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Abstract: A part of students learning in the classroom depends on how the teacher responds to their thinking. The literature has separately addressed teachers' feedback responses to errors and unexpected strategies that students put into play when solving tasks. We propose a framework to analyze these responses together based on three criteria: the focus of the answers (teacher or student), the type of knowledge (conceptual or procedural) that the teacher puts into play in the teacher-centered answers, and the types of actions (asking and proposing) involved in student-centered responses. We codified and analyzed the feedback responses of a group of mathematics teachers to a questionnaire that inquired about their curricular practices. We found similarities in their reports of responses to students' errors and unexpected strategies: two-thirds of teachers have a teacher-centered response. For the student-centered answers, the number of responses of the teacher in which he/she proposes activities is three times the number of responses in which he/she asks students questions. Furthermore, responses to unexpected strategies differ from responses to errors because teachers evaluate, correct, and accept those strategies.

Introduction

Teachers' decisions affect students' learning opportunities (Herbst et al., 2016). Understanding teachers' decision making can help teacher educators improve the quality of teaching (Bishop, 2008). There are various circumstances in the classroom that require teachers to make decisions spontaneously; that is, decisions that must be made in circumstances that teachers perceive as problematic in relation to the learning of their students and that are not necessarily part of their lesson script and/or their class routines. Two of these circumstances are errors and unexpected strategies of the students. Therefore, a part of the students' learning in the classroom depends on the way in which the teacher responds to their mistakes (Ball & Bass, 2003; Ma, 1999; Son & Sinclair, 2010) and to the unexpected strategies they bring into play when solving tasks (Son, 2016; Son & Crespo, 2009). What kind of feedback response do math teachers give to their students' errors and their strategies when approaching tasks?

In this article, we address this question by identifying and characterizing the feedback responses practicing teachers report to students' unexpected errors and strategies, and organizing these responses with respect to three criteria. The first is the focus of the responses: teacher or student. When the answer focuses on the teacher, we consider the type of knowledge that the teacher brings into play: conceptual or procedural. When the response is student-centered, we consider the type of action, for example, 'asking' or 'proposing', that the teacher performs. We also identify similarities and differences in teachers' feedback responses when reacting to errors or strategies.

We carried out an empirical study in which the participants were high school teachers who were starting a postgraduate training program. The students of these teachers ranged in age from 10 to 18 years old. We considered their answers to a questionnaire about one of their most recent lessons, in which they answered items about the feedback responses they had to errors and unexpected strategies during the teaching of a specific mathematics topic chosen by each teacher.

We organize this article into five sections. In the first section, we place the study in the context of the literature. Then, we develop the conceptual framework and present the objectives of the study. In the third section, we present the methodology and, in the fourth, the results. Finally, we present a discussion and conclusions section that includes the implications of the study.

Feedback Responses to Student Actions in the Classroom

Feedback can be understood as information provided by the teacher on aspects of student performance and understanding with reference to a goal (Hattie & Timperley, 2007; Voerman et al., 2012). In the literature, different types of feedback are identified. For example, Voerman et al. (2012) lay out two large categories: progress and discrepancy. Progress feedback emphasizes what has been accomplished of the learning expectations. Discrepancy feedback emphasizes what is missing from the learning expectations. In addition to these two categories, Voerman suggests that these same feedbacks can be positive or negative (if the student's performance is praised or criticized), and specific and non-specific (if the information addresses single aspects or not). Similarly, Guo and Wei (2019) propose five types of feedback: verification feedback (a judgment of a student's answers as correct or incorrect, such as grades or marks), directive feedback (telling students the direct answers or the complete solution to their questions or problems), scaffolding feedback (successive clues, hints, models, and hints provided to students or partial solutions that consist of breaking tasks into smaller or easier parts to make it easier for students to generate correct answers on their own), praise (praising the value of student learning performance, attitudes, or products), and criticism (negative responses to student performance, attitudes, or behaviors through expressions of disgust, disapproval, or rejection).

As we can see in these different typologies, the focus of the feedback response can be actions that the student must carry out (as is the case of scaffolding feedback) or information given by the teacher without necessarily implying immediate actions by the student (for example, progress, verification, praise, or criticism feedbacks). However, this approach does not discriminate the type of stimulus that causes the feedback response: error or unexpected strategy. For the purposes of this study, we complement these approaches to feedback with another perspective. Our perspective is based on the fact that student learning arises primarily from those teacher feedbacks that lead the student to do things and to face situations that lead him to overcome his mistakes. For this reason, we will use two broad groups to classify feedback response: student-centered and teacher-centered. Student-centered feedbacks are those that lead the student to perform activities in which he plays an active role and promote his/her learning. Consequently, feedbacks that contribute to learning correspond to student-centered reactions in which the learner acts, and feedback responses that contain only information, together with affective issues, in which the learner has a passive role, correspond to teacher-centered reactions.

In another approach on teachers' feedback responses to students' thinking, studies whose focus are responses to errors stand out. For example, in a study based on observations of mathematics, German, and economics classes, Tulis (2013) identified the following eleven categories of teacher response to student errors: the teacher ignores the error, corrects it, asks the same student who corrects redirects the question to another student, encourages the student to keep trying, gives the student time to reconsider his answer, promotes discussion with the whole class, avoids negative responses from classmates (teasing), criticizes or judges to the student, engages in demeaning behavior (for example, laughter) and expresses expressions of discomfort (disappointment or hopelessness). Subsequently, Sarkar Arani et al. (2017) took these categories and proposed an error management behavior framework for the mathematics teacher. This framework is organized into seven categories: ignoring the error, correction by the teacher, correction by the students, redirecting the question to another student, support after the errors, and negative response (expressing annoyance or humiliating behavior).

