FOUR PAGES ARE INDEED NECESSARY FOR PLANAR GRAPHS

Michael A. Bekos,[*] *Michael Kaufmann,*[†] *Fabian Klute,*[‡]
Sergey Pupyrev,[§] *Chrysanthi Raftopoulou,*[¶] *and Torsten Ueckerdt*[‖]

Abstract. An embedding of a graph in a book consists of a linear order of its vertices along the spine of the book and of an assignment of its edges to the pages of the book, so that no two edges on the same page cross. The book thickness of a graph is the minimum number of pages over all its book embeddings. Accordingly, the book thickness of a class of graphs is the maximum book thickness over all its members. In this paper, we address a long-standing open problem regarding the exact book thickness of the class of planar graphs, which previously was known to be either three or four. We settle this problem by constructing planar graphs that require four pages in all of their book embeddings, thus establishing that the book thickness of the class of planar graphs is four.

1 Introduction

Embedding graphs in books is a fundamental problem in graph theory, which has been the subject of intense research over the years mainly due to the numerous applications that it finds, e.g., in VLSI design, transportation planning and graph drawing [9, 11, 47, 14, 32, 36, 44, 40, 45]. Early results date back to the 70s by Ollmann [39], while several important milestones appear regularly over the years [8, 11, 14, 23, 26, 49]. In a *book embedding* of a graph, the vertices are restricted to a line, called the *spine* of the book, and the edges are assigned to different half-planes delimited by the spine, called *pages* of the book, so that no two edges on the same page cross; see Fig. 1. The *book thickness* (or *stack number* or *page number*) of a graph is the minimum number of pages required by any of its book embeddings.

Back in 1979, Bernhart and Kainen observed that the book thickness of a graph can be linear in the number of its vertices; for instance, the book thickness of the complete n-vertex graph K_n is $\lceil n/2 \rceil$; see [8]. Bounds on the book thickness that are sublinear in the number of vertices are known for several classes of graphs; see [35, 34, 18, 23, 12, 37, 3]. Planar graphs are the most notable such class, as is evident from the numerous papers that have been published on the topic over the years [13, 26, 31, 8, 38, 33, 16, 43, 22, 4, 30, 20, 24, 26, 25, 48, 49, 42, 1]. In particular, the graphs with book thickness one are precisely the outerplanar graphs [8]. The

[*] *Institut für Informatik, Universität Tübingen, Tübingen, Germany*, bekos@informatik.uni-tuebingen.de
[†] *Institut für Informatik, Universität Tübingen, Tübingen, Germany*, mk@informatik.uni-tuebingen.de
[‡] *Institute of Logic and Computation, Technische Universität Wien, Wien, Austria*, f.m.klute@uu.nl
[§] *Facebook, Inc., Menlo Park, CA, USA*, spupyrev@gmail.com
[¶] *School of Applied Mathematical & Physical Sciences, NTUA, Athens, Greece*, crisraft@mail.ntua.gr
[‖] *Institute of Theoretical Informatics, KIT, Karlsruhe, Germany*, torsten.ueckerdt@kit.edu

graphs with book thickness at most two are the subgraphs of planar Hamiltonian graphs [8], which include planar bipartite [22] and series-parallel graphs [43].

The study of the book thickness of general planar graphs was suggested by Leighton, who asked whether their book thickness is bounded by a constant; see [13]. Even though Bernhart and Kainen [8] initially conjectured that even the planar 3-trees (i.e., planar graphs with treewidth 3) have unbounded book thickness, the first positive answer to the question by Leighton was given by Buss and Shor [13], who proposed a simple recursive (on the number of separating triangles) algorithm to embed every planar graph in a book with nine pages; note that a planar graph without separating triangles is Hamiltonian [46], and thus embeddable in two pages.

The bound of nine pages by Buss and Shor was improved to seven by Heath [26], who introduced an important methodological foundation called *peeling-into-levels*[1], according to which the vertices of a planar graph are partitioned into levels such that (i) the vertices on the unbounded face are at level 0, and (ii) the vertices that are on the unbounded face of the subgraph induced by deleting all vertices of levels $\leq i-1$ are at level i $(0 < i < n)$. It is not difficult to see that the subgraph induced by the vertices at each level is outerplanar, and thus embeddable in a single page [8]. Hence, the main challenge is to embed the remaining edges, that is, those connecting vertices in consecutive levels.

Heath [26] managed to address this challenge with a relatively simple algorithm that uses six pages. In a subsequent work, which is probably the most cited in the field, Yannakakis [49] improved upon Heath's algorithm. Using the peeling-into-levels technique, he proposed a simple algorithm that yields embeddings in books with five pages (even though, the details of the algorithm are left to the reader). With a more complicated and involved algorithm, which is based on distinguishing different cases of the underlying order and the edges to be embedded, Yannakakis reduced the required number of pages to four, which is currently the best-known upper bound on the book thickness of planar graphs.

The best-known lower bound is usually attributed to Goldner and Harary [24], who proposed the smallest maximal planar graph that is not Hamiltonian, and therefore not embeddable in books with two pages; see Fig. 1a. However, this particular graph is a planar 3-tree and by a result of Heath [26], it is embeddable in a book with three pages; see Fig. 1b. Note that determining the exact book thickness of a planar graph turns out to be an $\mathcal{NP}$-complete problem, even for maximal planar graphs [46].

To the best of our knowledge, there is no planar graph described in the literature that requires more than three pages despite various efforts. In an extended abstract of [49], which appeared at STOC in 1986 [48], Yannakakis claimed the existence of such a graph and provided a sketch of a proof; notably the arguments in this sketch seem to be sound apart from the fact that some of the gadget-graphs that are central in the proof are not defined. The details of this sketch, however, never appeared in a paper. Furthermore, the proof-sketch was not part of the subsequent journal version [49] of the extended abstract [48]. Thus the problem of determining whether there exists a planar graph that requires four pages still remains unsolved, as also noted by Dujmović and Wood [18] in 2007, and clearly forms the most intriguing open problem in the field. Note that, in the same work, Dujmović and

[1] In the literature, sometimes this technique is erroneously attributed to Yannakakis [49].

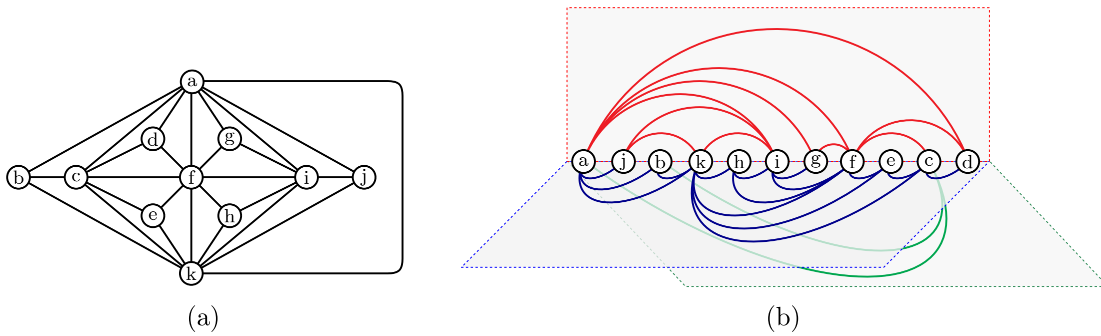


Figure 1: Illustration of (a) the Goldner-Harary graph and (b) its 3-page book embedding in which edges assigned to different pages are colored differently.

Wood proposed a planar graph that might require four pages in any of its book embeddings. However, they had overlooked a previous result by Heath [26] regarding the book thickness of planar 3-trees, which immediately implies that their claim was not valid. A more recent attempt to find a planar graph that requires four pages was made by Bekos, Kaufmann, and Zielke [7], who proposed a formulation of the problem of testing whether a given (not necessarily planar) graph admits an embedding into a book with a certain number of pages as a SAT instance, and systematically tested several hundred maximal planar graphs but without any particular success. Later Pupyrev [42] computed book embeddings of *all* maximal planar graphs of size $n \leq 18$ and found no instance that requires four pages.

Our contribution. In this paper, we address the aforementioned long-standing open problem. Our main result is summarized in the following theorem.

Theorem 1. *There exist planar graphs that do not admit* 3*-page book embeddings.*

Together with Yannakakis' upper bound of four [49], Theorem 1 implies the following corollary.

Corollary 1. *The book thickness of the class of planar graphs is four.*

We provide two proofs of Theorem 1. The first one is combinatorial (with some computer-aided prerequisites) and regards a significantly large planar graph. After recalling basic notions and results on book embeddings in Section 2, we describe the construction of this graph in Section 3, where we also present two properties of a particular subgraph of it, which have been verified by a computer (refer to Facts 1 and 2). In Section 4, we prove that the graph presented in Section 3 does not admit a 3-page book embedding. We give the main ingredients of this proof in Section 4.1, while in Section 4.2 we investigate a systematic analysis of cases of different underlying linear orders to conclude our main result.

We remark that our graph has treewidth 4. This is in contrast with planar graphs of treewidth 3 that always admit 3-page book embeddings [26]. Note that, in general, the class of graphs with treewidth k has book thickness k if $k \leq 2$ and $k+1$ if $k \geq 3$ [18, 23].

The second proof of Theorem 1 is purely computer-aided; see Section 5. With two independent implementations [5, 41] of the SAT formulation presented in [7], we confirm that

CC BY

a particular maximal planar graph with 275 vertices does not admit a 3-page book embedding; see Fig. 9 for an illustration of the graph. A key to our approach is the introduction of several symmetry-breaking constraints in the SAT instance. These constraints help to reduce the search space of possible satisfying assignments and made the instance verifiable using modern SAT solvers. We conclude in Section 6 with several open problems.

