The $\mathcal{N}=1$ effective action of F-theory compactifications

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ABSTRACT

The four-dimensional $\mathcal{N}=1$ effective action of F-theory compactified on a Calabi-Yau fourfold is studied by lifting a three-dimensional M-theory compactification. The lift is performed by using T-duality realized via a Legendre transform on the level of the effective action, and the application of vector-scalar duality in three dimensions. The leading order Kähler potential and gauge-kinetic coupling functions are determined. In these compactifications two sources of gauge theories are present. Space-time filling non-Abelian seven-branes arise at the singularities of the elliptic fibration of the fourfold. Their couplings are included by resolving the singular fourfold. Generically a $U(1)^r$ gauge theory arises from the R-R bulk sector if the base of the elliptically fibered Calabi-Yau fourfold supports $2r$ harmonic three-forms. The gauge coupling functions depend holomorphically on the complex structure moduli of the fourfold, comprising closed and open string degrees of freedom. The four-dimensional electro-magnetic duality is studied in the three-dimensional effective theory obtained after M-theory compactification. A discussion of matter couplings transforming in the adjoint of the seven-brane gauge group is included.

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Contents

1 Introduction

In connecting string theory with effectively four-dimensional observable physics one breaks the high amount of symmetry present in the ten-dimensional formulation on a compactification background. Demanding the existence of an effective four-dimensional $\mathcal{N} = 1$ supergravity theory improves the stability of the compactifications while still leading to interesting phenomenological scenarios. However, even if one is able to guarantee the presence of the observed particle spectrum, the symmetry breaking induces a massless moduli sector which would be in conflict with experiment. There has been vast progress to establish scenarios to stabilize these scalar fields [1, 2, 3]. This is particularly important due to the fact, that these fields determine the value of the four-dimensional couplings and scales. Global consistency conditions restrict valid scenarios, but are not believed to single out specific effective theories, or select a preferred vacuum. Thus, it is a crucial task to determine the realized $\mathcal{N} = 1$ effective theories with a realistic observable sector and evaluate their generic features.

In the last years there has been much progress in the study of Type II string compactifications with D-branes. These set-ups allow to localize non-Abelian gauge theories in the internal geometry, which arise on space-time filling D-branes or via light D-brane states on vanishing cycles in a singular geometry. Charged matter fields can arise on intersections of these branes. This allows for the possibility to construct set-ups resembling the four-dimensional particle spectrum and couplings of the MSSM [4, 2]. However, intersecting D-brane models do not naturally support GUT theories, since the required couplings are only generated at the non-perturbative level [5, 6]. Moreover, despite much progress, the implementation of moduli stabilization for D-brane deformations poses additional complications. Both of these issues are naturally addressed in F-theory compactifications which provides a geometrization of general seven-brane configurations.

In this work we will focus on Type IIB compactifications in which the gauge symmetries arises from space-time filling seven-branes or from vector zero modes of the R-R four-form. Type IIB compactifications with general seven-brane sources and varying dilaton-axion τ are known as F-theory compactifications [7]. F-theory provides a geometrization of the seven-branes by considering backgrounds which admit two extra dimensions confined on an auxiliary two-torus with complex structure parameter τ. The aim of this work is to study the four-dimensional effective supergravity actions arising in such F-theory compactifications. Demanding $\mathcal{N} = 1$ supersymmetry implies that F-theory has to be compactified on Calabi-Yau fourfolds X_4. The existence of the auxiliary two-torus then translates to the fact that X_4 has to be an elliptic fibration over some base B_3. The singularities of this fibration determine the four-cycles in B_3 wrapped by the seven-brane of Type IIB string theory on B_3.

In general F-theory compactifications the elliptic fibration X_4 can admit singularities associated with exceptional groups. This fact permits the existence of four-dimensional models with induced couplings following the selection rules of the representation theory of exceptional groups. This is crucial for many minimal constructions of GUT models with unified gauge-groups $SU(5)$ or $SO(10)$. This fact has revived a recent interest in the construction of realistic GUT models in F-theory starting with [8, 9, 10].[2] However, in the aim to connect these models with a moduli stabilizing sector the coupling to gravity is essential. Compact F-theory GUT models have been recently studied in [19, 20, 21, 22, 23], while F-theory up-lifts of orientifold models have been constructed in [24, 25]. In particular, the explicitly resolved Calabi-Yau fourfolds with a single non-Abelian gauge group of [20, 21, 22] can be viewed as simple examples for which the effective action computed in this work can be evaluated. A landscape of semi-realistic GUT models can thus be studied in an effective theory with a GUT sector coupled to a moduli stabilization sector.

Aiming to derive such a four-dimensional effective theory one immediately faces the problem that F-theory is not a fundamental theory with a twelve-dimensional weak coupling formulation. The detour which one has to take is to consider first a compactification of M-theory on the Calabi-Yau fourfold X_4 to three space-time dimensions [7]. Using the fact that X_4 is an elliptic fibration, one then identifies a limit which shrinks the fiber torus and grows the fourth non-compact dimension [3]. The basic idea is to fiberwise apply the duality between M-theory on T^2 and Type IIB string theory on S^1 after applying T-duality [26, 27]. Sending the volume of the T^2 to zero corresponds to the F-theory limit in which the S^1 becomes large. Since this will be our way to extract the four-dimensional effective action it will be crucial to formulate this limit very explicitly. M-theory vacua on Calabi-Yau fourfolds have been analyzed in refs. [28, 29].[3] The effective three-dimensional action of M-theory on a general, non-singular Calabi-Yau fourfold at large volume has been studied in ref. [30].[4] We simplify the computations of [30] and provide the tools to implement the F-theory limit.

To keep control of the M-theory reduction in the F-theory limit all Euclidean M5-branes wrapped on holomorphic six-cycles in X_4 have to be of large action. We argue that this requirement can be fulfilled even in the limit of a small elliptic fiber, and can be traced back to the fact that the appropriately identified vanishing T^2-volume R is

[2] In particular, phenomenological issues concerning the flavor structure have been addressed in local F-theory models. An incomplete list of recent refs. is [11, 12, 13, 14, 15, 16, 17, 18].

[3] In this paper we will neglect the effects of warping by working at very large compactification volume.

[4] See also refs. [31, 32], and references therein, for discussions on the F-theory action in compactifications with more supersymmetry.

connected with the action of a corresponding Euclidean M5-brane on B_3 via a Legendre transform. The F-theory limit can be obtained by following Euclidean M5-branes in M-theory which map to finite action Euclidean D3-branes in F-theory [33]. Furthermore, it will be crucial to use the existence of a well-defined four-dimensional theory arising after decompactification.

The F-theory limit will be extended to set-ups with non-Abelian gauge symmetry on seven-brane in which case a more subtle scaling of the fields has to be applied. The four-dimensional gauge field degrees of freedom in the Coulomb branch arise in the three-dimensional action as vector modes of the M-theory three-form paired with blow-up modes of the singular fibration of X_4. In three-dimensional theories with four supercharges vectors pair with real scalars into multiplets. We will use this fact and show that the four-dimensional Kähler potential and gauge-kinetic coupling function can be encoded by a single three-dimensional kinetic potential $\mathbf{K}^{\mathrm{M}}$. Expanding $\mathbf{K}^{\mathrm{M}}$ for small fiber volumes R of the elliptic Calabi-Yau fourfold, one can directly read of the four-dimensional F-theory Kähler potential and gauge-coupling functions. In this way we show that for a non-Abelian gauge group the gauge-coupling function is to leading order given by the volume of the wrapped seven-branes and explore the structure of further corrections. We show that matter transforming in the adjoint of the seven-brane gauge group can be coupled in the M-theory reduction. In the F-theory lift this matter corresponds to deformations of the seven-branes as well as Wilson line degrees of freedom. The D-terms together with the flux induced contribution are computed using the four-dimensional Kähler metric along the gauged directions. Also the flux induced superpotential together with a direct coupling of the adjoint deformations and Wilson lines on the seven-branes are discussed.

A second source of four-dimensional gauge symmetry arises from the scalars appearing as coefficients of an expansion of the M-theory three-form into a basis of non-trivial three-forms on B_3. In three dimensions, one uses the complex structure of X_4 to combine these scalars into complex fields which span a complex torus bundle $\mathbb{T}$ over the moduli space of complex structure deformations of X_4. In three dimensions massless vectors and scalars are dual, and we show that these scalars indeed lift to four-dimensional vectors in the F-theory limit. The holomorphic gauge coupling functions dependents on the complex structure moduli of the fourfold X_4, and encodes the geometry of the torus bundle $\mathbb{T}$. Four-dimensional electro-magnetic duality can be studied in the three-dimensional theory and allows to constrain the form of the gauge-couplings and non-perturbative superpotentials.

The paper is organized as follows. In section 2 we systematically introduce the lift of three-dimensional M-theory compactifications to four-dimensional F-theory compactifications. The Kähler moduli sector is discussed from a Type IIB and an M-theory perspective, and we introduce the usage of Legendre transforms in the effective theories. In section 3 we discuss non-Abelian seven-branes, and show how their couplings can be studied in an M-theory reduction. It will be crucial to resolve the singularities of the elliptic fibration and later define an appropriate scaling limit in the F-theory lift. The Abelian gauge theories arising from the R-R four-form of Type IIB string theory is introduced in section 4 using the M- to F-theory lift. We discuss the action of electro-magnetic duality on the three-dimensional variables and comment on the properties of the gauge coupling function and the three-dimensional superpotential. Finally, in section 5, matter transforming in the adjoint of the seven-brane gauge group is included. The corrections to the Kähler potential are determined in the F-theory lift. This allows to study the D-terms and to comment on the $\mathcal{N} = 1$ superpotential of the effective theory.

For the convienience of the reader, the main equations and results of this work are summarized in appendix A.

2 Systematics of F-theory compactifications

In this section we summarize the general strategy which we use to study F-theory compactifications. A first look at Calabi-Yau fourfolds with seven-branes in subsection 2.1 allows us to summarize the uncharged four-dimensional spectrum of a general F-theory compactification in subsection 2.2. In subsection 2.3 simple aspects of the Kähler moduli space are discussed from a Type IIB perspective. Our main tool in the determination of the effective action will be to understand F-theory compactifications as a limit in the M-theory Kähler moduli space in which a new non-compact dimension grows. This lift from a three- to a four-dimensional compactification is introduced in subsection 2.4, and will be extend in the following sections. Finally, in subsection 2.5 we summarize some basics about complex structure deformations on smooth Calabi-Yau fourfolds.

2.1 Type IIB string compactifications with seven-branes

Recall that ten-dimensional Type IIB string theory is believed to admit a non-perturbative $Sl(2, \mathbb{Z})$ symmetry. This group acts non-trivially on the dilaton-axion $\tau = C_0 + ie^{-\phi}$, where C_0 is the R-R axion and $e^{\langle\phi\rangle} = g_s$ is the string coupling. Since eight-dimensional branes couple to τ this implies that in addition to the well-known D7-branes also more general seven-branes obtained by an $Sl(2, \mathbb{Z})$ transformation can be included in a consistent Type IIB compactification. In this work we will focus on four-dimensional Type IIB string theory compactified on a complex three-dimensional manifold B_3. The seven-branes are wrapped on four-cycles, i.e. divisors, in B_3. In consistent solutions the tension of the seven-branes is locally canceled by the positive curvature of a Kähler base manifold B_3. All 7-branes are sources for τ and hence are identified by the behavior of the dilaton-axion τ profile on the compactification background. Close to the seven-branes τ can vary significantly and one is not generically at weak string coupling. The weak coupling limit is only approached when moving the seven-branes together to form O7-planes with the remaining branes being D7-branes [34]. In this limit the cancellation of seven-brane tension with the curvature of B_3 translates to the standard cancellation condition of the D7-brane tadpole.

Type IIB compactifications with general seven-brane sources and varying complex dilaton-axion τ are known as F-theory vacua [7]. F-theory provides a geometrization of the seven-branes by considering backgrounds $\mathbb{M}_{3,1} \times X_4$ which admit two auxiliary extra dimensions. Such extended solutions are constructed by attaching at each point of the original ten-dimensional Type IIB target space $\mathbb{M}_{3,1} \times B_3$ an auxiliary two-torus with complex structure parameter τ. The profile of τ translates to the non-trivial torus fibration structure of the fourfold X_4 when moving along B_3. The supersymmetry conditions and equations of motion for τ enforce this fibration to be an elliptic fibration, with τ varying holomorphically in the complex coordinates $\underline{u}$ of B_3. Note that in order for the four-dimensional effective theory to be $\mathcal{N} = 1$ supersymmetric the fourfold X_4 has to be a Calabi-Yau manifold.

Examples of such fourfolds can be represented by a complex polynomial constraints in a projective or toric ambient space. In particular, one can consider X_4 encoded by the Weierstrass equation

$$P_W = x^3 - y^2 + f(\underline{u}) x z^4 + g(\underline{u}) z^6 = 0 \ , \tag{2.1}$$

as well as a number of additional constraints $P_i(\underline{u}) = 0$. The coordinates $(x, y, z, \underline{u})$ are in general restricted by a number of scaling relations. In particular, for (y, x, z) one has the scaling relation $(y, x, z) \cong (\lambda^3 y, \lambda^2 x, \lambda z)$. Note that for f and g constant (2.1) indeed defines a a two-torus given by a degree 6 hypersurface in weighted projective space $\mathbb{P}_{3,2,1}$. This two-torus can degenerate over divisors in B_3. These degeneration loci precisely locate the seven-branes on the base, and are determined by the discriminate

$$\Delta = 27\, g^2 + 4\, f^3 \ . \tag{2.2}$$

The dilaton-axion profile $\tau(\underline{u})$ is then specified by the value of the classical $SL(2, \mathbb{Z})$ modular invariant j-function $j(\tau) = 4(24\, f)^3/\Delta$.

In general Δ can factorize into several components corresponding to different intersecting seven-branes. The singularities of the elliptic fibration over these seven-brane divisors in B_3 determine the gauge groups on the seven-branes. In the rest of the paper we will restrict to configurations with a single stack of seven-branes on a surface $\mathcal{S}$ leading to non-Abelian gauge group G. More precisely, we will restrict to examples where the class of Δ splits as

$$[\Delta] = rk(G)[\mathcal{S}] + [\Delta'] \ , \tag{2.3}$$

where $rk(G)$ is the number of seven-branes wrapped on $\mathcal{S}$. Such a non-trivial factorization might be imposed by tuning the complex structure of a smooth X_4 to appropriately degenerate the elliptic fibration over $\mathcal{S}$ to obtain the non-Abelian gauge symmetry. A number of examples of this type have been constructed in refs. [20, 21], with the aim to build compact $SU(5)$ GUT models.

It is important to note that in case one has a non-Abelian gauge group on $\mathcal{S}$ the degeneration of the elliptic fibration is so severe that the Calabi-Yau fourfold itself becomes singular. Let us denote this singular space by X_4^{sing}. In this case it is not possible to work with the singular space X_4^{sing} directly since the topological quantities such as the Euler characteristic and intersection numbers are not well-defined. In many cases, however, the singularities can systematically be blown up to obtain a smooth geometry $\hat{X}_4$ [35, 36, 37, 20, 21]. We will discuss this blow-up process in more detail in section 3.1. In summary, a possible way to construct examples is

$$X_4 \quad \xrightarrow{\text{fix complex str.}} \quad X_4^{\text{sing}} \quad \xrightarrow{\text{Kähler blow-up}} \quad \hat{X}_4 \ . \tag{2.4}$$

It is important to stress that many singular elliptic Calabi-Yau fourfolds X_4^{sing} might not admit a corresponding smooth X_4 obtained by complex structure deformation. In principle, this does not mean that F-theory on such spaces is not defined. Examples which do not admit a X_4 have a minimal gauge-group and have been studied intensively (see e.g. refs. [35, 36, 37], for Calabi-Yau threefold examples). For the discussion in this work, it will be necessary for X_4^{sing} to at least admit a resolution $\hat{X}_4$ as described in section 3.1.

Using such a set-up, we can determine the spectrum of the four-dimensional effective theory. The precise number of zero modes will be determined by the topological data of the Calabi-Yau fourfold and the surface $\mathcal{S}$ together with the non-trivial gauge-field configuration on $\mathcal{S}$. To summarize this spectrum will be the task of the next subsection.

2.2 The four-dimensional spectrum

In order to study F-theory compactifications, it is crucial to identify the fields which appear as the light degrees of freedom in the four-dimensional effective theory. In general, this is a hard task since the precise number does not only depend on the topological data of the Calabi-Yau fourfold $\hat{X}_4$, but also will require a knowledge of the flux background used in the reduction. To get a clue on the spectrum we take the standard strategy to first consider the case where all background fluxes are switched off. However, this will particularly be problematic when determining the chiral spectrum from the seven-branes, since four-dimensional chirality is only induced by a non-trivial flux background on the branes. We will need to return to this issue in section 5.

To summarize the spectrum in the absence of fluxes, we first summarize a few facts on the dimension of the cohomology groups of Calabi-Yau fourfolds $\hat{X}_4$. Let us denote by $h^{p,q}(B_3)$ and $h^{p,q}(\hat{X}_4)$ the Hodge numbers of the base B_3 and the resolved Calabi-Yau fourfold $\hat{X}_4$ respectively. Note that a Calabi-Yau fourfold has three independent non-trivial Hodge numbers $h^{1,1}(\hat{X}_4)$, $h^{2,1}(\hat{X}_4)$, and $h^{3,1}(\hat{X}_4)$. The remaining non-vanishing Hodge numbers are given by

$$\hat{X}_4: \quad h^{4,0} = h^{0,0} = h^{4,4} = 1\ , \qquad h^{2,2} = 2(22 + 2h^{1,1} + 2h^{3,1} - h^{2,1})\ . \tag{2.5}$$

For the basis B_3 one only finds two non-trivial Hodge numbers $h^{1,1}(B_3)$ and $h^{2,1}(B_3)$. Note that the fact that $h^{i,0}(\hat{X}_4) = 0$, $i = 1,2,3$ implies that also $h^{i,0}(B_3) = 0$.

In the absence of background flux the number of four-dimensional chiral multiplets with scalar components being a complex scalar can be given in terms of the Hodge numbers of X_4, $\hat{X}_4$ and B_3. The number of chiral multiplets is given by

$$n_{\rm c} = h^{3,1}(X_4) + h^{1,1}(B_3) + (h^{2,1}(\hat{X}_4) - h^{2,1}(B_3))\ . \tag{2.6}$$

To see this, one has to enter the precise prescription of the dimensional reduction. The first contribution $h^{3,1}(X_4)$ is readily seen to arise from the deformations of the complex structure of X_4.[5] These fields include the deformation moduli of the seven-branes and we will discuss this sector in subsection 2.5. The second contribution $h^{1,1}(B_3)$ arises from the zero modes of the Kähler form J of $\hat{X}_4$ expanded in harmonic two-forms on B_3. These are complexified by scalars arising from the R-R four-form C_4 of Type IIB string theory. The third contribution $h^{2,1}(\hat{X}_4) - h^{2,1}(B_3)$ is harder to identify in the Type IIB context, since it involves the full fourfold $\hat{X}_4$. We will come back to the explanation in the later parts of this work. In the next sections we will stepwise consider more and more general compactifications with fourfolds $\hat{X}_4$ with non-trivial $h^{2,1}(\hat{X}_4), h^{2,1}(B_3)$ and identify the corresponding fields and their effective couplings.

Let us also display the equation for the number of vector multiplets in the four-dimensional spectrum. As we will discuss in more detail below, the number of possible

[5] If a smooth X_4 does not exist, one can analyze $h^{3,1}(\hat{X}_4)$ complex structure deformations which preserve the singularities of $X_4^{\rm sing}$. Additional deformations can exist locally and are included as two-form variations on $\mathcal{S}$ as discussed in section 5.

$U(1)$ vector multiplets is given by

$$\tilde{n}_{\mathrm{v}} = (h^{1,1}(\hat{X}_4) - h^{1,1}(B_3) - 1) + h^{2,1}(B_3) \ . \tag{2.7}$$

Here n_{v} includes the number of $U(1)$'s in the non-Abelian gauge group G over S, as we recall in section 3. However, in the actual F-theory compactification the Coulomb branch of the seven-brane gauge theory is not accessible and these $U(1)$'s enhance to the full non-Abelian gauge connection of G. Hence, the actual number of $U(1)$ vector multiplets in a four-dimensional F-theory compactification is

$$n_{U(1)} = \tilde{n}_{\mathrm{v}} - \mathrm{rank}(G) \ , \tag{2.8}$$

with $\tilde{n}_{\mathrm{v}}$ as given in (2.7). Finally, it is straightforward to identify the $h^{2,1}(B_3)$ $U(1)$'s in (2.7) which arise by expanding the R-R four-form into $h^{2,1}(B_3)$ harmonic three-forms.

