Smooth critical points of planar harmonic mappings

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Abstract

In [9], Lyzzaik studies the local properties of light harmonic mappings. More precisely, he classifies their critical points and accordingly studies their topological and geometrical behaviour. One aim of our work is to shed some light on the case of smooth critical points, thanks to miscellaneous numerical invariants. Inspired by many computations, and with a crucial use of Milnor fibration theory, we get a fundamental and quite unexpected relation between three of these invariants. In the final part of the work we offer some examples providing significant differences between our harmonic setting and the real analytic one.

1 Introduction

A map $f : W \to \mathbb{R}^2$ defined on a domain W of $\mathbb{R}^2$ is called planar harmonic if both components of f are harmonic functions. When W is simply connected, identifying $\mathbb{R}^2$ with $\mathbb{C}$, the map f can also be written in the complex form $p(z) + \overline{q(z)}$ where p and q are holomorphic functions on W. We are interested on germs f of planar harmonic maps defined in a neighborhood of a point a and we will use mainly the local complex form.

For such a germ, we restrict ourselves to the situation given by the two following conditions:

1. The fiber at $f(a)$ is the single point a.
2. The critical set $\mathcal{C}_f$ is a smooth curve at the point a. (The critical set is the vanishing locus of the Jacobian).

The first condition means that f is light in the sense of Lyzzaik. In our paper, we will also present some remarks for the non light case. The second condition implies that $\mathcal{C}_f$ is not a single point. Otherwise, one can prove that up to C^1 change of coordinates, f is holomorphic or anti-holomorphic .

The critical value set $\mathcal{V}_f = f(\mathcal{C}_f)$ will play an important role in our work.

In our setting, the following natural equivalence relation is introduced: we say that two germs f, g of planar harmonic maps defined respectively at points a and b, are equivalent if there exists a germ of biholomorphism u between neighborhoods of a and b, and a real affine bijection ℓ such that: $\ell \circ f \circ u = g$. For this relation, we get four numerical invariants and miscellaneous normal forms. These tools allow to understand analytic and topological facts about germs of harmonic maps. In particular they shed a new light on the geometric models which appear in the work of Lyzzaik.

In order to fully understand these invariants, the study of the complexification of f plays a fundamental role.

Explicitly, the four numerical invariants are:

- the absolute value d of the local topological degree. In our situation, Lyzzaik's work shows that d is 0 or 1,
- the number m which is the lowest degree of a nonconstant monomial in the power series expansion of the harmonic germ f at the point a,
- the local multiplicity μ of the complexified map of f is defined as the cardinality of the generic fiber,
- the number j which is the valuation of the analytic curve $\mathcal{V}_f$, in a locally injective parametrization.

Inspired by Lyzzaik's models [9], we prove in a self-contained way that the germs f are classified topologically by the numbers m and d or equivalently by m and the parity of $m + j$. We give a simple description of these classes in terms of generalized folds and cusps.

A main result of our work is that the conditions $j \geq m$ and $\mu = j + m^2$ are necessary and sufficient for the existence of a harmonic planar germ satisfying the conditions 1. and 2. above.

The second relation was at first guessed, using many computations. Our proof is based on the theory of Milnor fibration for germs of holomorphic functions on $\mathbb{C}^2$.

The number j occurs also at another level. In fact we prove that the critical value set $\mathcal{V}_f$ can be parametrized by: $x(t) = Ct^j + h.o.t$ and $y(t) = C't^{j+1} + h.o.t$, where C and C' are nonzero constants. In other words, $\mathcal{V}_f$ is a curve with Puiseux pair $(j, j+1)$, and Puiseux theory gives a topological characterization of the complexification of the critical value set. This pair completes also Lyzzaik's description of $\mathcal{V}_f$ itself.

Every equivalence class of harmonic germs contains a normal form $p(z)^m - \overline{z}^m$, with p a holomorphic germ tangent to the identity at 0. When $m = 1$, μ is equal to the order at the origin of the holomorphic germ $\overline{p} \circ p$ where $\overline{p}(z)$ is defined as $\overline{p(\overline{z})}$. As a by-product, one gets an algorithm to compute μ in the polynomial case. We obtain also relations between the numerical invariants of $p(z) - \overline{z}$ and $p(z)^m - \overline{z}^m$.

The condition $m = 1$ for a harmonic germ means that the gradient of its Jacobian does'nt vanish at the point a. In an appendix, we generalize results obtained for numerical invariants in the harmonic case to real and complex analytic planar germs satisfying the previous Jacobian condition. In this situation, we get that $\mathcal{V}_f$ has still Puiseux pair $(j, j+1)$, with the same geometric consequences. Moreover, as for the harmonic case, $\mu = j + 1$.

In this analytic case, we get also an algorithm to compute μ inspired by a fundamental work of Whitney [12] on cusps and folds . At the end of the appendix, some examples are presented in order to compare the harmonic situation to the real analytic one.

Lyzzaik has also studied the case of non smooth critical sets. In a next work, we hope to extend some of our results to this more general situation.

2 Basic concepts and main results

Let $K = \mathbb{R}$ or $\mathbb{C}$. Let U be a domain in K^2. By convention a domain is a connected open set. Consider a planar mapping.

$$f : U \to K^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}.$$

We say that f is *a K-analytic map* if each of f_1 and f_2 can be expressed locally as convergent power series. In this case denote by
– J_f the jacobian of f
– $\mathcal{C}_f = \{ J_f = 0 \}$ the *critical set*
– $\mathcal{V}_f = f(\mathcal{C}_f)$ the *critical value set*.

We say that
– $z_0 \in \mathcal{C}_f$ is a *regular critical point* of f if $\nabla J_f(z_0) \neq (0,0)$;
– $z_0 \in \mathcal{C}_f$ is a *smooth critical point* of f if $\mathcal{C}_f$ is an 1-dimensional submanifold near z_0.

By implicit function theorem a regular critical point is necessarily a smooth critical point. But the converse is not true. We will see many examples in the following.

We say that f is *a planar harmonic map* if $K = \mathbb{R}$ and each of f_1, f_2 is C^2 with a laplacian equal to zero. Recall that $\Delta f_j(x,y) = (\partial_x^2 + \partial_y^2) f_j(x,y)$. Note that in this case each of f_i is locally the real part of a holomorphic map. Thus a planar harmonic map is in particular $\mathbb{R}$-analytic.

The *order* of an analytic map $f : K^m \to K^n$ at a point p in the source, is the lowest total degree on a non zero monomial in the coordinate-wise Taylor expansions of one of the components of $f - f(p)$ around p.

For an $\mathbb{C}$-analytic map $F : W \to \mathbb{C}^n$ with W an open set of $\mathbb{C}^n$ and for a point $w_0 \in W$, we define the *multiplicity* of F at w_0, by (see [3])

$$\mu(F, w_0) = \limsup_{w \to w_0} \# F^{-1} F(w) \cap X$$

where X is an open neighborhood of w_0 relatively compact in W such that $F^{-1}F(w_0) \cap \overline{X} = \{w_0\}$. If such X does not exist, set $\mu(F, w_0) = \infty$.

In the situation above there is an open neighborhood U of $F(w_0)$ and an open dense subset $U_1 \subset U$ such that for all $w \in U_1$, $\mu(F, w_0) = \# F^{-1}F(w) \cap X$. For a holomorphic map $\eta : U \to \mathbb{C}$ with U an open set of $\mathbb{C}$, and $w_0 \in U$, the two notions coincide.

For a $\mathbb{R}$-analytic map $f : U \to \mathbb{R}^2$, in particular a planar harmonic map, we define its multiplicity at a point to be the multiplicity of its holomorphic extension in $\mathbb{C}^2$. We will also frequently use the well known fact that for a holomorphic map $f : U \to \mathbb{C}^2$ its multiplicity at a point $p_0 = (x_0, y_0) \in U$ is equal to the codimension in the ring of power series of the ideal defined by its component :

$$\mu(f, p_0) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{u, v\}}{f_1(x_0 + u, y_0 + v), f_2(x_0 + u, y_0 + v)}.$$

See for example [4, theorem 6.1.4].

One objective of this work is to show that harmonic maps around a smooth critical point of a given order have only two types of topological behaviours, depending on the parity of the multiplicity.

Our investigation is based on Whitney's singularity theory on C^∞ planar mappings, multiplicity theory of holomorphic maps of two variables and Lyzzaik's work on light harmonic mappings.

A smooth 1-dimensional manifold in $\mathbb{R}^2$ admits a smooth parametrization. If the critical set of a harmonic mapping is smooth somewhere, there is actually a parametrization that is in some sense natural. This induces a natural parametrization $\beta(t)$ of the critical value set.

Definition 2.1. *We denote by $R_{\geq k}(s)$ a convergent power series on s whose lowest power in s is at least k. A planar K-analytic curve $\beta : \{|s| < \varepsilon\} \ni s \mapsto \beta(s) \in K^2$ is said to have the* order pair (j, k) *at $\beta(0)$, for some $1 \leq j < k \leq \infty$, if up to reparametrization in the source and an analytic change of coordinates in the range K^2, the curve takes the form $\beta(s) = \beta(0) + \begin{pmatrix} Cs^j + R_{\geq j+1}(s) \\ C's^k + R_{\geq k+1}(s) \end{pmatrix}$ with $C \cdot C' \neq 0$.*

Let us assume that k is not a multiple of j, and that the complexified parametrization of β is locally injective. This is the case in particular if (j, k) are co-prime. Then the order j and more generally the order-pair $(j, k) \in \mathbb{N}^2$ is an analytic invariant of the curve independently of such a parametrisation : j is the minimum, and k the maximum of the intersection multiplicities (β, γ) among all smooth K-analytic curves γ. This order-pair is also a topological invariant of the complexified curve because $\frac{1}{gcd(j,k)}(j, k)$ is its first Puiseux pair. Such a unique order-pair exists unless $j = 1$, or $j = +\infty$.

Our main goal is to establish a relationship between the order of the critical value curve and the multiplicity, and then to connect these invariants to Lyzzaik's topological models. More precisely, we will prove:

Theorem 2.2. *Let f be a planar harmonic map in a neighborhood of z_0 with z_0 as a smooth critical point.*

1. *(Critical value order-pair) The critical value curve has a natural parametrization and an order j at $f(z_0)$. It has an order-pair of the form $(1, \infty)$ if $j = 1$, and $(j, j + 1)$ if $1 < j < \infty$.*

2. *(Critical value order and multiplicity) The three invariants m order of f, j order of the critical values curve and μ multiplicity of the complexified map on $(\mathbb{C}^2, z_0)$ are related by the (in)equalities :*

$$\begin{cases} \infty \geq j \geq m \geq 1 \\ j + m^2 = \mu. \end{cases} \tag{2.1}$$

3. *(Topological model) Assume $\mu < \infty$. Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$. There is a neighborhood Δ of z_0, a pair of orientation preserving homeomorphisms $h_1 : \Delta \to \mathbb{D}$, $z_0 \mapsto 0$, $h_2 : \mathbb{C} \to \mathbb{C}$, $f(z_0) \mapsto 0$, and a pair of positive odd integers $2n^{\pm} - 1$ satisfying (2.2) below, such that*

$$h_2 \circ f \circ h_1^{-1}(re^{i\theta}) = \begin{cases} re^{i(2n^+-1)\theta} & 0 \leq \theta \leq \pi \\ re^{-i(2n^- -1)\theta} & \pi \leq \theta \leq 2\pi \ . \end{cases}$$

Moreover $\# f^{-1}(z) = n^+ + n^-$ or $n^+ + n^- - 2$ depending on whether z is in one sector or the other of $f(\Delta) \smallsetminus \beta$.

$$\boxed{\begin{array}{ll} \mu \textit{ even}, & \begin{pmatrix} 2n^+ - 1 \\ 2n^- - 1 \end{pmatrix} \in \left\{ \begin{pmatrix} m \\ m \end{pmatrix}, \begin{pmatrix} m+1 \\ m+1 \end{pmatrix} \right\} \\ \mu \textit{ odd}, & \begin{pmatrix} 2n^+ - 1 \\ 2n^- - 1 \end{pmatrix} \in \left\{ \begin{pmatrix} m+1 \\ m-1 \end{pmatrix}, \begin{pmatrix} m-1 \\ m+1 \end{pmatrix}, \begin{pmatrix} m \\ m+2 \end{pmatrix}, \begin{pmatrix} m+2 \\ m \end{pmatrix} \right\} \end{array}} \tag{2.2}$$

We want to emphasise that guessing and proving the relation $\mu = j + m^2$ was the main point of the work. Our starting point was the case $m = 1$, which corresponds to the critically regular case. We could establish then $\mu = j + 1$. But this case does not indicate a general formula. A considerable amount of numerical experiments have been necessary to reveal a plausible general relation, and then results in singularity theory about Milnor's fibres had to be employed to actually prove the relation.

We will deduce the topological model from our formula (2.1) and a result of Lyzzaik [9]. More precisely, we will parametrize the critical value curve β in a natural way and then express its derivative, as did Lyzzaik, in the form $\beta'(t) = Ce^{it/2} \cdot R(t)$, with $C \neq 0$ and $R(t)$ a real valued analytic function.

Lyzzaik defined in his Definition 2.2 the singularity to be of the *first kind* if $R(t)$ changes signs at 0, which is equivalent to $j > 0$ even, and of the *second kind* if $R(0) = 0$ and $R(t)$ does not change sign at 0, which is equivalent to $j \geq 1$ odd. He then deduced the local geometric shape of β (cusp or convex) in Theorem 2.3 following the kind. What we do here is to push further his calculation to determine the order-pair of the critical value curve β, which then gives automatically its shape (cusp or convex).

Lyzzaik then provided topological models in his Theorem 5.1 following the parity of an integer ℓ (which corresponds to our $m - 1$) and the kind (or the shape of β) of the singularity, corresponding in our setting to the parity of $m + j$. Thanks to our relation (2.1) , we may then express Lyzzaik's topological model in terms of the parity of μ.

Lyzzaik's proof relies on previous results of Y. Abu Muhanna and A. Lyzzaik [1]. We will reestablish his models with a self-contained proof.

As a side product, we obtain the following existence result (which was a priori not obvious):

Corollary 2.3. *Given any triple of integers (m, j, μ) satisfying (2.1), there is a harmonic map $g(z)$ with a smooth critical point z_0 such that $Ord_{z_0}(g) = m$, $\mu(g, z_0) = \mu$, and $(j, j + 1)$ is the order-pair of the critical value curve at $g(z_0)$.*

Given any pair of integers $n^\pm \geq 1$ satisfying $\begin{cases} n^+ = n^- & or \\ |n^+ - n^-| = 1 \end{cases}$, there are two consecutive integers $k, k + 1$ and harmonic maps with order $m = k$ and $m = k + 1$ respectively realising the topological model (2.2) for the pair $n^\pm$ and the order m.