Remark 1. *We remark that some weeks after we made our results available on ArXiv [6], a paper by Yannakakis [50, 51] appeared online independently proving Theorem 1. Although the flavor of the arguments are similar in both papers, our proofs are more concrete and provide exact estimations on the size of the constructed graphs. Additionally, we provide a fairly small planar graph, which is not 3-page book embeddable. Notably, this graph is the smallest known example that requires four pages.*

2 Preliminaries

A *vertex ordering* $\prec$ of a simple undirected graph $G = (V, E)$ is a total order of its vertex set V, such that for any two vertices u and v, $u \prec v$ if and only if u precedes v in the order. Two vertices u and v are said to be *on opposite sides* of an edge (x, y), where $u \prec v$ and $x \prec y$, if $u \prec x \prec v \prec y$ or $x \prec u \prec y \prec v$. Otherwise, u and v are *on the same side* of (x, y). We write $[v_1, v_2, \dots, v_k]$ to denote $v_i \prec v_{i+1}$ for all $1 \le i < k$. Let F be a set of $k \ge 2$ independent pairs of vertices $\langle s_i, t_i \rangle$, that is, $F = \{\langle s_i, t_i \rangle;\ i = 1, 2, \dots, k\}$. Assume without loss of generality that $s_i \prec t_i$, for all $1 \le i \le k$. If the order is $[s_1, \dots, s_k, t_k, \dots, t_1]$, then we say that the pairs of F form a *k-rainbow*, while if the order is $[s_1, t_1, \dots, s_k, t_k]$, then the pairs of F form a *k-necklace*. The pairs of F form a *k-twist* if the order is $[s_1, \dots, s_k, t_1, \dots, s_k]$. Note that since each edge is defined by a pair of vertices, the three definitions are directly extendable to independent edges; see Fig. 2. For this case, two independent edges that form a 2-twist (respectively, 2-rainbow, 2-necklace) are commonly referred to as *crossing* (respectively, *nested*, *disjoint*).

A *k-page book embedding* of a graph is a pair $\mathcal{E} = (\prec, \{E_1, \dots, E_k\})$, where $\prec$ is a vertex ordering of G and $\{E_1, \dots, E_k\}$ is a partition of E into sets of pairwise non-crossing edges, called *pages*. Equivalently, a k-page book embedding is a vertex ordering and a k-edge-coloring such that no two edges of the same color cross with respect to the ordering. The *book thickness* of a graph G is the minimum k such that G admits a k-page book embedding. As noted in several papers, a k-page book embedding, $\mathcal{E}$, can be transformed into a circular embedding, $C(\mathcal{E})$, with a k-edge-coloring in which all vertices appear on a circle in the same order as in $\mathcal{E}$ and the edges are drawn as straight-line segments in the interior of the circle, such that no two edges of the same color cross, and vice versa [8, 26]; see Fig. 5. The next lemma, whose proof is immediate, provides sufficient conditions for the non-existence of a 3-page book embedding.

Lemma 1. *A 3-page book embedding of a graph does not contain: (i) four edges that form a 4-twist in $\prec$, (ii) a pair of crossing edges that both cross two edges assigned to two different pages, and (iii) an edge that crosses three edges assigned to three different pages.*

The next result by Erdős and Szekeres [19] is used to simplify our case analysis.

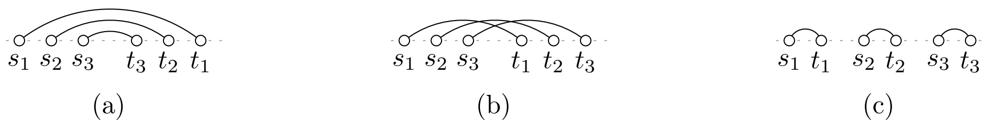


Figure 2: Illustration of three edges that form: (a) a 3-rainbow, (b) a 3-twist, and (c) a 3-necklace.

Lemma 2 (Erdős and Szekeres [19]). *Given $a, b \in \mathbb{N}$, every sequence of distinct real numbers of length at least $a \cdot b + 1$ contains a monotonically increasing subsequence of length $a + 1$ or a monotonically decreasing subsequence of length $b + 1$.*

Lemma 2 implies that, for every $r \geq 1$, if the input graph has sufficiently many independent edges, then one can always find r of them that form an r-rainbow or an r-twist or an r-necklace in every ordering $\prec$. To see this, assume that the graph contains r^3 independent edges. Represent each edge connecting the i-th with the j-th vertex in $\prec$ by a pair (i, j) with $i < j$. Consider the pairs sorted by the first coordinates, and apply Lemma 2 with $a = r^2$ and $b = r - 1$ to the second coordinates of the edges. Then, either (i) there exists $r^2 + 1$ edges such that every pair of them forms a 2-twist or a 2-necklace (corresponding to an increasing subsequence), which implies that r of them form an r-twist or an r-necklace [2], or (ii) there exists an r-rainbow (corresponding to a decreasing subsequence). Note that the same argument can be applied to r^3 designated pairs of vertices (not necessarily connected by an edge); thus we have the following corollary.

Corollary 2. *For every vertex ordering, $\prec$, of a graph with r^3 designated pairs of vertices, one can identify r pairs that form either an r-rainbow or an r-twist or an r-necklace in $\prec$.*

3 The Basic Graph Structure

The graph used to prove Theorem 1 is built using a sequence of gadgets—planar graphs that do not admit a 3-page book embedding under certain conditions. To define a gadget, denoted by Q_k, we recall the operation of the *stellation of a face* f, that is, the addition of a vertex in the interior of f connected to all vertices delimiting f. Accordingly, the operation of stellating a plane graph consists of stellating all its bounded faces.

For $k \geq 2$, graph Q_k is a plane graph, which contains as a subgraph the complete bipartite graph $K_{2,k}$ with bipartition $\{\{A, B\}, \{t_0, \dots, t_{k-1}\}\}$; see Fig. 3a. We choose the embedding of Q_k such that the faces of $K_{2,k}$ are $F_i = \langle A, t_i, B, t_{i+1} \rangle$ for $i = 0, \dots, k-1$ (indices taken modulo k) with F_{k-1} being its outerface. We refer to vertices A and B as the *poles* of Q_k, and to the vertices $t_0, \dots, t_{k-1}$ as the *terminals* of Q_k. For $i = 0, \dots, k-2$, we call terminals t_i and t_{i+1} of Q_k *consecutive*; notice that t_0 and t_{k-1} are not consecutive by the definition.

Let $i \in \{0, \dots, k-2\}$. In Q_k, vertices A and B are connected by a path of length 3 which is embedded within F_i and consists of the following three edges: (A, b_i), (b_i, a_i) and (a_i, B); see Fig. 3b. We refer to the two vertices a_i and b_i of this path as the *satellites* of the (consecutive) terminals t_i and t_{i+1}; accordingly, we refer to the edge connecting a_i and b_i as the

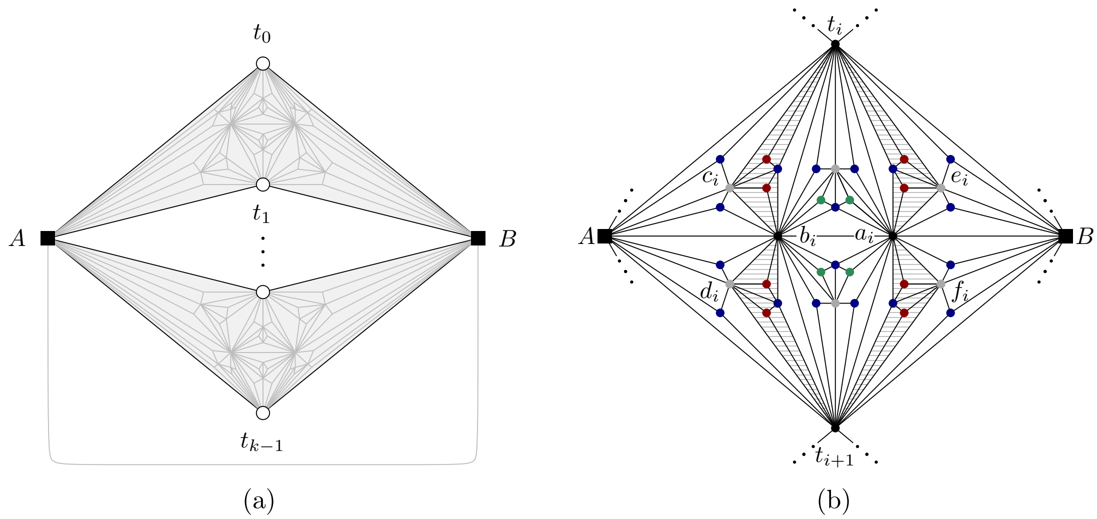


Figure 3: Illustration for the construction of graph Q_k.

satellite edge of t_i and t_{i+1}. Observe that we do not embed any path in F_{k-1}. The two faces on the opposite sides of the path embedded in F_i are triangulated by the edges $(t_i, a_i), (t_i, b_i)$ as well as (t_{i+1}, a_i) and (t_{i+1}, b_i). We proceed by stellating the graph constructed so far twice (refer to the gray and blue vertices in Fig. 3b, respectively). Let c_i, d_i, e_i and f_i be the vertices that stellated $\langle A, t_i, b_i\rangle$, $\langle A, b_i, t_{i+1}\rangle$, $\langle B, t_i, a_i\rangle$ and $\langle B, a_i, t_{i+1}\rangle$ in the first round of stellation. Let c'_i, d'_i, e'_i and f'_i be the vertices that stellated $\langle c_i, t_i, b_i\rangle$, $\langle d_i, b_i, t_{i+1}\rangle$, $\langle e_i, t_i, a_i\rangle$ and $\langle f_i, a_i, t_{i+1}\rangle$ in the second round of stellation; refer to the blue-colored vertices that lie within the gray-shaded regions of Fig. 3b. We proceed by stellating faces $\langle c_i, c'_i, t_i\rangle$, $\langle c_i, c'_i, b_i\rangle$, $\langle d_i, d'_i, b_i\rangle$, $\langle d_i, d'_i, t_{i+1}\rangle$, $\langle e_i, e'_i, t_i\rangle$, $\langle e_i, e'_i, a_i\rangle$, $\langle f_i, f'_i, a_i\rangle$ and $\langle f_i, f'_i, t_{i+1}\rangle$; refer to the red-colored vertices of Fig. 3b. The satellite edge (a_i, b_i) delimits two faces, each of which is neighboring two other faces that we stellate; refer to the green-colored vertices of Fig. 3b. Edge (A, B) completes the construction of Q_k. Note that graph Q_2, the first member in the described family, consists of 42 vertices and 126 edges.