2.3 Remarks on the Type IIB dimensional reduction

The aim of this work is to study the four-dimensional effective action of a Type IIB compactification with seven-branes as sketched in sections 2.1 and 2.2. More precisely, we aim to determine the $\mathcal{N}=1$ characteristic data in the general four-dimensional supergravity action [38]

$$S^{(4)} = -\int \tfrac{1}{2}R_4{*}1 + K_{I\bar{J}}\mathcal{D}M^I\wedge{*}\mathcal{D}\bar{M}^{\bar{J}} + \tfrac{1}{2}\mathrm{Re}f_{\Lambda\Sigma}\, F^\Lambda\wedge{*}F^\Sigma + \tfrac{1}{2}\mathrm{Im}f_{\Lambda\Sigma}\, F^\Lambda\wedge F^\Sigma + V{*}1 \ , \tag{2.9}$$

where the scalar potential is given by

$$V = e^K\big(K^{I\bar{J}}D_IWD_{\bar{J}}\bar{W} - 3|W|^2\big) + \tfrac{1}{2}\,(\mathrm{Re}\ f)^{-1\ \Lambda\Sigma}D_\Lambda D_\Sigma \ . \tag{2.10}$$

The complex fields M^I are the bosonic fields of chiral multiplets, and might be gauged by vectors A^Λ in the derivative $\mathcal{D}M^I$. Such gaugings will lead to the appearance of D-terms D_Λ in V (2.10). Note that $K_{I\bar{J}}$ and $K^{I\bar{J}}$ are the Kähler metric and its inverse, where locally one has $K_{I\bar{J}} = \partial_I\bar{\partial}_{\bar{J}}K(M,\bar{M})$ with $\partial_I = \partial/\partial M^I$. The scalar potential is expressed in terms of the Kähler-covariant derivative $D_IW = \partial_IW + (\partial_IK)W$.

In order to study the effective four-dimensional dynamics of an F-theory compactification one first might attempt to start with Type IIB supergravity and perform a dimensional reduction on B_3. However, one immediately encounters the problem that the fields

$$\tau = C_0 + ie^{-\phi} \ , \qquad G_2 = C_2 - \tau B_2 \ , \tag{2.11}$$

cannot be used in a Kaluza-Klein expansion, since they vary non-trivially over the threefold B_3. This is due to the fact that these fields transform under the $Sl(2,\mathbb{Z})$ symmetry of Type IIB string theory as

$$\tau \;\to\; \frac{a\tau+b}{c\tau+d}\;, \qquad G_2 \;\to\; \frac{G_2}{c\tau+d}\;, \tag{2.12}$$

where the matrix $(a,b|c,d)$ is an element of $Sl(2,\mathbb{Z})$. There will in general be effectively invariant modes on the fourfold X_4 and we will show in the following sections that one can systematically perform a Kaluza-Klein reduction of F-theory via M-theory.

Type IIB string theory, however, also admits D3-branes which couples to the R-R four-form C_4. Neglecting the induced lower-dimensional brane charges on their world-volume, these D3-branes are invariant under the $Sl(2,\mathbb{Z})$ transformations. Using such D3-brane probes one thus is able to study at least part of the four-dimensional effective action. For example, let us consider an Euclidean D3-brane wrapped on some divisor $D^{\rm b}_\alpha$ of B_3. At large volume of B_3 this brane will have a classical instanton action

$$T^{\rm b}_\alpha = \frac{1}{2\ell_s^4}\int_{D^{\rm b}_\alpha} J_{\rm b}\wedge J_{\rm b} + i\int_{D^{\rm b}_\alpha} C_4\;, \tag{2.13}$$

where $J_{\rm b}$ is the Kähler form on B_3 in the ten-dimensional Einstein frame in units ℓ_s^2. To render $T^{\rm b}_\alpha$ dimensionless one has to multiply the first term by ℓ_s^{-4}. Clearly, this expression is only valid up to corrections in the fields (2.11). A key observation is, that the volumes of four-cycles together with the R-R four-form C_4 appear as complex fields in the four-dimensional $\mathcal{N}=1$ effective theory [39, 40]. The volumes $v^\alpha_{\rm b}$ of two-cycles in the usual expansion $J_{\rm b} = v^\alpha_{\rm b}\omega_\alpha$, with a basis ω_α of $H^2(B_3,\mathbb{Z})$, are in fact scalars in dual linear multiplets $(v^\alpha, C^{(2)}_\alpha)$, where the $C^{(2)}_\alpha$ are two-forms (see ref. [40] for a detailed discussion). This general fact implies that $T^{\rm b}_\alpha$ can be obtained by a Legendre transformation from a *kinetic potential* $\tilde{K}^{\rm F}$ as

$$T^{\rm b}_\alpha = \partial_{L^\alpha_{\rm b}}\tilde{K}^{\rm F} + i\int_{D^{\rm b}_\alpha} C_4\;, \qquad L^\alpha_{\rm b} = \frac{v^\alpha_{\rm b}}{\mathcal{V}_{\rm b}}\;, \tag{2.14}$$

where $\mathcal{V}_{\rm b}$ is the quantum volume of B_3. In general, the kinetic potential $\tilde{K}^{\rm F}$ depends on L^α and the other complex fields M^I of the effective theory. It is related to the Kähler potential $K^{\rm F}$ via

$$K^{\rm F}(T,\bar{T}|M,\bar{M}) = \tilde{K}^{\rm F} - \tfrac{1}{2}(T^{\rm b}_\alpha + \bar{T}^{\rm b}_\alpha)L^\alpha_{\rm b}\;, \tag{2.15}$$

where the right-hand side has to be evaluated as a function of $T^{\rm b}_\alpha$ by solving (2.14) for $L^\alpha_{\rm b}$. Clearly, to reproduce the simple form (2.13) with $L^\alpha_{\rm b}$ as given in (2.14), one has to take

$$\tilde{K}^{\rm F} = -2\log\mathcal{V}_{\rm b} = \log\big(\tfrac{1}{3!}L^\alpha_{\rm b}L^\beta_{\rm b}L^\gamma_{\rm b}\mathcal{K}_{\alpha\beta\gamma}\big)\;, \qquad \mathcal{V}_{\rm b} = \frac{1}{3!}\int_{B_3} J_{\rm b}\wedge J_{\rm b}\wedge J_{\rm b} \tag{2.16}$$

where $\mathcal{K}_{\alpha\beta\gamma}$ is the triple intersection of three divisors $D_\alpha, D_\beta, D_\gamma$. Note that the simple forms (2.13), (2.16) of $T^{\rm b}_\alpha$, $K^{\rm F}$ are obviously not complete. This can already be inferred by comparing these expressions with the ones obtained in the orientifold picture [39, 40]. There various known corrections to both $T^{\rm b}_\alpha$ and $K^{\rm F}$, which do, however, depend on the dilaton-axion τ. It will be crucial to understand how these corrections are captured in an F-theory compactification. One of the aims of this paper is to compute part of these corrections to the Kähler potential $K^{\rm F}$ (or $\tilde{K}^{\rm F}$), and $T^{\rm b}_\alpha$ including the other light fields arising as complex structure deformations, seven-brane moduli, matter fields, and fields from the combinations of the form (2.11). In order to deal with varying dilaton-axion τ it thus will be necessary to analyze the higher-dimensional geometry $\mathbb{M}_{3,1} \times X_4$ which turn out to be tractable in an M-theory framework.

2.4 From M-theory to F-theory

In this subsection we describe four-dimensional F-theory vacua as special limits of M-theory compactifications. To begin with, let us recall the basic steps to link M-theory and Type IIB string theory [3]. Consider M-theory compactified on a two-torus T^2, naming one of the one-cycles, the A-cycle, and the other one-cycle, the B-cycle. The metric background is thus of the form

$$ds^2_{11} = \frac{v^0}{\operatorname{Im}\tau}\big((dx + \operatorname{Re}\tau dy)^2 + (\operatorname{Im}\tau)^2 dy^2\big) + ds^2_9 \ , \tag{2.17}$$

where τ is the complex structure modulus of the T^2, and v^0 describes its volume. If the volume v^0 of the two-torus is small, one can pick one of the one-cycles, say the A-cycle, to obtain type IIA string theory. T-duality along the B-cycle leads to the corresponding Type IIB set-up, and identifies $\tau = C_0 + ie^{-\phi}$. One can then decompactify the T-dualized B-cycle to grow an extra non-compact direction.

This construction can be applied fiber-wise for the elliptically fibered Calabi-Yau fourfold X_4. Hence, one considers M-theory on X_4, which leads to a *three-dimensional* theory with $\mathcal{N} = 2$ supersymmetry. The reduction and T-duality on the elliptic fiber leads to Type IIB string theory on $B_3 \times S^1$, where B_3 is the base. A fourth non-compact dimension is grown by decompactifying the S^1. Due to the T-duality operation this limit corresponds to sending the T^2 volume $v^0 \to 0$. One thus finds the duality

$$\begin{array}{rcl} \text{M-theory on } X_4 & \rightarrow & \text{Type IIB on } B_3 \times S^1 \text{ with varying } \tau \\ \text{M-theory on } X_4 \text{ with } v^0 \to 0 & \rightarrow & \text{F-theory on } X_4 \end{array} \tag{2.18}$$

Let us note that this duality can only be performed in such a simple way, at points of the fibration were the two-torus does not degenerate. If singularities appear, one has to carefully identify the corresponding M-theory and F-theory non-perturbative constituents. In particular, singularities of the elliptic fibration in M-theory lead to Kaluza-

Klein monopoles descending to Type IIA six-branes, while they yield seven-branes in the Type IIB set-up.

From the M-theory perspective it is not surprising that the theory can be trusted for a varying τ, since this parameter simply encodes the complex structure of an actual two-torus in the eleven-dimensional space. The expansion parameters kept small in the dimensional reduction will turn out to be the inverse volumes of the six-cycles in the Calabi-Yau fourfold X_4. Despite the fact that a complete formulation of M-theory is not known, one can attempt to include corrections which are known either via duality or in specific limits. For example, this indirect approach is used in the study of compactifications on singular fourfolds $X_4^{\rm sing}$, where M2 branes on vanishing cycles are believed to complete enhancements to non-Abelian gauge groups. However, also the limit from M-theory to F-theory is in general subtle. Note that a supersymmetric M-theory compactifications demands that one works with a Ricci flat metric on the Calabi-Yau fourfold. These metric properties are inherited by Type IIB on B_3, and it is non-trivial that an extension to the fiber directions exists. Fortunately, we will not need to make use of an explicit metric on X_4 or B_3. Nevertheless even on the level of cohomology and deformations a non-trivial mixing of open and closed string degrees of freedom will arise.

To make some first steps in analyzing the effective action, let us consider M-theory on a non-singular Calabi-Yau fourfold X_4 with an elliptic fibration. Since X_4 is smooth there will be no non-Abelian gauge symmetries in three dimensions. On such a fibration there is a natural set of divisors which span $H_6(X_4,\mathbb{R})$. Firstly, one has the section of the fibration which is homologous to the base B_3. Secondly, there is the set of *vertical divisors* D_α which are obtained as $D_\alpha = \pi^{-1}(D^{\rm b}_\alpha)$, where $D^{\rm b}_\alpha$ is a divisor of B_3 and π is the projection to the base $\pi : X_4 \rightarrow B_3$. For these smooth elliptic fibrations one has $h^{1,1}(B_3) = h^{1,1}(X_4) - 1$ such divisors. One can now attempt to use a probe M5-brane to analyze Kähler coordinates. At large volume the naive action for an M5-brane on such a $D_{\mathcal{A}} = (D_0, D_\alpha)$ then reduces as

$$T_{\mathcal{A}} = \frac{1}{6}\int_{D_{\mathcal{A}}} J\wedge J\wedge J + i\int_{D_{\mathcal{A}}} C_6 \ , \qquad (2.19)$$

where v^0 is the volume of the elliptic fiber. In order to make $T_{\mathcal{A}}$ dimensionless one would need to multiply the first term in (2.19) with ℓ_M^{-6}. We will suppress units in most of the equations below. To determine the Kähler potential $K^{\rm M}$ for the fields T_0, T_α one analyses the Weyl rescaling to the three-dimensional Einstein frame. In a large volume compactification, only the classical volume $\mathcal{V}$ arises as pre-factor of the Einstein-Hilbert term. Comparing this with the $e^{K^{\rm M}}$ pre-factor in the scalar potential, on infers [30]

$$K^{\rm M} = -3\log\mathcal{V} \ , \qquad \mathcal{V} = \frac{1}{4!}\int_{X_4} J\wedge J\wedge J\wedge J \ . \qquad (2.20)$$

To evaluate K^{M} as a function of $T + \bar{T}$ one first expands the Kähler form J as

$$J = v^0 \omega_0 + v^\alpha \omega_\alpha \ . \tag{2.21}$$

where ω_0, ω_α are the two-forms Poincaré dual to B_3, D_α. Using this expansion on has to solve (2.19) for the modes v^0, v^α of J and insert the result into (2.20). This evaluation is more conveniently performed in a dual picture, which we explain in more generality next.

In general, one notes that the v^α, v^0 appear as elements of vector multiplets (v^α, A^α) and (v^0, A^0) with the vectors arising in the expansion of the M-theory three-form C_3 as

$$C_3 = A^0 \wedge \omega_0 + A^\alpha \wedge \omega_\alpha \ . \tag{2.22}$$

Hence, again one expects the $T_{\mathcal{A}} = (T_0, T_\alpha)$ to be given by

$$T_{\mathcal{A}} = \partial_{L^{\mathcal{A}}} \tilde{K}^{\mathrm{M}} + i\rho_{\mathcal{A}} \ , \tag{2.23}$$

where $\rho_{\mathcal{A}}$ generalize the imaginary parts in (2.19), and $L^{\mathcal{A}} = (R, L^\alpha)$ are defined as

$$R = \frac{v^0}{\mathcal{V}} \ , \qquad L^\alpha = \frac{v^\alpha}{\mathcal{V}} \ . \tag{2.24}$$

This is the analog of (2.14), but now for some M-theory kinetic potential $\tilde{K}^{\mathrm{M}}$, and $\mathcal{V}$ being the quantum volume of X_4. Again, the Kähler potential K^{M} is related to the kinetic potential via the Legendre transform

$$K^{\mathrm{M}}(T, \bar{T}|M, \bar{M}) = \tilde{K}^{\mathrm{M}} - \tfrac{1}{2}(T_{\mathcal{A}} + \bar{T}_{\mathcal{A}})L^{\mathcal{A}} \ , \tag{2.25}$$

where M^I are other complex scalars in the three-dimensional theory. It is straightforward to evaluate $\tilde{K}^{\mathrm{M}}$ at large volume to obtain the simple expression (2.19) for $T_{\mathcal{A}} = (T_0, T_\alpha)$. One finds

$$\tilde{K}^{\mathrm{M}} = \log(R) + \log\left(\tfrac{1}{3!}L^\alpha L^\beta L^\gamma \mathcal{K}_{\alpha\beta\gamma} + \tfrac{1}{2} R L^\alpha L^\beta \mathcal{K}_{\alpha\beta} + \tfrac{1}{2} R^2 L^\alpha \mathcal{K}_\alpha + R^3 \mathcal{K}\right) \ . \tag{2.26}$$

The intersection numbers we introduced are

$$\mathcal{K}_{\alpha\beta\gamma} = B_3 \cdot D_\alpha \cdot D_\beta \cdot D_\gamma = \int_{B_3} \omega_\alpha \wedge \omega_\beta \wedge \omega_\gamma \ , \tag{2.27}$$

and similarly for $\mathcal{K}_{\alpha\beta} = \mathcal{K}_{00\alpha\beta}$ and the remaining terms. Note that for an elliptic fibration the intersection numbers satisfy

$$\mathcal{K}_{\alpha\beta\gamma\delta} = D_\alpha \cdot D_\beta \cdot D_\gamma \cdot D_\delta = 0 \ , \tag{2.28}$$

for the vertical divisors. This allows us to split of the factor $\log(R)$ in (2.26). Furthermore, it implies that for an elliptic fibration one has

$$T_0 = \frac{1}{R} + p(R) + i\rho_0 \ , \qquad T_\alpha = \frac{v^0}{2}\int_{D^{\rm b}_\alpha} J\wedge J + i\rho_\alpha \ , \tag{2.29}$$

where $p(R)$ is a power-series in R regular in the limit $R\to 0$. Observe that $\mathrm{Re}T_0$ starts precisely with an inverse of R, with R being proportional to the volume of the elliptic fiber v^0 as introduced in (2.17) and (2.21). This will be the key to study the F-theory limit.

Let us now turn to the discussion of the F-theory limit. This limit will identify the M-theory compactification on X_4 to three dimensions with a four-dimensional F-theory compactification. Let us consider Type IIB string theory on $S^1\times B_3$. Before taking the limit there is one distinguished dimension which corresponds to one of the torus directions in the elliptic fiber of X_4. Labeling this fourth dimension by x^3 the Type IIB metric is of the form

$$ds^2_{\rm IIB} = r^{-2} g^3_{\mu\nu} dx^\mu dx^\nu + r^2 (dx^3 + A^0_\mu dx^\mu)^2 + ds^2_{B_3} \ , \qquad \mu,\nu = 0,1,2 \ , \tag{2.30}$$

where r is the radius of the fourth dimension, $g^3_{\mu\nu}$ is the three-dimensional Einstein frame metric, and A^0_μ is a three-dimensional vector. As recalled above, the F-theory limit is obtained by performing the reduction and T-duality, and sending $r\to\infty$ to decompactify the fourth dimension. Note that A_0 in (2.30) is identified in the M-theory to F-theory lift with the vector A_0 in the multiplet (R, A_0) introduced in (2.22) and (2.24). Also A_0 in the dimensional reduction with metric (2.30) is in a vector multiplet (r^{-2}, A_0). Hence, one identified the radius r in (2.30) with R in (2.24) as

$$R = r^{-2} \ . \tag{2.31}$$

One thus realizes that the shrinking of the elliptic fiber $R\to 0$ corresponds to growing an extra dimension. Furthermore, due to the Legendre transform from R to T_0 one sees that $R\to 0$ corresponds to $\mathrm{Re}T_0\to\infty$. This pushes the analysis into a regime, where Euclidean M5-branes wrapped on the base B_3 become very massive and do not correct the $\mathcal{N}=1$ data [33].

The F-theory lift can be studied further by realizing that M5-branes on vertical divisors will turn into D3-branes wrapped on the four-cycles $D^{\rm b}_\alpha$ in B_3 with finite action [33]. Hence, in the F-theory limit, one indeed identifies the $T^{\rm b}_\alpha$ introduced in section 2.3 and T_α of section 2.4, given e.g. in (2.13) and (2.19). Equivalently, one can use the fact that $L^{\mathcal{A}}$ in the M-theory reduction are elements of vector multiplets. Since, $T^{\rm b}_\alpha$ introduced in (2.23) remains finite in the F-theory limit, also L^α defined in (2.24) has to remain finite.

Hence, we conclude that the F-theory limit is more accurately given by

$$R = \ell_M^6 \cdot \frac{v^0}{\mathcal{V}} \quad \to \quad 0 \ , \qquad\qquad L^\alpha = \ell_M^6 \cdot \frac{v^\alpha}{\mathcal{V}} \quad \to \quad L^\alpha_{\rm b} = \ell_s^4 \cdot \frac{v^\alpha_{\rm b}}{\mathcal{V}_{\rm b}} \quad \text{finite} \ , \tag{2.32}$$

or, equivalently, by

$$\mathrm{Re}T_0 \quad \to \quad \infty \ , \qquad\qquad\qquad T_\alpha \quad \to \quad T^{\rm b}_\alpha \quad \text{finite} \ , \tag{2.33}$$

where we have restored the ℓ_s and ℓ_M dependence to elucidate the limit. Let us stress that (2.32) implies a non-trivial scaling of the v^α. Taking the volume $\mathcal{V}$ to be quartic in v^0, v^α one finds for $v^0 \propto \epsilon$ that $v^\alpha \propto 1/\sqrt{\epsilon}$ and $\mathcal{V} \propto 1/\sqrt{\epsilon}$ in the limit $\epsilon \to 0$. Using (2.32) for a quartic $\mathcal{V}$ the elliptic fiber volume R scales in the F-theory limit as $R \cong \mathcal{V}_{\rm b}^2/\mathcal{V}^3$. The claim is that the limit and identification (2.32) also holds if one includes further corrections and other moduli. In this case, however, one has to replace $\mathcal{V}$ with the appropriate quantum volume, as we will discuss below.

The key objects we will study in the following sections are the two three-dimensional kinetic potentials $\tilde{K}^{\rm M}$ and $\mathbf{K}^{\rm M}$ and their four-dimensional lifts. The latter potential $\mathbf{K}^{\rm M}$ is obtained from $\tilde{K}^{\rm M}$ via a Legendre transform of *only* the vector multiplets (L^α, A^α), since these vectors lift to four-dimensional chiral multiplets $T^{\rm b}_\alpha$. $\mathbf{K}^{\rm M}$ is given by

$$\mathbf{K}^{\rm M} = \tilde{K}^{\rm M} - \tfrac{1}{2}(T_\alpha + \bar{T}_\alpha) L^\alpha \ , \tag{2.34}$$

where L^α is replaced by its Legendre transform $\mathrm{Re}T_\alpha = \partial_{L^\alpha} \tilde{K}^{\rm M}$. This will be discussed in more detail in section 3.3. Even in the presence of vector multiplets and further complex scalars, we will argue that the full M-theory kinetic potential $\mathbf{K}^{\rm M}$ admits in the limit (2.32) the expansion

$$\mathbf{K}^{\rm M} = \log R + K^{\rm F} - \frac{1}{R} g + \mathcal{O}(R) \ , \tag{2.35}$$

where $K^{\rm F}$ is the Kähler potential of the four-dimensional F-theory compactification and the real function g will encode the dynamics of four-dimensional vector fields. The expression (2.35) is readily checked when inserting the kinetic potential $\tilde{K}^{\rm M}$ given in (2.26) into (2.34), together with $K^{\rm F}$ given in (2.15), (2.16). Clearly, in this simple case one has $g = 0$.

