Small eigenvalues of the low temperature linear relaxation Boltzmann equation with a confining potential

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Abstract

We study the linear relaxation Boltzmann equation, a simple semiclassical kinetic model. We provide a resolvent estimate for an associated non-selfadjoint operator as well as an estimate on the return to equilibrium. This is done using a scaling argument and non-semiclassical hypocoercive estimate.

Keywords and phrases: Boltzmann equation, hypocoercivity, quasimode, non-selfadjoint operator.
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1 Introduction

The spectral study of inhomogeneous kinetic equations and the trend to the equilibrium of the related system of particles are natural subjects of interest and some progress has been made in the past decade in the spirit of the so-called 'hypocoercivity'. In this article we are interested in the study at low temperature of a simple (from the kinetic point of view) but difficult (from the spectral analysis point of view) linear model where collisions between particles are not of diffusion type, but of (non-local) relaxation type. This system as already been studied by Hérau in [7] with improvements by Dolbeault *et al.* in [1, 2], but at fixed temperature.

The final purpose is to study the existence of metastable states and a possible tunnel effect for the system, which implies very long relaxation time to the equilibrium. Here we provide first spectral results for the low-lying eigenvalues and the return to the equilibrium at low temperature for the following simple linear relaxation Boltzmann model:

$$\begin{cases} \partial_t f + v.\partial_x f - \frac{1}{m}\partial_x V.\partial_v f = Q(f) \\ f_{|t=0} = f_0 \end{cases},$$

where the unknown $f(t,x,v)$ is the density of probability of the system of particles at time $t \in \mathbb{R}_+$, position $x \in \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$. We will assume

that, for all $t \geq 0$, $f(t, \cdot, \cdot)$ belongs to L^2. Here the collision kernel Q models interactions between particles in the gas and is given by

$$Q(f) = \gamma \left[\left(\int_{\mathbb{R}^d_v} f(t,x,v) \, dv \right) m_\beta - f \right],$$

where

$$m_\beta(v) = \frac{e^{-\frac{m\beta v^2}{2}}}{\left(\frac{2\pi}{m\beta} \right)^{d/2}}$$

is the $L^1(\mathbb{R}^d_v)$-normalized Maxwellian in the velocity direction with $\beta = 1/kT$ where k is the Boltzmann constant, T the temperature of the system, and γ is the friction coefficient. The potential $V \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ only depends on the position x. We are interested in the low temperature regime of the system and develop for this a semiclassical framework. We put $h = kT = 1/\beta$, we set $m = 1$, $\gamma = 1$, and pose

$$\mu_h = m_{1/h} = \frac{1}{(2\pi h)^{d/2}} e^{-\frac{v^2}{2h}}.$$

We also introduce the spatial Maxwellian and the global Maxwellian given by

$$\rho_h(x) = \frac{e^{-\frac{V(x)}{h}}}{\displaystyle\int_{\mathbb{R}^d_x} e^{-\frac{V(x)}{h}}}. \qquad \mathcal{M}_h(x,v) = \rho_h(x) \mu_h(v).$$

It is immediate to check that $\mathcal{M}_h$ is the only distributional steady state of the system up to renormalization (see [5]). This equation describes a system of large number of particles submitted to an external force coming from the potential V and interacting according to the collision kernel Q, whose effect is a simple relaxation toward the local Maxwellian $\left(\int_{\mathbb{R}^d_{v'}} f(t,x,v') \, dv' \right) \mu_h(v)$.

We multiply our equation by h which becomes

$$\begin{cases} h\partial_t f + v.h\partial_x f - \partial_x V.h\partial_v f = hQ(f), \\ f_{|t=0} = f_0. \end{cases}$$

The semiclassical limit $h \to 0^+$ corresponds to the low temperature regime of the system.

Hypothesis 1.1. *The potential V is a Morse function with n_0 local minima and with derivatives of order 2 or more bounded. Moreover $e^{-\frac{V}{h}} \in L^1$ for all $h \in [0, h_0[$ and there exists $C > 0$ such that $|\nabla V(x)| \geq \dfrac{1}{C}$ for $|x| > C$.*

A standard Hilbert space for the study of the time independent equation is the weighted space

$$\mathcal{H} = \left\{ f \in \mathcal{D}' \mid \frac{f}{\mathcal{M}_h^{1/2}} \in L^2 \right\}$$

and we see from the Cauchy-Schwarz inequality an Hypothesis 1.1 that $\mathcal{H}$ is a subset of $L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v, dxdv)$. This is then more convenient to work with the rescaled function

$$u(t,x,v) = \mathcal{M}_h^{-1/2} f(t,x,v) \in \mathcal{C}(\mathbb{R}_+, L^2)$$

(the continuity will be a consequence of an associated semigroup property) and the new equation reads

$$\begin{cases} h\partial_t u + v.h\partial_x u - \partial_x V.h\partial_v u + h(\mathrm{Id} - \Pi_h)u = 0 \\ u_{|t=0} = u_0, \end{cases}$$

where Π_h is the orthogonal projection in $\mathcal{H}$ (with t as a parameter) on the space $E_h = \left\{ \rho\mu_h^{1/2}, \ \rho \in L^2(\mathbb{R}_x^d) \right\}$ of local equilibria (note that E_h is closed). In deed, we have

$$Q(\mathcal{M}_h^{1/2}) = \mathcal{M}^{1/2}(u - \langle u, \mu_h \rangle_{L^2(\mathbb{R}_v^d)} \mu_h)$$

and

$$\rho_h(x)^{-1/2} v.\partial_x \rho_h^{1/2}(x) - \mu_h(v)^{-1/2} \partial_x V.\partial_v \mu_h^{1/2}(v) = 0$$

