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Abstract

We consider a Schrödinger operator with a Hermitian 2x2 matrix-valued potential which is lattice periodic and can be diagonalized smoothly on the whole R^n. In the case of potential taking its minimum only on the lattice, we prove that the well-known semiclassical asymptotic of first band spectrum for a scalar potential remains valid for our model.

Keywords : semiclassical asymptotic, spectrum, eigenvalues, Schrodinger, periodic potential, BKW method, width of the first band, magnetic field.
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Contents

1 Introduction

Schrödinger operators with periodic matrix-valued potentials appear in many models in physics. Such models have been used recently to describe the motion of an atom in optical fields ([Co], [Co-Da], [Da-al]), see also [Ca-Yu]. The aim of this paper is to investigate their spectral properties using semiclassical analysis. We focus on the first spectral band and assume that the potential has a non degenerate minimum. The Schrödinger operators with a non-Abelian gauge potential are Hamiltonian operators on $L^2(\mathbb{R}^n;\mathbb{C}^m)$ of the following form :

$$H^h \;=\; h^2 \textstyle\sum_{k=1}^{n}(D_{x_k}I - A_k)^2 + V + hQ + h^2R = P^h(x,hD)\;. \tag{1.1}$$

The classical symbol of $P^h(x,hD)$, $P^h(x,\xi)$, for $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$, is given by

$$P^h(x,\xi) = \sum_{k=1}^{n}\big\{(\xi_k I - hA_k(x))^2 + ih^2\partial_{x_k}A_k(x)\big\} + V(x) + hQ(x) + h^2R(x)\;, \tag{1.2}$$

I is the identity $m\times m$ matrix, $V,\ Q,\ R$ and the A_k are hermitian $m\times m$ matrix with smooth coefficients and Γ periodic:

$$\left.\begin{array}{l} A_k = (a_{k,ij}(x))_{1\le i,j\le m}, \quad V = (v_{ij}(x))_{1\le i,j\le m},\\ Q = (q_{ij}(x))_{1\le i,j\le m}, \quad R = (r_{ij}(x))_{1\le i,j\le m},\\ a_{k,ij},\ v_{ij},\ q_{ij},\ r_{ij}\in C^\infty(\mathbb{R}^n;\mathbb{C}),\\ \overline{a_{k,ji}} = a_{k,ij},\ \overline{v_{ji}} = v_{ij},\ \overline{q_{ji}} = q_{ij},\ \overline{r_{ji}} = r_{ij}\\ a_{k,ij}(x-\gamma) = a_{k,ij}(x),\ v_{ij}(x-\gamma) = v_{ij}(x),\\ q_{ij}(x-\gamma) = q_{ij}(x) \text{ and } r_{ij}(x-\gamma) = r_{ij}(x)\ \forall\gamma\in\Gamma\;; \end{array}\right\} \tag{1.3}$$

Γ is a lattice of $\mathbb{R}^n$, $\Gamma = \{\sum_{k=1}^{n} m_k\beta_k;\ m_k\in\mathbb{Z}\}$,
$\beta_1,\beta_2,\ldots,\beta_n\in\mathbb{R}^n$ form a basis, $det(\beta_1,\beta_2,\ldots,\beta_n)\neq 0$.
We use the notation $D = (D_{x_1},\ldots,D_{x_n})$ where $D_{x_k} = -i\partial_{x_k},\ k = 1...n$, so $D^2 = -\Delta$ is the Laplacian operator on $L^2(\mathbb{R}^n)$.

The dual basis $\{\beta_1^\star,\ldots,\beta_n^\star\}$ of the reciprocal lattice $\Gamma^\star$, is the basis of $\mathbb{R}^n$ defined by the relations

$$\beta_j^\star.\beta_k = 2\pi\delta_{jk}\ :\quad \Gamma^\star = \{\sum_{k=1}^{n} m_k\beta_k^\star;\ m_k\in\mathbb{Z}\}.$$

The fundamental cell, the Wigner-Seitz cell,

$$\mathbb{W}^n = \{\sum_{k=1}^{n} x_k\beta_k\;;\ x_k\in]-\frac{1}{2},\frac{1}{2}[\}\,,$$

will be identified with the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\Gamma$ and the dual cell, the Brillouin zone, is defined by

$$\mathbb{B}^n = \{\sum_{k=1}^{n} \theta_k \beta_k^\star ; \theta_k \in] -\frac{1}{2}, -\frac{1}{2}[\} \ .$$

We will identify $L^2(\mathbb{T}^n;\mathbb{C}^m)$ with Γ periodic functions of $L^2_{loc}(\mathbb{R}^n;\mathbb{C}^m)$ provided with the norm of $L^2(\mathbb{W}^n;\mathbb{C}^m)$. In the same way the Sobolev space $W^k(\mathbb{T}^n;\mathbb{C}^m)$, with $k \in \mathbb{N}$, may be identified with Γ periodic functions of $W^k_{loc}(\mathbb{R}^n;\mathbb{C}^m)$ provided with the norm of $W^k(\mathbb{W}^n;\mathbb{C}^m)$.

By Floquet theory, (see [Ea] or [Re-Si]), we have

$$H^h = \int_{\mathbb{B}^n}^{\oplus} H^{h,\theta} d\theta \ ,$$

with $H^{h,\theta}$ the partial differential operator $P_h(x, h(D-\theta))$ on $L^2(\mathbb{T}^n;\mathbb{C}^m)$.
The ellipticity of $P_h(x, h(D-\theta))$ implies that the spectrum of $H^{h,\theta}$ is discrete

$$\mathrm{sp}(H^{h,\theta}) = \{\lambda_j^{h,\theta};\ j \in \mathbb{N}^\star\},\ \lambda_1^{h,\theta} \leq \lambda_2^{h,\theta} \leq \ldots \leq \lambda_j^{h,\theta} \leq \lambda_{j+1}^{h,\theta} \leq \ldots \tag{1.4}$$

each $\lambda_j^{h,\theta}$ is an eigenvalue of finite multiplicity and each eigenvalue is repeated according to its multiplicity.
(When $m = 1$ and $V = Q = R = A_k = 0$, $(\frac{1}{\sqrt{|\mathbb{T}^n|}} e^{i\omega.x})_{\omega\in\Gamma^\star}$ is the Hilbert basis of $L^2(\mathbb{T}^n)$ which is composed of eigenfunctions of $h^2(D-\theta)^2$).

The Floquet theory guarantees that

$$\mathrm{sp}(H^h) = \bigcup_{\theta\in\mathbb{B}^n} \mathrm{sp}(H^{h,\theta}) = \bigcup_{j=1}^{\infty} b_j^h \ , \tag{1.5}$$

where b_j^h denotes the j-th band $b_j^h = \{\lambda_j^{h,\theta}, \theta \in \mathbb{B}^n\}$.

In the sequel h_0 will be a non negative small constant, h will be in $]0, h_0[$, and any non negative constant which doesn't depend on h will invariably be denoted by C.

2 Preliminary: the artificial gauge model

We will be interested in the model of artificial gauge considered in [Co], [Co-Da] and [Da-al]

$$\left.\begin{array}{l} m = 2,\ V = vI + W,\ A_k = Q = R = 0,\ \forall k, \\ W = w.\sigma,\ \text{with } w = (w_1, w_2, w_3),\ v \text{ and the } w_j \text{ are in } C^\infty(\mathbb{R}^n;\mathbb{R}), \end{array}\right\} \tag{2.1}$$

we denote $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, where the σ_j are the Pauli matrices.