Let us remark that in principle one should explicitly evaluate the M-theory Kähler potential around the F-theory limit (2.32), including corrections to the large volume expressions. To some extend this is indeed possible by using mirror symmetry for Calabi-Yau fourfolds. The basic strategy is to construct the mirror X_4' to X_4 and rewrite the Kähler potential $K^{\rm M} = -3 \log \mathcal{V}$ using the mirror periods of the mirror $(4,0)$ form Ω' using the techniques described in subsection 2.5. In other words, one needs to compute

$$K^{\rm M} = -3 \log \int_{X_4'} \Omega' \wedge \bar{\Omega}' \ . \tag{2.36}$$

However, it is important to stress, that this potential has to be restricted to a *real submanifold* of dimension $h^{1,1}(X_4)$ in the mirror complex structure moduli space. This is analog to the description of Type IIA orientifolds [41]. Moreover, K^{M} will need to be evaluated as a function of the coordinates $T_{\mathcal{A}}$, with real parts given by the real parts of certain Ω' periods. Giving a precise formulation of this mirror identification is beyond the scope of this work. However, let us note that certain corrections have already been computed in [42]. In particular, it was shown that the Calabi-Yau fourfold volume $\mathcal{V}$ of X_4, appearing in (2.20), is corrected by the terms

$$\Delta\mathcal{V} = \frac{5\zeta(4)}{2^4(2\pi i)^6}\int_{X_4} c_2(X_4)^2 + k_1\int_{X_4} J\wedge c_3(X_4) + k_2\int_{X_4} J^2\wedge c_2(X_4) + \dots\ , \tag{2.37}$$

where c_2, c_3 is the second and third Chern class of TX_4, and k_i are some numerical constants. In particular, there is no correction proportional to the Euler number $\chi(X_4)$.[6] This seems to be crucial when taking the F-theory limit in the corrected $\mathcal{N}=1$ coordinates $T_{\mathcal{A}}$.

2.5 Complex structure deformations

In this section we recall some basic facts about the complex structure deformations of a non-singular Calabi-Yau fourfold $\hat{X}_4$ mainly following [43, 44, 45]. Around a fixed background complex structure these arise as metric deformations with purely holomorphic and anti-holomorphic indices

$$\delta g_{\bar{\imath}\bar{\jmath}} = -\frac{1}{3||\Omega||^2}\bar{\Omega}_{\bar{\imath}}{}^{klm}(\chi_{\mathcal{K}})_{klm\bar{\jmath}}\,\delta z^{\mathcal{K}}\ , \tag{2.38}$$

where Ω is holomorphic $(4,0)$-form on $\hat{X}_4$. Hence, the complex structure deformations are counted by the basis $\chi_{\mathcal{K}}, \mathcal{K}=1,\dots,h^{3,1}(\hat{X}_4)$ of $H^{3,1}(\hat{X}_4)$. As for Calabi-Yau threefolds the infinitesimal deformations $\delta z^{\mathcal{K}}$ are unobstructed in the absence of background fluxes and can be extended to a complex $h^{3,1}(\hat{X}_4)$-dimensional moduli space $\mathcal{M}^{\mathrm{cs}}$. The metric on this moduli space is given by

$$G_{\mathcal{K}\bar{\mathcal{L}}} = \partial_{z^{\mathcal{K}}}\partial_{\bar{z}^{\mathcal{L}}}K^{\mathrm{cs}} = \frac{\int_{\hat{X}_4}\chi_{\mathcal{K}}\wedge\bar{\chi}_{\mathcal{L}}}{\int_{\hat{X}_4}\Omega\wedge\bar{\Omega}}\ , \qquad K^{\mathrm{cs}} = -\log\int_{\hat{X}_4}\Omega\wedge\bar{\Omega}\ , \tag{2.39}$$

where we also recalled that $G_{\mathcal{K}\bar{\mathcal{L}}}$ is Kähler and is thus locally given by the derivative of the Kähler potential K^{cs}.

The Kähler potential K^{cs} for the complex structure deformations $z^{\mathcal{K}}$ can be expressed through the periods of Ω as we discuss momentarily. It is important to stress that in

[6] This is in contrast to the case of Calabi-Yau threefolds, where the Euler number of the threefold corrects the threefold volume.

the fourfold case the variations of the $(4,0)$ form Ω with respect to the complex structure deformations do not span the full cohomology $H^4(\hat{X}_4)$, but rather only a subspace $H^4_H(\hat{X}_4)$, known as the primary horizontal subspace of $H^4(\hat{X}_4)$ [43]. It takes the form

$$H^4_H(\hat{X}_4, \mathbb{C}) = H^{4,0} \oplus H^{3,1} \oplus H^{2,2}_H \oplus H^{1,3} \oplus H^{0,4} \ , \tag{2.40}$$

where $H^{2,2}_H$ consists of the elements in $H^{2,2}$ which can be obtained as second variation of Ω with respect to the complex structure on X_4.[7] In the fourfold case, however, one has to introduce a special basis $\gamma^{(i)}_a$ of $H^H_4(\hat{X}_4, \mathbb{C})$ which inherits the integrality properties of a mirror dual basis of $\oplus_q H^{q,q}(\hat{X}'_4, \mathbb{Z})$ [43], where $\hat{X}'_4$ is the mirror of $\hat{X}_4$. This allows to define the periods

$$\begin{aligned} \Pi^{(i)\, a_i} &= \int_{\gamma^{(i)}_{a_i}} \Omega \ , \qquad i = 0, \ldots, 4 \ , \qquad a_i = 1, \ldots, h^{4-i,i}_H(\hat{X}_4) \ , && (2.41) \\ \Pi &\equiv (\Pi^{(0)}, \Pi^{(1)a}, \Pi^{(2)\alpha}, \Pi^{(3)a}, \Pi^{(4)}) \equiv (X^0, X^a, \mathcal{G}^\alpha, \mathcal{F}^a, \mathcal{F}^0) \ , && (2.42) \end{aligned}$$

where $h^{4-i,i}_H(\hat{X}_4)$ denote the dimensions of the respective cohomologies in (2.40). The basis $\gamma^{(i)}_{a_i}$ can be chosen to admit the intersections

$$\Sigma \equiv \left(\gamma^{(i)}_{a_i} \cap \gamma^{(j)}_{b_j}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \eta & 0 \\ 0 & 0 & Q & 0 & 0 \\ 0 & \eta^T & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.43}$$

which only yields for $j = 4 - i$ the non-zero intersection matrices $\eta_{ab}, Q_{\alpha\beta}$. The group preserving Σ will be denoted by G_Σ:

$$N \in G_\Sigma : \quad N^T \Sigma N = \Sigma \ . \tag{2.44}$$

Inserting (2.41) and (2.43) into (2.39) the Kähler potential $K^{\rm cs}$ can be expressed in terms of the periods Π using

$$\int_{\hat{X}_4} \Omega \wedge \bar{\Omega} = \Pi^T \Sigma \bar{\Pi} = X^0 \bar{\mathcal{F}}^0 + \eta_{ab} X^a \bar{\mathcal{F}}^b + Q_{\alpha\beta} \mathcal{G}^\alpha \bar{\mathcal{G}}^\beta + c.c \ . \tag{2.45}$$

Using the (p,q)-structure of forms obtained as derivative of Ω one derives a number of vanishing conditions which translate into non-trivial conditions on the periods Π.[8] However, in contrast to the Calabi-Yau threefold case there does not exist a prepotential which determines the periods.

[7]The fact that not all $H^4(\hat{X}_4)$ can be reached as variation of Ω is in contrast to the Calabi-Yau threefold case. In the threefold case one can simply define the periods of the holomorphic $(3,0)$-form by introducing an integral homology basis of $H_3(Y_3, \mathbb{Z})$.

[8]See, e.g., refs. [44, 46] for a summary of these conditions.

In principle the periods Π can be computed explicitly as a function of the complex structure deformations $z^{\mathcal{K}}$ by methods discussed, for example, in refs. [43, 44, 45]. More precisely, given specific constraints (2.1) which determine X_4 as a hypersurface or complete intersection in a toric or projective ambient space, one can compute a set of differential equations, the Picard-Fuchs equations, which admit a linear combination of the Π as solution. The precise linear combination of the solutions to Picard-Fuchs equations at a given point in moduli space $\mathcal{M}^{\text{cs}}$ can be fixed by analytic continuation and the analysis of monodromies around special loci in $\mathcal{M}^{\text{cs}}$ (see [47] for the original work on Calabi-Yau threefolds, and [42] for extensions to Calabi-Yau fourfolds). While technically rather involved this gives, at least for Calabi-Yau fourfolds with few complex structure moduli, a prescription to compute K^{cs} explicitly at various points in the moduli space for a given $\hat{X}_4$.

In the computation of the periods Π it is crucial to have a detailed understanding of the *global structure* of the moduli space $\mathcal{M}^{\text{cs}}$. As already mentioned this structure is largely captured by the monodromies around the special loci, such as the fourfold conifold [42][9], the large complex structure point, and the orbifold locus. More precisely one has to determine the monodromy matrices M and their generated group G^{sym}

$$\Pi \ \rightarrow \ M\Pi, \quad M \in G^{\text{sym}} \subset G_{\Sigma} \ , \tag{2.46}$$

when encircling the special loci of $\mathcal{M}^{\text{cs}}$. G^{sym} is typically only defined via the specifications of its generators M, and encodes the global symmetries of $\mathcal{M}^{\text{cs}}$.

Let us end this section by noting that the complex structure deformations of $\hat{X}_4$ can be obstructed when allowing for non-trivial background fluxes [1, 2, 3]. More precisely, in an M-theory reduction on $\hat{X}_4$, the complex structure moduli of $\hat{X}_4$ are obstructed by a non-trivial flux background G_4 appearing in the Gukov-Vafa-Witten superpotential [49]

$$W = \int_{\hat{X}_4} \Omega \wedge G_4 \ . \tag{2.47}$$

This superpotential can be computed explicitly by evaluating the periods Π in an integral basis [42, 50]. However, this is only the correct W for a compactification on a non-singular $\hat{X}_4$ of M-theory. In the F-theory limit the superpotential (2.47) will be further refined due to the appearance of non-Abelian gauge symmetries at singularities of X_4^{sing}.

[9] The deformed fourfold conifold is also known as Stenzel space [48].

3 Non-Abelian seven-branes in F-theory compactifications

In this section we systematically include non-Abelian gauge groups into the discussion of the four-dimensional F-theory effective action. Recall that in compactifications with multiple seven-branes on a divisor $\mathcal{S}$ in B_3 the gauge-theory on their world-volume will be a non-Abelian group G. In the following we will concentrate on simply laced gauge groups which reside in the ADE series and can be obtained by singularities of the elliptic fibration of a Calabi-Yau fourfold X_4^{sing}. Hence, we will concentrate on groups $SU(N), SO(N)$ and the exceptional groups E_6, E_7, E_8. More precisely, we consider a stack of seven-branes on the divisor $\mathcal{S}$ on the base. We denote by $F = dA + A \wedge A$ the eight-dimensional field-strength on their world-volume $\mathcal{W} = \mathbb{M}_{3,1} \times \mathcal{S}$. The gauge field transforms in the adjoint of the group G, and an overall $U(1)$-factor might split off, as familiar for the case $U(N) = SU(N) \times U(1)$. Such $U(1)$ factors often play a special role, and can be included in the analysis of the effective action as described in [51]. In order to determine the effective four-dimensional theory one splits F into contributions with two four-dimensional indices, mixed indices and two indices on $\mathcal{S}$:

$$F = F_4 + F_w + F_{\text{flux}} \ . \tag{3.1}$$

The modes of the first set F_4 correspond to four-dimensional gauge fields and will be discussed in more detail in subsections 3.2 and 3.3. The second set F_w in (3.1) are defined to be one-forms on $\mathcal{S}$ and one-forms in $\mathbb{M}_{3,1}$ and hence capture the Wilson line degrees of freedom as discussed in subsection 5.3. The last set F_{flux} captures non-trivial flux configurations on the seven-branes as briefly discussed in section 5. Note that in an F-theory compactification the underlying group theory is actually encoded geometrically, due to the presence of the singularities of the fibration over $\mathcal{S}$. In subsection 3.1 we discuss that this remains to be the case after the resolution of these singularities.

3.1 Singularity resolutions for seven-brane gauge theories

Recall that the elliptic fibration will be singular over the discriminant Δ given in (2.2). If the discriminant Δ factorizes the components will correspond to different seven-branes. As already noted in (2.3) we will restrict to the case that Δ has two components $[\Delta] = rk(G)[\mathcal{S}] + [\Delta']$. Here $rk(G)$ is the rank non-Abelian gauge group on the $rk(G)$ seven-branes wrapped on $\mathcal{S}$. The gauge groups over the divisors $\mathcal{S}$ can be determined explicitly using generalizations of the Tate formalism [36]. Let us split the basis of vertical divisors D_α, $\alpha = 1, \dots h^{1,1}(B_3)$ introduced in section 2 as

$$D_\alpha = (S, D'_\beta) \ . \tag{3.2}$$

This split will be convenient in the following, since S plays a distinguished role in the analysis of the gauge symmetries.

In case of non-Abelian gauge groups the elliptic fibration and the Calabi-Yau fourfold $X_4^{\rm sing}$ itself becomes singular as in (2.4). The singularities can systematically be blown up to obtain a smooth geometry [35, 36, 37, 20, 21]. In the following we will consider cases where $X_4^{\rm sing}$ admits a split simultaneous resolution

$$\pi: \quad \hat{X}_4 \ \rightarrow \ X_4^{\rm sing} \ , \tag{3.3}$$

where π is the blow-down map from the smooth fourfold $\hat{X}_4$ to $X_4^{\rm sing}$. The existence of a split simultaneous resolution implies that one can identify rank(G) irreducible divisors $\hat{D}_i$ describing the resolution of the ADE singularity over $\mathcal{S}$. Each of these divisors is a $\mathbb{P}^1$ bundle over $\mathcal{S}$, and the $\hat{D}_i$ intersect at generic points in $\mathcal{S}$ as the Dynkin diagram of G in the elliptic fiber of $\hat{X}_4$. Note that it is important to shift

$$S \ \rightarrow \ S' = S + \sum_i a^i \hat{D}_j \ , \tag{3.4}$$

where a^i are the Dynkin numbers associated to the Dynkin node $\hat{D}_i$. In the following we will use the basis

$$D_\alpha = (S', D'_\beta) \ , \tag{3.5}$$

which, by a slight abuse of notation, replace the D_α introduced in (3.2). The redefinitions (3.4) and (3.5) are performed to ensure that the intersection numbers satisfy

$$\mathcal{K}_{i\alpha\beta\gamma} \equiv \hat{D}_j \cdot D_\alpha \cdot D_\beta \cdot D_\gamma = 0 \ . \tag{3.6}$$

We will show below, that the existence of such a condition in an appropriately chosen basis is in accord with constraints of four-dimensional $\mathcal{N} = 1$ supersymmetry. On the resolved geometry of the ADE singularity one then has

$$(\hat{D}_i \cdot \hat{D}_j + C_{ij}\, S' \cdot B_3) \cdot D_\alpha \cdot D_\beta = 0 \ , \tag{3.7}$$

for all vertical divisors D_α in (3.5). Here C_{ij} is the Cartan matrix of G. Furthermore, one can now include in the intersection form (3.7) the extended node

$$\hat{D}_0 = S' - \sum_i a^i \hat{D}_i = S \ , \tag{3.8}$$

such that

$$(\hat{D}_i \cdot \hat{D}_j + C_{ij}\, S' \cdot B_3) \cdot D_\alpha \cdot D_\beta = 0 \ , \qquad i,j = 0, ..., rk(G) \ , \tag{3.9}$$

where now C_{ij} is the Cartan matrix of the extended Dynkin diagram. If $\hat{X}_4$ is realized as a hypersurface or complete intersection in a projective/toric ambient space, a resolution of the ambient space itself can consistently resolve the enhanced ADE singularities on $\mathcal{S}$ [35, 37, 20]. In these examples one can check explicitly by computing the intersection numbers that (3.7) and (3.9) are satisfied in the properly chosen topological phase.

The equations (3.7), (3.6) and (3.9) can also be written in terms of two-forms which are Poincaré dual to $B_3, \hat{D}_i, D_\alpha$. We list them for completeness:

1. A two-form ω_0 Poincaré dual to the base B_3 of the elliptic fibration.
2. Two-forms $\omega_\alpha, \alpha = 1, \ldots h^{1,1}(B)$ on X_4 which are Poincaré dual to vertical divisors $D_\alpha = \pi^{-1}(D^{\mathrm{b}}_\alpha)$, as in section 2.4.
3. Two-forms $\mathrm{w}_i, i = 1, \ldots, \mathrm{rank}(G)$ which are Poincaré dual to the blow-up divisors $\hat{D}_i$ which were introduced after (3.3). The two-form w_0 corresponding to the extended node of the Dynkin diagram $\hat{D}_0$ can be canonically included in the discussion [51, 52]. w_0 is not linearly independent of the two-forms so far, and thus is not needed to form a basis of $H^{1,1}(\hat{X}_4)$.

The intersection conditions (3.7) and (3.9) translate to

$$\mathcal{K}_{ij\alpha\beta} \equiv \int_{\hat{X}_4} \mathrm{w}_i \wedge \mathrm{w}_j \wedge \omega_\alpha \wedge \omega_\beta = -C_{ij} \int_{\mathcal{S}} \omega_\alpha \wedge \omega_\beta \equiv -C_{ij}\, \mathcal{K}_{S|\alpha\beta} \ , \tag{3.10}$$

where we have used in the second equality that $\omega_0, \omega_{S'}$ are Poincaré dual to B_3, S'. The last equality defines the intersection form $\mathcal{K}_{S|\alpha\beta}$ on $\mathcal{S}$ of the D_α of the ambient space. It is a well-known fact that there can be additional non-trivial two-cycles on $\mathcal{S}$ which are not induced from intersections of $\mathcal{S}$ with the D_α. These elements will be included later on, but do not alter the intersection analysis presented here.

It is important to stress that the singularity of the elliptic fibration can vary over $\mathcal{S}$ and enhance to groups larger than G along complex curves and points. In this case the resolution of the singularity becomes more involved, as discussed in detail in refs. [53]. However, if $\hat{X}_4$ is realized as a hypersurface or complete intersection in a projective/toric ambient space, a resolution of the ambient space itself can consistently resolve the enhanced ADE singularities at all co-dimensions on $\mathcal{S}$ [35, 37, 20, 21]. In this case one systematically resolves the ambient space with divisors $\tilde{D}_i, i = 1, \ldots, rk(G)$ which at generic points of $\mathcal{S}$ restrict to the resolving divisors $\hat{D}_i$ introduced in this subsection. In accord with the analysis of the effective action, this resolution increases the number of two-forms on $\hat{X}_4$ by $rk(G)$ forms w_i. The resolution of further enhancements along curves and points does not change $h^{1,1}(\hat{X}_4)$. However, one will find new four-forms on $\hat{X}_4$ which are not a wedge of two $(1,1)$-forms. These new four-forms are crucial in defining the G_4 fluxes determining the chiral spectrum.

3.2 Non-Abelian gauge groups in M-theory

To study the F-theory lift we first discuss the appearance of non-Abelian gauge groups in the M-theory picture. In order to do that, we recall that in M-theory $U(1)$ vector fields can arise from the three-form C_3 with field strength $G_4 = dC_3$. Considering M-theory on the resolved fourfold $\hat{X}_4$ will correspond to the Coulomb branch of the gauge theory where G is broken to $U(1)^{rk(G)}$ over $\mathcal{S}$. To study the Coulomb branch one replaces the general expression (3.1) by an expansion into the Cartan generators $\mathcal{T}_i$ of the adjoint representation of G. We choose these to obey

$$\mathrm{Tr}(\mathcal{T}_i \mathcal{T}_j) = C_{ij} \ , \qquad i, j = 1, \ldots, rk(G) \ , \tag{3.11}$$

The expansion (3.1) translates into components of the four-form G_4 by writing schematically

$$F = F^i \, \mathcal{T}_i \qquad \rightarrow \qquad G_4|_{\mathrm{brane}} \cong F^i \wedge \mathrm{w}_i = (F_4^i + F_{\mathrm{flux}}^i + F_w^i) \wedge \mathrm{w}_i \ . \tag{3.12}$$

Note that this cannot be a precise statement, since the M-theory reduction is to three rather then four space-time dimensions. It will be the task of the next subsection to make the lift more explicit. We start with the vectors with field strength F_4^i, for which the lift (3.12) is most directly applicable.

To obtain the complete set of massless three-dimensional vectors let us consider the expansion three-form C_3 into two-forms of $H^{1,1}(\hat{X}_4)$ as

$$C_3|_{\mathrm{vector}} = A^0 \wedge \omega_0 + A^\alpha \wedge \omega_\alpha + A^i \wedge \mathrm{w}_i \ , \tag{3.13}$$

where A^0, A^α, A^i are three-dimensional vector fields. In the three-dimensional $\mathcal{N} = 2$ theory these vectors are combined with the real scalars arising in the expansion of the Kähler form J of $\hat{X}_4$ into vector multiplets. One expands

$$J = v^0 \omega_0 + v^\alpha \omega_\alpha + \mathrm{v}^i \mathrm{w}_i \ , \tag{3.14}$$

where the v^i measure the volumes of the blow-up $\mathbb{P}^1$'s. As noted above, the A^i are the r $U(1)$ vector fields which correspond to the Cartan generators of the non-Abelian gauge group G. The non-Abelian gauge symmetry arises in the limit in which the volumes of the blow-ups go to zero. Then M2 branes wrapped on chains of resolving $\mathbb{P}^1$ fibers become massless and provide the missing gauge degrees of freedom to form a non-Abelian gauge group G. We will return to the restoration of G in the discussion of the F-theory lift in the next section.