3 Normal forms for planar harmonic mappings

Recall that any real harmonic function on a simply connected domain in $\mathbb{C}$ is the real part of some holomorphic function. Therefore, if $U \subset \mathbb{C}$ is simply connected, and $f : U \to \mathbb{C}$ is a harmonic mapping, then $f = p + \overline{q}$ where p and q are holomorphic functions in U that are unique up to additive constants. We will say that $p + \overline{q}$ is a *local expression* of f. In a study around a point z_0 we will often take the unique local expression in the form $f(z) = f(z_0) + p(z) + \overline{q(z)}$ with $p(z_0) = q(z_0) = 0$.

3.1 Existence and unicity of the normal forms

Definition 3.1. A *natural equivalence relation. For Z, W open sets in $\mathbb{C}$, with $z_0 \in Z$, $w_0 \in W$, and for harmonic mappings $f : Z \to \mathbb{C}$ and $g : W \to \mathbb{C}$, we say that (f, z_0) and (g, w_0) are equivalent and we write*

$$(f, z_0) \sim (g, w_0)$$

if there is a bijective $\mathbb{R}$-affine map $H : \mathbb{C} \mapsto \mathbb{C}, z \mapsto az + b\overline{z} + c$ and a biholomorphic map $h : W' \to Z'$ with $z_0 \in Z' \subset Z$, $w_0 \in W' \subset W$ such that $h(w_0) = z_0$ and $g = H \circ f \circ h$ on W'.

Lemma 3.2. *Let f be a non-constant harmonic map defined on a neighborhood of z_0. Then*

$$(f, z_0) \sim (g, 0) \quad \text{for some} \quad g(z) = z^m - \overline{z^n(1 + O(z))} \tag{3.1}$$

with $\infty \geq n \geq m \geq 1$ (here $O(z)$ denotes a holomorphic map near 0 vanishing at 0).

Moreover if another map $G(z) = z^M - \overline{z^N(1 + O(z))}$ with $\infty \geq N \geq M \geq 1$ satisfies $(G, 0) \sim (g, 0)$ then $(M, N) = (m, n)$. If $m < n$ then $g(z) = \dfrac{1}{c^m} G(cz)$ for c an $(m + n)$-th root of unity.

Proof. We may assume $z_0 = 0$ and $f(0) = 0$.

I. We may assume that f is harmonic on a simply connected open neighborhood V of 0. One can thus write $f(z) = p(z) - \overline{q(z)}$ with p, q holomorphic on V. Replacing f by $f - f(0)$ we may assume $f(0) = 0$, and we may also assume $p(0) = q(0) = 0$.

Case 0. Assume $p \equiv 0$ or $q \equiv 0$. Replacing $f(z)$ by $\overline{f(z)}$ if necessary we may assume $q \equiv 0$. In this case $p(z) = az^m(1 + O(z))$ with $a \neq 0$ and there is a bi-holomorphic map h so that $p(z) = (h(z))^m$. Therefore $f(z) = g(h(z))$ with $g(w) = w^m$.

II. Assume now that none of p, q is a constant function. Replacing $f(z)$ by $\overline{f(z)}$ if necessary we may assume $p(z) = az^m(1+O(z))$ and $q(z) = bz^n(1+O(z))$ with $\infty > n \geq m \geq 1$ and $a \cdot b \neq 0$.

Replacing f by $(\overline{b\lambda})^{-n} f(\lambda z)$ changes a to $(\overline{b\lambda})^{-n} a \cdot \lambda^m$ and b to 1. We may thus assume $f(z) = Az^m(1 + O(z)) - \overline{z^n(1 + O(z))}$, $A \neq 0$.

Case 1. $m < n$. Choose ρ so that $\dfrac{A \cdot \rho^m}{\overline{\rho}^n} = 1$. Replace f by $\dfrac{f(\rho z)}{\overline{\rho}^n}$ we may assume $f(z) = z^m(1 + O(z)) - \overline{z^n(1 + O(z))}$.

Case 2. $m = n$. We choose τ so that $\dfrac{A \cdot \tau^m}{\overline{\tau}^n} = \dfrac{A \cdot \tau^m}{\overline{\tau}^m} \in \mathbb{R}_+^*$. We may thus assume

$$f(z) = cz^m(1 + O(z)) - \overline{z^m(1 + O(z))}, \quad c > 0.$$

If $c = 1$ we stop. Assume $c \neq 1$. Then $H(z) := z + \frac{1}{c}\overline{z}$ is an invertible linear map. And as c is real, we get easily :

$$H(f(z)) = (c - \frac{1}{c})z^m(1 + O(z)) - \overline{O(z^{m+1})} \qquad \text{with } c - \frac{1}{c} \neq 0\ .$$

Replacing f by $H \circ f$ we are reduced to Case 0 or Case 1.

Therefore in any case we may assume

$$f(z) = z^m(1 + O(z)) - \overline{z^n(1 + O(z))}, \quad 1 \leq m \leq n,\ m < \infty,\ n \leq \infty.$$

Now there is a holomorphic map h with $h(0) = 0$, $h'(0) = 1$ defined in a neighborhood of 0 so that the holomorphic part of f can be expressed as $h(z)^m$. Then

$$f \circ h^{-1}(z) = z^m - \overline{z^n(1 + O(z))}, \quad 1 \leq m \leq n,\ m < \infty,\ n \leq \infty$$

on some neighborhood of 0. This establishes the existence of normal forms.

Let us now take a map $G(z) = z^M - \overline{z^N(1 + O(z))}$ with $\infty \geq N \geq M \geq 1$ so that $(G, 0) \sim (g, 0)$ with $g(z) = z^m - \overline{z^n(1 + O(z))}$ and $m \leq n$. It is easy to see that $M = m$. Let now $h(z) = cz(1 + O(z))$ be a holomorphic map with $c \neq 0$ and $H(z) = az + b\overline{z}$ so that $H \circ G \circ h(z) = g(z)$. Then

$$a \cdot (h(z))^m - a \cdot \overline{(h(z))^N(1 + O(z))} + b \cdot \overline{(h(z))^m} - b \cdot (h(z))^N(1 + O(z)) = z^m - \overline{z^n(1 + O(z))}.$$

Assume $n > m = M$. If $N = m$ then the terms z^m and $\overline{z}^m$ have coefficients $(a-b)c^m$ and $(b-a)\bar{c}^m$ on the left hand side, and $(1,0)$ on the right hand side. This is impossible. So $N > m$ as well.

Comparing the $\overline{z}^m$ term on both sides we get $b = 0$, and then the z^m term we get $ac^m = 1$. Comparing then the holomorphic part of both sides we get $h(z) = cz$. Now the anti-holomorphic part gives $N = n$ and $\overline{a}c^n = 1$. It follows that $\overline{c}^{-m}c^n = 1$. So $|c| = 1$, $c^{m+n} = 1$ and $c^{-m}G(cz) = g(z)$. q.e.d.

We remark that in the case $m = n$ the normal form is not unique. Here is an example:

Let $G(z) = z + iz^2 - \overline{z}$. For any $\Re b \neq -\frac{1}{2}$ the map is equivalent to $G(z) + bG(z) + b\overline{G(z)} = (z + iz^2 + biz^2) - \overline{z + \overline{b}iz^2} = w + O(w^2) - \bar{w} =: g(w)$ for $w = z + \overline{b}iz^2$.

3.2 Criterion and normal forms for critically smooth points

We say that a subset set Q of $\mathbb{C}$ is a **locally regular star** at z_0 of ℓ-arcs if there is a neighborhood U of z_0 and a univalent holomorphic map $\phi : U \to \mathbb{C}$ with $\phi(z_0) = 0$ so that $Q \cap U = \{z, \phi(z)^\ell \in \mathbb{R}\}$. If $\ell = 1$ then Q is a smooth arc in U.

Lemma 3.3. *Let f be a harmonic map in a neighborhood of z_0. The following conditions are equivalent:*

1) $\mathcal{C}_f$ is a non-constant smooth $\mathbb{R}$-analytic curve in a neighborhood of z_0.

2) For $m := Ord_{z_0}(f)$, in a local expression $f(z) = p(z) + \overline{q(z)}$, we have $m = Ord_{z_0}(p) = Ord_{z_0}(q) < \infty$, the map $\psi(z) := \dfrac{p'(z)}{q'(z)}$ extends to a holomorphic map at z_0, with $|\psi(z_0)| = 1$ and $\psi'(z_0) \neq 0$.

3) $(f, z_0) \sim (g, 0)$ with

$$g(z) = z^m + bz^{m+1} + O(z^{m+2}) + \overline{z^m}, \quad |b| = 1; \tag{3.2}$$

Every equivalence class of such (f, z_0) has a representative in any of the following forms (with any choice of signs): $(f, z_0) \sim (h, 0)$ with

$$h(z) = \pm z^m + bz^{m+1} + O(z^{m+2}) \pm \overline{z^m} \quad \text{or} \quad h(z) = \pm\Big(z + bz^2 + O(z^2)\Big)^m \pm \overline{z^m}, \quad |b| = 1. \tag{3.3}$$

Furthermore, z_0 is a regular critical point if and only if $m = 1$.

Proof. Assume at first $f(z) = p(z) + \overline{q(z)}$, with

$$p(z) = z^m + bz^{m+k} + O(z^{m+k+1}), \quad q(z) = z^m, \quad k \geq 1,\ b \neq 0.$$

Note that $J_f = |p'|^2 - |q'|^2$. Set $\psi(z) = \dfrac{p'(z)}{q'(z)}$. We have

$$\mathcal{C}_f = \{J_f = 0\} = \{q' = 0\} \cup \{|\psi| = 1\} = \{0\} \cup \psi^{-1}(S^1) = \psi^{-1}(S^1)\ .$$

But $\psi^{-1}(S^1)$ is a locally regular star at 0 of k-arcs. So $\mathcal{C}_f$ is smooth at 0 if and only if $k = 1$, or equivalently, $\psi'(0) \neq 0$.

This proves in particular the implication 3)$\Longrightarrow$1).

Let us prove 1)$\Longrightarrow$3). We may assume f is in the local normal form (3.1) . If $m \neq n$ then it is easy to see that z_0 is an isolated point of $\mathcal{C}_f$. This will not happen under the smoothness assumption of $\mathcal{C}_f$. So $m = n$.

Replace f by $\overline{f(az)/a^m}$ with $a^{2m} = -1$ we have $f(z) = p(z) + \overline{q(z)}$, with

$$p(z) = z^m + bz^{m+k} + O(z^{m+k+1}), \quad q(z) = z^m, \quad k \geq 1, \ b \neq 0.$$

Since $\mathcal{C}_f$ is smooth at 0 by the argument above we have $k = 1$. We may then replace f by $f(\lambda z)/\lambda^m$ for $\lambda = \dfrac{1}{|b|} > 0$ to get a normal form so that $|b| = 1$. This is (3.2) .

The rest of the proof is similar. We leave the details to the reader. q.e.d.

4 Order $j(f, z_0)$ of the critical value curve for a harmonic map

For a harmonic map near a smooth critical point, we will introduce what we call the natural parametrization of the critical value curve, and then compute its order-pair in this coordinate.

Points 3, 4 and 5 of the following result are due to Lyzzaik, [9]. Just to be self-contained we reproduce Lyzzaik's proof here (with a somewhat different presentation).

Lemma 4.1. *Assume $f(z) = p(z) + \overline{q(z)}$ is an harmonic mapping and is critically smooth at z_0. Set $\psi(z) = \dfrac{p'(z)}{q'(z)}$ and $m = Ord_{z_0} f$.*

1. *We have $\lambda := \psi(z_0) \in S^1$ and $\psi'(z_0) \neq 0$. The critical set $\mathcal{C}_f$ in a neighborhood of z_0 coincides with $\psi^{-1}(S^1)$, is locally a smooth arc. We endow this arc what we call the natural parametrization by $\gamma(t) := \psi^{-1}(\lambda e^{it})$;*

2. *We then endow the critical value set what we call its natural parametrization by $\beta(t) := f(\gamma(t))$. Set $j = Ord_0(\beta(t))$. Either $\beta(t) \equiv \beta(0) = f(z_0)$, in which case $j = +\infty$ by convention, or $\infty > j \geq m$.*

3. *For the line $L = \{f(z_0) + s\sqrt{\lambda}, s \in \mathbb{R}\}$, the set $f^{-1}(L)$ is a locally regular star at z_0 with $2(m+1)$ branches.*

4. *We have $\beta'(t) = \sqrt{\lambda e^{it}} R(t)$, with $R(t) = 2\Re\Big(\sqrt{\lambda e^{it}} \dfrac{d}{dt} q(\gamma(t))\Big)$, an $\mathbb{R}$-analytic real function of t.*

5. *In the case $\beta'(t) \not\equiv 0$, the curve $t \mapsto \beta(t)$ is locally injective, has a strictly positive curvature in a punctured neighborhood of* 0*, turns always to the left, is tangent to* L *at* $\beta(0)$*.*

6. *We have $j-1 = Ord_0(R)$ and $\infty \geq j \geq m$. Either $j = \infty$ and $\beta \equiv \beta(0)$, or the curve β has the order-pair $(j, j+1)$ at* 0*.*

Proof. Point 1. We have

$$\mathcal{C}_f = \{J_f = 0\} = \{q' = 0\} \cup \{|\psi| = 1\} = \{q' = 0\} \cup \psi^{-1}(S^1) \ .$$

But $q(z)$ is not constant (otherwise $\psi \equiv \infty$) we know that $\{q' = 0\}$ is discrete and avoids a punctured neighborhood of z_0. Therefore, reducing U if necessary, we have $\{J_f = 0\} \cap U = \psi^{-1}(S^1) \cap U$, and we may choose a holomorphic branch of $\sqrt{\psi(z)}$ for $z \in U$. From Lemma 3.3 we know that $\psi(z_0) \in S^1$, $\psi'(z_0) \neq 0$ and so ψ is locally injective. Reducing U further if necessary, we see that $\{J_f = 0\} \cap U$ is a smooth arc. We call the parametrization $\gamma(t) = \psi^{-1}(\psi(z_0) \cdot e^{it})$ the natural parametrization of $\mathcal{C}_f$.

Point 2. We endow the critical value set with the natural image parametrization $\beta(t) = f(\gamma(t))$.