Let us remark that

$$V = vI + W, \quad W = w.\sigma, \quad W^2 = |w|^2 I \ . \tag{2.2}$$

In the sequel we will assume that

$$\left.\begin{array}{l} |w| > 0 \\ v(x) - |w(x)| \text{ has a unique non degenerate minimum on } \mathbb{T}^n. \end{array}\right\} \tag{2.3}$$

Due to the invariance of the Laplacian by translation and by the action of $\mathbb{O}(n)$, we can assume, up to a composition by a translation of the potentials, that

$$\left.\begin{array}{l} v(\gamma) - |w(\gamma)| < v(x) - |w(x)|, \ \forall x \in \mathbb{R}^n \setminus \Gamma \text{ and } \forall \gamma \in \Gamma, \\ v(x) - |w(x)| = E_0 + \displaystyle\sum_{k=1}^{n} \tau_k^2 x_k^2 + \mathbf{O}(|x|^3), \ \text{as } |x| \to 0, \end{array}\right\} \tag{2.4}$$

$(\tau_k > 0, \ \forall k)$.

There exists $U \in \mathbb{U}(2)$, (a unitary 2×2 matrix), such that

$$U^\star V U = \widetilde{V} = \left[\begin{array}{cc} v - |w| & 0 \\ 0 & v + |w| \end{array} \right] . \tag{2.5}$$

As $|w|$ never vanishes, $U = U(x)$ can be chosen smooth and Γ periodic:

$$U = \left(\begin{array}{cc} u_{11} & u_{12} \\ u_{21} & u_{22} \end{array} \right) \in C^\infty(\mathbb{T}^n; \mathbb{U}(2));$$

for example $u_{11} = \dfrac{1}{2\sqrt{|w|(|w| - \mathrm{Re}((w_1 + iw_2)e^{-i\theta}))}}(w_3 - |w| + e^{i\theta}(w_1 - iw_2)),$

$u_{21} = \dfrac{1}{2\sqrt{|w|(|w| - \mathrm{Re}(w_1 + iw_2)e^{-i\theta}))}}(w_1 + iw_2 - e^{i\theta}(w_3 + |w|)),$

$u_{12} = \overline{u_{21}}, \ u_{22} = -\overline{u_{11}}$ and $\theta = \chi(\dfrac{w_2^2 + w_3^2}{|w|^2})\dfrac{\pi}{2},$

where $\chi(t)$ is a smooth function on the real line, $0 \le \chi(t) \le 1$,

$\chi(t) = 1$ when $|t| \le 1/4$ and $\chi(t) = 0$ when $|t| \ge 1/2$. So

$$U = (\alpha, \beta, \rho).\sigma + i\delta\sigma_0\, , \quad \text{with } (\alpha, \beta, \rho, \delta) \in C^\infty(\mathbb{T}^n; \mathbb{S}^3); \tag{2.6}$$

σ_0 is the 2×2 identity matrix and $\mathbb{S}^3$ is the unit sphere of $\mathbb{R}^4$.

When $w_1 + iw_2 \neq 0$ or when $w_3 < 0$, one can choose U such that $\delta = 0$ by taking $(\alpha, \beta, \rho) = \dfrac{1}{\sqrt{2|w|}}(-\dfrac{w_1}{\sqrt{|w| - w_3}}, -\dfrac{w_2}{\sqrt{|w| - w_3}}, \sqrt{|w| - w_3}).$

Firstly let us expand the formula of the operator

$$\widetilde{H}^h = U^\star H^h U = h^2 D^2 I + U^\star V U - 2ih^2 \sum_{k=1}^{n} \left[(U^\star \partial_{x_k} U) D_{x_k} - h^2 U^\star \partial_{x_k}^2 U \right]$$

which can be rewritten as

$$\widetilde{H}^h = U^\star H^h U = h^2 \sum_{k=1}^{n} (D_{x_k} I - A_k)^2 + \widetilde{V} + h^2 R\,, \tag{2.7}$$

where $A_k = iU^\star \partial_{x_k} U$:

$$A_k = [(\partial_{x_k}\alpha, \partial_{x_k}\beta, \partial_{x_k}\rho)\wedge(\alpha,\beta,\rho) + (\delta\partial_{x_k}\alpha - \alpha\partial_{x_k}\delta, \delta\partial_{x_k}\beta - \beta\partial_{x_k}\delta, \delta\partial_{x_k}\rho - \rho\partial_{x_k}\delta)].\sigma\,, \tag{2.8}$$

and

$$R = \sum_{k=1}^{n} \left\{ (U^\star \partial_{x_k} U)^2 + (\partial_{x_k} U^\star).(\partial_{x_k} U) \right\}\ . \tag{2.9}$$

So we can assume that H^h is of the form (1.1) with $m = 2,\ Q = 0,\ A_k$ and R given by (2.8) and (2.9), with U defined by (2.6), and $V = \widetilde{V}$ a diagonal matrix given by (2.5).

Theorem 2.1 *Under the above assumptions, the first bands $b_j^h,\ j = 1, 2, \ldots,$ of H^h are concentrated around the value $h\mu_j + E_0$ $j = 1, 2, \ldots,$ in the sense that, there exist $N_0 > 1$ and $h_0 > 0$ such that*

$$distance(h\mu_j + E_0, b_j^h) \le Ch^2, \quad \forall j < N_0 \text{ and } \forall h,\ 0 < h < h_0,$$

where $\mu_j = \sum_{k=1}^{n} (2j_k + 1)\tau_k,\ j_k \in \mathbb{N}$, the $(\mu_\ell)_{\ell\in\mathbb{N}^\star}$ is the increasing sequence of the eigenvalues of the harmonic oscillator $-\Delta + \sum_{k=1}^{n} \tau_k^2 x_k^2$.

3 Proof of Theorem 2.1

Proof. According to the above discussion, we can assume that

$$H^h = P^h(x, hD)\,, \quad \text{with } P^h(x, hD) = \begin{pmatrix} P_{11}^h(x, hD) & P_{12}^h(x, hD) \\ P_{21}^h(x, hD) & P_{22}^h(x, hD) \end{pmatrix}, \tag{3.1}$$

with

$$\left.\begin{array}{l} P_{11}^h(x, hD) = h^2(D - a_{.,11}(x))^2 + v(x) - |w(x)| + h^2 r_{11}(x) \\ P_{22}^h(x, hD) = h^2(D + a_{.,11}(x))^2 + v(x) + |w(x)| + h^2 r_{22}(x) \\ P_{12}^h(x, hD) = -h^2 a_{.,12}(x).(D + a_{.,11}(x)) - h^2 a_{.,12}(x).(D - a_{.,11}(x)) \\ \qquad\qquad + ih^2 div(a_{.,12}(x)) + h^2 r_{12}(x) \\ P_{21}^h(x, hD) = -h^2 a_{.,21}(x)(D - a_{.,11}(x)) - h^2 a_{.,21}(x).(D + a_{.,11}(x)) \\ \qquad\qquad + ih^2 div(a_{.,21}(x)) + h^2 r_{21}(x)\,. \end{array}\right\} \tag{3.2}$$

($D = (D_{x_1}, D_{x_2}, \dots, D_{x_n})$ and $a_{.,ij}(x) = (a_{1,ij}(x), a_{2,ij}(x), \dots, a_{n,ij}(x))$.)
(We used that $a_{.,22} = -a_{.,11}$).

