain a clearer picture of the wrappings of the D6-branes as indicated in Table 5.3.

	D6	D3	D6	D7	D6	D5
T <sup>3</sup>	×			×		×
	×			×	×	
	×		×		×	
Base						
			×	×		
			×	×	×	×

Table 5.3: It is summarized how mirror symmetry acts on different brane configurations. The table shows the six dimensions of the Calabi-Yau manifold, split into base and fiber.  $\times$  indicate the directions wrapped by each brane. Mirror symmetry acts as T-duality on all directions of the  $T^3$ -fiber. It exchanges Dirichlet and Neumann boundary conditions, while it does not act on the base. Different wrappings of a D6-brane correspond to different branes in the Type IIB side.

## Inclusion of D7 brane moduli

Let us now discuss mirror symmetry for the D7-brane case. The effective action for a pair of moving D7-branes was computed in [21]. In this setup, the brane wraps a four-cycle  $S^{(1)}$  while its orientifold image wraps a non-intersecting  $S^{(2)}$ . One can view the whole configuration as a single D7-brane wrapping a divisor  $S_+ = S^{(1)} + S^{(2)}$ . Brane deformations and Wilson line moduli can be expanded in terms of

$$\begin{aligned}\chi &= \chi^A s_A + \bar{\chi}^{\bar{A}} \bar{s}_{\bar{A}}, & A = 1, \dots, h_-^{(2,0)}(S_+), \\ a &= a^I \gamma_I + \bar{a}^{\bar{I}} \bar{\gamma}_{\bar{I}} & I = 1, \dots, h_-^{(0,1)}(S_+),\end{aligned}\tag{5.27}$$

where  $s_A$  and  $\gamma_I$  are complex normal vectors to  $S^{(1)}$  and  $(0,1)$ -forms on  $S^{(1)}$ , respectively. The complex type of  $s_A$  and  $\gamma_I$  is induced by the complex structure of  $\tilde{Y}$ . Moreover, one can use the holomorphic  $(3,0)$ -form  $\Omega$  on  $\tilde{Y}$  to map the  $s_A$  to  $(2,0)$ -forms  $\mathcal{S}_A = s_A \lrcorner \Omega$  on  $S^{(1)}$ . The four-dimensional fields are thus the  $h_-^{(2,0)} + h_-^{(1,0)}$  complex scalars  $\chi^A$  and  $a^I$ , respectively.

Including the open string degrees of freedom, the chiral coordinates  $(\tau, G_a, T_\lambda^B)$  are shifted to [21]

$$\begin{aligned}S &= \tau + \mathcal{L}_{A\bar{B}} \chi^A \bar{\chi}^{\bar{B}}, & G^k &= c^k - \tau b^k, \\ T_\lambda^B &= \frac{1}{2} e^{-\phi_B} \mathcal{K}_{\lambda\rho\sigma} v^\rho v^\sigma + i\rho_\lambda - i\frac{1}{2} \mathcal{K}_{\lambda kl} b^k G^l + i\mathcal{C}_{\lambda I \bar{J}} a^I \bar{a}^{\bar{J}}.\end{aligned}\tag{5.28}$$

The coupling functions  $\mathcal{L}_{A\bar{B}}$  and  $\mathcal{C}_{\lambda I \bar{J}}$  for the basis of brane deformations and Wilson line moduli on the four-cycle are given by

$$\mathcal{L}_{A\bar{B}} = \frac{\int_{S_+} \mathcal{S}_A \wedge \bar{\mathcal{S}}_{\bar{B}}}{\int_{\tilde{Y}} \Omega \wedge \bar{\Omega}}, \quad \mathcal{C}_{\lambda I \bar{J}} = \int_{S_+} \omega_\lambda \wedge \gamma_I \wedge \bar{\gamma}_{\bar{J}}.\tag{5.29}$$

Since the closed moduli are the same, we proceed in the same way as we did for the closed and the D3-brane cases, identifying the coordinates as (5.16). Analogously to the D3-brane case, we expand up to second order in the open moduli and match both theories by

$$\partial_{V^\lambda} (e^{2DA} G_{ij}) u^i u^j \cong i\mathcal{C}_{\lambda I \bar{J}} a^I \bar{a}^{\bar{J}}, \quad \partial_{V_0} (e^{2DA} G_{ij}) u^i u^j \cong i\mathcal{L}_{A\bar{B}} \chi^A \bar{\chi}^{\bar{B}}, \quad \partial_{V_k} (e^{2DA} G_{ij}) u^i u^j \cong 0,\tag{5.30}$$

where we have indicated that as in the D3-brane case one will need to make the shift symmetry manifest before finding complete match. Crucially one has to split the Type IIA coordinates into two sets  $\zeta^I$  and  $\zeta^A$  and identify

$$\zeta^I \cong a^I, \quad \zeta^A \cong \chi^A.\tag{5.31}$$

One notes that Wilson line moduli and brane deformations do not mix on the Type IIB side which seems to be in contrast to the general form on the Type IIA side. We will argue later how this splitting can be understood from the SYZ-picture of mirror symmetry.

As already suggested in (5.25) one expects that the open corrections to the  $\mathcal{N} = 1$  coordinates can again be given in terms of chain integrals. Let us first give the expression for  $K_o$  which encodes upon differentiation with respect to  $V^\lambda, V_0, V_k$  the corrections in  $T_\lambda, N^0, N^k$ . Explicitly, we propose

$$K_o = \frac{i}{4} \int_{\mathcal{C}_5} s_A \lrcorner \text{Im} \Phi^{\text{ev}} \int_{\mathcal{C}_5} \bar{s}^A \wedge \Omega + \frac{i}{4} \int_{\mathcal{C}_5} \mathcal{F}_{\text{D7}} \wedge \gamma_I \wedge \text{Im} \Phi^{\text{ev}} \int_{\mathcal{C}_5} \mathcal{F}_{\text{D7}} \wedge \bar{\gamma}^I + c.c. , \quad (5.32)$$

where  $\Phi^{\text{ev}}$  is given in (5.9). Here we have used a five-chain  $\mathcal{C}_5$  ending on the D7-brane and a reference four-cycles  $S_+^0$ . Note that similar to the D6-brane case we have to introduce a dual basis  $s_A$  and  $s^A$ . To do that we use the fact that no-where vanishing (3,0)-form  $\Omega$  provides an identification

$$\Omega : NS_+ \rightarrow TS_+^* \wedge TS_+^* , \quad (5.33)$$

of normal vectors with two-forms of  $S_+$ . Hence, in the Type IIB setting we adopt this basis to the complex structure by demanding that  $s_A$  is a complex normal vector in  $H_+^0(NS_+)$  and  $s^A$  is a (2,0)-form in  $H_-^{(2,0)}(S_+)$  on  $S_+^0$ . Similarly,  $\gamma_I$  is a (0,1)-form as introduced above and  $\gamma^J$  is a (1,2)-form in  $H_-^{(1,2)}(S_+^0)$ . These forms are defined to be dual and hence satisfy

$$\int_{S_+^0} \bar{s}^A \wedge (s_B \lrcorner \Omega) = \delta_B^A , \quad \int_{S_+^0} \gamma_I \wedge \bar{\gamma}^J = \delta_I^J . \quad (5.34)$$

As in the D6-brane case we have to extend these forms to the chain. It is interesting to note that the expression (5.32) indeed reproduces the leading order corrections after differentiating with respect to  $V^\lambda, V_0, V_k$ .