Write $f(z) = p(z) + \overline{q(z)}$, $p(z) = a(z-z_0)^m + h.o.t.$ and $q(z) = q(z_0) + A(z-z_0)^m + h.o.t.$ for some $a, A \neq 0$. Due to the smoothness of the critical set at z_0, we have $\gamma(t) = z_0 + \gamma'(0) \cdot t + h.o.t.$ with $\gamma'(0) \neq 0$. So

$$\beta(t) = f(\gamma(t)) = p(\gamma(t)) + \overline{q(\gamma(t))} = \beta(0) + (a\gamma'(0)^m + \overline{A\gamma'(0)^m})t^m + h.o.t \ .$$

It follows that $j = Ord_0 \beta(t)$ satisfies $m \leq j \leq +\infty$.

Point 3. Without loss of generality we may assume $z_0 = 0$ and $f(z_0) = 0$. Choose a local expression $f(z) = p(z) + \overline{q(z)}$ so that $p(0) = q(0) = 0$. Then $p(z) = az^m + O(z^{m+1})$ and $q(z) = bz^m + O(z^{m+1})$ for some $a, b \neq 0$. We have $\dfrac{a}{b} = \psi_f(0) = \lambda$. Rewrite now f in the form

$$f(z) = \sqrt{\lambda}\Big(P(z) + \overline{Q(z)}\Big) = \sqrt{\lambda}\Big(P(z) - Q(z) + Q(z) + \overline{Q(z)}\Big)$$

with $P(z) = p(z)/\sqrt{\lambda}$. Then $P(z)$ and $Q(z)$ have identical coefficient for the term z^m and $Ord_0(P) = Ord_0(Q) = m$. Set

$$F(z) = P(z) - Q(z), \ r(z) = Q(z) + \overline{Q(z)} \quad \text{so that} \quad f(z) = \sqrt{\lambda}\Big(F(z) + r(z)\Big). \qquad (4.1)$$

Note that $r(z)$ is real-valued, and $F(z)$ is holomorphic with multiplicity greater than m. Write F in the form $F(z) = cz^{m+n}(1 + O(z))$ with $c \neq 0$ and $n \geq 1$. As $P(z) = Q(z) + F(z)$, we have

$$\psi_f(z) = \frac{\sqrt{\lambda}P'(z)}{\overline{\sqrt{\lambda}}Q'(z)} = \lambda\Big(1 + \frac{F'(z)}{Q'(z)}\Big).$$

It follows that $n = Ord_0 \psi_f$. But $Ord_0 \psi_f = 1$ by the smoothness assumption of the critical set. So $n = 1$ and F takes the form $F(z) = cz^{m+1}(1 + O(z))$ with $c \neq 0$.

Finally $f^{-1}L = \{f(z) \in \sqrt{\lambda}\cdot\mathbb{R}\} = \{F(z)+r(z)\in\mathbb{R}\} = \{F(z)\in\mathbb{R}\} = F^{-1}\mathbb{R}$. This set is therefore a locally regular star of $2(m+1)$ branches.

Point 4. We follow the calculation of Lyzzaik. Let $z \in \mathcal{C}_f$. Then $|\psi(z)| = 1$. So there are two choices of $\sqrt{\psi(z)}$. Fix a choice of the square root.

$$\begin{aligned} Df|_z &= p'(z)dz + \overline{q'(z)}d\overline{z} \\ &= q'(z)\psi(z)dz + \overline{q'(z)}d\overline{z} \\ &= \sqrt{\psi(z)}\left(\sqrt{\psi(z)}q'(z)dz + \overline{\sqrt{\psi(z)}q'(z)dz}\right) \\ &= \sqrt{\psi(z)}\,\Re\left(2\sqrt{\psi(z)}q'(z)dz\right). \end{aligned}$$

As $\gamma(t)$ is defined by $\psi(\gamma(t)) = \lambda e^{it}$, we have

$$\beta'(t) = Df|_{\gamma(t)}(\gamma'(t)) = \sqrt{\lambda e^{it}}\,R(t), \quad \text{where} \quad R(t) = \Re\left(2\sqrt{\lambda e^{it}}\frac{d}{dt}q(\gamma(t))\right).$$

Points 5 and 6. Assume that β is not constant. Then $Ord_0(\beta) = j < \infty$, $R(t) \not\equiv 0$ and $Ord_0 R = j-1$. So $\dfrac{\beta'(t)}{R(t)} \to \sqrt{\lambda}$ as $t \to 0$. It follows that $\beta(t)$ is tangent to L at $\beta(0)$. Furthermore,

$$R(t) = C(t^{j-1} + bt^j + O(t^{j+1}))\ ,\ C \in \mathbb{R}^*,\ b \in \mathbb{R}\ .$$

A simple calculation shows that $\beta''(t) = \left(\dfrac{R'(t)}{R(t)} + \dfrac{i}{2}\right)\beta'(t)$. As $\dfrac{R'(t)}{R(t)}$ is real, we see already that the oriented angle from β' to β'' is in $]0,\pi[$. One can also check the sign of the curvature of β:

$$\kappa_\beta(t) = \frac{\Im\left(\overline{\beta'(t)}\cdot\beta''(t)\right)}{|\beta'(t)|^3} = \frac{1}{2|\beta'(t)|} > 0, \quad t \in]-\delta,\delta[\setminus\{0\}\ . \tag{4.2}$$

This shows that there is some $\delta > 0$ such that $\beta(t)$ is on the left of its tangent for any $t \in]-\delta,\delta[\setminus\{0\}$ if $\beta'(0) = 0$ and for any $t \in]-\delta,\delta[$ if $\beta'(0) \neq 0$.

Moreover,

$$\beta(t) = \beta(0) + \int_0^t \beta'(s)ds = \beta(0) + C\sqrt{\lambda}\int_0^t e^{is/2}\left(s^{j-1} + bs^j + h.o.t.\right)ds$$

$$= \beta(0) + C\sqrt{\lambda}\int_0^t \left(s^{j-1} + (b+\frac{i}{2})s^j + h.o.t.\right)ds.$$

So

$$\Re\frac{\beta(t)-\beta(0)}{C\sqrt{\lambda}} = \int_0^t s^{j-1}(1+O(s^j))ds, \quad \Im\frac{\beta(t)-\beta(0)}{C\sqrt{\lambda}} = \int_0^t \frac{s^j}{2}(1+O(s^{j+1}))ds.$$

It follows that $t \mapsto \beta(t)$ is locally injective and β has the order pair $(j, j+1)$ at 0. q.e.d.

Definition 4.2. *Let f be a harmonic map and z_0 be a smooth critical point. We denote by $j(f, z_0)$ the integer so that the critical value curve has the order-pair $(j(f, z_0), j(f, z_0)+1)$ in its natural parametrization. We will call $j(f, z_0)$ the critical value order of f at z_0.*

Let us notice that $j(f, z_0)$ is an analytic invariant hence is a fortiori invariant under our equivalence relation on harmonic maps.

5 Between critical value order and multiplicity

The objective here is to prove the following

Theorem 5.1. *Given a harmonic map G together with a smooth critical point z_0, the three local analytic invariants m order of f, j order of the critical values curve and μ multiplicity of the complexified map on $(\mathbb{C}^2, z_0)$ are related by the (in)equalities :*

$$\begin{cases} \infty \geq j \geq m \geq 1 \\ j + m^2 = \mu. \end{cases}$$

5.1 A formula for the multiplicity μ

For $p(z) = \sum a_i z^i$ we use $\overline{p}(z)$ to denote the power series $\overline{p}(z) = \sum \overline{a}_i z^i$. The following lemma provides a formula for the multiplicity, which in the case of a polynomial p leads to an algorithm.

Lemma 5.2. *Let $p(z)$ be a holomorphic map with $p(0) = 0$. Let $f(z) = p(z) - \bar{z}$ and $g(z) = p(z)^m - \overline{z}^m$ (with $m \geq 1$ an integer). Then $\mu(f, 0) = Ord_0(\bar{p} \circ p(z) - z)$ and more generally :*

$$\mu(g, 0) = \sum_{\xi^m = \eta^m = 1} Ord_0\Big(\eta\, \overline{p}(\xi\, p(z)) - z\Big).$$

Proof. Consider the following holomorphic extensions of f and g in $\mathbb{C}^2$:

$$M_f : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} p(u) - v \\ \overline{p}(v) - u \end{pmatrix}, \quad M_g : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} p(u)^m - v^m \\ (\overline{p}(v))^m - u^m \end{pmatrix}.$$

By definition $\mu(f, 0) = \mu(M_f, \mathbf{0})$ and $\mu(g, 0) = \mu(M_g, 0)$. Let us work directly with M_g. It is known for example by [4, theorem 6.1.4] that $\mu(g, 0) < \infty$ if and only if the germs of planar curves $p(u)^m - v^m = 0$ and $p(v))^m - u^m$ have no branch in common. This condition means that there is a neighborhood of $(0, 0)$ in $\mathbb{C}^2$, in which $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the only solution of the system of equations $M_f \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since this system is equivalent to the existence of ξ, η, such that

$$\xi^m = \eta^m = 1, \text{ and } v = \xi p(u), \quad \eta \overline{p}(\xi p(u)) - u = 0$$

the condition $\mu(g,0)=\infty$ is indeed equivalent to the finiteness of the order in the right-hand side of the statement of lemma 5.2.

We denote $\mu_1 = \sum_{\xi^m=\eta^m=1} Ord_0\Big(\eta\,\overline{p}(\xi\,p(z)) - z\Big)$ this order. Solving the equation $M_g\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}0\\t\end{pmatrix}$, we get

$$\left\{\begin{array}{r} v-\xi p(u)=0\\ \displaystyle\prod_{\xi^m=\eta^m=1}(\eta\overline{p}(\xi p(u))-u)=t\end{array}\right.$$

There are μ_1 distinct solutions in the variable u for the second equation, hence μ_1 solutions for the system which merge at a single solution $(0,0)$ when $t\to 0$. These solutions are all simple which means that M_g is locally invertible. Applying again [4, theorem 6.1.4], this proves that $\mu(M_g,\mathbf{0})=\mu_1$. q.e.d.

Note that $|p'(0)|\neq 1$ iff $\mu(f,0)=1$. Otherwise $\mu(f,0)\geq 2$.

5.2 Normalizations

Lemma 5.3. *Any harmonic map G near a smooth critical point z_0 is equivalent to $(g,0)$ with $g(z)=p(z)^m-\overline{z}^m$ for some integer $m\geq 1$ and some holomorphic function $p(z) = z+bz^2+O(z^3)$, $|b|=1$. Furthermore, setting $f_\xi(z)=\xi\cdot p(z)-\overline{z}$, $\xi\in\mathbb{C}$, we have*

$$\mu(G,z_0)=\mu(g,0)=\left\{\begin{array}{ll} m^2+m & \text{if } (-\mathrm{b}^2)^{\mathrm{m}}\neq 1\\ \mu(f_{-b^2},0)+(m-1)(m+2)>m^2+m & \text{otherwise.}\end{array}\right.$$

Note that in the particular case $m=1$, the above formula becomes

$$\mu(G,z_0)=\mu(g,0)=\left\{\begin{array}{ll} 2 & \text{if } b^2\neq -1\\ \mu(f_1,0)>2 & \text{otherwise.}\end{array}\right.$$

Proof. The existence of the model map g follows from Lemma 3.3. In the following the sums are over the m-th roots of unity for both η and ξ. By Lemma 5.2,

$$\begin{aligned}
\mu(g,0) &= \sum_{\xi^m=\eta^m=1} Ord_0\Big(\eta\,\overline{p}(\xi\,p(z))-z\Big)\\
&= \Bigg[\sum_{\eta\xi\neq 1}+\sum_{\eta\xi=1}\Bigg] Ord_0\Big(\eta\,\overline{p}(\xi\,p(z))-z\Big)\\
&= \sum_{\eta\xi\neq 1} Ord_0\Big(\eta\,\overline{p}(\xi\,p(z))-z\Big)+\sum_{\xi^m=1} Ord_0\Big(\overline{\xi p}(\xi\,p(z))-z\Big)\\
&\overset{Lem.5.2}{=} \sum_{\eta\xi\neq 1} Ord_0\Big(\eta\,\overline{p}(\xi\,p(z))-z\Big)+\sum_{\xi^m=1,\xi\neq -b^2} Ord_0\Big(\overline{\xi p}(\xi\,p(z))-z\Big)+C\mu(f_{-b^2},0)
\end{aligned}$$

where $C=0$ if $-b^2$ does not coincide with any m-th root of unity, and $C=1$ otherwise.

There are $m(m-1)$ pairs of (η,ξ) with $\eta^m = 1 = \xi^m$ and $\eta\xi \neq 1$. For each pair of them, $Ord_0\Big(\eta\,\overline{p}(\xi\, p(z)) - z\Big) = 1$. This gives $m(m-1)$ for the first sum above.

Now for any ξ with $\xi^m = 1, \xi \neq -b^2$, we have $Ord_0\Big(\overline{\xi p}(\xi\, p(z)) - z\Big) = 2$. If $-b^2$ does not equal to any m-th root of unity, there are m terms in the middle sum above, so $\mu(g,0) = m(m-1) + 2m = m^2 + m$. Otherwise there are $m-1$ terms, so $\mu(g,0) = m(m-1) + 2(m-1) + \mu(f_{-b^2}, 0) = m^2 + m - 2 + \mu(f_{-b^2}, 0)$. In this case one can check easily that $\mu(f_{-b^2}, 0) > 2$. So $\mu(g,0) > m^2 + m$. q.e.d.

Consider now

$$g(z) = p(z)^m - \overline{z}^m = \Big(z + bz^2 + O(z^3)\Big)^m - \overline{z}^m, \quad |b| = 1.$$

A direct calculation using the first term of $\gamma(t) = \psi_g^{-1}(-e^{it})$ shows that the critical value curve β in its natural parametrization satisfies $\beta(t) = \dfrac{2i}{(m+1)^m}\Im\Big(\dfrac{i^m}{b^m}\Big)t^m + o(t^m)$.

Clearly $m = Ord_0(g)$. Let j be the ordre of $\beta(t)$ at 0, and μ the multiplicity of g at 0. We want to prove

$$j \geq m \quad \text{and} \quad \mu = j + m^2.$$

Note that for $|b| = 1$,

$$(-b^2)^m = 1 \Longleftrightarrow \Big(\frac{i}{\overline{b}}\Big)^{2m} = 1 \Longleftrightarrow \Big(\frac{i}{b}\Big)^{2m} = 1 \Longleftrightarrow \Big(\frac{i}{b}\Big)^{m} = \pm 1 \Longleftrightarrow \Im\Big(\frac{i^m}{b^m}\Big) = 0\ .$$

This, together with Lemma 5.3, gives:

Corollary 5.4. *(The generic case) For $p(z) = z + bz^2 + O(z^3)$, $|b| = 1$ with $(-b^2)^m \neq 1$, and $g(z) = p(z)^m - \overline{z}^m$, we have*

$$j = m \quad \textit{and} \quad \mu = j + m^2 = m + m^2.$$

If $(-b^2)^m = 1$ then $j > m$.

