OBJECTUAL BELIEFS OF TWO PRESERVICE MATHEMATICS TEACHERS ABOUT TEACHING GEOMETRIC TRANSFORMATIONS WITH GEOMETER'S SKETCHPAD

CRENÇAS OBJETUAIS DE DOIS PROFESSORES DE MATEMÁTICA EM FORMAÇÃO SOBRE O ENSINO DE TRANSFORMAÇÕES GEOMÉTRICAS COM O GEOMETER'S SKETCHPAD

Received: 06 December 2017
Accepted: 31 July 2018

Shashidhar Belbase

shashidhar.belbase@zu.ac.ae

Zayed University

ABSTRACT

The purpose of this paper is to present pre-service high school mathematics teachers' objectual beliefs about the use of Geometer's Sketchpad (GSP) for teaching geometric transformations (GTs). The participants of the study were two senior undergraduate pre-service high school mathematics teachers at a public university in the United States of America. The study comprised a series of ten task-based interviews, five with each participant. I conceptualized radical constructivist grounded theory (RCGT) with five assumptions to guide the research process outlining the relationship between the researcher and participants, as well as emergent process of data construction, analysis, and interpretation. The results include three in vivo categories concerning GSP as an object of teaching GTs. The categories were – interface between algebra and geometry, the semantics of GTs with GSP, and syntactic of GTs with GSP. I addressed pedagogical implications of these categories at the end.

Keywords: Mathematics Teachers' Beliefs, Objectual Beliefs, Technology Integration, Radical Constructivist Grounded Theory (RCGT), Geometer's Sketchpad (GSP), Syntactics of GTs with GSP, Semantics of GTs with GSP.

RESUMO

O objetivo deste artigo é apresentar as crenças objetuais dos professores de matemática de ensino médio em formação sobre o uso do Geometer's Sketchpad (GSP) para o ensino de transformações geométricas (TG). Os participantes do estudo foram dois alunos dos últimos períodos de licenciatura em matemática para o ensino médio de uma universidade pública dos Estados Unidos da América, em prática de ensino. O estudo compreendeu uma série de dez entrevistas baseadas em tarefas, cinco com cada participante. Eu conceituei a teoria fundamentada no construtivismo radical (TFCR) com cinco pressupostos para guiar o processo de pesquisa delineando a relação entre o pesquisador e os participantes, bem como o processo emergente de construção, análise e interpretação de dados. Os resultados incluem três categorias in vivo relativas ao GSP como objeto de ensino das TG. As categorias foram - interface entre álgebra e geometria, a semântica das TG com GSP, e a sintática das TG com GSP. No final, abordei as implicações pedagógicas dessas categorias.

Palavras-chave: Crenças de professores de matemática, crenças objetuais, integração de tecnologia, teoria fundamentada no construtivismo radical (TFCR), Geometer's Sketchpad (GSP), sintática de transformações geométricas com GSP, semântica de transformações geométricas com GSP.

1. Introduction

At first, I introduce the context of technology integration in mathematics education, geometric transformation in school mathematics, application of Geometer's Sketchpad (GSP) in school mathematics, technology integration in mathematics education, beliefs about technology integration, beliefs about the use of GSP, and framed purpose and research question. Second, I discuss the theoretical frame of radical constructivist grounded theory (RCGT) by introducing five assumptions. Third, I outline the methodological trail. Fourth, I present the results with three categories of beliefs. Finally, I present conclusion and address some pedagogical implications of the study.

1.1 Context of Technology Integration

There is an ongoing debate about technology in every sector of life including business, industry, transportation, tourism, science and engineering, and education. Mathematics education is not beyond the dialogue about what technology to use in teaching and learning mathematics, how to use the technological tools for meaningful teaching and learning, and why to invest in resources for technological advances in mathematics education. In this context, the National Council of Teachers of Mathematics (NCTM, 2000) stated in its technology principle that use of technology enhances student learning. The Common Core State Standards for Mathematics (CCSSM) emphasized technology integration in mathematics education for making sense of mathematical problems, reasoning about the problems abstractly, modelling those problems, and enhancing algebraic thinking (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

Technology integration in mathematics teaching and learning has been prescribed in all European countries for investigations, reasoning, data analysis, developing mathematics concepts, teacher professional development, independent learning, and assessment (Parveva, Noorani, Ranguelov, Motiejunaite, & Kerpanova, 2011). The Singapore government emphasized technology integration in education by establishing technology-rich schools known as 'Future Schools' and helping them in leading educational institutions in the country (Lye & Churchill, 2013). The Mathematics Course Advisory Committee (CAC) of Western Australia's School Curriculum and Standards Authority (SCSA) emphasized computer algebra system (CAS) for algebraic manipulations (Kissane, McConney, & Ho, 2015). These examples indicate growing popularity of technology integration in mathematics education all over the world.

1.2 Geometric Transformation in School Mathematics

School mathematics curricula in different countries include geometric transformations. In the USA, geometric transformation for school mathematics has been recommended by NCTM (2000) and CCSSM (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). In India, the curriculum of the Central Board of Secondary Education has prescribed geometric transformations to be taught in school mathematics (CBSE, 2016). In the United Kingdom, the General Certificate of Secondary Education (GCSE) mathematics curricula also includes transformation geometry as an integral part of school mathematics (Department of Education, 2014). School mathematics curricula of Nepal has mandated teaching and learning of geometric transformations and its types in grades 6-8 at the introductory level in general mathematics and more depth in optional mathematics (CDC, 2012). These examples clearly demonstrate the scope of geometric transformation in school mathematics in different countries.

1.3 Application of GSP in School Mathematics

Many researchers have explored the importance and effect of using GSP in teaching mathematics. A study by Nordin et al. (2010) used pedagogical usability criteria of student control, activities, objective oriented, application, motivation, value added, knowledge, and flexibility and response. This study found that the application of GSP for teaching and learning mathematics promoted students' higher order thinking. Rahim (2000) discussed classroom uses of GSP in pre-service mathematics teacher education in Ontario, Canada with the support of the Ministry of Education. Participants in the study found GSP useful for graphical representation of different mathematical functions, for example, the derivative of a function as a slope.

Use of GSP for teaching geometry or other mathematics contents by teachers depends on their ability and confidence to use the tool, and general beliefs about the tool (Shafer, 2004). In a recent study by Kim (2016), the pre-service mathematics teachers believed that GSP is convenient for the construction of geometric objects (e.g., polygons), measurements, and proofs. These studies and others (e.g., Ramli & Mustapha, 2014) reported a positive effect of using GSP on students' learning of mathematics, and it serves as a vehicle to transform teacher beliefs and practice (Jiang, 2011; Shafer, 2004). Hence, studies show there is growing motivation to use GSP for teaching and learning mathematics.

1.4 Technology Integration in Mathematics Education

Technology integration in education in general, and mathematics education in particular, has increased over the last few decades. The meaningful use of technology for teaching and learning largely depends on teachers' knowledge and beliefs about the tools. Some researchers (e. g., Ertmer, 2006) agreed that teachers' beliefs affect the way they use technological tools for teaching mathematics. They also found that proper use of technological tools plays a decisive role in mathematics education. Use of technology in teaching makes a lesson interesting to students with a deeper sense of problem-solving, creative thinking, and reasoning (Ertmer et al., 2012). Therefore, use of technological tools such as computers, calculators, iPads, and other mobile devices make abstract mathematical concepts more accessible to students than may be possible with equations or formulas (Foley & Ojeda, 2007 as cited in Belbase, 2017). Utilization of these tools may help students to collaborate with each other or perform independent problem-solving. Hence, technology may play a significant role in helping students learn mathematics in a more meaningful way by developing positive attitude and images of the subject with fewer anxieties (Belbase, 2013).