Let us next discuss the effective action of three-dimensional vector multiplets (A^0, v^0), (A^i, v^i) and (A^α, v^α). Since we will be more general later on, we will include in the following expressions also a number of three-dimensional chiral multiplets M^I. As in (2.24) we first introduce the rescaled variables which actually appear in the three-dimensional $\mathcal{N}=2$ vector multiplets,

$$R=\frac{v^0}{\mathcal{V}}\ , \qquad \xi^i = R\cdot\zeta^i=\frac{\mathrm{v}^i}{\mathcal{V}}\ , \qquad L^\alpha=\frac{v^\alpha}{\mathcal{V}}\ , \tag{3.15}$$

where we have defined also ζ^i by splitting off a factor of R.[10] Note that in the Kähler cone one generically has $\xi^i\leq 0$ as v^i appears in front of a two-form Poincaré dual to an exceptional divisor $\hat{D}_i$ in (3.14). This ensures, in particular, that the volumes of the divisors $\hat{D}_i$ are positive as we will see below. The three-dimensional action for the vector multiplets $(A^{\hat\Lambda},\xi^{\hat\Lambda})\cong(A^0,R),(A^i,\xi^i),(A^\alpha,L^\alpha)$ is of the form

$$\begin{aligned} S^{(3)} \;&=\; \int -\tfrac{1}{2}R_3 * \mathbf{1} - \tilde{K}_{I\bar{J}}\, dM^I\wedge * d\bar{M}^J + \tfrac{1}{4}\tilde{K}_{\hat\Lambda\hat\Sigma}\, d\xi^{\hat\Lambda}\wedge * d\xi^{\hat\Sigma} \\ &\qquad -\tfrac{1}{4}\tilde{K}_{\hat\Lambda\hat\Sigma}\, F^{\hat\Lambda}\wedge * F^{\hat\Sigma} + F^{\hat\Lambda}\wedge \mathrm{Im}(\tilde{K}_{\hat\Lambda I}\, dM^I)\ , \end{aligned} \tag{3.16}$$

where the kinetic terms of the vectors and scalars are determined by a single real function, the kinetic potential $\tilde{K}(M^I,\bar{M}^J|\xi^{\hat\Lambda})$ with

$$\tilde{K}_{I\bar{J}}=\partial_{M^I}\partial_{\bar{M}^J}\tilde{K}\ , \qquad \tilde{K}_{\hat\Lambda\hat\Sigma}=\partial_{\xi^{\hat\Lambda}}\partial_{\xi^{\hat\Sigma}}\tilde{K}\ , \qquad \tilde{K}_{\hat\Lambda I}=\partial_{\xi^{\hat\Lambda}}\partial_{M^I}\tilde{K}\ . \tag{3.17}$$

In the M-theory reduction we will denote the kinetic potential by $\tilde{K}^{\mathrm{M}}$ as above.

In a next step we aim to find the leading kinetic potential which captures the new degrees of freedom extending $\tilde{K}^{\mathrm{M}}$ given in (2.26). One still has $\tilde{K}^{\mathrm{M}}=-3\log\mathcal{V}+K^{\mathrm{cs}}$, but now one needs to include the fields ξ^i in the volume expansion. They appear as

$$\tilde{K}^{\mathrm{M}}=\log[\tfrac{1}{6}RL^\alpha L^\beta L^\gamma\mathcal{K}_{\alpha\beta\gamma}-\tfrac{1}{4}\xi^i\xi^j\, C_{ij}\, L^\alpha L^\beta\mathcal{K}_{S|\alpha\beta}+\mathcal{O}(R^3,\xi^3)]+K^{\mathrm{cs}}(z)\ , \tag{3.18}$$

with K^{cs} given in (2.39). In this expression we have used (2.27) for $\mathcal{K}_{\alpha\beta\gamma}$, and (3.10) to obtain the term involving $\mathcal{K}_{S|\alpha\beta}$. The expansion in the logarithm (3.18) does not contain a linear term in ξ^i as ensured by (3.6) on the resolved $\hat{X}_4$. This turns out to be crucial for the M-theory compactification to lift to a four-dimensional theory with gauge group on $\mathcal{S}$. Let us stress that the expression (3.18) is a large volume expression, since it simply arose by expanding the quadruple intersections on a Calabi-Yau fourfold. Various corrections to this expression are expected as discussed briefly in section 2.4. However, by simply using (3.18) and performing a Taylor expansion for small R,ξ^i one nevertheless finds non-trivial match with the three- and four-dimensional expectations from the gauge theory and the gravity background. It would be very interesting to understand this match in more detail and to extend the following considerations to include further corrections.

To prepare for the discussion of the F-theory limit, let us now Taylor expand the above results for small $R,|\zeta^i|=|\xi^i|/R$. The three-dimensional gauge kinetic coupling

[10] This split will be convenient in the discussion of the F-theory lift.

function in (3.16) can be determined as the second derivative of $\tilde{K}^{\rm M}$ with respect to the scalars which are in three-dimensional vector multiplets. In particular, one finds that

$$\tilde{K}^{\rm M}_{ij} = -\frac{C_{ij}\,\mathcal{K}_{S|\beta\gamma}L^\beta L^\gamma}{2\,R\mathcal{V}_L} + \mathcal{O}(R,\zeta^2)\ , \qquad \mathcal{V}_L \equiv \tfrac{1}{6}\mathcal{K}_{\alpha\beta\gamma}L^\alpha L^\beta L^\gamma\ . \tag{3.19}$$

Furthermore, using $\tilde{K}^{\rm M}$ one readily evaluates the real parts of the dual Kähler coordinates T_α, T_i, T_0 via (2.23). Firstly, we have

$$\text{Re}T_\alpha = \frac{1}{2}\frac{\mathcal{K}_{\alpha\beta\gamma}L^\beta L^\gamma}{\mathcal{V}_L} + \mathcal{O}(R)\ , \tag{3.20}$$

where we recall from (3.5) that one of these coordinates is $T_{S'}$ corresponding to the divisor S'. Furthermore, one evaluates by using (3.18) and (3.15) that

$$\text{Re}T_i = -\frac{C_{ij}\zeta^j\,\mathcal{K}_{S|\beta\gamma}L^\beta L^\gamma}{2\,\mathcal{V}_L} + \mathcal{O}(R)\ , \qquad \text{Re}T_0 = \frac{1}{R} + \mathcal{O}(R,\zeta^2)\ . \tag{3.21}$$

Note that in the Kähler cone one has $\zeta^i \leq 0$ and one has $\text{Re}T_i \geq 0$, and $\text{Re}T_\alpha \geq 0$, $\text{Re}T_0 \geq 0$.

The expressions for T_α, T_i exactly match the expectations from the point of view of a reduction from a four-dimensional to a three-dimensional gauge theory as discussed in [54, 55, 56]. To make this more precise, we will denote by T_S the complex scalar corresponding to the divisors $S \subset X_4$ introduced in (3.4), (3.8) such that

$$T_S = T_{S'} - \sum_{i=1}^{rk(G)} a^i T_i\ . \tag{3.22}$$

One can qualitatively analyze the three-dimensional superpotential obtained in the M-theory compactification from M5-brane instantons wrapped on divisors in $\hat{X}_4$. In ref. [56] it was argued that M5-branes on the blow-up divisors $\hat{D}_i$ as well as the extended node $\hat{D}_0$, defined in (3.8), satisfy the necessary criteria [33] to yield a non-trivial instanton correction to the superpotential. In addition also an M5-brane wrapped on the base B_3 satisfies these criteria [33]. Hence, the superpotential is expected to contain the terms

$$W^{\rm M} = \sum_{i=1}^{rk(G)} \mathcal{A}_i e^{-T_i} + \mathcal{B}e^{-T_{S'}+a^i T_i} + \mathcal{C}e^{-T_0}\ , \tag{3.23}$$

where $\mathcal{A}_i, \mathcal{B}, \mathcal{C}$ generically depend holomorphically on the other complex scalars, e.g. the complex structure deformations, of the compactification. The terms with pre-factors $\mathcal{A}_i$ correspond to the gauge theory instantons discussed in ref. [54, 55], while the term with $\mathcal{B}$ is associated with a four-dimensional gauge instanton [55] and vanishes in the

3d limit $r^2 = 1/R \to 0$. To see this one identifies $\tilde{K}^{\rm M}_{ij} \propto -1/g_3^2$, where g_3 is the three-dimensional gauge coupling constant. Comparing (3.19) with (3.20),(3.21) one then finds $\text{Re}T_i \propto -\xi^i/g_3^2$, and $\text{Re}T_{S'} \propto R/g_3^2 = 1/(r^2 g_3^2)$.[11] Finally, one can also interpret the last term in (3.23). One first notes that $\text{Re}T_0 \propto r^2$. Following the duality described in section 2.4 one notes that this instanton correction is due to a universal gravitational instanton in a four-dimensional theory on $\mathbb{M}_{2,1} \times S^1$ [57]. In fact, the M5-brane becomes an NS5-brane in going from M-theory to Type IIA. This NS5-brane T-dualizes into a four-dimensional Taub-NUT geometry. The gravitational action is indeed proportional to r^2 [58, 59].

3.3 Seven-brane gauge theory in the F-theory lift

It is crucial to stress that in F-theory limit one necessarily takes the limit in which the resolution of the singularities is blown down. In other words, while one is able to access the Coulomb branch in M-theory this is no longer possible in F-theory. One thus has to extend the F-theory limit (2.32) to include the blow-up volumes. This leads us to replace (2.32) by

$$\begin{aligned} \text{singular:} \qquad & R \ \to\ 0\ , \qquad\qquad \zeta^i = \xi^i/R \ \to\ 0\ , \\ \text{finite:} \qquad & L^\alpha \ \to\ L^\alpha_{\rm b}\ , \end{aligned} \tag{3.24}$$

or

$$\text{Re}T_0 \ \to\ \infty\ , \qquad \text{Re}T_i \ \to\ 0\ , \qquad T_\alpha \ \to\ T^{\rm b}_\alpha\ . \tag{3.25}$$

This implies the scaling of the $v^{\mathcal{A}}$ as $v^0 \propto \epsilon$, $\text{v}^i \propto \epsilon^2$ and $v^\alpha \propto 1/\sqrt{\epsilon}$ in the limit $\epsilon \to 0$. In other words, the low-energy expansion of the F-theory effective action is around the special point (3.24) in the Kähler moduli space. However, the variations $\delta R, \delta\zeta^i$ are not un-physical but rather will appear as degrees of freedom in four-dimensional fields. In particular, four-dimensional vector A^i_4 fields are given by

$$A^i_4 = (A^i + \delta\zeta^i A^0, \delta\zeta^i)\ , \tag{3.26}$$

containing ζ^i defined in (3.15). The lift of the three sets of three-dimensional vector multiplets is: (v^0, A^0) lifts to the metric of the fourth non-compact dimension $g_{33}, g_{3\mu}$, (v^α, A^α) lift to chiral multiplets in four dimensions, and (v^i, A^i) lift to $U(1)$ vectors corresponding to the Cartan generators of G. This is summarized in table 3.1.

[11] The expression for $\text{Re}T_S$ appears to differ by a factor of r from the results of [55, 56]. However, this difference is readily explained by noting that in our analysis we work with a three-dimensional action Weyl rescaled to the three-dimensional Einstein frame with canonically normalized Einstein-Hilbert term. In refs. [55, 56] such a Weyl rescaling was not performed, which explains the different dependence on r. Also note that we have set $M_p = 1$ in three dimensions.

3-dim multiplet	4-dim F-theory	
$(L^{\mathcal{A}}, A^{\mathcal{A}})$	$h^{1,1}(X_4) - h^{1,1}(B_3) - 1$ vector mult.	$(\xi^i, A^i) \to A_4$ adjoint
	$h^{1,1}(B_3)$ chiral multiplets	$(L^\alpha, A^\alpha) \to T_\alpha$
	extra dimension	$(R, A^0) \to (g_{33}, g_{\mu 3})$

Table 3.1: *The F-theory lift of the fields arising from the Kähler form of* $\hat{X}_4$.

To proceed we first recall that the multiplets (v^α, A^α) actually lift to four-dimensional chiral multiplets. It is therefore convenient to rather work with $\mathbf{K}^{\rm M}(T_\alpha|\xi, R)$ given by

$$\mathbf{K}^{\rm M}(T_\alpha + \bar{T}_\alpha, M^I|\xi^i, R) = \tilde{K}^{\rm M} - \tfrac{1}{2}(T_\alpha + \bar{T}_\alpha)L^\alpha \ , \tag{3.27}$$

where L^α is replaced by its Legendre transform

$$T_\alpha = \partial_{L^\alpha}\tilde{K}^{\rm M} + i\int_{D_\alpha} C_6 \ . \tag{3.28}$$

Note that we have included additional scalars M^I in the expressions for $\mathbf{K}^{\rm M}$ and T_α, since their inclusion does not change the discussion presented here. These complex scalars are specified later, and include complex structure moduli, Wilson line moduli, matter fields, etc. The expressions (3.28) and (3.27) are very similar to the discussion to (2.23) and (2.25). However, it is crucial to stress, that in $\mathbf{K}^{\rm M}$, we have kept the vector multiplets containing R, ξ^i, and only dualized the multiplets (v^α, A^α). It is straightforward to check that

$$\frac{\partial \mathbf{K}^{\rm M}}{\partial T_\alpha} = -\frac{1}{2}L^\alpha \ , \qquad \frac{\partial \mathbf{K}^{\rm M}}{\partial M} = \frac{\partial \tilde{K}^{\rm M}}{\partial M} \ , \qquad M \in (M^I, \xi^i, R) \ . \tag{3.29}$$

Note that the right-hand sides of these expressions are evaluated by first taking derivatives of $\tilde{K}^{\rm M}$ viewed as a function of (L^α, ξ^i, R) and M^I, and then use (3.28) to express the result as a function of T_α, R, ξ^i, M^I. Note that by differentiating (3.28) one also finds

$$\frac{\partial L^\alpha}{\partial T_\beta} = \tilde{K}^{{\rm M}\ L^\alpha L^\beta} \ , \qquad \frac{\partial L^\alpha}{\partial M} = -\tilde{K}^{{\rm M}\ L^\alpha L^\beta}\partial_M \tilde{K}^{\rm M}_{L^\beta} \ , \qquad M \in (M^I, \xi^i, R) \ . \tag{3.30}$$

In order to study the F-theory lift one has to evaluate the kinetic potential $\mathbf{K}^{\rm M}$ in the limit (3.24). One thus performs a Taylor expansion of $\mathbf{K}^{\rm M}$ for small R, ξ^i around the strict F-theory limit $R = \xi^i = 0$. In the following we will denote the restriction of a function $f(R,\xi)$ to the F-theory limit by $f|_*$. The reasoning that there exists a four-dimensional $\mathcal{N} = 1$ supergravity theory in the F-theory limit (3.24) significantly restricts the form of the expansion of $\mathbf{K}^{\rm M}$. Namely, from a reduction of a four-dimensional $\mathcal{N} = 1$

supergravity theory to three dimensions, as recalled in appendix B, one infers that the kinetic potential has to admit the form

$$\mathbf{K}^{\mathrm{M}} = \log R + K^{\mathrm{F}}(T,M) - \frac{1}{2R}\,\mathrm{Re}f_{ij}(T,M)\,\xi^i\xi^j + \dots\ , \tag{3.31}$$

which resembles (B.4) in appendix B. Here $K^{\mathrm{F}}(T,M)$ is the four-dimensional Kähler potential depending on the complex scalars T_α, M^I, and $f_{ij}(T,M)$ is the four-dimensional holomorphic gauge coupling function. Note that the two terms involving R are precisely the singular terms in the limit $R \to 0$. Further terms in the (R,ξ^i)-expansion will not be relevant in our computation of the four-dimensional effective theory. The expression (3.31) implies that

$$\partial_{\xi^j}\mathbf{K}^{\mathrm{M}}\big|_* = \partial_{\xi^j}\tilde{K}^{\mathrm{M}}\big|_* = 0\ , \tag{3.32}$$

in a set-up which consistently lifts to a four-dimensional $\mathcal{N}=1$ F-theory compactification. Similarly, comparing (3.31) with the general Taylor expansion, one concludes that the four-dimensional gauge-coupling function is given by

$$\begin{aligned}\mathrm{Re}\,f_{ij} &= -R\cdot\mathbf{K}^{\mathrm{M}}_{\xi^i\xi^j}\big|_* = -R\cdot\big(\tilde{K}^{\mathrm{M}}_{\xi^i\xi^j} - \tilde{K}^{\mathrm{M}}_{\xi^i L^\alpha}\tilde{K}^{\mathrm{M}\ L^\alpha L^\beta}\tilde{K}^{\mathrm{M}}_{L^\beta\xi^j}\big)\big|_* \\ &= -R\cdot K^{\mathrm{M}}_{\xi^i\xi^j}\big|_*\ ,\end{aligned} \tag{3.33}$$

where one has to apply (3.30) and (3.32) to evaluate the last two equalities. f_{ij} is the holomorphic gauge-coupling function of the vectors A^i_4.

In general, the holomorphic gauge coupling function f_{ij} can be of the form

$$\mathrm{Re}\,f_{ij} = \mathrm{Re}(\mathcal{C}^\alpha_{ij}\,T_\alpha - \tilde{f}_{ij}(M) + \mathcal{O}(e^{-T}))\ , \tag{3.34}$$

where $\mathcal{C}^\alpha_{ij}$ are real constants and $\tilde{f}_{ij}(M)$ is a homomorphic function in the remaining complex scalars M^I. Note that the perturbative shift symmetry for T_α in three dimensions prevents T_α to appear with additional perturbative contributions. This can be traced back to the fact that T_α arises by dualizing a three-dimensional vector multiplet (A^α, v^α). We next evaluate (3.33) and (3.34), neglecting T_α instanton corrections. Using the expression (3.28) one obtains the differential equation

$$\big(\mathcal{C}^\alpha_{ij}\partial_{L^\alpha} + R\partial_{\xi^i}\partial_{\xi^j}\big)\tilde{K}^{\mathrm{M}} = \mathrm{Re}\,\tilde{f}_{ij}(M)\ , \qquad \text{for } R,\zeta^i \to 0\ . \tag{3.35}$$

This equation poses constraints on the M-theory kinetic potential to ensure that this three-dimensional theory can be obtained by dimensional reduction from a four-dimensional effective theory with holomorphic gauge-coupling function f_{ij}. It is not hard to check that $\tilde{K}^{\mathrm{M}}$ as given in (3.18) satisfies (3.35) for $C^{S'}_{ij} = C_{ij}$ with all other C^α_{ij} vanishing and $\tilde{f}_{ij} = 0$.

In the limit (3.24) the A_i^3, ξ^i are expected to combine into a non-Abelian potential $F_4 = dA_4 + [A_4, A_4]$ for the group G. Here the missing degrees of freedom appear due to M2-branes wrapped on the vanishing $\mathbb{P}^1$'s in the fibers of the $\hat{D}_i$. This implies that

$$\int_{\mathbb{M}_{2,1}} \mathrm{Re} f_{ij}\, F^i \wedge *_3 F^j \quad \to \quad \int_{\mathbb{M}_{3,1}} \mathrm{Re} f_G \,\mathrm{Tr}(F_4 \wedge *_4 F_4) \ , \tag{3.36}$$

where f_G is the holomorphic gauge coupling function of the four-dimensional non-Abelian gauge theory. Let us now apply the F-theory lift, to the large volume expression (3.18) for $\tilde{K}^{\mathrm{M}}$. Performing the Legendre transform and a R, ξ^i Taylor expansion one finds that $\mathbf{K}^{\mathrm{M}}$ is given by

$$\mathbf{K}^{\mathrm{M}} = \log R + \log[\mathcal{V}_L(T + \bar{T})] + K^{\mathrm{cs}}(z) - \frac{1}{2R} \mathrm{Re} T_{S'}\, C_{ij} \xi^i \xi^j + \dots \ . \tag{3.37}$$

where we have to use (3.20) to evaluate $L^\alpha(T)$ as a function of $T_\alpha + \bar{T}_\alpha$. The term proportional to R^{-1} in this expansion is directly evaluated by using (3.33) and (3.19). Comparing (3.37) to the general expression (3.31) one easily determines the four-dimensional Kähler potential K^{F} and gauge coupling function f_{ij},

$$\begin{aligned} K^{\mathrm{F}}(z,T) &= \log \mathcal{V}_L(T+\bar{T}) + K^{\mathrm{cs}}(z) \\ &= -2\log \mathcal{V}_{\mathrm{b}} - \log \int_{\hat{X}_4} \Omega \wedge \bar{\Omega} \ , \\ f_G(T) &= T_{S'} \ , \end{aligned} \tag{3.38}$$

where we inserted (2.39), and $L^\alpha_b = v^\alpha_{\mathrm{b}}/\mathcal{V}_{\mathrm{b}}$ after using (3.24). Note that this precisely agrees with the expectation for a seven-brane wrapped on the divisor $\mathcal{S}$ in the base B_3 [60, 9].