It remains to work on the degenerate case $(-b^2)^m = 1$.

Lemma 5.5. *Any harmonic map of the form $g(z) = (z + bz^2 + o(z^3))^m - \overline{z}^m$ with $(-b^2)^m = 1$ is equivalent to a map of the form $(z + iz^2 + o(z^2))^m - \overline{z}^m$.*

Proof. One just need to replace g by $g(\lambda z)/\overline{\lambda}^m$ for $\lambda = 1/(-ib)$. q.e.d.

5.3 The normalised degenerate case

The following statement will complete the proof of Theorem 5.1. This is by far the hardest case.

Theorem 5.6. *Let $p(z) = z + iz^2 + O(z^3)$ be a holomorphic map in a neighborhood of* 0 *and $m \geq 1$ be an integer. Set $g(z) = p(z)^m - \overline{z}^m$. Then g is a harmonic map with* 0 *as a smooth critical point. Let j be the ordre of the critical value curve at $g(0) = 0$ in its natural parametrization, and μ the multiplicity of g at* 0*. Then*

$$j > m \quad \textit{and} \quad \mu = j + m^2.$$

Proof. We know already that 0 is a smooth critical point of g and $j > m$ (Corollary 5.4). Let's look at the complexification of g:

$$G\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} p(u)^m - v^m \\ -u^m + \overline{p}(v)^m \end{pmatrix} = \begin{pmatrix} u^m(1 + iu + o(u))^m - v^m \\ -u^m + v^m(1 - iv + o(v))^m \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}.$$

The critical set in $\mathbb{C}^2$ of G contains the set $\{(uv)^{m-1} = 0\}$ which consists of two branches $u = 0$ and $v = 0$.

The corresponding critical value branches are

$$G\begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} -v^m \\ v^m(1 - iv + O(v^2))^m \end{pmatrix}, \; G\begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} u^m(1 + iu + O(u^2))^m \\ -u^m \end{pmatrix}.$$

Both are plane curves with order pair $(m, m + 1)$. The other branch gives a critical value curve β with order pair $(j, j + 1)$, as we already know from the real calculation. By comparing the two parametrizations, an elementary calculation shows that these two branches are distinct. They are also distinct from the third branch since we shall prove that $j > m$ hence that they have different first Puiseux pairs.

The local behavior of G at each of these critical branches, off the origin, is given by the following:

Lemma 5.7. *The multiplicity of G at a real critical branch point (off the origin) is 2, and the multiplicity of G at a non-real critical branch point (off the origin) is m.*

Proof. The expression of G at the point $\begin{pmatrix} 0 \\ v_0 \end{pmatrix}$ in local coordinates $\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} u \\ v - v_0 \end{pmatrix}$ is

$$G\begin{pmatrix} u \\ v_0 + w \end{pmatrix} - \begin{pmatrix} -v_0^m \\ \overline{p}(v_0)^m \end{pmatrix} = \begin{pmatrix} p(u)^m - mv_0^{m-1}w + O(w^2)) \\ -u^m + Q(v_0)w + O(w^2) \end{pmatrix}.$$

By using to Taylor formula for $\overline{p}(v_0 + w)^m$ we find $Q(v_0) = m\overline{p}(v_0)^{m-1}\overline{p}'(v_0)$. In order to see that the germ of G at the point $\begin{pmatrix} 0 \\ v_0 \end{pmatrix}$ is equivalent by analytic coordinates changes to the germ $\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} \to \begin{pmatrix} \hat{u}^m \\ \hat{v} \end{pmatrix}$, it is sufficient to check that $Q(v_0) \neq mv_0^{m-1}$ for any small enough non zero v_0. The proof for the branch $u \to G\begin{pmatrix} u \\ 0 \end{pmatrix}$ is similar. q.e.d.

The preimage $G^{-1}(S)$, of S a small sphere centered at the origin, is a smooth 3-manifold, and in fact we are going to prove a stronger result stating that the pair

$(G^{-1}(B), G^{-1}(S))$ is diffeomorphic to the pair made of the standard ball and the standard sphere.

We notice that $G^{-1}(S)$ is defined by the equation $N(u,v) := \left\| F\begin{pmatrix} u \\ v \end{pmatrix} \right\|^2 = \epsilon^2$, and is the boundary of $G^{-1}(B) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \mid \left\| F\begin{pmatrix} u \\ v \end{pmatrix} \right\|^2 \leq \epsilon^2 \right\}$. The above result will then follow from a general statement about a function N defined on an open set of $\mathbb{R}^n$ given in the next lemma.

Lemma 5.8. *Let $N : W \to \mathbb{R}$ be a positive real analytic map defined in a neighborhood of $0 \in \mathbb{R}^n$ such that $N^{-1}(0) = \{0\}$. Then there is $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$ the pair of sets*

$$\big(\{x \in \mathbb{R}^n \mid N(x) \leq \epsilon^2\}, \{x \in \mathbb{R}^n \mid N(x) = \epsilon^2\}\big)$$

is diffeomorphic to the standard ball and the standard sphere.

Proof. First we prove that N is a submersion outside the origin if we restrict to a small enough neighborhood of 0 : there is a constant $r_0 > 0$ such that if $0 < \|x\| \leq r_0$ we have

$$\mathrm{grad}N(x) := \left(\frac{\partial N}{\partial x_1}, \ldots, \frac{\partial N}{\partial x_n}\right)(x) \neq 0.$$

Indeed if this was not true, we could by the curve selection lemma [10, lemma 3.1] find an analytic path $\gamma : [0, \eta_0[\longrightarrow W$ such that $\gamma(0) = 0$ and $\gamma(t) \neq 0$ for $t \in]0, \eta_0[$, and $\mathrm{grad}N(\gamma(t)) = 0$. But then we would have $\frac{d}{dt}(N(\gamma(t))) = \langle \gamma'(t), \mathrm{grad}N(\gamma(t))\rangle = 0$. But then $N(\gamma(t))$ would be constant equal to $N(\gamma(0)) = 0$ and this contradicts $\gamma(t) \neq 0$ for $t \neq 0$.

In a second step we show that the gradient of N tends to point out from 0 when $t \to 0$. More precisely this means that given an analytic path $\gamma : [0, \eta_0[$ such that $\gamma(0) = 0$ and $\gamma(t) \neq 0$ for $t \in]0, \eta_0[\to W$, we have:

$$\lim_{t \to 0} \frac{\langle \gamma(t), \mathrm{grad}N(\gamma(t))\rangle}{\|\gamma(t)\| \cdot \|\mathrm{grad}N(\gamma(t))\|} \geq 0.$$

Indeed let α, β be the valuations of γ and $\mathrm{grad}N \circ \gamma$. We have power series expansions with initial vector coefficients $a, b \in \mathbb{R}^4$:

$$\gamma(t) = at^\alpha + o(t^\alpha), \quad \mathrm{grad}N(\gamma(t)) = bt^\beta + o(t^\beta)$$

and the limit above is $\frac{\langle a,b\rangle}{\|a\|\|b\|}$. The expansion of the derivative of γ is $\gamma'(t) = \alpha a t^{\alpha-1} + o(t^{\alpha-1})$, and therefore $\frac{d}{dt}(N(\gamma(t))) = \langle \gamma'(t), \mathrm{grad}N(\gamma(t))\rangle = \alpha\langle a, b\rangle t^{\alpha+\beta-1} + o(t^{\alpha+\beta-1})$. Since $N(\gamma(t)) > 0$ for small enough positive t, this forces the inequality $\langle a, b\rangle \geq 0$ and we are done.

We deduce a quantified version of this behaviour of the gradient vector field, showing that the angle of the vectors $x, \mathrm{grad}N(x)$ is bounded away from π. Precisely making the constant r_0 above smaller if necessary we may assume that for $0 < \|x\| \leq r_0$:

$$\frac{\langle x, \mathrm{grad}N(x)\rangle}{\|x\| \cdot \|\mathrm{grad}N(x)\|} \geq -\frac{1}{2}.$$

This claim is a consequence of the curve selection lemma, because the limit property of $\mathrm{grad}N(x)$ implies that 0 cannot be in the closure of the semi analytic set

$$Z := \{x \in W \mid 0 < \|x\| \leq r_0, \quad \langle x, \mathrm{grad}N(x)\rangle < -\frac{1}{2}\|x\| \cdot \|\mathrm{grad}N(x)\|\}.$$

Our third and last step is to show that we have a homotopy between $\Sigma = N^{-1}(\epsilon^2)$ and the standard ball $\|x\|^2 = \epsilon^2$ because the gradient of interpolations between N and $\|x\|^2$ never vanishes outside the origin. Indeed the choice we made for r_0 has the following consequence: for any $t \in [0,1]$, we have $2tx + (1-t)\mathrm{grad}N(x) \neq 0$ and this implies that the the relative gradient with respect to $(x_1, \ldots, x_n)$ of the deformation $N(t,x) := t\|x\|^2 + (1-t)N(x)$ is non zero for any $x \neq 0$:

$$\forall t \in [0,1], \forall x, 0 < \|x\| \leq r_0, \quad \mathrm{grad}_x N(t,x) = \left(\frac{\partial N}{\partial x_1}, \ldots, \frac{\partial N}{\partial x_n}\right)(t,x) \neq 0.$$

Using the continuity of N let us choose $\epsilon_0 < r_0$ such that $N(x) \leq \epsilon_0^2 \implies \|x\| < r_0$. Then the property we obtained on the gradient shows that for each $t \in [0,1]$ the set $\Sigma_t := N_t^{-1}(\epsilon^2))$ (resp $\Sigma := N^{-1}(\epsilon^2))$ is a submanifold of the open ball $B(0, r_0)$ (resp. of the product $[0,1] \times B(0, r_0))$. The set $B_t = N_t^{-1}([0, \epsilon_0^2])$ is a manifold with boundary S_t and interior an open set of $\mathbb{R}^n$. Similarly Σ is a part of the boundary of $B = N^{-1}([0, \epsilon_0]) \subset [0,1] \times B(0, r_0)$ to be completed by $B_0 \cup B_1$[1]. We notice that (B_1, Σ_1) is the standard ball of radius ϵ_0 with its boundary.

Finally the restriction to Σ of the projection $(t,x) \longrightarrow t$ is a submersion. This implies by the version with boundary of a well known theorem of Ehresmann [5] that the pair (B, Σ) is locally trivial above $[0,1]$ which means that we have a diffeomorphism

$$(B, \Sigma) \longrightarrow [0,1] \times (B_0, \Sigma_0).$$

In particular we have a diffeomorphism $(B_0, \Sigma_0) \longrightarrow (B_1, \Sigma_1)$ as expected. q.e.d.

Let us now come back to the map G. Take a small round closed ball D of radius ϵ_0 and its preimage B so that $G : B \to D$ is a covering of degree μ outside the critical value curves, and that ∂D is transverse to the critical value set. It follows from lemma 5.8 applied to $N = \|G\|^2$ that ∂B is a smooth 3-variety diffeomorphic to a sphere. We take r_0 as in this lemma and denote : $B_\epsilon = \{x \mid N(x) \leq \epsilon^2\} \subset D_{r_0}$, $\Sigma_\epsilon = \partial B_\epsilon$ for all $0 < \epsilon \leq \epsilon_0$.

Let $\ell : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto ax_1 + bx_2$ be a generic linear form. For ϵ_0 small enough the disc $(\ell = 0) \cap D$ intersects the critical value set $\mathcal{V}$ only at the origin. Therefore for t a small

[1] we might avoid easily to consider a manifold with a corner along $S_0 \cup S_1$ by enlarging slightly the range of t to an open interval $]-\eta, 1+\eta[$.

enough non zero complex number, the line L_t with equation $\ell(u,v) = t$ is transversal to the boundary of D and $L_t \cap D$ is a disc Δ_t. Furthermore if $t \neq 0$ L_t, intersects the critical value set $\mathcal{V}$ at $j + 2m$ points contained in the interior of D.

Set $Y_t := \{\ell(G_1, G_2) = t\} = G^{-1}(\{\ell = t\})$.

Proposition 5.9. *With well chosen r_0, ϵ_0, as in the proof of lemma 5.8 and $t \neq 0$ small enough, $X_t := Y_t \cap B_{\epsilon_0}$ is diffeomorphic to the Milnor fiber of the function $\ell(G_1, G_2)$.*

Proof. In the proof of lemma 5.8 we may choose if necessary a smaller r_0 to guarantee that the standard ball $B'_{r_0} = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \mid \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| \leq r_0 \right\}$ is a Milnor ball which means that X_0 is transverse to the standard sphere $\partial B'_r$ for each $r \in]0, r_0]$ and the Milnor fiber is by definition $X_t \cap B'_{r_0}$ for $0 < |t| \leq \eta_0$, with η_0 small enough. By this very definition B'_r is also a Milnor ball and $X_t \cap B'_r$ a Milnor fiber provided that we restrict the condition on t to $0 < |t| \leq \eta$ for an appropriate $\eta < \eta_0$. In fact for such a t the inclusion $X_t \cap B'_r \subset X_t \cap B'_{r_0}$ yields a deformation retract between two diffeomorphic varieties. Now we have the inclusion $B_{\epsilon_0} \subset B'_{r_0}$ and choosing r small enough to get $B'_r \subset B_{\epsilon_0}$ we can perform again the construction of lemma 5.8 and we get the chain of inclusions:

$$B_{\epsilon_0} \subset B'_r \subset B_{\epsilon_0} \subset B'_{r_0}. \tag{5.1}$$

Let us choose η_0 small enough both for the validity of the Milnor fibration and for the transversality of the intersections $L_t \cap \partial D$ as described above, with D of radius ϵ_0. We have to notice also that L_0 is transverse to D_ϵ for all $\epsilon \leq \epsilon_0$. Then at any point $y \in \partial X_t = Y_t \cap \Sigma_{\epsilon_0}$, the two varieties Y_t and Σ_{ϵ_0} are also transversal, and so are Y_0 and Σ_ϵ for $0 < \epsilon \leq \epsilon_0$. Indeed at such a point y we have avoided $\mathcal{V}$ and the map G is a local diffeomorphism.