The technological tools, for example, computers and iPads, help teachers to design lessons in a creative way by integrating audio, visual, and animation. Teachers can develop constructive and flexible mathematics learning tools for students (Garry, 1997). However, there is a challenge of technology integration in mathematics education without a change of teachers' mindset to the benefit of using technological tools (Chai, Wong, & Teo, 2011; Leatham, 2002; Wachira & Keengwe, 2011). This is because their beliefs are not uniform and not consistent with their classroom practices (Belbase, 2015a; Ertmer et al., 2012).

1.5 Beliefs about Technology Integration

There are different views about categorizing teacher beliefs about technology integration in teaching and learning mathematics. Some studies on teacher beliefs about technology integration (e. g., Lin, 2008) labelled technology related beliefs as-- no technology use, pre-mastery use, post-mastery use, and exploratory use (Misfeldt et al., 2016). Other studies (e. g., Erens & Eichler, 2015) identified four categories of teacher beliefs about technology integration-- initiator, explorer, reinforcer, and symbiotic collaborator. Chen's (2011) categories of teacher beliefs about technology integration are-- instrumental (technology is just an instrument to solve mathematics problems) and empowerment (technology enhances the power of visualizing, representing, and complex problem-solving). Most of the studies on mathematics teachers' beliefs about technology integration discussed positive and negative beliefs or constructivist and instrumentalist beliefs (Belbase, 2017; Goldin, 2002). Also, there are studies which examined teacher beliefs based on functions of technology in the mathematics classroom, for example, technology for thinking and reasoning, technology for meaning, technology for an extension, and technology for contextualization of content and pedagogy (Polly, 2015 as cited in Belbase, 2017).

A positive belief about technology integration in mathematics teaching and learning may originate from teachers' knowledge of content of mathematics, pedagogical knowledge, and knowledge of using technology. According to Mishra and Koehler (2006), this combination forms technological-pedagogical-content-knowledge (TPACK) which is considered as an integral part of classroom practices. With such a need, TPACK becomes an important aspect of teacher education and training to enhance efficient use of technological tools in the classroom (Hunter, 2015). Cuevas (2010) suggests that use of technology in the mathematics classroom may take students "into the domain of nonroutine tasks" (p. 374) with TPACK. Teachers should be able to integrate technology with paper and pencil activities so that students have the opportunity to learn concepts and procedures both by hands-on and technological constructions. While doing this, teachers may use technology in mathematics teaching as an exploring, connecting, and thinking tool (Cuevas, 2010). These skills may play a significant role in developing positive beliefs about technology integration among mathematics teachers for the meaningful teaching of mathematics (Belbase, 2015b & 2017; Misfeldt et al., 2016).

1.6 Beliefs about the Use of GSP

There are a few studies about pre-service or in-service mathematics teachers' beliefs about the use of GSP for teaching and learning mathematics. Leatham (2002) used GSP as one of the technologies among others to study teacher beliefs. He found that pre-service teachers considered using technological tools, including GSP, for dynamic illustration of the relationship between variables and facilitation of exploring mathematical concepts. Building on Leatham (2002), Shafer (2004) studied the effectiveness of using GSP in teaching mathematics. Results indicated that teaching with GSP was not significantly different from teaching mathematics in the traditional setting, although the participants had a positive feeling about the use of GSP in the classroom. The pre-service teachers who have positive beliefs about the use of GSP consider that GSP helps students in exploring, reasoning and proving, constructing, measuring, and calculating (Kim, 2016). However, these beliefs may not guarantee the effective use of the tool for teaching and learning mathematics (Belbase, 2015b; Kim, 2016).

1.7 Purpose and Research Question

The studies mentioned above are not sufficient to understand teacher beliefs about technological tools as objects of teaching and learning mathematics to build inter and intra-subject matter connections, differentiation of meaning, and structural organization of the concepts. This study aims to complement the literature on teacher beliefs about technology integration in mathematics education. The research question for the study is - What beliefs do pre-service high school mathematics teachers hold about teaching geometric transformations using Geometer's Sketchpad? This paper draws on a significant part of the author's doctoral dissertation research. Therefore, the method of the study, theoretical frame, data, and result come from the same work.

2. Theoretical Framework

What is the study about? What is the source of information? How are data constructed/collected? How are the data analysed? How are the findings interpreted? What is the relationship of the researcher with the participants? What is the relationship of the researcher with the data and findings? These questions seek a structure of thought process to guide the study input, process, and outcome. The structure to guide the inquiry process is a theoretical framework. Anfara and Mertz (2006) defined a theoretical framework "as any empirical or quasi-empirical theory of social and/or psychological processes, at a variety of levels, that can be applied to the understanding of phenomena" (p. xxvii). This definition of theoretical framework excludes the notion of paradigms and methodological issues because it focuses on the social and psychological processes. Such theoretical framework may originate from the integration of different epistemologies, methodologies, and processes. For this study, I conceptualized a theoretical framework of radical constructivist grounded theory (RCGT) by synthesizing five assumptions from radical constructivism (Von Glasersfeld, 1991 & 1995) and grounded theory (Charmaz, 2006; Strauss & Corbin, 1998). These assumptions are related to symbiosis, voice, cognition, adaptation, and praxis (Belbase, 2013, 2014, 2015a, 2015b, 2016, & 2017). I discuss each of them in the following sub-sections.

2.1 Symbiosis

First, I assumed that the researcher and participants have a symbiotic relationship with mutualism, so both will benefit both from the research process. Epistemology of radical constructivism (Steffe & Thompson, 2000; Von Glasersfeld, 1991 & 1995) and methodology of grounded theorists (Charmaz, 2006; Corbin & Strauss, 2008; Glaser & Strauss, 1967) integrated to define a mutual collaboration between the researcher and participants, thus creating a symbiosis of the research process. In qualitative studies, a researcher may construct data through a variety of methods-- teaching experiments (Steffe, 2002; Steffe & Thompson, 2000), clinical task-based interviews (Goldin, 2000; Maher, 1998), ethnographic observations, etc. to name a few. In these processes, there is a transactional relationship between the researcher and participants. In this study, I applied clinical task-based interviews to gather data, and the participants gained experience and knowledge of using GSP for teaching GTs.

2.2 Voice

Second, I assumed that a qualitative research embraces researcher and participants' voices. The participants' voice is presented through vignette, protocols, narratives, life stories, to name a few, in the forms of different expressions. The researcher's voice is presented through interpretive accounts and reflexivity (Hertz, 1997). Here, reflexivity of the researcher may relate to his or her sense, awareness, and consciousness to the issues through a deeper abstraction of meanings (Belbase, 2015). Both grounded theorists (e. g., Bergkamp, 2010; Warfield, 2013) and constructivists (Charmaz 2006; Warfield, 2013) consider researcher's reflexivity as an integral part of interpreting and theorizing of concepts from data while keeping the participants' voice up front (Mauther & Doucet, 2003; Pierre, 2009).

2.3 Cognition

Third, I assumed that doing qualitative research is a cognitive function because it involves constructing data, coding the data, categorizing concepts, implementing theoretical sampling, comparing codes and categories, and memoing as active mental processes (Morse, 1994). The researcher performs in situ data construction (Friedhoff et al., 2013) and coding and categorizing of data side by side (Clarke, 2005). The participants build up their experiences through a series of interactive task situations. The researcher keeps on performing theoretical memoing (Charmaz, 2006) adding new questions and updating tasks for the participants to explore new categories as cognitive processes (Bailyn, 1977).

2.4 Adaptation

Fourth, I assumed that research process, in a radical sense, is an emerging process in which both participants and researcher adapt to new experiences, contexts, and challenges (Belbase, 2015 a & b, 2016 & 2017). The conceptualization of meaning to construct new codes and categories is a continuous self-adaptive process until there is a time when no new concepts emerge with additional data (Glaser & Strauss, 1967; Lichtenstein, 2000). In this sense, a study in RCGT is associated with construction of conceptual categories from data as an adaptation to new meanings grounded on data (Layder, 1998; Welsh, 2009).