Several studies have established comparisons between feedback responses to errors in different countries. For example, in a study of teacher responses to student errors in mathematics classrooms in the United States and China, Schleppenbach et al. (2007) found that teachers in both countries responded to errors differently. Teachers in the United States made more statements about errors (indicating that answers are incorrect, giving the correct answers, giving explanations, and encouraging students to correct themselves) than teachers in China, who instead asked more follow-up questions on errors (repeat the question, redirect the question, expand the question, ask for agreement with a statement, and ask for an addition to the answers). On the other hand, Santagata (2005) examined the responses that teachers in the United States and Italy gave to students' errors during mathematics classes. For example, in the United States, he found that when teachers addressed an error, they provided the correct answers or chose the correct answers from a series of student responses. When the teacher asked the student who had made the mistake to correct himself, the teacher repeated the question to the same student, asked him to explain why he gave that answer, or added a suggestion to help the student arrive at the correct answers. When the teacher asked another student to make the correction, the teacher redirected the question to a different student or added a hint while redirecting the question. Finally, on some occasions, the teacher asked students to identify a partner's mistake, or some students corrected a partner's mistake on their own initiative before the teacher intervened.

Son and Sinclair (2010), in a study that analyzes how teachers interpret and respond to students' geometric errors, identified some response patterns. The dominant patterns are the generalization of properties (the teacher does not directly address the error and, in his intervention, explains general properties in which the error is framed), the return to the basics (the teacher atomizes the error into more basic elements, concepts or procedures, to help the student to overcome the difficulty) and the Plato and the slave approach (the teacher assumes that the student knows how to solve the task, but, since the student does not remember, the teacher must help him or her to do it). In the first two cases, the teacher says or shows the basic properties or elements, and in the third case, the teacher usually guides the student by asking for more information.

In addition to the feedback responses to the errors, we are also interested in the responses of the teachers to the unexpected strategies of the students when they solve tasks in the classroom. Campbell et al. (1998) propose three criteria that mathematics teachers should consider when faced with strategies invented by students and which, therefore, could guide teachers' responses to unexpected solution strategies. The first is validity. The teacher must assess whether the strategy works for the given task. The second criterion is generalizability. If the strategy works, you need to consider whether the strategy works for similar cases. And the third criterion is effectiveness. It consists of determining if the strategy is more appropriate than another (for example, the one suggested by the teacher) for the solution of the task in question.

In a study focused on prospective teachers' reasoning and responses to a student's nontraditional strategy of dividing fractions, Son and Crespo (2009) found that a small percentage of High School teachers (18%) and almost half of Primary teachers (41%) encouraged students to explain and justify their strategy. The other participants, on the other hand, focused their responses on their own explanations and justifications and gave little space and opportunity for the students to discuss their responses. Similarly, Son (2016), in a study on future teachers' reasoning and responses to students' informal and formal strategies for integer subtraction, found that more than half of the teachers (about 60%) focused their responses on the teacher: the teacher is who says or explains why the strategy works. This happened regardless of whether the students used traditional methods or self-generated methods.

As we have explained up to this point, most studies have addressed mathematics teachers' feedback responses to errors and unexpected strategies separately. Research on errors dominates. A significant proportion of these studies have been aimed at establishing correlations between the answers and the knowledge of the teacher and have been framed in specific topics of school mathematics. The different studies coincide in describing these responses to student thinking in terms of student actions and teacher actions and, on occasion, consider affective issues or teacher attitudes. These antecedents show the interest of deepening the characterization of the teacher's answers with different purposes.

In this article, we identify and characterize math teachers' feedback responses to students' errors and unexpected strategies. We study these two types of feedback responses and identify similarities and differences between them. The teachers in the study freely expressed what they did during the development of a recent lesson corresponding to a mathematical content chosen by them. We structure their feedback responses through reference frameworks that we developed based on previous results (Boaler & Brodie, 2004; Brodie, 2011; Santagata, 2005; Sarkar Arani et al., 2017; Schleppenbach et al., 2007; Son, 2013, 2016; Son & Crespo, 2009; Son & Sinclair, 2010; Tulis, 2013). The frames of reference are based on three criteria. First, we consider the focus of the responses identified: teacher or student. In the case of teacher-centered answers, we consider the type of knowledge that the teacher puts into play: conceptual or procedural. And, in the case of student-focused answers, we consider the type of action: ask or propose. The results obtained allow us to reflect on the tendencies that teachers show in their feedback responses and to conjecture about the origin of the answers in the general beliefs of teachers about the teaching and learning of mathematics.

Theoretical Framework

We organize the theoretical framework of this study in three sections. In the first place, we present a section in which we expose the notions of error and unexpected strategy. We then develop student- and teacher-centered approaches. Next, we present the distinction between conceptual knowledge and procedural knowledge, as a particular category to address teacher-centered responses, and the distinction between asking and proposing actions, as a particular category to address student-centered responses. Finally, we describe the frames of reference that we will use to classify the teachers' responses.