Because of these transversalities we can construct Milnor fibrations with Milnor fiber $Y_t \cap B_\epsilon$ using "pseudo Milnor balls" B_ϵ which make a basis of neighborhoods of 0. The arguments are exactly the same as with the standard Milnor fibration. It is known (see[[7], Theorem 3.3) that this Milnor fiber is diffeomorphic to the standard one. The proof uses the chain of inclusions (5.1). Indeed we choose t small enough for the intersections of Y_t with the four terms in (5.1), to be Milnor fibers. The two inclusions $Y_t \cap B_\epsilon \subset Y_t \cap B_{\epsilon_0}$ and $Y_t \cap D_r \subset Y_t \cap D_{r_0}$ are homotopy equivalences. Therefore, in the sequence of maps

$$H^i(Y_t \cap B_\epsilon) \xrightarrow{\alpha_1} H^i(Y_t \cap D_r) \xrightarrow{\alpha_2} H^i(Y_t \cap B_{\epsilon_0}) \xrightarrow{\alpha_3} H^i(Y_t \cap D_{r_0})$$

$\alpha_3 \circ \alpha_2$ and $\alpha_2 \circ \alpha_1$ are isomorphisms and this forces the middle arrow to be an isomorphism for $i = 0, 1$. Since we work on surfaces with boundaries this is enough to obtain that they are diffeomorphic. q.e.d.

Proposition 5.10. *The surface X_t is connected and $\chi(X_t) = 2m - m^2$. Furthermore its boundary has m connected components and its genus is $g(X_t) = \frac{(m-1)(m-2)}{2}$.*

Proof. Since X_t is a smooth real surface with boundary its Euler characterictic is $\chi(X_t) = 1 - \dim(H^1(X_t), \mathbb{C})$ because it is connected by [10]. The first statement in the proposition is equivalent to the fact that the Milnor number $\mu(\ell \circ G) = \dim(H^1(X_t), \mathbb{C})$ $(m-1)^2$. To check this fact recall that $\mu(\ell \circ G)$ is an analytic invariant (and even a topological one) of the function. Let us calculate a standard form up to an analytic change of coordinates, for $L := \ell(G_1, G_2)$:

$$\begin{aligned} L(u,v) &= a(p(u) - v^m) + b(-u^m + \overline{p}(v) \\ &= (a-b)u^m(1+O(u)) - (a-b)v^m(1+O(v)) = U^m - V^m \end{aligned}$$

where $\Phi : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \varphi(u) \\ \psi(v) \end{pmatrix}$ is a diagonal change of coordinates. We can now check that $\mu(\ell(G_1, G_2)) = (m-1)^2$ by the formula for the Milnor number as the codimension of the Jacobian ideal : $\mu(L) = \dim_{\mathbb{C}} \mathbb{C}\{u,v\}/(\frac{\partial L}{\partial u}, \frac{\partial L}{\partial v}) = \dim_{\mathbb{C}} \mathbb{C}\{U,V\}/(U^{m-1}, V^{m-1})$. The last statement follows since the number of components of the boundary is the number of irreducible local components of the curve $L(u,v) = 0$. q.e.d.

Now we are ready to finish the proof of theorem 5.6. We already know that $F : X_t \to \Delta_t$ is a ramified cover of degree μ with $j + 2m$ critical values and that above each critical value there is exactly one critical point.

By the proof of 5.7 we know that the germ of the map F, at a critical point different from $(0,0)$, is up to analytic changes of coordinates, equivalent to one of the two germs $(z_1, z_2) \to (z_1, z_2^2)$ or $(z_1, z_2) \to (z_1, z_2^m)$. Since the disc Δ_t is transversal to the critical value curve, we deduce that for $F : X_t \to \Delta_t$ the critical points are simple on the smooth branch, and of local multiplicity m (therefore counts as $m-1$ critical points), above the fantom curves.

By Riemann-Hurwitz, $\chi(X_t) + \#\{\text{critical points}\} = \mu\chi(\Delta) = \mu$. So $1 - (m-1)^2 + (j + 2m(m-1)) = \mu$. That is $\mu = j + m^2$. q.e.d.

Combining with Lemma 5.3, in which we plug in $b = i$, $\mu(f_{-b^2}, 0) = \mu(f, 0)$ we get:

Corollary 5.11. *For $m \geq 1$, $f(z) = (z + iz^2 + O(z^3)) - \overline{z}$ and $g(z) = (z + iz^2 + O(z^3))^m - \overline{z}^m$, the four quantities $j(f), \mu(f), j(g), \mu(g)$ at 0 are related as follows:*

$$\mu(g) = \mu(f) + m^2 + m - 2, \quad j(g) = \mu(g) - m^2 = j(f) + m - 1 = \mu(f) + m - 2.$$

In particular each of these number determines the three other ones.

6 Topological models for harmonic smooth critical points

Notice that due to the equality $\mu = j + m^2$, the integers μ and $m + j$ have the same parity. In this section we will reformulate Lyzzaik's topological model in terms of the parity of $m + j$. We provide a self-contained proof.

We then show examples of harmonic maps with prescribed numerical invariants or with prescribed local models.

6.1 Local models

Theorem 6.1. *(topological model, inspired by Lyzzaik, [9]) Let f be a harmonic map with z_0 a smooth critical point. Set $m = Ord_{z_0}(f)$. Let j be the integer so that the critical value curve β at z_0 has the order-pair $(j, j+1)$. Assume $j < \infty$.*

In this case, define $n^\pm$ by the following table:

$\begin{pmatrix} 2n^+ - 1 \\ 2n^- - 1 \end{pmatrix}$	*β convex ($m \le j$ odd)*	*β cusp ($m \le j$ even)*
m odd	$\begin{pmatrix} m \\ m \end{pmatrix}$	$\begin{pmatrix} m+2 \\ m \end{pmatrix}$ *or* $\begin{pmatrix} m \\ m+2 \end{pmatrix}$
m even	$\begin{pmatrix} m+1 \\ m-1 \end{pmatrix}$ *or* $\begin{pmatrix} m-1 \\ m+1 \end{pmatrix}$	$\begin{pmatrix} m+1 \\ m+1 \end{pmatrix}$

(6.1)

Set $R_{n^+,n^-}(z) = R_{n^+,n^-}(re^{i\theta}) = \begin{cases} re^{i(2n^+-1)\theta} & 0 \le \theta \le \pi \\ re^{-i(2n^--1)\theta} & \pi \le \theta \le 2\pi \end{cases}.$

Then one of the choices of $R_{n^+,n^-}(z)$ (the choice is unique if $m + j$ is odd) is a local topological model of f, in the following sense: There is a neighborhood U of 0, two orientation preserving homeomorphisms: $h_1 : U \to \mathbb{D}$, $0 \mapsto 0$, $h_2 : \mathbb{C} \to \mathbb{C}$, $0 \mapsto 0$, such that

$$h_2 \circ f \circ h_1^{-1}(z) = R_{n^+,n^-}(z).$$

Moreover $\#f^{-1}(z) = n^+ + n^-$ or $n^+ + n^- - 2$ depending on whether z is in one sector or the other of $f(U) \setminus \beta$.

Notice that only the parity but not the size of j comes into account, and $n^+ - n^- = 0, 1$ or -1.

Proof. By Lemma 3.3 we can assume $z_0 = 0$ and f takes the form $f(z) = p(z) + \overline{q(z)}$ with

$$p(z) = z^m + bz^{m+1} + O(z^{m+2}), \quad q(z) = z^m, \quad |b| = 1.$$

In this case $\psi(z_0) = 1$. From lemma 4.1, we know that $t \mapsto \beta(t)$ is locally injective and the local shape of β corresponds to that of $u(t^j + it^{j+1})$. Therefore β is a convex curve on one half plane if j is odd and is a cusp of the first kind tangent to $\mathbb{R}$ if j is even, then has its tangent lines on the right. See Figure 1.

Write $f(z) = p(z) - q(z) + 2\Re q(z) = b(\kappa(z))^{m+1} + 2\Re q(z)$ with κ a holomorphic map tangent to the identity at 0. We may take $\kappa(z)$ as coordinate and transform f into the following holomorphic+real normal form

$$f(z) = e^{i\theta} z^{m+1} + r(z) = F(z) + r(z) \quad \text{with } F(z) = e^{i\theta} z^{m+1}, r(z) = 2\Re(z^m + O(z^{m+1})). \tag{6.2}$$

Claim 0. In this form the critical value curve β is either a convex curve on one half plane or is a cusp of the first kind tangent to $\mathbb{R}$.

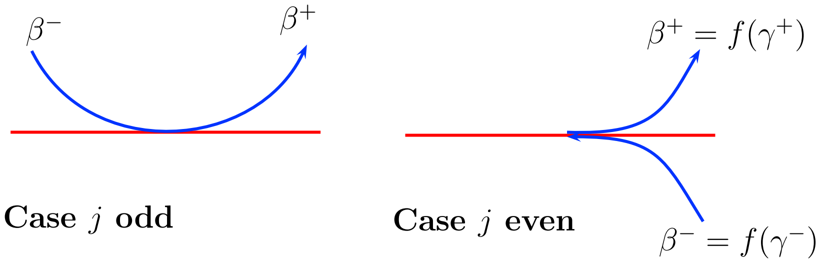

Figure 1: The shape of the critical value curve

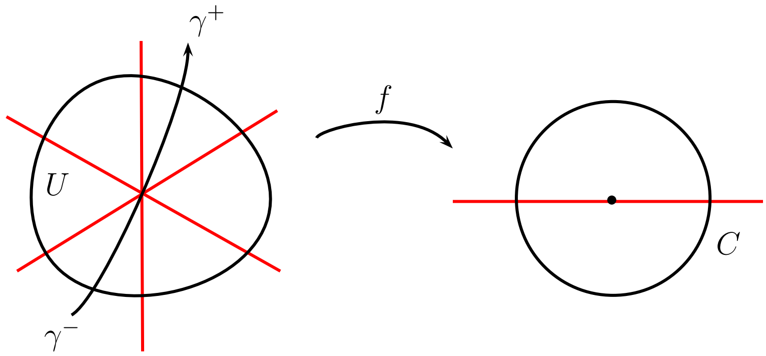

Figure 2: The domain U and $F^{-1}(\mathbb{R})$

Proof. We have only changed the variable in the source plane. So this new normal form has the same critical value curve as before.

Claim 1. We give here a specific proof to be compared to lemma 5.8. For a small enough round circle $C = \{|z| = s\}$ in the range, its preimage by f contains a Jordan curve connected component bounding a neighborhood U of 0, with $f(U) \subset D_s$ (not necessarily equal) and $f : U \to D_s$ proper (see Figure 2).

Notice that the tangent of γ at 0 depends on the choice of θ in $b = e^{i\theta}$, whereas the tangent of β at 0 does not depend on θ.

Proof. By assumption on $j < \infty$ the point 0 is an isolated point in $f^{-1}(0)$. So there is $r > 0$ such that $\{|z| \leq r\}$ is contained in the domain of definition Ω of f and $0 \notin f(\{|z| = r\})$.

There is therefore a small round open disc D centred at 0 in the range such that $D \cap f(\{|z| = r\}) = \emptyset$.

Let W be an open connected subset of D containing 0.

As f is continuous $f^{-1}(W)$ is open in Ω. Let V be the connected component of $f^{-1}(W)$ containing 0. Then V is an open neighborhood of 0 with $V \subset \{|z| < r\} \subset\subset \Omega$.

We now claim that $f|_V : V \to W$ is proper.

Proof. Let $V \ni z_n \to z \in \partial V$. We need to show $f(z_n) \to \partial W$. As $z \in \partial V \subset \Omega$ the map f is defined and continuous at z. It follows that $W \ni f(z_n) \to f(z) \in \overline{W} = W \sqcup \partial W$.

If $f(z) \in W$, then by continuity f maps a small disc neighborhood B of z into W, consequently

$$B \cup V \text{ is } \begin{cases} \text{connected} \\ \text{strictly larger than } V\text{, and} \\ \text{a subset of } f^{-1}(W). \end{cases}$$

This contradicts the choice of V as a connected component of $f^{-1}(W)$ and ends the proof of the claim. We now choose W a small enough disc such that $t \to |\beta(t)|$ is strictly increasing (resp. decreasing) as along as $t > 0$ (resp. $t < 0$) and $\beta(t) \in W$ and consider the proper map $f := f|_V : V \to W$.

Fix now $C = \{|z| = s\}$ contained in W in the range. The map f is a local homeomorphism at every point of $f^{-1}C \setminus \gamma$. Due to the local fold model at points of γ^* we may conclude that $f^{-1}C$ is a 1-dimensional topological manifold, which is actually piecewise smooth. It is also compact by properness, so has only finitely many components, each is a Jordan curve.

Let I be an island, i.e. an open Jordan domain in V bounded by a curve in $f^{-1}C$. We claim that $f(I) \subset D_s := \{|z| < s\}$.

Assume $f(I) \setminus \overline{D}_s \neq \emptyset$. Then $|f|$ on the compact set $\overline{I}$ reaches its maximum at an interior point $x \in I$. Then x can not be outside γ as f is locally open outside γ. But if $x \in \gamma$ then $f(x) \in \beta$ and $|f|$ restricted to γ can not reach a local maximum since $|\beta(t)|$ is locally monotone. This is not possible.

So $f(I) \subset \overline{D}_s$. But if for some $x \in I$ we have $f(x) \in C = \partial D_s$, then I contains a component (so a Jordan curve) of $f^{-1}C$. Choose a point x' in this curve but disjoint from γ. Then f is a local homeomorphism on a small disc B centred at x' with $B \subset I$ and $f(B)$ contains points outside $\overline{D}_s$. This is not possible by the previous paragraph. So we may conclude that $f(I) \subset D_s$.

We claim now $0 \in I$. Otherwise $0 \notin f(I)$ and we may argue as above using the minimum of $|f|$ on $\overline{I}$ to reach a contradiction.

It follows that $f^{-1}C$ has only one component in V bounding a Jordan domain U containing 0 and $f(U) \subset D_s$. As f maps the boundary into the boundary (not necessarily onto), $f : U \to D_s$ is proper.

Claim 2. The set $F^{-1}\mathbb{R}^*$ is a regular star of $2(m+1)$ radial branches from 0 to ∞ and $F^{-1}\mathbb{R} \cap U$ is connected (see Figure 2).

Otherwise there is a segment $L \subset F^{-1}\mathbb{R}^*$ connecting two boundary points of U. As $f(s) = F(s) + r(s)$ with r real, $f(L) \subset \mathbb{R}$. But $f^{-1}(0) = 0$. So $f(L)$ is a segment in $\mathbb{R}^*$ by Intermediate Value Theorem. Now as f has no turning points (critical points) in L, it maps L bijectively onto a real segment with constant sign, and the two ends are in $f(\partial U) = C$. This contradicts the choice that C is a round circle.

Claim 3. Each sector S of $U \setminus F^{-1}\mathbb{R}$ is mapped by both f and F into the same upper half plane. Each branch ℓ of $F^{-1}\mathbb{R}^*$ is mapped by f to a real segment with constant sign (but not necessarily equal to the sign of $F(\ell)$). Two consecutive branches on the same side of γ have images under f with opposite signs, and two consecutive branches separated by γ have images under f with the same sign.

