A CONCEPT-BASED LEARNING PROGRESSION FOR RATIONAL NUMBERS

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ABSTRACT

Rational number understanding is viewed as fundamental and critical to developing future knowledge and skills, and is therefore essential for success in the 21st century world. This report describes a provisional learning progression for rational numbers, specifically as embodied in fractions and decimals, that was designed to be useful towards the development of formative assessment.

Keywords: Assessment; Cognition; Rational number; Fractions.

RESUMO

O entendimento dos números racionais e visto como fundamental e crítico para o desenvolvimento futuro de conhecimentos e habilidades, e é por isso essencial para o sucesso no mundo do século 21. Este artigo descreve uma progressão de aprendizagem provisória para os números racionais, especialmente representados na forma fracionária e decimal, que foi projetada para ser útil para o desenvolvimento da avaliação formativa.

Palavras-chave: Avaliação; Cognição; Número racional; Frações.

Learning progressions (LPs) articulate a trajectory of learning and understanding in a domain, and in doing so, they can provide the big picture of what is to be learned, support instructional planning, and act as a guide for formative assessment (Heritage, 2008). There is evidence that superior teachers use a conceptual structure similar to a LP (Clements & Sarama, 2004). For example, in one study of a reform-based curriculum, the teachers who had the most valuable in-class discussions saw themselves not as moving through a curriculum but as helping students move through a progression or range of solution methods (Fuson, Carroll, & Drueck, 2000); that is, they were simultaneously using and modifying a type of learning trajectory (Clements & Sarama, 2004). Simon (1995) discussed the knowledge of a *hypothetical learning trajectory* (an empirically-based model of pedagogical thinking) as being essential to developing pedagogical thinking. Simon elaborated on this notion in 2004, demonstrating how thinking about the learning process and engaging in reflective abstraction promotes student learning (Simon, 2004).

The Rational Number Project at the University of Minnesota (see Post et al., 1998) named fractions and decimals as topics that lie at the heart of rational number reasoning and, therefore, the heart of elementary mathematics. Analyses of the components of the concept of rational number suggest that this concept is connected to most other topics in mathematics school learning. Siegler et al. (2012) have found that elementary school students' knowledge of fractions and of division uniquely predicts those students' knowledge of algebra and overall mathematics achievement in high school. This prediction stands even after statistically controlling for other types of mathematical knowledge, general intellectual ability, working memory, and family income and education. Thus, a LP for rational number understanding that is accessible to teachers would be a significant tool for the advancement of mathematical thinking in the classroom.

1. Some Background on Learning Progressions

The terms "learning progressions," "learning trajectories," and "developmental models" are often used interchangeably although have slightly different definitions. In the present paper, we adopt the term *provisional learning progression*[1] which we define as:

> A description of qualitative change in a student's level of sophistication for a key concept, process, strategy, practice, or habit of mind. Change in student standing on such a progression may be due to a variety of factors, including maturation and instruction. Each progression is presumed to be modal--i.e., to hold for most, but not all, students. Finally, it is provisional, subject to empirical verification and theoretical challenge (Educational Testing Service, 2012).

In this paper, we bring together the literature on fraction and decimals to create one comprehensive LP for rational number understanding. We did this because of the vast number of studies concerning rational numbers that point to (either explicitly or implicitly) the difficulties in understanding that different representations of the same quantity (decimal, fraction, percent) are, in fact the same number and that there are infinite equivalent fractions and different meanings (subconstructs) of the same fraction (see. Behr et al., 1983; Behr et al., 1993; Behr & Post, 1992; Behr, Wachsmuth, Post, & Lesh, 1984; Kieren, 1995; Ohlsson, 1988). For example, if previously a student knew that the number 2 stood for a group of two objects, now the symbol 2/5 is (a) part/whole (i.e., 2 out of 5); (b) division (i.e., 2 items divided between 5 people); (c) ratio (i.e., 2 to 5 ratio); (d) a measure (i.e., 0.4; fixed quantity, number line representation); and

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[1] This is the term and definition adopted by the Cognitively Based Assessment *as*, *of*, and *for* Learning (CBAL) research initiative at the Educational Testing Service. CBAL assessments are built around learning progressions that are deeply rooted in psychological, cognitive, and educational research (see http://www.ets.org/research/topics/cbal/initiative/).