## 5.2 Mirror symmetry for O5-orientifolds and D5-branes

Let us now discuss the second Type IIB set-up which is obtained by an involution with O5-planes as fix-point set. The bulk  $\mathcal{N} = 1$  coordinates of the moduli space  $\mathcal{M}^{\text{Q}}$  are given as functions of the zero-modes in the expansion

$$\begin{aligned} J &= v^k \omega_k , & C_2 &= \tilde{C}_2 + c^k \omega_k , & k &= 1, \dots, h_+^{(1,1)}(\tilde{Y}) , \\ B_2 &= b^\lambda \omega_\lambda , & C_4 &= \rho_\lambda \tilde{\omega}^\lambda , & \lambda &= 1, \dots, h_-^{(1,1)}(\tilde{Y}) . \end{aligned} \quad (5.35)$$

Note the difference that we have used forms of different  $\sigma$ -parity in the expansion for the R-R-fields,  $C_2$  and  $C_4$  as required for the second orientifold projection in (5.2). While  $C_0$  has been projected out  $C_2$  now contains a four-dimensional two-form  $\tilde{C}_2(x)$  which together with the dilaton  $\phi_B$  form the bosonic content of a linear multiplet. However,  $\tilde{C}_2$  can be dualized to a scalar field  $h$  and form with  $\phi_B$  a chiral multiplet. The  $\mathcal{N} = 1$  coordinates which span  $\mathcal{M}^{\text{Q}}$  are thus the  $h^{(1,1)} + 1$  complex fields

$$\begin{aligned} t^k &= e^{-\phi_B} v^k - i c^k , & P_\lambda &= \mathcal{K}_{\lambda\rho k} b^\rho t^k + i \rho_\lambda , \\ S &= e^{-\phi_B} \mathcal{V} + i h - \frac{i}{2} \rho_\lambda b^\lambda - \frac{1}{2} P_\lambda b^\lambda , \end{aligned} \quad (5.36)$$

Formally the Kähler potential is the same as in the O3/O7-case given in (5.7). However, it now has to be evaluated as a function of the coordinates  $t'^k, P_\lambda$  and  $S$  by using there explicit form (5.36). Similar to (5.8) we can write

$$- \text{Im} \Phi^{\text{ev}} + i \sum_n e^{B_2} \wedge C_{2n} = -t'^k \omega_k + P_\lambda \tilde{\omega}^\lambda + S \text{vol}_{\tilde{Y}}. \quad (5.37)$$

Let us turn to the discussion of the mirror Type IIA side to this construction. As explained above the second set-up with O5-planes is obtained by choosing the three-form  $\alpha_0$  for the fundamental period  $X^0$  to lie in the negative eigenspace  $H_-^3(\tilde{Y})$ . Again we will perform a rescaling of  $\Omega$  setting the coefficient of  $\alpha_0$  to be constant. The special coordinates are then given by

$$g_A^{-1} = 2\text{Im}CX^0, \quad q^k = \frac{\text{Re}CX^k}{\text{Im}CX^0}, \quad q^\lambda = \frac{\text{Im}CX^\lambda}{\text{Im}CX^0}. \quad (5.38)$$

Now the rescaled prepotential  $f$  is given by  $\mathcal{F}(2CX) = -i(2\text{Im}CX^0)^2 f(q^k, q^\lambda)$ . This allows us to rewrite  $C\Omega$  in the rescaled coordinates as

$$2C\Omega = g_A^{-1} \left[ q^k \alpha_k + i\alpha_0 + iq^\lambda \alpha_\lambda + f_\lambda \beta^\lambda - (-2f + q^k f_k + q^\lambda f_\lambda) \beta^0 + i f_k \beta^k \right]. \quad (5.39)$$

Moreover, we can use the special coordinates to write  $(N'^k, T'_\lambda{}^A, T'_0{}^A)$  as

$$N'^k = g_A^{-1} q^k + i\xi^k, \quad T'_0{}^A = g_A^{-1} (-2f + q^\lambda f_\lambda + q^k f_k) + i\tilde{\xi}_0, \quad T'_\lambda{}^A = -g_A^{-1} f_\lambda + i\tilde{\xi}_\lambda. \quad (5.40)$$

With  $f$  in the large complex structure limit

$$f(q) = \frac{1}{6} \kappa_{klm} q^k q^l q^m - \frac{1}{2} \kappa_{l\mu\rho} q^l q^\mu q^\rho. \quad (5.41)$$

this allows us to write

$$T'_0{}^A = g_A^{-1} \left( \frac{1}{6} \kappa_{klm} q^k q^l q^m - \frac{1}{2} \kappa_{\mu\lambda k} q^\mu q^\lambda q^k \right) + i\tilde{\xi}_0, \quad T'_\lambda{}^A = g_A^{-1} \kappa_{\lambda\mu k} q^\mu q^k + i\tilde{\xi}_\lambda. \quad (5.42)$$

The mirror mapping is then realized by

$$t'^k \leftrightarrow N'^k \quad \text{and} \quad (S, P_\lambda) \leftrightarrow (T'_0{}^A, T'_\lambda{}^A). \quad (5.43)$$

In terms of the Kaluza-Klein modes this amounts to the identification of the closed moduli

$$\begin{aligned} g_A^{-1} &= e^{-\phi_B}, & q^k &= v^k, & q^\lambda &= b^\lambda, \\ \tilde{\xi}_0 &= h - \rho_\lambda b^\lambda + \frac{1}{2} \mathcal{K}_{l\lambda\kappa} c^l b^\lambda b^\kappa, & \xi^k &= -c^k, & \tilde{\xi}_\lambda &= \rho_\lambda - \mathcal{K}_{\lambda\kappa l} c^l b^\kappa. \end{aligned} \quad (5.44)$$

The identification of the basis elements on the Type IIA and Type IIB side is given in Table 5.4.

$H^3(Y)$	$H^{\text{even}}(\tilde{Y})$
$\alpha_0 \in H_-^3(Y)$	1
$\alpha_k \in H_+^3(Y)$	$\omega_k \in H_+^2(\tilde{Y})$
$\alpha_\lambda \in H_-^3(Y)$	$\omega_\lambda \in H_-^2(\tilde{Y})$
$\beta^k \in H_-^3(Y)$	$\tilde{\omega}^k \in H_+^4(\tilde{Y})$
$\beta^\lambda \in H_+^3(Y)$	$\tilde{\omega}^\lambda \in H_-^4(\tilde{Y})$
$\beta^0 \in H_+^3(Y)$	$\mathcal{V}^{-1} \text{vol}_{\tilde{Y}}$

Table 5.4: The mirror mapping from the basis of  $H^3(Y)$  to the basis of even cohomologies of the mirror Calabi-Yau  $\tilde{Y}$  in  $O5/O9$  orientifold setups.

### Inclusion of D5 brane moduli

We now consider a pair of D5-branes on curves  $\Sigma^{(1)}$  and  $\Sigma^{(2)}$  which are interchanged under the orientifold involution. We call the positive union of  $\Sigma^{(1)}$  and  $\Sigma^{(2)}$  by  $\Sigma_+ = \Sigma^{(1)} + \Sigma^{(2)}$ . Again we view this as a single D5-brane on the quotient space. The open moduli for a single D5-brane, corresponding to complex brane deformations  $\chi^A$ ,  $A = 1, \dots, \dim H_-^0(N\Sigma_+)$  and Wilson line moduli  $a^I$ ,  $I = 1, \dots, h_-^{(0,1)}(\Sigma_+)$ , correct the  $\mathcal{N} = 1$  coordinates according to [22]

$$\begin{aligned}
t^k &= t^k + \mathcal{L}_{AB}^k \chi^A \bar{\chi}^{\bar{B}}, \\
P_\lambda &= \mathcal{K}_{\lambda\rho k} b^\rho t^k + i\rho_\lambda, \\
S &= e^{-\phi_B} \mathcal{V} + ih - \frac{i}{2} \rho_\lambda b^\lambda - \frac{1}{2} P_\lambda b^\lambda + \mathcal{C}_{I\bar{J}} a^I \bar{a}^{\bar{J}}.
\end{aligned} \tag{5.45}$$

Here we have introduced the couplings

$$\mathcal{L}_{AB}^k = -i \int_{\Sigma_+} s_{A\downarrow} \bar{s}_{B\downarrow} \tilde{\omega}^k, \quad \mathcal{C}_{I\bar{J}} = i \int_{\Sigma_+} \gamma_I \wedge \bar{\gamma}_{\bar{J}} \tag{5.46}$$

The Kähler potential now has to be evaluated as a function of  $t^k, P_\lambda, S$  as well as the open coordinates  $\chi^A$  and  $a^I$ .