The following two facts that hold for certain members of the constructed graph family have been verified by a computer using the SAT-formulation proposed in [7]; we provide further details in Section 5. We use these facts in the combinatorial proof of Theorem 1.

Fact 1. *Graph Q_k with $k \geq 7$ does not admit an embedding in a book with three pages, $\mathcal{B}$lue, $\mathcal{R}$ed and $\mathcal{G}$reen, under the following restrictions: (i) the poles A and B are consecutive in the ordering, (ii) all edges from A to the terminals of Q_k belong to $\mathcal{B}$lue, and (iii) all edges from B to the terminals of Q_k belong to $\mathcal{R}$ed or $\mathcal{G}$reen.*

Fact 2. *Graph Q_k with $k \geq 10$ does not admit an embedding in a book with three pages, $\mathcal{B}$lue, $\mathcal{R}$ed and $\mathcal{G}$reen, under the following restrictions: (i) all terminals of Q_k are on the same side of (A, B), (ii) all edges from A to the terminals of Q_k belong to $\mathcal{B}$lue, and (iii) all edges from B to the terminals of Q_k belong to $\mathcal{R}$ed.*

Note that Fact 1 imposes stronger restrictions in the vertex ordering than Fact 2, while Fact 2 imposes stronger restrictions to the edges adjacent to A and B. In the remainder, we

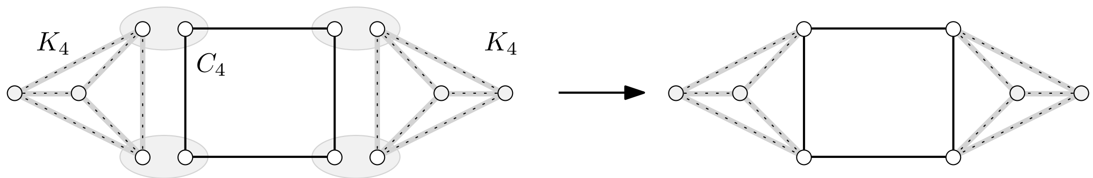


Figure 4: Attaching two copies of the complete graph K_4 along two edges of a 4-cycle C_4.

denote by Q the smallest member of the constructed family of graphs for which both Facts 1 and 2 hold:

$$Q := Q_{10}$$

Consider a plane graph G and let H be a plane graph with two designated vertices A and B that appear consecutively along its outerface. The operation of *attaching* H along an edge (u, v) of G consists of removing (u, v) from G and of introducing H into G by identifying vertex A of H with vertex u of G and vertex B of H with vertex v of G; see Fig. 4. The obtained graph is clearly planar, since both G and H are planar and simple due to removal of (u, v) from G.

4 A Combinatorial Proof with Computer-Aided Prerequisites

In this section, we construct a planar graph G containing several copies of Q. Using Facts 1 and 2, we explore certain properties of graph G (Section 4.1) to prove that it does not admit a 3-page book embedding by analyzing possible vertex orderings (Section 4.2).

4.1 The Idea

We prove Theorem 1 by contradiction, that is, by assuming that G admits a book embedding $\mathcal{E}$ with three pages denoted by $\mathcal{B}lue$, $\mathcal{R}ed$, and $\mathcal{G}reen$. Graph G contains as a subgraph a *base graph*, which we denote by G_N, consisting of a large number $N \gg 1$ of copies of graph Q that share the same pair of poles, A and B, and edge (A, B). Hence, graph G_N is symmetric with respect to A and B. Let n_Q and m_Q be the number of vertices and edges in Q, and let b_Q be the number of 3-page book embeddings of graph Q. Clearly b_Q is upper bounded by $3^{m_Q} \cdot n_Q!$; it follows that if N is at least $\kappa \cdot 3^{m_Q} \cdot n_Q!$, then by pigeonhole principle G_N contains κ copies of graph Q with the *majority property*, that is, corresponding vertices of Q in each of these κ copies appear in the same relative order in $\mathcal{E}$, and additionally the edges that connect these vertices in each of the copies are assigned to the same pages. We refer to two vertices that correspond to the same vertex in Q and that belong to different copies satisfying the majority property as *twin vertices*. Accordingly, two edges connecting twin vertices are called *twin edges*.

Lemma 3. *A pair of independent twin edges either form a 2-rainbow or a 2-necklace in $\mathcal{E}$.*

Proof. Observe that two independent twin edges cannot form a 2-twist, as they are assigned to the same page in $\mathcal{E}$ by the majority property. □

Next we further increase N to guarantee an additional property, called the *monotonic property*, for the κ copies of graph Q that comply with the majority property. Denote by p_Q the number of pairs of vertices in Q, that is, $p_Q = \frac{n_Q(n_Q-1)}{2}$. By Corollary 2, if N is at least $\kappa^{3 \cdot p_Q} \cdot 3^{m_Q} \cdot n_Q!$, then one can identify κ copies of Q in G_N complying with the majority property, such that, for each pair of vertices of Q, the corresponding pairs of vertices in these κ copies form a κ-rainbow or κ-twist or a κ-necklace in $\mathcal{E}$. We specify κ in the case analysis of Section 4.2.

While we mainly focus on the base graph G_N, to facilitate our analysis in cases in Section 4.2, we perform an augmentation step that completes the construction of G. Let H_N be a copy of the base graph G_N. We attach a copy of H_N along every satellite edge of the base graph G_N. We refer to the obtained graph as the *final graph* G, which by construction is biconnected; the poles of the base graph G_N and the endvertices of each of its satellite edges are separation pairs in G. Next we investigate all possible vertex orderings of G in its 3-page book embedding $\mathcal{E}$.

4.2 Case analysis

Consider the base graph G_N and let $\mathcal{Q}_1, \dots, \mathcal{Q}_\kappa$ be the κ copies of graph Q that comply with the majority and the monotonic properties. Assuming that A is the first vertex in $\mathcal{E}$, we consider two main cases in our proof:

C.1. There exist two terminals of $\mathcal{Q}_1$ that are on opposite sides of edge (A, B) in $\mathcal{E}$.
C.2. All terminals of $\mathcal{Q}_1$ are on the same side of (A, B) in $\mathcal{E}$.

Case C.1: We first rule out Case C.1 in which there exist two terminals of $\mathcal{Q}_1$, say $\langle x_1, y_1\rangle$, that are on opposite sides of edge (A, B) in $\mathcal{E}$. Observe that in this case it is not a loss of generality to assume that x_1 and y_1 are consecutive in the sequence of terminals of $\mathcal{Q}_1$. By the majority property, the corresponding terminals $\langle x_2, y_2\rangle, \dots, \langle x_\kappa, y_\kappa\rangle$ of $\mathcal{Q}_2, \dots, \mathcal{Q}_\kappa$ are also on opposite sides of edge (A, B). Let $\langle a_1, b_1\rangle, \dots, \langle a_\kappa, b_\kappa\rangle$ be the corresponding satellite vertices of $\langle x_1, y_1\rangle, \dots, \langle x_\kappa, y_\kappa\rangle$. W.l.o.g., we further assume that $x_1 \prec \dots \prec x_\kappa$, which by the monotonic property implies that either $y_1 \prec \dots \prec y_\kappa$ or $y_\kappa \prec \dots \prec y_1$. Since G_N is symmetric with respect to A and B, we may further assume that the ordering of the vertices in $\mathcal{E}$ is either $[A \dots x_1 \dots x_\kappa \dots B \dots y_1 \dots y_\kappa]$ or $[A \dots x_1 \dots x_\kappa \dots B \dots y_\kappa \dots y_1]$. We next prove that both patterns are forbidden, assuming $\kappa = 3$. Since G_N is symmetric with respect to A and B, by the majority property we may further assume w.l.o.g. that a_i and x_i are on the same side of (A, B), namely, $A \prec a_i \prec B$ holds, for each $i = 1, \dots, \kappa$.

Forbidden Pattern 1. $[A \dots x_1 \dots x_2 \dots x_3 \dots B \dots y_1 \dots y_2 \dots y_3 \dots]$

Proof. By the monotonic property, it follows that either $a_1 \prec a_2 \prec a_3$ or $a_3 \prec a_2 \prec a_1$ holds, and that $b_1 \prec b_2 \prec b_3$ or $b_3 \prec b_2 \prec b_1$ holds. We start with a few auxiliary propositions.

Proposition 1. $A \prec x_3 \prec a_3 \prec a_2 \prec a_1 \prec B$.

CC BY

Proof. If $a_1 \prec a_2 \prec a_3$, then the twin edges (a_1, y_1), (a_2, y_2) and (a_3, y_3) form a 3-twist in $\mathcal{E}$, which contradicts Lemma 3. Hence, $a_3 \prec a_2 \prec a_1$ must hold. Assume now that $a_1 \prec x_1$, which by the majority property implies that $a_i \prec x_i$, for each $i = 1, 2, 3$. Since $a_3 \prec a_2 \prec a_1$ holds, it follows that the relative order is $[A \dots a_3 \dots a_2 \dots a_1 \dots x_1 \dots x_2 \dots x_3 \dots B]$. Hence, edges (a_1, y_1), (a_2, B), (a_3, x_3) and (A, x_2) form a 4-twist in $\mathcal{E}$, which is a contradiction by Lemma 1.i. Thus, $x_1 \prec a_1$ must hold, which by the majority property implies that $x_i \prec a_i$, for each $i = 1, 2, 3$. Since $x_3 \prec a_3$ and $a_3 \prec a_2 \prec a_1$ holds, the proposition follows. □

Similarly, we can prove the following.