4 Gauge theories from the R-R sector

In this section we discuss the gauge theory arising from vector fields obtained by the reduction of the R-R four-form C_4 in an F-theory compactification on a Calabi-Yau fourfold with $2r = 2h^{2,1}(B_3)$ harmonic three-forms. We begin to review some general facts about the four-dimensional $U(1)^r$ gauge theory in subsection 4.1. The F-theory gauge couplings are determined by lifting a three-dimensional M-theory compactifications on X_4 to four dimensions in subsection 4.2. In the three-dimensional theory the vector fields are dual to complex scalars. In subsection 4.3 we comment on the action of four-dimensional electro-magnetic duality on the three-dimensional effective theory constraining the form of the superpotential.

4.1 Type IIB perspective and the four-dimensional action

Let us first recall some facts about the four-dimensional gauge theory arising from the massless vector modes of C_4. Explicitly the gauge fields arise in the expansion

$$C_4|_{vec} = V^\kappa \wedge \alpha_\kappa - \tilde{V}_\kappa \wedge \beta^\kappa \ , \qquad \kappa = 1, \ldots, r \ , \tag{4.1}$$

where we have displayed the 'electric' and 'magnetic' four-dimensional vectors $V^\kappa, \tilde{V}_\kappa$. The real symplectic basis $(\alpha_\kappa, \beta^\kappa)$ of $H^3(B_3, \mathbb{Z})$ obeys

$$\int_{B_3} \alpha_\lambda \wedge \beta^\kappa = \delta^\kappa_\lambda \ , \qquad \int_{B_3} \alpha_\kappa \wedge \alpha_\lambda = \int_{B_3} \beta^\kappa \wedge \beta^\lambda = 0 \ . \tag{4.2}$$

Note that C_4 in Type IIB supergravity has a self-dual five-form field strength $F_5 = *_{10} F_5$ and hence only half of the vectors in (4.1) parametrize independent degrees of freedom. As we argue in the next subsection 4.2 one expects that this self-duality is generalized in an F-theory compactification. Despite this generalization the choices for splitting the gauge fields into sets V^κ and $\tilde{V}_\kappa$ will be related by symplectic rotations in $Sp(2r, \mathbb{Z})$ of the basis $(\alpha_\kappa, \beta^\kappa)$ preserving (4.2). On the level of the four-dimensional effective action these rotations correspond to electro-magnetic rotations as we recall next.

Let us summarize some general facts about electro-magnetic duality rotations. It is well-known that the four-dimensional $\mathcal{N} = 1$ action for $U(1)$ gauge fields is of the form

$$S^4_{U(1)} = -\int_{\mathbb{M}_{3,1}} \tfrac{1}{2} \mathrm{Re} f_{\kappa\lambda}\, \mathcal{F}^\kappa \wedge *_4 \mathcal{F}^\lambda + \tfrac{1}{2} \mathrm{Im} f_{\kappa\lambda}\, \mathcal{F}^\kappa \wedge \mathcal{F}^\lambda \tag{4.3}$$

where $\mathcal{F}^\kappa = dV^\kappa$, and $f_{\kappa\lambda}$ is a holomorphic function of the complex scalars in the chiral multiplets. The electro-magnetic rotations mix Bianchi identities and equations of motion for the gauge fields V^κ. The elements of the symplectic group satisfy

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2r, \mathbb{Z}) \ , \qquad \begin{array}{c} D^T A - B^T C = 1 \ , \\ A^T C = C^T A, \ \ B^T D = D^T B \ , \end{array} \tag{4.4}$$

where A, B, C, D are $r \times r$ matrices. The matrix (4.4) acts linearly on the vector $(\mathcal{G}_\kappa, \mathcal{F}^\kappa)$, where

$$\mathcal{G}_\kappa = -\delta S^4_{U(1)} / \delta \mathcal{F}^\kappa = \mathrm{Re} f_{\kappa\lambda} *_4 \mathcal{F}^\lambda + \mathrm{Im} f_{\kappa\lambda}\, \mathcal{F}^\lambda \ . \tag{4.5}$$

Note that the gauge fields couple to the complex scalars of the theory via the holomorphic function $f_{\kappa\lambda}$. Hence, a electro-magnetic rotation can be induced by a transformation of the scalars and has to be accompanied with a rotation of the coupling matrix $\mathbf{f}$ with entries $\mathbf{f}_{\kappa\lambda} \equiv -i f_{\kappa\lambda}$ as

$$\mathbf{f} \ \rightarrow \ (A\mathbf{f} + B)(C\mathbf{f} + D)^{-1} \ . \tag{4.6}$$

The factor $-i$ arises from the convention that $\mathrm{Re} f_{\kappa\lambda} = (1/g^2)_{\kappa\lambda}$ is positive definite since it defines the inverse gauge-coupling.

In Type IIB supergravity $\mathcal{G}_\kappa$ given in (4.5) admits an identification as the field strength of $\tilde{V}_\kappa$. However, in a general F-theory compactification it is not expected that this can be inferred from the classical self-duality of F_5, since $f_{\kappa\lambda}$ will in general also depend on other moduli, such as the deformations of the seven-branes. We determine the $f_{\kappa\lambda}$ in terms of the geometry of the Calabi-Yau fourfold in the next subsection.

4.2 Bulk gauge theory in the F-theory lift

In order to study the F-theory dynamics and couplings of the vectors in (4.1) in more detail we again have to perform an M- to F-theory lift. In order to simplify the discussion, we restrict ourselves in this section to Calabi-Yau fourfolds with

$$r \equiv h^{2,1}(\hat{X}_4) = h^{2,1}(B_3) \; . \tag{4.7}$$

Moreover, recall that $h^{3,0}(\hat{X}_4) = 0$ such that the whole third cohomology splits as

$$H^3(\hat{X}_4, \mathbb{C}) = H^{2,1}(\hat{X}_4) \oplus H^{1,2}(\hat{X}_4) = H^{2,1}(B_3) \oplus H^{1,2}(B_3) \; . \tag{4.8}$$

This split depends on the complex structure on $\hat{X}_4$ and hence varies non-trivially over the complex structure moduli space $\mathcal{M}^{\mathrm{cs}}$ discussed in section 2.5. Moreover, in the quantum theory one actually has to consider the torus bundle $\mathbb{T} \to \mathcal{M}^{\mathrm{cs}}$ with complex r-torus fibers

$$\mathbb{T}_z = H^{2,1}(\hat{X}_4)/H^3(\hat{X}_4, \mathbb{Z}) \cong H^{2,1}(B_3)/H^3(B_3, \mathbb{Z}) \; , \tag{4.9}$$

and base $\mathcal{M}^{\mathrm{cs}}$. At special points in $\mathcal{M}^{\mathrm{cs}}$ the fibers $\mathbb{T}_z$ can become singular signaling that the effective theory was not properly determined since light degrees of freedom have been improperly discarded. At generic points in the moduli space one finds an $\mathcal{N} = 1$, $U(1)^r$ gauge theory after the lift of the three-dimensional M-theory compactification to a four-dimensional F-theory compactification. Note that the study of such torus fibration has recently attracted much attention in the context of reductions of $\mathcal{N} = 2$ gauge theories from four to three dimensions (see refs. [61] for recent progress and further references). Clearly, our $\mathcal{N} = 1$ set-up is much less constraint by supersymmetry. While in $\mathcal{N} = 2$ the torus bundle is of dimension $4r$ and admits a Hyperkähler metric the $\mathcal{N} = 1$ torus bundle $\mathbb{T}$ is of dimension $2r + 2h^{3,1}(\hat{X}_4)$ and has a Kähler metric. One of the tasks of this section is to determine the classical form of this metric in a theory coupled to gravity.

As already noted in section 2.2 imposing (4.7) ensures that all non-trivial three-forms on $\hat{X}_4$ descend to four-dimensional vector multiplets in the F-theory limit. In the M-theory reduction these vectors arise from expanding the M-theory three-form C_3 into

$(\alpha_\kappa, \beta^\kappa)$ via the Kaluza-Klein Ansatz

$$\begin{aligned} C_3 &= A^{\mathcal{A}} \wedge e_{\mathcal{A}} + \tilde{a}^\kappa \, \alpha_\kappa - \tilde{b}_\kappa \, \beta^\kappa \\ &= A^{\mathcal{A}} \wedge e_{\mathcal{A}} + \mathcal{N}_\kappa \, \psi^\kappa + \bar{\mathcal{N}}_\kappa \, \bar{\psi}^\kappa \ , \end{aligned} \tag{4.10}$$

where $A^{\mathcal{A}} = (A^0, A^i, A^\alpha)$ are vectors which we already included in (3.13), and $(\tilde{a}^\kappa, \tilde{b}_\kappa)$, $\mathcal{N}_\kappa$ are real and complex scalars in three dimensions. In three-dimensions massless scalars are dual to vectors and we will show momentarily, that after dualizing $\tilde{b}_\kappa$ into a vector V_3^κ, the fields (V_3^κ, a^κ) comprise the degrees of freedom of a four-dimensional vector V^κ in (4.1).

In (4.10) we have also introduced a complex basis of (2,1)-forms ψ^κ of $H^{2,1}(\hat{X}_4)$. The basis elements ψ^κ depend on the complex structure of $\hat{X}_4$ and naturally define complex coordinates $\mathcal{N}_\kappa$. Let us explore the relation between the complex and real basis for $H^3(\hat{X}_4)$. In general, one can identify

$$\psi^\kappa = \tfrac{1}{2} \mathrm{Re} f^{\kappa\lambda} (\alpha_\lambda - i \bar{f}_{\lambda\mu} \beta^\mu) \ , \qquad \psi^\kappa - \bar{\psi}^\kappa = -i \beta^\kappa \ , \tag{4.11}$$

for a complex function $f_{\kappa\lambda}$ of the complex structure moduli $z^{\mathcal{K}}$, with $\mathrm{Re} f^{\kappa\lambda} \equiv (\mathrm{Re} f_{\kappa\lambda})^{-1}$ being the inverse of the real part of $f_{\kappa\lambda}$. One can now show that for an appropriate choice of ψ^κ the function $f_{\kappa\lambda}(z)$ is holomorphic in $z^{\mathcal{K}}$. This can be deduced from the fact that for a complex manifold $\hat{X}_4$ the filtration $F^3(\hat{X}_4) = H^{3,0}$, $F^2(\hat{X}_4) = H^{3,0} \oplus H^{2,1}$, etc. consists of holomorphic bundles $F^i(\hat{X}_4)$ over the space of complex structure deformations [62]. Since $H^{3,0}$ is trivial one finds that $F^2 = H^{2,1}$ is a holomorphic bundle and one can locally choose a basis $\psi^\kappa(z)$ as in (4.11). The matrix $f_{\kappa\lambda}$ is readily extracted from ψ^κ using (4.2). As an immediate consequence of (4.10) and (4.11) one concludes that

$$\mathcal{N}_\kappa = f_{\kappa\lambda}(z) \, \tilde{a}^\lambda - i \tilde{b}_\kappa = -i(\tilde{b}_\kappa - \mathbf{f}_{\kappa\lambda} \tilde{a}^\lambda) \ , \tag{4.12}$$

where $\mathbf{f}_{\kappa\lambda} = -i f_{\kappa\lambda}$ as in (4.6). These expressions, together with the analysis of the couplings of $\mathcal{N}_\kappa$, allows to identify the $\mathcal{N}_\kappa$ as arising from four-dimensional vectors V^κ after reduction to three dimensions on a circle. Moreover, the function $f_{\kappa\lambda}(z)$ is the holomorphic four-dimensional gauge coupling function in (4.3). The reduction from four to three dimensions is reviewed in appendix B. To fully justify this identification under reduction one also has to analyze the couplings of the scalars $\mathcal{N}_\kappa$, and hence derive the three-dimensional kinetic potential including the $\mathcal{N}_\kappa$.

To identify the appearance of the moduli $\mathcal{N}_\kappa$ in the M-theory kinetic potential one dimensionally reduces the eleven-dimensional kinetic term for $G_4 = dC_3$ and the Chern-

Simons term

$$S^{11}_{G_4} = -\int \tfrac{1}{4} G_4 \wedge * G_4 + \tfrac{1}{12} C_3 \wedge G_4 \wedge G_4 \tag{4.13}$$
$$= -\int_{\mathbb{M}_{2,1}} \mathcal{G}^{\kappa\lambda} D\mathcal{N}_\kappa \wedge * \overline{D\mathcal{N}_\lambda} - \tfrac{1}{2} d_{\mathcal{A}}{}^{\kappa\bar{\lambda}} F^{\mathcal{A}} \wedge \mathrm{Im}(\bar{\mathcal{N}}_\kappa D\mathcal{N}_\lambda) + \dots \ ,$$

where $D\mathcal{N}_\kappa = d\mathcal{N}_\kappa - \mathrm{Re}\mathcal{N}_\lambda \, \mathrm{Re} f^{\lambda\mu} \, df_{\mu\kappa}$. The metric on the space of three-forms is given by

$$\mathcal{G}^{\kappa\lambda} = \frac{1}{2\mathcal{V}} \int_{\hat{X}_4} \psi^\kappa \wedge * \bar{\psi}^\lambda = -\frac{v^{\mathcal{A}} d_{\mathcal{A}}{}^{\kappa\bar{\lambda}}}{2\mathcal{V}} \ , \qquad d_{\mathcal{A}}{}^{\kappa\bar{\lambda}} = i \int_{\hat{X}_4} \omega_{\mathcal{A}} \wedge \psi^\kappa \wedge \bar{\psi}^\lambda \ , \tag{4.14}$$

where we have used $*\bar{\psi}^\kappa = -iJ \wedge \bar{\psi}^\kappa$. The coupling $d_{\mathcal{A}}{}^{\kappa\bar{\lambda}}$ depends on the complex structure moduli through the complex three-forms ψ^κ. The crucial point to note is that (4.13) also induces a coupling of $\mathcal{N}_\kappa$ and the complex structure moduli $z^{\mathcal{K}}$ to the three-dimensional vectors $A^{\mathcal{A}}$. Note that many of the above statements, in particular equations (4.11) and (4.13), are independent of the restriction (4.7). However, in case we restrict to geometries where all non-trivial three-forms arise from the base B_3, i.e. (4.7) is obeyed, one can further deduce that the only non-vanishing $d_{\mathcal{A}}{}^{\kappa\bar{\lambda}}$ is along ω_0, the two-form Poincaré dual to B_3 in $\hat{X}_4$. Explicitly, one has

$$\mathcal{G}^{\kappa\lambda} = -\tfrac{1}{2} R \cdot d_0{}^{\kappa\bar{\lambda}} \ , \qquad d_0{}^{\kappa\bar{\lambda}} = i \int_{B_3} \psi^\kappa \wedge \bar{\psi}^\lambda = -\tfrac{1}{2} \mathrm{Re} f^{\kappa\lambda} \ , \tag{4.15}$$

where we inserted (4.11). By comparing (4.13) with (3.16) one thus infers that $\mathcal{N}_\kappa$ appears in the kinetic potential $\mathbf{K}^{\mathrm{M}}$ as

$$\mathbf{K}^{\mathrm{M}} = \log R + K^F(z,T) - \frac{1}{2R} \mathrm{Re} T_{S'} C_{ij} \xi^i \xi^j + \frac{1}{2} R \cdot \mathrm{Re} f^{\kappa\lambda}(z) \, \mathrm{Re}\mathcal{N}_\kappa \, \mathrm{Re}\mathcal{N}_\lambda \ , \tag{4.16}$$

where $K^F(z,T)$ is the $\mathcal{N}_\kappa$-independent four-dimensional Kähler potential (3.38). Note that this kinetic potential reproduces correctly the first term in the reduction of (4.13). The second term in the reduction (4.13) is only reproduced up to a total derivative. It is now readily checked that (4.16) indeed encodes the dynamics of four-dimensional vectors V^κ. Comparing (4.16) to the general expression (B.9), obtained by dimensional reduction of the general $\mathcal{N} = 1$ four-dimensional action, one finds

$$f_G = T_{S'} \ , \qquad f^{\mathrm{RR}}_{\kappa\lambda} = f_{\kappa\lambda}(z) \ . \tag{4.17}$$

where $f^{\mathrm{RR}}_{\kappa\lambda}$ is the four-dimensional gauge coupling function of the R-R $U(1)$'s.

4.3 Electro-magnetic duality in the three-dimensional action

Having established the couplings of the $\mathcal{N}_\kappa$ in the three-dimensional action, it is interesting to investigate the action of the symplectic group (4.4) when acting linearly on the basis vector $(\alpha_\kappa, \beta^\kappa)$ and $(\tilde{b}_\kappa, \tilde{a}^\kappa)$ in (4.10). To explore that in more detail, we compute

$$T_0 = \partial_R \mathbf{K}^{\mathrm{M}} + ic_0 = \tilde{T}_0 + \tfrac{1}{4}\mathrm{Re} f^{\kappa\lambda}\mathcal{N}_\kappa(\mathcal{N}_\lambda + \bar{\mathcal{N}}_\lambda) \ , \qquad (4.18)$$

where $\tilde{T}_0$ is invariant under the transformations (4.4).[12] It is not hard to check that $\mathbf{f}_{\kappa\lambda} = -if_{\kappa\lambda}$ transforms as in (4.6). We evaluate the transformations of $\mathcal{N}_\kappa, T_0$ under (4.4) by using the invariance of $\tilde{T}_0$ as

$$\mathcal{N}_\kappa \to (C\mathbf{f} + D)^{-1\lambda}_{\kappa}\mathcal{N}_\lambda \ , \qquad T_0 \ \to \ T_0 + \tfrac{i}{2}C^{\mu\kappa}(C\mathbf{f}+D)^{-1\lambda}_{\kappa}\mathcal{N}_\lambda\mathcal{N}_\mu \ . \qquad (4.19)$$

As a further set of transformations one can evaluate the behavior under integral shifts n^κ of $\tilde{a}^\kappa$. One finds that

$$\mathcal{N}_\kappa \ \to \ \mathcal{N}_\kappa + 2\pi i\, \mathbf{f}_{\kappa\lambda}n^\kappa \ , \qquad T_0 \ \to \ T_0 - 2\pi n^\kappa \mathcal{N}_\kappa - 2\pi^2 i\, \mathbf{f}_{\kappa\lambda}n^\kappa n^\lambda \ . \ . \qquad (4.20)$$

Note that the non-trivial shifts (4.19) and (4.20) for T_0 are expected from an analysis of the M5-brane action [63, 64].

The transformation properties (4.6), (4.19) and (4.20) can be used to constrain the couplings. In particular, a subgroup $H \subset Sp(2r,\mathbb{Z})$ might provide an actual symmetry group of the four-dimensional gauge theory. This group has to be determined by studying the monodromies of the torus fibration (4.9) over the complex structure moduli space of $\hat{X}_4$. In other words, since $f_{\kappa\lambda}$ depends on the complex structure deformations of $\hat{X}_4$, a symmetry of $\mathcal{M}^{\mathrm{cs}}$ can induce an H action on the gauge-fields. Of particular interest are the monodromy symmetries (2.46) of $\mathcal{M}^{\mathrm{cs}}$. Using intuition from the orientifold limes one expects that there exist Calabi-Yau fourfolds with a natural lattice embedding

$$H^3(\hat{X}_4,\mathbb{Z}) \ \hookrightarrow \ H^4(\hat{X}_4,\mathbb{Z}) \ . \qquad (4.21)$$

This implies an action of the monodromy group (2.46) of $\mathcal{M}^{\mathrm{cs}}$ on the gauge fields from the R-R sector such that $H \subset G^{\mathrm{sym}}$. This would be reminiscent of an underlying $\mathcal{N}=2$ theory. However, in contrast to the $\mathcal{N}=2$ theory the Kähler moduli sector discussed in sections 2 and 3 can directly correct the gauge coupling functions in this $\mathcal{N}=1$ setting. Nevertheless, as we show in a forthcoming publication, it is interesting to explore the link between the geometry of the complex structure moduli space $\mathcal{M}^{\mathrm{cs}}$ and the $\mathcal{N}=1$ gauge theory of this section.

[12] Note that as in [65] we have shifted the imaginary part c_0 to $\mathrm{Im}\tilde{T}_0 = c_0 - \frac{1}{2}\mathrm{Re} f^{\kappa\lambda}\mathrm{Im}\mathcal{N}_\kappa\mathrm{Re}\mathcal{N}_\lambda$ claiming that this $\tilde{T}_0$ is the correct invariant combination.