Errors and Unexpected Strategies

An error is the visible manifestation of a learning difficulty. The error is directly observable in the student's incorrect answers to specific questions and tasks required by the teacher. Due to the fact that many of the errors arise from partial knowledge that has worked for the student in other situations, they tend to be systematic, sporadic, persistent and not questionable by the student (Rico, 1995). Excluded from this characterization are those erroneous manifestations of students that are produced by learning problems (i.e., dyscalculia), chance or are the result of inattention or negligence (González & Gómez, 2018; Movshovitz-Hadar et al., 1987). Errors can prevent the student from progressing in solving a task.

On the other hand, an unexpected strategy is a sequence of students' actions that the teacher did not plan as an alternative solution to a proposed task. Unexpected strategies can range from basic algorithms (i.e., performing addition or multiplication of natural numbers by methods other than those taught) to sophisticated procedures to solve a particular situation (i.e., finding solutions to systems of equations by the graphic method when the teacher has taught the algebraic method).

Student-Focused and Teacher-Focused Approaches

Teachers' feedback responses to student errors and unexpected strategies can be organized in terms of who gives the information, who gives an explanation or justification, and who ultimately decides on the appropriateness of the ideas presented. In the literature, two main approaches are proposed, depending on who is the subject of the action to be carried out: student-centered and teacher-centered. Student-centered responses can be seen, for example, when a teacher asks students to explain, present, and justify their response (mistake or strategy), or asks and guides students to discover whether responses are valid or not. In contrast, in teacher-centered answers, the teacher clarifies, explains, evaluates, or shows whether the answer is valid or not, and provides direct information about the answers (Son, 2013, 2016; Son & Crespo, 2009; Son & Sinclair, 2010).

Conceptual Knowledge and Procedural Knowledge

In the field of cognition, there is a certain consensus in distinguishing two types of mathematical knowledge: conceptual and procedural (Son & Sinclair, 2010). Conceptual knowledge refers to the understanding of the principles that structure a topic and the interrelationships between its elements (Rittle-Johnson & Alibali, 1999). Examples of this type of knowledge are the properties and definitions of mathematical concepts. Procedural knowledge refers to the ability to carry out action sequences, algorithms and procedures to respond to a task (Hiebert & Lefevre, 1986; Rittle-Johnson & Alibali, 1999). During the learning process, these two types of knowledge are in constant interaction, and it is not always clear in what order they are acquired (Cañadas et al., 2018; Star, 2005).

In the context of this study, the distinction between these two types of knowledge has been used in some works to typify responses classified as teacher-centered. For example, Son and Crespo (2009) established that future mathematics teachers tend to focus on reiterating procedures to students to overcome an error, and leave aside the conceptual aspects that may be behind the error. However, when they are asked to identify the source of the error, they mostly interpret it conceptually (Son & Sinclair, 2010). In a more recent study, Son (2013) distinguishes between two profiles of teachers: those with a conceptual approach (who use verbs such as "recognize" or "see" characteristics or properties in their responses to errors) and teachers who have a procedural approach (using verbs like "use", "calculate" or "configure").

Two Types of Actions: Ask and Propose

Within the teacher's feedback responses to errors and unexpected strategies focused on the student, we distinguish the actions that seek to gather additional information about the student's thinking, from the actions in which the teacher directly proposes that the student carry out alternative activities. The first, which will be identified by the verb *to ask*, consists of asking the student for more information about the task carried out, through questions, or asking him to indicate how he solved the task. These responses have the purpose of verbalizing the student's thinking (Boaler & Brodie, 2004; Santagata, 2005; Sarkar Arani et al., 2017; Schleppenbach et al., 2007; Son & Sinclair, 2010; Tulis, 2013). For example, the teacher asks a student to explain how they arrived at that answer, How did you do it?, which focuses on what the task asks for, What information is requested?, or relate their ideas to a known concept or procedure, Where is the "x" in the diagram?

The answers identified by the verb *to propose* are intended for the student to do something additional to the original task so that he can contrast what was done. For example, the teacher proposes to a student who has made an error to discuss his solution with a classmate, solve a new task or solve an alternative task that leads him to identify his error.

Frameworks for Organizing Teachers' Answers

Based on these conceptual references and the literature review we have presented, we propose the following frameworks for organizing teacher feedback responses to errors and unexpected strategies. These frames of reference are conceived as structures that will allow us to apply the criteria to classify the teacher's answers in a structured way. To develop these frameworks, we have selected those criteria that have been proposed by at least two of the studies we reviewed. The first level distinguishes who is the subject of the action to be carried out, student or teacher. The second level corresponds to the set of possible actions that can be carried out by the student or the teacher. When the response focuses on the student, we will use the distinction between the actions of asking (if the student is asked to provide additional information) and propose (if the student is asked to perform a new task). When the response focuses on the teacher, we will distinguish the response as conceptual or procedural, depending on the type of knowledge that the teacher brings into play. In Figure 1, we present our synthesis of these responses to errors.

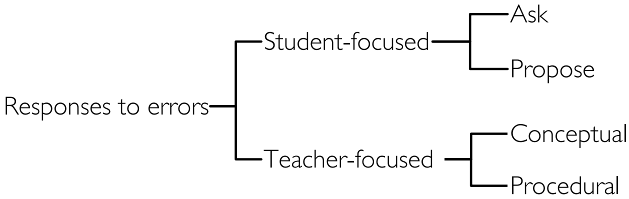

Figure 1. Error feedback response framework

In feedback responses to unexpected strategies, in addition to the above criteria, the literature reports those actions of teachers related to evaluating the strategy suggested by the student (Brodie, 2011; Campbell et al., 1998). In Figure 2, we present our synthesis of these responses to unexpected strategies.

