Multimodality and mathematical meaning-making: Blind students' interactions with Symmetry

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ABSTRACT

In this paper, we examine the claim that mathematical cognition is embodied by exploring the co-ordinations of multimodal resources which characterise dialogues between researchers and blind mathematics students. We begin by considering approaches to understanding the relationship between perception and conception in different academic fields, notably philosophy and neuroscience. This leads us to adopt an embodied perspective on mathematical cognition, with roots in the Phenomenology of Merleau-Ponty. We illustrate how the lived-in bodies of the two blind students interviewed impinged upon their mathematical activities as they worked on a series of task involving symmetrical figures and geometrical transformations – and how the practices of the student with no visual memories differed from those of the student who had more recently lost his sight.

Keywords: Mathematics Education, Visual impairment, Embodied cognition, Phenomenology, Symmetry, Geometrical transformations.

RESUMO

Neste artigo, examinamos a premissa de que a cognição matemática é corporificada por meio da exploração coordenada de recursos multimodais que caracterizam diálogos entre pesquisadores e aprendizes cegos. Começamos considerando algumas abordagens para compreender a relação entre percepção e concepção em diferentes campos acadêmicos, particularmente da Filosofia e da Neurociência. Isso nos levou a adotar uma perspectiva corporificada da cognição matemática com base na Fenomenologia de Merleau-Ponty. Ilustramos como os corpos de dois alunos cegos foram afetados por suas atividades matemáticas enquanto trabalhavam em uma série de tarefas envolvendo figuras simétricas e transformações geométricas – e como as práticas de aprendizes que não tem memória visual diferem daqueles que perderam a visão mais recentemente.

Palavras-chave: Educação Matemática, Cegos, Cognição corporificada, Fenomenologia, Simetria, Transformação Geométrica.

Introduction

> *One of our company bethought him of asking our blind man if he would like to have eyes. "If it were not for curiosity" he replied, "I would just as soon have long arms: it seems to me that my hands would tell me more of what goes on in the moon than your eyes or your telescope; and besides, eyes cease to see sooner than hands to touch. I would be as well off if I perfected the organ I possess, as if I obtained the organ I am deprived of." (Diderot, 1916/1749, p.76-77*).

We live in an antagonist society in which we can only discuss the real if we know the imaginary, the concrete if we have a conception of the abstract, the eternal if we are capable of thinking of the ephemeral and the ideal if we can judge the possible. In the area of education, we also confront antagonistic situations. Such situations arise frequently in the school context and involve especially its principal actors – students and teachers. On the one hand are the teachers, who might ask themselves how they can learn to use their eyes and their ears, trained as they are to recognize the so called "normal", to see and to hear the diversity that composes their classes. On the other hand we have the students, who, wishing to achieve that which is expected by their teachers, their families and by society at large might enquire of themselves "how can we learn to see and to hear that which we are supposed to be perceiving?"

In our studies with students with disabilities, we seek parameters through which we ourselves might learn to see and to hear mathematical practices even when they do not sound or look exactly as we have come to expect. Our searches have led us to try to understand how the body influences the process that it sets off in the brain and, reciprocally, how the body feels cerebrally initiated processes. For us, the importance and the role of the body for and in cognition are unquestionable, but we still know relatively little about how to exploit the potential of the body to facilitate processes associated with mathematical cognition of those who do not have all the human perceptual and sensory-motor channels at their disposal. The perspectives of researchers who explore the embodied nature of cognition (for example, Barsalou, 2008; Gallese & Lakoff, 2005; Gallese, 2005; Oakley, 2007; Damásio, 2005, 2007), which are increasingly permeating the field of mathematics education, offer what appear to us to be ways of interpreting the processes of appropriating mathematical practices of such learners. We take from these embodied perspectives a basic premise which guides all our work: what we know is a result of our constant acting in and interacting with the world we inhabit, in ways that are both enabled and constrained by our bodys' capacities.