2.5 Praxis

Finally, I assumed that the outcomes of analysis and interpretation of data should resonate with the phenomenon under study. The quality of research outcomes as categorical or thematic constructions can be examined with criteria of fit and viability, which is praxis. The criterion of fit demonstrates the applicability of categories to describe and interpret the phenomenon under study (Charmaz, 2006; Glaser & Strauss, 1967; Strauss & Corbin, 1998; Von Glasersfeld, 1991

& 1995). The criterion of viability demonstrates the usefulness of categories in a similar context (Rodwell, 1998; Von Glasersfeld, 1995).

3. Method

3.1 Participants

I purposefully selected two participants who were pre-service high school mathematics teachers in a university in the Rocky Mountain Region of the U. S. There were no selection criteria other than access to them, their availability for interviews, and their interest to volunteer in the study. One of them was a male participant (Jack) who was a returning student to the College of Education after working for a private company for more than five years. He wanted to be a mathematics teacher for middle grades. Another was a female participant (Cathy) who was a continuing or regular senior undergraduate student. She was an English language learner in her primary grades. Her ideal grade level for teaching is senior high school level. None of them had prior teaching experience other than during micro teaching in the methods of teaching mathematics class and later during the practicum. I used Jack and Cathy as pseudonyms to protect their identity.

3.2 Interviews

I constructed an interview protocol to help me in the interview process. The protocol was a primary guide to start the interview questions based on some tasks and interactions throughout the tasks (Goldin, 2000; Maher & Sigley, 2014). There were four major tasks in the separate interviews. The first was the construction of an object and its image under reflection on a line by using paper and pencil, geoboard, and then by using GSP. The second task was the construction of an object and its image under rotation on a point by using paper and pencil and then by using GSP. The third task was the construction of an object and its image under translation with directed line segment (a vector) by using paper and pencil, geoboard, and then by using GSP. The fourth task was the construction of an object and its images under composite of reflection followed by a rotation, reflection followed by a translation, and rotation followed by a translation by using paper and pencil and then by using GSP. Figures 1a and 1b indicate examples of tasks done by the participants-- paper-pencil, geoboard, and then in GSP. Each of these analytical tasks was followed by a subsequent interaction and an interview with each participant. The fifth interview was to confirm their beliefs expressed during the earlier four interviews. Each interview lasted between 30 and 90 minutes. I conducted the first two interviews myself. A fellow graduate student helped me to perform the rest of three interviews so that I could observe the process and add to the interview questions as necessary. I video recorded the interviews and transcribed verbatim for analysis and interpretation.

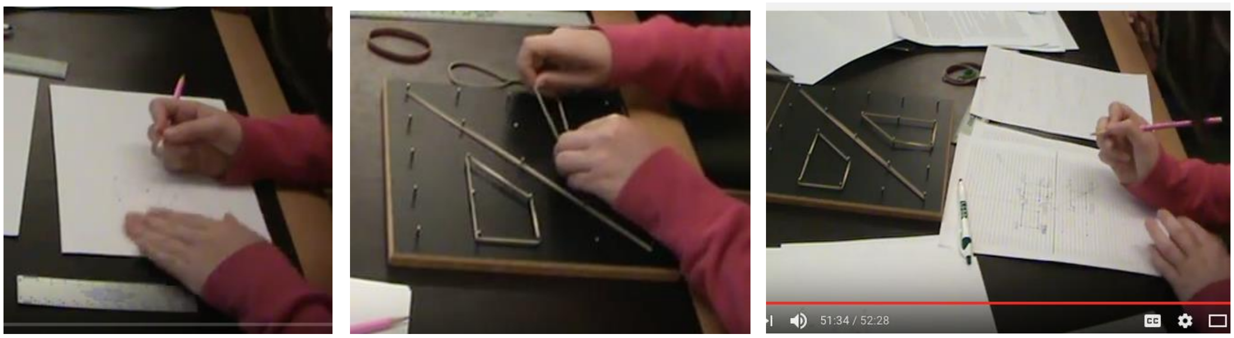

Figure 1a. Exploring Geometric Transformations with Paper-Pencil and Geoboard (by Cathy)

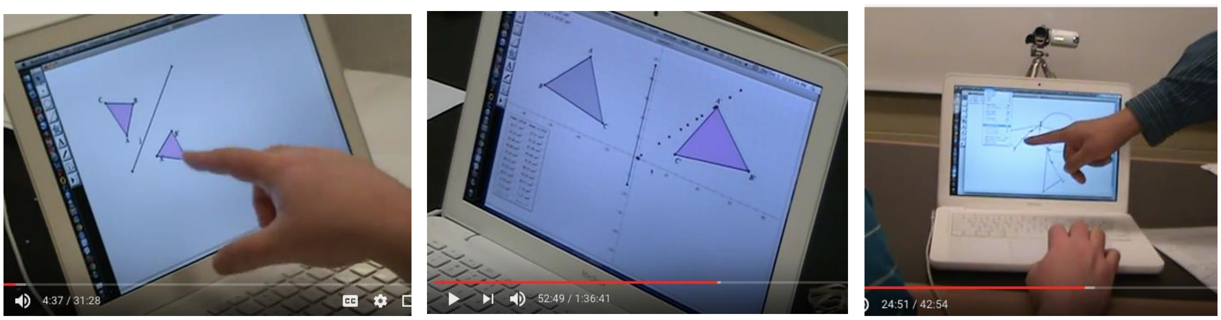

Figure 1b. Exploring Geometric Transformations with Geometer's Sketchpad (by Jack)

3.3 Memoing

After each task-based interview, I wrote a reflective theoretical note with major points, nonverbal expressions, and my reflection on the interview (Lempert, 2007). After each interview, I stepped back and asked myself – "What is going on here? How can I make sense of it?" (Thornber & Charmaz, 2014, p. 163). The process of memoing complemented the analysis of data by identifying preliminary codes in the data.

3.4 Analysis

The analysis of the raw data went through the process of identifying the primary codes based on the related meanings coming from the pieces (words, sentences, and paragraphs) of data. I arranged the primary codes in a group of secondary codes to form categories based on their conceptual similarities (Charmaz, 2005; Strauss & Corbin, 1998). I also used the preliminary codes from the interview memos to link other codes into categories. I constructed the primary codes from the first interview data before conducting the second interview. I modified some questions in the second interview to focus on the categories that I constructed from the first interview. Likewise, I completed the analysis of an interview data before the next interview so that I could add questions, remove unnecessary questions, and modify some questions so that data from subsequent interviews helped in saturating the codes. The analysis was an iterative process (Morse, 1994; Thornber & Charmaz, 2014). Therefore, analysis of data first involved the process of constructing primary codes by breaking down data into conceptual pieces. These conceptual pieces were names with specific codes. This was a reductive approach to surgically zooming onto the concepts within the body of data. After segregating the codes with similar meaning, I reorganized these codes into groups of codes and integrated them into categories. I integrated similar categories to final categories (Charmaz, 2006; Clarke, 2005). From this inductive process of combining concepts to codes and codes to categories, I was able to construct three final categories as themes. These categories were-- interface between algebra and geometry, the semantics of GTs with GSP, and syntactic of GTs with GSP.