The time independent operator is now

$$\begin{aligned} P_h &= v.h\partial_x - \partial_x V.h\partial_v + h(\mathrm{Id} - \Pi_h) \\ &= X_0^h + h(\mathrm{Id} - \Pi_h), \end{aligned}$$

and the aim of this paper is to prove the following theorem:

Theorem 1.2. *Suppose that V satisfies hypothesis 2.2. Then P_h has 0 as simple eigenvalue and there exists $h_0 > 0$, and $\delta > 0$ such that*

i) *for all $h \in]0, h_0]$, $\mathrm{Spec}\, P_h \cap B(0, \delta h)$ consists of exactly n_0 (counted for multiplicity) real eigenvalues which are exponentially small with respect to $\frac{1}{h}$,*

ii) *for all $\delta_1 \in]0, \delta]$, there exists $C > 0$ such that, for all $h \in]0, h_0]$, if $\delta_1 h \leq |z| \leq \delta h$ then*

$$\|(P_h - z)^{-1}\| \leq \frac{C}{h}.$$

This work is a first step towards getting semiclassical expansions of the smallest eigenvalues of our operator in the spirit of Helffer *et al.* in [4] for the Witten Laplacian and by Hérau *et al.* in [12, 8, 9, 10] for the Kramers-Fokker-Planck operator. In those four last papers, the authors obtained asymptotic expansions for low-lying eigenvalues of the operator using pseudodifferential technics. Here, one of the major problem is that the projection Π_h in P_h does not obey good symbol estimates uniformly in h. To overcome this problem, we use a scaling S_h defined below in (2.1) and h-free hypocoercive estimates in the spirit of [7, 1, 2] for the linear relaxation Boltzmann equation or [11] for the Fokker-Planck equation. Since we work in a non-selfadjoint case, some particular tools have to be used. We first need to get resolvent estimates allowing control of the norm of the spectral projection onto the generalized eigenspace associated with all the eigenvalues with modulus smaller than δh. We also need the notion of a PT-like symmetry (cf. [10]) which is a powerful tool that we use like in [10] in order to show that there are no Jordan blocks in the action of the operator on the generalized eigenspaces associated with the low-lying eigenvalues.

The plan of the paper is the following: in the second section we prove a hypocoercive estimate from which we deduce a resolvent estimate. In the third section we finish proving the main result using the PT symmetry property. The last part is dedicated, in the spirit of Boltzmann H-theorem, to getting a

convergence result of the solution of the linear relaxation Boltzmann equation on the generalized eigenspace associated with the low-lying eigenvalues.

Notation: For any multi-index $\alpha \in \mathbb{N}^d$, we write $\partial^\alpha = \partial_{x_1}^{\alpha_1}...\partial_{x_d}^{\alpha_d}$ (and similarly for v). We also denote $u = \widetilde{O}(\mathrm{e}^{-\frac{\alpha}{h}})$ for $u \in L^2$ when there exist $C, N > 0$ such that $\|u\| \leq h^{-N}\mathrm{e}^{-\frac{\alpha}{h}}$

2   Hilbertian hypocoercivity

Hilbertian hypocoercivity (cf. [1, 7, 14]) refers to way to get coercive estimates by using a slight modification of the scalar product or the operator. We shall first discuss the maximal accretivity of our operator, so that we can apply our spectral results to describe the properties of the semigroup. As P_h is the sum of a non-negative selfadjoint operator (an orthogonal projection) and a skewadjoint operator, so P_h is accretive. Let equip P_h with the domain $D = \left\{u \in L^2 \mid X_0^h u \in L^2\right\}$, we then get a maximal accretive operator. Indeed X_0^h is maximal accretive on D and $\mathrm{Id} - \Pi_h$ is a bounded operator on L^2. So $P_h = X_0^h + h(\mathrm{Id} - \Pi_h)$ is maximal accretive.

As announced in the introduction, the proof of our estimate relies on a scaling argument. If we conjugate P_h with the dilatation operator,

$$S_h : \left\{ \begin{array}{ccc} L^2(\mathbb{R}_x^d \times \mathbb{R}_v^d) & \to & L^2(\mathbb{R}_x^d \times \mathbb{R}_v^d) \\ u & \mapsto & h^{-d/2}u(\frac{\cdot}{\sqrt{h}}, \frac{\cdot}{\sqrt{h}}) \end{array} \right. , \tag{2.1}$$

then $S_h^{-1}P_h S_h = hP$ where P is the operator

$$\begin{array}{rcl} P & = & v.\partial_x - \partial_x V_h(x).\partial_v + (\mathrm{Id} - \Pi_1) \\ & = & X_0 + (\mathrm{Id} - \Pi_1), \end{array}$$

where $V_h(x) = \frac{1}{h}V(\sqrt{h}x)$. Notice that P depend on h because the potentials V_h depend on h. Nevertheless we chose a h-less notation, since we will have estimates for P that are uniform w.r.t. h. More precisely, these estimates will only depend on the L^∞ norm of second order or higher order derivatives of V_h. Using that derivatives of order two or more of V are bounded, we have (if $h \leq 1$) for $k \geq 2$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| = k$,

$$\|\partial_x^\alpha V_h\|_\infty = h^{(k-2)/2}\|\partial_x^\alpha V\|_\infty \leq \|\partial_x^\alpha V\|_\infty.$$

Thus we get uniformity w.r.t h on the L^∞ norm of second order or higher order derivatives of V_h.