In the first part of this paper, we consider positions from both philosophy and neuroscience concerning the relationships between what is perceived through the body's senses and what is conceived by the brain. In sequence, we present two episodes which illustrate that, in the case of the learners involved in this study – who were blind – both their doings and imaginings evoked multimodal resources utilized in previous experiences as they negotiated the demands of a novel learning situation. The mathematical practices in question involve aspects of plane geometry, more specifically transformations of the Euclidean plane. To make sense of the mathematical practices which emerged, we draw from Piaget and Garcia's epistemological categorisation of geometrical activity into three phases, intrafigural, interfigural and transfigural. We end by reflecting on how the particular bodily resources of the blind students, and the means through which they came to visualize the objects in question, involved them in transiting between the phases of the triad in different ways.

The Body and Cognition

One of the principles of the embodied perspective we are adopting is that cognitive activity is not confined to cerebral activity, and, hence, it is not possible to disassociate experience and perception from cognition. This point of view is consistent with the thesis presented by certain philosophers that posits as an inherent potential of human beings the tendency to transform the sensations perceived by the sense organs into ideas – using signs – which can be elaborated into knowledge. Historically, the forms through which these sensations are transformed have provoked considerable controversy – not least as concerns the role of the body in the cognitive process. The debate took hold in earnest at the end of the 16th Century, a time marked by social transformations which emerged in opposition to the Teocentrismo which had dominated the middle ages. Lines were drawn in the fields of philosophical and scientific investigation as two methodological camps emerged: the empiricist perspective proposed by Francis Bacon which argued for a science based on observation and experimentation and the rationalist perspective defended by Descartes, which posited reason as the source for scientific certainty (Descartes, 1979, p. IV-X).

For the rationalists, such as Descartes (1979), the methods through which knowledge can be ascertained are not sensory, but rely on the intellect and on deductive reasoning. For Damásio (1994; p.248), Descartes's celebrated phrase – *I think therefore I am* – clearly demonstrates his view of the relationship between body and brain. He argues that this phrase:

> suggests that thinking, and awareness of thinking, are the real substrate of being. And since we know that Descartes imagined thinking as an activity quite separate from the body, it does celebrate the separation of mind, the 'thinking thing' (*res cogitans*) from the nonthinking body, that which has extension and mechanical parts (*res extensa*).

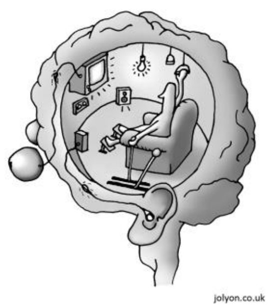


Figure 1: The homunculus

Descartes (1979, p. 229), considered a very small gland in the centre of the brain, the pineal gland, to be site of the soul and the location in which information from the sense organs was transformed in order that the actions of the body might be controlled. This mechanist view has been criticism by many including the phenomenologist Merleau-Ponty (1964/1945) and the neurologist Damásio (1994) as implying the residing in the brain is a "little person" or homunculus controlling the body (Figure 1[1]).

In contrast to the rationalists, empiricists such as Locke and Diderot defended the idea that all knowledge is a consequence of experience. In 1690, Locke published *An Essay concerning Human Understanding*, in which he presented his view of the human soul as a "blank sheet" at the moment of birth, on which nothing is initially "written". In his conclusions, Locke postulated that if the human adult possesses knowledge, it is the

[1] Available at http://www.jolyon.co.uk/illustrations/ConsciousnessIntroduction/images/Image%20(8).jpg

fruit of ideas generated on the basis of sensed experience and reflection (Locke, 1991, p.XII). The dichotomous position defended by Locke, in which, on the one hand, we are passive subjects, born ignorant, and recipients of all our sensory experiences and, on the other, were are active subjects who reflect upon these experiences to elaborate simple ideas (or perceptions) that compose the intellect, has also been criticized by Merleau-Ponty and Damásio.