Proposition 2. *If $A \prec b_1 \prec B$, then $A \prec b_3 \prec b_2 \prec b_1 \prec x_1 \prec B$.*
Proposition 3. *If $B \prec b_1$, then $B \prec b_3 \prec b_2 \prec b_1 \prec y_1$.*

Let $i \in \{1, 2, 3\}$. We consider two cases, depending on whether a_i and b_i are on the same or on different sides of (A, B). Assume first the former case. Since $A \prec a_i \prec B$, it follows that $A \prec b_i \prec B$. By Propositions 1 and 2, the relative order is $[A \dots b_3 \dots b_2 \dots b_1 \dots x_1 \dots x_3 \dots a_3 \dots a_2 \dots a_1]$. Hence, edges (a_1, b_1), (a_2, b_2), (a_3, b_3) and (A, x_1) form a 4-twist; a contradiction by Lemma 1.i. Assume a_i and b_i are on different sides of (A, B). By the majority property, $B \prec y_i$. By Propositions 1 and 3, the relative order is $[A \dots a_3 \dots a_2 \dots a_1 \dots B \dots b_3 \dots b_2 \dots b_1]$, which implies that edges (a_1, b_1), (a_2, b_2), (a_3, b_3) and (A, B) form a 4-twist; a contradiction by Lemma 1.i. □

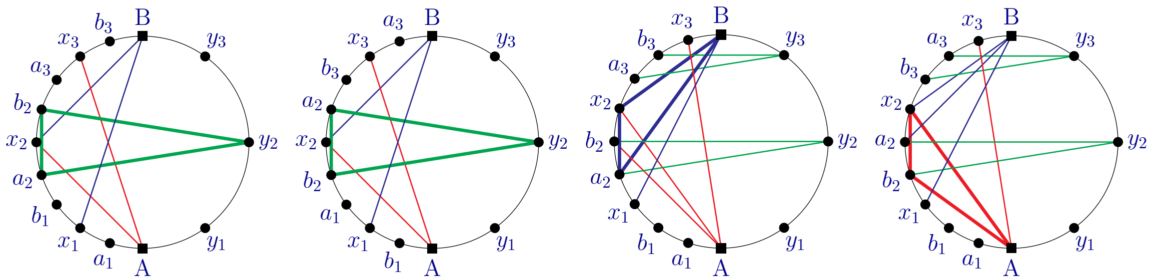
Forbidden Pattern 2. $[A \dots x_1 \dots x_2 \dots x_3 \dots B \dots y_3 \dots y_2 \dots y_1]$

Proof. Let $i \in \{1, 2, 3\}$. By the monotonic property, either $a_1 \prec a_2 \prec a_3$ or $a_3 \prec a_2 \prec a_1$ holds, and either $b_1 \prec b_2 \prec b_3$ or $b_3 \prec b_2 \prec b_1$ holds. Since $A \prec a_i \prec B$, it follows that if $a_3 \prec a_2 \prec a_1$, then the twin edges (a_1, y_1), (a_2, y_2) and (a_3, y_3) form a 3-twist, which a contradiction by Lemma 3. Hence, $A \prec a_1 \prec a_2 \prec a_3 \prec B$ holds.

We proceed by distinguishing two subcases depending on whether the satellite vertices a_i and b_i are on the same or different sides of (A, B). We first consider the former case. Since $A \prec a_1 \prec a_2 \prec a_3 \prec B$ holds, it follows that either $A \prec b_1 \prec b_2 \prec b_3 \prec B$ or $A \prec b_3 \prec b_2 \prec b_1 \prec B$ holds. If $b_3 \prec b_2 \prec b_1$, then the twin edges (b_1, y_1), (b_2, y_2) and (b_3, y_3) form a 3-twist, which is a contradiction by Lemma 3. Hence, $A \prec b_1 \prec b_2 \prec b_3 \prec B$ must hold. By the monotonic property, the partial order of vertices A, B and of the vertices in $\{x_i, y_i, a_i, b_i;\ i = 1, 2, 3\}$ is one of the following FP2.1-FP2.4; note that the cases that corresponds to FP2.3 and FP2.4 in which the terminal x_i precedes the satellite vertices a_i and b_i, are symmetric to FP2.4 and FP2.3, respectively, due to the symmetry of G_N with respect to A and B.

FP2.1 $[A \dots a_1 \dots x_1 \dots b_1 \dots a_2 \dots x_2 \dots b_2 \dots a_3 \dots x_3 \dots b_3 \dots B \dots y_3 \dots y_2 \dots y_1]$

Refer to Fig. 5a. Since edges (A, x_3), (x_1, B) and (a_2, y_2) form a 3-twist, they are assigned to different pages in $\mathcal{E}$. By the majority property, we may assume that $(A, x_i) \in \mathcal{R}ed$, $(a_i, y_i) \in \mathcal{G}reen$ and $(x_i, B) \in \mathcal{B}lue$. It follows that $(b_i, y_i) \in \mathcal{G}reen$ and $(a_i, b_i) \in \mathcal{G}reen$. Consider now vertex s_2^{aby} of G_N that was introduced due to



(a) (b) (c) (d)

Figure 5: Illustrations for (a) FP2.1, (b) FP2.2, (c) FP2.3, and (d) FP2.4.

the stellation of face $\langle a_2, b_2, y_2 \rangle$ in G_N. Due to edge (b_2, s_2^{aby}), vertex s_2^{aby} can be neither in $[A \dots a_2]$ nor in $[y_2 \dots A]$, as otherwise (b_2, s_2^{aby}) crosses $(x_2, B) \in \mathcal{B}lue$, $(a_2, y_2) \in \mathcal{G}reen$ and either $(A, x_2) \in \mathcal{R}ed$ or $(A, x_3) \in \mathcal{R}ed$, respectively, which is a contradiction by Lemma 1.iii. Similarly, due to edge (y_2, s_2^{aby}), vertex s_2^{aby} cannot be in $[a_2 \dots b_2]$. Finally, due to edge (a_2, s_2^{aby}), vertex s_2^{aby} cannot be in $[b_2 \dots y_2]$. Hence, there is no feasible placement of s_2^{aby} in $\mathcal{E}$, which is a contradiction.

FP2.2 $[A \dots b_1 \dots x_1 \dots a_1 \dots b_2 \dots x_2 \dots a_2 \dots b_3 \dots x_3 \dots a_3 \dots B \dots y_3 \dots y_2 \dots y_1]$

Refer to Fig. 5b. This case can be led to a contradiction following the reasoning of FP2.1.

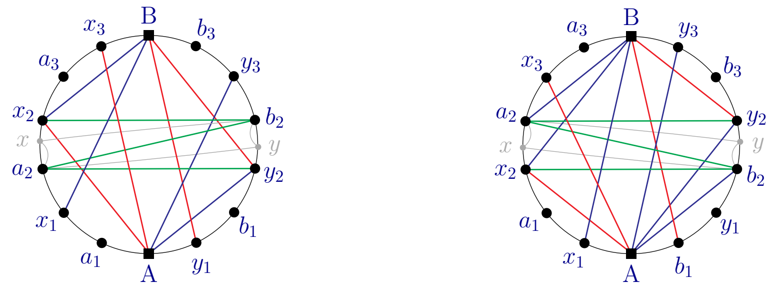
FP2.3 $[A \dots a_1 \dots b_1 \dots x_1 \dots a_2 \dots b_2 \dots x_2 \dots a_3 \dots b_3 \dots x_3 \dots B \dots y_3 \dots y_2 \dots y_1]$

Refer to Fig. 5c. Since edges (A, x_3), (x_1, B) and (a_2, y_2) form a 3-twist, we can assume that $(A, x_i) \in \mathcal{R}ed$, $(a_i, y_i) \in \mathcal{G}reen$ and $(x_i, B) \in \mathcal{B}lue$, which implies that $(b_i, y_i) \in \mathcal{G}reen$, $(a_i, B) \in \mathcal{B}lue$, $(A, b_i) \in \mathcal{R}ed$ and $(a_i, x_i) \in \mathcal{B}lue$. Consider now vertex s_2^{Bax} of G_N that was introduced due to the stellation of face $\langle B, a_2, x_2 \rangle$ in G_N. Due to edge (a_2, s_2^{Bax}), vertex s_2^{Bax} cannot be in $[x_2 \dots y_2]$. Analogously, vertex s_2^{Bax} cannot be in $[y_2 \dots a_2]$, due to edge (x_2, s_2^{Bax}). Finally, vertex s_2^{Bax} cannot be in $[a_2 \dots x_2]$, due to edge (B, s_2^{Bax}). Hence, there is no feasible placement of s_2^{Bax} in $\mathcal{E}$; a contradiction.

FP2.4 $[A \dots b_1 \dots a_1 \dots x_1 \dots b_2 \dots a_2 \dots x_2 \dots b_3 \dots a_3 \dots x_3 \dots B \dots y_3 \dots y_2 \dots y_1]$

Refer to Fig. 5d. Since edges (A, x_3), (x_1, B) and (a_2, y_2) form a 3-twist, we can assume that $(A, x_i) \in \mathcal{R}ed$, $(a_i, y_i) \in \mathcal{G}reen$ and $(x_i, B) \in \mathcal{B}lue$. Hence, $(a_i, B), (x_i, B) \in \mathcal{B}lue$, $(b_i, y_i) \in \mathcal{G}reen$ and $(A, b_i), (b_i, x_i) \in \mathcal{R}ed$. It is not hard to see that there is no feasible placement for vertex s_2^{Abx} of G_N introduced due to the stellation of face $\langle A, b_2, x_2 \rangle$ in $\mathcal{E}$.

We now consider the case in which the satellite vertices a_i and b_i are on different sides of (A, B). Since $A \prec a_1 \prec a_2 \prec a_3 \prec B$, either $B \prec b_1 \prec b_2 \prec b_3$ or $B \prec b_3 \prec b_2 \prec b_1$ holds. If $b_1 \prec b_2 \prec b_3$, then a 3-twist is formed by the twin edges (b_1, x_1), (b_2, x_2) and (b_3, x_3), which is a contradiction by Lemma 3. Hence, $b_3 \prec b_2 \prec b_1$ must hold. By the monotonic property,



(a) (b)

Figure 6: Illustrations for (a) FP2.7 and (b) FP2.8.

the partial order of vertices A, B and of the vertices in $\{x_i, y_i, a_i, b_i;\ i = 1, 2, 3\}$ is one of the following:

FP2.5 $[A \dots x_1 \dots a_1 \dots x_2 \dots a_2 \dots x_3 \dots a_3 \dots B \dots b_3 \dots y_3 \dots b_2 \dots y_2 \dots b_1 \dots y_1]$

Edges (A, x_3), (x_1, B), (x_2, b_2), (a_2, y_2) form a 4-twist; a contradiction by Lemma 1.i.