We now will follow [7] to get a hypocoercive estimate on P. We introduce the two following differential operators:

$$a_j = (\partial_{x_j} + \partial_{x_j}V_h/2), \qquad b_j = (\partial_{v_j} + v_j/2) ,$$

and their formal adjoints

$$a_j^* = (-\partial_{x_j} + \partial_{x_j}V_h/2), \qquad b_j^* = (-\partial_{v_j} + v_j/2) .$$

We put

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} ,$$

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and

$$\Lambda^2 = a^*a + b^*b + 1.$$

Notice that $a^*a = -\Delta_x + |\partial_x V_h(x)|^2/4 - \Delta V_h(x)/2$ is nothing but the Witten Laplacian (in position) and $b^*b = -\Delta_v + v^2/4 - d/2$ the harmonic oscillator (in velocity). Under our hypotheses (see [5]), $\mathscr{S}(\mathbb{R}^{2d})$ is a core for Λ^2 and Λ^r which is well-defined for all $r \in \mathbb{R}$ and the operators a, b, and Λ^2 are continuous on $\mathcal{S}$ and $\mathcal{S}'$. We have the following relation between a, b, and P:

$$P = b^*a - a^*b + (\mathrm{Id} - \Pi_1)$$

We also introduce the semiclassical scaling operator acting on functions only depending on x (to be related to (2.1)), that is

$$T_h : \begin{cases} L^2(\mathbb{R}_x^d) & \to & L^2(\mathbb{R}_x^d) \\ u & \mapsto & h^{-d/4}u(\frac{\cdot}{\sqrt{h}}) \end{cases} .$$

Notice that $W_h = hT_h a^*a T_h^{-1}$ is then the semiclassical Witten Laplacian:

$$W_h = -h^2\Delta_x + |\partial_x V(x)|^2/4 - h\Delta V(x)/2.$$

Under our hypotheses 1.1 on V and from [4], this Witten Laplacian W_h has n_0 exponentially small real eigenvalues and there exists $0 < \tau \le 1$ fixed from now on such that the remaining part of the spectrum is included in $[\tau h, +\infty[$ for $h \in]0,1]$. We recall from [4] that there exist well-chosen cut-off functions $\chi_j \in \mathcal{C}_0^\infty$, $1 \le j \le n_0$ each localizing to a neighborhood of jth minimum of V, for which

$$e_j(x) = \chi_j(x)e^{-\frac{V(x)}{2h}} \tag{2.2}$$

serves as a quasimode for W_h in the sense that, for some $c > 0$,

$$W_h e_j = O(e^{-\frac{c}{h}}\|e_j\|).$$

We defined the associated non-semiclassical quasimodes

$$f_j(x) = T_h^{-1}e_j(x), \qquad g_j(x,v) = f_j(x)\mu_1^{1/2}(v),$$

(that still depend on h) and introduce the associated vector spaces

$$F = \mathrm{Vect}\,\{f_j\,,\ j \in [\![1...n_0]\!]\} \subset L^2(\mathbb{R}_x^d),$$
$$G = \mathrm{Vect}\,\{g_j\,,\ j \in [\![1...n_0]\!]\} \subset L^2(\mathbb{R}_x^d \times \mathbb{R}_v^d).$$

We also put for all j

$$g_j^h = S_h g_j \tag{2.3}$$

We have the following lemma:

Lemma 2.1. *The family $(g_j^h)_j$ is almost orthogonal and consists of exponentially small quasimodes for P_h (respectively P_h^*), meaning that there exists $\alpha > 0$ such that for all $j, k \in [\![1...n_0]\!]$, $j \ne k$,*

$$(g_j^h, g_k^h) = O(e^{-\frac{\alpha}{h}}\|g_j^h\|\|g_k^h\|)$$

. and for all $j \in [\![1...n_0]\!]$,

$$P_h g_j^h = \widetilde{O}(e^{-\frac{\alpha}{h}}\|g_j^h\|)$$

(respectively $P_h^* g_j^h = \widetilde{O}(e^{-\frac{\alpha}{h}}\|g_j^h\|)$).

Proof. According to the expression of the quasimodes in (2.2), we have

$$g_j^h(x,v) = e_j(x)\mu_h^{1/2}(v) = \chi_j(x)\mathrm{e}^{-\frac{V(x)}{2h}}\mathrm{e}^{-\frac{v^2}{4h}}.$$

We immediately deduce the relation of almost orthogonality of the family $(g_j^h)_j$ from the one of $(e_j)_j$ (see Proposition (6.1) in [4]). From the expression of g_j^h, we deduce that $P_h g_j^h = X_0^h g_j^h = v.\nabla\chi_j \mathrm{e}^{-\frac{V(x)}{2h}}\mathrm{e}^{-\frac{v^2}{4h}}$. From the estimates on χ_j (see proof of Proposition (6.1) in [4]), we get that $\|X_0^h g_j^h\| = O(\mathrm{e}^{-\frac{\alpha}{h}})$ which shows the assertion for P_h Since $P_h^* = -X_0^h + h(1-\Pi_h)$, we get the same result for P_h^*. $\qquad\square$

The following proposition is the core of the hilbertian hypocoercivity and expresses a coercivity property of the operator P using a small bounded perturbation involving the fundamental auxiliary operator

$$L = \Lambda^{-2}a^*b.$$

Proposition 2.2. *There exist ε, A, $h_0 > 0$ such that for all $h \leq h_0$ and $u \in \mathcal{S}\cap G^\perp$*

$$\mathrm{Re}\,(Pu,(\mathrm{Id}+\varepsilon(L+L^*))u) \geq \frac{1}{A}\|u\|^2,$$

where A can be chosen to depend explicitly on the second and third derivatives of V and $\|\varepsilon L\| \leq 1$.

Remark 2.3. In [7], the case of one dimensional G is treated. Here V_h satisfy similar hypotheses, but the spectral gap is dramatically (exponentially) small due to the fact that the minima of V_h are at a distance of order $\frac{1}{\sqrt{h}}$ from one another. Using $G^\perp$ allows us to get a bound in proposition 2.2 which is uniform with respect to h sufficiently small.