Merleau-Ponty (1962/1945, p.55-56) agreed with Locke as regards the origins of knowledge, that is, he also posited that our knowledge comes from our experiences. He strongly disagreed in respect to the notion of a passive consciousness, an "empty space", which for him suggested that consciousness was being treated as simply a deposit of information. In the empiricist perspective, knowledge of the external world is thus reduced to sensory data: first sensations are experienced and these associate to compose perceptions which, in their turn, associate to form abstract ideas housed in the mind (Carmo, 2007). Damásio (1994, p.256) also criticized the empiricist perspective, claiming that perceiving does not imply only in "making the brain receive direct signals from a determined stimulus, much less direct photographic images". For him, every organism[2] alters itself "in order to obtain the best possible interface. The body is not passive". We might associate Damásio's view with the idea of intentionality, central to the phenomenological perspective and hence with Merleau-Ponty's criticism that "Empiricism cannot see that we need to know what we are looking for, otherwise we would not be looking for it" (Merleau-Ponty, 1962/1945, p.33).

Assuming a material conception[3], Diderot published his *Letter on the blind for the use of those who can see* in 1749 in which he wrote about an interview in someone blind from birth. In trying to respond to the question of how a man born blind can form ideas of figures, he attributed importance to the body and the role of tactile perceptions in the process of knowing:

> By the movements of his own body and stretching his hand in various directions, by passing his fingers continuously over an object, he gets an idea of space. []. In a more general sense, by repeated usage of the sense of touch, he has a memory of sensations experienced at different points; and he is capable of combining these sensations or points and forming figures. (Diderot, 1749/1916, p.83-84).

Diderot (ibid.) argued that the blind acquire knowledge through "imagination that is nothing more that the facility to record and combine sensations of palpable things" which enables them to perceive "things in a form much more abstract" than those who can see, since abstraction, from the empiricist point of view, involves "separating in thinking the perceptible qualities of the bodies, either from one another or from the body itself" (Diderot, 1916/1749, p.87-88). Like Diderot, Merleau-Ponty (1962/1945, p. 422) also stressed the function of tactile perception "the movement of the body itself is, for touch, that which illumination is for vision", however he rejected the dualist mind-body separation which characterises Diderot's description of abstraction.

Merleau-Ponty's position corresponds to that of those now working in the field of embodied cognition. For example, according to Barsalou (2008), it is our perceptive

[2] For Damásio, body and brain form an inseparable and dynamic organism which interacts intensely with its social and physical environments through its movements and its sensory apparatus.
[3] A form of empiricism in which it is claimed that all things are composed of material and all phenomena are the result of material interactions.

experiences that permit us to formulate multimodal representations, which are stored in our memories and reactivated to simulate states of perception, action and introspection associated with an object, even when it is not present. That is, the states associated with past experiences of a category of knowledge are relived through simulation. For him "simulations represent abstract concepts" (2009, p.1282) and hence an abstraction involves no stripping of the perceived experiences that gave rise to it. Oakley (2007) and Damásio (2005) call these multimodal representations "images", condensed sensory-motor modalities resulting from perceptual experiences of the visual, auditory, haptic, motor, olfactory and gustatory systems. Regardless of the term used – multimodal representations or images – these are the resources which permit simulation. Gallese (2005, p.41) defines simulation as "a functional process that possesses a certain representational content, typically focusing on possible states of its target object."

Gallese (2005, p.27) also suggests that some of the results now being obtained in neuroscientific investigations offer support to the phenomenological approach to perception. He states "that everything we see, we simultaneously also see it as a tactile object", that is, we associate the perceptions offered by vision (or by touch in the case of the blind) with sensations of a subjective nature, "as something which is directly related to the alive body, but not by virtue of its visibility". It is the "tactile alive body" which sustains our cognitive apparatus. This connects to the idea that both doing something and imagining doing the same something involve a shared neural substrate. Gallese and Lakoff (2005) offer neurological evidence for this position, arguing that circuitry across brain regions link different sensorial modalities "infusing each with the properties of others" (p.456). In the case of mathematics learning, for some at least, accepting that doing and imagining involve the same cognitive resources, might go somewhat against the vein. Mathematical imaginings are frequently revered as the means of engaging with the abstract, while doing is seen as a less sophisticated interaction with concrete situations. It also raises questions about what we understand as mathematical concepts. If cognition is multimodal and if imagining involves reliving – and re-feeling – previous doings, then concepts cannot be seen as mental representations in which the abstract, logical universal properties of an object are stored in a somehow transcendental form stripped of then particularities of the settings in which it was encountered.