FP2.6 $[A \dots a_1 \dots x_1 \dots a_2 \dots x_2 \dots a_3 \dots x_3 \dots B \dots y_3 \dots b_3 \dots y_2 \dots b_2 \dots y_1 \dots b_1]$

Edges (A, x_3), (x_1, B), (x_2, b_2), (a_2, y_2) form a 4-twist; a contradiction by Lemma 1.i.

FP2.7 $[A \dots a_1 \dots x_1 \dots a_2 \dots x_2 \dots a_3 \dots x_3 \dots B \dots b_3 \dots y_3 \dots b_2 \dots y_2 \dots b_1 \dots y_1]$

As opposed to FP2.1–FP2.6, we do not directly rule out this case. Instead, we identify a copy of G_N in the final graph G (see Section 4.1) for which the preconditions of Case C.2 hold. Thus, we reduce this case to C.2, for which a direct contradiction is shown below.

Refer to Fig. 6a. Since edges (A, x_3), (x_1, B) and (a_2, y_2) form a 3-twist, by the majority property, we may assume that $(A, x_i) \in \mathcal{Red}$, $(a_i, y_i) \in \mathcal{Green}$ and $(x_i, B) \in \mathcal{Blue}$. Hence, $(b_i, x_i) \in \mathcal{Green}$ and $(a_i, b_i) \in \mathcal{Green}$. Since $(a_i, y_i) \in \mathcal{Green}$ and since edges (A, y_3), (y_1, B) and (a_2, y_2) also form a 3-twist, by the majority property, we may further assume that either $(A, y_i) \in \mathcal{Blue}$ and $(y_i, B) \in \mathcal{Red}$, or $(A, y_i) \in \mathcal{Red}$ and $(y_i, B) \in \mathcal{Blue}$. In the following, we discuss the former case; the latter is analogous.

Consider the copy H_N of graph G_N that is attached along the satellite edge (a_2, b_2) in the final graph G, and let $Q_{(a_2,b_2)}$ be *any* copy of graph Q in H_N. We prove that no two terminals of $Q_{(a_2,b_2)}$ are on opposite sides of (a_2, b_2). Assume the contrary, which implies that there exist two consecutive terminals, say x and y, of $Q_{(a_2,b_2)}$ that are on opposite sides of (a_2, b_2). It is not difficult to see that either x is in $[a_2 \dots x_2]$ and y is in $[b_2 \dots y_2]$, or vice versa; see Fig. 6a. By construction of graph $Q_{(a_2,b_2)}$, vertices x and y are connected by a path of length 2 in $Q_{(a_2,b_2)} \setminus \{a_2, b_2\}$. Let z be the intermediate vertex of this path. Due to edge (x, z), vertex z can be only in $[x_1 \dots x_2]$, which implies that its second edge (z, y) crosses three edges of different colors; a contradiction by Lemma 1.iii. Hence, all terminals of $Q_{(a_2,b_2)}$ are on the same side of (a_2, b_2). As this property holds for all copies of graph Q in H_N, Case C.2 applies for graph H_N, as we mentioned above.

FP2.8 $[A \ldots x_1 \ldots a_1 \ldots x_2 \ldots a_2 \ldots x_3 \ldots a_3 \ldots B \ldots y_3 \ldots b_3 \ldots y_2 \ldots b_2 \ldots y_1 \ldots b_1]$

Refer to Fig. 6b. This case can be reduced to C.2 closely following the reasoning of FP2.7.

This concludes the discussion of Case C.1 in which there exist two terminals of $\mathcal{Q}_1$ (and thus, of $\mathcal{Q}_2, \ldots, \mathcal{Q}_\kappa$) that are on opposite sides of edge (A, B) in $\mathcal{E}$. □

Case C.2: We next rule out the case in which all terminals of $\mathcal{Q}_1$ (and, thus of $\mathcal{Q}_2, \ldots, \mathcal{Q}_\kappa$) are on the same side of (A, B) in $\mathcal{E}$. By Fact 2 applied on $\mathcal{Q}_1$, we may assume that there exist two terminals, and thus two consecutive terminals $\langle x_1, y_1 \rangle$, of $\mathcal{Q}_1$ such that either edges (A, x_1) and (A, y_1), or edges (B, x_1) and (B, y_1) have been assigned to different pages in $\mathcal{E}$. Assume w.l.o.g. that $(B, x_1) \in \mathcal{Red}$ and $(B, y_1) \in \mathcal{Green}$. Since G_N is symmetric with respect to A and B, we may further assume w.l.o.g. that $A \prec x_1 \prec y_1 \prec B$. By the majority property, the corresponding terminals $\langle x_2, y_2 \rangle, \ldots, \langle x_\kappa, y_\kappa \rangle$ of $\mathcal{Q}_2, \ldots, \mathcal{Q}_\kappa$ are also between A and B in $\prec$, and $(B, x_i) \in \mathcal{Red}$ and $(B, y_i) \in \mathcal{Green}$, for each $i = 1, \ldots, \kappa$. W.l.o.g., let $x_1 \prec \ldots \prec x_\kappa$. Finally, let $\langle a_1, b_1 \rangle, \ldots, \langle a_\kappa, b_\kappa \rangle$ be the corresponding satellite vertices of $\langle x_1, y_1 \rangle, \ldots, \langle x_\kappa, y_\kappa \rangle$. By Lemma 2, there are three subcases to consider, namely, the pairs $\langle x_1, y_1 \rangle, \ldots, \langle x_\kappa, y_\kappa \rangle$ can form a κ-twist, a κ-rainbow, or a κ-necklace; refer to Forbidden Patterns 3, 4 and 5, respectively. To rule out the first two, it suffices to assume $\kappa = 3$. However, for the last one we use a larger value for κ.

Forbidden Pattern 3. $[A \ldots x_1 \ldots x_2 \ldots x_3 \ldots y_1 \ldots y_2 \ldots y_3 \ldots B]$

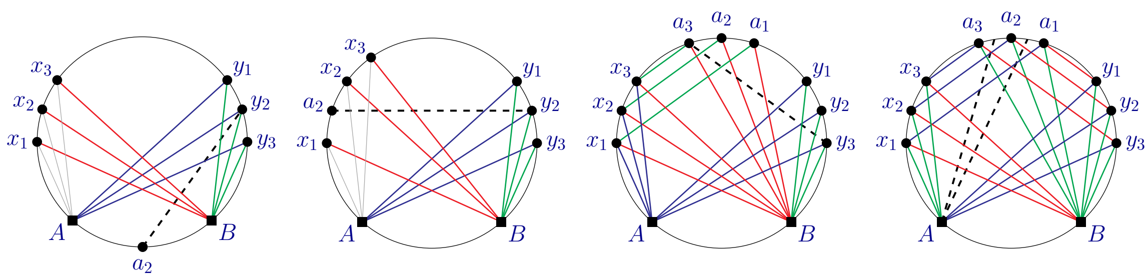
Proof. Let $i \in \{1, 2, 3\}$. Since edge (A, y_2) crosses both $(B, x_1) \in \mathcal{Red}$ and $(B, y_1) \in \mathcal{Green}$, by the majority property that $(A, y_i) \in \mathcal{Blue}$. Similarly, $(A, x_i) \in \mathcal{Blue}$ or $(A, x_i) \in \mathcal{Green}$.

Proposition 4. *$x_1 \prec a_i \prec y_3$ and $x_1 \prec b_i \prec y_3$*

Proof. Assume to the contrary that $a_i \prec x_1$ or $y_3 \prec a_i$. If $a_i \prec x_1$ or $B \prec a_i$, edge (a_2, y_2) crosses $(A, y_1) \in \mathcal{Blue}$, $(B, x_1) \in \mathcal{Red}$, $(B, y_1) \in \mathcal{Green}$, a contradiction by Lemma 1.iii; see Fig. 7a. Otherwise $(y_3 \prec a_i \prec B)$, edge (a_2, x_2) crosses $(A, y_1) \in \mathcal{Blue}$, $(B, x_3) \in \mathcal{Red}$, $(B, y_1) \in \mathcal{Green}$, a contradiction by Lemma 1.iii. The proof of the other claim is analogous. □

Proposition 5. *$x_3 \prec a_3 \prec a_2 \prec a_1 \prec y_1$ and $x_3 \prec b_3 \prec b_2 \prec b_1 \prec y_1$*

Proof. We argue for the former; the latter is analogous. If the twin edges (a_1, x_1), (a_2, x_2) and (a_3, x_3) form a 3-necklace, then a_2 is in $[x_1 \ldots x_3]$, which implies that edge (a_2, y_2) crosses $(A, y_1) \in \mathcal{Blue}$, $(B, x_3) \in \mathcal{Red}$ and $(B, y_1) \in \mathcal{Green}$ (see Fig. 7b); a contradiction by Lemma 1.iii. Hence by Lemma 3, (a_1, x_1), (a_2, x_2) and (a_3, x_3) form a 3-rainbow. Similarly, we argue that edges (a_1, y_1), (a_2, y_2) and (a_3, y_3) also form a 3-rainbow. Since the two 3-rainbows share a_1, a_2 and a_3, the proof follows from Proposition 4. □



(a) (b) (c) (d)

Figure 7: Illustrations for Forbidden Pattern 3.