(e) multiplicative operator (i.e., operator that reduces [2/5] or enlarges [5/2] the size of another quantity). It's also been found numerous times that translation between representations does not come easily (Markovits & Sowder, 1991, 1994; Moss, 2005; Sowder, 1995).

To our knowledge, the only other attempt to bring together the literature of fractions and decimals toward one comprehensive learning progression was Callingham and Watson's (2004) trajectory for mental calculation with rational numbers. This was notably different in that it dealt purely with mental calculations and not the conceptual understanding behind the numbers and operations with them. Thus, our goal was to build a comprehensive learning progression for the understanding of rational numbers that would be accessible for teachers of grades 3-5 but also informative for researchers and learning theorists in clearly laying out the main components of rational number understanding, including prerequisite skills and knowledge, conceptual shifts, and misconceptions.

2. The emerging understanding of fractions and decimals

2.1 Fractions review

We began this work with a comprehensive review of the literature in fraction and decimal understanding. We identified several main approaches to study the understanding as well as the teaching of fractions: (1) through understanding of the fraction as representing different types of entities, such as part/whole, part/group, point on the number-line (see Novillis, 1976), (2) through the types of representations one can use to explain fractions (see Andrade, 2011; Common Core Standards Writing Team, 2011), and (3) through the ability to explain fractions and operations on fractions in context (see Ma, 1999). Graeber & Triosh (1990) and Lamon (1999, 2001, 2007) showed that the understanding of a/b as indicating a quotient is rare amongst students. Lamon (2001) further noted that poor understanding of a/b as indicating division can lead to many problems in high school calculus. According to Smith (2002) the essential mistake of many students is to interpret quotients as pairs of whole numbers. For example, they see 3/5 as 3 and 5. As a result, they may think that 3/5 = 5/7, because the difference between 3 and 5 is 2, which is the same between 5 and 7.

Toluk (1999, as cited in Oksuz and Middleton, 2007) studied four children in a series of parallel individual teaching experiments. She found that children progress from seeing whole-number division and fair sharing as two different domains (division and fractions), to seeing fractions in terms of division. In her analysis, she came up with a model describing her students' development: First, the children had wholly separate conceptions of Fractions-as-Part/Whole and Division-as-Whole-Number Quotients, then when students were asked to subdivide a remainder in fair-sharing contexts, they eventually came to see the possibilities of fractional quotients describing cases where the numerator is larger than the denominator. Finally, over time, the situational and notational analogies presented for Fractions-as-Fair Share and Division-as-Fractions allowed students to conceive of both fractions and division as being the same thing. However, Toluk instructed these children towards this progression and this naturally occurs very rarely with students. In fact, Ma (1999) showed that many teachers in the United States are unable to construct the situational analogies necessary for this transformation.

One of the bodies of work that has been particularly influential is Novillis (1976) that created a hierarchical structure depicting the foundations of fractional understanding. The Novillis model, entitled “A Hierarchy of Selected Subconcepts of the Fraction Concept (HSSFC)” associates various concepts of fractions with their related models. The hierarchy was tested on the basis of a “Fraction Concept Test” created by the author. Connections were then confirmed or unconfirmed on the basis of which questions were correct conditionally on the basis of other questions being correct. Based on these results, she concluded that associating fractions with part-whole and part-group models was a prerequisite to associating a fraction with a point on the number line as well as a number of other prerequisite constructs. She also noted that many students can associate the fraction 1/5 with a set of five objects, one of which is shaded, but most cannot associate 1/5 with a set of ten objects, two of which are shaded.