One can now proceed as in refs. [63, 64, 65] and constrain the form of the last term in three-dimensional superpotential (3.23) as

$$W^{\mathrm{M}}(z,\mathcal{N},T_0) = \tilde{\mathcal{C}} \cdot \Theta(\mathbf{f},\mathcal{N}) e^{-T_0}\ , \tag{4.22}$$

where $\tilde{C}$ can still be a holomorphic function in $z^{\mathcal{K}}$. The key point to note is that $\Theta(\mathbf{f},\mathcal{N})$ transforms under (4.4) and the integral shifts of $(a^\kappa, \tilde{b}_\kappa)$ such that it cancels the shifts of T_0. This allows to identify $\Theta(\mathbf{f},\mathcal{N})$ to be a Jacobi form [68]. It is an interesting task to determine explicitly the form of $\Theta(\mathbf{f},\mathcal{N})$ for a given Calabi-Yau fourfold with $h^{2,1}(B_3) > 0$. Moreover, it would be interesting to investigate a four-dimensional interpretation of (4.22), by recalling that the leading term in T_0 is the action of a Taub-NUT gravitational instanton [57].[13]

Let us end this section by giving one simple example of a Calabi-Yau fourfold in which the above formalism can be applied. Namely, one can consider the elliptic fibration over the cubic hypersurface B_3 in $\mathbb{P}^4$. The cubic is a Fano threefold with $h^{2,1}(B_3) = 5$, $h^{1,1}(B_3) = 1$. The corresponding elliptically fibered Calabi-Yau fourfold X_4 is constructed as complete intersection as summarized in appendix A of ref. [21]. Its Hodge data are

$$h^{1,1}(X_4) = 2\ , \quad h^{2,1}(X_4) = 5\ , \quad h^{3,1} = 1483\ , \tag{4.23}$$

where the two $(1,1)$-forms correspond to the hyperplane class in B_3 and the fiber of X_4. Note that X_4 is generically non-singular, so that no non-Abelian gauge symmetry arises from space-time filling seven-branes.[14] The cubic threefold and its intermediate Jacobian $H^{2,1}(B_3)/H^3(B_3,\mathbb{Z})$ has been studied in detail in ref. [69]. In this case only the dependence on the complex structure moduli of B_3 was included. It would be interesting to extend this analysis to the whole fourfold X_4. Moreover, such an analysis is particularly interesting in the case that the R-R gauge theory can be traced through heterotic F-theory duality. It was argued in refs. [70, 71, 72] that the $U(1)$'s in F-theory arise in the heterotic dual as gauge fields on heterotic M5-branes wrapped on a curve $\mathcal{C}$ in the heterotic compactification manifold.[15] We hope to report on progress along these lines in a future publication.

5 On matter couplings on seven-branes

In this section we discuss how certain matter couplings on seven-branes are encoded in the $\mathcal{N}=1$ F-theory effective action. In subsection 5.1 we review briefly aspects of the local

[13] See refs. [66, 67], for recent progress on F-theory instantons.
[14] By tuning the complex structure non-trivial gauge-theories arise as discussed in [21].
[15] See refs. [73, 50] for recent discussions on this duality.

seven-brane world volume theory. Adjoint matter encoding deformations of the seven-branes is discussed in subsection 5.2, where we also present a detailed determination the D-term supersymmetry conditions. Adjoint Wilson line moduli and scalars from the Type IIB R-R and NS-NS two-forms are discussed in subsection 5.3.

5.1 Seven-brane world volume theory

In a local analysis of the world-volume theory on a stack of seven-branes wrapped on $\mathcal{S}$ it was shown in ref. [13, 8, 9] that the zero modes can be obtained by solving eight-dimensional F- and D-term equations.

To make this more explicit, we first specify a background seven-brane configuration extracted from X_4^{sing} or $\hat{X}_4$. Recall from section 3.1 that the local gauge group at a point p in $\mathcal{S}$ is determined by the ADE degeneration of the elliptic fibration X_4^{sing} at p. Let us denote by G^{max} the maximal local gauge group which appears when considering all points on $\mathcal{S}$. We will consider in the following cases where all other points of $\mathcal{S}$ have local gauge groups inside G^{max}. The actual physical gauge group G is obtained at generic points in $\mathcal{S}$ where no further enhancement takes place. Further enhancements will arise over complex *matter curves* obtained at the intersection of $\mathcal{S}$ with the locus Δ' given in (2.3), as well as *Yukawa points* on $\mathcal{S}$ where matter curves meet. This information can be explicitly extracted for a given X_4^{sing}, or $\hat{X}_4$. To specify the background of the local field theory on the seven-brane, this data is conveniently encoded by a $(2,0)$ form $\langle\varphi\rangle$ on $\mathcal{S}$. In the simplest case, when the breaking of G^{max} over the matter curves can be captured by a vev in the Cartan subalgebra $\mathfrak{h}_{\text{max}}$ to G^{max} one has

$$\langle\varphi\rangle \;\in\; \mathfrak{h}_{\text{max}} \otimes K\mathcal{S}\ , \tag{5.1}$$

which is specified for a fixed complex structure of X_4^{sing}. Each non-trivial entry of $\langle\varphi\rangle$ specifies a breaking of G^{max}. Due to a minimal gauge group G on $\mathcal{S}$ one has at least $rk(G)$ vanishing entries in the background configuration. Moreover, note that over the matter curves additional vanishing conditions apply, which ensure that the gauge group locally enhances further. Further data specifying the background are given by $\langle A\rangle$, the background value of the gauge field A on $\mathcal{S}$. The four-dimensional effective theory is computed for the fluctuations φ' and A' around such a background configuration, i.e. one expands

$$\varphi = \langle\varphi\rangle + \varphi'\ , \qquad A = \langle A\rangle + A'\ . \tag{5.2}$$

At low energies only the zero-modes for φ', A' will appear. These are determined by solving the eight-dimensional F- and D-term equation expanded around the background as we recall next.

Let us denote by $\mathcal{A}$ the $(1,0)$-part of the seven-brane gauge field A in a fixed complex structure of X_4^{sing}. The eight-dimensional F- and D-term equations in this complex structure are given by

$$\text{8d F-term :} \qquad \bar{\partial}_{\bar{\mathcal{A}}}\varphi = 0 \ , \qquad F^{0,2} = 0 \ , \tag{5.3}$$

$$\text{8d D-term :} \qquad \omega \wedge F + \tfrac{i}{2}[\bar{\varphi}, \varphi] = 0 \ , \tag{5.4}$$

where $\omega = cJ_{\text{b}}|_{\mathcal{S}}$ with c constant on $\mathcal{S}$ as determined below, $F = dA + A \wedge A$, and $\partial_{\mathcal{A}} = \partial + \mathcal{A}\wedge$ is the gauge-covariant derivative on $\mathbb{M}_{3,1} \times \mathcal{S}$. Note that the F-term equations admit a gauge invariance

$$\bar{\mathcal{A}} \ \to \ g^{-1}\bar{\mathcal{A}}g + g^{-1}\bar{\partial}g \ , \qquad \varphi \ \to \ g^{-1}\varphi g \ . \tag{5.5}$$

In order to extract the light modes appearing in the four-dimensional effective action one first expands the F- and D-term equations to linear order in the fluctuations φ', A' as

$$\bar{\partial}_{\langle A \rangle}\bar{\mathcal{A}}' = 0 \ , \qquad \bar{\partial}_{\langle A \rangle}\varphi' + [\bar{\mathcal{A}}', \langle\varphi\rangle] = 0 \ , \tag{5.6}$$

$$\omega \wedge \partial_{\langle A \rangle}\bar{\mathcal{A}}' + \tfrac{i}{2}[\langle\bar{\varphi}\rangle, \varphi'] = 0 \ . \tag{5.7}$$

It was argued in ref. [13] that finding the zero modes for the F-term equations (5.6), can be studied by determining the cohomology to a particular differential operator. Each class is independent of the Kähler form due to the gauge invariance (5.5). The linearized D-term conditions (5.7) then determine a specific representative in a cohomology class. Of course, that is similar to, for example, the standard Kaluza-Klein Ansatz for the M-theory three-form (2.22), (3.13) and (4.10), where the expansion forms are representing cohomology classes and are independent of the Kähler moduli. Despite such a cohomology theory it is not immediately clear how to include the zero modes A', φ' into a Kaluza-Klein reduction. One of the complications is the coupling of φ' and A' zero modes in the second equation of (5.6).

In the following we will restrict to the simplest situation and consider the special case of matter φ' and A' transforming in the adjoint of G, and set $\langle A \rangle = 0$. Since the background $\langle\varphi\rangle$ has zero entries along the adjoint of G on $\mathcal{S}$, the conditions (5.6) reduce to $\bar{\partial}\bar{\mathcal{A}}' = 0$ and $\bar{\partial}\varphi' = 0$. These conditions are satisfied if $\bar{\mathcal{A}}', \varphi'$ are expanded in a basis of $H^{0,1}(\mathcal{S})$ and $H^{2,0}(\mathcal{S})$ with coefficients transforming in the adjoint representation. In the remaining two subsections we study this Kaluza-Klein Ansatz in an F- and M-theory reduction.

5.2 Adjoint matter on the seven-brane world volume

In this subsection we discuss the inclusion of adjoint matter φ' localized on the world volume of a seven-brane with gauge group G. In the four-dimensional effective theory φ' is expanded into zero modes ρ_ν forming a basis of $H^{2,0}(\mathcal{S})$, with coefficients being the four-dimensional matter fields φ'^ν. The degrees of freedom captured by φ can be turned into additional complex structure deformations if one moves from the resolved Calabi-Yau fourfold $\hat{X}_4$ to the deformed Calabi-Yau fourfold X_4 as in (2.4). This suggests that at leading order the deformations captured by φ' appear in the Kähler potential as

$$K^{\rm cs}(z,\bar{z},\varphi',\bar{\varphi}') = -\log\big[\int_{\hat{X}_4}\Omega\wedge\bar{\Omega} - \int_{\mathcal{S}}\mathrm{Tr}(\varphi'\wedge\bar{\varphi}')\big]\ , \tag{5.8}$$

where the trace is in the adjoint of G. In the following we compute the four-dimensional D-term induced via the minimal coupling to adjoint matter and a non-trivial background flux.

One first notes that there is now charged adjoint matter φ' coupling via the covariant derivative $D\varphi'^\nu = d\varphi'^\nu + [A_4,\varphi'^\nu]$. To compute the D-term arising due to this gauging, we recall some general facts about supergravity theories with four supercharges. Denoting by M^I all complex scalars in chiral multiplets, the Killing vector $X^I_i(M,\bar{M})$ of vectors A^i appear in the minimal coupling $DM^I = dM^I + iX^I_i A^i$. The D-term for the vector multiplet with A^i is now evaluated using the general four-dimensional supergravity identity [38]

$$\partial_{M^I} D_i = K_{I\bar{J}}\bar{X}^{\bar{J}}_i\ , \qquad K_{I\bar{J}} = \partial_{M^I}\partial_{\bar{M}^J}K\ . \tag{5.9}$$

One thus has to evaluate the Killing vector $X^{\varphi^\nu}_{A_4}$ and the derivative of the Kähler potential for φ'^ν. Using (5.8) one finds at leading order

$$X^{\varphi^\nu}_{A_4} = -i[\cdot\,,\varphi'^\nu]\ , \qquad \partial_{\varphi'^\nu}\partial_{\bar{\varphi}'^\mu}K^{\rm cs} = \frac{\int_{\mathcal{S}}\rho_\nu\wedge\bar{\rho}_\mu}{\int_{X_4}\Omega\wedge\bar{\Omega}}\ . \tag{5.10}$$

This yields the leading order D-term

$$D^{\varphi'}_G = \frac{i}{\int_{X_4}\Omega\wedge\bar{\Omega}}\int_{\mathcal{S}}[\varphi',\bar{\varphi}']\ . \tag{5.11}$$

Let us next include the G_4 flux. The scalar identity (5.9) reduces trivially to three space-time dimensions and can be directly evaluated for the M-theory reduction [30, 74]. One focuses on the following terms in the reduction of eleven-dimensional supergravity action

$$S_{CS} = -\frac{1}{12}\int C_3\wedge G_4\wedge G_4 = -\frac{1}{2}\int_{\mathbb{M}_{2,1}}A^i\wedge dA^\alpha\int_{\hat{X}_4}\omega_\alpha\wedge \mathrm{w}_i\wedge G_4 + \dots\ , \tag{5.12}$$

where we inserted the expansion (3.13) for C_3 and one of the $G_4 = dC_3$. The integral over $\hat{X}_4$ arises due to the non-trivial flux. One realizes that this provides an additional coupling involving the vector multiplets A^i. Recall that the A^i lift to four-dimensional vectors, while the A^α lift to four-dimensional two-forms and are dualized into scalars $\mathrm{Im}\, T_\alpha$, via the Legendre transform (3.28). Hence, the term (5.12) is of Stückelberg type and induces a gauging of T_α via the covariant derivative

$$DT_\alpha = dT_\alpha + iX_{\alpha i}A^i \ , \qquad X_{\alpha i} = \frac{1}{2}\int_{\hat{X}_4} \omega_\alpha \wedge \mathrm{w}_i \wedge G_4 \ , \tag{5.13}$$

where $X_{\alpha i}$ is the flux-dependent Killing vector of the gauged shift symmetry. Since $X_{\alpha i}$ is independent of the chiral multiplets one can integrate (5.9) to $D_i = K_{T_\alpha} X_{\alpha i}$. Using $K_{T_\alpha} = -\frac{1}{2}L^\alpha$ for $K = \mathbf{K}^{\mathrm{M}}$ as given in (3.29), one finds

$$D_i = -\frac{1}{4}L^\alpha \int_{\hat{X}_4} \omega_\alpha \wedge \mathrm{w}_i \wedge G_4 = \frac{1}{4} C_{ij}\, L^\alpha \int_{\mathcal{S}} \omega_\alpha \wedge F^j_{\mathrm{flux}} \ , \tag{5.14}$$

where we inserted the G_4 flux given in (3.12), used the identity (3.10), and applied Poincaré duality. It is worthwhile noting that the calculation of this D-term did not depend on the precise form of the Kähler potential, since $K_{T_\alpha} = -\frac{1}{2}L^\alpha$ is fixed by the Legendre transform (3.27), (3.28) and corrections to $\tilde{K}^{\mathrm{M}}$ will alter the definition of T_α rather then K_{T_α}. The D-term potential is then evaluated as

$$V_D = \tfrac{1}{2}\mathrm{Re} f^{-1\, ij} D_i D_j \ , \tag{5.15}$$

where $\mathrm{Re} f_{ij}$ is the real part of the holomorphic gauge coupling function of the seven-branes on $\mathcal{S}$ given in (3.34), (3.37). It is not hard to check that this potential term precisely arises from the dimensional reduction of the term $\int G_4 \wedge * G_4$ appearing in the eleven-dimensional supergravity theory. In the F-theory lift, one thus combines (5.11) and the lifted version of (5.14). This yields the D-term for a non-Abelian group G on $\mathcal{S}$:

$$D_G = \frac{1}{4\mathcal{V}_{\mathrm{b}}} \int_{\mathcal{S}} J_{\mathrm{b}} \wedge F_{\mathrm{flux}} + \frac{i}{\int_{X_4} \Omega \wedge \bar{\Omega}} \int_{\mathcal{S}} [\varphi', \bar{\varphi}'] \ . \tag{5.16}$$

Note that this agrees with the expression (5.4) obtained in ref. [9] for a seven-brane decoupled from gravity if one includes the pre-factors $\mathcal{V}_{\mathrm{b}}$ and $\int_{X_4} \Omega \wedge \bar{\Omega}$.

5.3 Wilson lines and R-R and NS-NS two-form moduli

In this subsection we will include the degrees of freedom corresponding to Wilson line moduli on the seven-branes on a divisor $\mathcal{S}$ as well as the moduli from the Type IIB R-R and NS-NS two-forms.

The Wilson line degrees of freedom are present if the Hodge-numbers $h^{1,0}(\mathcal{S})$ are non-zero. More generally they can arise for a non-trivial flux background, or be localized along matter curves as briefly discussed in subsection 5.1. Here we will focus on the simplest case where the Wilson lines transform in the adjoint of G and arise as $h^{1,0}(\mathcal{S})$ zero modes. In the Kaluza-Klein zero mode expansion of the seven-brane gauge fields A the complex Wilson line scalars N^b appear as

$$A|_{\text{scalar}} = \mathcal{A} + \bar{\mathcal{A}} = \bar{\mathrm{N}}^b\,\gamma_b + \mathrm{N}^b\,\bar{\gamma}_b\ , \tag{5.17}$$

where γ_b is a basis of $H^{1,0}(\mathcal{S})$. Note that we have used the split of the first cohomology group on $\mathcal{S}$ by using the induced complex structure from $\hat{X}_4$. One can introduce a basis $(\hat{\alpha}_a, \hat{\beta}^b)$ of $H^1(\mathcal{S},\mathbb{Z})$ and write

$$\gamma_b = \hat{\alpha}_b - i\bar{f}_{bc}\hat{\beta}^c\ , \tag{5.18}$$

for $f_{cb}(z)$ being a holomorphic function in the complex structure moduli $z^{\mathcal{K}}$ of $\hat{X}_4$. Note that (5.17) and (5.18) are consistent with the holomorphicity properties of the superpotential induced in the presence of a non-Abelian gauge theory. In was argued in refs. [9, 8], that the four-dimensional flux superpotential (2.47) is corrected by $W_{\text{brane}} = \int_{\mathcal{S}} \mathrm{Tr}(F\wedge\varphi)$. Thus, the effective four-dimensional superpotential is given by

$$W^{\mathrm{F}} = \int_{\hat{X}_4} G_4 \wedge \Omega(z) + \int_{\mathcal{S}} \mathrm{Tr}(F_{\text{flux}} \wedge \varphi') + Y_{ab\nu}\mathrm{Tr}(\mathrm{N}^a\mathrm{N}^b\varphi'^{\nu})\ , \tag{5.19}$$

with Yukawa couplings

$$Y_{ab\nu}(z) = \int_{\mathcal{S}} \bar{\gamma}_a \wedge \bar{\gamma}_c \wedge \rho_\nu\ . \tag{5.20}$$

Inserting (5.18) one sees that $Y_{ab\nu}$ is holomorphic in the complex structure deformations for a holomorphically varying basis ρ_ν of $(2,0)$ forms on $\mathcal{S}$. By using the duality with the M-theory set-up, we discuss in the following how the Wilson line moduli N^a appear in the Kähler potential.

To begin with, let us note that the base B_3 has $h^{1,0}(B_3) = 0$, such that the one-forms in $H^{1,0}(\mathcal{S})$ have to be trivial in B_3. On the level of the Poincaré dual three-cycles $\mathcal{A}_3^b \in H_3(\mathcal{S})$ this implies that $\mathcal{A}_3^a$ has to be trivial in B_3. Hence there exist four-chains which admit $\mathcal{A}_3^a$ as boundaries. In the Calabi-Yau fourfold $\hat{X}_4$, one has to check whether or not the singular elliptic fibration makes the four-chains into five-cycles in $H_5(\hat{X}_4)$. This can happen since the generic elliptic fiber admits one-cycles which pinch on the location of the seven-branes. This construction is natural for $\mathcal{S}$, since one has more than one brane in the presence of a non-Abelian gauge group. These branes have been moved apart once one blows up the singularity introducing new divisors $\hat{D}_i$ and corresponding two-forms w_i. Using the complex structure of $\hat{X}_4$ to split $H^{1,0}(\mathcal{S}) \oplus H^{0,1}(\mathcal{S})$ this yields a map $H^{1,0}(\mathcal{S}) \hookrightarrow H^{2,1}(\hat{X}_4)$. Locally, one can write these $(2,1)$ forms as

$$\Psi_{a\,i} \cong \gamma_a \wedge \mathrm{w}_i, \tag{5.21}$$

where w_i are $(1,1)$ forms on $\hat{X}_4$ already introduced above. The local expressions for $\Psi_{a\,i}$ might patch together to harmonic forms representing elements of $H^{2,1}(\hat{X}_4)$. The Wilson line scalars $\mathrm{N}^{a\,i}$ in the M-theory reduction carry an extra index labeling the seven-brane. Recall that in section 4.2 we discussed the $h^{2,1}(B_3)$ vectors which correspond to scalars $\mathcal{N}_\kappa$ in three dimensions. Here we will set $h^{2,1}(B_3) = 0$ and show that the remaining fields in $H^{2,1}(\hat{X}_4)$ correspond to complex scalars N^a in four dimensions. We will discuss the general case in appendix C, where we include the full split

$$N^I = (\mathcal{N}_\kappa, \mathrm{N}^a) \ , \tag{5.22}$$

with $\mathcal{N}_\kappa$ descending to vectors and N^a descending to scalars in the F-theory lift.

There is also a second set of scalars which is counted by elements in $H^{2,1}(\hat{X}_4)$. Namely, also degrees of freedom from the R-R and NS-NS two-form C_2, B_2, combined into a complex G_2 in (2.11), can patch together and yields scalar fields in the dimensional reduction. To make this more precise, let us introduce the $\nu^{(\kappa)}$ complex one-forms on the elliptic fiber

$$\nu^{(\kappa)} = \tfrac{i}{2}(\mathrm{Im}\tau)^{-1}(dx + \bar{\tau} dy) \ . \tag{5.23}$$

These are defined in a patch $\mathcal{U}_\kappa$ on B_3 labeled by κ. On the overlap $\mathcal{U}_\kappa \cap \mathcal{U}_\lambda$ of two such patches these one-forms can transform with an $Sl(2,\mathbb{Z})$ transformation as evaluated using (2.12). We denote by $C_2^{(\kappa)}, B_2^{(\kappa)}$ the two-forms in the patch $\mathcal{U}_\kappa$. Locally one can now introduce a three-form

$$B_2^{(\kappa)} \wedge dx + C_2^{(\kappa)} \wedge dy = G_2^{(\kappa)} \wedge \nu^{(\kappa)} + \bar{G}_2^{(\kappa)} \wedge \bar{\nu}^{(\kappa)} \ . \tag{5.24}$$

This three-form transforms invariantly when moving from patch to patch on the base B_3, and hence can lead to zero modes which have to be included in an F-theory reduction.