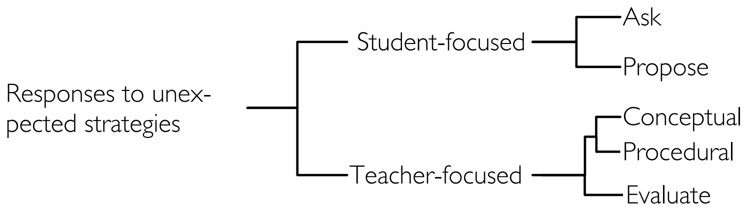

Figure 2. Framework of feedback response to unexpected strategies

We will rely on these schemes to establish the categories and codes with which we will analyze the responses of the teachers who participated in our study. We seek to corroborate these frameworks and complement them with a more detailed level of evidence.

Research Objectives

Based on the notions that we have presented in the previous sections, we establish the following objectives of this study.

To check that the frames of reference allow to organize Mathematics teachers' feedback responses to students' unexpected errors and strategies.

To identify similarities and differences in reports of math teachers' feedback responses to students' unexpected errors and strategies.

Method

The participants in this study were in-service mathematics teachers who had just started a postgraduate training program. Given the number of participants (94 teachers) and the spatial location of the schools where they worked (a radius of more than 200 km), conditions that made it difficult to observe lessons, a questionnaire was designed that, based on the proposed curriculum model in the program, could be used to describe and characterize their implementation practices in the classroom. Next, we present the context in which the study was carried out, the description of the sample of teachers and the phases of data collection, the coding, and the analysis.

Context and Sample

The sample consisted of 94 mathematics teachers from public and private secondary schools in Bogotá, Cundinamarca and Tolima (Colombia). Just over half of the group (53%) were male, the majority (85%) worked in public schools, and 86% had more than five years of experience. The data was collected through a questionnaire that was answered by all the teachers who participated in the postgraduate program between 2014 and 2018. To avoid the influence of the program on the answers, these were collected at the beginning of the program and part of the researchers were not educators of the program. This information was provided voluntarily and had no effect on their grades. Therefore, this was a convenience sample of teachers interested in improving their teaching practices.

The training program covers the planning, implementation, and evaluation of mathematics curriculum designs. In this program, teachers are expected to develop deep pedagogical knowledge about mathematical content aimed at supporting their decisions for planning, implementing and assessing lessons.

Data Collection

We designed a questionnaire in which teachers reported what they did during the development of a recent lesson. Being a lesson that had just been implemented, the information provided was reliable (Desimone, 2009; Ross, McDougall, Hogaboam -Gray, & LeSage , 2003). We do not ask for the opinion of teachers; we asked them to report on the actions they actually took. Desimone (2009) has shown, based on several studies, that surveys that ask behavioral and descriptive, not evaluative, questions have been shown to have good validity and reliability. In addition, it turns out to be a powerful instrument to collect information from large groups of teachers (Uysal, 2012). The selection of the topic and the group of students was left to the discretion of the teacher. The items of the questionnaire that we consider in this study are the following.

> *Did your students make mistakes when they tried to solve the tasks? If you answered yes to the previous item, how did you react to this situation?*
> *Was there a student who solved a task differently from what you explained in class? If you answered yes to the previous item, how did you react to this situation*?

These items went through multiple revisions and pilot tests were carried out with volunteer teachers who were interviewed to verify possible interpretations of the questions. As a result of these pilot tests, the wording of several questions was adjusted to make language clearer for the interviewee. The teachers responded individually, without interaction with the researchers, by filling out an online form. Since the teachers responded spontaneously based on what happened during the implementation, the feedback responses correspond to verbal and synchronous responses. In addition, the teachers did not necessarily provide information on the context in which their response arose since the questions focused on “how did you react”.

This study is built from the reports that teachers gave about their response to an error or an unspecified strategy. As can be seen in the questions asked, we did not ask the teacher to describe in detail what the situation was that he interpreted as a mistake or an unforeseen strategy of a student. The teachers' responses correspond to the “teachers’ perspective,” as the data are obtained from retrospective subjective perspectives. Therefore, the report made by the teacher is not only the action performed, but also his interpretation of what he did (Varela & Shear, 1999). In other words, we weren’t necessarily expecting answers like “I got up from the table, walked over to your desk, looked at your notebook for three minutes, and asked, why did you do that?” But we did expect answers like “I wanted to know why he did it that way and I asked him”. This leads our coding to focus on the actions described and, in a complementary way, on the intentions.

Data Coding and Analysis

Of the 94 teachers who responded to the questionnaire, 85 reported that at least one student made an error and 54 indicated that at least one student used unexpected strategies. The mathematical content and the school level of the group in which the class was held were not the focus of attention for coding. Our coding focused on the actions that were part of the teacher’s feedback responses, without reference to specific mathematical content. The teachers’ responses were coded by text segments, that is, by the teacher’s sentences that refer to a single action that is part of their response. Therefore, the same answer to the questionnaire could be coded with more than one code if the teacher mentioned different actions in his answer and, consequently, his answer was separated into different text segments.