2.2 Decimals Review

In the decimals literature, we identify two main approaches to study students’ understandings and difficulties with decimals: one which focuses the “misconceptions” of decimal notations and the other which studies decimal numbers as part of the wider rational number system. Studies that focus the decimal notation, identify notation-errors as indications of conceptual difficulties originated from either whole numbers or fractions (e.g. Sackur-Grisvard & Leonard,1985; Nesher and Peled, 1986; Resnick, Nesher, Leonard, Magone, Omanson and Peled, 1989; Stacey & Steinle, 1999; Roche, 2005; Roche, 2010). Two main kinds of error are observed when students are asked to compare decimal numbers: “longer is larger” (e.g., deciding 0.125 is larger than 0.3 because the “longer” the decimal portion the larger the number) and “shorter is larger” (e.g., deciding 0.3 is larger than 0.496, because in the first number the whole is divided into tens and in the later the whole is divided into thousands). Research has found both of these errors to be repeated, though persistence is more prevalent from the first kind (assumed to be originated from whole number understanding). Lesser detected is the second kind that may be derived from fraction understanding. Sackur-Grisvard & Leonard (1985) suggested steps in the acquisition of comparison rules for decimal numbers in which children are assumed to acquire first understanding of the presence of the point, to be followed by understanding of the property of the place value and finally understanding of the property of zero.

The main conclusion from these studies of decimal notation misconceptions is the need to connect between the decimal concept and the fraction concept. One suggestion is to use fractional language in the teaching of decimals (Roche, 2010). For example, speak of 2.75 as 2 and 75 hundredths (instead of two point seventy five), and avoid rules that encourage whole number thinking.

Moss and Case (1999), as well as Confrey (1994) argue that “the order of teaching fraction-decimal-percent is more arbitrary and that what matters is that the general sequence of coordinations remains progressive and closely in tune with children's original understanding” (p. 125). Moss and Case went further to suggest a curriculum where the order of teaching is first percents, followed by decimals and ending with fractions. In an experiment they found that the students who received the experimental curriculum showed a deeper understanding of rational numbers than those in the control group, as well as less reliance on whole number strategies when solving novel problems. In a different study, Resnick et al. (1989) found similar results when comparing students from the US, Israel and France, where each country had a different curriculum. Specifically, the French curriculum at the time and place where the experiment was conducted included preceding decimals to fractions.

2.3 A Concept-Based Approach

Although there have been previous efforts at defining separate learning progressions for (a) fractions and (b) decimals (see Confrey and Maloney, 2010; Steffe, 2004; Kalchman, Moss, and Case, 2001), it is clear from the literature that it is the connections amongst these skills that is most indicative of mathematical success, not just in the early grades but also into high school (see Thompson & Saldanha, 2003). For this reason, we decided on a structure that allows these learning progressions to connect to each other to create one larger map that is focused on the concepts behind the understanding of rational number representations and the corresponding representations. Another concept-based progression of rational number is Confrey et.al.'s (2009) trajectory for equipartitioning. We did incorporate some elements of this progression into our own LP, specifically some conceptual shifts between levels 1, 2, and 3 (see section *The Provisional Learning Progression*, later in this document) that deal specifically with changes in the concept of equipartitioning.

3. The Rational Number Learning Progression

The model of students' understanding of rational number has two central concepts that develop through the stages: *a shift from a part/whole representation into a single number understanding*, and *an integration of decimal and fraction notations and representations*. We begin with a detailed description of each level in regards to what students understand at that level, what students can do, and what they might have trouble doing.

We acknowledge that change in student standing in the progression relies on both maturation and instruction and thus it is possible for a student to show evidence of one level with fractions, for instance, and a lower level with decimals because they have not yet received decimal instruction. The notion, however, is that the cognitive underpinnings necessary to achieve standing on a level may be present, even if instruction has not yet made possible the ability to perform at a certain level with both fractions and decimals. For a complete review of the literature that was reviewed in the preparation of this LP, see Arieli-Attali & Cayton-Hodges, (2014).