If $(A, x_i) \in \mathcal{B}lue$, then $(x_i, a_i) \in \mathcal{G}reen$ and $(B, a_i) \in \mathcal{R}ed$, which imply that edge (a_3, y_3) crosses $(A, y_1) \in \mathcal{B}lue$, $(x_1, a_1) \in \mathcal{G}reen$ and $(B, a_2) \in \mathcal{R}ed$ (see Fig. 7c); a contradiction by Lemma 1.iii. Hence, $(A, x_i) \in \mathcal{G}reen$. This implies that $(x_2, a_2) \in \mathcal{B}lue$ and $(a_2, y_2) \in \mathcal{R}ed$; see Fig. 7d. Since by majority property $(x_i, a_i) \in \mathcal{B}lue$ and $(a_i, y_i) \in \mathcal{R}ed$, it follows that $(B, a_i) \in \mathcal{G}reen$. We next argue about b_2. By Proposition 5, b_2 is in $[x_3 \dots y_1]$. In the presence of a_1, a_2 and a_3 in the same interval, we can further restrict the placement of b_2 either in $[a_3 \dots a_2]$ or in $[a_2 \dots a_1]$. However, in both cases edge (A, b_2) crosses three edges of different colors, namely, $(B, a_3) \in \mathcal{G}reen$, $(x_1, a_1) \in \mathcal{B}lue$ and $(a_3, y_3) \in \mathcal{R}ed$, which is a contradiction by Lemma 1.iii. □

Forbidden Pattern 4. $[A \dots x_1 \dots x_2 \dots x_3 \dots y_3 \dots y_2 \dots y_1 \dots B]$

Proof. Since $(B, x_i) \in \mathcal{R}ed$ and $(B, y_i) \in \mathcal{G}reen$, as in the proof of Forbidden Pattern 3, we prove that $(A, y_i) \in \mathcal{B}lue$, and that either $(A, x_i) \in \mathcal{B}lue$ or $(A, x_i) \in \mathcal{G}reen$. As in the proof of Proposition 5, we can further prove that a 3-rainbow is formed both by the twin edges (a_1, x_1), (a_2, x_2) and (a_3, x_3) and by the twin edges (a_1, y_1), (a_2, y_2) and (a_3, y_3). Since both rainbows share a_1, a_2 and a_3, it is not possible that they exist simultaneously due to the underlying order $[x_1 \dots x_2 \dots x_3 \dots y_3 \dots y_2 \dots y_1]$. □

Forbidden Pattern 5. $[A \dots x_1 \dots y_1 \dots x_2 \dots y_2 \;\; \dots \;\; x_\kappa \dots y_\kappa \dots B]$

Proof. Let $i \in \{1, 2 \dots, \kappa\}$. Recall that $(B, x_i) \in \mathcal{R}ed$ and $(B, y_i) \in \mathcal{G}reen$. Since each of (A, x_2) and (A, y_2) crosses both $(B, x_1) \in \mathcal{R}ed$ and $(B, y_1) \in \mathcal{G}reen$, by majority property it follows that $(A, x_i) \in \mathcal{B}lue$ and $(A, y_i) \in \mathcal{B}lue$; see Fig. 8. To rule out this case, we assume that κ is even, such that $\kappa > d_Q + 4$, where d_Q denotes the length of the maximum shortest path between a terminal of graph Q and every other vertex of it that passes neither through A nor$B$. Note that $d_Q \neq n_Q$. Consider the copy $\mathcal{Q}_{\kappa/2}$ of graph Q, to which the terminals $x_{\kappa/2}$ and $y_{\kappa/2}$ belong. By Case C.2, all terminals of $\mathcal{Q}_{\kappa/2}$ are in $[A \dots B]$ in $\mathcal{E}$. In the following, we show that all the vertices of $\mathcal{Q}_{\kappa/2}$ that are different from A and B are in $[y_1 \dots x_\kappa]$. This implies that each of the terminals of $\mathcal{Q}_{\kappa/2}$ is connected to A through an edge of the $\mathcal{B}lue$ page (as it is involved in crossings with $(B, x_1) \in \mathcal{R}ed$ and $(B, y_1) \in \mathcal{G}reen$), and to B through an edge of either the $\mathcal{R}ed$ or of the $\mathcal{G}reen$ page (as it is involved in a crossing with $(A, y_1) \in \mathcal{B}lue$). The contradiction is obtained by Fact 1 applied to $\mathcal{Q}_{\kappa/2}$, whose preconditions (i)–(iii) are met as discussed above.

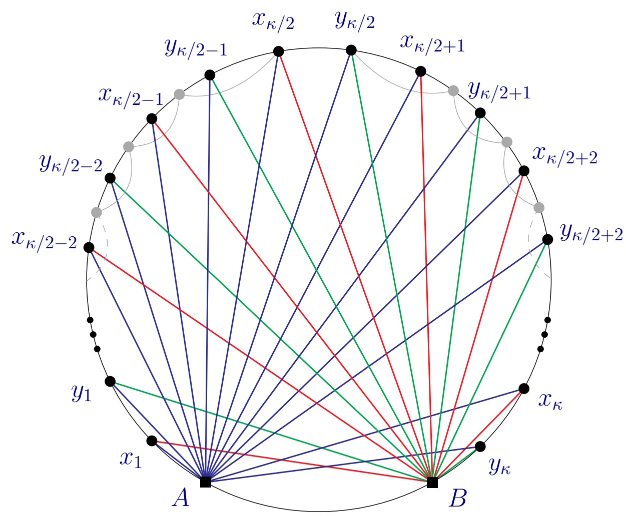


Figure 8: Illustration for Forbidden Pattern 5.

To complete the proof, we observe that all the vertices of $\mathcal{Q}_{\kappa/2}$ that are different from A and B and at distance 1 either from $x_{\kappa/2}$ or from $y_{\kappa/2}$ lie in $[x_{\kappa/2-1} \dots y_{\kappa/2+1}]$, as otherwise an edge incident to $x_{\kappa/2}$ or $y_{\kappa/2}$ is inevitably crossing three edges of different colors; a contradiction by Lemma 1.iii. By induction, we obtain that all the vertices that are different from A and B and at distance j either from $x_{\kappa/2}$ or from $y_{\kappa/2}$ lie in $[y_{\kappa/2-j/2-1} \dots x_{\kappa/2+j/2+1}]$, if j is even, and in $[x_{\kappa/2-\lfloor j/2\rfloor-1} \dots y_{\kappa/2+\lfloor j/2\rfloor+1}]$, if j is odd. By the definition of d_Q, any vertex of $\mathcal{Q}_{\kappa/2}$ different from A and B is in $[x_{\kappa/2-\lfloor d_Q/2\rfloor-1} \dots y_{\kappa/2+\lfloor d_Q/2\rfloor+1}]$, which by the choice of κ is in $[x_2 \dots y_{\kappa-1}]$, and thus in $[y_1 \dots x_k]$, as desired. □

By Cases C.1 and C.2, it follows that graph G does not admit a 3-page book embedding, which completes the proof of Theorem 1.

We conclude this section with some insights on the size of graph G. For most of the patterns that we proved to be forbidden, the value of κ is 3. However, in Forbidden Pattern 5, this value is increased to $d_Q + 5$, which equals 28. Using this value, one can compute the number N of copies of graph Q in the base graph G_N with $n_Q = 354$, $m_Q = 1{,}056$ and $p_Q = 62{,}481$. Since each of the N copies of graph Q in the base graph G_N gives rise to nine copies of the base graph in the final graph G, the size of graph G is enormously large. In the next section, we present a considerably smaller graph that serves as a certificate to Theorem 1.

5 A Computer-Aided Proof

In this section, we first briefly recall an efficient automatic approach for computing book embeddings with certain number of pages that was first proposed in [7]. Then, we apply this approach (with appropriate modifications) to find a medium-sized planar graph that requires four pages, and to verify Facts 1 and 2 for Q_7 and Q_{10}, respectively.

To formulate the book embedding problem as a SAT instance, Bekos et al. [7] use three different types of variables, denoted by σ, ϕ and χ, with the following meanings: (i) for a pair of vertices u and v, variable $\sigma(u, v)$ is `true`, if and only if u is to the left of v along the spine, (ii) for an edge e and a page ρ, variable $\phi_\rho(e)$ is `true`, if and only if edge e is assigned to page ρ of the book, and (iii) for a pair of edges e and e', variable $\chi(e, e')$ is `true`, if and only if e and e' are assigned to the same page. Hence, there exist in total $O(n^2 + m^2 + pm)$ variables, where n denotes the number of vertices of the graph, m its number of edges, and p the number of available pages. A set of $O(n^3 + m^2)$ clauses ensures that the underlying order is indeed linear, and that no two edges of the same page cross; for details we point the reader to [7].

Using the above SAT formulation, we are able to test various planar graphs on 3-page embeddability. One that does not admit a 3-page book embedding (see Fig. 9) is constructed from graph Q_8 by removing the edge connecting its poles A and B and by identifying its opposite terminals, t_0 and t_7. Formally, start with an embedded $Q_8 = (V_8, E_8)$ having the outerface $\langle A, t_0, B, t_7\rangle$ after the removal of (A, B). Then, *contract* t_0 and t_7, that is, create a new graph $Q_8^\circ = (V_8^\circ, E_8^\circ)$ in which (i) $V_8^\circ = V_8 \setminus \{t_8\}$, (ii) for every edge $(u, v) \in E_8$ such that $u \neq t_7, v \neq t_7$, there exists a corresponding edge $(u, v) \in E_8^\circ$, and (iii) for every edge $(v, t_7) \in E_8$, there exists a corresponding edge $(v, t_0) \in E_8^\circ$; refer to Fig. 9 for an illustration. It is easy to see that the contraction of t_0 and t_8 can be done in a planarity-preserving way, and hence, Q_8° is maximal planar with 275 vertices and 819 edges.

Our early attempts to verify 3-page embeddability of Q_8° were unsuccessful due to an enormous search space of possible satisfying assignments. To reduce the search space, we introduce several *symmetry-breaking* constraints, that is, variable assignments that preserve the satisfiability of an instance:

– we choose a particular vertex as the first one along the spine: $\sigma(A, v)$ for every $v \in V_8^\circ \setminus \{A\}$;

– since Q_8° is symmetric with respect to the terminals, we select t_0 to be the first among the terminals in the vertex ordering: $\sigma(t_0, t_i)$ for all $0 < i \leq 6$;

– a vertex ordering can be reversed without affecting its book embeddability; we introduce a rule so that the SAT instance contains only one of the two possible solutions: $\sigma(t_1, t_2)$;

– to break symmetries of page assignments, we fix an edge to a particular page: $\phi_1(A, t_0)$;

– similarly, another edge can be assigned to one of the first two pages: $\phi_1(B, t_0) \vee \phi_2(B, t_0)$;

– since K_4 is not 1-page book embeddable (as it is not outerplanar), we impose for every K_4 subgraph of Q_8° that not all its edges are assigned to the same page, namely, for every such a subgraph with edges $e_1, \ldots, e_6$ we set: $\neg\phi_\rho(e_1) \vee \ldots \vee \neg\phi_\rho(e_6)$, for every $1 \leq \rho \leq 3$.