Let us denote by $H_{11}^{h,\theta}$ and $H_{22}^{h,\theta}$ the operators associated with $P_{11}^h(x, h(D-\theta))$ and $P_{22}^h(x, h(D-\theta))$ on $L^2(\mathbb{T}^n; \mathbb{C})$.
Then, if $c_0 = \min |w(x)|$ and $c_1 = \max \|R(x)\|$,

$$\mathrm{sp}(H_{11}^{h,\theta}) \subset [E_0 - h^2 c_1, +\infty[\quad \text{and} \quad \mathrm{sp}(H_{22}^{h,\theta}) \subset [E_0 - h^2 c_1 + 2c_0, +\infty[.$$

To prove the theorem it is then enough to prove the proposition below.

Proposition 3.1 *Let us consider a constant c, $0 < c < c_0$. Then there exists $C_0 > 0$ such that, for any $E^h \in]-\infty, E_0 + 2c[$, we have*

$$\left.\begin{array}{l} E^h \in \mathrm{sp}(H^{h,\theta}) \;\Rightarrow\; distance(E^h, \mathrm{sp}(H_{11}^{h,\theta}) \le C_0 h^2 \\ E^h \in \mathrm{sp}(H_{11}^{h,\theta}) \;\Rightarrow\; distance(E^h, \mathrm{sp}(H^{h,\theta})) \le C_0 h^2 \,. \end{array}\right\} \tag{3.3}$$

Proof. For such E^h, $(H_{22}^{h,\theta} - E^h)^{-1}$ exists and, thanks to semiclassical pseudodifferential calculus of [Ro] (see also [Di-Sj]), for $h_0 > 0$ small, if $0 < h < h_0$ then $\|(H_{22}^{h,\theta} - E^h)^{-1}\|_{L^2(\mathbb{T}^n)} + \|h(D-\theta)(H_{22}^{h,\theta} - E^h)^{-1}\|_{L^2(\mathbb{T}^n)}$
$+\|(H_{22}^{h,\theta} - E^h)^{-1} h(D-\theta)\|_{L^2(\mathbb{T}^n)} + \|h(D-\theta)(H_{22}^{h,\theta} - E^h)^{-1} h(D-\theta)\|_{L^2(\mathbb{T}^n)} \le C$, and then

$$\|P_{12}^h(x, h(D-\theta))(H_{22}^{h,\theta} - E^h)^{-1} P_{21}^h(x, h(D-\theta))\|_{L^2(\mathbb{T}^n)} \le h^2 C.$$

So if $E^h \in \mathrm{sp}(H^{h,\theta})$, then $u^h = (u_1^h, u_2^h) \neq (0,0)$ is an eigenfunction of $H^{h,\theta}$ associated with E^h iff

$$\left.\begin{array}{l} H_{11}^{h,\theta} u_1^h + P_{12}^h(x, h(D-\theta)) u_2^h = E^h u_1^h \\ u_2^h = -(H_{22}^{h,\theta} - E^h I)^{-1} P_{21}^h(x, h(D-\theta)) u_1^h \,. \end{array}\right\} \tag{3.4}$$

In fact $E^h \in]-\infty, E_0 + c[$ will be an eigenvalue of $H^{h,\theta}$ iff there exists u_1^h in the Sobolev space $W^2(\mathbb{T}^n; \mathbb{C})$, $\|u_1^h\|_{L^2(\mathbb{T}^n)} \neq 0$, such that

$$H_{11}^{h,\theta} u_1^h - P_{12}^h(x, h(D-\theta))(H_{22}^{h,\theta} - E^h I)^{-1} P_{21}^h(x, h(D-\theta)) u_1^h = E^h u_1^h \,,$$

then we get the first part of Proposition 3.1.

If E^h is an eigenvalue of $H_{11}^{h,\theta}$ satisfying the assumption of Proposition 3.1, and u_1^h an associated eigenfunction, then with $u^h = (u_1^h, -(H_{22}^{h,\theta} - E^h)^{-1} P_{21}^h(x, h(D-\theta)) u_1^h)$, one has

$$\begin{array}{l} \|(H^{h,\theta} - E^h I) u^h\|_{L^2(\mathbb{T}^n; \mathbb{C}^2)} \\ = \|P_{12}^h(x, h(D-\theta))(H_{22}^{h,\theta} - E^h)^{-1} P_{21}^h(x, h(D-\theta)) u_1^h\|_{L^2(\mathbb{T}^n; \mathbb{C})} \\ \le h^2 C \|u^h\|_{L^2(\mathbb{T}^n; \mathbb{C}^2)} \,, \end{array}$$

we get the second part of Proposition 3.1.

Theorem 2.1 follows from Proposition 3.1 and [Si-1], [Si-2], [He-Sj-1] and [He-Sj-2] results, (see also [He]), which guarantee that the sequence of eigenvalues of $H_{11}^{h,\theta}$, $(\lambda_j(H_{11}^{h,\theta}))_{j \in \mathbb{N}^\star}$ satisfies $\forall N_0 > 1$, $\exists h_0 > 0$, $C_0 > 0$ *s.t.* $\forall h$, $0 < h < h_0$ and $\forall j \le N_0$, $|\lambda_j(H_{11}^{h,\theta}) - (h\mu_j^h + E_0)| \le C_0 h^2$ □

4 Asymptotic of the first band

For any real Lipschitz Γ periodic function ϕ, and for any $u \in W^2(\mathbb{T}^n;\mathbb{C}^2)$, we have the identity

$$\left.\begin{array}{l} \mathrm{Re}\left(< P^h(x,h(D-\theta))u \,|\, e^{2\phi/h}u >_{L^2(\mathbb{T}^n;\mathbb{C}^2)}\right) = \\ \displaystyle\sum_{k=1}^n h^2 \|((D_{x_k}-\theta_k)I - A_k)e^{\phi/h}u\|^2_{L^2(\mathbb{T}^n;\mathbb{C}^2)} \\ \quad + < (\widetilde{V} - |\nabla\phi|^2 I + h^2 R)u \,|\, e^{2\phi/h}u >_{L^2(\mathbb{T}^n;\mathbb{C}^2)} . \end{array}\right\} \tag{4.1}$$

This identity enables us to apply the method used in [He-Sj-1], (see also [He] and [Ou]). We define the Agmon [Ag] distance on $\mathbb{R}^n$

$$d(y,x) = \inf_\gamma \int_0^1 \sqrt{v(\gamma(t)) - |w(\gamma(t))| - E_0}\,|\dot\gamma(t)|dt \ , \tag{4.2}$$

the inf is taken among paths such that $\gamma(0) = y$ and $\gamma(1) = x$.
For common properties of the Agmon distance, one can see for example [Hi-Si].

We will use that, for any fixed $y \in \mathbb{R}^n$, the function $d(y,x)$ is a Lipschitz function on $\mathbb{R}^n$ and $|\nabla_x d(y,x)|^2 \leq v(x) - |w(x)| - E_0$ almost everywhere on $\mathbb{R}^n$.

Using that the zeros of $v(x) - w(x) - E_0$ are the elements of Γ and are non degenerate, we get that the real function $d_0(x) = d(0,x)$ satisfies, (see [He-Sj-1]), $|\nabla d_0(x)|^2 = v(x) - |w(x)| - E_0$ in a neighbourhood of 0.