Proof: We follow partially [7] and take care of the uniform dependence with respect to h. Let $u \in L^2$ and $\varepsilon > 0$. We have

$$\begin{aligned}
&\mathrm{Re}\,(Pu,(\mathrm{Id}+\varepsilon(L+L^*))u)\\
&= \mathrm{Re}\,((\mathrm{Id}-\Pi_1)u,(\mathrm{Id}+\varepsilon(L+L^*))u) + \mathrm{Re}(X_0u,(\mathrm{Id}+\varepsilon(L+L^*))u)\\
&= \|(\mathrm{Id}-\Pi_1)u\|^2 + \varepsilon\mathrm{Re}\,((\mathrm{Id}-\Pi_1)u,(L+L^*)u) + \varepsilon\mathrm{Re}\,(X_0u,(L+L^*)u)\\
&= I + II + III,
\end{aligned}$$

where we used that X_0 is skewadjoint for the last term. We first get a rough lower bound for the two first terms with the Cauchy-Schwarz inequality:

$$I + II \geq \frac{1}{2}\|(\mathrm{Id}-\Pi_1)u\|^2 - \frac{1}{2}\varepsilon^2\|(L+L^*)u\|^2 \geq \frac{1}{2}\|(\mathrm{Id}-\Pi_1)u\|^2 - \varepsilon^2\|L\|^2\|u\|^2.$$

Let now study the last term more carefully:

$$III = \varepsilon\mathrm{Re}\,(X_0u,(L+L^*)u) = \varepsilon\mathrm{Re}\,([L,X_0]u,u),$$

since X_0 is skewadjoint. Using the definition of $L = \Lambda^{-2}a^*b$ and the commutation relations between a, b, Λ^2, and X_0 (cf. [7]), we get

$$\begin{aligned}
[L,X_0] &= [\Lambda^{-2}a^*b,X_0]\\
&= [\Lambda^{-2},X_0]a^*b + \Lambda^{-2}[a^*,X_0]b + \Lambda^{-2}a^*[b,X_0]\\
&= -\Lambda^{-2}[\Lambda^2,X_0]\Lambda^{-2}a^*b - \Lambda^{-2}b^*\mathrm{Hess}\,V_h b + \Lambda^{-2}a^*a.
\end{aligned}$$

We used the algebraic relation $[A, B^{-1}] = -B^{-1}[A, B]B^{-1}$. We put

$$\mathcal{A} = -\Lambda^{-2}[\Lambda^2, X_0]\Lambda^{-2}a^*b - \Lambda^{-2}b^*\operatorname{Hess}Vb.$$

We are now going to prove to two intermediate lemmas.

Lemma 2.4. *The operators $\mathcal{A}$ and L are bounded on L^2 (uniformly in h). Moreover their norm can be explicitly bounded in terms of the second and third order derivatives of V.*

Remark 2.5. In fact these bounds are with respect to the second and third order derivatives of V_h, but this bounds are uniform in h, because the second and third order derivatives of V_h are uniformly bounded with respect to h.

Proof: We recall some ideas of the proof from [7]. Using that $L = \Lambda^{-2}a^*b$, we compute the commutator

$$[\Lambda^2, X_0] = -b^*(\operatorname{Hess}V_h - \operatorname{Id})a - a^*(\operatorname{Hess}V_h - \operatorname{Id})b,$$

which gives

$$\mathcal{A} = \quad \Lambda^{-2}b^*(\operatorname{Hess}V_h - \operatorname{Id})a\Lambda^{-2}a^*b + \Lambda^{-2}a^*(\operatorname{Hess}V_h - \operatorname{Id})b\Lambda^{-2}b^*a$$
$$-\Lambda^{-2}b^*\operatorname{Hess}V_hb.$$

We see that it is sufficient to prove that for any real $d \times d$ matrix $M(x)$ which depends only on x and which is bounded with its first derivative bounded, the operators

$$\Lambda^{-2}b^*M(x)a, \quad \Lambda^{-2}b^*M(x)b, \quad \Lambda^{-2}a^*M(x)b$$

are bounded on L^2. We only prove boundedness for the first operator, since the proofs for the other two operators are similar and easier. It is sufficient to show boundedness for the adjoint $a^*M(x)b\Lambda^{-2}$. For $u \in \mathcal{S}$, we write

$$
\begin{aligned}
\|a^*M(x)b\Lambda^{-2}u\| &\leq \sum_{j,k}\|a_j^*M_{j,k}(x)b_k\Lambda^{-2}u\| \\
&\leq \sum_{j,k}\|M_{j,k}(x)a_j^*b_k\Lambda^{-2}u\| + \sum_{j,k}\|(\partial_j M_{j,k}(x))b_k\Lambda^{-2}u\| \\
&\leq (\|M\|_{L^\infty} + \|\nabla M\|_{L^\infty}) \\
&\quad \times \sum_{j,k}(\|a_j^*b_k\Lambda^{-2}u\| + \|b_k\Lambda^{-2}u\|),
\end{aligned}
$$

where we used that $[a_j^*, M_{j,k}] = -\partial_j M_{j,k}$. As $b^*b \leq \Lambda^2$ and $1 \leq \Lambda^2$, we easily check that $\|b_k\Lambda^{-2}u\| \leq \|u\|$. For the term $\|a_j^*b_k\Lambda^{-2}u\|$, we write

$$
\begin{aligned}
\|a_j^*b_k\Lambda^{-2}u\|^2 &= (a_ja_j^*b_k\Lambda^{-2}u, b_k\Lambda^{-2}u) \\
&\leq (a_j^*a_jb_k\Lambda^{-2}u, b_k\Lambda^{-2}u) + ((\partial_j^2V_h)b_k\Lambda^{-2}u, b_k\Lambda^{-2}u) \\
&\leq (\Lambda^2b_k\Lambda^{-2}u, b_k\Lambda^{-2}u) + \|\operatorname{Hess}V_h\|_{L^\infty}(b_k\Lambda^{-2}u, b_k\Lambda^{-2}u).
\end{aligned}
$$