In order to discuss mirror symmetry to the D6-brane set-up we again compare the form of the  $\mathcal{N} = 1$  coordinates. Expanding to second order in the open corrections we find

$$-\partial_{V_k}(e^{2D_A} G_{ij}) u^i u^j \cong \mathcal{L}_{AB}^k \chi^A \bar{\chi}^{\bar{B}}, \quad -\partial_{V_0}(e^{2D_A} G_{ij}) u^i u^j \cong \mathcal{C}_{I\bar{J}} a^I \bar{a}^{\bar{J}}, \quad -\partial_{V_\lambda}(e^{2D_A} G_{ij}) u^i u^j \cong 0. \tag{5.47}$$

More interestingly, we can also directly compare the open Kähler potential  $K_o$ . To do that, we give a chain integral expression for the D5-brane case. We introduce a the three-chain  $\mathcal{C}_3$  ending on a reference cycle  $\Sigma_+^0$  and the two-cycle to which the brane has moved. The open Kähler potential then takes the form

$$K_o = -\frac{i}{4} \int_{\mathcal{C}_3} s_{A\downarrow} \text{Re} \Phi^{\text{ev}} \int_{\mathcal{C}_3} \bar{s}^A \cdot \Omega - \frac{i}{4} \int_{\mathcal{C}_3} \mathcal{F}_{D5} \wedge \gamma_I \wedge \text{Re} \Phi^{\text{ev}} \int_{\mathcal{C}_3} \mathcal{F}_{D5} \wedge \bar{\gamma}^I + c.c., \tag{5.48}$$

where  $\Phi^{\text{ev}}$  is given in (5.9). Note that this expression has a similar structure as (5.32). However, due to the lower dimensionality of the chain the four-form part of  $\text{Re } \Phi^{\text{ev}}$  is picked up in the first term of (5.48), while the zero-form part of  $\text{Re } \Phi^{\text{ev}}$  contributes in the second term of (5.48). In the case of a D5-brane the (3, 0)-form  $\Omega$  on  $\tilde{Y}$  provides a map

$$\Omega : N\Sigma_+ \otimes N\Sigma_+ \rightarrow T\Sigma_+^* , \quad (5.49)$$

taking two normal vectors to a one-form on  $\Sigma_+$ . This allows us to introduce a basis  $s^A$  of  $H^0(T\Sigma_+^0 \otimes \overline{N\Sigma_+^0})$  which is dual to the normal vectors  $s_A$ . Hence, the  $\cdot$  in (5.48) indicates that the vector part of  $s^A$  is inserted, while the form part of  $s^A$  is wedged with  $\Omega$ . We also introduce complex one-forms  $\gamma^J$  on  $\Sigma_+^0$  which are dual to the (0, 1)-forms  $\gamma_I$  used in the expansion determining the complex Wilson line scalars  $a^I$ . Explicitly, the  $s^A, \gamma^I$  have to satisfy on the reference  $\Sigma_+^0$  that

$$\int_{\Sigma_+^0} s_A \lrcorner \bar{s}^B \cdot \Omega = \delta_A^B , \quad \int_{\Sigma_+^0} \gamma_I \wedge \bar{\gamma}^J = \delta_I^J , \quad (5.50)$$

As in the D6-brane case the basis forms and vectors have to be extended trivially to the chain  $\mathcal{C}_3$  to evaluate the open Kähler potential (5.48). One can now check that the expansion (5.48) leads upon differentiation with respect to  $V_k, V^0, V^\lambda$  the leading order corrections in (5.45).

### 5.3 General remarks on the structure of the couplings

In this subsection we address the question if there is a simple way to understand the mappings of (5.47), (5.21) and (5.30) using the SYZ-picture of mirror symmetry. For example for D5-branes the  $(\partial_{V_k}(e^{2D_A} G_{ij}), \partial_{V_0}(e^{2D_A} G_{ij}))$  correct the coordinates  $t^k$  and  $S$  by brane deformations and Wilson line moduli as demanded by the mirror identification (5.47). In contrast, the coordinates  $P_\lambda$  do not receive any contributions from open moduli and hence  $\partial_{V_\lambda}(e^{2D_A} G_{ij})$  has to vanish in the D6-brane set-up mirror dual to a D5-brane. To analyze this question in the SYZ-picture, first let us look at the gauge coupling functions. In the limit of vanishing open string moduli they are given by the analogous to the D6-brane gauge coupling function  $f_{\text{D6}} = N^k \int_L \alpha_k - T_\lambda \int_L \beta^\lambda$ ,

$$f_{\text{D3}} = \tau , \quad f_{\text{D5}} = t^\Sigma \int_{\Sigma_+} \omega_\Sigma , \quad f_{\text{D7}} = T_S \int_{S_+} \tilde{\omega}^S , \quad (5.51)$$

where  $\Sigma_+(S_+)$  is the curve(divisor) wrapped by the D5(D7)-brane, and they are obtained from a basis of homology by

$$\begin{aligned} [\Sigma_+] &= n^k [\Sigma_k] , & \Sigma_k &\in H_2^+(Y) \quad \text{and} \\ [S_+] &= n_\lambda [S^\lambda] , & S^\lambda &\in H_4^+(Y) . \end{aligned} \quad (5.52)$$

Therefore the forms appearing in (5.51) are, in terms of the cohomology basis,  $\omega_\Sigma = n^k \omega_k$  and  $\tilde{\omega}^S = n_\lambda \tilde{\omega}^\lambda$ .



From the four internal dimensions the D7-brane wraps, locally two of them are along the  $T^3$ -fiber and the other two on the base, as seen from table 5.3. The mirror D6-brane, on the other hand, wraps one dimension on  $T^3$ -fiber and two dimensions on the base. It is also inferred from the gauge coupling function of the D7-brane (5.51) that  $\tilde{\omega}^\lambda$  sits on the brane, therefore having two “legs” on the 3-Torus and two on the base. We define thus the notation  $\tilde{\omega}^\lambda : (bbtt)$ , where  $b$  and  $t$  correspond to base and torus components. Table 5.2 shows that  $\tilde{\omega}^\lambda$  on the Type IIB side is mapped on the Type IIA side to  $\beta^\lambda$ . Therefore, from table 5.3, since  $\beta^\lambda$  must sit on the mirror D6-brane, it should satisfy  $\beta^\lambda : (bbt)$ .  $\beta^\lambda$  must be dual to  $\alpha_\lambda$  on the Calabi-Yau manifold  $Y$ , thus  $\alpha_\lambda : (btt)$ . A similar analysis can be done for the D5 and D3-Branes, from where we obtain  $\alpha_k : (btt)$ ,  $\beta^k : (bbt)$ ,  $\beta^0 : (bbb)$  and  $\alpha_0 : (ttt)$ .

One can now analyze the open moduli corrections to the  $\mathcal{N} = 1$  chiral coordinates from the metric derivatives  $\partial_{V_0}\widehat{\mathcal{G}}_{ij}$ ,  $\partial_{V_k}\widehat{\mathcal{G}}_{ij}$  and  $\partial_{V^\lambda}\widehat{\mathcal{G}}_{ij}$ . As a simple example we consider the D3-brane case. We can rewrite the corrections in terms of the normal deformations  $\eta^i$

$$\text{Re}(N'^0 - N^0) = \partial_{V_0}(e^{2D_A}\widehat{\mathcal{G}}_{ij})\eta^i\eta^j = \frac{1}{2} \int_{L_0} \hat{\alpha}_k \wedge \eta_{\lrcorner} \beta^0 \int_{L_0} \hat{\beta}^k \wedge \eta_{\lrcorner} J . \quad (5.53)$$

Since the brane wraps the three-torus, both integrands in (5.53) must be of the form  $(ttt)$ . The normal directions of this D6-brane are all on the base, so  $\eta_{\lrcorner} \beta^0 : (bb)$ , making the first integral vanish. Therefore there is no correction to  $N'^0 = i\tau$  coming from  $\partial_{V_0}\widehat{\mathcal{G}}_{ij}$ , as was already seen in (5.21). By repeating the analysis to  $\partial_{V^\lambda}\widehat{\mathcal{G}}_{ij}$  and  $\partial_{V_k}\widehat{\mathcal{G}}_{ij}$  one shows that only the latter can be non-vanishing, and analysing in the same fashion the corrections for the D5 and the D7 cases we obtain the same relations as (5.47) and (5.30).

One can realize then that brane deformations with normal direction  $\eta$  along one cycle of the 3-torus on the Type IIA side are mapped to Wilson line moduli along the T-dual cycle on the Type IIB side, while brane deformations along the base are mapped to brane deformations on the Type IIB side, also along the base. In the opposite direction, brane deformations on the Type IIB side along the 3-torus are mapped to Wilson line moduli along the dual cycle on the Type IIA side.