3.1 Progress Variables

Progress variables are dimensions of knowledge that develop through the progression. We defined five progress variables to be used in this Learning Progression: Fractional Units, Measure/Fraction as number, Additive Structure, Multiplicative Structure, and Strategic Thinking/Flexibility. These variables are defined below.

1. *Fractional Units:* This progress variable refers to how the student perceives the relationship between quantities, between the part and the whole, between the unit of the partitioned whole and a quantity that consists from several such units (see Steffe, 2004).

2. *Measure/Fraction as number:* The notion of fractions as number is not evident in early stages of the development of a fraction concept. The first step in the development of the notion of a fraction as a measure is the ability to place a fraction on a number line of size [0,1]. In fact, this is not yet a measure understanding, but rather a perception of partitioning the line [0,1] into n equal parts and finding the n^{th} part according to the convention of starting to count from the left (the zero). The next level of development is conceiving of a fraction on a number-line longer than 1 (e.g., placing a ¼ [0,2] or [0,5] etc.) and of an improper fraction on that line (e.g., placing 1¼ on [0,2] or [0,5]). Only after this do children perceive of the fraction as a measure independently of the specific scale of the number line.

3. Additive Structure: Vergnaud (1994) defines additive structures as "the capacity that a person has to identify, understand and tackle situations where addition and subtraction operations are applicable." This progress variable refers to how students understand and apply the applicability of addition and subtraction to fractions.

4. *Multiplicative Structure:* Similar to additive structures, this progress variable refers to how students understand and apply the applicability of multiplication and division to fractions. Multiplicative structures are first apparent when students show early division concepts, such as being able to solve for one-half of one-quarter. However, this early understanding of fraction as operator is the only multiplicative structure available to the student until they recognize fractions as numbers, setting the path towards multiplicative multiplication and division.

5. Strategic thinking/Flexibility: Students differ in their ability to apply different strategies at problem solving. While at early stages of understanding, a student may have only one strategy available to solve a specific problem, when his/her understanding progresses, more strategies are at hand, and the ability to choose the more "efficient" one, the one that will solve the problem quicker and easier, may be an indication of a higher level of understanding. Moreover, in real world problems part of the difficulty is sometimes to figure out exactly what the problem is, and how to "model" it mathematically. Kilpatrick, Swafford & Bradford (2001) termed this ability as "strategic competence".

3.2 The Provisional Learning Progression

Prior Knowledge (Level 0) – The concept of half; halving or splitting into two equal parts

Level 0 in our progression is the prerequisite or the basic prior knowledge that students possess before entering elementary school and formal teaching. It is assumed that most students entering elementary school already have a colloquial notion of halving from everyday experience.

Level 1 – Early Part/Whole Understanding - The beginning of part/whole and part/group understanding; repeated halving, and equipartitioning into number of parts 2^n; separate understandings of a part/whole relation as exhibited in fractions in equipartitioning context and in decimals as related to money

At this level, students understand the relation between the parts that are *smaller than* and *embedded in* the whole. Students at this level also know and understand that the parts should be *equal* – though the meaning of "equal" may not be fully established, that is, some students may identify equal with congruent shapes. Although they understand what it means to equipartition to any number of parts, partitioning to an even number of parts and specifically to 2^n number of parts is easier and allows for repeated halving strategy.

At the same time, with decimals, students may have basic understanding of money units, and the part-whole relationship they exhibit (e.g., that there are 4 quarters in one dollar, that 50 cents are half of a dollar, etc.), and may be able to work (add, subtract) with them using visual or physical representation, *without* understanding of the place value conception or the meaning of the decimal point. Their knowledge of part-whole in money context is a separate construct than their emerging concept of part-whole in the context of formal teaching of fractions.

Progress Variables: Levels of Units: Students at this level have one level of units (wholes). *Measure/Fraction as number:* Students at this level do not yet have a concept of fraction as number. *Additive Structure:* Students at this level do not yet see the additive structure of fractions. *Multiplicative Structure:* Students at this level do not yet see the multiplicative structure of fractions. *Strategic Thinking/Flexibility:* Students at this level are unable to work strategically and flexibly with fractions.