We summarize the properties of the Agmon distance we will need:

$$\left.\begin{array}{ll} i) & \exists\, R_0 > 0 \text{ s.t. } d_0(x) \in C^\infty(B_0(R_0)) \\ ii) & |\nabla d_0(x)|^2 = v(x) - |w(x)| - E_0 \ , \ \forall x \in B_0(R_0) \\ iii) & |\nabla d_0(x)|^2 \leq v(x) - |w(x)| - E_0 \\ iv) & |\nabla d_\Gamma(x)|^2 \leq v(x) - |w(x)| - E_0 \end{array}\right\} \tag{4.3}$$

where $d_0(x) = d(0,x)$, $B_0(r) = \{x \in \mathbb{R}^n;\ d_0(x) < r\}$
and $d_\Gamma(x) = d(\Gamma, x) = \min\limits_{\omega\in\Gamma} d(\omega, x)$.

The least Agmon distance in Γ is

$$S_0 = \inf_{1\leq k\leq n} d_0(\beta_k) = \inf_{\rho\neq\omega,\ (\omega,\rho)\in\Gamma^2} d(\omega,\rho) \ . \tag{4.4}$$

The Agmon distance on $\mathbb{T}^n$, $d^{\mathbb{T}^n}(.,.)$, is defined by its Γ-periodic extension on $(\mathbb{R}^n)^2$

$$d^{\mathbb{T}^n}(y,x) = \min_{\omega\in\Gamma} d(y, x+\omega).$$

Then

$$\frac{S_0}{2} = \sup_r\{r > 0 \ s.t. \ \{x \in \mathbb{T}^n;\ d^{\mathbb{T}^n}(x_0,x) < r\} \text{ is simply connected}\} \ , \tag{4.5}$$

where x_0 is the single point in $\mathbb{T}^n$ such that $v(x_0) - |w(x_0)| = E_0$. The Γ-periodic function on $\mathbb{R}^n$, $d_\Gamma(x)$ is the one corresponding to the extension of $d^{\mathbb{T}^n}(x_0, x)$.

If $\lambda^{h,\theta}$ is an eigenvalue of $H^{h,\theta}$ and if $u^{h,\theta}$ is an associated eigenfunction, then by (4.1) one gets as in the scalar case considered in [He-Sj-1]) and [He-Sj-2],

$$\left.\begin{array}{l} \sum_{k=1}^{n} h^2 \|((D_{x_k} - \theta_k)I - A_k) e^{\phi/h} u^{h,\theta}\|^2_{L^2(\mathbb{T}^n;\mathbb{C}^2)} \\ \quad + < [\widetilde{V} - |\nabla\phi|^2 I + h^2 R - \lambda^{h,\theta} I]_+ u^{h,\theta} \,|\, e^{2\phi/h} u^{h,\theta} >_{L^2(\mathbb{T}^n;\mathbb{C}^2)} \\ = < [\widetilde{V} - |\nabla\phi|^2 I + h^2 R - \lambda^{h,\theta} I]_- u^{h,\theta} \,|\, e^{2\phi/h} u^{h,\theta} >_{L^2(\mathbb{T}^n;\mathbb{C}^2)}, \end{array}\right\} \tag{4.6}$$

so, when $\phi(x) = d^{\mathbb{T}^n}(x_0, x)$, necessarily $\lambda^{h,\theta} - E_0 + \mathbf{O}(h^2) > 0$,
and if $h/C < \lambda^{h,\theta} - E_0 < hC$, then $u^{h,\theta}$ is localized in energy near x_0, for any $\eta \in]0, 1[$, $\exists C_\eta > 0$ such that

$$\left.\begin{array}{l} \sum_{k=1}^{n} h^2 \|((D_{x_k} - \theta_k)I - A_k) e^{\eta d^{\mathbb{T}^n}(x_0,x)/h} u^{h,\theta}\|^2_{L^2(\mathbb{T}^n;\mathbb{C}^2)} \\ \quad +(1-\eta^2) < (v - |w| - E_0) u^{h,\theta} \,|\, e^{2\eta d^{\mathbb{T}^n}(x_0,x)/h} u^{h,\theta} >_{L^2(\mathbb{T}^n;\mathbb{C}^2)} \\ \leq hC_\eta \int_{\{x \in \mathbb{T}^n;\ d^{\mathbb{T}^n}(x_0,x) < \sqrt{h}C\}} |u^{h,\theta}(x)|^2 dx. \end{array}\right\} \tag{4.7}$$

Let $\Omega \subset \mathbb{T}^n$ an open and simply connected set with smooth boundary satisfying, for some η, $0 < \eta < S_0/2$,

$$\{x \in \mathbb{T}^n;\ d^{\mathbb{T}^n}(x_0, x) < \frac{S_0 - \eta}{2}\} \subset \Omega \subset \{x \in \mathbb{T}^n;\ d^{\mathbb{T}^n}(x_0, x) < S_0/2\} \tag{4.8}$$

Let H^h_Ω be the selfadjoint operator on $L^2(\Omega; \mathbb{C}^2)$ associated with $P^h(x, hD)$ with Dirichlet boundary condition. We denote by $(\lambda_j(H^h_\Omega))_{j\in\mathbb{N}^\star}$ the increasing sequence of eigenvalues of H^h_Ω. Using the method of [He-Sj-1], we get easily the following results

Theorem 4.1 *For any η, $0 < \eta < S_0/2$, there exist $h_0 > 0$ and $N_0 > 1$ such that, if $0 < h < h_0$ and $j \leq N_0$, then*

$$\forall\theta \in \mathbb{B}^n\,, \quad 0 \,<\, \lambda_j(H^h_\Omega) \,-\, \lambda^{h,\theta}_j \,\leq\, Ce^{-(S_0-\eta)/(2h)}\,; \tag{4.9}$$

so the length of the band b^h_j satisfies $|b^h_j| \leq Ce^{-(S_0-\eta)/(2h)}$.

For the first band, we have the following improvement

$$|b^h_1| \leq Ce^{-(S_0-\eta)/h}. \tag{4.10}$$

Sketch of the proof.

As Ω is simply connected and the one form θdx is closed, there exists a smooth real function $\psi_\theta(x)$ on $\overline{\Omega}$ such that $e^{-i\psi_\theta(x)} P^h(x, hD) e^{i\psi_\theta(x)} = P^h(x, h(D-\theta))$: the Dirichlet operators on Ω, H^h_Ω and $H^{h,\theta}_\Omega$ associated to $P^h(x, hD)$ and $P^h(x, h(D-\theta))$ are gauge equivalent, so they have the same spectrum.

Therefore the min-max principle says that

$$0 < \lambda_j(H_\Omega^{h,\theta}) - \lambda_j^{h,\theta} = \lambda_j(H_\Omega^h) \; - \; \lambda_j^{h,\theta}.$$

But the exponential decay of the eigenfunction $\varphi_j^{h,\theta}(x)$ associated with $\lambda_j^{h,\theta}$, given by (4.7) implies that

$$\|(P^h(x,h(D-\theta)) - \lambda_j^{h,\theta})\chi\varphi_j^{h,\theta}(x)\|_{L^2(\Omega;\mathbb{C}^2)} \leq Ce^{-(S_0-\eta+\epsilon)/(2h)},$$

for some $\epsilon > 0$, and for a smooth cut-off function χ supported in Ω and $\chi(x) = 1$ if $d^{\mathbb{T}^n}(x_0,x) \leq (S_0 - \eta + \epsilon)/2$.