3.5 Interpretation

I interpreted the categories in three layers. In the first layer, I reconstructed interview transcripts, bringing the pieces of interviews to the respective categories. While doing this, I reconstructed the participants' narrative by bringing the fragmented conceptual units together in a vignette for each category. In the second layer, I identified key points from the vignette and discussed them. In the third layer, I linked the major concepts from the vignettes to the relevant literature as an extended interpretation in the discussion of the categories. Hence, I employed emphatic interpretation by entering myself into the context through these layers to understand meanings, concepts, and their interrelationship from within the data (Willig, 2014). The interpretation was focused on making sense of their beliefs about the use of GSP as an object in teaching GTs. Such beliefs have been termed as objectual beliefs in the literature, because they emphasize properties of the object of beliefs, here in this research it was GSP for teaching GTs (Audi, 2015). Construction of these categories was not a straightforward linear process. I tried to observe and dig deeper into the meaning of belief concepts, codes, and categories while constructing the final categories of objectual beliefs from the data. The construction of these categories needed the vision to see beyond what is visible in front of me as the data. It took me to a position to search meaning that was not obvious, explicit, and crystal clear. I embraced vagueness, complexity, and uncertainty throughout this journey.

4. Results

The results of the entire study included six kinds of belief constructs related to action, affect, attitude, cognition, environment, and object. In this paper, I reported only one construct-- beliefs associated with the object of teaching GTs with GSP in the form of objectual beliefs. This construct has three categories – interface between geometry and algebra, the syntactic of GTs with GSP, and the semantics of GTs with GSP (Fig. 2). Other constructs have been commissioned in other publications. I discussed each of these categories in the following sub-sections.

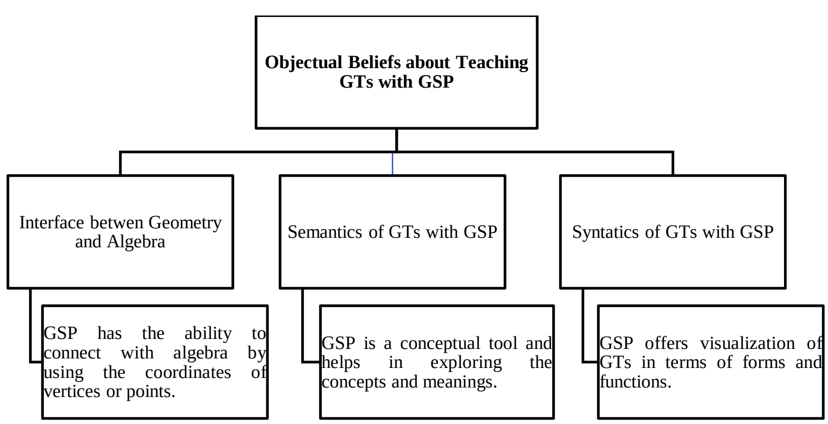

Figure 2. Categorization of Objectual Beliefs and Their Characteristics

4.1 Interface between Geometry and Algebra

The participants expressed their beliefs about the interface between geometry and algebra in terms of the hands-on side while working with geometry and minds-on side while manipulating algebraic relationship associated with processes of GTs during teaching GTs with the use of GSP. Some of the leading questions that I asked the participants within this category were—You constructed reflection of a polygon on a grid in GSP and in a paper. How would you represent reflection in algebra? Alright, so tell us something about GSP. Can we relate some properties of reflection transformations to algebra with GSP? Do you think you could explore some algebraic properties that are related to this example (Rotation)? What properties could we explore from these examples (of the polygon and its image under rotation or composite GTs)? Then, I constructed belief narratives of the participants about the use of GSP as a tool for teaching GTs where GSP could have both geometric and algebraic manipulations with the tools available in it.

4.1.1 Cathy's Belief Narrative on the Interface

I think, in the reflection of a point on Y-axis, a point (x, y) would change into (-x, y). If I write it into a matrix, it would be a negative one--zero (in the first row), and zero--one in the second row. In the reflection on X-axis, the y coordinate changes to negative. I think, the matrix would be zero- negative one (in the first column), one – zero in the second column. It (GSP) has abilities to show the picture side or the hands-on side of geometry. I haven't really explored algebra with it. I think it has the ability of the algebra. I don't know how far that goes. But, I think in geometry it has all of what we need, honestly, to teach geometry class. And, you can do like without a book. Instead of a book, we tell them what you are doing. You don't even need the book because they can create every single thing. Or, you can create and give them that document that they can play with it. Everything it would be in geometry class and possibly in algebra class. In a class, however, I have not discovered that far in algebra with GSP. We didn't do algebraic manipulations. I think it would really be helpful. I have explored a ton of algebra except it is in GSP. I have done that more in GeoGebra. I think this (GSP) does have enough that you could be able to do that (algebraic manipulation). I can relate (x, y) coordinates to (-x, y) as a generalisation of reflection under Y-axis. It is possible for other transformations too.

4.1.2 Jack's Belief Narrative on the Interface

We can write coordinates of points to reflect. It is then connected to algebra. You can put them in a matrix. I am not sure what matrix could be, but yes, there should be one for each. [After putting some vertices and their images in coordinate points into a matrix, he struggles to find the exact matrix for a while. With trial and error for a while, he gets matrices of reflection on the Y-axis and X-axis.] I was not introduced to the matrix for transformation in high school besides simple adding, subtracting and multiplying. If students are ready for the matrix with some knowledge of basic stuff, then it is possible for them to discover it (GT). If they don't have experience of basics matrices, absolutely, it is not possible for them to make this connection in GSP. Matrices are awesome ways to bridge algebra and geometry.

Oh, I like GSP. Have you guys used GeoGebra? GSP is a lot cooler than GeoGebra. It costs more, cause GeoGebra is free, but just the thing you can do with it, like overlaying the coordinate grid in GSP and visualizing everything right on it. I really liked that. I like how you can do the detail of the constructing things like on the Ferris wheel. Just how you can show them, show how all those stuffs are constructive. If I made right angles, you could do Pythagorean theorems or stuffs like that. But, you can just talk about distances, solving for a side, setting them equal. You can do all sorts of different things. I mean, the fact is that if you like to do a measurement and those stuffs, it's cool. You can see what that algebra means to, without focusing much on the process so much. We can do a little conceptual understanding. That means they still need to take time to explore the process of where that comes from. They need to be comfortable with that before you give it to them. They (students) will be able to use the coordinates. Um, you know, think about how long I took just to count those tiny squares. Here (in GSP), you can go to the coordinates. Then, all of a sudden, you can get to the algebra. You can do it as a part of doing algebraic manipulation. You can solve for distance or side lengths as algebraic manipulation with GSP.

Cathy believes that reflection transformation could be understood with folding a paper into a half and the line of fold works as a mirror. She thinks that the coordinates of reflection can be expressed in a matrix algebra. After defining reflection and working on GSP to construct coordinates and writing them into a matrix for reflection on X and Y-axes, she demonstrated her consolidated belief that an algebraic manipulation of reflection is possible with GSP. However, forming the matrix was not the role of GSP. She wrote the matrices for reflections manipulating relationship between object and image points in terms of coordinates. Cathy found that GTs functioned as an interface between visual and mental process with the use of GSP. She referred that GSP had the ability to demonstrate the picture side of the structure or the hands-on side of manipulating geometry. Her view of the picture side was related to a physical aspect of doing GTs with visualization. Next, her view about the hands-on side was related to the mental aspect of physical manipulation of geometric constructions of GTs with GSP. She was not quite sure about how algebra was related to GTs. She even did not know how far the connection to algebra could go in the classroom. Nonetheless, she accepted that GSP is useful in algebraic manipulation of GTs. However, she was not confident in making this connection. She believed that she was not ready to use GSP for teaching concepts of a GT as an interface between geometry and algebra. She explored coordinates and matrices of reflection and translation. She related (x, y) coordinates to (-x, y) as a generalisation of reflection under Y-axis. She considered that such exploration is possible for other transformations too.