Level 2 – Fraction as Unit - The establishment of part/whole concept of a fraction; equipartitioning with all numbers; unit fraction concept and common fractions smaller than one whole (proper fractions); separate understanding of decimals mostly in context of money

At this level, students understand the concept of a *unit fraction* as a separate unit that belongs to and gets its meaning from the partitioned whole. They can name or use simple notation using the term "*out of*", like in "1 out of 4". Evidence that this level of understanding is robust can be found in students' reaction to improper fraction (they may say that 5/3 cannot be, because there cannot be 5 out of 3, see Olive & Steffe, 2002). They can see a common fraction of *a/b* as built from "combining" a unit fraction *1/b a* times. This "combining" of unit fractions, seems multiplicative, yet it is a result of an additive structure (because it is still limited to the size of the whole, and is dependent on that whole). Thus, they understand at this stage *addition and subtraction* of fractions as joining and separating parts referring to the same whole. They have emergent understanding of equivalent fractions for special cases, using visual or physical models. In the decimal context, student still develop separate but parallel understanding. They understand the partition of the money unit (one dollar) to 100 parts, and understand the meaning of any fractional part of that whole money unit (e.g., $2.35). The basic fractional parts of the money system (quarter, dime, nickel, penny) receive a special meaning as “unit” decimal, which can be combined or operated with concrete meaning.

Progress Variables: Levels of Units: Students at this level have two levels of units (units of units). This includes early anticipation/imagination of parts and is the start of symbolic representation. *Measure/Fraction as number:* Students at this level do not yet have a concept of fraction as number. *Additive Structure:* Students at this level have the beginning understanding of an additive structure. *Multiplicative Structure:* Students at this level have an early division concept, such as “What is ½ of ¼ ?” *Strategic Thinking/Flexibility:* Students at this level start to see different options in equipartitioning. Using anticipation/imagination allows for early (yet limited) flexibility and strategic thinking. However, they may only see different options with certain benchmark fractions (that is, while with 1/2, 1/4 or even 1/8 they may see equivalent ways of partitioning, it may not be the case with 1/6 or 1/21).

Level 3 – Fraction as Single Number - The shift to the concept of fraction as a single number; a number-line representation; emergent understanding of improper fraction; early integration of fraction and decimal notation; fraction as a measure

At this level students are able to conceive of a fraction as a *single number* in its own right, and understand the meaning of an improper fraction. The transition from viewing 5/8 as 5 parts out of 8, to viewing it as 5 times 1/8 that started to develop in the previous level, continues and crystallizes here with the understanding that one can iterate this unit any number of times even beyond the original partitioned whole. Thus, at this level students can hold "three levels" of units (see Steffe, 2002) in their head: the whole, the unit fraction, and the improper fraction/ mixed number. At this level student have a notion of *magnitude* attached to the fraction, they understand the concept of equivalent fractions, and that different labels and notations can refer to the same magnitude/measure/value. A fraction as a measure is established.

In the decimal arena, students are ready to conceptualize decimals as numbers, detached from money. They understand that the decimal notation is an expansion of the whole number notation, and they see it in measurement context where a measure of an item is denoted as 2.45 cm, etc.

Progress Variables: Levels of Units: Students at this level have three levels of units (units of units of units). This is the highest number of levels documented and opens a wide variety of possibilities in terms of embedding and disembedding fractional parts. *Measure/Fraction as number:* Students at this level are able to see a fraction as a single number and work with it as a measure in its own right. *Additive Structure:* Students at this level have a more advanced additive structure that allows for a sum greater than one and flexibility with the whole. *Multiplicative Structure:* Students at this level are able to move beyond the early division concept to include fractions as numbers and not just operators. *Strategic Thinking/Flexibility:* Students at this level are able to exhibit some variety in the ways they solve problems, due to their ability to refer to numbers as *quantities*. The ability to locate a fraction and a decimal on the number line opens up the possibility to see them as different representations of the same quantity.