We now use that $[\Lambda^2, b_k] = -b_k$ and continue to derive our series of inequalities:

$$
\begin{aligned}
(\Lambda^2b_k\Lambda^{-2}u, b_k\Lambda^{-2}u) &+ \|\operatorname{Hess}V_h\|_{L^\infty}(b_k\Lambda^{-2}u, b_k\Lambda^{-2}u) \\
&\leq (b_ku, b_k\Lambda^{-2}u) + (\|\operatorname{Hess}V_h\|_{L^\infty} + 1)(b_k\Lambda^{-2}u, b_k\Lambda^{-2}u) \\
&\leq (\|\operatorname{Hess}V_h\|_{L^\infty} + 2)\|u\|^2,
\end{aligned}
$$

because $b_k^*b_k \leq \Lambda^2$ and $1 \leq \Lambda^2$. And this ends the proof of the lemma thanks to the previous remark. $\qquad\square$

Let us go back to the proof of proposition 2.2. We have $E_1 \subset \ker b$ because b is the annihilation operator in velocity. Because b appears on the right of every term in the definition of $\mathcal{A}$, we therefore get

$$E_1 \subset \ker \mathcal{A}.$$

So we can write $\mathcal{A} = \mathcal{A}(\mathrm{Id} - \Pi_1)$ and we have for $u \in G^\perp \cap \mathcal{S}$

$$
\begin{aligned}
III &= \varepsilon \mathrm{Re}\,(\mathcal{A}(\mathrm{Id} - \Pi_1)u, u) + \varepsilon \mathrm{Re}\,(\Lambda^{-2}a^*au, u) \\
&\geq -\tfrac{1}{4}\|(\mathrm{Id} - \Pi_1)u\|^2 - \varepsilon^2 \|\mathcal{A}\|^2 \|u\|^2 + \varepsilon \mathrm{Re}\,(\Lambda^{-2}a^*au, u).
\end{aligned}
$$

It is clear that Λ^2, a^*a, and Π_1 commute and we can write

$$
\begin{aligned}
\varepsilon \mathrm{Re}\,(\Lambda^{-2}a^*au, u) &= \varepsilon \mathrm{Re}\,(\Lambda^{-2}a^*a\Pi_1 u, u) + \varepsilon \mathrm{Re}\,(\Lambda^{-2}a^*a(\mathrm{Id} - \Pi_1)u, u) \\
&= \varepsilon \mathrm{Re}\,(\Lambda^{-2}a^*a\Pi_1 u, \Pi_1 u) \\
&\quad + \varepsilon \mathrm{Re}\,(\Lambda^{-2}a^*a(\mathrm{Id} - \Pi_1)u, (\mathrm{Id} - \Pi_1)u) \\
&\geq \varepsilon \mathrm{Re}\,(\Lambda^{-2}a^*a\Pi_1 u, \Pi_1 u) - \varepsilon \|(\mathrm{Id} - \Pi_1)u\|^2,
\end{aligned}
$$
(2.4)

where we used that $a^*a \leq \Lambda^2$. Here we slightly diverge from [7]: we are going to work with the quasimodes of the Witten Laplacian instead of the first eigenvector of Λ^2 and this will give us exponentially small remainder terms in our estimates. We now need an intermediate lemma.

Lemma 2.6. *For all $u \in G^\perp \cap \mathcal{S}$ then, $\mathrm{Re}\,(\Lambda^{-2}a^*a\Pi_1 u, \Pi_1 u) \geq \frac{\tau}{4}\|\Pi_1 u\|^2$.*

Proof : Let $\mathbb{P}_h$ denote the spectral projection onto the eigenspaces associated with the n_0 smallest eigenvalues of the semiclassical Witten laplacian $W_h = hT_h a^*a T_h^{-1}$(which are the same as the eigenvalues of ha^*a). According to [4] and [6], we have for all $w \in \mathcal{S}(\mathbb{R}_x^d)$

$$(hT_h a^*a T_h^{-1}(1 - \mathbb{P}_h)T_h w, (1 - \mathbb{P}_h)T_h w) \geq \tau h \|(1 - \mathbb{P}_h)T_h w\|^2$$

We then put $\mathbb{P} = T_h^{-1}\mathbb{P}_h T_h$ the (orthogonal) projection on the spectral subspace associated with the n_0 smallest eigenvalues of the Witten laplacian a^*a, so the previous inequality becomes (since T_h is unitary)

$$(a^*a(1 - \mathbb{P})w, (1 - \mathbb{P})w) \geq \tau \|(1 - \mathbb{P})w\|^2$$

Moreover, we have

$$
\begin{aligned}
(a^*aw, w) &= (a^*a[1 - \mathbb{P} + \mathbb{P}]w, [1 - \mathbb{P} + \mathbb{P}]w) \\
&= (a^*a(1 - \mathbb{P})w, (1 - \mathbb{P})w) + (a^*a\mathbb{P}w, \mathbb{P}w) \\
&\quad + (a^*a(1 - \mathbb{P})w, \mathbb{P}w) + (a^*a\mathbb{P}w, (1 - \mathbb{P})w) \\
&= (a^*a(1 - \mathbb{P})w, (1 - \mathbb{P})w) + (a^*a\mathbb{P}w, \mathbb{P}w),
\end{aligned}
$$

because the ranges of $1 - \mathbb{P}$ and $\mathbb{P}$ are stable under a^*a. By definition of $\mathbb{P}$, we get that there exists $\alpha > 0$ such that for all j

$$(a^*a\mathbb{P}w, \mathbb{P}w) = O(\mathrm{e}^{-\frac{\alpha}{h}})\|w\|^2.$$

Now, if we take $w \in F^\perp$, we have $\|(1 - \mathbb{P})w\|^2 = \|w\|^2 + O(\mathrm{e}^{-\frac{\alpha}{h}})\|w\|^2$. We therefore get

$$(a^*aw, w) \geq \tau \|w\|^2 + O(\mathrm{e}^{-\frac{\alpha}{h}})\|w\|^2.$$

Since $(a^*a + 1)^{-1/2}w \in F^\perp$, we get by the max-min principle (since a^*a is selfadjoint)

$$(a^*a(a^*a + 1)^{-1/2}w, (a^*a + 1)^{-1/2}w) \geq \frac{\tau}{1+\tau}\|w\|^2 + O(e^{-\frac{\alpha}{h}})\|w\|^2.$$