Clearly, the expression (5.24) will be part of the M-theory three-form when describing F-theory via the M-theory lift. Hence, we have to perform a reduction very similar to the one of section 4.2. We thus expand C_3 as

$$C_3 = A^{\mathcal{A}} \wedge e_{\mathcal{A}} + a^a\, \alpha_a + b_a\, \beta^a = A^{\mathcal{A}} \wedge e_{\mathcal{A}} + \bar{\mathrm{N}}^a\, \Psi_a + \mathrm{N}^a\, \bar{\Psi}_a \ , \tag{5.25}$$

where (a^a, b_a) are three-dimensional real scalars which combine into complex scalars N^a. The three-forms (α_a, β^a), $a = 1, \ldots, h^{2,1}(\hat{X}_4)$ in (5.25) comprise a basis of $H^3(\hat{X}_4, \mathbb{Z})$. In contrast to the basis $(\alpha_\kappa, \beta^\kappa)$ introduced in section 4.2, the basis (α_a, β^a) is not canonically symplectic, since it is generically not supported only on one divisor in $\hat{X}_4$. However, if N^a correspond exclusively to Wilson line degrees of freedom on $\mathcal{S}$, one can make contact with the discussion from the begin of this subsection. The Wilson lines $\mathrm{N}^{a\,i}$ are labeled with an extra index i counting the number of branes on $\mathcal{S}$. The corresponding $(2,1)$

3-dim multiplet	4-dim F-theory	
N^a	$h^{2,1}(\hat{X}_4) - h^{2,1}(B_3)$ chiral multiplet	Wilson line scalars
		B_2, C_2 scalars

Table 5.1: *Chiral multiplets arising in the F-theory lift of complex scalars in C_3.*

forms $\Psi_{a\,i}$ are near $\mathcal{S}$ of the form (5.21). However, in the following we will keep the analysis general and only comment on the Wilson line case at the end of the section.

As in section 4.2 the complex structure defining N^a is induced by the complex structure on $\hat{X}_4$, as captured by the $(2,1)$-forms Ψ_a. However, one now choses a basis to expand Ψ_a as

$$\Psi_a = \alpha_a - i\bar{f}_{ab}\beta^b \ , \qquad \mathrm{Im}\Psi_a = -\mathrm{Re}f_{ab}\beta^b \ , \tag{5.26}$$

for a holomorphic function f_{ab} of the complex structure moduli $z^{\mathcal{K}}$. These are the analogs of the forms introduced in (5.18). As an immediate consequence of (5.25) and (5.26) one concludes that

$$\mathrm{N}^a = \tfrac{1}{2}\mathrm{Re}f^{ab}(\bar{f}_{ab}\, a^b + i b_a) \ , \tag{5.27}$$

with $\mathrm{Re}f^{ab} \equiv (\mathrm{Re}f_{ab})^{-1}$ being the inverse of the real part of f_{ab}. The couplings of the fields N^a are, as in (4.14), captured by the complex structure dependent function

$$d_{\mathcal{A}a\bar{b}} = i \int_{\hat{X}_4} \omega_{\mathcal{A}} \wedge \Psi_a \wedge \bar{\Psi}_b \ . \tag{5.28}$$

Note that in contrast to the case of section 4.2 the Ψ_a do not have all indices on B_3 since we have assumed $h^{2,1}(B_3) = 0$. Hence, the only non-vanishing couplings are actually $d_{\alpha a\bar{b}}$, since only ω_α has only indices on the base B_3. Hence, a computation similar to the one leading to (4.16), yields the kinetic potential

$$\tilde{K}^{\mathrm{M}} = \tilde{K}^{\mathrm{M}}_o(R, L, \xi) + L^\alpha d_{\alpha a\bar{b}}(z,\bar{z})\, \mathrm{ReN}^a \mathrm{ReN}^b + \dots \ , \tag{5.29}$$

where $\tilde{K}^{\mathrm{M}}_o(R, L, \xi)$ is the original kinetic potential independent of the fields N^a, and L^α are the scalars in the three-dimensional vector multiplets (A^α, L^α) which dualize to four-dimensional complex scalars T_α.

In considering the F-theory lift to four dimensions one realizes that (5.29) cannot be the complete correction. To see this, one notes that the real part of the holomorphic gauge coupling function f_{ij} has to obey (3.34). Since the corrections term in (5.29) modifies the definition of the coordinates T_α, holomorphy of the four-dimensional gauge coupling implies that that (5.29) is missing a correction proportional to ξ^2. To ensure

(3.35) with $\tilde{f} = 0$, one finds that the correct expression for $\tilde{K}^{\rm M}$ is

$$\tilde{K}^{\rm M} = \tilde{K}^{\rm M}_o(R, L, \xi) + (L^\alpha - R^{-1}\, C^\alpha_{ij}\, \xi^i \xi^j)\, d_{\alpha a \bar{b}}(z, \bar{z})\, {\rm ReN}^a {\rm ReN}^b\ . \tag{5.30}$$

Note that the new term in $\tilde{K}^{\rm M}$ which is linear in L^α is removed by the Legendre transform to coordinates T_α and kinetic potential $\mathbf{K}^{\rm M}$ as in (3.28) and (3.31),(3.34). More precisely, one has

$$T_\alpha = \partial_{L^\alpha} \tilde{K}^{\rm M}_o(R, L, \xi) + d_{\alpha a \bar{b}}(z, \bar{z})\, {\rm ReN}^a {\rm ReN}^b + i\rho_\alpha\ , \tag{5.31}$$

and

$$\mathbf{K}^{\rm M} = \log R + K^{\rm F} - \frac{1}{2R} {\rm Re} T_{S'}\, \xi^i \xi^j + \mathcal{O}(R)\ . \tag{5.32}$$

In this expression $K^{\rm F}$ is readily evaluated from the original $K^{\rm F}_o$ by replacing $T_\alpha \to T_\alpha + d_{\alpha a \bar{b}} {\rm ReN}^a {\rm ReN}^b$. It is straightforward to evaluate this expression for the $\tilde{K}^{\rm M}_o$ given in (3.18).

In the F-theory lift of the expressions (5.32) one uses again the fact that gauge fields descent to a non-Abelian theory removing the indices i, j and replacing the gauge coupling function as in (3.38). Moreover, if all ${\rm N}^a$ correspond to Wilson line moduli ${\rm N}^{a\,i}$ the expression (5.31) is lifted to

$${\rm Re} T_\alpha = \partial_{L^\alpha} \tilde{K}^{\rm M}_o(R, L, \xi) + d_{\alpha a \bar{b}} {\rm Tr}({\rm ReN}^a\, {\rm ReN}^b)\ , \tag{5.33}$$

where the four-dimensional ${\rm N}^a$ transform in the adjoint of G, and we have inserted

$$d_{\alpha a \bar{b}} = i \int_{\mathcal{S}} \omega_\alpha \wedge \gamma_a \wedge \bar{\gamma}_b\ , \tag{5.34}$$

as can be evaluated using (5.21), (5.28), and an identity similar to (3.10). It is not hard to check that this lift leads to the correct local expression for the kinetic terms of the Wilson line moduli studied in [9]. Furthermore, (5.32) together with (5.33) yields the correct expression in the orientifold limes [60] and orbifold limits [75, 2].

6 Conclusions

In this paper we studied the four-dimensional effective action of F-theory compactified on an elliptically fibered Calabi-Yau fourfold X_4. Many of the important equations and results of this paper are summarized in appendix A. The main tool was to use a non-trivial scaling limit to lift the three-dimensional supergravity theory obtained by compactification of M-theory on X_4. We have shown explicitly how the massless Kaluza-Klein modes arising in the reduction of the M-theory three-form and the Kähler form along the elliptic fiber of X_4 encode the massless metric degrees of freedom along an

S^1 used in a compactification from four to three dimensions. The guiding principle to formulate the M- to F-theory lift has been to demand finiteness of M5-brane actions wrapped on vertical divisors. An M5-brane wrapped on the base B_3 has infinite action in the F-theory limit as it scales as the square of the S^1 radius.

The M- to F-theory lift can be extended to singular elliptic fibrations if the singularity has been resolved by blow-up yielding a fourfold $\hat{X}_4$. The degrees of freedom induced by the new blow-up forms correspond to the massless degrees of freedom in the reduction of four-dimensional $U(1)$ vectors. The gauge-group enhances to a non-Abelian group due to massless M2-branes on resolving cycles of vanishing volumes. In the four-dimensional F-theory picture this corresponds to a stack of seven-branes with non-Abelian gauge group on their world-volume. We considered the F-theory limit using the large volume expressions for the resolved Calabi-Yau fourfold $\hat{X}_4$ to compute the leading gauge-coupling function for the seven-branes. It is of interest to study the corrections to these expressions. In particular, the splitting of the gauge-couplings in the orientifold limit observed in ref. [77], has to be investigated in the M- to F-theory lift. However, it is crucial to note that these corrections depend on the dilaton-axion τ, and hence are expected to be generalized via a holomorphic function of the complex structure moduli as in section 5.

From the M-theory reduction we were able to extract the four-dimensional Kähler potential and gauge coupling functions. In three dimensions both are encoded by a single function, the kinetic potential $\mathbf{K}^{\mathrm{M}}$, which has to be expanded around the F-theory limit in the Kähler moduli space. The expressions found in the F-theory reduction specialize to the results found in the orientifold limit in refs. [40, 60]. It will be crucial to examine corrections to these leading order results. While the KKLT scenario [78] of stabilizing Kähler moduli has an implementation in F-theory, it remains to be shown how the large volume compactifications of [79] can be realized. This is again due to the fact that the corrections used in [79] dependent on the dilaton-axion and lift non-trivially to F-theory.

In addition to the seven-brane gauge theories we have also investigated the gauge dynamics of the Abelian gauge theory arising in the reduction of the R-R four-from. Massless vectors arise in the F-theory reduction if the base of the fourfold $\hat{X}_4$ admits harmonic three-forms. The gauge-coupling function is determined by a holomorphic function $f_{\kappa\lambda}(z)$ encoding the fibration of a torus bundle $\mathbb{T}$ over the complex structure moduli space of $\hat{X}_4$. It is an interesting task to compute this holomorphic function explicitly for Calabi-Yau fourfold examples. There will be special loci in the complex structure moduli space at which this Abelian theory enhances to a non-Abelian gauge group, just as in Seiberg-Witten theory. To study the R-R gauge theory at various points in open-closed complex structure moduli space might yield new insights for $\mathcal{N} = 1$ gauge theories. In particular, it will be interesting to explore how the singularities for non-

Abelian seven-branes are seen by the R-R gauge theory.

In the last section we have focused on matter corresponding to seven-brane deformations and Wilson line moduli which transform in the adjoint of the non-Abelian gauge group. Again we were able to determine their couplings in the $\mathcal{N} = 1$ Kähler potential. This allowed us to derive the matter D-term including the flux correction. The Wilson line moduli non-trivially correct the Kähler moduli. We argued that in M-theory these fields arise as zero modes along harmonic three-forms with one leg on the seven-brane surface $\mathcal{S}$. It will be a crucial task to extend the analysis to localized charged matter arising at intersections of the seven-brane on $\mathcal{S}$ with other seven-branes. These fields can be chiral if fluxes on the world-volume are turned on. It is conceivable that the results of section 5 naturally extend to this case if one considers fields φ', A' transforming in the adjoint of $G^{\max}$ introduced in this section. However, the zero mode conditions appear to mix contributions from the two sectors, and it will be interesting to establish a formulation where this mixing is canonically captured. Let us conclude by noting that questions concerning moduli stabilization and chirality both require a detailed understanding of fluxes in F-theory, and the full picture needs yet to be explored.[16]

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Appendices

A Summary of results

In this appendix we summarize some of the key results of this work. Most of the details are skipped and can be found in the main text. The general results for the Kaluza-Klein reduction from four to three dimensions can be found in appendix B.

[16]For a recent discussion using heterotic/F-theory duality, see refs. [42, 73, 50, 80].

General expressions for the M-theory reduction

Considering M-theory on a resolved Calabi-Yau fourfold $\hat{X}_4$ the Kähler potential in three dimensions is

$$\begin{aligned} K^{\mathrm{M}}(z,T,N) &= K^{\mathrm{cs}}(z) - 3\log\mathcal{V} \\ &= -\log\int_{\hat{X}_4}\Omega\wedge\bar{\Omega} - 3\log\int_{\hat{X}_4'}\Omega'\wedge\bar{\Omega}' \ , \end{aligned} \tag{A.1}$$

as discussed in (2.20), (2.36) and (2.39). The quantum volume $\mathcal{V}$ is determined by mirror symmetry, where $\hat{X}_4'$ is the mirror of $\hat{X}_4$, and Ω' is the mirror $(4,0)$ form. K^{M} depends on the complex structure moduli $z^{\mathcal{K}}$ through Ω. The second term has to be evaluated on a real Lagrangian slice of dimension $h^{1,1}(\hat{X}_4) = h^{3,1}(\hat{X}_4')$. We choose real coordinates $v^{\mathcal{A}}$ to parameterize this slice. The real part of the Kähler moduli $\mathrm{Re}\, T_{\mathcal{A}}$ are defined as real parts of the mirror periods $\Pi^{(3)}_{\mathcal{A}}(v)$ corresponding to the mirror of six-cycles in $\hat{X}_4$ (see e.g. (4.18), (5.31)):

$$T_{\mathcal{A}} = \mathrm{Re}\,\Pi^{(3)}_{\mathcal{A}}(v) + d_{\mathcal{A}IJ}(v,z)N^I\mathrm{Re}N^J + i\tilde{\rho}_{\mathcal{A}} \ , \tag{A.2}$$

where $d_{\mathcal{A}IJ}$ is in general a function of $v^{\mathcal{A}}, z^{\mathcal{K}}$. Chiral multiplets N^I with axion-like $\mathrm{Im}N^I$ appear quadratically in $T_{\mathcal{A}}$. There can be other moduli corresponding to further matter multiplets. The Kähler potential (A.1) has to be evaluated as a function of $z^{\mathcal{K}}, T_{\mathcal{A}}, N^I$. In particular, one has to solve

$$2\,\mathrm{Re}\,\Pi^{(3)}_{\mathcal{A}}(v) = T_{\mathcal{A}} + \bar{T}_{\mathcal{A}} - 2\,d_{\mathcal{A}IJ}(v,z)\mathrm{Re}N^I\mathrm{Re}N^J \tag{A.3}$$

for $v^{\mathcal{A}}$ and insert the result into (A.1). If N^I transforms as an adjoint under a non-Abelian gauge group one has to replace $\mathrm{Re}N^I\mathrm{Re}N^J \to \mathrm{Tr}(\mathrm{Re}N^I\mathrm{Re}N^J)$. Depending on the type of six-cycle one has the split

$$T_{\mathcal{A}} = (T_0, T_\alpha, T_i) \ , \tag{A.4}$$

where T_0 corresponds to the base B_3, the T_α correspond to vertical divisors D_α, and the T_i are associated with resolution divisors $\hat{D}_i$ (subsection 3.1). If $\mathrm{Im}T_{\mathcal{A}}$ has a shift symmetry all $T_{\mathcal{A}}$ can be dualized to three-dimensional vector multiplets

$$(L^{\mathcal{A}}, A^{\mathcal{A}}) \ = \ \big((R, A^0)\,,\ (L^\alpha, A^\alpha)\,,\ (\xi^i, A^i)\big) \ , \tag{A.5}$$

in accord with the split (A.4). This dualization corresponds to performing a Legendre transform

$$\tilde{K}^{\mathrm{M}}(L,z,M) = K^{\mathrm{M}} - \frac{1}{2}(T_{\mathcal{A}} + \bar{T}_{\mathcal{A}})L^{\mathcal{A}} \ , \qquad \frac{\partial K^{\mathrm{M}}}{\partial T_{\mathcal{A}}} = -\frac{1}{2}L^{\mathcal{A}} \ . \tag{A.6}$$

$\tilde{K}^{\mathrm{M}}$ is a kinetic potential encoding the kinetic terms of the complex scalars $z^{\mathcal{K}}, N^I$ as well as the vector multiplets $(L^{\mathcal{A}}, A^{\mathcal{A}})$ in the three-dimensional action (3.16). It is often useful to work with $\tilde{K}^{\mathrm{M}}$ since the dependence on $L^{\mathcal{A}}$ can be much simpler as the $T_{\mathcal{A}}$-dependence of K^{M}. This is the case at large volume.

M-theory reduction at large volume

If all volumes in $\hat{X}_4$ are large the classical expressions for the mirror periods are inserted into (A.1) and (A.2):

$$\mathcal{V} = \frac{1}{4!}\int J^4\ , \qquad T_{\mathcal{A}} = \frac{1}{3!}\int_{D_{\mathcal{A}}} J^3 + d_{\mathcal{A}IJ}(z) N^I \mathrm{Re} N^J + i\tilde{\rho}_{\mathcal{A}}\ . \tag{A.7}$$

For an elliptic fibration with a resolved G-singularity over $\mathcal{S}$ one has the following conditions for the quadruple intersections:

$$D_\alpha \cdot D_\beta \cdot D_\gamma \cdot D_\delta = \hat{D}_i \cdot D_\beta \cdot D_\gamma \cdot D_\delta = 0\ , \tag{A.8}$$

$$(\hat{D}_i \cdot \hat{D}_j + C_{ij}\, S' \cdot B_3) \cdot D_\alpha \cdot D_\beta = 0\ , \tag{A.9}$$

where C_{ij} is the Cartan matrix for G. The $d_{\mathcal{A}IJ}$ in (A.7) are independent of $v^{\mathcal{A}}$ at large volume, and given by

$$d_{\mathcal{A}IJ} = i\int_{\hat{X}_4} \omega_{\mathcal{A}} \wedge \psi_I \wedge \bar{\psi}_J\ , \tag{A.10}$$

where ψ_I are $(2,1)$ forms on $\hat{X}_4$. These $(2,1)$-forms vary over the complex structure moduli space. The complex structure dependence is encoded by a holomorphic function $f_{IJ}(z)$ as in (4.11) and (5.26). The kinetic potential $\tilde{K}^{\mathrm{M}}$ at large volume is (see (3.18) and (5.29))

$$\begin{aligned}\tilde{K}^{\mathrm{M}} \;\; &= \;\; \log[\tfrac{1}{6} R L^\alpha L^\beta L^\gamma \mathcal{K}_{\alpha\beta\gamma} - \tfrac{1}{4}\xi^i \xi^j\, C_{ij}\, L^\alpha L^\beta \mathcal{K}_{S|\alpha\beta} + \mathcal{O}(R^3, \xi^3)] + K^{\mathrm{cs}}(z) \\ &\quad + L^{\mathcal{A}} d_{\mathcal{A}IJ} \mathrm{Re} N^I \mathrm{Re} N^J + \mathcal{O}(\xi^2, R^2)\ .\end{aligned} \tag{A.11}$$

Using the index structure of the $(2,1)$-forms on $\hat{X}_4$ one splits $N^I = (\mathrm{N}^a, \mathcal{N}_\kappa)$ as in (5.22), and appendix C, and has

$$L^{\mathcal{A}} d_{\mathcal{A}IJ} N^I N^J = L^\alpha d_{\alpha ab} \mathrm{N}^a \mathrm{N}^b + R {d_0}^{\kappa\lambda} \mathcal{N}_\kappa \mathcal{N}_\lambda + 2\xi^i {d_{ia}}^\kappa \mathrm{N}^a \mathcal{N}_\kappa\ . \tag{A.12}$$

The ψ^κ in ${d_0}^{\kappa\lambda}$ have all indices in the base B_3 as in (4.11). The Ψ_a are the remaining $(2,1)$ forms in (5.26).

F-theory lift, moduli matching, and the $\mathcal{N}=1$ characteristic data

The F-theory limit is a limit in Kähler moduli space given by

$$\begin{aligned} R &\to 0\ , & L^\alpha &\to L^\alpha_{\rm b}\ , & \xi^i/R &\to 0\ , \\ {\rm Re}T_0 &\to \infty\ , & T_\alpha &\to T^{\rm b}_\alpha\ , & {\rm Re}T_i &\to 0\ . \end{aligned} \tag{A.13}$$

This limit sets the background values of the fields R, ξ^i. Their variations appear in the effective action. The three-dimensional M-theory compactification expanded around this limit has to be compared with the dimensional reduction of a general four-dimensional $\mathcal{N}=1$ supergravity theory reduced to three dimensions on an S^1 of radius r. The elliptic fiber volume R is identified as $R = r^{-2}$. The three-dimensional fields lift non-trivially to four dimensions as summarized in table A.1.