Jack, on the other hand, considered that GSP was a complete tool for teaching and learning of GTs. He accepted that the use of GSP could even replace the textbook. A teacher could use GSP for doing all kinds of activities including construction and manipulation of GTs without even opening the textbook. He believed that GSP had a possibility of being used in an algebra class, too. In that sense, he even asserted that GSP could be used for the algebraic manipulation of different GT processes. However, he did not feel confident in using the tool because of the limited experience in exploring algebra with GSP. He believed that GSP could provide a constructive teaching and learning environment with options to show or hide technological details of the interface between visual and mental. He also thought that GSP could help in building algebraic manipulation of GTs with coordinates. Jack anticipated that Pythagorean theorem could be used as a part of doing the algebraic manipulation of GTs with GSP. He considered solving for distance or side lengths as algebraic manipulation with GSP.

Both Cathy and Jack constructed object and image of polygons under different GTs with GSP. They expressed that algebraic manipulation could help students make sense of the geometry of a GT. Further, they considered that it was possible to do algebraic manipulation of GTs with GSP with basic knowledge of coordinate geometry and matrices. Therefore, they believed that manipulating different GT process with GSP could provide an interface between geometry and algebra.

4.2 The Semantics of GTs with GSP

The research participants shared their beliefs about the semantics of GTs with GSP with the researcher, and these beliefs were related to procedural, conceptual, and inter-connected meanings of the GT processes with the dynamic feature of GSP. Some of the leading questions related to this category were— What does reflection transformation mean to you? Do you like or dislike the features in GSP for teaching Reflection or any GTs? So, what features about GSP do you like? Where does it fit better, towards a procedural tool or is it a conceptual tool? The meaning of one's actions of teaching and learning GT with GSP is related to the semantics of the processes associated with them. The meanings of different GTs with GSP are an integral part of using the tool for conceptual and procedural clarity of the processes.

4.2.1 Cathy's Belief Narrative on Semantics of GTs with GSP

I think a reflection is like folding a paper into half, and you get a line (She folds a sheet of paper to demonstrate a line of fold). One side of the paper is an image of the other side under a reflection under the line. So, then draw a half of a butterfly at one side, fold it to other side and mark on the object to get the image as a reflection on it. Another example could be two halves of a sandwich. I like them (features in GSP) a lot. The only thing I don't want is that it does not tell you what you are doing really. It is just rotating, whatever that means, and it is just rotating. Um, I really like all of them. Honestly, the constructions are constructions, and they are not freehand. Something like you can go to the square, it's square. I can go and see that it is, in fact, a square. Like, you can't fool any (person). It's construction. That is probably the biggest thing that I like about it (GSP). I like that you can get immediate responses. So, I guess without having to go through it and find the area of each triangle, that's not the point. You can get all that stuff. There is like a lot of built in software (program) in GSP. I would only use it as a conceptual tool because I want the procedure to have been down (to be understood) before using it. Because it escapes the procedure really, it's short-cut. It's definitely the concept. Have the procedure down, now let's discover the concept. I would say conceptual because it is not doing the procedure.

4.2.2 Jack's Belief Narrative on Semantics of GTs with GSP

Reflection is about constructing an image of an object to the other side of a line, like looking on a mirror. While introducing reflection in a class, probably, I would fold a paper and show them (students). The crease of fold is like a line of reflection. It is a simple way to introduce reflection before using GSP. With GSP we can do a lot more. I love the animation feature in GSP. Because you can actually show them (students) what's happening. I like how you also can construct. It's not just, oh I make a circle. There is a not a square button (tool). You know, you have to build on it. It does not eliminate and give them a free thing. They still have to learn the ideas. It is useful with the pencil and paper and then they can do more with it. So, I really like that. I think it's more conceptual. The fact that they can see the concepts and like animation and moving, that's the bigger role I think. You can teach construction (procedure) with a compass and a ruler. This is a lot you could speak it up, see it, and hide it. Not erase it. GSP is a tool for conceptual understanding. It skips a lot of steps. It has short-cuts. GSP has a learning curve, just they (students) have to know what they trying to do with it. If you show stuff (GTs) in coordinate grid over, it is a lot easier for them (students) cause they can see it on the computer. If you have to choose one of from paper-pencil, geoboard, and GSP to do geometric transformation, I would choose GSP because you can do more with it. When you are translating this (a polygon on GSP), you can even see at what time it was here or there. I think, you can see the details of moving this to this position. Series of movement.

Cathy's definition of reflection (and other GTs) and processes (meanings) associated with them consolidated her beliefs from what she did in the task-situations. Cathy could make sense of GSP as a conceptual tool for teaching and learning GTs. She believed that construction and manipulation were parts of creating meaning of GT processes with GSP. The built-in program in GSP was helpful for making this connection among GTs, both visually and mentally. She considered that building an integrated tool in GSP for the direct construction was an additional creative and constructive feature in GSP which was not available in other similar programs, for example, GeoGebra. However, she believed that GSP tools are not freehand, but one needs to build on them. Hence, she believed that GSP had a semantic domain (e. g., construction, manipulation, and animation) leading to the conceptual understanding of GTs.

On the other hand, Jack liked the dynamic animation feature in GSP. With this feature, he believed that a teacher could show the students what was happening with GTs. He believed that some of the tools in GSP were related to constructions that were not a free thing, for example, construction of a square without starting from the points and lines, but by using the constructing tool for a square. He believed that there was not a square button (tool) to click directly, but it could be created by the user. He considered that GSP could be more useful in conjunction with physical manipulation of paper and pencil activities. However, he thought that manipulation of GT concepts could be easier with GSP compared to the physical hands-on materials. Hence, he believed that developing conceptual understanding (meaning) of GTs was the biggest role of GSP.

Cathy considered that constructions were simply constructions without meaning unless one could interpret the process. For her, GSP was a semantically neutral tool because, for her, it was not the GSP that constructs meaning, but it was teacher or students to create meanings out of GT processes with GSP. However, she accepted that the meaning of a GT would be more explicit with GSP because, for her, it is a conceptual tool. On the other hand, Jack believed that animation and other dynamic features in GSP could help students make sense of a GT. That means, for him, the use of GSP could provide a greater semantic interpretation of the processes related to different GTs. Therefore, Cathy and Jack demonstrated their consolidated beliefs about semantics of GT with GSP.

4.3 The Syntactic of GTs with GSP

The participants expressed their beliefs about teaching GTs with the use of GSP regarding the structural organization. The syntactic of GTs with GSP has been considered as the internal features and structures that provide flexible, dynamic, and constructive environment inherent in GSP. Some of the leading questions that led the construction of this category were— What else we can do with GSP? Is GSP a tool for doing constructions or doing drawings? Why? Do you think that the construction of an object and image under a GT with GSP is more visible, meaningful, and maybe powerful? The syntactic of GTs with GSP can help teachers and students to understand the nature and functions of organizational structures within the processes of a GT.

4.3.1 Cathy's Belief Narrative on Syntactic of GTs with GSP

I think we can plot points in GSP. With GSP, you don't have to see and count points, I mean the distance. I can count, but you don't have to take time to count everything. GSP does it for you and it is easier to see what's going on. You can move this (holds a vertex of a triangle and drags it) and see the changes. It (GSP) should be a tool for the construction and demonstration of structures of GTs. It is totally possible that it is used for drawing to show patterns within a GT. Constructions are a lot more in geometry, and that's what it is gonna make it and work throughout. I think, it might be more visible. But, I don't think it is more meaningful or more powerful. I think you constructing it with your hands with paper and pencil is an awesome fit, I guess. I think that would be more rewarding to the learner, actually make this instead of clicking a button on the computer and make it. I think, that's why I would go back to the paper and pencil to see that they actually get the conceptual. It won't take you that long to do that with paper and pencil. With a paper and pencil, you have to absolutely know how to do it. I would show them paper and pencil activity and solidify. GSP helps in multiple transformations to create designs. The multiple rotations and then reflection on a line is like an art. Since there are already built in tools for reflection, rotation, and translation, you don't have to rediscover them, but you are just using them for practice. So, it is a practice tool. It just demonstrates what you already know.