Moreover we clearly have

$$1 \geq \frac{\tau}{1+\tau} \geq \frac{\tau}{2},$$

because $\tau \leq 1$. We get the result by taking as w the function defined for almost every v by $x \mapsto \Pi_1 u(x, v)$ and h small enough, since $\Pi_1 u \in E_1$ and $\Pi_1 u(\,\cdot\,, v) \in F^\perp$. $\qquad\square$

End of proof of proposition 2.2: We can now put the result of Lemma 2.6 in inequality (2.4) and we get

$$\varepsilon \mathrm{Re}\,(\Lambda^{-2}a^*au, u) \geq \varepsilon\frac{\tau}{4}\|\Pi_1 u\|^2 - \varepsilon\|(\mathrm{Id} - \Pi_1)u\|^2.$$

We then get

$$III \geq -\frac{1}{4}\|(\mathrm{Id} - \Pi_1)u\|^2 - \varepsilon^2\|\mathcal{A}\|^2\|u\|^2 + \varepsilon\frac{\tau}{4}\|\Pi_1 u\|^2 - \varepsilon\|(\mathrm{Id} - \Pi_1)u\|^2.$$

Bringing together our estimates for $I + II$ and III, this yields to

$$\mathrm{Re}\,(Pu, (\mathrm{Id} + \varepsilon(L + L^*))u)$$
$$\geq \tfrac{1}{2}\|(\mathrm{Id} - \Pi_1)u\|^2 - \varepsilon^2\|L\|^2\|u\|^2$$
$$-\tfrac{1}{4}\|(\mathrm{Id} - \Pi_1)u\|^2 - \varepsilon^2\|\mathcal{A}\|^2\|u\|^2 + \varepsilon\tfrac{\tau}{4}\|\Pi_1 u\|^2 - \varepsilon\|(\mathrm{Id} - \Pi_1)u\|^2$$
$$\geq \tfrac{1}{8}\|(\mathrm{Id} - \Pi_1)u\|^2 + \varepsilon\tfrac{\tau}{4}\|\Pi_1 u\|^2 - \varepsilon^2(\|\mathcal{A}\|^2 + \|L\|^2)\|u\|^2.$$

by taking $\varepsilon \leq 1/8$. Using that Π_1 is an orthogonal projection and that $\varepsilon \leq 1/8$ we get

$$\mathrm{Re}\,(Pu, (\mathrm{Id} + \varepsilon(L + L^*))u) \geq \left(\varepsilon\frac{\tau}{4} - \varepsilon^2(\|\mathcal{A}\|^2 + \|L\|^2)\right)\|u\|^2.$$

Taking ε/τ small enough, we get for a constant A sufficiently large (uniform in h)

$$\mathrm{Re}\,(Pu, (\mathrm{Id} + \varepsilon(L + L^*))u) \geq \frac{\tau^2}{A}\|u\|^2.$$

This ends the proof of the proposition. $\qquad\square$

Corollary 2.7. *There exists $c > 0$ and $h_0 > 0$ sufficiently small, such that for all $h \in [0, h_0[$, for all $z \in \mathbb{C}$ with $\mathrm{Re}\,z \leq ch$ and $v \in \mathcal{S} \cap (S_h G)^\perp$*

$$\|(P_h - z)v\| \geq ch\|v\|. \tag{2.5}$$

Proof: We just proved that for all $u \in \mathcal{S} \cap G^\perp$

$$\mathrm{Re}\,(Pu, (\mathrm{Id} + \varepsilon(L + L^*))u) \geq \frac{\tau^2}{A}\|u\|^2.$$

We then get by multiplying the inequality by h and putting $v = S_h u$

$$\mathrm{Re}\,(P_h v, S_h(\mathrm{Id} + \varepsilon(L + L^*))u) \geq \frac{\tau^2 h}{A}\|u\|^2.$$

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For all $z \in \mathbb{C}$, we can write

$$\operatorname{Re}\left((P_h - z)v, S_h(\operatorname{Id} + \varepsilon(L + L^*))u\right) \geq \frac{\tau^2 h}{A}\|u\|^2 - \operatorname{Re}\left(z(v, S_h(\operatorname{Id} + \varepsilon(L + L^*))u)\right).$$

First notice that if we choose $\varepsilon < \dfrac{1}{2\|L\|}$, we then have $(v, S_h(\operatorname{Id} + \varepsilon(L + L^*))u) \geq 0$. So by the Cauchy-Schwarz inequality, we get

$$\|(P_h - z)v\|\, \|S_h(\operatorname{Id} + \varepsilon(L + L^*))u\|$$
$$\geq \frac{\tau^2 h}{A}\|u\|^2 - \min(0, \operatorname{Re} z)\,\|v\|\,\|S_h(\operatorname{Id} + \varepsilon(L + L^*))u\|.$$

Since S_h is unitary and L is bounded, we get for all $v \in \mathcal{S} \cap (S_h G)^{\perp}$

$$\|(P_h - z)v\|\,(1 + 2\varepsilon\|L\|)\,\|v\|$$
$$\geq \frac{\tau^2 h}{A}\|v\|^2 - \min(0, \operatorname{Re} z)(1 + 2\varepsilon\|L\|)\,\|v\|^2.$$