Proof. As $f(s) = F(s) + r(s)$ with r real, and $F(S)$ is either on the upper or lower half plane, the same is true for $f(S)$ with the same imaginary sign.

The fact that $F(\ell) \subset \mathbb{R}^*$ implies $f(\ell) \subset \mathbb{R}$. But $f^{-1}(0) = 0$. So $f(\ell)$ is a segment in $\mathbb{R}^*$ by Intermediate Value Theorem. Now as f has no turning points (critical points) in ℓ, it maps ℓ bijectively onto a real segment with constant sign.

We now prove by contradiction that two consecutive branches on the same side of γ have images under f with opposite signs. Let W be a small closed sector neighborhood of 0 bounded by two consecutive branches on the same side of γ and a small arc α. Assume f maps the two branches to the same segment in $\mathbb{R}$, say $[0, \varepsilon]$. As $W \cap f^{-1}\mathbb{R}^* = W \cap F^{-1}\mathbb{R}^* = \emptyset$, the connected set $f(W)$ is disjoint from $\mathbb{R}^-$. And $f(\alpha)$ is disjoint from 0. Since $f(W)$ is not entirely contained in $\mathbb{R}^+$, one can find $v \in \partial f(W) \smallsetminus \Big(f(\alpha) \cup \mathbb{R}^+ \cup \{0\} \Big)$. So $v = f(w)$ for some interior point w of W. This contradicts that f is a local homeomorphism.

We may prove similarly that two consecutive branches of $F^{-1}\mathbb{R}^*$ separated by γ have images under f with the same sign, using the fact that f realises a fold along γ^*.

Claim 4. Let S be a sector of $U \smallsetminus F^{-1}\mathbb{R}$ disjoint from γ. Then f maps S homeomorphically onto one of the half discs $\{|z| < s, \Im z > 0\}, \{|z| < s, \Im z < 0\}$, and in S the number of branches of $f^{-1}(f(\gamma))$ is equal to the number of branches of $F^{-1}(F(\gamma))$ (see Figure 5).

Proof. The previous claim says that f is a local homeomorphism on S, and $f(S)$ is contained in one of the half discs, say $\{|z| < s, \Im z > 0\}$. We also know that $f : S \to \{|z| < s, \Im z > 0\}$ is proper, so is in fact a covering. As S is simply connected, we conclude that f on S is a homeomorphism onto its image. We also need to prove that $f(S)$ is one of the half discs bounded by $C \cup \mathbb{R}$.

For $t \in]0, \varepsilon[$, set $\gamma^\pm(t) = \gamma(\pm t)$. Consider $\delta^\pm(t) = F(\gamma^\pm(t))$ and $\beta^\pm(t) = f(\gamma^\pm(t))$,

By (6.2) we know that $\delta^-(t)$ and $\beta^-(t)$ are in the same half plane of $\mathbb{C} \smallsetminus \mathbb{R}$, idem for the pair $\delta^+(t)$ and $\beta^+(t)$. Comparing with the shape of β relative to $\mathbb{R}$ we know that $\delta^\pm(t)$ are in the same half plane if β is convex and in opposite half planes otherwise.

Claim 5. The map f sends each S of the two sectors of $U \smallsetminus F^{-1}\mathbb{R}$ intersecting γ onto one small sector χ with 0 angle at 0 of $\mathbb{C} \smallsetminus (C \cup \beta \cup \mathbb{R})$, and $S \cap f^{-1}(\beta) \subset \gamma$ (see Figure 3).

This is due to the harmonicness: f folds a small neighborhood of $z \in \gamma^*$ onto a half neighborhood of $f(z)$ on the concave side of β (see Figure 3). As χ does not contain the other branch of β, the preimage S contains no other co-critical points than γ.

Claim 6. The critical curve γ separates the branches of $F^{-1}\mathbb{R}^*$ into two parts whose numbers depend on the shape of β, by the following table:

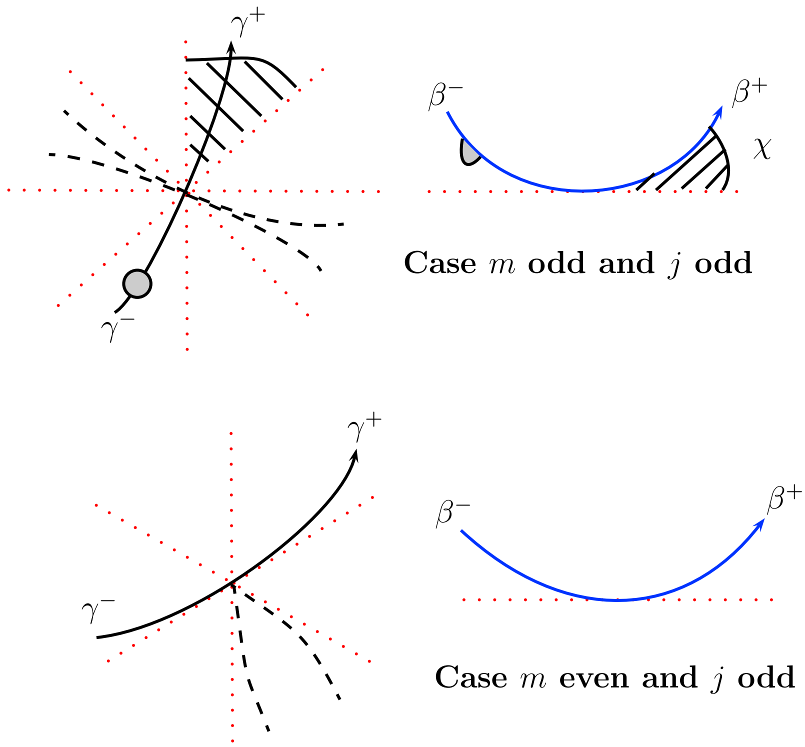

Figure 3: The folding sides for harmonic maps f

$\begin{pmatrix} \#\text{right branches of } F^{-1}\mathbb{R}^* \\ \#\text{left branches of } F^{-1}\mathbb{R}^* \end{pmatrix}$	β convex	β cusp
m odd	equal number	differ by 2
m even	differ by 2	equal number

Proof. For $t \in]0, \varepsilon[$, we have $\gamma^\pm(t) = \gamma(\pm t)$, $\delta^\pm(t) = F(\gamma^\pm(t))$ and $\beta^\pm(t) = f(\gamma^\pm(t))$. We need to know the relative positions between $\delta^\pm(t)$ and $\mathbb{R}$ in order to get the relative positions between $\gamma \subset F^{-1}(\delta^\pm(t))$ and $F^{-1}\mathbb{R}^*$.

We know that $\delta^\pm(t)$ are in the same half plane if β is convex and in opposite half planes otherwise.

On the other hand, the two curves $\gamma^\pm(t), t \in [0, \varepsilon[$ make an angle π at $\gamma(0)$. As $F(z) = e^{i\theta} z^{m+1}$,

$$\text{angle}_0(\delta^\pm(t)) = (m+1) \cdot \text{angle}_0(\gamma^\pm(t)) = (m+1)\pi \mod 2\pi = \begin{cases} 0 & \text{if } m \text{ is odd} \\ \pi & \text{if } m \text{ is even.} \end{cases}$$

Now pullback these shapes by $F(z) = e^{i\theta} z^{m+1}$, we get the claim. See Figure 4.

Claim 7. In any case, the number of sectors in $U \smallsetminus f^{-1}\beta$ is odd in each side of γ. Denoting them by $2n^\pm - 1$, with $+$ for the right-side of γ and $-$ the left side, one can related them to the numbers of branches of $F^{-1}\mathbb{R}^*$ separated by γ by:

	β convex, j odd	β cusp, j even
$\begin{pmatrix}2n^+-1\\ 2n^--1\end{pmatrix} =$	$\begin{pmatrix}\#\text{right branches of } F^{-1}\mathbb{R}^*-1\\ \#\text{left branches of } F^{-1}\mathbb{R}^*-1\end{pmatrix}$	$\begin{pmatrix}\#\text{right branches of } F^{-1}\mathbb{R}^*\\ \#\text{left branches of } F^{-1}\mathbb{R}^*\end{pmatrix}$

Proof. The shape of β is determined by the parity of j in its order-pair $(j, j+1)$: If j is odd then β is convex, if j is even then β is a cusp. In the following only the shape of β is relevant, but not the value of j. It follows that if $m+j$ is odd, $F^{-1}\mathbb{R}$ contains the tangent line of γ at 0.

See Figure 5.

Now as the total number of branches of $F^{-1}\mathbb{R}^*$ is $2(m+1)$, we get, by Claim 6,

$\begin{pmatrix}\#\text{right branches of } F^{-1}\mathbb{R}^*\\ \#\text{left branches of } F^{-1}\mathbb{R}^*\end{pmatrix}$	β convex, j odd	β cusp, j even
m odd	$\begin{pmatrix}m+1\\ m+1\end{pmatrix}$	$\begin{pmatrix}m+2\\ m\end{pmatrix}$ or $\begin{pmatrix}m\\ m+2\end{pmatrix}$
m even	$\begin{pmatrix}m+2\\ m\end{pmatrix}$ or $\begin{pmatrix}m\\ m+2\end{pmatrix}$	$\begin{pmatrix}m+1\\ m+1\end{pmatrix}$

We get (6.1) .

Claim 8. Now we forget about $F^{-1}\mathbb{R}$ and consider only the sectors in U partitioned by $f^{-1}\beta$. The same arguments as above show that f maps each sector homeomorphically onto one of the two sectors in $D_s \setminus \beta$ in the range.

To construct coordinate changes h_1, h_2 from f to R_{n^+,n^-}, one proceeds as follows:

Define at first an orientation preserving homeomorphisms $h_2 : \overline{D}_s \to \overline{\mathbb{D}}$ mapping 0 to 0 and $\beta \cap \overline{D}_s$ onto $[-1, 1]$. Note that $R^{-1}_{n^+,n^-}[-1, 1]$ partitions $\overline{\mathbb{D}}$ into the same number of sectors as the partition of U by $f^{-1}\beta$. We just need now to construct h_1 sector on sector so that $R_{n^+,n^-} \circ h_1 = h_2 \circ f$ on that sector and h_1 is an orientation preserving mapping from $\gamma \cap \overline{D}_s$ onto $[-1, 1]$. We can see that h_1 is a homeomorphism from U to $\mathbb{D}$. q.e.d.

Notice that the local topological degree of f can be expressed in the following table:

	$\beta(t)$ convex, $m \le j$ odd		$\beta(t)$ cusp, $m \le j$ even	
	m odd	m even	m odd	m even
$f_{z=0} \sim \begin{pmatrix}z^{2n^+-1}\\ \overline{z}^{2n^--1}\end{pmatrix}$	$\begin{pmatrix}z^m\\ \overline{z}^m\end{pmatrix}$	$\begin{pmatrix}z^{m+1}\\ \overline{z}^{m-1}\end{pmatrix}$ or $\begin{pmatrix}z^{m-1}\\ \overline{z}^{m+1}\end{pmatrix}$	$\begin{pmatrix}z^{m+2}\\ \overline{z}^{m}\end{pmatrix}$ or $\begin{pmatrix}z^{m}\\ \overline{z}^{m+2}\end{pmatrix}$	$\begin{pmatrix}z^{m+1}\\ \overline{z}^{m+1}\end{pmatrix}$
$\deg(f, 0) =$	0	± 1	± 1	0
$\# f^{-1}(z) =$	$m+1, m-1$	$m+1, m-1$	$m+2, m$	$m+2, m$
$\mu(f, 0) =$	$j + m^2$			

Corollary 6.2. *In the generic case $f(z) = (z + bz^2 + O(z^3))^m - \overline{z}^m$ with $(-b^2)^m \neq 1$, we have*

	$\beta(t)$ *convex, $m = j$ odd*	$\beta(t)$ *cusp, $m = j$ even*
$f_{z=0} \sim \begin{pmatrix} z^{2n^+-1} \\ \overline{z}^{2n^--1} \end{pmatrix}$	$\begin{pmatrix} z^m \\ \overline{z}^m \end{pmatrix}$	$\begin{pmatrix} z^{m+1} \\ \overline{z}^{m+1} \end{pmatrix}$
$\deg(f, 0) =$	0	0
$\#f^{-1}(z) =$	$m+1, m-1$	$m+2, m$
$\mu(f, 0) =$	$m + m^2$	

6.2 Prescribing numerical invariants or local models for harmonic mappings

Now we are ready to prove Corollary 2.3. Due to the equality $j + m^2 = \mu$, we only need to prove that given two integers μ, m satisfying $m \geq 1$ and $\mu \geq m^2 + m$ there exist harmonic maps of the form $g(z) = p(z)^m - \bar{z}^m$ such that $\mu(g, 0) = \mu$.

Assume that $p(z) = z + bz^2 + o(z^2)$ with $|b| = 1$.

In the case $\mu = m^2 + m$, one can take p such that $(-b^2)^m \neq 1$ and apply Lemma 5.3.

Assume now $\mu > m^2 + m$, in particular $\mu > 2$. Choose $b = i$. Then $-b^2 = 1$ is always a m-th root of unity. And $f_{-b^2}(z) = f_1(z) = p(z) - \bar{z}$. Choose p such that $\mu(f_1, 0) - 2 = \mu - (m^2 + m)$ and apply Lemma 5.3.

Now given a pair of positive integers $n^{\pm}$ with $n^+ = n^-$, resp. with $|n^+ - n^-| = 1$, one can use the table (6.1) to find a suitable pair m and j, or the table (2.2) to find a suitable pair m and μ, and proceed as above to find an harmonic map realising the model. q.e.d.

Here are some concrete examples realizing a given pair (μ, m) with $\infty \geq \mu \geq m^2 + m$.

If $\infty > \mu = m^2 + m$, take any $p(z) = z + bz^2$ with $|b| = 1$ and $(-b^2)^m \neq 1$. Then $\mu(p(z)^m - \bar{z}^m, 0) = \mu$.

If $\infty > \mu > m^2 + m$, set $\nu = \mu - (m^2 + m) + 2 = \mu - (m-1)(m+2)$ and $p_\nu(z) = z \sum_{s=0}^{\nu-2} (iz)^s + az^\nu$ with $\Re a \neq 0, \pm 1$ and $g_\nu(z) = (p_\nu(z))^m - \overline{z}^m$. Then $\mu(g, 0) = \mu$.

If $\mu = \infty$, set $p(z) = -\dfrac{z}{1-z}$ and $g(z) = p(z)^m - \overline{z}^m$. We have $\overline{p} \circ p(z) = p \circ p(z) = z$, and

$$\mu(g, 0) = \sum_{\xi^m=1, \eta^m=1, \xi,\eta\neq 1} Ord_0\Big(\eta\, \overline{p}(\xi\, p(z)) - z\Big) + Ord_0\Big(\overline{p}(p(z)) - z\Big) = \infty.$$

One can also check by hand that $j(g, 0) = \infty$.