## 6 Conclusions

In this paper we have derived the four-dimensional  $\mathcal{N} = 1$  effective action of IIA and IIB Calabi-Yau orientifolds including single space-time filling  $Dp$ -branes by performing a Kaluza-Klein reduction. In particular, we derived the  $\mathcal{N} = 1$  characteristic data of the open-closed system for space-time filling D6-branes in an Type IIA Calabi-Yau orientifold. In the determination of the Kähler potential  $K$  we showed that the complex  $\mathcal{N} = 1$  open coordinates appear in  $K$  only through a redefinition of the closed coordinates.  $K$  itself can be viewed as a function of real three- and two-forms. In the presence of D6-branes these forms have localized corrections with the open coordinates. In addition to the kinetic terms of the scalars we have also determined the holomorphic gauge coupling function

of the brane and bulk  $U(1)$  gauge fields including possible mixed terms.

We have discussed the  $\mathcal{N} = 1$  characteristic data of the orientifold compactifications both for a finite as well as for the infinite dimensional case. In a fixed background Calabi-Yau geometry a D6-brane on a special Lagrangian cycle  $L_0$  has  $b^1(L_0)$ . The scalar potential vanishes for these deformations. Considering a general normal deformation of  $L_0$  the superpotential (4.14) and D-terms (4.24) are induced. These have been determined explicitly and were shown to be given in terms of chain integrals over a four-chain  $\mathcal{C}_4$  ending on the reference cycle  $L_0$  and the deformed cycle  $L_\eta$ . We also argued that the corrections to the closed string coordinates can be formulated as chain integrals. In particular, we introduced a Kähler potential  $K_o$  which depends on both open and closed deformations and encodes the corrections to the  $\mathcal{N} = 1$  closed coordinates.  $K_o$  as given in (4.8) contains two chain integrals involving both the Kähler form  $J$  as well as the holomorphic three-form  $\text{Im}C\Omega$ . When restricting to a finite dimensional deformation space  $K_o$  was shown to restrict to the Kähler potential introduced by Hitchin on the moduli space of special Lagrangian submanifolds with  $U(1)$  connection.

In the last part of the paper we related our Type IIA results to the  $\mathcal{N} = 1$  data for Type IIB orientifold compactifications with D3-, D5-, or D7-branes by using mirror symmetry. The SYZ proposal to view the internal manifold as  $T^3$  fibration, with possibly resolved singular fibers, allowed us the match of the  $\mathcal{N} = 1$  data for branes and orientifold planes of different dimensionalities with the D6/O6 set-up. The mirror map has been evaluated in special limits of the closed and open moduli space. It will be interesting to extend this analysis to the interior of the open-closed moduli space. The general chain integral expressions for the  $\mathcal{N} = 1$  coordinates, Kähler potential and gauge coupling function might allow to compute quantum corrections using geometric methods on one side of the mirror correspondence and applying the mirror map.

There are various further directions in which our results can be extended. It is well-known that D6-branes in Type IIA string theory are obtained from specific geometries, so-called Taub-NUT spaces, in an M-theory. More precisely, one expects that the D6/O6 compactifications considered in this work naturally lift to a compactification of M-theory on a  $G_2$  manifold. The  $\mathcal{N} = 1$  data found in this paper will naturally embed into the  $\mathcal{N} = 1$  data of non-singular  $G_2$  reduction found in [40, 41, 42]. One expects that similar issues as for D7/O7 compactifications embedded into F-theory arise [47]. Also for the D6/O6 compactifications it will be interesting to understand the origin of the flux independent Stückelberg gaugings as it was found in [48] for F-theory set-ups. Moreover, it will be interesting to generalize the set-up to intersecting D6-branes including the possibility of fundamental matter. This will modify the  $\mathcal{N} = 1$  data in both orientifold and M-theory compactifications [2, 23].

In flux compactifications the backreaction is often so strong that the compactification manifold cannot be a Calabi-Yau manifold. This implies that one has to compute the effective action by looking at variations around a new non-Calabi-Yau solution. These often can be described using

generalized geometry as discussed for  $\mathcal{N} = 1$  vacua, for example in [49, 13, 50], and the review [51]. It would be interesting to extend our results to such a generalized setting. Note that formally this is rather straightforward by replacing  $e^{Jc}$  and  $\text{Re}(C\Omega)$  by general pure spinors in all our expressions. However, it will be desirable to show if one still can explicitly compute the  $\mathcal{N} = 1$  data by finding non-trivial example threefolds which are described by generalized geometric methods and cannot be analyzed in either symplectic or complex geometry.

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## Appendices

### A Derivation of the Kähler metric

Let us now discuss the derivation of the Kähler metric and compare the result with the effective action for the D6-brane found by dimensional reduction, (2.75) and (2.77). Firstly, we note that the metrics for  $\text{Re}M^K$  and the pure  $\xi^K$  terms match the result found from the reduction of the closed string action, since  $\tilde{K}^{KL} = (G_{kl}, G^{\lambda\kappa}, G_k^\lambda)$ , as described in [11]. We need then to check the terms involving open string moduli  $\zeta^i$ . From the reduction of the action the metrics  $\mathcal{G}_{ij}$  and  $\hat{\mathcal{G}}_{ij}$  are

$$\hat{\mathcal{G}}_{ij} = \mu_{ki} \lambda_j^k, \quad \mathcal{G}_{ij} = \mu_{ik} (\lambda^{-1})_j^k, \quad (\text{A.1})$$

where, recalling equations (3.11) and (2.16),

$$e^{-\phi} \theta_i = \lambda_i^j \tilde{\alpha}_j, \quad \theta_i = s_i \lrcorner J|_{L_0},$$

$$\frac{1}{2} e^{-\phi} * \theta_i = \mu_{ji} \tilde{\beta}^j, \quad * \theta_i = -2e^\phi s_i \lrcorner \text{Im}(C\Omega)|_{L_0}.$$

The coefficients  $\mu_{ij}$  and  $\lambda_i^j$  are calculated to be

$$e^{2D}\mu_{ij} = \frac{1}{2} \int_L \tilde{\alpha}_i \wedge s_{j\lrcorner} (V^\kappa \alpha_\kappa + V_k \beta^k), \quad \lambda_i^j = \int_L \tilde{\beta}^j \wedge s_{i\lrcorner} J, \quad (\text{A.2})$$

also making use of the relations  $\int \tilde{\alpha}_i \wedge \tilde{\beta}^j = \delta_i^j$ . To leading order, the  $V$  derivatives of  $\mu_{ij}$  are

$$\frac{\partial}{\partial V^\lambda} (e^{2D}\mu_{ij}) = \frac{1}{2} \int_L \tilde{\alpha}_i \wedge s_{j\lrcorner} \alpha_\lambda, \quad \frac{\partial}{\partial V_k} (e^{2D}\mu_{ij}) = \frac{1}{2} \int_L \tilde{\alpha}_i \wedge s_{j\lrcorner} \beta^k, \quad (\text{A.3})$$

On the other hand,  $\lambda_i^j$  is independent of  $(V^\lambda, V_k)$ , at least for leading order complex structure deformations. This implies using (3.34), (3.21) and (A.1) that

$$\tilde{K}_{\zeta^i \bar{\zeta}^j} = \frac{\partial (e^{2D}\mathcal{G}_{ij})}{\partial V_K} V^K = e^{2D}\mathcal{G}_{ij}, \quad (\text{A.4})$$

which is in accord with the result (2.75) found from dimensional reduction. The derivatives of the metric with respect to  $(V^\lambda, V_k)$  are given explicitly by (for first order deformations)

$$\partial_{V^\lambda} (e^{2D}\mathcal{G}_{ij}) = \frac{1}{2} \int_L \tilde{\alpha}_i \wedge s_{l\lrcorner} \alpha_\lambda \left( \int_L \tilde{\beta}^j \wedge s_{l\lrcorner} J \right)^{-1}, \quad \partial_{V_k} (e^{2D}\mathcal{G}_{ij}) = \frac{1}{2} \int_L \tilde{\alpha}_i \wedge s_{l\lrcorner} \beta^k \left( \int_L \tilde{\beta}^j \wedge s_{l\lrcorner} J \right)^{-1}. \quad (\text{A.5})$$