So $distance(\lambda_j^{h,\theta}, \mathrm{sp}(H_\Omega^h)) \leq Ce^{-(S_0-\eta+\epsilon)/(2h)}$.
This achieves the proof of (4.9).

Let us denote $E^{h,\theta}$, (respectively $E^{h,\Omega}$), the first eigenvalue $\lambda_1^{h,\theta}$, (respectively $\lambda_1(H_\Omega^h)$), and $\varphi^{h,\theta}(x)$, (respectively $\varphi^{h,\Omega}(x)$), the associated normalized eigenfunctions. Let χ be a cut-off function satisfying the same properties as before. Then $P^h(x,h(D-\theta))(e^{-i\psi_\theta(x)}\chi(x)\varphi^{h,\Omega}(x)) = \lambda_1(H_\Omega^h)e^{-i\psi_\theta(x)}\chi(x)\varphi^{h,\Omega}(x) + e^{-i\psi_\theta(x)}r_0^h(x)$ with, thanks to the same identity as (4.7) for Dirichlet problem on Ω,

$$\|r_0^h\|_{L^2(\mathbb{T}^n;\mathbb{C}^2)} \leq Ce^{-\frac{S_0-\eta}{2h}}.$$

The same argument used in [He-Sj-1], (see also [He]), gives this estimate

$$|E^{h,\theta} - E^{h,\Omega} - < r_0^h \,|\, \chi\varphi^{h,\Omega} >_{L^2(\mathbb{T}^n;\mathbb{C}^2)} | \leq Ce^{-(S_0-\eta)/h}.$$

As $\tau_h =< r_0^h \,|\, \chi\varphi^{h,\Omega} >_{L^2(\mathbb{T}^n;\mathbb{C}^2)}$ does not depend on θ, so

$$\forall\theta \in \mathbb{B}^n \; , \quad |E^{h,\theta} - E^{h,\Omega} - \tau_h| \leq Ce^{-(S_0-\eta)/h}$$

this estimate ends the proof of (4.10) □

As for the tunnel effect in [He-Sj-1] and [Si-2], we have an accuracy estimate for the first band, like the scalar case in [Si-3] and in [Ou] (see also [He]).

Theorem 4.2 *There exists $h_0 > 0$ such that, if $0 < h < h_0$ then*

$$|b_1^h| \leq Ce^{-S_0/h}.$$

Sketch of the proof. Instead of comparing $H^{h,\theta}$ with an operator defined in a subset of $\mathbb{T}^n$, we have to work on the universal cover $\mathbb{R}^n$ of $\mathbb{T}^n$.

We take $\Omega \subset \mathbb{R}^n$ an open and simply connected set with smooth boundary satisfying, for some η_0, $0 < \eta_0 < \eta_1 < S_0/2$,

$$B_0((S_0 + \eta_0)/2) \; \subset \; \Omega \; \subset \; B_0((S_0 + \eta_1)/2). \tag{4.11}$$

So Ω contains the Wigner set $\mathbb{W}^n$, more precisely

$$\mathbb{W}^n \; \subset \; \Omega \; \subset \; 2\mathbb{W}^n \quad \text{and} \quad \Omega \cap \Gamma = \{0\}.$$

We let also denote H^h_Ω the Dirichlet operator on $L^2(\Omega;\mathbb{C}^2)$ associated with $P^h(x,hD)$, and E^h_Ω its first eigenvalue. The associated eigenfunction is also denoted by $\varphi^{h,\Omega}(x)$.

In the same way as to get (4.7), we have

$$\sum_{k=1}^n h^2\|(D_{x_k}I - A_k)e^{d_0(x)/h}\varphi^{h,\Omega}\|^2_{L^2(\Omega;\mathbb{C}^2)} \le hC\int_{B_0(\sqrt{h}C)} |\varphi^{h,\Omega}(x)|^2 dx, \qquad (4.12)$$

then the Poincaré estimate gives

$$\int_\Omega e^{2d_0(x)/h}|\varphi^{h,\Omega}(x)|^2 dx \le h^{-1}C\int_{B_0(\sqrt{h}C)} |\varphi^{h,\Omega}(x)|^2 dx. \qquad (4.13)$$

Let χ a smooth cut-off function satisfying

$$\chi(x) = 1 \text{ if } d_0(x) \le (S_0+\eta_0)/2 \quad \text{and} \quad \chi(x) = 0 \text{ if } x \notin \Omega.$$

Then the function

$$\varphi^{h,\theta}(x) = \sum_{\omega\in\Gamma} e^{i\theta(x-\omega)}\chi(x-\omega)\varphi^{h,\Omega}(x-\omega)$$

is Γ-periodic and satisfies

$$\left.\begin{array}{l} (P^h(x,h(D-\theta)) - E^h_\Omega)\varphi^{h,\theta}(x) = r^{h,\theta} \text{ and} \\ \|r^{h,\theta}\|_{L^2(\mathbb{W}^n;\mathbb{C}^2)} \le Ce^{-(S_0+\eta_0)/(2h)}\|\varphi^{h,\theta}\|_{L^2(\mathbb{W}^n;\mathbb{C}^2)} \end{array}\right\} \qquad (4.14)$$

and
$< r^{h,\theta} \,|\, \varphi^{h,\theta} >_{L^2(\mathbb{W}^n;\mathbb{C}^2)} =$

$$\sum_{\omega,\rho\in\Gamma_0} e^{i\theta(\rho-\omega)}\int_{\mathbb{W}^n} ([P^h(x,hD);\chi]\varphi^{h,\Omega})(x-\omega).\overline{(\chi\varphi^{h,\Omega})}(x-\rho)dx$$

with $\Gamma_0 = \{0, \pm\beta_1, \dots, \pm\beta_n\}$ and

$$[P^h(x,hD);\chi] = -2h^2 i\sum_{k=1}^n \partial_{x_k}\chi(D_{x_k}I - A_k) - h^2\Delta\chi I.$$

So

$$\left|\frac{1}{\|\varphi^{h,\theta}\|^2_{L^2(\mathbb{W}^n;\mathbb{C}^2)}} < r^{h,\theta} \,|\, \varphi^{h,\theta} >_{L^2(\mathbb{W}^n;\mathbb{C}^2)}\right| \le Ce^{-S_0/h}. \qquad (4.15)$$

The proof comes easily from (4.12) and (4.13) as in [Ou] or in [He].

Using the same argument of [He-Sj-1] as in the proof of (4.10), we get that

$$|E^{h,\theta} - E^h_\Omega - \tau^{h,\theta}| \le Ce^{-(S_0+\eta_0)/h)}, \qquad (4.16)$$

with $\tau^{h,\theta} = \dfrac{1}{\|\varphi^{h,\theta}\|^2_{L^2(\mathbb{W}^n;\mathbb{C}^2)}} < r^{h,\theta} \,|\, \varphi^{h,\theta} >_{L^2(\mathbb{W}^n;\mathbb{C}^2)}$.