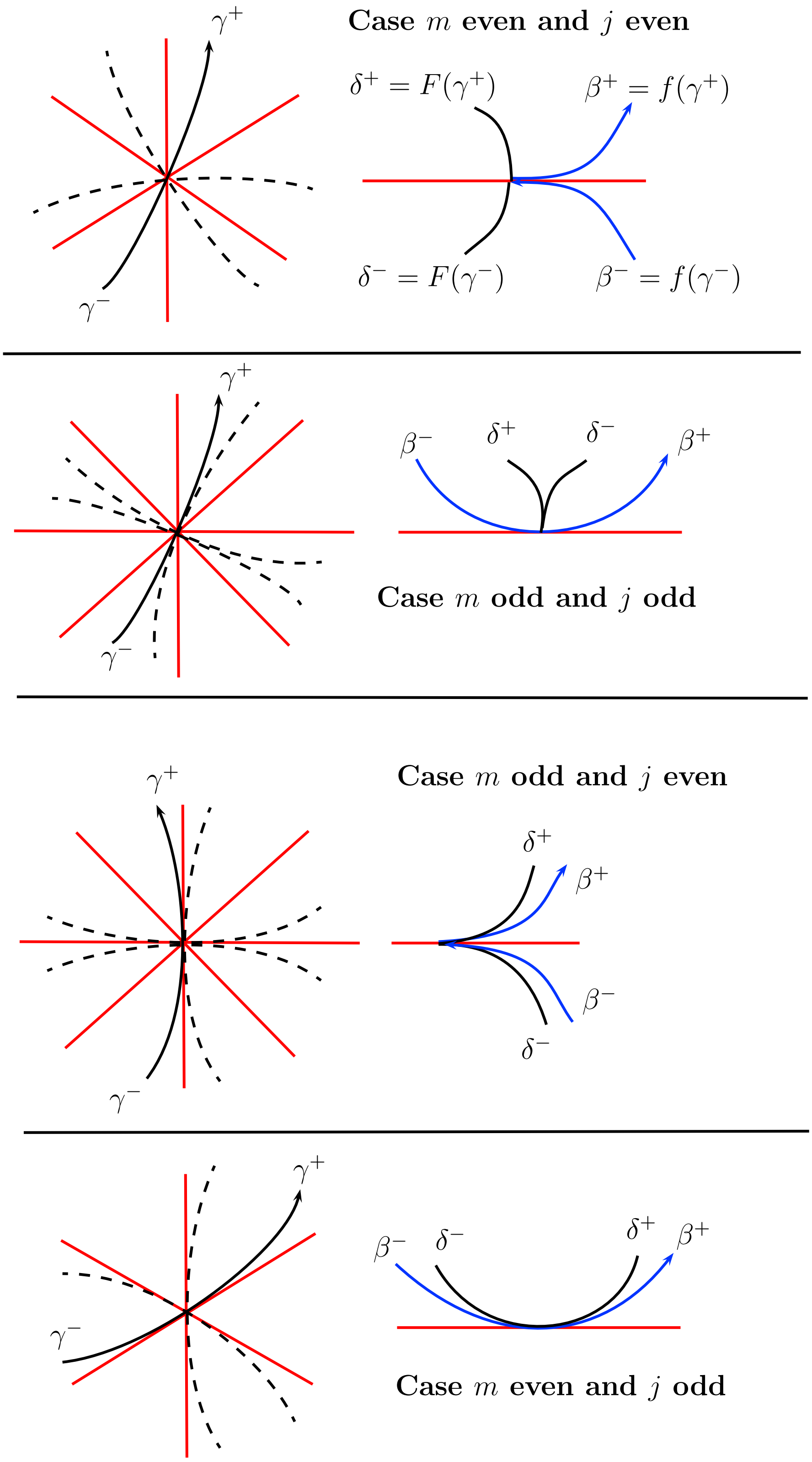

Figure 4: The left hand figures are $F^{-1}(\mathbb{R})$ (in red) and $F^{-1}(F(\gamma))$ (in black). The shape of $\beta^{\pm}$ is determined by the parity of j. The curves $\delta^{\pm}$ are in the same half planes as $\beta^{\pm}$ due to the fact that $F - f$ is real. The angle between $\delta^{\pm}$ is determined by the parity of m, as $F(z) = e^{i\theta} z^{m+1}$.

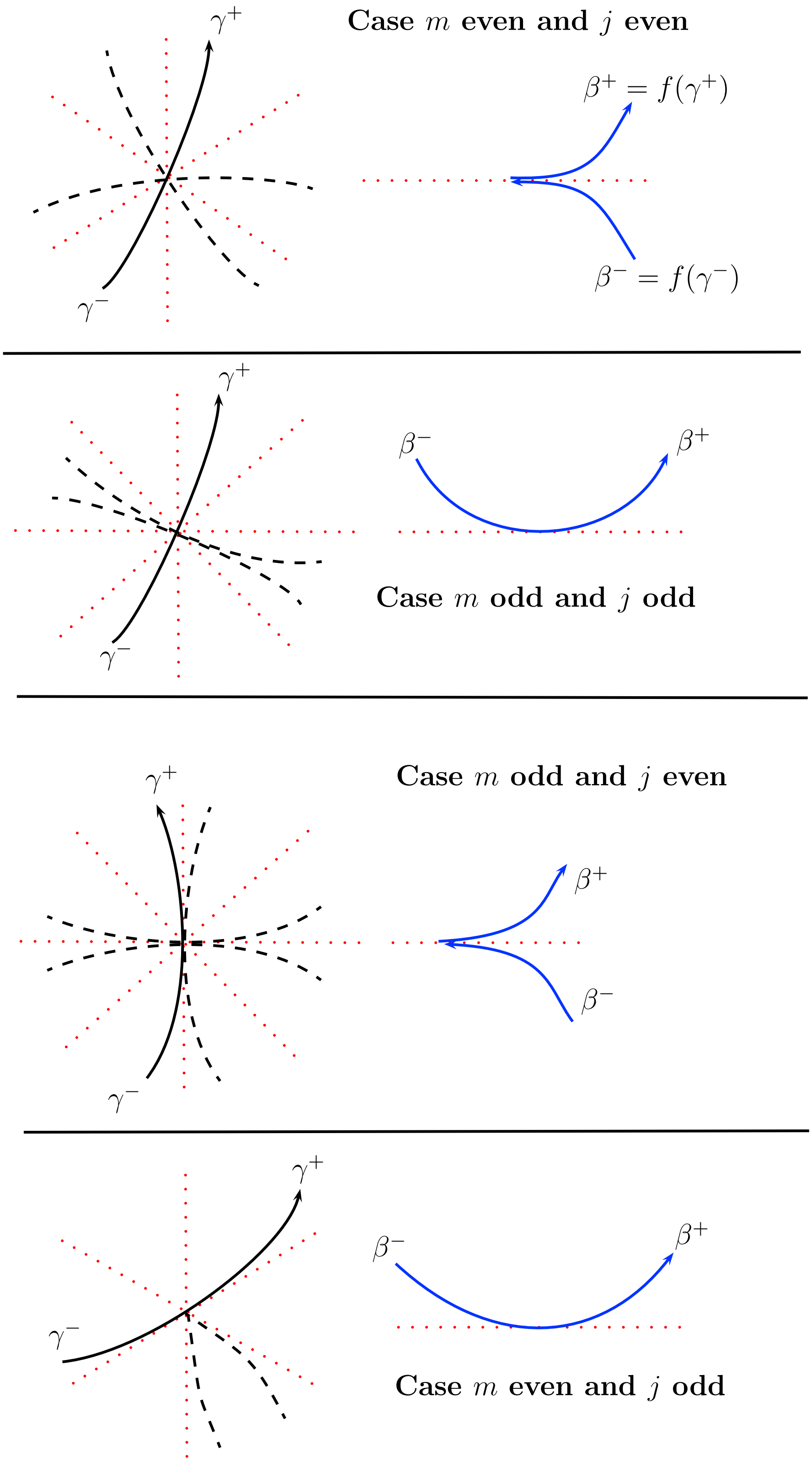

Figure 5: The cocritical set $f^{-1}(f(\gamma)) = f^{-1}(\beta)$. We have kept the red lines for reference. In each sector S bounded by red lines, the number of branches of $f^{-1}(f(\gamma))$ is equal to that of $F^{-1}(F(\gamma))$ (refer to Figure 4), except in the two sectors containing $\gamma^{\pm}$, where $f^{-1}(f(\gamma)) \subset \gamma$.

A Analytic planar maps at a regular critical point

Let $K = \mathbb{R}$ or $\mathbb{C}$. Let $f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto f\begin{pmatrix} x \\ y \end{pmatrix}$ be a K-analytic map with a as a regular critical point. The critical set $\mathcal{C}_f$, as a level set of J_f, is everywhere orthogonal to the gradient vector field $(\partial_x J_f, \partial_y J_f)$. The unique curve $\Gamma(t)$ satisfying

$$\Gamma'(t) = \begin{pmatrix} -\partial_y J_f(\Gamma(t)) \\ \partial_x J_f(\Gamma(t)) \end{pmatrix}, \quad \Gamma(0) = a$$

thus parametrizes the critical curve $\{J = 0\}$. Now the map f transports the curve $\Gamma(t)$ to the critical value set, inducing thus a natural local parametrization $t \mapsto \Sigma(t) = f(\Gamma(t))$.

In this section we prove that the critical value curve of a K-analytic map at a regular critical point takes always a pair $(j, j+1)$ as its order-pair, and $\mu = j + 1$. We then give a recursive algorithm computing j, thus μ.

A.1 Critical value order-pair and multiplicity

Theorem A.1. *Let $K = \mathbb{R}$ or $\mathbb{C}$. Let $W \subset K^2$ be an open neighborhood of $w_0 \in K^2$, and $F : W \to K^2$ a K-analytic mapping with w_0 as a regular critical point. Then,*

1. *(critical value order-pair) Let j be the order at $F(w_0)$ of the critical value curve in its natural parametrization. This curve has an order-pair of the form $(1, \infty)$ if $j = 1$, and $(j, j+1)$ if $1 < j < \infty$.*
2. *(critical value order and multiplicity) The order j is related to the multiplicity by the formula*
$$j + 1 = \mu(F, w_0). \tag{A.1}$$
3. *(topological model in the reals) In the case $K = \mathbb{R}$ and $\mu(F, w_0) < \infty$, $\begin{cases} \mu(F, w_0) \text{ even, or} \\ \mu(F, w_0) \text{ odd} \end{cases}$ iff there is a pair of topological local changes of coordinates h, H of $\mathbb{R}^2$, so that $H \circ F \circ h$ takes the standard $\begin{cases} \text{fold form } \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y^2 \end{pmatrix}, \text{ or} \\ \text{cusp form } \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ xy + y^3 \end{pmatrix}. \end{cases}$ In particular, outside the critical value set, the number of preimages is either 0 or 2 in the fold case, and 1 or 3 in the cusp case.*

Proof. We will make a sequence of analytic changes of coordinates to F. This will lead to new maps whose critical value curves differ from that of F by analytic changes of coordinates. We will see that in some suitable coordinates the critical value curve has an order pair in the form $(j, j+1)$. If $j > 1$ then $j, j+1$ are co-prime and the pair becomes then an analytic invariant. It follows that the critical value curve of our original map has also the same order-pair.

Precompose F by a translation if necessary we may assume $w_0 = \mathbf{0}$. Denote by $DF_{\mathbf{0}}$ the differential of F at $\mathbf{0}$.

Precompose and post-compose F by some rotations if necessary we may assume that $Ker(DF_{\mathbf{0}})$ is the y-axis and $Image(DF_{\mathbf{0}})$ is the x-axis. Divide F by a non-null constant in K if necessary we may further assume $DF_{\mathbf{0}}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$.

It follows that the Jacobian matrix $Jac_F(\mathbf{0})$ is $\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$.

Write now $F\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}f(x,y)\\g(x,y)\end{pmatrix}$ and set $\phi\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}f(x,y)\\y\end{pmatrix}$. Then $Jac_\phi(\mathbf{0}) = Id$. It follows that ϕ is a local diffeomorphism.

Replace now F by $F\circ\phi^{-1}$ we may assume F takes the form

$$F\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\g(x,y)\end{pmatrix}, \quad DF_{\mathbf{0}} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} .$$

By assumption $(\nabla J_F)_{\mathbf{0}} \neq (0,0)$. Set $(\nabla J_F)_{\mathbf{0}} = (b,b')$. Note that $J_F = \dfrac{\partial g}{\partial y}$. We get thus the following local expansion

$$\frac{\partial g}{\partial y} = 0 + bx + b'y + R_{\geq 2}(x,y) = x(b + a(x,y)) + y^j(c + R_{\geq 1}(y))$$

for some $\infty \geq j \geq 1$ ($\infty \geq j \geq 2$ if $b' = 0$) and $b\cdot c \neq 0$, where the function $a(x,y)$ has no constant term.

The case $b' \neq 0$. The curve β can be parametrized by x and has the order-pair $(1,\infty)$ at 0.

details ??

We will only treat the case $b\cdot b' \neq 0$. Then $j \geq 2$. We will prove that the critical value curve has $(j, j+1)$ as its order-pair at $\mathbf{0}$.

Set $B(x) = g(x,0)$. We have $B(0) = 0$. Post-compose now F by $\begin{pmatrix}u\\v\end{pmatrix} \mapsto \begin{pmatrix}u\\v - B(u)\end{pmatrix}$ we may assume

$$F\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\g(x,y)\end{pmatrix} \quad \text{with} \quad g(x,y) = \int_0^y \frac{\partial g}{\partial y}(x,y)dy. \tag{A.2}$$

Case $j = \infty$. We see that $\mathcal{C}_F$ is locally the y-axis and $g(x,y) = b\,xy\cdot A(x,y)$ with $A(0,0) = 1$. So

$$F\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\b\,xy\cdot A(x,y)\end{pmatrix}, \quad A(0,0) = 1 \tag{A.3}$$

and $F|_{\mathcal{C}_F}$ is locally constant. We may also make a change of variable $\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}x\\b\,y\cdot A(x,y)\end{pmatrix} = \Phi\begin{pmatrix}x\\y\end{pmatrix}$. Clearly Φ is locally invertible and $F\circ\Phi^{-1}\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}x_1\\x_1y_1\end{pmatrix}$.