The derivatives of the metric  $\hat{\mathcal{G}}_{ij}$  are, in turn,

$$\partial_{V^\lambda} (e^{2D}\hat{\mathcal{G}}_{ij}) = \frac{1}{2} \int_L \tilde{\alpha}_l \wedge s_{i\lrcorner} \alpha_\lambda \int_L \tilde{\beta}^l \wedge s_{j\lrcorner} J, \quad \partial_{V_k} (e^{2D}\hat{\mathcal{G}}_{ij}) = \frac{1}{2} \int_L \tilde{\alpha}_l \wedge s_{i\lrcorner} \beta^k \int_L \tilde{\beta}^l \wedge s_{j\lrcorner} J. \quad (\text{A.6})$$

To also check the mixing terms of the Wilson lines  $a^i$  with the scalars  $\zeta^K$  we expand

$$\frac{\partial K_o}{\partial \zeta^i} = \frac{1}{2} \mu_{ij} |_{\text{fix}} \eta^j + \dots, \quad (\text{A.7})$$

to lowest order in the  $\eta^i$ . This yields the lowest order expression for  $\tilde{K}_{\zeta^i}^K$  evaluated to be

$$\tilde{K}_{\zeta^i}^k = \hat{\mathcal{I}}_i^k, \quad \tilde{K}_{\zeta^i}^\lambda = \hat{\mathcal{I}}_{i\lambda}, \quad (\text{A.8})$$

where were used equations (2.61) and (2.62)

$$\hat{\mathcal{I}}_i^k = \int_L \tilde{\alpha}_i \wedge \eta_{\lrcorner} \beta^k + \dots, \quad \hat{\mathcal{I}}_{i\lambda} = \int_L \tilde{\alpha}_i \wedge \eta_{\lrcorner} \alpha_\lambda + \dots. \quad (\text{A.9})$$

## B Supergravity with several linear multiplets

In this appendix we want to show, in a step by step way, how does the dualization from linear to chiral multiplets work, following [11]. We want to relate the effective action in terms of linear multiplets  $(V_K, C_K^2)$ , obtained by generalizing a result in [53],

$$\begin{aligned} \mathcal{L} = & -\tilde{K}_{\zeta^i \bar{\zeta}^j} d\zeta^i \wedge *d\bar{\zeta}^j + \frac{1}{4} \tilde{K}_{V_K V_L} dV_K \wedge *dV_L \\ & + \tilde{K}_{V_K V_L} dC_K^2 \wedge *dC_L^2 - i dC_K^2 \wedge (\tilde{K}_{V_K \zeta^i} d\zeta^i - \tilde{K}_{V_K \bar{\zeta}^i} d\bar{\zeta}^i), \end{aligned} \quad (\text{B.1})$$

with the one with chiral multiplets, (3.35),

$$\begin{aligned} \mathcal{L}^{\text{kin}} &= -(\tilde{K}_{\zeta^i \bar{\zeta}^j} + \tilde{K}_{\zeta^i}^K \tilde{K}_{KL} \tilde{K}_{\bar{\zeta}^j}^L) d\zeta^i \wedge *d\bar{\zeta}^j \\ &\quad + \tilde{K}_{KL} (d\text{Re}M^I \wedge * \text{Re}M^J + d\xi^K \wedge *d\xi^J) - 2\tilde{K}_{KL} \tilde{K}_{\zeta^i}^L (d\text{Re}M^I \wedge *du^j + d\xi^I \wedge *da^j) . \end{aligned}$$

In (B.1)  $\tilde{K}(V, \zeta, \bar{\zeta})$  is a function of the scalars  $V_K$  and the chiral multiplets  $\zeta^i$ . The function  $\tilde{K}$  encodes the dynamics of the fields, and we would like to relate it to the Kähler potential from (3.35). The standard procedure is to eliminate the fields  $C_K^2$  in favor of its duals  $\xi^K$  by introducing an appropriate term to the action

$$\mathcal{L} \rightarrow \mathcal{L} + \delta\mathcal{L} , \quad \delta\mathcal{L} = -2\xi^K dC_K^3 = -2C_K^3 \wedge d\xi^K , \quad (\text{B.2})$$

where  $\xi^K(x)$  is a Lagrange multiplier. By solving the equations of motion for  $\xi^K$  one finds  $dC_K^3 = 0$  such that locally  $C_K^3 = dC_K^2$ , giving  $\delta\mathcal{L} = 0$  as expected. One can use the equations of motion of  $C_K^3$ ,

$$*C_K^3 = \tilde{K}^{V_K V_L} \left( d\xi^L + \frac{i}{2} (\tilde{K}_{V_L \zeta^i} d\zeta^i - \tilde{K}_{V_L \bar{\zeta}^i} d\bar{\zeta}^i) \right) \quad (\text{B.3})$$

to eliminate it from (B.1),

$$\begin{aligned} \mathcal{L} &= -\tilde{K}_{\zeta^i \bar{\zeta}^j} d\zeta^i \wedge *d\bar{\zeta}^j + \frac{1}{4} \tilde{K}_{V_K V_L} dV_K \wedge *dV_L \\ &\quad + \tilde{K}^{V_K V_L} \left( d\xi^L - \text{Im}(\tilde{K}_{V_L \zeta^j} d\zeta^j) \right) \wedge * \left( d\xi^L - \text{Im}(\tilde{K}_{V_L \bar{\zeta}^i} d\bar{\zeta}^i) \right) . \end{aligned} \quad (\text{B.4})$$

For our particular case, we can further simplify this equation. Comparing (2.56) with the Chern-Simons action (2.53), one can notice that the field  $C^2$  couples, to first order, with the imaginary part of  $\zeta^i$ , namely  $a^i$ . We can assume that  $\tilde{K}$  is a function only of  $V_L$  and the real part of  $\zeta^i$ ,  $\text{Re}\zeta^i = u^i$ . We will see shortly that this assumption agrees with our results (indications that  $\tilde{K}$  depends only on  $\text{Re}\zeta^i$  can be inferred from section 3, as in equation (3.19)). The effective Lagrangian (B.4) thus simplifies to

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} \tilde{K}_{u^i u^j} d\zeta^i \wedge *d\bar{\zeta}^j + \frac{1}{4} \tilde{K}_{V_K V_L} dV_K \wedge *dV_L \\ &\quad + \tilde{K}^{V_K V_L} \left( d\xi^K - \frac{1}{2} \tilde{K}_{V_K u^i} d\text{Im}\zeta^i \right) \wedge * \left( d\xi^L - \frac{1}{2} \tilde{K}_{V_L u^j} d\text{Im}\zeta^j \right) . \end{aligned} \quad (\text{B.5})$$

We would like to relate this  $N = 1$  Lagrangian to the standard Lagrangian of chiral multiplets  $\Phi = (M^I, \zeta^i)$

$$\begin{aligned} \mathcal{L} &= -K_{\Phi\bar{\Phi}} d\Phi \wedge *d\bar{\Phi} \\ &= -K_{\zeta^i \bar{\zeta}^j} d\zeta^i \wedge *d\bar{\zeta}^j - K_{M^I \bar{M}^J} (d\text{Re}M^I \wedge * \text{Re}M^J + d\xi^K \wedge *d\xi^J) \\ &\quad - 2K_{M^I \bar{\zeta}^j} (d\text{Re}M^I \wedge *du^j + d\xi^I \wedge *da^j) . \end{aligned} \quad (\text{B.6})$$

and relate the Kähler metrics  $K_{\Phi\bar{\Phi}}$  with derivatives of the function  $\tilde{K}$ , as in equation (3.35). This is obtained by performing a Legendre transform with respect to the fields  $M^K$ ,

$$K(M, \zeta) = \tilde{K}(V, \zeta + \bar{\zeta}) + (M^K + \bar{M}^K) V_K \quad (\text{B.7})$$

where  $V_K(\zeta, M)$  is written as a function of the complex fields  $\zeta^i$  and implicitly of new field  $M^K$ , defined as

$$M^K = -\frac{1}{2}\tilde{K}_{V_K} + i\xi^K. \quad (\text{B.8})$$

One can see  $(M^K + \bar{M}^K)$  as the conjugate coordinate to  $V_K$ . To see that equations (B.6) and (B.5) are indeed related by this Legendre transformation, one has to calculate the derivatives of  $K$  in terms of the derivatives of  $\tilde{K}$ . One starts by differentiating (B.8),

$$\begin{aligned} \frac{\partial V_K}{\partial M^L} &= -\tilde{K}^{V_K V_L}, \\ \frac{\partial V_K}{\partial \zeta^j} &= \frac{1}{2} \frac{\partial V_K}{\partial M^L} \frac{\partial M^L}{\partial u^j} = \frac{1}{2} \tilde{K}^{V_K V_L} \tilde{K}_{V_L u^j}. \end{aligned} \quad (\text{B.9})$$