We found answers made up of up to six text segments. For example, in the answer "I guided them and I asked them questions about the topic so they could solve it", we have two text segments. The first segment of the text "I guided them" focuses on the actions of the teachers; the second, "I asked them questions about the topic so they could solve it" focuses on the actions of the students. In addition, we found answers from teachers that we did not code because they did not correspond to any response. This is the case of answers like "I knew that there were different methods to carry out this type of exercise", where no action is indicated by the teacher or the students. This type of answers was less than two percent of the total answers.

Based on the frames of reference, three researchers separately coded the answers to the questionnaires. From this first coding, each researcher adjusted the code structure of the frames of reference and complemented them with additional codes when the existing ones were insufficient or when they could be further refined. Then, in discussion sessions, the researchers compared their new code structures and agreed to formulate a unique organization of these codes, both for errors and for unexpected strategies. The codes and their description can be downloaded from https://bit.ly/UED_TF_Codes.

With the new code structures, the researchers performed a second encoding. Inter-investigator reliability analyzes were performed using the Kappa coefficient to determine inter-investigator consistency. A Kappa coefficient of 0.87 ($p<0.05$) was obtained and the mean percentage of agreement between researchers was 96%. Differences between investigators were resolved in a discussion of the results, which included the criteria used and more detailed descriptions of the codes.

Once all the text segments had been coded, the proportion of each of the codes over the total number of text segments was calculated. Any of the text segments could be coded at least in the first two levels contemplated in the frameworks: response centered on the teacher or the student in the first level; and, for the second level, type of conceptual or procedural knowledge, if the response was centered on the teacher, and ask or propose, if the response was centered on the student. In the case of teacher-centered responses to unexpected strategies, the second level also included the evaluation of those strategies.

Since the text segments could be encoded in at least the first two levels, the percentages of the text segments encoded in those levels add up 100%. However, when refining the levels, the percentages may not add up to 100%, since not all text segments contain enough information to encode them with the refined codes.

Results

Next, we present the results obtained for each research objective. We first show the structure of the feedback responses to the errors and unexpected strategies, and the percentages of each type of response, providing examples of the teachers' responses. Then, we present the similarities and differences in the teachers' responses according to the errors and unexpected strategies.

Teachers' Feedback Response to Student Errors

In Figure 3, we present the structure of teachers' feedback responses to student errors and the respective percentages.

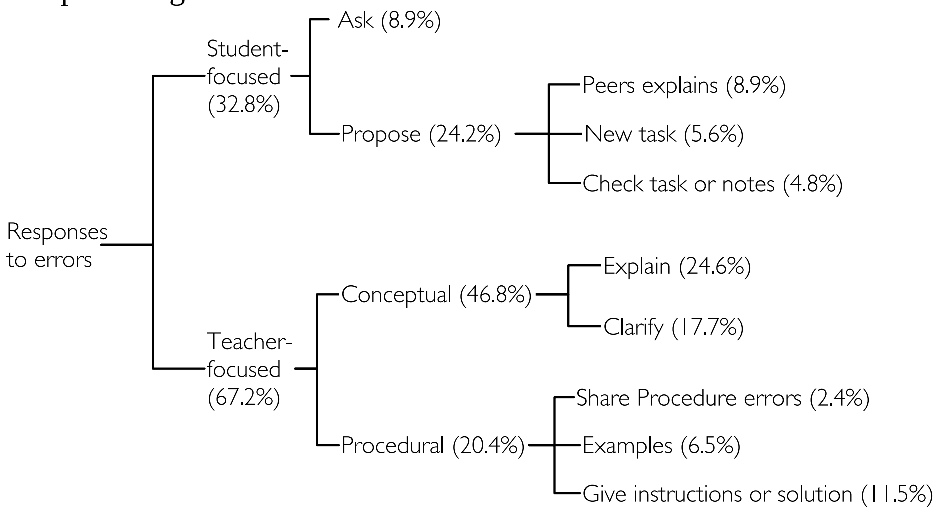

Figure 3. Structure and percentages of the teacher's feedback responses to student errors

About a third (32.8%) of the feedback responses to the errors were focused on students, and about two-thirds (67.1%) focused on teachers. In the first of these categories, we identified those answers related to asking students questions to gather additional information (8.9%) and those in which they proposed to do something new (24.2%). Within the Asking category we found questions aimed at identifying the difficulty in the student, seeking for students to express their opinion and definitions of the mathematical concepts involved in the solution. Examples of this type of response are: "I asked them questions to find the difficulty", "I asked them which solution they considered correct" and "I asked the students the definition of each trigonometric ratio". The Propose category was subdivided by whether the student is asked to explain to his classmates what he did (8.9%), whether he is asked to propose new, more basic or similar tasks (5.6%) and whether he is asked to review what he did or to review his notes (4.8%). For example, the latter case corresponds to responses such as "[I asked him] to redo the exercises".

Teacher-centered responses to errors were organized according to the type of knowledge involved: conceptual (46.8%) and procedural (20.4%). Within the conceptual answers, the explanations made by the teacher dominate (24.6%). These explanations may consist of presenting the topic in different ways (different contexts, giving feedback or reinforcing the topic or concepts), explaining the same thing (but in more detail) or explaining the specific situation. This is the case of responses such as "I tried to explain them further" and "I looked for another strategy to explain the topic". Teachers also reacted by clarifying students' doubts, solving questions or solving the task with the students (17.7%). For example, a teacher said "[I used] different words to clarify doubts". In the procedural responses, there are actions aimed at sharing calculation or procedural errors with the whole group (2.4%), giving examples with similar cases (6.5%) and giving specific instructions to solve the task or give the correct solution (11.5%). The latter is the case of responses such as "write the beginning of the procedure so that they finish the rest of the procedure".