With two independent implementations of [7] and using the above extra rules, we are able to verify that graph Q_8° is not 3-page embeddable, thus providing an alternative proof to Theorem 1. The source codes of both implementations are available to the community

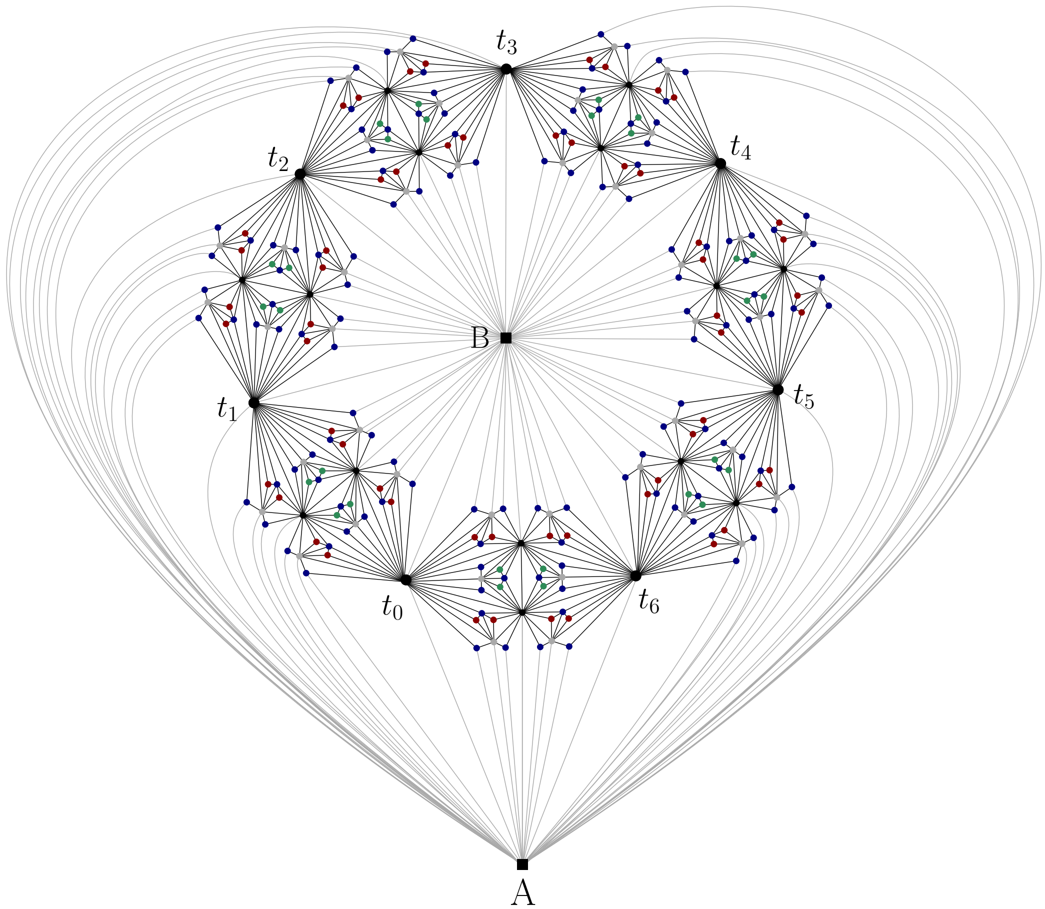


Figure 9: Illustration of graph Q_8° consisting of 275 vertices and 819 edges.

at [5, 41]. The first implementation [41] was executed on a dual-node 28-core 2.4 GHz Intel Xeon E5-2680 machine with 256GB RAM. To verify unsatisfiability, we used the `plingeling` [10] parallel SAT solver, which needed approx. 48 hours using 56 available threads. The second implementation [5] was executed on a much weaker single-node 4-core 3.3 GHz Intel Core i5-4590 machine with 16GB RAM. Since the machine is weaker, to verify unsatisfiability, we split the actual problem into subproblems depending on the number of terminals between A and B. Since the graph is symmetric with respect to A and B, it is enough to assume that there exist 0, 1, 2 or 3 terminals between A and B. For each of the cases, we further distinguish subcases depending on the relative order of these terminals. In total, we consider 28 subproblems, which we solved using the `lingeling` [10] SAT solver on a single thread. The total time needed to verify unsatisfiability of these subproblems was approx. 35 hours.

We emphasize that Q_8° is likely the minimal graph from the considered family that requires four pages. For example, an analogously constructed Q_7°, as well as non-contracted variants, Q_k, with up to $k = 10$, do admit book embeddings in three pages. Similarly, performing fewer stellations yields 3-page embeddable instances.

Unlike computationally expensive processing of Q_8°, our approach is very efficient for verification of Fact 1 and Fact 2. The main reason is that the generated SAT instances contain more constraints, which significantly reduce the search space of possible solutions. We use the same two implementations to verify that Q_7 does not admit a 3-page book embedding under the restrictions of Fact 1 and that Q_{10} does not admit a 3-page book embedding under the restrictions of Fact 2. Both implementations are able to process the graphs within several minutes, even when a single-threaded SAT solver is utilized. Again we stress that the two graphs are minimal in the considered family that satisfy the properties of the facts.

6 Conclusion

By closing the gap between the lower bound and the upper bound on the book thickness of planar graphs, we resolved a problem that remained open for more than thirty years. We mention three interesting research directions that are related to our work.

1. There exist several subclasses of planar graphs with book thickness two proposed in the literature. For example, 4-connected planar graphs [38], planar graphs without separating triangles [33], Halin graphs [16], series-parallel graphs [43], bipartite planar graphs [22], planar graphs of maximum degree 4 [4], triconnected planar graphs of maximum degree 5 [30], and maximal planar graphs of maximum degree 6 [20]. On the other hand, the planar graphs with book thickness three are less studied, and to the best of our knowledge only include the class of planar 3-trees [26]. Recently, Guan and Yang [25] suggested an algorithm to embed general (that is, not necessarily triconnected) planar graphs of maximum degree 5 in books with three pages, but it is not known whether there exist such graphs that require three pages (an open problem of independent research interest). Here, we suggest to study other natural subclasses of planar graphs with book thickness three. Two candidates are: (a) the class of planar Laman graphs, and (b) the class of planar graphs with bounded maximum degree $\Delta \geq 7$. Note that both classes contain members that are not 2-page book embeddable.

2. In the literature, book embeddings are also known as *stack layouts*, since the edges assigned to the same page (called *stack* in this context) follow the last-in-first-out model in the underlying linear order. The "dual" concept of a book embedding is the so-called *queue layout* in which the edges assigned to the same page (called *queue* in this context) follow the first-in-first-out model. A recent breakthrough result by Dujmović et al. [17] suggests that planar graphs admit queue layouts with at most 49 queues. Here, we are asking whether planar graphs admit *mixed* layouts with s stacks and q queues for some $s < 4$ and $q < 49$? Such mixed layouts partition the edges of a graph into s stacks and q queues, while using a common vertex ordering; they have been introduced by Heath, Leighton and Rosenberg [27]. Pupyrev [42] showed that one stack and one queue do not suffice for planar graphs, while de Col et al. [15] proved that testing the existence of a 2-stack 1-queue layout of general (non-planar) graphs is $\mathcal{NP}$-complete.

3. Finally, we would like to see progress on the book thickness of planar directed acyclic graphs (DAGs). Note that in the directed version of the book embedding problem, the edge directions must be consistent with the constructed vertex ordering. Heath et al. [29, 28] asked whether the book thickness of upward planar DAG is bounded by a constant, and they provided constant bounds for directed trees, unicyclic DAGs, and series-parallel DAGs. Frati et al. [21] extended their results in the upward planar triangulations of bounded diameter or of bounded maximum degree. However, the general question remains open.

Acknowledgment. Part of this research was conducted during GNV'19 workshop (30 June - 5 July, 2019, Heiligkreuztal, Germany). We thank the participants for fruitful discussions.

References

[1] Alam, J.M., Bekos, M.A., Gronemann, M., Kaufmann, M., Pupyrev, S.: On dispersable book embeddings. In: Brandstädt, A., Köhler, E., Meer, K. (eds.) Graph-Theoretic Concepts in Computer Science. LNCS, vol. 11159, pp. 1–14. Springer (2018), https://doi.org/10.1007/978-3-030-00256-5_1

[2] Alam, J.M., Bekos, M.A., Gronemann, M., Kaufmann, M., Pupyrev, S.: Queue layouts of planar 3-trees. In: Biedl, T.C., Kerren, A. (eds.) Graph Drawing and Network Visualization. LNCS, vol. 11282, pp. 213–226. Springer (2018), https://doi.org/10.1007/978-3-030-04414-5_15

[3] Bekos, M.A., Bruckdorfer, T., Kaufmann, M., Raftopoulou, C.N.: The book thickness of 1-planar graphs is constant. Algorithmica 79(2), 444–465 (2017), https://doi.org/10.1007/s00453-016-0203-2

[4] Bekos, M.A., Gronemann, M., Raftopoulou, C.N.: Two-page book embeddings of 4-planar graphs. Algorithmica 75(1), 158–185 (2016), https://doi.org/10.1007/s00453-015-0016-8

[5] Bekos, M.A., Haug, M., Kaufmann, M., Männecke, J.: An online framework to interact and efficiently compute linear layouts of graphs. CoRR abs/2003.09642 (2020), http://arxiv.org/abs/2003.09642, online version http://algo.inf.uni-tuebingen.de/linearlayouts; source code available at https://github.com/linear-layouts/SAT

[6] Bekos, M.A., Kaufmann, M., Klute, F., Pupyrev, S., Raftopoulou, C.N., Ueckerdt, T.: Four pages are indeed necessary for planar graphs. CoRR abs/2004.07630 (2020), https://arxiv.org/abs/2004.07630

[7] Bekos, M.A., Kaufmann, M., Zielke, C.: The book embedding problem from a SAT-solving perspective. In: Giacomo, E.D., Lubiw, A. (eds.) Graph Drawing and Network Visualization. LNCS, vol. 9411, pp. 125–138. Springer (2015), https://doi.org/10.1007/978-3-319-27261-0_11