Using these expressions one easily calculates the first derivatives of the Kähler potential (B.7) as

$$K_{M^K} = V_K, \quad K_{\zeta^i} = \frac{1}{2} \tilde{K}_{u^i}. \quad (\text{B.10})$$

Applying the equations (B.9) once more when differentiating (B.10) one finds the Kähler metrics

$$\begin{aligned} K_{M^K \bar{M}^L} &= -\tilde{K}^{V_K V_L}, \quad K_{M^K \bar{\zeta}^i} = \frac{1}{2} \tilde{K}^{V_K V_L} \tilde{K}_{V_L u^i}, \\ K_{\zeta^i \bar{\zeta}^j} &= \frac{1}{4} \tilde{K}_{u^i u^j} + \frac{1}{4} \tilde{K}_{u^i V_K} \tilde{K}^{V_K V_L} \tilde{K}_{V_L u^j}, \end{aligned} \quad (\text{B.11})$$

with inverses

$$\begin{aligned} K^{M^K \bar{M}^L} &= -\tilde{K}_{V_K V_L} + \tilde{K}_{u^i V_K} \tilde{K}^{u^i u^j} \tilde{K}_{V_L u^j}, \\ K^{M^K \bar{\zeta}^j} &= 2\tilde{K}^{u^i u^j} \tilde{K}_{u^i V_K}, \quad K^{\zeta^i \bar{\zeta}^j} = 4\tilde{K}^{u^i u^j}. \end{aligned} \quad (\text{B.12})$$

Finally, one checks that  $K(T, N)$  is indeed the Kähler potential for the Lagrangian (B.5). This is done by inserting in the definition of  $T_\kappa$  and the Kähler metrics obtained above into (B.6), yielding back (B.5).

## C Mixing of brane and bulk $U(1)$ vectors

In this Appendix we analyze the 4D effective action for all the massless spacetime vector fields that appear after dimensional reduction. They are the  $A^\alpha$  and  $A_\alpha$  components coming from the combination of RR and  $B_2$  bulk fields (2.53), and  $A$ , the massless vector component of the  $U(1)$  field  $A_{D6}$  on the brane, (2.38). The duality relation between  $C_3$  and  $C_5$  implies a electric-magnetic duality between  $A^\alpha$  and  $A_\alpha$ . To avoid the overcounting of degrees of freedom, we consider both fields, but each weighted by a factor of one half, as in [21]. This procedure gives the action

$$\begin{aligned} S_{\text{vec}}^{(4)} &= -\int \frac{1}{2} \text{Re} f_r F \wedge *F + \frac{1}{2} \text{Im} f_r F \wedge F \\ &+ \frac{1}{4} (\text{Im} \mathcal{N}_{\alpha\beta} + \text{Re} \mathcal{N}_{\alpha\gamma} \text{Im} \mathcal{N}^{\gamma\delta} \text{Re} \mathcal{N}_{\delta\beta}) dA^\alpha \wedge *dA^\beta \\ &+ \frac{1}{4} \text{Im} \mathcal{N}^{\alpha\beta} dA_\alpha \wedge *dA_\beta - \frac{1}{2} \text{Re} \mathcal{N}_{\alpha\gamma} \text{Im} \mathcal{N}^{\gamma\beta} dA_\beta \wedge *dA^\alpha - \Delta_\alpha dA^\alpha \wedge F - \tilde{\mathcal{J}}^\alpha dA_\alpha \wedge F, \end{aligned} \quad (\text{C.1})$$

where  $F = dA$ ,  $\Delta_\alpha = (a^j \Delta_{j\alpha} + \Gamma_\alpha)$  and

$$\text{Im}\mathcal{N}_{\alpha\beta} = - \int_Y \omega_\alpha \wedge * \omega_\beta \quad \text{Im}\mathcal{N}^{\alpha\beta} = (\text{Im}\mathcal{N}_{\alpha\beta})^{-1} = - \int_Y \tilde{\omega}^\alpha \wedge * \tilde{\omega}^\beta \quad \text{Re}\mathcal{N}_{\alpha\beta} = -b^a \mathcal{K}_{a\alpha\beta} . \quad (\text{C.2})$$

Recalling the duality relation (2.5) for the  $\mathcal{A}$  fields

$$e^B d\mathcal{A}|_6 = - *_{10} (e^B d\mathcal{A})|_4 , \quad (\text{C.3})$$

we obtain, for  $A^\alpha$  and  $A_\alpha$ ,

$$d(A_\alpha \tilde{\omega}^\alpha) + dA^\beta b^a \omega_a \wedge \omega_\beta = - * dA^\gamma *_{6} \omega_\gamma . \quad (\text{C.4})$$

We take the wedge product of the above expression with  $\omega_\alpha$  and integrate to obtain the duality relation

$$dA_\alpha = \text{Im}\mathcal{N}_{\alpha\beta} * dA^\beta + \text{Re}\mathcal{N}_{\alpha\beta} dA^\beta . \quad (\text{C.5})$$

From the variation of action (C.1), we obtain the equations of motion for  $A_\alpha$  and  $A^\alpha$ ,

$$\begin{aligned} \frac{1}{2}(\text{Im}\mathcal{N}_{\alpha\beta} + \text{Re}\mathcal{N}_{\alpha\gamma} \text{Im}\mathcal{N}^{\gamma\delta} \text{Re}\mathcal{N}_{\delta\beta}) d * dA^\beta - \frac{1}{2} \text{Re}\mathcal{N}_{\alpha\gamma} \text{Im}\mathcal{N}^{\gamma\beta} d * dA_\beta - \Delta_\alpha dF = 0 , \quad (\text{C.6}) \\ \frac{1}{2} \text{Im}\mathcal{N}^{\alpha\beta} d * dA_\alpha - \frac{1}{2} \text{Re}\mathcal{N}_{\alpha\gamma} \text{Im}\mathcal{N}^{\gamma\beta} d * dA^\alpha - \tilde{\mathcal{J}}^\beta dF = 0 . \end{aligned}$$

However, if one takes the exterior derivative of equation (C.5) and compare with (C.6), one notes that the equations are not compatible. That is, the equations of motion and the duality constraints cannot be simultaneously satisfied. In order to make the duality relation consistent, one should modify the field strengths as

$$dA^\alpha \rightarrow G^\alpha = dA^\alpha - 2\tilde{\mathcal{J}}^\alpha F , \quad dA_\alpha \rightarrow G_\alpha = dA_\alpha + 2\Delta_\alpha F , \quad (\text{C.7})$$

as well as the duality relation (C.5) by the same redefinition. This modified action becomes then

$$\begin{aligned} S_{\text{vec}}^{(4)} \rightarrow & - \int \frac{1}{4}(\text{Im}\mathcal{N}_{\alpha\beta} + \text{Re}\mathcal{N}_{\alpha\gamma} \text{Im}\mathcal{N}^{\gamma\delta} \text{Re}\mathcal{N}_{\delta\beta}) G^\alpha \wedge * G^\beta - \frac{1}{2} \text{Re}\mathcal{N}_{\alpha\gamma} \text{Im}\mathcal{N}^{\gamma\beta} G_\beta \wedge * G^\alpha \quad (\text{C.8}) \\ & + \frac{1}{4} \text{Im}\mathcal{N}^{\alpha\beta} G_\alpha \wedge * G_\beta + \frac{1}{2} \text{Re}f_{\text{r}} F \wedge * F + \frac{1}{2} \text{Im}f_{\text{r}} F \wedge F - \Delta_\alpha G^\alpha \wedge F - \tilde{\mathcal{J}}^\alpha G_\alpha \wedge F . \end{aligned}$$

The equations coming from this action are

$$\begin{aligned} dG^\alpha = -2\tilde{\mathcal{J}}^\alpha dF , \quad dG_\alpha = 2\Delta_\alpha dF , \quad G_\alpha = \text{Im}\mathcal{N}_{\alpha\beta} * G^\beta + \text{Re}\mathcal{N}_{\alpha\beta} G^\beta , \quad (\text{C.9}) \\ \frac{1}{2}(\text{Im}\mathcal{N}_{\alpha\beta} + \text{Re}\mathcal{N}_{\alpha\gamma} \text{Im}\mathcal{N}^{\gamma\delta} \text{Re}\mathcal{N}_{\delta\beta}) d * G^\beta - \frac{1}{2} \text{Re}\mathcal{N}_{\alpha\gamma} \text{Im}\mathcal{N}^{\gamma\beta} d * G_\beta - \Delta_\alpha dF = 0 , \\ \frac{1}{2} \text{Im}\mathcal{N}^{\alpha\beta} d * G_\alpha - \frac{1}{2} \text{Re}\mathcal{N}_{\alpha\gamma} \text{Im}\mathcal{N}^{\gamma\beta} d * G^\alpha - \tilde{\mathcal{J}}^\beta dF = 0 . \end{aligned}$$

The first two equations follow directly from (C.7), the third is the imposed duality condition, and the two remaining are the equations of motion for  $A^\alpha$  and  $A_\alpha$ . One can check that they are now consistent, by starting with the equation of motion for one of the fields and obtaining the equation for the dual field after imposing the duality conditions.