4.3.2 Jack's Belief Narrative on Syntactic of GTs with GSP

It (GSP) actually forces you to construct things. Yeah, I mean they are, not a lot powerful, but it's definitely more visual. I don't know that it is meaningful. I think more powerful, and it is more visual. You can hit meaning with a lot of different things. So, I don't know if GSP makes more on those, but it's definitely useful. I like the visualization. How you can visualize the angles, like you can put the angles in here so that they can see it right out there, and side lengths. I think that's cool. I think it really can help build on it. I think it helps with the concept of angles, especially when you draw something like this, where it is seen, which angle which, and what's happening. I think that helps with the concept of rotation. Most of them know what a rotation is just by the definition of the word. But, can they visualize it? Being able to draw the lines and measure these angles up here I think that really helps. The thing you can do with it, like overlaying the coordinate grid in GSP and visualizing everything right on it. I like how you can do the details in constructing things, like Ferris wheel.

Cathy considered that GSP made it easy to see points and how they change in structure (or form) when the user applied 'click and drag' on a vertex of an object. She appeared to believe that GSP made a change of the object and image size or distance easy and fast. She believed that the processes of GTs involved a lot of constructions with the use of tools in GSP. She further considered that constructions of a GT with GSP could demonstrate the concepts (of a content) more visibly. She reflected that GSP might not change the meaning of a concept, but it might help in understanding the structure through visualization. That means, for her, use of GSP does not make semantic differentiation of GT process, but it made the syntactic structures explicit (more visible). Hence, she believed that GSP was useful to demonstrate process structures (forms) of a GT. The structural phenomenon of GTs is visible with dynamic 'click and drag' feature in GSP.

Jack also believed that GSP was a syntactical tool because one could overlay the coordinate grid on GSP and visualize everything right on it. He believed that GSP was a tool for the construction, and it might force one to construct things in the dynamic environment to understand the underlying structures and processes of GTs. He considered that with GSP one can construct things to use it as an integrated tool. For example, he thought that one could build an actual square on GSP and save it as a tool to use later to construct other squares directly. He believed that the different features of GSP could be built on, hid within, and made it visible as necessary. For him, GSP as a teaching tool was not necessarily a powerful, but a meaningful visual tool.

Although Cathy thought that GSP was a tool for the construction and visualization, she did not believe that use of this tool was necessary to make sense of GTs. She contended that the use of GSP provided a better meaning and power of GTs. Jack also accepted that GSP was not a lot powerful and meaningful, but it was more visual tool for GTs. Hence, they seemed to believe that the use of GSP could offer a syntactic differentiation of the GT processes with visualization of forms, rather than changing power and meaning of GTs.

5. Discussion

The participants expressed beliefs can be re-interpreted by making extension and connection to the literature and theory of teacher beliefs. Some key points from their beliefs about the interface between algebra and geometry are related to the visualization with pictures and hands-on sides of geometry with physical manipulatives, and the ability to integrate algebra with coordinates and matrices to the geometry. The other key point related to this category was about using GSP for algebraic manipulation by developing algebraic structures of a GT with problems around sides, angles, and others. Some researchers (e.g., Erbas et al., 2005) offered ideas about algebraic manipulation using GSP and showed how algebra and geometry overlap and produce multiple solutions. They explored the emerging patterns of mathematical concepts for visualizing and modelling the problems in the algebra-geometry interface. Other researchers considered that students could use GSP to explore linear relationships among different properties of curves (e. g., parabola, ellipse, and hyperbola), and the gradients of functions (Meng, 2009; Meng & Sam, 2013). The students can learn the interplay between geometrical and algebraic properties and representations of GTs by using GSP (Steketee & Scher, 2012).

Some key points from participants' beliefs about the semantics of GTs with GSP were related to cognitive abilities associated with semantics to construct the meaning of a GT with GSP. There are some semantic processes of reflection, rotation, translation, and a composite of them regarding “object of reference” (Hoong & Khoh, 2003, p. 46). Therefore, there should be a reference to an object and corresponding image regarding a line of reflection, a point of rotation, and a vector of translation to make sense of the GT processes. However, this aspect was not explicit in the participants' beliefs about properties of GTs at the beginning discussions during task-based interviews. It surfaced as an issue when we discussed these references in the last interview with backward thinking of GT process, for example finding the centre of rotation (Belbase, 2015b). Hence, we can use GSP for different semantic structures of GTs available within the tool (Bell et al., 1989; Nesher, 1988; Vergnaud, 1988). While teaching and learning GTs with GSP, the teacher can use the tool at three semantic layers. First, he or she can demonstrate the construction of object and image under a transformation. Second, he or she can help students identify the transformation process by observing and interpreting the relationship between object and image. And, third, he or she can help students to operate and connect multiple transformations (De Corte & Verschaffel, 1987; Wearne & Hiebert, 1988).

The participants' beliefs about the syntactic of GTs with GSP were related to the use of GSP in making syntactic of GTs explicit. However, Cathy believed that she could do it without using the technology. Therefore, for her, the use of GSP made the processes of GTs more visual, but not necessarily more meaningful and powerful. The participants' views showed a contrast in their beliefs about the use of GSP for understanding the syntactic of GTs. The use of GSP for demonstrating GTs, the processes of any GT breaks down into visual steps and structures with the help of 'click and drag' features that help students move from concrete to abstract through visualization (Wearne & Hiebert, 1988). Although the details in the background of the software is not visible to the users, the tools in GSP help them develop conceptual and procedural understanding of processes of GTs (De Corte & Verschaffel, 1987). Hence, GSP functions as a syntactic tool or artefact for meaningful teaching and learning of GTs.

6. Limitations

This study had some limitations in terms of data source, analytical approach, and scope of generalisation of findings. There were only two participants in this study. I conducted five task-based interviews with each participant to explore their beliefs about teaching GTs with GSP. In this context, the data was limited to two participants who did not represent general beliefs of other pre-service mathematics teachers. I used RCGT as a theoretical framework to guide the research input, process, and outcome. This framework profoundly guided the layered interpretive process. While doing this, I chose to embrace subtleties coming from the process with complexity and uncertainty and influenced by my ontological, epistemological, and axiological biases. Hence, the findings in the forms of three dimensions (or categories) of objectual beliefs -- interface, semantics, and syntactic of GTs with GSP have limited scope of generalisation as a theory of pre-service teachers' objectual beliefs.

7. Conclusion

I concluded from the study that both the participants had beliefs that GSP provides an interface between geometry and algebra. They worked both paper-pencil and GSP to observe the transition from geometry to algebra using the coordinate system and consolidated their beliefs about geometry-algebra interface. Their beliefs about the semantics of GTs with GSP evolved from their constructive action during backward moving in composite transformations. The meaning of the composite transformation became explicit with the identification of the centre of rotation, line of reflection, and vector of translation. Their beliefs about GSP as a procedural tool was weak and as a conceptual tool was strong. The beliefs about syntactic of GTs with GSP got more consolidated when they saw the structure of multiple transformations to form a pattern (like an art) or linear relation (with a constant rate of change) between the measurement of object and image size.

8. Implication

The results with the three categories have some pedagogical implications for moving across interfaces, semantic interpretations of GT processes, and syntactic of GT processes with GSP.