Theorem 4.2 follows from (4.15) and (4.16) □

5 B.K.W. method for the Dirichlet ground state

Let Ω be an open set satisfying (4.8). more precisely $\Omega \subset \mathbb{R}^n$ an open, bounded and simply connected set with smooth boundary satisfying, for some η_1 and η_2, $0 < \eta_1 < \eta_2 < S_0/2$,

$$\{x \in \mathbb{R}^n;\ d_0(x) < \frac{S_0 - \eta_2}{2}\} \subset \Omega \subset \{x \in \mathbb{R}^n;\ d_0(x) < \frac{S_0 - \eta_1}{2}\} \tag{5.1}$$

Theorem 5.1 *The first eigenvalue $E^{h,\Omega} = \lambda_1(H^h_\Omega)$ of the Dirichlet operator H^h_Ω admits an asymptotic expansion of the form*

$$E^{h,\Omega} \simeq \sum_{j=0}^{\infty} h^j e_j \ ,$$

and if $S_0 - \eta_1$ is small enough, the associated eigenfunction $\varphi^{h,\Omega}$ has also an asymptotic expansion of the form

$$\varphi^{h,\Omega} = e^{-\phi/h}(f_h^+, f_h^-)\ , \quad f_h^\pm \simeq \sum_{j=0}^{\infty} h^j f_j^\pm\ , \quad (f_0^- = 0)\,.$$

As usual

$$e_0 = E_0\,, \quad e_1 = \tau_1\,, \quad e_2 = r_{11}(0) + \sum_{k=1}^{n} |a_{k,11}(0)|^2\,, \tag{5.2}$$

and ϕ is the real function satisfying the eikonal equation

$$|\nabla\phi(x)|^2 = v(x) - |w(x)| - E_0\,, \tag{5.3}$$

equal to $d(x)$ in a neighbourhood of 0.
(r_{11} and the $a_{k,11}$ are defined by (1.1) and (1.3). E_0 and τ_1 are defined by (2.2) and (2.4)).

Proof. When the gauge potential matrix is identified with the one form

$$Adx \ = \sum_{k=1}^{n} A_k(x)dx_k\ ,$$

its curvature form appears to be the related magnetic field $B = dA + A \wedge A$:

$$B = \sum_{1\le j<k\le n} (\partial_{x_j}A_k(x) - \partial_{x_k}A_j(x))dx_j \wedge dx_k + \sum_{1\le j<k\le n} (A_j(x)A_k(x) - A_k(x)A_j(x))dx_j \wedge dx_k.$$

For our purpose, only the vector magnetic potential $a_{.,11}$ is significant. We will work with Coulomb vector gauge $a_{.,11}$:

$$div(a_{.,11}(x)) = \sum_{k=1}^{n} \partial_{x_k} a_{k,11}(x) \ = \ 0\,. \tag{5.4}$$

It is feasible thanks to the existence of a smooth real and Γ periodic function $\psi(x)$ such that $\Delta\psi(x) = div(a_{.,11}(x))$.
Let $\mathcal{O}$ be any open set of $\mathbb{R}^n$ (or one can take also $\mathcal{O} = \mathbb{T}^n$). Conjugation of $P^h(x, hD)$ by the unitary operator J_ψ on $L^2(\mathcal{O}; \mathbb{C}^2)$:

$$J_\psi = \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix}, \tag{5.5}$$

leads to changing $a_{.,11}(x)$ for $a_{.,11}(x) - \nabla\psi(x)$ and $a_{.,21}(x)$ for $e^{2i\psi(x)}a_{.,21}(x)$; the new $a_{.,22}(x)$ is equal to minus the new $a_{.,11}(x)$, and the new $a_{.,12}(x)$ remains the conjugate of the new $a_{.,21}(x)$. So by (2.8) we have

$$\left.\begin{array}{l} a_{.,11} = \beta\nabla\alpha - \alpha\nabla\beta + \delta\nabla\rho - \rho\nabla\delta - \nabla\psi\,, \\ a_{.,21} = e^{2i\psi}[(\rho\nabla\beta - \beta\nabla\rho + \delta\nabla\alpha - \alpha\nabla\delta) + i(\alpha\nabla\rho - \rho\nabla\alpha + \delta\nabla\beta - \beta\nabla\delta)] \\ a_{.,22} = -a_{.,11}\,, \quad a_{.,12} = \overline{a_{.,21}} \\ \Delta\psi = div(\beta\nabla\alpha - \alpha\nabla\beta + \delta\nabla\rho - \rho\nabla\delta) \end{array}\right\} \tag{5.6}$$

Let us write

$$e^{-\phi/h}P^h(x, hD)(e^{-\phi/h}f_h) = W_0(x)f_h \,+\, hW_1(x, D)f_h \,+\, h^2W_2(x, D)f_h \tag{5.7}$$

with

$$W_0(x) = V(x) - |\nabla\phi(x)|^2 I$$

$$W_1(x, D) = \Delta\phi I + 2i\sum_{k=1}^{n}\partial_{x_k}\phi(D_{x_k}I - A_k)$$

$$W_2(x, D) = \sum_{k=1}^{n}(D_{x_k}I - A_k)^2 \,+\, R(x).$$

So

$$W_1(x, D) = \begin{pmatrix} \Delta\phi - 2i\nabla\phi.(i\nabla + a_{.,11}) & -2i\nabla\phi.\overline{a_{.,21}} \\ -2i\nabla\phi.a_{.,21} & \Delta\phi - 2i\nabla\phi.(i\nabla - a_{.,11}) \end{pmatrix},$$

and

$$W_2(x, D) = \begin{pmatrix} (i\nabla + a_{.,11})^2 + r_{11} & (i\nabla + a_{.,11}).\overline{a_{.,21}} + \overline{a_{.,21}}.(i\nabla - a_{.,11}) + r_{12} \\ a_{.,21}.(i\nabla + a_{.,11}) + (i\nabla + a_{.,11}).a_{.,21} + r_{21} & (i\nabla - a_{.,11})^2 + r_{22} \end{pmatrix}$$

We look for an eigenvalue $E^h \simeq \sum_{j=0}^{\infty} h^j e_j$
and an associated eigenfunction $f_h \simeq \sum_{j=0}^{\infty} h^j f_j$, so

$$e^{\phi/h}(P^h(x, hD) - E^h I)(e^{-\phi/h}f_h) \simeq \sum_{j=0}^{\infty} h^j\kappa_j$$

with

$$
\begin{aligned}
\kappa_0(x) &= (W_0(x) - e_0 I) f_0(x) \\
\kappa_1(x) &= (W_1(x,D) - e_1 I) f_0(x) + (W_0(x) - e_0 I) f_1(x) \\
\kappa_2(x) &= (W_2(x,D) - e_2 I) f_0(x) + (W_1(x,D) - e_1 I) f_1(x) + (W_0(x) - e_0 I) f_2(x) \\
\kappa_j(x) &= -e_j f_0(x) - \textstyle\sum_{\ell=1}^{j-3} e_{j-\ell} f_\ell(x) + (W_2(x,D) - e_2 I) f_{j-2}(x) \\
&+ \quad (W_1(x,D) - e_1 I) f_{j-1}(x) + (W_0(x) - e_0 I) f_j(x),\ (j > 2).
\end{aligned}
$$

We recall that $f_j = (f_j^+, f_j^-)$ and we want that $\kappa_j(x) = 0,\ \forall j$.