3-dim. multiplet	4-dim. F-theory	
$(L^{\mathcal{A}}, A^{\mathcal{A}})$	$h^{1,1}(\hat{X}_4) - h^{1,1}(B_3) - 1$ vector multiplets	$(\xi^i, A^i) \to A^i$
	$h^{1,1}(B_3)$ chiral multiplets	$(L^\alpha, A^\alpha) \to T_\alpha$
	extra dimension	$(R, A^0) \to (g_{33}, g_{\mu 3})$
N^I	$h^{2,1}(B_3)$ vector multiplets	$\mathcal{N}_\kappa \to V^\kappa$
	$h^{2,1}(\hat{X}_4) - h^{2,1}(B_3)$ chiral multiplets	$\mathrm{N}^a \to \mathrm{N}^a$
$z^{\mathcal{K}}$	$h^{3,1}(\hat{X}_4)$ chiral multiplets	$z^{\mathcal{K}} \to z^{\mathcal{K}}$

Table A.1: *The four-dimensional F-theory spectrum without matter fields.*

Since among the $T_{\mathcal{A}}$ in (A.4) only T_α lifts to a complex scalar in four dimensions, the key object to work with is the kinetic potential

$$\mathbf{K}^{\rm M}(T_\alpha, z, N|\xi, R) = \tilde{K}^{\rm M} - \tfrac{1}{2}(T_\alpha + \bar{T}_\alpha)L^\alpha\ , \qquad {\rm Re}\, T_\alpha = \partial_{L^\alpha}\tilde{K}^{\rm M}\ . \tag{A.14}$$

This potential depends on the scalars ξ^i, R which are in vector multiplets. The $\mathcal{N}_\kappa$ lift to four-dimensional vectors, but appear as complex scalars in $\mathbf{K}^{\rm M}$. One expands $\mathbf{K}^{\rm M}$ around the F-theory limit (A.13), ${\rm Re}\mathcal{N}_\kappa \to 0$, as

$$\mathbf{K}^{\rm M} = \log R + K^{\rm F}(z, T, {\rm N})|_* + \tfrac{1}{2}\tilde{K}^{\rm M}_{\xi^i\xi^j}|_*\xi^i\xi^j + \tfrac{1}{2}\tilde{K}^{\rm M}_{{\rm Re}\mathcal{N}_\kappa {\rm Re}\mathcal{N}_\lambda}|_*{\rm Re}\mathcal{N}_\kappa{\rm Re}\mathcal{N}_\lambda + \ldots\ , \tag{A.15}$$

where $|_*$ indicates evaluation in the strict limit. There is no linear term in $\mathcal{N}_\kappa$ due to its quadratic appearance in (A.2). Linear terms in ξ^i have to be absent to ensure match with a kinetic potential obtained by dimensional reduction. Using appendix B a *general* $\mathbf{K}^{\mathrm{M}}$ obtained by reduction is

$$\mathbf{K}^{\mathrm{M}} = \log R + K(M,\bar{M}) - \tfrac{1}{2}R^{-1}\Delta_{ij}\,\xi^i\xi^j + \tfrac{1}{2}R\,\Delta^{\kappa\lambda}\,\mathrm{Re}\mathcal{N}_\kappa\mathrm{Re}\mathcal{N}_\lambda - \,\Delta_i^\lambda\xi^i\,\mathrm{Re}\mathcal{N}_\lambda\ , \quad \text{(A.16)}$$

where K is the four-dimensional Kähler potential and Δ is given in (B.10). For $\Delta_i^\lambda = 0$ one has: $\Delta_{ij} = \mathrm{Re}f_{ij}$, $\Delta^{\kappa\lambda} = \mathrm{Re}f^{\kappa\lambda}$, where f_{ij} and $f_{\kappa\lambda}$ are holomorphic gauge coupling functions. In order that T_α can arise holomorphically in f_{ij} the kinetic potential $\tilde{K}^{\mathrm{M}}$ has to satisfy

$$\big(\mathcal{C}^\alpha_{ij}\partial_{L^\alpha} + R\partial_{\xi^i}\partial_{\xi^j}\big)\tilde{K}^{\mathrm{M}} = \mathrm{Re}\,\tilde{f}_{ij}(M)\ , \qquad \text{for } R, \xi^i/R \to 0\ . \qquad \text{(A.17)}$$

Here $\tilde{f}_{ij}$ is a holomorphic function which can in general appear in the gauge-coupling function. It was found to be zero at leading order.

Since the indices i,j parameterize the blow-up divisors of a G singularity over $\mathcal{S}$, they label Cartan $U(1)'s$ in G. In the F-theory limit (A.13) the indices i run over all generators, and the gauge group enhances as $U(1)^{rk(G)} \to G$, due to light M2-branes. Similarly, one can treat matter deformations φ' and Wilson line moduli transforming in the adjoint of G, as discussed in section 5. To display the formulas, we assume that all N^a are Wilson line moduli, since otherwise we have to introduce new indices and split the set N^a. For the simple $\tilde{K}^{\mathrm{M}}$ given in (A.11) one compares (A.14) with (A.16). One then finds

$$K^{\mathrm{F}}(z,T,\varphi',\mathrm{N}) \;=\; -2\log\mathcal{V}_{\mathrm{b}} - \log\Big[\int_{\hat{X}_4}\Omega\wedge\bar{\Omega} - \int_{\mathcal{S}}\mathrm{Tr}(\varphi'\wedge\bar{\varphi}')\Big]\ , \qquad \text{(A.18)}$$

$$T_\alpha \;=\; \frac{1}{2!}\int_{D^{\mathrm{b}}_\alpha} J_{\mathrm{b}}\wedge J_{\mathrm{b}} + d_{\alpha ab}(z)\mathrm{Tr}(\mathrm{N}^a\mathrm{Re}\mathrm{N}^b) + i\tilde{\rho}_\alpha\ , \qquad \text{(A.19)}$$

together with the gauge coupling functions

$$f_G = T_{S'}\ , \qquad\qquad f^{\mathrm{RR}}_{\kappa\lambda} = f_{\kappa\lambda}(z)\ . \qquad \text{(A.20)}$$

Note that the couplings $d_{\alpha ab}(z)$, $f_{\kappa\lambda}(z)$ as well as the correction $\varphi'\wedge\bar{\varphi}'$ depend on all complex structure moduli of $\hat{X}_4$. In particular, this includes couplings to seven-brane moduli. Note that various additional corrections can be computed using this formalism. For example, we have indicated volume corrections to K^{F} in (2.37). Finally, the superpotential was given in (5.19), (5.20) and reads

$$W^{\mathrm{F}} = \int_{\hat{X}_4} G_4\wedge\Omega(z) + \int_{\mathcal{S}}\mathrm{Tr}(F_{\mathrm{flux}}\wedge\varphi') + Y_{ab\nu}\mathrm{Tr}(\mathrm{N}^a\mathrm{N}^b\varphi'^\nu)\ , \qquad \text{(A.21)}$$

while the leading order D-term is computed in section 5.2 and reads

$$D_G = \frac{1}{4\mathcal{V}_{\mathrm{b}}}\int_{\mathcal{S}} J_{\mathrm{b}}\wedge F_{\mathrm{flux}} + \frac{i}{\int_{X_4}\Omega\wedge\bar{\Omega}}\int_{\mathcal{S}}[\varphi',\bar{\varphi}']\ . \qquad \text{(A.22)}$$

B $4d \to 3d$ reduction and scalar-vector duality

In this appendix we discuss the reduction of the bosonic $\mathcal{N}=1$ action (2.9) to three space-time dimensions on a circle S^1 of radius r (see, e.g., refs. [30]). The four-dimensional metric, and the four-dimensional vectors split as

$$g^{(4)}_{\mu\nu} = \begin{pmatrix} g^{(3)}_{pq} + r^2 A^0_p A^0_q & r^2 A^0_q \\ r^2 A^0_p & r^2 \end{pmatrix} , \qquad A^\Lambda_\mu = (A^\Lambda_p + A^0_p \zeta^\Lambda, \zeta^\Lambda) , \tag{B.1}$$

where $A^0_p,\ A^\Lambda_p,\ p=0,1,2$ are vectors and ζ^Λ as well as r are scalars in three dimensions. Note that we will use the same symbol A^Λ for four- and three-dimensional vectors. However, it should be clear by the context whether we are working in four or three dimensions. Performing the Kaluza-Klein reduction of the action (2.9) and employing a Weyl rescaling to the three-dimensional Einstein frame, the three-dimensional action can be brought into the standard form

$$\begin{aligned} S^{(3)} \;\; = \;\; \int & -\tfrac{1}{2} R_3 * \mathbf{1} - \tilde{K}_{I\bar{J}}\, dM^I \wedge * d\bar{M}^J + \tfrac{1}{4} \tilde{K}_{\hat{\Lambda}\hat{\Sigma}}\, d\xi^{\hat{\Lambda}} \wedge * d\xi^{\hat{\Sigma}} \\ & -\tfrac{1}{4} \tilde{K}_{\hat{\Lambda}\hat{\Sigma}}\, F^{\hat{\Lambda}} \wedge * F^{\hat{\Sigma}} + F^{\hat{\Lambda}} \wedge \mathrm{Im}(\tilde{K}_{\hat{\Lambda} I}\, dM^I) , \end{aligned} \tag{B.2}$$

where the kinetic terms of the vectors and scalars are determined by a single real function, the kinetic potential $\tilde{K}(M^I, \bar{M}^J | \xi^{\hat{\Lambda}})$, as $\tilde{K}_{I\bar{J}} = \partial_{M^I} \partial_{\bar{M}^J} \tilde{K}$, $\tilde{K}_{\hat{\Lambda}\hat{\Sigma}} = \partial_{\xi^{\hat{\Lambda}}} \partial_{\xi^{\hat{\Sigma}}} \tilde{K}$, and $\tilde{K}_{\hat{\Lambda} I} = \partial_{\xi^{\hat{\Lambda}}} \partial_{M^I} \tilde{K}$. In order to do that, we identify

$$R = r^{-2} , \qquad \xi^{\hat{\Lambda}} = (R, R\zeta^\Lambda) , \quad A^{\hat{\Lambda}} = (A^0, A^\Lambda) . \tag{B.3}$$

It is straightforward to determine the kinetic potential $\tilde{K}$. Clearly, it will contain the four-dimensional Kähler potential $K(M^I, \bar{M}^J)$, and additional terms encoding the kinetic terms of the vector multiplets $(A^{\hat{\Lambda}}, \xi^{\hat{\Lambda}})$. Explicitly it takes the form[17]

$$\tilde{K} = K(M, \bar{M}) + \log R - \frac{1}{2R} \mathrm{Re} f_{\Lambda\Sigma}(M)\, \xi^\Lambda \xi^\Sigma . \tag{B.4}$$

It is important to stress that in three dimensions massless vectors are dual to real scalars with Peccei Quinn shift symmetries. However, in the kinetic potential (B.4) one can still distinguish the four-dimensional origin of the term by considering the power n of the R^n pre-factor. In fact, four-dimensional scalars carry no pre-factor in the $D=3$ kinetic potential and action, while three-dimensional scalars $\zeta^\Lambda \equiv \xi^\Lambda / R$ which arise from $D=4$ vector multiplets carry a pre-factor R^{-1} in (B.4).

To study the F-theory reduction it turns out to be convenient to dualize some of the vectors A^Λ in (B.3) into scalars. Note that only those vectors are dualizable which do

[17] Note that we have included a factor 1/2, by rescaling the vector fields.

not gauge any scalar fields in the effective theory. For example, in an F-theory reduction these correspond to the $U(1)$ vectors A^κ which arise in the reduction of the R-R four-form C_4 as discussed in section 4. We thus split

$$A^\Lambda = (A^\kappa, A^i) \ , \qquad \xi^\Lambda = (\xi^\kappa, \xi^i) \ , \qquad f_{\Lambda\Sigma} = (f_{ij}, f_{\kappa j}, f_{\kappa\lambda}) \ . \tag{B.5}$$

To dualize the vectors A^κ into scalars $\tilde{\xi}_\kappa$ one adds a Lagrange multiplier term $\propto F^\kappa \wedge d\tilde{\xi}_\kappa$ to the effective action (B.2) and eliminates F^κ by its equation of motion. At the level of the kinetic potential $\tilde{K}$ and coordinates this amounts to performing a Legendre transformation with respect to $\tilde{K}$. The scalars $(\xi^\kappa, \tilde{\xi}_\kappa)$ then combine into a complex coordinates $\mathcal{N}_\kappa$. More precisely, one introduces new complex coordinates

$$\mathcal{N}_\kappa = -\partial_{\xi^\kappa} \tilde{K} - i\tilde{\xi}_\kappa \ , \tag{B.6}$$

and transforms the kinetic potential as

$$\tilde{K}^{(1)}(M, \bar{M}, \mathcal{N} + \bar{\mathcal{N}} | L, \xi^i) = \tilde{K} - \tfrac{1}{2}(\mathcal{N}_\kappa + \bar{\mathcal{N}}_\kappa)\xi^\kappa \ . \tag{B.7}$$

It is important to stress that in this expression one has to evaluate ξ^κ as function of $\mathcal{N}_\kappa, M^I$ and R, ξ^i by solving (B.6).

The explicit evaluation of N_k and $\tilde{K}^{(1)}$ is straightforward since $\tilde{K}$, given in (B.4), has such a simple form in this situation. Evaluating $\partial_{\xi^\kappa}\tilde{K}$ one finds

$$\mathcal{N}_\kappa = R^{-1}\mathrm{Re} f_{\kappa\Lambda}\xi^\Lambda - i\tilde{\xi}_\kappa = f_{\kappa\Lambda}\zeta^\Lambda - i\tilde{b}_\kappa \ , \tag{B.8}$$

where we defined $\tilde{b}_\kappa = \tilde{\xi}_\kappa + \mathrm{Im} f_{\kappa\Lambda}\zeta^\Lambda$. Solving for ξ^κ and inserting the result in (B.7) yields the expression

$$\tilde{K}^{(1)} = K(M, \bar{M}) + \log R - \tfrac{1}{2}R^{-1}\Delta_{ij}\, \xi^i\xi^j + \tfrac{1}{2}R\, \Delta^{\kappa\lambda}\, \mathrm{Re}\mathcal{N}_\kappa \mathrm{Re}\mathcal{N}_\lambda - \, \Delta_i^\lambda \xi^i\, \mathrm{Re}\mathcal{N}_\lambda \ , \tag{B.9}$$

where

$$\Delta_{ij} = \mathrm{Re} f_{ij} - \mathrm{Re} f_{\kappa i}\mathrm{Re} f^{\kappa\lambda}\mathrm{Re} f_{\lambda j} \ , \qquad \Delta^{\kappa\lambda} = \mathrm{Re} f^{\kappa\lambda} \ , \qquad \Delta_i^\lambda = \mathrm{Re} f^{\kappa\lambda}\, \mathrm{Re} f_{\kappa i} \ . \tag{B.10}$$

Using this kinetic potential in (B.2) one obtains the three-dimensional effective action for the chiral multiplets with complex scalars $(M^I, \mathcal{N}_\kappa)$ and the vector multiplets (ξ^i, A^i). The four-dimensional $\mathcal{N} = 1$ Kähler potential $K(M, \bar{M})$ and gauge-kinetic coupling function $f_{\Lambda\Sigma}(M)$ are then determined comparing the F-theory kinetic potential with the general expression (B.9).

C Compactifications with general three-forms on the Calabi-Yau fourfold

In order to proceed we need to dualize some of the vector multiplets $(L^{\mathcal{A}}, A^{\mathcal{A}})$ into chiral multiplets. We will precisely pick those which correspond to divisors D_α. We thus split

$$
\begin{aligned}
L^{\mathcal{A}} &= (L^\alpha | \xi^i, R) \ , && \alpha = 1, \ldots, h^{1,1}(B_3) \\
A^{\mathcal{A}} &= (A^\alpha | A^i, A^0) \ , && i = h^{1,1}(B_3) + 1, \ldots, h^{1,1}(\hat{X}_4) - 1 \ .
\end{aligned}
\tag{C.1}
$$

Our aim is to dualize the vector multiplets (L^α, A^α) into chiral multiplets with two scalars (L^α, ρ_α) which combine into complex coordinates T_α. We also split the set of complex scalars N^I as

$$
\begin{aligned}
N^I = (\mathcal{N}_\kappa, \mathrm{N}^a) \ , \qquad & \kappa = 1, \ldots, h^{2,1}(B_3) \ , \\
& a = h^{2,1}(B_3) + 1, \ldots, h^{2,1}(\hat{X}_4) \ .
\end{aligned}
\tag{C.2}
$$

This is again done to distinguish three-dimensional multiplets which descend to chiral or vector multiplets in four dimensions. More precisely, the N^a will descend to complex scalars while the $\mathcal{N}_\kappa$ become vectors in four space-time dimensions. We summarize this spectrum in table A.1.

The chiral multiplets and their corresponding kinetic potential are obtained by a Legendre transformation similar to the analysis of section B. One thus defines the new complex coordinates

$$
T_\alpha = \partial_{L^\alpha} \tilde{K}^{\mathrm{M}} + i \rho_\alpha \ , \tag{C.3}
$$

and evaluates the effective action starting with a new kinetic potential

$$
\tilde{K}^{\mathrm{F}}(z, \bar{z}, N + \bar{N}, T + \bar{T} | \xi^i, R) = \tilde{K}^{\mathrm{M}} - \tfrac{1}{2}(T_\alpha + \bar{T}_\alpha) L^\alpha \ , \tag{C.4}
$$

such that

$$
\frac{\partial \tilde{K}^{\mathrm{F}}}{\partial T_\alpha} = -\frac{1}{2} L^\alpha \ , \qquad \frac{\partial \tilde{K}^{\mathrm{F}}}{\partial M} = \frac{\partial \tilde{K}^{\mathrm{M}}}{\partial M} \ , \qquad M \in (N^I, \xi^i, R, z) \ . \tag{C.5}
$$

Note that the right-hand sides of these expressions are evaluated by first taking derivatives of $\tilde{K}^{\mathrm{M}}$ viewed as a function of $L^{\mathcal{A}} = (L^\alpha, \xi^i, R)$ and N^I, and then use (C.3) to express the result as a function of T_α, R, ξ^i, N^I. One thus uses, for example, the identity

$$
\begin{aligned}
\partial_{\xi^i} \big[\tilde{K}^{\mathrm{M}}(T, N | \xi) \big] &= \partial_{\xi^i} \big[\tilde{K}^{\mathrm{M}}(N | \xi, L(T, \xi, N)) \big] \\
&= \partial_{\xi^i} \big[\tilde{K}^{\mathrm{M}}(N | \xi, L) \big] + \partial_{L^\alpha} \big[\tilde{K}^{\mathrm{M}}(N | \xi, L) \big] \frac{\partial L^\alpha}{\partial \xi^i} \ ,
\end{aligned}
\tag{C.6}
$$

where we have suppressed the dependence on z and R to make the expressions more readable. Note that by differentiating (C.3) one also finds

$$\frac{\partial L^\alpha}{\partial T_\beta} = \tilde{K}^{\mathrm{M}\ L^\alpha L^\beta} \ , \qquad \frac{\partial L^\alpha}{\partial M} = -\tilde{K}^{\mathrm{M}\ L^\alpha L^\beta}\partial_M \tilde{K}^{\mathrm{M}}_{L^\beta} \ , \qquad M \in (N^I, \xi^i, R, z) \ . \tag{C.7}$$

However, since the kinetic potential $\tilde{K}^{\mathrm{M}}$ before dualization is in general very complicated the resulting expression for the dual theory turns out to be rather involved. Let us make some observations first. Note that ξ^i only appears linearly and quadratically in the general expression (B.9) since ξ^i arises as the fourth component of $D=4$ vectors and we only included the standard Yang-Mills terms for these fields. Hence, we expand $\tilde{K}^{\mathrm{F}}$, given in (C.4), to quadratic order around $\xi^i = 0$, $\mathcal{N}_\kappa = 0$ and note that it should take the form

$$\tilde{K}^{(1)} = \tilde{K}^{\mathrm{F}}\big|_* - \tfrac{1}{2}R^{-1}\ \Delta_{ij}\,\xi^i\,\xi^j + \tfrac{1}{2}R\ \Delta^{\kappa\lambda}\,\mathrm{Re}\mathcal{N}_\kappa\,\mathrm{Re}\mathcal{N}_\lambda - \Delta_i^\kappa\,\xi^i\,\mathrm{Re}\mathcal{N}_\kappa \ . \tag{C.8}$$

where the $*$ indicates that the expression has to be evaluated at $\xi^i = 0, \mathrm{Re}\mathcal{N}_\kappa = 0$. Here we have abbreviated

$$\begin{aligned}
\Delta_{ij} &= -R\,\partial_{\xi^i}\partial_{\xi^j}\tilde{K}^{\mathrm{F}}\big|_* &&\overset{!}{=} \mathrm{Re}f_{ij} - \mathrm{Re}f_{i\kappa}\ \mathrm{Re}f^{\kappa\lambda}\ \mathrm{Re}f_{\lambda j}\ , \\
\Delta_i^\kappa &= -\partial_{\xi^i}\partial_{\mathrm{Re}\mathcal{N}_\kappa}\tilde{K}^{\mathrm{F}}\big|_* &&\overset{!}{=} \mathrm{Re}f_{i\kappa}\ \mathrm{Re}f^{\kappa\lambda}\ , \\
\Delta^{\kappa\lambda} &= R^{-1}\,\partial_{\mathrm{Re}\mathcal{N}_\kappa}\partial_{\mathrm{Re}\mathcal{N}_\lambda}\tilde{K}^{\mathrm{F}}\big|_* &&\overset{!}{=} \mathrm{Re}f^{\kappa\lambda}\ .
\end{aligned} \tag{C.9}$$

Here the expressions after the second equal signs in each line are the expected expressions obtained by comparison with (B.9). It will be the kinetic potential (C.8), which one compares to the expression (B.9) to extract the $\mathcal{N}=1, D=4$ characteristic data of F-theory on a Calabi-Yau fourfold. Since there is no linear term in (B.9) we thus expect to find

$$\partial_{\xi^j}\tilde{K}^{\mathrm{F}}\big|_* = \partial_{\xi^j}\tilde{K}^{\mathrm{M}}\big|_* = 0\ , \qquad \partial_{\mathcal{N}_\kappa}\tilde{K}^{\mathrm{F}}\big|_* = \partial_{\mathcal{N}_\kappa}\tilde{K}^{\mathrm{M}}\big|_* = 0\ , \tag{C.10}$$

in a set-up which consistently lifts to a four-dimensional $\mathcal{N}=1$ F-theory compactification. Similarly one evaluates the second derivatives and translates the derivatives of $\tilde{K}^{\mathrm{F}}$ into derivatives of $\tilde{K}^{\mathrm{M}}$.

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