The interface between algebra and geometry can help teachers and students to develop a meaningful connection between these two content domains in school mathematics. Use of technological tools, for example, GSP for teaching GTs or any other related content may provide layers of semantic interpretations of such interfaces. The syntactic of a mathematics content and role of technology in this visual organization of the concepts can help in creating a positive teaching-learning environment. The findings and discussion also revealed that teaching and learning of mathematics with technology allows students and teachers to have a hands-on experience of the operational functions of technological tools, for example, GSP (Stols et al., 2008). Such experiences are related to constructions of geometrical figure associated with a math concept; movement of any part of the figure using the powerful tool to visualize the changes related to the idea; measurements of different variables; and manipulations of math concepts in the technological environment. The minds-on activity associated with a math concept can be conversions or transformation of one kind of geometric, algebraic, or functional structure to another through observation and interpretation of relationships among variables, consolidation of outcomes of changes, and debriefing and understanding different mathematical operations. Use of technology may function as an interface between dualities of different domains (e. g., hands-on and minds-on; algebra and geometry) (Cuoco & Goldenberg, 1997; Hoong, 2003) that eventually contributes to consolidate teacher and students' positive beliefs about the semantics and syntactic of any content of mathematics with a technological tool, for example GSP.

Acknowledgment

I would like to acknowledge Dr. Larry L. Hatfield and Dr. Linda S. Hutchison for their mentorship, care and support for this study during my graduate study at the University of Wyoming, Laramie, Wyoming, USA. I would like to thank my 'then classmates' Dr. Tracey Gorham Blanco and Dr. Wandee Kasemsukpipat for their help at different levels during this study. I would like to thank Christopher R. Holtkamp for careful proofreading and commenting on the manuscript. I am very grateful to the anonymous reviewers for their comments and feedback.

References

Anfara, V. A., & Mertz, N. T., Jr. (Eds.). (2006). *Theoretical frameworks in qualitative research*. Thousand Oaks, CA: Sage Publications, Inc.

Audi, R. (2015). *Rational beliefs: Structure, grounds, and intellectual virtue*. Oxford: Oxford University Press.

Bailyn, L. (1977). Research as a cognitive process: Implications for data analysis. *Quality and quantity, 11*, 97-107.

Belbase, S. (2013). Images, anxieties, and attitudes toward mathematics. *International Journal of Education in Mathematics, Science, and Technology, 1*(4), 230-237. Accessed on July 20, 2017 from: http://dergipark.ulakbim.gov.tr/ijemst/article/view/5000036019

Belbase, S. (2014). *Radical constructivist grounded theory: A hybrid epistemology*. A presentation at the College of Education Research Symposium, University of Wyoming, Laramie, Wyoming (March 7, 2014). Online accessed on May 23, 2017 from: http://repository.uwyo.edu/education_symp/2014/Schedule/4/

Belbase, S. (2015a). A preservice secondary mathematics teacher's beliefs about teaching mathematics with technology. *International Journal of Research in Education and Science, 1*(1), 31-44. Accessed on June 15, 2017 from: http://ijres.net/index.php/ijres/article/view/15

Belbase, S. (2015b). *Pre-service secondary mathematics teachers' beliefs about teaching geometric transformations using Geometer's Sketchpad*. A doctoral dissertation, University of Wyoming, Laramie, WY, USA.

Belbase, S. (2016). Reflective and reflexive beliefs of two pre-service secondary mathematics teachers about teaching geometric transformations using Geometer's Sketchpad. *European Journal of Educational and Social Sciences, 1*(1), 34-65. Online accessed on May 28, 2017 from: http://ejess.eu/EJESS_vol1_1_MANID243_34-65.pdf

Belbase, S. (2017). Attitudinal and cognitive beliefs of two pre-service secondary mathematics teachers. *International Journal of Research in Education and Science (IJRES), 3*(2), 307-326. Online accessed from: http://ijres.net/index.php/ijres/article/view/231

Bell, A., Greer, B., Grimison, L., & Mangan, C. (1989). Children's performance on multiplicative word problems: Elements of a descriptive theory. *Journal for Research in Mathematics Education, 20*(5), 434–449.

Bergkamp, J. (2010). *The paradox of emotionality and competence in multicultural competency training: A grounded theory*. A doctoral dissertation, Antioch University, Seattle, WA, USA.

Central Board of Secondary Education (CBSE, India). (2016). *Mathematics: Coordinate geometry and transformations*. Delhi: CBSE.

Chai, C. S., Wong, B., & Teo, T. (2011). Singaporean pre-service teachers' beliefs about epistemology, teaching and learning, and technology. *Teacher Development: An International Journal of Teachers' Professional Development, 15* (4), 485-498.

Charmaz, K. (2006). *Constructing grounded theory: A practical guide through qualitative analysis*. Thousand Oaks, CA: Sage Publications, Inc.

Chen, R. J. (2011). Pre-service mathematics teachers' ambiguous views of technology. *School Science and Mathematics, 111*(2), 56-67.

Clarke, A. E. (2005). *Situational analysis: Grounded theory after the postmodern turn.* Thousand Oaks, CA: Sage Publications, Inc.

Corbin, J. M., & Strauss, A. L. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). Los Angeles, CA: Sage Publications, Inc.

Cuevas, G. J. (2010). Integrating technology in the mathematics classroom. In K. Cennamo, J. Ross, & P. Ertmer (Eds.), *Technology integration for meaningful classroom use: A standard-based approach* (pp. 369-386). Belmont, CA: WADSWORTH.

Cuoco, A. A., & Goldenberg, E. P. (1997). Dynamic geometry as a bridge from Euclidean Geometry to Analysis. In J. R. King & D. Schattschneider (Eds.), *Geometry turned on!: Dynamic software in learning, teaching, and research* (pp. 33-44). Washington, D. C.: The Mathematical Association of America.

Curriculum Development Center (CDC, Nepal). (2012). *Basic education curriculum*. Sanothimi, Bhaktapur, Nepal: CDC.

De Corte, E., & Verschaffel, L. (1987). The effect of semantic structure on first graders' strategies for solving addition and subtraction word problems. *Journal of Research in Mathematics Education, 18*(5), 363-381.

Department for Education (DfE, UK). (2014). *National curriculum in England: Mathematics programmes of study (July 16, 2014)*. Manchester, UK: DfE.

Erbas, A. K., Ledford, S. D., Orrill, C. H., & Polly, D. (2005). Promoting problem solving across geometry and algebra by using technology. *Mathematics Teacher, 98*(9), 599-603.

Erens, R., & Eichler, A. (2016). Beliefs and technology. In C. Bernack-Schüler, R. Erens, A. Eichler, & T. Leuders (Eds.), *Views and beliefs in mathematics education: Results of the 19th MAVI Conference* (pp. 133 – 144). Germany: Springer Spektrum.

Ertmer, P. A. (2006). *Teacher pedagogical beliefs and classroom technology use: A critical link*. Online resource document retrieved on May 24, 2013 from: http://www.edci.purdue.edu/ertmer/docs/AERA06_TchrBeliefs.pdf

Ertmer, P. A., Ottenbreit-Leftwich, A. T., Sadik, O., Sendurur, E., & Sendurur, P. (2012). Teacher beliefs and technology integration practices: A critical relationship. *Computer & Education, 59*(2012), 423-435. DOI: 10.1016/j.compedu.2012.02.001

Friedhoff, S., Zu Verl, C. M., Pietsch, C., Meyer, C., Vomprass, J., & Liebig, S. (2013). Social research data: Documentation, management, and technical implementation within the SFB 882. *SFB 882 Working Paper Series, 16* (February). Online available from: http://www.sfb882.uni-bielefeld.de/

Garry, T. (1997). Geometer's Sketchpad in the classroom. In J. R. King & D. Schattschneider (Eds.), *Geometry turned on! Dynamic software in learning, teaching, and research* (pp. 55-62). Washington, D. C.: The Mathematical Association of America.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research.* New York, NY: Aldine de Gruyter.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A.E. Kelly & R.A. Lesh (Eds.), *Handbook of research design in mathematics and science education (pp. 35-44)*. Mahwah, NJ: Lawrence Erlbaum Associates.