As was mentioned, the duality condition implies that the degrees of freedom for the fields are not independent. To eliminate the dependence of  $A_\alpha$  in favor of its dual, we now treat the field strength  $G_\alpha$  as an independent field, and add to the action the term

$$\delta S = -\frac{1}{2}dA^\alpha \wedge (G_\alpha - 2\Delta_\alpha F) + \lambda(dG_\alpha - 2\Delta_\alpha dF) , \quad (\text{C.10})$$

where  $\lambda$  is an auxiliary field acting as a Lagrange multiplier. The equations for this modified action are the same as (C.9), but now they all come from variations on the fields  $A^\alpha$ ,  $G_\alpha$  and  $\lambda$ . Having the equations for  $G_\alpha$ , we now substitute them back into the action, and obtain

$$\begin{aligned} S_{\text{vec}}^{(4)} &= -\int \frac{1}{2}\text{Re}f_{\text{r}} F \wedge *F + \frac{1}{2}\text{Im}f_{\text{r}} F \wedge F \\ &\quad + \frac{1}{2}dA^\alpha \wedge (\text{Im}\mathcal{N}_{\alpha\beta} *G^\beta + \text{Re}\mathcal{N}_{\alpha\beta}G^\beta - 2\Delta_\alpha F) \\ &\quad - \Delta_\alpha(dA^\alpha - 2\tilde{\mathcal{J}}^\alpha F) \wedge F - (\text{Im}\mathcal{N}_{\alpha\beta} *G^\beta + \text{Re}\mathcal{N}_{\alpha\beta}G^\beta)\tilde{\mathcal{J}}^\alpha \wedge F \\ &= -\int \frac{1}{2}(\text{Re}f_{\text{r}} + 4\text{Im}\mathcal{N}_{\alpha\beta}\tilde{\mathcal{J}}^\alpha\tilde{\mathcal{J}}^\beta) F \wedge *F + \frac{1}{2}(\text{Im}f_{\text{r}} + 4\Delta_\alpha\tilde{\mathcal{J}}^\alpha + 4\text{Re}\mathcal{N}_{\alpha\beta}\tilde{\mathcal{J}}^\beta\tilde{\mathcal{J}}^\alpha)F \wedge F \\ &\quad - 2\text{Im}\mathcal{N}_{\alpha\beta}\tilde{\mathcal{J}}^\beta dA^\alpha \wedge *F - 2(\Delta_\alpha + \tilde{\mathcal{J}}^\beta\text{Re}\mathcal{N}_{\alpha\beta})dA^\alpha \wedge F \\ &\quad + \frac{1}{2}\text{Im}\mathcal{N}_{\alpha\beta}dA^\alpha \wedge *dA^\beta + \frac{1}{2}\text{Re}\mathcal{N}_{\alpha\beta}dA^\alpha \wedge dA^\beta , \end{aligned} \quad (\text{C.11})$$

from where we can extract a corrected gauge coupling function  $f_{\text{cor}}$  for the brane U(1) gauge fields,

$$\text{Re}f_{\text{cor}} = \text{Re}f_{\text{r}} + 4\text{Im}\mathcal{N}_{\alpha\beta}\tilde{\mathcal{J}}^\alpha\tilde{\mathcal{J}}^\beta , \quad \text{Im}f_{\text{cor}} = \text{Im}f_{\text{r}} + 4\Delta_\alpha\tilde{\mathcal{J}}^\alpha + 4\text{Re}\mathcal{N}_{\alpha\beta}\tilde{\mathcal{J}}^\beta\tilde{\mathcal{J}}^\alpha , \quad (\text{C.12})$$

a gauge coupling function  $f_\alpha$  for the mixing between brane and bulk gauge bosons,

$$\text{Re}f_\alpha = -4\text{Im}\mathcal{N}_{\alpha\beta}\tilde{\mathcal{J}}^\beta , \quad \text{Im}f_\alpha = -4(\Delta_\alpha + \tilde{\mathcal{J}}^\beta\text{Re}\mathcal{N}_{\alpha\beta}) , \quad (\text{C.13})$$

and the gauge coupling function for the vector field  $A^\alpha$  from the bulk (3.38),

$$f_{\alpha\beta} = -i\tilde{\mathcal{N}}_{\alpha\beta} . \quad (\text{C.14})$$

## D Symmetries and moment maps

In this appendix we will have a closer look at the symmetries of the symplectic form (4.3). This is crucial for the determination of the first derivatives of the Kähler potential  $K_o$ .

Before entering the detailed study of our set-up, let us quickly recall some general facts about moment maps. Denote by  $G$  a Lie group preserving the symplectic form  $\varphi$  on a manifold  $\mathcal{V}_o$ , and by  $\mathfrak{g}$  the Lie algebra of  $G$ . There is a map identifying an element  $\xi \in \mathfrak{g}$  with a vector field  $X(\xi)$ , and by the invariance of  $\varphi$  under  $G$  and the fact that  $d\varphi = 0$  one has

$$\mathcal{L}_{X(\xi)}\varphi = d(X(\xi)\lrcorner\varphi) = 0 . \quad (\text{D.1})$$



The moment map is a function  $\mu : \mathcal{V}_0 \rightarrow \mathfrak{g}^*$ , where  $\mathfrak{g}^*$  is the dual to  $\mathfrak{g}$  under some pairing  $\langle \cdot, \cdot \rangle$ , which satisfies

$$d\langle \mu, \xi \rangle = X(\xi) \lrcorner \varphi . \quad (\text{D.2})$$

In our example there are two Lie group actions, which will help us to study  $K_o$ . As before we will be mostly interested in a local analysis around  $L_0$ . Therefore it suffices to specify the associated Lie algebras  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$

$$\mathfrak{g}_1 = \Omega_{\text{ex}}^2(L_0) , \quad \mathfrak{g}_2 = \Omega_{\text{ex}}^1(L_0) , \quad (\text{D.3})$$

where  $\Omega_{\text{ex}}^i(L_0)$  are the exact  $i$ -forms on  $L_0$ . In order to check that these indeed preserve the symplectic form  $\varphi$  given in (4.3) we have to specify the maps from  $\mathfrak{g}_i$  to tangent vectors  $T_{(L_0, A_0)}\mathcal{V}_o$ . So given an exact two-form  $\xi$  in  $\mathfrak{g}_1$  and an exact one-form  $\tilde{\xi}$  in  $\mathfrak{g}_2$  one defines a tangent vector  $\tau(\xi)$  and a normal vector  $\eta(\tilde{\xi})$  by demanding that

$$(\tau(\xi) \lrcorner \Omega_1)|_{L_0} = \xi , \quad (\eta(\tilde{\xi}) \lrcorner J)|_{L_0} = \tilde{\xi} . \quad (\text{D.4})$$

It turns out to be useful to identify also the tangent vectors  $\tau(\xi)$  with normal vectors to  $L_0$  using the complex structure  $I$  on  $Y$ . One first notes that the normal bundle to  $L_0$  admits the split

$$NL_0 = (NL_0)^{\text{harm}} \oplus (NL_0)^{\text{ex}} \oplus (NL_0)^{\text{cex}} . \quad (\text{D.5})$$

This split is performed in such a way that, e.g.  $X \in (NL_0)^{\text{harm}}$  yields a harmonic one-form  $(X \lrcorner J)|_{L_0}$ . Similarly, one defines  $(NL_0)^{\text{ex}}$  and  $(NL_0)^{\text{cex}}$  corresponding to exact and co-exact one-forms. By Hodge-decomposition of one-forms each normal vector has a unique decomposition under (D.5). Returning to the two Lie algebras, one has

$$\eta(\tilde{\xi}) \in (NL_0)^{\text{ex}} , \quad I\tau(\xi) \in (NL_0)^{\text{cex}} , \quad (\text{D.6})$$

for  $\tilde{\xi} \in \mathfrak{g}_2$  and  $\xi \in \mathfrak{g}_1$ . The latter follows from the fact that  $(I\tau(\xi) \lrcorner J)|_{L_0} = -2e^\phi * (\tau(\xi) \lrcorner \Omega_1)|_{L_0} = -2e^\phi * \xi$ , using (2.18) and (D.4). Since  $\xi$  is exact one concludes that  $(I\tau(\xi) \lrcorner J)|_{L_0}$  is co-exact, i.e. is annihilated by  $d^*$ .