In summary, these results show that most of the teachers who participated in this study reacted to errors with responses focused on their own performance (67.1%): they apparently believe that offering new explanations or clarifying doubts is what they should do to help students overcome their mistakes. When teachers' responses are student-centered, they slightly prefer to propose new tasks, and actions are focused on guiding students to help other students overcome mistakes.

Teachers’ Feedback Response to Unexpected Student Strategies

In Figure 4, we present the structure of the different teachers’ feedback responses to unexpected strategies and indicate the respective percentage.

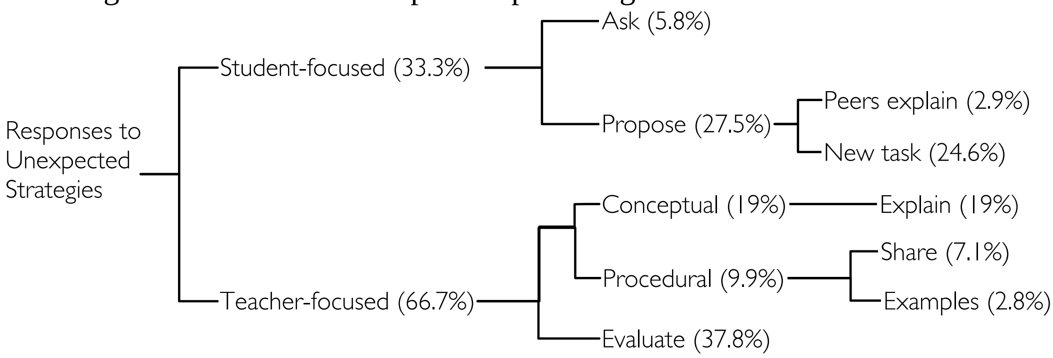

Figure 4. Teachers’ feedback responses to unexpected student strategies

In feedback responses to unexpected student-centered strategies (33.3%), we found that, in a small portion, the teacher asks students questions to express their opinion or to explain the advantages and disadvantages of the new strategy (5 .8%). Examples of these answers are “I asked how you solved it” and “I asked the group about the advantages and disadvantages of the peer method”. Likewise, in this category, the teacher makes different types of proposals to the students (27.5%): they propose to explain the strategy to their classmates (2.9%) and they propose new situations that confirm the validity of the strategy (24.6%). For example, the answer “I suggested to the student that he show his strategy to present different possibilities” is representative of the proposal aimed at explaining the strategy to his classmates.

Teacher-focused responses to unexpected strategies (66.7%) were coded as conceptual knowledge-focused (19%), procedural knowledge-focused (9.9%), or assessment-related (37.8%). Within the conceptual responses we find the explanations made by the teacher to show that there are different strategies to solve the task. An example of this type of response is “I showed [students] why there are multiple ways to get to the same result.” In the procedural answers we found situations of sharing the new procedure with the whole group (7.1%) and giving students examples of other cases in which, the strategy works (2.8%). This is the case of responses such as “I present it to the group as an alternative solution.”

In the answers related to the evaluation (37.8%), the teacher determines if the new strategy proposed by the students is adequate. For example, one teacher said, “I check the strategy to see if it’s correct,” and another said, “I showed [students] where there was a mistake.”

In summary, these results show that most teachers responded to unexpected strategies with responses focused on their own performance (66.7%), especially aimed at evaluating proposals for new strategies before sharing them with the whole group (37.8%). The dominant type of knowledge is conceptual (19%) as evidenced through the explanations given by the teacher.

Similarities in Feedback Responses to Students' Errors and Unexpected Strategies

We found similarities in teachers' feedback responses to errors and unexpected strategies. The first finding is the similarity of proportions in the responses centered on the student (32.9% and 33.3%, respectively) and on the teacher (67.1% and 66.7%, respectively). When comparing proportions, we found that there is no statistically significant evidence to reject the hypothesis of equality of these proportions (P= 0.009).

In the student-centered responses, we observed that, for errors and unexpected strategies, the percentage of Ask responses (8.9% and 5.8%, respectively) is less than half of the proposed responses (24.2% and 27. 5%, respectively).

In the responses focused on the teacher, we observed that, in errors and unexpected strategies, the percentage of Conceptual responses (46.8% and 19%, respectively) is very close to double the percentage of Procedure responses (20.4% and 9.9%, respectively).

This similarity analysis provides evidence that error and unexpected strategies feedback responses are two sides of the same coin. In both cases, the teacher responds to the student's thinking based on his planning, his experience, his beliefs, and his knowledge. In this study, we show that these responses are essentially the same, regardless of the type of stimulus that generates them.

For the group of teachers who participated in this study, these results show a tendency to respond to errors and unexpected strategies in a teacher-centered way: they focused their responses on their explanations and justifications and provided limited opportunities for students to discuss their mistakes and strategies.

Differences in Feedback Responses to Students' Unexpected Errors and Strategies

Teachers' feedback responses differ by errors and unexpected strategies only when a high level of encoding detail is reached. At this level of detail, we found relevant differences in three aspects. The first is that, in responses to unexpected strategies, the evaluation percentage is higher than the conceptual and procedural percentages. The second aspect is the explanations. In responses to errors, this response represents slightly more than half of the conceptual responses. However, in responses to unexpected strategies, explanations are less used by teachers, who are the ones who disseminate student proposals. The last difference between the answers refers to the explanation to the other students. In response to mistakes, this response is more frequent than in response to unexpected strategies.