Therefore, taking $\operatorname{Re} z \leq ch = \dfrac{\tau^2 h}{2(1 + 2\varepsilon\|L\|)A}$ leads to

$$\|(P_h - z)v\| \geq ch\|v\|.$$

$\square$

Since we are working with a non-selfadjoint operator, we have to be careful in order to get a resolvent estimate on the whole space. Thanks to lemma 2.1, we see that $S_h G = \operatorname{Vect} g_j^h$ vanishes under P_h up to a $O(\mathrm{e}^{-\frac{\alpha}{h}})$ and that $(S_h G)^{\perp}$ is stable under P_h up to a $O(\mathrm{e}^{-\frac{\alpha}{h}})$, meaning that there exists $\alpha > 0$ such that for all $u \in S_h G$ and $v \in (S_h G)^{\perp}$,

$$P_h u = \widetilde{O}(\mathrm{e}^{-\frac{\alpha}{h}}\|u\|),$$

and there exists $v' \in L^2$ with $v' = \widetilde{O}(\mathrm{e}^{-\frac{\alpha}{h}}\|v\|)$ such that

$$P_h v - v' \in (S_h G)^{\perp}.$$

Denoting by Π the orthogonal projection on $S_h G$, the two previous relations and Pythagoras' theorem allow us to write that for all $u \in L^2$

$$\|(P_h - z)(\operatorname{Id} - \Pi)u + (P_h - z)\Pi u\|^2 = \|(P_h - z)(\operatorname{Id} - \Pi)u\|^2 + \|(P_h - z)\Pi u\|^2 + O(\mathrm{e}^{-\frac{2\alpha}{h}})\|u\|^2.$$

We then have for all $u \in L^2$

$$\begin{aligned}
\|(P_h - z)u\|^2 &= \|(P_h - z)(\operatorname{Id} - \Pi)u + (P_h - z)\Pi u\|^2 \\
&= \|(P_h - z)(\operatorname{Id} - \Pi)u\|^2 + \|(P_h - z)\Pi u\|^2 \\
&\quad + O(\mathrm{e}^{-\frac{2\alpha}{h}})\|u\|^2 \\
&\geq \tfrac{c^2 h^2}{2}\|(1 - \Pi)u\|^2 + \|(P_h - z)\Pi u\|^2 + O(\mathrm{e}^{-\frac{2\alpha}{h}})\|u\|^2,
\end{aligned}$$

where we used the estimate (2.5) and choose z smaller than $\frac{c}{\sqrt{2}}$. Thanks to the triangle inequality and the fact that $P_h \Pi u = O(\mathrm{e}^{-\frac{\alpha}{h}})\|u\|$ by definition of Π, we get

$$\begin{aligned}
\|(P_h - z)u\|^2 &\geq \tfrac{c^2 h^2}{2}\|(1 - \Pi)u\|^2 + |z|^2\|\Pi u\|^2 + O(\mathrm{e}^{-\frac{2\alpha}{h}})\|u\|^2 \\
&\geq \min(\tfrac{c^2 h^2}{2}, |z|^2)(\|(1 - \Pi)u\|^2 + |z|^2\|\Pi u\|^2),
\end{aligned}$$

Therefore, for all $\delta_1 < \frac{c}{2}$, if we take $\delta_1 h \leq |z| \leq \frac{c}{2}h$ and h small enough we get

$$\|(P_h - z)u\| \geq \frac{\delta_1}{2}\|u\|$$

We conclude that there exists $\delta > 0$ such that for all $0 < \delta_1 \leq \delta$, if $\delta_1 h \leq |z| \leq \delta h$, we have the resolvent estimate

$$\|(P_h - z)^{-1}\| \leq \frac{C_{\delta_1}}{h}.$$

This completes the proof of part $ii)$ of Theorem 1.2.

3   PT-symmetry

Let H be the generalised eigenspace associated with the eigenvalues of modulus smaller than δh, where δ is sufficiently small as in Theorem 1.2, that is the range of the spectral projection $\Pi_0 = \frac{1}{2i\pi}\int_{|z|=\delta h}(z - P_h)^{-1}\,\mathrm{d}z$. We can write the proposition

Proposition 3.1. *We have* $\dim H = n_0$*, where* n_0 *is the number of local minima of the potential* V.

Proof :   From proposition 2.2, the coercive estimate on the orthogonal of a n_0 dimensional subspace gives us that $\dim H \leq n_0$. Moreover we saw that $(z - P_h)g_j^h = z g_j^h + \widetilde{O}(\mathrm{e}^{-\frac{\alpha}{h}})$; for z with $|z| = \delta h$, we therefore get $(z - P_h)^{-1}g_j^h = \frac{1}{z}g_j^h + \widetilde{O}(\mathrm{e}^{-\frac{\alpha}{h}})$. By integrating along the circle centered in 0 and with radius δh, we finally get

$$\Pi_0 g_j^h = g_j^h + \widetilde{O}(\mathrm{e}^{-\frac{\alpha}{2h}})$$

Since the g_j^h are almost orthogonal, they are linearly independent. We deduce that $\dim H \geq n_0$, so $\dim H = n_0$. $\square$

It remains to show that H contains no Jordan blocks for P_h which we do by using an extra symmetry of our operator that is PT-symmetry: P referring to parity (that is velocity reversal) and T to time reversal. Let $\kappa : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be given by $\kappa(x, v) = (x, -v)$. We then put $U_\kappa u = u \circ \kappa$, $u \in L^2$, so that U_κ is unitary and selfadjoint. We also introduce the non-definite Hermitian form

$$(u, v)_\kappa = (U_\kappa u, v),\ u, v \in L^2$$

Note that

$$P_h^* = U_\kappa^{-1}P_h U_\kappa,$$

so P_h is formally selfadjoint with respect to the Hermitian form $(.,.)_\kappa$.