Next we need to check the invariance (D.1) of the symplectic form  $\varphi$ . We do that by first noting that on a special Lagrangian  $L_0$  the form  $\varphi$  can be written as

$$\varphi(X, Y) = \int_{L_0} (Y \lrcorner J)|_{L_0} \wedge (X \lrcorner \Omega_1)|_{L_0} - (X \lrcorner J)|_{L_0} \wedge (Y \lrcorner \Omega_1)|_{L_0} , \quad (\text{D.7})$$

which is deduced by inserting  $X$  and  $Y$  into  $J \wedge \Omega_1 = 0$ . We have to check that  $d(\tau(\xi) \lrcorner \varphi) = 0$  for all  $\xi \in \mathfrak{g}_1$ , and similarly for the action of  $\mathfrak{g}_2$ . This is straightforward when using  $\varphi$  in the form (D.7). For  $\tau(\xi) \lrcorner \varphi$  only the term in the expression (D.7) containing  $(\tau(\xi) \lrcorner \Omega_1)|_{L_0}$  is non-zero since  $(\tau(\xi) \lrcorner J)|_{L_0}$  vanishes on the Lagrangian cycle  $L_0$ . Together with the fact that  $L_0$  is compact this yields the desired invariance under the action of  $\mathfrak{g}_1$ . Similarly one checks invariance under the

action of  $\mathfrak{g}_2$  using the term in (D.7) containing  $(\eta(\tilde{\xi}) \lrcorner J)|_{L_0}$  and the fact that  $(\eta(\tilde{\xi}) \lrcorner \Omega_1)|_{L_0} = 0$ . The invariance ensures the existence of two moment maps  $\mu_1$  and  $\mu_2$  for the respective Lie algebras.

In a next step we determine the dual Lie algebras  $\mathfrak{g}_1^*$  and  $\mathfrak{g}_2^*$  with respect to the pairing  $\langle \alpha, \beta \rangle = \int_{L_0} \alpha \wedge \beta$ . This implies that  $\mathfrak{g}_i^*$  is the space of non-closed  $i$ -forms, i.e.

$$\mathfrak{g}_1^* = \frac{\Omega^1(L_0)}{\Omega_{\text{cl}}^1(L_0)}, \quad \mathfrak{g}_2^* = \frac{\Omega^2(L_0)}{\Omega_{\text{cl}}^2(L_0)}, \quad (\text{D.8})$$

where  $\Omega_{\text{cl}}^i(L_0)$  are closed  $i$ -forms. Finally, one determines the moment maps  $\mu_i$  of  $\mathfrak{g}_i$  which obey (D.2) by direct calculation

$$\mu_1 = [\hat{\mu}_1], \quad \mu_2 = [\hat{\mu}_2], \quad (\text{D.9})$$

where  $\hat{\mu}_i$  are the non-closed  $i$ -forms which have been introduced in (2.22), and the brackets  $[\cdot]$  mean that these maps are only defined up to closed forms as required in (D.8). This is checked by evaluating

$$\begin{aligned} \frac{d}{dz^I} \int_{L_0} \hat{\mu}_1 \wedge \xi &= \int_{L_0} (s_I \lrcorner J)|_{L_0} \wedge \xi = \varphi(\tau(\xi), \partial_{z^I}), \\ \frac{d}{dz^I} \int_{L_0} \hat{\mu}_2 \wedge \tilde{\xi} &= \int_{L_0} (s_I \lrcorner \Omega_2)|_{L_0} \wedge \tilde{\xi} = \varphi(\eta(\xi), \partial_{z^I}). \end{aligned} \quad (\text{D.10})$$

For the first equalities we have evaluated the derivative in the direction  $\partial_{z^I}$  by taking the Lie derivative of the expression under the integral with respect to  $\partial_{z^I}$ . In evaluating the second equalities we used the form (D.7) of  $\varphi$  and the fact that  $(\partial_{z^I} \lrcorner \Omega_1)|_{L_0} = i(\partial_{z^I} \lrcorner \Omega_2)|_{L_0}$ . A rather compact way to rewrite the moment maps is by using the chain  $\mathcal{C}_4$  introduced in (2.50). Since  $J$  and  $\Omega_2$  vanish on  $L_0$  one thus has

$$\langle \mu_1, \xi \rangle = \mathcal{I}(J, \xi), \quad \langle \mu_2, \tilde{\xi} \rangle = \mathcal{I}(\Omega_2, \tilde{\xi}), \quad (\text{D.11})$$

where  $\mathcal{I}$  is the chain integral introduced in (2.59).

We have just found an explicit characterization of the symmetries of  $\mathcal{V}_o$  around  $L_0$ . In the following want to make contact to the parameterization used in section 2 for the Kaluza-Klein modes of the D6-brane. Let us recall that using a Hodge-decomposition with respect to the induced metric the mode expansion for  $A_{\text{D6}}$  reads

$$A_{\text{D6}} = a^i \tilde{\alpha}_i + a_{\text{ex}}^I dh_I + a_{\text{cex}}^J d^* \gamma_J. \quad (\text{D.12})$$

Note that the Hodge-star metric orthogonally splits under this decomposition as in (2.43). We also want to split the normal vectors  $s_I$  appearing in (2.36). As in the decomposition (D.5) one picks a basis of normal vectors  $s_i, s_I^{\text{ex}}, s_J^{\text{cex}}$  and corresponding one-forms  $\theta_I = (s_I \lrcorner J)|_{L_0}$  such that  $\theta_i = \tilde{\alpha}_i$ ,  $\theta_I^{\text{ex}} = dh_I$  and  $\theta_J^{\text{cex}} = d^* \gamma_J$ . This implies that the metrics (2.47) and (2.48) are identical in this basis. Moreover, from (4.5) we inferred that

$$\partial_{z^I} \partial_{\bar{z}^J} K_o = \frac{1}{2} e^{-\phi} \int_{L_0} \theta_I \wedge * \theta_J. \quad (\text{D.13})$$

which can be adapted to the Hodge-decomposition of the one-forms  $\theta_I$  under an appropriate split of complex coordinates  $z^I = (z^i, z_{\text{ex}}^I, z_{\text{cex}}^J)$ . In this leading order analysis we thus find that  $\text{Im}z^i = a^i$ ,  $\text{Im}z_{\text{ex}}^I = a_{\text{ex}}^I$  and  $\text{Im}z_{\text{cex}}^J = a_{\text{cex}}^J$ .

One can proceed and determine the first derivatives of the Kähler potential in the coordinates  $z^I$  by using the moment map analysis of the previous section. In fact, one infers from (D.10) that

$$\partial_{z_{\text{ex}}^I} K_o = \int_{L_0} \mu_2 \wedge \theta_I^{\text{ex}}, \quad \partial_{z_{\text{cex}}^J} K_o = \int_{L_0} *\mu_1 \wedge \theta_J^{\text{cex}}. \quad (\text{D.14})$$

It is straightforward to evaluate  $\partial_{z_{\text{ex}}^I} K_o$  and  $\partial_{z_{\text{cex}}^J} K_o$  at leading order in the deformations using (2.24). They take the simple form

$$\partial_{z^I} K_o = \int_{L_0} \theta_\eta \wedge *\theta_I. \quad (\text{D.15})$$

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