Discussion and Conclusions

In this study, we identified and structured the feedback responses that practicing mathematics teachers report to their students' errors and unexpected strategies. The participating teachers were middle and high school teachers who were starting a training program. These teachers answered questions from a questionnaire that asked about their feedback responses to unexpected mistakes and strategies of a particular group of students in a recent lesson on a specific math topic. This study is framed in the teachers' feedback responses in real classroom situations.

We built reference frameworks to organize teachers' feedback responses based on the results of other studies (Santagata, 2005; Sarkar Arani et al., 2017; Schleppenbach et al., 2007; Son, 2013, 2016; Son & Crespo, 2009; Son & Sinclair, 2010; Tulis, 2013). These frames of reference are centered on student-focused and teacher-focused responses. Each of these categories is broken down, in turn, into two, depending respectively on whether the response is a question or a proposal, if it is student-centered; and the type of conceptual or procedural knowledge, if it is centered on the teacher. By coding teachers' feedback responses, we were able to organize their answers with these frameworks, and new categories emerged from the evidence with which we supplemented and refined these frameworks. With this, we achieved our first objective, in which we tried to corroborate if the frames of reference allowed us to order the teachers' feedback responses to the students' unexpected errors and strategies.

For our second objective, we established a joint analysis of teachers' feedback responses to errors and unexpected strategies. Our results show certain aspects of teachers' responses that, being common to errors and strategies, suggest that they may be associated with their beliefs about what mathematics is and how mathematics is learned and taught (Ernest, 1989). The results show a tendency to respond to errors and unexpected strategies in a teacher-centered way (explanations and justifications given by the teacher), thus, limiting the opportunities for students to discuss their mistakes and strategies. This type of response, from this group of teachers, can be associated with a view of the teacher's role as "explainer" and a view of learning as knowledge reception (Ernest, 1989). Therefore, a contribution of this study is to show that these feedback responses of mathematics teachers are independent of the stimulus that generated the response to the error or the strategy, and that, therefore, although they are immediate responses that arise spontaneously in the classroom, they obey certain general patterns of teacher behavior.

These results are consistent with those obtained by Son (2013), who found that future teachers, faced with student errors, do not usually listen to their students and prefer to give information and repeat the procedure until the students identify their errors. Similar results were obtained in studies on student responses to strategies (Son & Crespo, 2009), in which most future teachers focused their responses on explaining why the strategy worked.

Likewise, Son and collaborators (Son, 2013; Son & Crespo, 2009) claim that teachers' responses to unexpected errors and strategies are a consequence of the fact that the subjects of their studies were future teachers. Since our sample consisted of practicing teachers, one contribution of this study is to show that these reactions do not differ whether they are teachers in training or in service.

Unlike other studies (Sarkar Arani et al., 2017; Tulis, 2013), in this study, no negative responses on the part of the teachers (i.e., ignoring the error or criticizing the student) have been identified. Surely, the use of self-reports can induce the non-existence of this type of response.

Some authors question the reliability and validity of self-reports to investigate teachers' practices. However, as we argued above, these instruments have proven to be very reliable when teachers are asked about specific topics of their classroom practices. In this study, we asked 94 participating teachers about what happened in a recent lesson and about two specific aspects, and we obtained good quality information. However, we recognize that a limitation of the study is that the teachers did not necessarily provide us with information on the context in which their response arose, since the questions focused on "how did you react". It would also be important to know why he reacted that way. Typically, a teacher will respond to each interaction with their students differently, as teachers strive to meet the individual needs of students.

Our results also confirm the importance of mathematics teacher training programs to help teachers interpret and respond to unexpected situations in the classroom, assuming that these responses are of great importance in learning. This information is useful for teacher educators. It tells trainers that teachers need to incorporate better feedback into their teaching.

In particular, teacher training programs should train teachers to help students overcome their difficulties, through their responses to student errors and unexpected strategies.

Finally, this study also has implications for future lines of research. We intend to continue with studies that compare teachers' feedback responses to errors and unexpected strategies, before and after going through training programs, to show to what extent these responses are modified. We also intend to delve into the relationship between the feedback responses identified and the types of teachers' beliefs, a topic that will be of great interest for the improvement of in-service teacher training programs.

References

Ball, D. L., & Bass, H. (2003). Knowing mathematics for teaching. In R. Strässer, G. Brandell, & B. Grevholm (Eds.), *Educating for the future. Proceedings of an international symposium on mathematics teacher education* (pp. 159-178). Royal Swedish Academy of Sciencies.

Bishop, A. J. (2008). Decision-making, the intervening variable. In P. Clarkson & N. Presmeg (Eds.), *Critical issues in mathematics education* (pp. 29-35). Springer. https://doi.org/https://doi.org/10.1007/978-0-387-09673-5_3

Boaler, J., & Brodie, K. (2004). The importan, nature and impact of teacher questions. *North American Chapter of the International Group for the Psychology of Mathematics Education October 2004 Toronto, Ontario, Canada*, 774.

Brodie, K. (2011). Working with learners' mathematical thinking: Towards a language of description for changing pedagogy. *Teaching and Teacher Education*, *27*(1), 174-186. https://doi.org/https://doi.org/10.1016/j.tate.2010.07.014

Campbell, P. F., Rowan, T. E., & Suarez, A. R. (1998). What criteria for student-invented algorithms. *The teaching and learning of algorithms in school mathematics*, 49-55.