Goldin, G. A. (2002). Affect, meta-affect, and mathematical beliefs structures. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 59-72). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Hertz, R. (Ed.). (1997). *Reflexivity and voice*. Thousand Oaks, CA: Sage Publications, Inc.

Hoong, L. Y. (2003). Use of the Geometer's Sketchpad in secondary schools. *The Mathematics Educator*, *7*(2), 86-95.

Hoong, L. Y., & Khoh, L.-T. S. (2003). Effects of Geometer's Sketchpad on spatial ability and achievement in transformation geometry among secondary two students in Singapore. *The Mathematics Educator, 7*(1), 32-48.

Hunter, J. (2015). *Technology integration and high possibility classrooms: Building from TPACK*. New York, NY: Routledge.

Jiang, Z. (2011). Participant research essay for TTAME Research Team. In S. A. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e*. Laramie, WY: College of Education, University of Wyoming.

Kim, S. (2016). *Relationship between pre-service secondary mathematics teachers' beliefs, knowledge, and technology use.* A doctoral dissertation, University of Georgia, Athens, GA.

Kissane, B., McConney, A., & Ho, K. F. (2015). *Review of the use of technology in mathematics education*. Perth, Western Australia: Government of Western Australia School Curriculum and Standards Authority.

Layder, D. (1998). *Sociological practice: Linking theory and social research*. London: Sage Publications, Inc.

Leatham, K. R. (2002). *Pre-service secondary mathematics teachers' beliefs about teaching with technology*. A doctoral dissertation, University of Georgia, Athens, GA.

Lempert, L. B. (2007). Asking questions of the data: Memo writing in the grounded theory tradition. In A. Bryant & K. Charmaz (Eds.), *The SAGE handbook of grounded theory (pp. 245-264).* Los Angeles, CA: Sage Publications, Inc.

Lichtenstein, B. B. (2000). *The matrix of complexity: A multi-disciplinary approach for studying emergence in coevolution*. Online retrieved on January 20, 2014 from: http://www.hsdinstitute.org/learn-more/library/articles/MatrixOfComplexity.pdf

Lye, S. Y., & Churchill, D. (2013). Teaching with technology in a Future School in Singapore: A mathematics teacher's experience. In L. Y. Tay & C. P. Lim (Eds.), *Creating holistic technology-enhanced learning experiences: Tales from a Future School in Singapore* (pp. 39-58). Rotterdam, The Netherlands: Sense Publishers.

Maher, C. A. (1998). Constructivism and constructivist teaching: Can they co-exist? In O. Bjorkqvist (Ed.), *Mathematics teaching from a constructivist point of view* (pp. 29 – 42). Finland: Abo Akademi.

Maher, C. A., & Sigley, R. (2014). Task-based interviews in mathematics education. *Encyclopedia of Mathematics Education* (pp. 579-582). DOI: 10.1007/978-94-007-4978-8_147.

Mauther, N. S., & Doucet, A. (2003). Reflexive accounts and accounts of reflexivity in qualitative data analysis. *Sociology, 37*(3), 413-431.

Meng, C. C. (2009). Engaging students' geometric thinking through phase-based instruction using Geometer's Sketchpad: A case study. *Journal Pendidik dan Pendidikan, 24*, 89- 107.

Meng, C. C., & Sam, L. C. (2013). Developing pre-service teachers' technological pedagogical content knowledge for teaching mathematics with the Geometer's Sketchpad through lesson study. *Journal of Education and Learning*, *2*(1), 1-8.

Misfeldt, M., Jankvist, U. T., & Aguilar, M. S. (2016). Teacher beliefs about the discipline of mathematics and the use of technology in the classroom. *Mathematics Education, 11*(2), 395-419. DOI: 10.12973/iser.2016.2113a

Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A new framework for teacher knowledge. *Teachers College Record, 108*(6), 1017-1054.

Morse, J. M. (1994). Emerging from the data: The cognitive process of analysis in qualitative inquiry. In J. Morse (Ed.), *Critical issues in qualitative research methods* (pp. 23–43). Thousand Oaks, CA: Sage Publications, Inc.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics.* Washington, DC: Authors.

Nesher, P. (1988). Multiplicative school word problems: Theoretical approaches and empirical findings. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 19–40). Hillsdale, NJ: Erlbaum.

Nordin, N., Zakaria, E., Mohamed, N. K., & Embi, M. A. (2010). Pedagogical usability of the Geometer's Sketchpad (GSP) digital module in the mathematics teaching. *The Turkish Online Journal of Educational Technology, 9*(4), 113-117.

Parveva, T., Noorani, S., Ranguelov, S., Motiejunaite, A., & Kerpanova, V. (2011). *Mathematics education in Europe: Common challenges and national policies*. Education, Audiovisual and Culture Executive Agency, European Commission.

Pierre, E. A. (2009). Decentering voice in qualitative inquiry. In A. Y. Jackson & L. A. Mazzei (Eds.), *Voice in qualitative inquiry: Challenging conventional, interpretive, and critical conceptions in qualitative research* (pp. 221-236). New York, NY: Routledge.

Polly, D. (Ed.). (2015*). Cases on technology integration in mathematics education*. Hershey, PA: Information Science Reference.

Rahim, M. H. (2000). *A classroom use of the Geometer's Sketchpad in a mathematics pre-service teacher education program.* Thunder Bay, Ontario: Faculty of Education, Lakehead University.

Ramli, R., & Mustapha, R. (2014). An investigation on the GSP implementation in the teaching of mathematics at a Malaysian Technical School. *Journal of Asian Vocational Education and Training, 7*, 74-83.

Rodwell, M. K. (1998). *Social work constructivist research*. New York, NY: Garland Publishing Inc.

Shafer, K. G. (2004). *Two high school teachers' initial use of Geometer's Sketchpad: Issues of implementation.* A doctoral dissertation, Western Michigan University, Kalamazoo, MI.

Steffe, L. P. (2002). The constructivist teaching experiment: Illustrations and implications. In E. Von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 177 – 194). New York, NY: Kluwer Academic Publishers.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267 – 307). Hillsdale, NJ: Erlbaum.

Steketee, S., & Scher, D. (2012). Sensations to teach composition of functions. *Mathematics Teacher, 106*(4), 261-268.

Strauss, A. L., & Corbin, J. M. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks, CA: Sage Publications, Inc.

Thornberg, R., & Charmaz, K. (2014). Grounded theory and theoretical coding. In U. Flick (Ed.), *The SAGE handbook of qualitative data analysis* (pp. 153-169). Los Angeles: Sage Publications, Inc.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141–162). Hillsdale, NJ: Erlbaum.

Von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. New York, NY: Routledge Falmer.

Von Glasersfeld, E. (Ed.). (1991). *Radical constructivism in mathematics education*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Wachira, P., & Keengwe, J. (2011). Technology integration barriers: Urban school mathematics teachers' perspective. *Journal of Science Education and Technology, 20*(1), 17-25. DOI 10.1007/s10956-010-9230-y

Warfield, H. A. (2013). *The therapeutic value of pilgrimage: A grounded theory study North Carolina State University*. A doctoral dissertation, Raleigh, North Carolina.

Welsh, R. (2009). *International barriers to small business development: A study of independent retailers from the Edinburgh South Asian Community*. A doctoral dissertation, Queen Margaret University.

Willig, C. (2014). Interpretation and analysis. In U. Flick (Ed.), *The SAGE handbook of qualitative data analysis* (pp. 136-150). Los Angeles, CA: Sage Publications, Inc.