[8] Bernhart, F., Kainen, P.C.: The book thickness of a graph. J. Comb. Theory, Ser. B 27(3), 320–331 (1979), https://doi.org/10.1016/0095-8956(79)90021-2

[9] Biedl, T.C., Shermer, T.C., Whitesides, S., Wismath, S.K.: Bounds for orthogonal 3D graph drawing. J. Graph Algorithms Appl. 3(4), 63–79 (1999), http://dx.doi.org/10.7155/jgaa.00018

[10] Biere, A.: Lingeling, Plingeling and Treengeling entering the SAT Competition 2013. In: Balint, A., Belov, A., Heule, M., Järvisalo, M. (eds.) SAT Competition 2013. Department of Computer Science Series of Publications B, vol. B-2013-1, pp. 51–52. University of Helsinki (2013), source code available at https://github.com/arminbiere/lingeling

[11] Binucci, C., Da Lozzo, G., Giacomo, E.D., Didimo, W., Mchedlidze, T., Patrignani, M.: Upward book embeddings of st-graphs. In: Symposium on Computational Geometry. LIPIcs, vol. 129, pp. 13:1–13:22. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik (2019), https://doi.org/10.4230/LIPIcs.SoCG.2019.13

[12] Blankenship, R.L.: Book Embeddings of Graphs. Ph.D. thesis, Louisiana State University (2003)

[13] Buss, J.F., Shor, P.W.: On the pagenumber of planar graphs. In: DeMillo, R.A. (ed.) ACM Symposium on Theory of Computing. pp. 98–100. ACM (1984), http://doi.acm.org/10.1145/800057.808670

[14] Chung, F.R.K., Leighton, F.T., Rosenberg, A.L.: Embedding graphs in books: A layout problem with applications to VLSI design. SIAM Journal on Algebraic and Discrete Methods 8(1), 33–58 (1987), http://dl.acm.org/citation.cfm?id=21936.25439

[15] de Col, P., Klute, F., Nöllenburg, M.: Mixed linear layouts: Complexity, heuristics, and experiments. In: Archambault, D., Tóth, C.D. (eds.) Graph Drawing and Network Visualization. LNCS, vol. 11904, pp. 460–467. Springer (2019), https://doi.org/10.1007/978-3-030-35802-0_35

[16] Cornuéjols, G., Naddef, D., Pulleyblank, W.R.: Halin graphs and the travelling salesman problem. Math. Program. 26(3), 287–294 (1983), https://doi.org/10.1007/BF02591867

[17] Dujmović, V., Joret, G., Micek, P., Morin, P., Ueckerdt, T., Wood, D.R.: Planar graphs have bounded queue-number. In: Zuckerman, D. (ed.) FOCS. pp. 862–875. IEEE Computer Society (2019), https://doi.org/10.1109/FOCS.2019.00056

[18] Dujmović, V., Wood, D.R.: Graph treewidth and geometric thickness parameters. Discrete & Computational Geometry 37(4), 641–670 (2007), https://doi.org/10.1007/s00454-007-1318-7

[19] Erdös, P., Szekeres, G.: A combinatorial problem in geometry. Compositio Mathematica 2, 463–470 (1935), http://www.numdam.org/item/CM_1935__2__463_0

[20] Ewald, G.: Hamiltonian circuits in simplicial complexes. Geometriae Dedicata 2(1), 115–125 (1973), https://doi.org/10.1007/BF00149287

CC BY

[21] Frati, F., Fulek, R., Ruiz-Vargas, A.J.: On the page number of upward planar directed acyclic graphs. J. Graph Algorithms Appl. 17(3), 221–244 (2013), https://doi.org/10.7155/jgaa.00292

[22] de Fraysseix, H., de Mendez, P.O., Pach, J.: A left-first search algorithm for planar graphs. Discrete & Computational Geometry 13, 459–468 (1995), https://doi.org/10.1007/BF02574056

[23] Ganley, J.L., Heath, L.S.: The pagenumber of k-trees is $O(k)$. Discrete Applied Mathematics 109(3), 215–221 (2001), https://doi.org/10.1016/S0166-218X(00)00178-5

[24] Goldner, A., Harary, F.: Note on a smallest nonhamiltonian maximal planar graph. Bulletin of the Malaysian Math. Sci. Soc. 1(6), 41–42 (1975)

[25] Guan, X., Yang, W.: Embedding planar 5-graphs in three pages. Discret. Appl. Math. (2019), https://doi.org/10.1016/j.dam.2019.11.020

[26] Heath, L.S.: Embedding planar graphs in seven pages. In: FOCS. pp. 74–83. IEEE Computer Society (1984), https://doi.org/10.1109/SFCS.1984.715903

[27] Heath, L.S., Leighton, F.T., Rosenberg, A.L.: Comparing queues and stacks as mechanisms for laying out graphs. SIAM J. Discrete Math. 5(3), 398–412 (1992), https://doi.org/10.1137/0405031

[28] Heath, L.S., Pemmaraju, S.V.: Stack and queue layouts of directed acyclic graphs: Part II. SIAM J. Comput. 28(5), 1588–1626 (1999), https://doi.org/10.1137/S0097539795291550

[29] Heath, L.S., Pemmaraju, S.V., Trenk, A.N.: Stack and queue layouts of directed acyclic graphs: Part I. SIAM J. Comput. 28(4), 1510–1539 (1999), https://doi.org/10.1137/S0097539795280287

[30] Hoffmann, M., Klemz, B.: Triconnected planar graphs of maximum degree five are subhamiltonian. In: Bender, M.A., Svensson, O., Herman, G. (eds.) ESA. LIPIcs, vol. 144, pp. 58:1–58:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2019), https://doi.org/10.4230/LIPIcs.ESA.2019.58

[31] Istrail, S.: An algorithm for embedding planar graphs in six pages. Iasi University Annals, Mathematics-Computer Science 34(4), 329–341 (1988)

[32] Jacobson, G.: Space-efficient static trees and graphs. In: Symposium on Foundations of Computer Science. pp. 549–554. IEEE Computer Society (1989), https://doi.org/10.1109/SFCS.1989.63533

[33] Kainen, P.C., Overbay, S.: Extension of a theorem of Whitney. Appl. Math. Lett. 20(7), 835–837 (2007), https://doi.org/10.1016/j.aml.2006.08.019

[34] Malitz, S.M.: Genus g graphs have pagenumber $O(\sqrt{g})$. J. Algorithms 17(1), 85–109 (1994), https://doi.org/10.1006/jagm.1994.1028

[35] Malitz, S.M.: Graphs with E edges have pagenumber $O(\sqrt{E})$. J. Algorithms 17(1), 71–84 (1994), https://doi.org/10.1006/jagm.1994.1027

[36] Munro, J.I., Raman, V.: Succinct representation of balanced parentheses and static trees. SIAM J. Comput. 31(3), 762–776 (2001), https://doi.org/10.1137/S0097539799364092

[37] Nesetril, J., de Mendez, P.O.: Sparsity - Graphs, Structures, and Algorithms, Algorithms and combinatorics, vol. 28. Springer (2012), https://doi.org/10.1007/978-3-642-27875-4

[38] Nishizeki, T., Chiba, N.: Planar Graphs: Theory and Algorithms, chap. 10. Hamiltonian Cycles, pp. 171–184. Dover Books on Mathematics, Courier Dover Publications (2008)

[39] Ollmann, T.: On the book thicknesses of various graphs. In: Hoffman, F., Levow, R., Thomas, R. (eds.) Southeastern Conference on Combinatorics, Graph Theory and Computing. Congressus Numerantium, vol. VIII, p. 459 (1973)

[40] Pratt, V.R.: Computing permutations with double-ended queues, parallel stacks and parallel queues. In: Aho, A.V., Borodin, A., Constable, R.L., Floyd, R.W., Harrison, M.A., Karp, R.M., Strong, H.R. (eds.) ACM Symposium on Theory of Computing. pp. 268–277. ACM (1973), https://doi.org/10.1145/800125.804058

[41] Pupyrev, S.: A SAT-based solver for constructing optimal linear layouts of graphs, source code available at https://github.com/spupyrev/bob

[42] Pupyrev, S.: Mixed linear layouts of planar graphs. In: Frati, F., Ma, K. (eds.) Graph Drawing and Network Visualization. LNCS, vol. 10692, pp. 197–209. Springer (2017), https://doi.org/10.1007/978-3-319-73915-1_17

[43] Rengarajan, S., Madhavan, C.E.V.: Stack and queue number of 2-trees. In: Du, D., Li, M. (eds.) COCOON. LNCS, vol. 959, pp. 203–212. Springer (1995), https://doi.org/10.1007/BFb0030834

[44] Rosenberg, A.L.: The diogenes approach to testable fault-tolerant arrays of processors. IEEE Trans. Computers 32(10), 902–910 (1983), https://doi.org/10.1109/TC.1983.1676134

[45] Tarjan, R.E.: Sorting using networks of queues and stacks. J. ACM 19(2), 341–346 (1972), https://doi.org/10.1145/321694.321704

[46] Wigderson, A.: The complexity of the Hamiltonian circuit problem for maximal planar graphs. Tech. Rep. TR-298, EECS Department, Princeton University (1982)

[47] Wood, D.R.: Degree constrained book embeddings. J. Algorithms 45(2), 144–154 (2002), https://doi.org/10.1016/S0196-6774(02)00249-3

[48] Yannakakis, M.: Four pages are necessary and sufficient for planar graphs (extended abstract). In: Hartmanis, J. (ed.) ACM Symposium on Theory of Computing. pp. 104–108. ACM (1986), https://doi.org/10.1145/12130.12141

[49] Yannakakis, M.: Embedding planar graphs in four pages. J. Comput. Syst. Sci. 38(1), 36–67 (1989), https://doi.org/10.1016/0022-0000(89)90032-9

[50] Yannakakis, M.: Planar graphs that need four pages. Journal of Combinatorial Theory, Series B 145, 241–263 (2020), https://doi.org/10.1016/j.jctb.2020.05.008

[51] Yannakakis, M.: Planar graphs that need four pages. CoRR abs/2005.14111 (2020), https://arxiv.org/abs/2005.14111