Cañadas, M. C., Gómez, P., & Pinzón, A. (2018). Análisis de contenido. In P. Gómez (Ed.), *Formación de profesores de matemáticas y práctica de aula: conceptos y técnicas curriculares* (pp. 53-112). Universidad de los Andes. http://funes.uniandes.edu.co/11904/

Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. *Educational Researcher*, *38*(3), 181-199. https://doi.org/https://doi.org/10.3102/0013189X08331140

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics Teaching: The State of the Art* (pp. 249-254). Falmer Press.

González, M. J., & Gómez, P. (2018). Análisis cognitivo. In P. Gómez (Ed.), *Formación de profesores de matemáticas y práctica de aula: conceptos y técnicas curriculares* (pp. 113-196). Universidad de los Andes. http://funes.uniandes.edu.co/11905/

Guo, W., & Wei, J. (2019). Teacher feedback and students' self-regulated learning in mathematics: A study of Chinese secondary students. *The Asia-Pacific Education Researcher*, *28*(3), 265-275. https://doi.org/https://doi.org/10.1007/s40299-019-00434-8

Hattie, J., & Timperley, H. (2007). The power of feedback. *Review of Educational Research*, *77*(1), 81-112. https://doi.org/https://doi.org/10.3102/003465430298487

Herbst, P., Chazan, D., Kosko, K. W., Dimmel, J., & Erickson, A. (2016). Using multimedia questionnaires to study influences on the decisions mathematics teachers make in instructional situations. *ZDM*, *48*(1), 167-183. https://doi.org/10.1007/s11858-015-0727-y

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-23). Lawrence Erlbaum Associates.

Ma, L. (1999). *Knowing and teaching mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Lawrence Erlbaum Associates. https://doi.org/https://doi.org/10.4324/9781410602589

Movshovitz-Hadar, N., Zaslavsky, O., & Inbar, S. (1987). An empirical classification model for errors in high school mathematics. *Journal For Research in Mathematics Education*, *18*(1), 3-14. https://doi.org/https://doi.org/10.2307/749532

Rico, L. (1995). *Conocimiento numérico y formación del profesorado*. Universidad de Granada.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology*, *91*(1), 175-189. https://doi.org/https://doi.org/10.1037/0022-0663.91.1.175

Santagata, R. (2005). Practices and beliefs in mistake-handling activities: A video study of Italian and US mathematics lessons. *Teaching and Teacher Education*, *21*(5), 491-508. https://doi.org/https://doi.org/10.1016/j.tate.2005.03.004

Sarkar Arani, M. R., Shibata, Y., Sakamoto, M., Iksan, Z., Amirullah, A. H., & Lander, B. (2017). How teachers respond to students' mistakes in lessons. *International Journal for Lesson and Learning Studies*, *6*(3), 249-267. https://doi.org/10.1108/IJLLS-12-2016-0058

Schleppenbach, M., Flevares, L. M., Sims, L. M., & Perry, M. (2007). Teachers' responses to student mistakes in Chinese and US mathematics classrooms. *The Elementary school journal*, *108*(2), 131-147. https://doi.org/https://doi.org/10.1086/525551

Son, J.-W. (2013). How preservice teachers interpret and respond to student errors: ratio and proportion in similar rectangles. *Educational Studies in Mathematics*, *84*(1), 49-70. https://doi.org/https://doi.org/10.1007/s10649-013-9475-5

Son, J.-W. (2016). Moving beyond a traditional algorithm in whole number subtraction: Preservice teachers' responses to a student's invented strategy. *Educational Studies in Mathematics : An International Journal*, *93*(1), 105-129. https://doi.org/10.1007/s10649-016-9693-8

Son, J.-W., & Crespo, S. (2009). Prospective teachers' reasoning and response to a student's non-traditional strategy when dividing fractions. *Journal of Mathematics Teacher Education*, *12*(4), 235-261. https://doi.org/https://doi.org/10.1007/s10857-009-9112-5

Son, J.-W., & Sinclair, N. (2010). How Preservice Teachers Interpret and Respond to Student Geometric Errors. *School Science and Mathematics*, *110*(1), 31-46. https://doi.org/https://doi.org/10.1111/j.1949-8594.2009.00005.x

Star, J. R. (2005). Reconceptualizing Procedural Knowledge. *Journal, for Research in Mathematics Education. 36*(5), 404-411.

Tulis, M. (2013). Error management behavior in classrooms: Teachers' responses to student mistakes. *Teaching and Teacher Education*, *33*, 56-68. https://doi.org/http://dx.doi.org/10.1016/j.tate.2013.02.003

Uysal, H. H. (2012). Evaluation of an in-service training program for primary-school language teachers in Turkey. *Australian Journal of Teacher Education*, *37*(7), 14-29, Article 2. https://doi.org/http://dx.doi.org/10.14221/ajte.2012v37n7.4

Varela, F. J., & Shear, J. (1999). First-person methodologies: What, why, how. *Journal of Consciousness studies*, *6*(2-3), 1-14.

Voerman, L., Meijer, P. C., Korthagen, F. A., & Simons, R. J. (2012). Types and frequencies of feedback interventions in classroom interaction in secondary education. *Teaching and Teacher Education*, *28*(8), 1107-1115. https://doi.org/https://doi.org/10.1016/j.tate.2012.06.006