Level 4 – Representational Fluency - Increased representational fluency including: smooth translating between different notations; partitioning multiple units; using partitioning in flexible ways; established multiplication of fractions with fractions; fraction as an operator

At this level, students have a concept of multiple notations. That is, they master the notion that a referent (value) can be expressed in different ways (fraction, decimal, simplified fraction, mixed number). Student at this level have a strong conception of addition and subtraction of fractions and decimals; their meaning and their application in different context. They have an early multiplicative structure and they can identify multiplicative relations and patterns to use for problem solving. They understand partitioning in a deeper way that allows them to use it even in complex number combinations and exhaustively (partition the remainder). They understand the concept of a fraction as *operator*, and thus see a mapping between a fraction and a decimal, for example, $(1/4) \times 5$ read as a quarter of 5. Students have an increased symbolic fluency.

At this level, students may have trouble with contextualizing and modeling division problems with fractions and decimals as well as generalizing from concrete problem solving strategies to general strategies, for example, although they can flexibly move between equivalent expressions like, $3 \times (2/5)$ equivalent to $6 \times (1/5)$, they would not see the general case where $n \times (a/b) = (n \times a)/b$ OR $k \times (a/b) = k \times a\ (1/b)$.

Progress Variables: Levels of Units: Students at this level have three levels of units (units of units of units) as in Level 3. *Measure/Fraction as number:* Students at this level are able to see a fraction as a single number as in Level 3. *Additive Structure:* Students at this level have a more advanced understanding of the additive properties of fractions. They are able to manipulate fractions additively with ease. *Multiplicative Structure:* Students at this level are able to multiplicatively relate any two fractions. For instance, they can solve the problem “transform 2/3 into 2/5 multiplicatively”. They are also able to apply the distributive property to fractions. *Strategic Thinking/Flexibility:* Students at this level have an advanced level of strategic thinking, whereby they have several different strategies to solve a problem, and they can choose the more efficient one for the specific problem. They are also able to recognize the mathematical model within a (real-world) problem in most cases, but not all.

Level 5 – General Model - General model of fraction; contextual fluency; fraction as quotient

At this level students understand the multiple faces of fractions, and specialize and generalize across contexts (conceptual depth and breadth/ contextual fluency), that is, some context require conceiving the fraction as a single number, where others may require viewing it as a relation between two quantities, i.e., in ratio situations. Students at this level have a strong connection between the numerical representation (the fraction, the decimal, an expression that includes both) and the context. They can decontextualize from complex word problems to expressions and equations, as well as contextualize (find a context) appropriately for an equation or expression involving rational number of any sort, including division of fractions.

Progress Variables: Levels of Units: Students at this level have three levels of units (units of units of units) as in Levels 3 and 4. *Measure/Fraction as number:* Students at this level are able to see a fraction as a single number as in Levels 3 and 4. *Additive Structure:* Students at this level have the same additive structure as that of Level 4. *Multiplicative Structure:* Students at this level have an advanced multiplicative concept, enabling them to use symbolic notation in a rich and meaningful way. *Strategic Thinking/Flexibility:* Students at this level show the greatest amount of flexibility. They have several different strategies to solve a problem, and they can choose the more efficient one for the specific problem, and they can recognize the mathematical model within any problem. They are able to answer such questions as "How many 3/4 are in 2/5?" and other problems of rescaling.

4. Conclusion

A great deal has been written about rational number development. This progression is meant to be situated among that literature, with the conclusion that it is possible to create a comprehensive progression that encompasses both fraction and decimal understanding, a notion important in both assessment and teaching. We believe that it is the interaction amongst these topics that is vital toward a holistic understanding of rational number. It is these interactions that often prove to be the most important in later mathematics achievement. This progression was confirmed and refined through individual cognitive interviews and is currently being used towards the development of formative assessments that both provide information to teachers about student understanding and also exemplify how the levels of the learning progression are demonstrated through tasks. That is, the tasks help teachers internalize the progression so that they can use it more flexibly in their own practice.

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