1) Term of order 0

As $\kappa_0(x) = 0 \Leftrightarrow \begin{cases} (-|\nabla\phi(x)|^2 + v(x) - |w(x)| - e_0) f_0^+(x) = 0 \\ (-|\nabla\phi(x)|^2 + v(x) + |w(x)| - e_0) f_0^-(x) = 0\,, \end{cases}$

choosing ϕ satisfying (5.3), then

$e_0 = E_0$ and $-|\nabla\phi(x)|^2 + v(x) + |w(x)| - e_0 = 2|w(x)| > 0$ implies that

$$
f_0^-(x) \;=\; 0\,, \quad e_0 = E_0 \quad \text{and} \quad W_0(x) - e_0 I = \begin{pmatrix} 0 & 0 \\ 0 & 2|w(x)| \end{pmatrix}. \tag{5.8}
$$

2) Term of order 1.

The components of $\kappa_1(x) = (\kappa_1^+(x), \kappa_1^-(x))$ become

$$
\begin{aligned}
\kappa_1^+(x) &= (\Delta\phi(x) - e_1 - 2ia_{.,11}(x).\nabla\phi(x)) f_0^+(x) + 2\nabla\phi(x).\nabla f_0^+(x) \\
\kappa_1^-(x) &= 2|w(x)| f_1^-(x) \;-\; 2i(a_{.,21}(x).\nabla\phi(x)) f_0^+(x)\,,
\end{aligned}
$$

As $|\nabla\phi(x)|$ has a simple zero at $x_0 = 0$, the equation $\kappa_1^+(x) = 0$ can be solved only when $e_1 = \Delta\phi(0)$. In this case there exists a unique function $f_0^+(x)$ such that $f_0^+(0) = 1$ and $\kappa_1^+(x) = 0$. We can conclude that the study of the term of order 1 leads to

$$
\left.
\begin{aligned}
& e_1 = \Delta\phi(0)\,, \\
& 2\nabla\phi(x).\nabla f_0^+(x) = [e_1 - \Delta\phi(x) + 2ia_{.,11}(x).\nabla\phi(x)] f_0^+(x)\,, \\
& f_1^-(x) = \tfrac{i}{|w(x)|}(\nabla\phi(x).a_{.,21}(x)) f_0^+(x)\,.
\end{aligned}
\right\} \tag{5.9}
$$

3) Term of order 2.

The components of $\kappa_2(x) = (\kappa_2^+(x), \kappa_2^-(x))$ become

$$
\left.
\begin{aligned}
\kappa_2^+(x) &= (\,(i\nabla + a_{.,11})^2 + + r_{11}(x) - e_2) f_0^+(x) \\
&\quad + (\Delta\phi(x) - e_1 - 2ia_{.,11}(x).\nabla\phi(x)) f_1^+(x) + 2\nabla\phi(x).\nabla f_1^+(x) \\
&\quad - 2i(\overline{a_{.,21}(x)}.\nabla\phi(x)) f_1^-(x) \\
\kappa_2^-(x) &= 2|w(x)| f_2^-(x) - 2i(a_{.,21}(x).\nabla\phi(x)) f_1^+(x) \\
&\quad + 2\nabla\phi(x).\nabla f_1^-(x) + 2ia_{.,11}(x).\nabla\phi(x) f_1^-(x) + 2ia_{.,21}(x).\nabla f_0^+(x) \\
&\quad + i(div(a_{.,21}(x)) f_0^+(x) + r_{21}(x) f_0^+(x)\,,
\end{aligned}
\right\} \tag{5.10}
$$

The unknown function $f_1^+(x)$ must give $\kappa_2^+(x) = 0$ in (5.10). This equation, with the initial condition $f_1^+(0) = 0$, can be solved only if

$$
e_2 = (i\nabla + a_{.,11})^2 f_0^+(0) + r_{11}(0).
$$

(We used that $f_0^+(0)=1$). So $\kappa_2=0$ implies

$$
\begin{array}{l}
e_2 \;=\; (i\nabla + a_{.,11})^2 f_0^+(0) + r_{11}(0)\,,\\
2\nabla\phi(x).\nabla f_1^+(x) \;=\; -(\Delta\phi(x) - e_1 - 2ia_{.,11}(x).\nabla\phi(x))f_1^+(x)\\
\quad -(\,(i\nabla + a_{.,11})^2 + e_2 - r_{11}(x))f_0^+(x) + 2i(a_{.,12}(x).\nabla\phi(x))f_1^-(x)\\
f_2^-(x) \;=\; \frac{1}{2|w(x)|}[2i(a_{.,21}(x).\nabla\phi(x))f_1^+(x) - 2\nabla\phi(x).\nabla f_1^-(x) - 2i(a_{.,11}(x).\nabla\phi(x))f_1^-(x)\\
\qquad -2ia_{.,21}(x).\nabla f_0^+(x) - i(div(a_{.,21}(x))f_0^+(x) - r_{21}(x)f_0^+(x)]\,.
\end{array}
$$

4) Term of order j> 2.
We assume that e_ℓ for $\ell=0,1,\dots,j-1$, the functions $f_\ell^\pm(x)$ for $\ell=0,1,\dots,j-2$, and the one $f_{j-1}^-(x)$ are well-known, $f_\ell^+(0)=0$ when $0<\ell<j-1$.

The equation $\kappa_j^+=0$ becomes

$$
\left.\begin{array}{l}
2\nabla\phi(x).\nabla f_{j-1}^+(x) \;+\; (\Delta\phi(x) - e_1 - 2ia_{.,11}(x).\nabla\phi(x))f_{j-1}^+(x) \;=\\
2i(\overline{a_{.,21}(x)}.\nabla\phi(x))f_{j-1}^-(x) + \displaystyle\sum_{\ell=0}^{j-3} e_{j-\ell}f_\ell^+(x)\\
-(\,(i\nabla + a_{.,11})^2 - e_2 + r_{11}(x)\,)f_{j-2}^+(x)\\
-(\,r_{12}(x) + idiv(\overline{a_{21}})\,)f_{j-2}^-(x) - 2i\overline{a_{21}}.\nabla f_{j-2}^-(x)
\end{array}\right\} \tag{5.11}
$$

This equation has a unique solution $f_{j-1}^+(x)$ such that $f_{j-1}^+(0)=0$ iff

$$
\left.\begin{array}{l}
e_j \;=\; (\,(i\nabla + a_{.,11})^2 \;+\; r_{11}(0)\,)f_{j-2}^+(0) + r_{12}(0))f_{j-2}^-(0)\\
-2i\overline{a_{.,21}(0)}.\nabla)f_{j-2}^-(0) - \sum_{\ell=0}^{j-3} e_{j-\ell}f_\ell^+(0)\,.
\end{array}\right\} \tag{5.12}
$$

The equation $\kappa_j^-=0$ gives

$$
f_j^-(x) \;=\; \frac{1}{2|w(x)|}\times
$$

$$
[-2\nabla\phi(x).\nabla f_{j-1}^-(x) + (e_1 - 2ia_{.,11}(x).\nabla\phi(x))f_{j-1}^-(x) + 2ia_{.,21}(x).\nabla\phi(x)f_{j-1}^+(x)
$$

$$
+(e_2 - (i\nabla - a_{.,11})^2)f_{j-2}^-(x) - idiv(a_{.,21}(x))f_{j-2}^+(x) - 2ia_{.,21}(x).\nabla f_{j-2}^+(x) + \sum_{\ell=0}^{j-3} e_{j-\ell}f_\ell^-(x)]
$$

5) End of the proof.
Let $\chi(x)$ be a cut-off function equal to 1 in a neighbourhood of 0 and supported in Ω. Then taking $\chi(x)f_j(x)$ instead of $f_j(x)$, we get a function $\varphi^{h,\Omega}$ satisfying Dirichlet boundary condition and Theorem 5.1. The self-adjointness of the related operator ensures that the computed sequence (e_j) is real □

6 Sharp asymptotic for the width of the first band

Returning to the proof of Theorem 4.2, we have to study carefully the $\tau^{h,\theta}$ defined in (4.16), using the method of [He-Sj-1] performed in [He] and [Ou].