Proposition 3.2. *For h small enough, the restriction of $(.,.)_\kappa$ to $H \times H$ is positive definite uniformly with respect to h.*

Proof :   Let us recall that $g_j^h(x, v) = \chi_j(x)\mathcal{M}_h^{1/2}(x, v)$ and that $\Pi_0 g_j^h = g_j^h + \widetilde{O}(\mathrm{e}^{-\frac{\alpha}{2h}})$. So $g_j^h \circ \kappa = g_j^h$ and the family $(g_j^h)_{1 \leq j \leq n_0}$ is almost orthonormal

for the Hermitian form $(.,.)_\kappa$ Then there exists a basis $(g_{0,j})_{1\leq j\leq n_0}$ of H such that

$$(g_{0,j}, g_{0,j})_\kappa = 1 + O(e^{-\frac{\alpha}{h}}), \quad (g_{0,j}, g_{0,j'})_\kappa = O(e^{-\frac{\alpha}{h}}) \text{ pour } j \neq j'.$$

Then for $H \ni u = \sum_{k=1}^{n_0} u_k g_{0,k}$, we get

$$(u, u)_\kappa = \sum_{k=1}^{n_0} (1 + O(e^{-\frac{\alpha}{h}}))|u_k|^2 \geq \|u\|^2/C.$$

This ends the proof. $\qquad\square$

Thanks to the formal selfadjointness of P_h with respect to $(.,.)_\kappa$, we get the following proposition.

Proposition 3.3. *For h small enough, the restriction of $P_h : H \to H$ is selfadjoint with respect to the scalar product $(.,.)_\kappa$ on H. Moreover P_h has exactly n_0 real eigenvalues counted for multiplicity smaller than δh and they are $O(e^{-\frac{\alpha}{h}})$.*

This implies in particular point *i)* of Theorem 1.2.

4   Return to equilibrium

To conclude, we translate our spectral estimates into estimates for the semigroup decay. Let us recall that Corollary 2.7 gives us that for all $u \in \mathcal{S} \cap (S_h G)^\perp$ and $\mathrm{Re}\, z \leq ch$

$$\|(P_h - z)u\| \geq ch\|u\|.$$

Let us rewrite the previous resolvent estimate as an estimate on $(\mathrm{Id} - \Pi_0)L^2$ for Π_0 the spectral projection associated with the n_0 smallest eigenvalues. We first note that $(g_j^h)_j$ are quasimodes for both P_h and P_h^*. We then have for all j

$$\Pi_0 g_j^h = \Pi_0^* g_j^h = g_j^h + \widetilde{O}(e^{-\frac{\alpha}{h}}).$$

So if we take $u \in L^2$, we get for all j

$$(g_j^h, (\mathrm{Id} - \Pi_0)u) = ((\mathrm{Id} - \Pi_0^*)g_j^h, (\mathrm{Id} - \Pi_0)u) = O(e^{-\frac{\alpha}{h}})\|(\mathrm{Id} - \Pi_0)u\|$$

We conclude that one can write for all $u \in L^2$

$$(\mathrm{Id} - \Pi_0)u = v + \sum_j a_j g_j^h$$

with $v \in G^\perp$, $\|v\| = \|(\mathrm{Id} - \Pi_0)u\| + O(e^{-\frac{\alpha}{h}})\|(\mathrm{Id} - \Pi_0)u\|$ and for all j, $|a_j| = O(e^{-\frac{\alpha}{h}})\|(\mathrm{Id} - \Pi_0)u\|$. We then have with $\mathrm{Re}\, z \leq ch$, thanks to our previous resolvent estimate,

$$\|(P_h - z)(\mathrm{Id} - \Pi_0)u\| \geq ch\|v\| - O(e^{-\frac{\alpha}{h}})\|(\mathrm{Id} - \Pi_0)u\|$$

We therefore get for h small enough and a new c the resolvent estimate on $(\mathrm{Id} - \Pi_0)L^2$ with $\mathrm{Re}\, z \leq ch$

$$\|(P_{h|(\mathrm{Id}-\Pi_0)L^2} - z)^{-1}\| \leq \frac{1}{ch}.$$

Therefore we have obtained an uniform resolvent estimate on the left half-plane $\{\mathrm{Re}\, z \leq ch\}$. As $\Pi_0 L^2$ and $(\mathrm{Id} - \Pi_0)L^2$ are stable under P_h (by definition of Π_0), the Gearhart-Prüss theorem (cf. [3]) for $P_{h\,|(\mathrm{Id}-\Pi_0)L^2}$ allows us to write that for all $t > 0$

$$\mathrm{e}^{-tP_h} = \mathrm{e}^{-tP_h}\Pi_0 + O(e^{-cht}).$$

Despite the fact that the projection is not orthogonal, since we have $\Pi_0 = \frac{1}{2i\pi} \int_{\mathcal{C}(0,\delta h)} (z - P_h)^{-1}\, \mathrm{d}z$, our resolvent estimate gives us $\|\Pi_0\| = O(1)$. Now if we denote by $(\mu_j)_{j=1\ldots n_0}$ the exponentially small eigenvalues (counted with multiplicity) and Π_j the associated spectral projections, we have that $\|\Pi_j\| = O(1)$. Indeed, we saw that the restriction $P_{h\,|H}$ is selfadjoint with respect to $(.,.)_\kappa$, which is equivalent (on H) to the ambient scalar product, so the projections are bounded. Thus, we can sum up all of this into the following Proposition.

Proposition 4.1. *There exists $\delta_1 > 0$ and $\alpha > 0$ such that for all , $t \geq 0$ and h small enough*

$$\mathrm{e}^{-tP_h} = \sum_{j=1}^{n_0} \mathrm{e}^{-t\mu_j}\Pi_j + O(e^{-\delta_1 ht}),$$

with $\|\Pi_j\| = O(1)$ and $\mu_j = O(\mathrm{e}^{-\frac{\alpha}{h}})$.

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