Case $j < \infty$. The map F takes the form

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \dfrac{y^{j+1}}{j+1}(c + s(y)) + b\,xy \cdot A(x,y) \end{pmatrix}, \quad A(0,0) = 1.$$

Let y_1 be the analytic function in y tangent to the identity at 0 so that $y_1^{j+1} = y^{j+1}(1+s(y)/c)$ and change the variable y to y_1, one can further reduce F to the following form (by abuse of notation we use again y to denote the new variable):

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \dfrac{c\,y^{j+1}}{j+1} + b\,xy \cdot \hat{A}(x,y) \end{pmatrix}, \quad \hat{A}(0,0) = 1. \tag{A.4}$$

Note that $J_F(x,y) = \dfrac{\partial}{\partial y}\left(\dfrac{c\,y^{j+1}}{j+1} + b\,xy \cdot \hat{A}(x,y)\right) = c\,y^j + b \cdot x(1 + C(x,y))$ for some function $C(x,y)$ that vanishes at $(0,0)$.

Solving now the implicit equation $J_F(x,y) = 0$, we see that the critical set $\mathcal{C}_F$ is locally parametrized by $y \mapsto \gamma(y) = \begin{pmatrix} x(y) \\ y \end{pmatrix}$ with $x(y) = -\dfrac{c}{b} y^j + R_{\geq j+1}(y)$.

We may now compute the critical value curve in this coordinate :

$$\beta(y) = F(\gamma(y)) = F\begin{pmatrix} x(y) \\ y \end{pmatrix} = \begin{pmatrix} x(y) \\ \dfrac{c\,y^{j+1}}{j+1} - b\dfrac{c\,y^j}{b} y + R_{\geq j+2}(y) \end{pmatrix} = \begin{pmatrix} -\dfrac{c\,y^j}{b} + R_{\geq j+1}(y) \\ \dfrac{-c\,j}{j+1} y^{j+1} + R_{\geq j+2}(y) \end{pmatrix}.$$

It follows that β has the order-pair $(j, j+1)$ at 0. This proves Point 1.

Let us prove that the multiplicity $\mu(F, \mathbf{0})$ of F at $\mathbf{0}$ is $j+1$. This multiplicity is equal to

$$\limsup_{(x,y)\to\mathbf{0}} \#\left(U \cap F_{\mathbb{C}}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

with U a small neighborhood of $\mathbf{0}$ in $\mathbb{C}^2$.

By (A.4) if $x = 0$ and y is close to 0 but $y \neq 0$, then $\#F^{-1}\begin{pmatrix} 0 \\ y \end{pmatrix} = j+1$. This is also true for $\begin{pmatrix} x \\ y \end{pmatrix}$ close to $\mathbf{0}$ by Rouché's theorem applied to the second coordinate function of F as a function of y. We know that

$$\forall \varepsilon > 0,\ \exists \eta > 0\ s.t.\ \forall |s|, |t| < \eta,\ |g(s,y) - t|_{|y|=\varepsilon} \neq 0\ .$$

This proves Point 2.

Point 3. Assume $K = \mathbb{R}$. All functions below will have real coefficients. One can write $x(y) = -\dfrac{c}{b}(y + R_{\geq 2}(y))^j$.

If $j+1$ is even the function $x(y)$ is a local homeomorphism of an interval about 0 so has a unique inverse. It follows that on each vertical line $x=c$ for c small, the map F has a unique critical point. Since F sends the line into itself, it must be a fold.

The case $j+1$ odd: $x(y)$ is a convex curve staying on one half plane, say the left half plane. It follows that for every $c>0$ small, the map F sends the vertical line $x=c$ homeomorphically to itself. And for $c<0$ small F on the line $x=c$ behaves topologically as $ay+y^3$ with $a<0$. So F is a topological cusp.

The rigorous constructions of the changes of coordinates are very similar to Claims 1 and 8 in the proof of Theorem 6.1. As our map F here preserves vertical lines, we may instead choose to pull back small rectangles $]-r,r[\times]-s,s[$ so that the upper and lower boundary segments do not intersect the critical and co-critical sets. We omit the details. q.e.d.

Remark that in the proof we have also established a collapsing model in K: We have $\mu(F,w_0)=\infty$ iff there is a pair of K-analytic changes of coordinates φ and Φ so that $\Phi\circ F\circ\varphi$ takes the standard collapsing form $\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}x\\xy\end{pmatrix}$.

Remark also that Whitney has given a geometric model in K for the 'stable singularity' cases: $\begin{cases}\mu(F,w_0)=2\\ \mu(F,w_0)=3\end{cases}$ iff there is a pair of K-analytic changes of coordinates h,H so that $H\circ F\circ h$ takes the standard $\begin{cases}\text{fold form }\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}x\\y^2\end{pmatrix}\\ \text{cusp form }\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}x\\xy+y^3\end{pmatrix}.\end{cases}$

A.2 A recursive algorithm computing j

This subsection is inspired by a conversation with H.H. Rugh.

For a C^∞ planar mapping f, let J be the jacobien of f. In the following both our domaine and range planes will be $\mathbb{R}^2$ identified with $\mathbb{C}$. In this spirit the jacobien will also be considered as a map with range in $\mathbb{R}$.

Consider now a map f from U to $\mathbb{C}$, we define

$$\nabla_{\mathbb{R}}f=(f_x,f_y)\text{ and }\nabla f=(f_z,f_{\bar z}):=\Big(\frac{1}{2}(f_x-if_y),\frac{1}{2}(f_x+if_y)\Big).$$

Mimicking Whitney's definition for folds and cusps, we set recursively

$$M_1=\begin{vmatrix}\nabla_{\mathbb{R}}J\\ \nabla_{\mathbb{R}}f\end{vmatrix},\ M_2=\begin{vmatrix}\nabla_{\mathbb{R}}J\\ \nabla_{\mathbb{R}}M_1\end{vmatrix},\cdots,\ M_k=\begin{vmatrix}\nabla_{\mathbb{R}}J\\ \nabla_{\mathbb{R}}M_{k-1}\end{vmatrix},\cdots; \tag{A.5}$$

$$L_1=\begin{vmatrix}\nabla f\\ \nabla J\end{vmatrix},\ L_2=\begin{vmatrix}\nabla L_1\\ \nabla J\end{vmatrix},\cdots,\ L_k=\begin{vmatrix}\nabla L_{k-1}\\ \nabla J\end{vmatrix},\cdots. \tag{A.6}$$

Proposition A.2. *Let $f : (\mathbb{R}^2, a) \to (\mathbb{C}, f(a))$ be a smooth map. We have,*

$$\forall\, n \geq 1, M_n = (2i)^n L_n.$$

Let $\Gamma(t)$ the trajectory of the vector field $(-J_y(z), J_x(z))$ with initial point a, and set $\Sigma(t) = f(\Gamma(t))$. We have

$$\forall\, n \geq 1,\ \Sigma^{(n)}(t) = M_n(\Gamma(t)) = (2i)^n L_n(t).$$

In particular $\Sigma^{(n)}(0) = M_n(a) = (2i)^n L_n(a)$.

Proof. Let $G, H : (\mathbb{R}^2, a) \to (\mathbb{C}, G(a))$ be two C^∞ smooth mappings.

I. We claim first

$$\begin{vmatrix} \nabla_{\mathbb{R}} H \\ \nabla_{\mathbb{R}} G \end{vmatrix} = 2i \begin{vmatrix} \nabla G \\ \nabla H \end{vmatrix}. \tag{A.7}$$

Proof. Recall that $\nabla_{\mathbb{R}} H = (H_x, H_y)$ and $\nabla_{\mathbb{R}} G = (G_x, G_y)$. It follows from $G_z = \dfrac{1}{2}(G_x - iG_y)$ and $G_{\overline{z}} = \dfrac{1}{2}(G_x + iG_y)$ that $G_x = G_z + G_{\overline{z}}$ and $G_y = i(G_z - G_{\overline{z}})$. So

$$\begin{vmatrix} \nabla_{\mathbb{R}} H \\ \nabla_{\mathbb{R}} G \end{vmatrix} = \begin{vmatrix} H_x & H_y \\ G_x & G_y \end{vmatrix} = i \begin{vmatrix} H_z + H_{\overline{z}} & H_z - H_{\overline{z}} \\ G_z + G_{\overline{z}} & G_z - G_{\overline{z}} \end{vmatrix} = -2i \begin{vmatrix} H_z & H_{\overline{z}} \\ G_z & G_{\overline{z}} \end{vmatrix} = -2i \begin{vmatrix} \nabla H \\ \nabla G \end{vmatrix}.$$

II. Apply now (A.7) to $G = f$ and $H = J$, we get $M_1 = (2i)L_1$, then to $G = M_1$ we get

$$M_2 = \begin{vmatrix} \nabla_{\mathbb{R}} J \\ \nabla_{\mathbb{R}} M_1 \end{vmatrix} = (2i) \begin{vmatrix} \nabla_{\mathbb{R}} J \\ \nabla_{\mathbb{R}} L_1 \end{vmatrix} \overset{(A.7)}{=} (2i)^2 \begin{vmatrix} \nabla L_1 \\ \nabla J \end{vmatrix} = (2i)^2 L_2\ .$$

By induction

$$M_n = (2i)^n L_n, \quad \forall\, n \geq 1\ . \tag{A.8}$$

III. We claim now

$$\frac{d}{dt} G(\Gamma(t)) = \begin{vmatrix} \nabla_{\mathbb{R}} J \\ \nabla_{\mathbb{R}} G \end{vmatrix}_{\Gamma(t)} . \tag{A.9}$$

Proof. Using the fact that $\Gamma'(t) = (-J_y, J_x)|_{\Gamma(t)}$, for any $v \in \mathbb{C}^2$ we have

$$\langle v, \Gamma'(t) \rangle = \begin{vmatrix} J_x\ J_y \\ v \end{vmatrix}_{\Gamma(t)} = \begin{vmatrix} \nabla_{\mathbb{R}} J \\ v \end{vmatrix}_{\Gamma(t)} .$$

Write $G : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} G_1(x,y) \\ G_2(x,y) \end{pmatrix}$ (by identifying the range plane to $\mathbb{R}^2$). Then

$$\frac{d}{dt} G(\Gamma(t)) = DG|_{\Gamma(t)}(\Gamma'(t)) = \begin{pmatrix} \langle \nabla_{\mathbb{R}} G_1(\Gamma(t)), \Gamma'(t) \rangle \\ \langle \nabla_{\mathbb{R}} G_2(\Gamma(t)), \Gamma'(t) \rangle \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} \nabla_{\mathbb{R}} J \\ \nabla_{\mathbb{R}} G_1 \end{vmatrix} \\ \begin{vmatrix} \nabla_{\mathbb{R}} J \\ \nabla_{\mathbb{R}} G_2 \end{vmatrix} \end{pmatrix}_{\Gamma(t)} .$$

Identify $G(x,y)$ with $G_1(x,y)+iG_2(x,y)$. We have

$$\frac{d}{dt}G(\Gamma(t)) = \begin{vmatrix} \nabla_{\mathbb{R}}J \\ \nabla_{\mathbb{R}}G_1 \end{vmatrix}_{\Gamma(t)} + i \begin{vmatrix} \nabla_{\mathbb{R}}J \\ \nabla_{\mathbb{R}}G_2 \end{vmatrix}_{\Gamma(t)} = \begin{vmatrix} \nabla_{\mathbb{R}}J \\ \nabla_{\mathbb{R}}G \end{vmatrix}_{\Gamma(t)} .$$

IV. Apply now (A.9) inductively to $G = f, M_1, M_2, \cdots$, we get

$$\Sigma'(t) = \frac{d}{dt}f(\Gamma(t)) = \begin{vmatrix} \nabla_{\mathbb{R}}J \\ \nabla_{\mathbb{R}}f \end{vmatrix}_{\Gamma(t)} = M_1(\Gamma(t));$$

$$\Sigma''(t) = \frac{d}{dt}M_1(\Gamma(t)) = \begin{vmatrix} \nabla_{\mathbb{R}}J \\ \nabla_{\mathbb{R}}M_1 \end{vmatrix}_{\Gamma(t)} = M_2(\Gamma(t)) .$$

By induction $\Sigma^{(n)}(t) = M_n(\Gamma(t))$, and $\Sigma^{(n)}(0) = M_n(\Gamma(0)) = M_n(a)$.

Combining with (A.8) we get $\Sigma^{(n)}(0) = (2i)^n L_n(a)$ as well. q.e.d.

Corollary A.3. *The invariant j is the first integer n for which $L_n(0) \neq 0$.*

B Examples

These examples illustrate some differences between the harmonic and the general real analytic case.

1. General remarks:

a. In the harmonic case, by a theorem of Hans Lewy, the locus of non local injectivity is the same as the critical set. In particular, this implies that C_f and hence V_f have a topological meaning. This is no longer true in the real analytic case, as is shown for instance by the map: $(x,y) \to (x,y^3)$.

b. One can easily check that for a real analytic planar germ g from $(\mathbb{C},0)$ into itself, the critical set (and the locus of non injectivity) are the same for the germs g and g^n $(n \in \mathbb{N}^*)$ outside the origin.

c. As the case of a regular critical point was studied before, we give examples for which the gradient of the Jacobian vanishes at the origin.

Here are examples of planar analytic germs g at the origin (C_g is smooth and coincides with thelocus of non local injectivity).

A. In general, the condition $\mu = j + m^2$ is not satisfied in the analytic case. The map $g(x,y) = (x, x^2y^2 + y^4)$ is a simple example. The critical set is the x-axis. One has: $j = m = 1$ and $\mu = 4$.

B. Topological differences.

1. $g(x,y) = (x+iy^2)^2$ has local topological degree 0 at the origin, but V_f is $[0,+\epsilon[$. So the germ at the origin is not topologically equivalent to the germ of a harmonic map.

2. A real analytic germ with a smooth critical set can have any local topological degree. For instance, take a harmonic map f of degree 1 or -1 and put $g = f^n$. This map has local topological degree n or $-n$, and then, is not topologically equivalent to a harmonic germ for $n > 1$.

C. Problem for the parametrization of V_g.

$g(x, y) = (x + iy^2)^3$. The parametrization of V_g which comes from the parametrization $x = t, y = 0$ of C_f is of the non-injective form $x = t^3, y = 0$, when t is complex.

D. Examples with Puiseux pair $(2, 5)$

$g(x, y) = (x^2 + y^2, x^5 + cx^3y^2 + xy^4)$.

The Jacobian is equal to $2y((2c - 5)x^4 + (4 - 3c)x^2y^2 - y^4)$. Then For $4/3 < c < 5/2$, one gets an example satisfying the wanted conditions ($C_g = \{y = 0\}$). Moreover, it is topologically a fold.

One can also check that $m = j = 2$, $\mu = \infty$ for $c = 2$ and $\mu = 10$ otherwise.

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