Using (4.14)–(4.16), we have

$$\tau^{h,\theta} = \sum_{\omega\in\Gamma_0^+} \left(e^{-i\theta\omega}(\rho_\omega^+ + \rho_\omega^-) + e^{i\theta\omega}\overline{(\rho_\omega^+ + \rho_\omega^-)} \right) , \tag{6.1}$$

with $\Gamma_0^+ = \{\beta_1, \dots, \beta_n\}$,

$$\rho_\omega^+ = \frac{1}{\|\varphi^{h,\theta}\|^2_{L^2(\mathbb{W}^n;\mathbb{C}^2)}} \int_{\mathbb{W}^n} ([P^h(x,hD);\chi]\varphi^{h,\Omega})(x-\omega).\overline{(\chi\varphi^{h,\Omega})}(x)dx$$

and $\rho_\omega^- = \dfrac{1}{\|\varphi^{h,\theta}\|^2_{L^2(\mathbb{W}^n;\mathbb{C}^2)}} \displaystyle\int_{\mathbb{W}^n} ([P^h(x,hD);\chi]\varphi^{h,\Omega})(x).\overline{(\chi\varphi^{h,\Omega})}(x+\omega)dx\,.$
We get from the formula of $[P^h(x,hD);\chi]$ and from the estimate (4.13) that

$$\rho_\omega^+ = -\frac{h^2}{\|\varphi^{h,\theta}\|^2_{L^2(\mathbb{W}^n;\mathbb{C}^2)}} \int_{\mathbb{W}^n} (\nabla\chi(x-\omega)\nabla\varphi^{h,\Omega})(x-\omega)).\overline{(\chi\varphi^{h,\Omega})}(x)dx + \tag{6.2}$$

$$+\,\mathbf{O}(he^{-S_0/h})$$

and

$$\rho_\omega^- = -\frac{h^2}{\|\varphi^{h,\theta}\|^2_{L^2(\mathbb{W}^n;\mathbb{C}^2)}} \int_{\mathbb{W}^n} (\nabla\chi(x)\nabla\varphi^{h,\Omega})(x)).\overline{(\chi\varphi^{h,\Omega})}(x+\omega)dx \,+\, \mathbf{O}(he^{-S_0/h})\,.$$

Theorem 6.1 *Under the assumption of Theorem 4.2, if for any* $\omega \in \{\pm\beta_1, \dots, \pm\beta_n\}$ *such that the Agmon distance in* $\mathbb{R}^n$ *between* 0 *and* ω *is the least one, (i.e.* $d(0,\omega) = S_0$*), there exists one or a finite number of minimal geodesics joining* 0 *and* ω*, then there exists* $\eta_0 > 0$ *and* $h_0 > 0$ *such that*

$$b_1^h = \eta_0 h^{1/2} e^{-S_0/h} \left(1 \,+\, \boldsymbol{O}(h^{1/2})\right) \,, \quad \forall h \in]0, h_0[\,.$$

Sketch of the proof. Following the proof of splitting in [He-Sj-1] and [He], in (6.2) we can change $\mathbb{W}^n$ into $\mathbb{W}^n \cap \mathcal{O}$, where $\mathcal{O}$ is any neighbourhood of the minimal geodesics between 0 and the $\pm\beta_k$, such that $d(x) = d(0,x) \in C^\infty(\mathcal{O})$. In this case the B.K.W. method is valid in $\mathbb{W}^n \cap \mathcal{O}$. If $\varphi^h_{B.K.W.}$ is the B.K.W. approximation of $\varphi^{h,\Omega}$ in $\mathbb{W}^n \cap \mathcal{O}$, then, thanks to (4.1), for any $p_0 > 0$ there exists C_{p_0} such that

$$h\sum_{k=1}^n \|(D_{x_k}I - A_k)e^{d(x)/h}\chi_0(\varphi^{h,\Omega} - \varphi^h_{B.K.W.})\|^2_{L^2(\mathbb{W}^n;\mathbb{C}^2)}$$

$$+\|e^{d(x)/h}\chi_0(\varphi^{h,\Omega} - \varphi^h_{B.K.W.})\|^2_{L^2(\mathbb{W}^n;\mathbb{C}^2)} \,\leq\, h^{p_0}C_{p_0}\,, \tag{6.3}$$

where χ_0 is a cut-off function supported in $\mathbb{W}^n \cap \mathcal{O}$ and equal to 1 in a neighborhood of the minimal geodesics between 0 and the $\pm\beta_k$. We have assumed that $\|\varphi^{h,\Omega}\|_{L^2(\mathbb{W}^n;\mathbb{C}^2)} = 1$ and then $\|\chi_0\varphi^h_{B.K.W.})\|^2_{L^2(\mathbb{W}^n;\mathbb{C}^2)} - 1 = \mathbf{O}(h^p)$ for any $p > 0$.

As (6.2) remains valid if we change $\varphi^{h,\Omega}$ into $\chi_0\varphi^{h,\Omega}$, the estimate (6.3) allows also to change $\varphi^{h,\Omega}$ into $\chi_0\varphi^h_{B.K.W.}$. As a consequence, Theorem 6.1 follows easily, if in $\mathbb{W}^n \cap \mathcal{O}$, $\chi(x) = \chi_1(d(x))$ for a decreasing function χ_1 on $[0, +\infty[$ with compact support, equal to 1 in a neighborhood of 0. In this case (6.2) becomes

$$\rho^+_\omega = \frac{h^{(2-n)/2}}{(2\pi)^{1/2}\prod_{k=1}^n \tau_k^{1/2}} \times \tag{6.4}$$

$$\int_{\mathbb{W}^n\cap\mathcal{O}} \chi_1'(d(x-\omega))\chi_1(d(x))|\nabla d(x-\omega)|^2 f_0^+(x-\omega) f_0^+(x) e^{-(d(x-\omega)+d(x))/h} dx + \mathbf{O}(he^{-S_0/h})$$

and

$$\rho^-_\omega = \frac{h^{(2-n)/2}}{(2\pi)^{1/2}\prod_{k=1}^n \tau_k^{1/2}} \times$$

$$\int_{\mathbb{W}^n\cap\mathcal{O}} \chi_1'(d(x))\chi_1(d(x+\omega))|\nabla d(x)|^2 f_0^+(x) f_0^+(x+\omega) e^{-(d(x)+d(x+\omega))/h} dx + \mathbf{O}(he^{-S_0/h}).$$

We remind that for any y in a minimal geodesic joining 0 to $\pm\beta_k$, if $y \neq 0$ and $y \neq \pm\beta_k$, then the function $d(x) + d(x \mp \beta_k)$, when it is restricted to any hypersurface orthogonal to the geodesic through y, has a non degenerate minimum S_0 at y□

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