Geometrical Structures

The mathematical domain under consideration in this article is that of Geometry, an area usually associated with the visual field and hence one that brings particular challenges to blind mathematics learners. An examination of the psychogenesis of this domain was undertaken as part of the historical-epistemological study carried out by Piaget and Garcia (1987). For them, both the historical development of Geometry and the psychogenesis of geometrical structures are characterised by three stages of development: intrafigural, interfigural e transfigural.

In the intrafigural stages, learners do not attend to transformation of figures within a structured space (be it the two-dimensional plane or three-dimentional space). Rather, they are focused on the internal properties of isolated figures or on comparisons of the internal properties of two or more figures. Piaget and Garcia called the interfigural stage that in which learners use references internal to the system under analysis, that is, figures in a plane, or in space and the set of objects in question present characteristics of a totality. In this stage, geometrical transformations are treated as relationships through which a figure object is associated with its figure image, but not as transformations applied to all the points that compose the system in question: points are considered as supports or vertices of figures. An individual working at the interfigural stage considers that any change in the shape of a figure is a result of the dislocation of its parts, since interfigural comparisons involve comparisons between initial and final locations (Piaget and Garcia, 1987, p.118). They argue that following the interfigural stage "a third period begins, which we shall call the transfigural. It is characterised by the predominance of structures" (ibid., p.110). This stage is concerned not only with the transformation of one figure onto another, but involves operations with all the points of the plane (or three-dimensional space), and the verification of variations and invariants associated with different applications and conditions. The transfigural stage represents, above all, operations on a set of elements, in which all the transformations can be composed and decomposed, becoming objects in their own right.

In their conclusions, Piaget and Garcia stress that that stages of this triad compose a continuous process in which "the structures obtained at the transfigural level give way, in their turn, to intrafigural analyses which lead to a new interfigural phase, and after to the production of super transfigural structures and so on indefinitely" (ibid., p. 132). Thus the process of succession between the three levels follows, for them, an absolute hierarchy: intra-inter-trans.

Feeling symmetry

> Symmetry, which is perhaps a matter of pure convention among is, is certainly so in many respects between a blind man and the sighted. A blind man studies by his touch that disposition required between the parts of a whole to enable it to be called beautiful; and then at length attains to a just application of that term. But is saying "that is beautiful," he does not form an opinion, it is no more than repeating the judgement of those who see; (Diderot 1916/1749, p.70-71).

The experimental activities discussed in the remainder of this text were realised with two blind Mathematics learners (for whom we shall use the fictional names Lucas and Edson) and involve geometrical concepts related to symmetry and the geometrical transformation reflection. In order to establish an in-depth understanding of the cognitive processes of the subjects who participated in our research, we chose to make use of Vygotsky's method of double stimulation (Vygotsky, 1998), in which the subject is confronted with a task that exceeds his or her existing levels of knowledge and capabilities (Cole & Scribner, 1998). The task is proposed within a structured situation and the subject receives active guidance from the researcher, so that strategies, at the beginning not known to the subject, might eventually be constructed and the task resolved (Veer & Valsiner, 1996).

In the episodes we present here, the first stimuli took the form of material tools and the second stimuli were offered in the researchers interventions. In general, in planning activities and tactile tools for the blind students, we sought to offer multimodal[4] stimuli which might favour a repertoire of actions fruitful in the situation at hand, but also adaptable for use in other contexts. The two episodes involve the study of symmetry and reflection and have been selected from a series of individual task-based interviews carried out with one subject (Lucas) who had been blind since the age of two years and another who only lost his sight completely at the age of fifteen years (Edson). The tasks were planned to be applied in accord with the diagram presented in Figure 2.

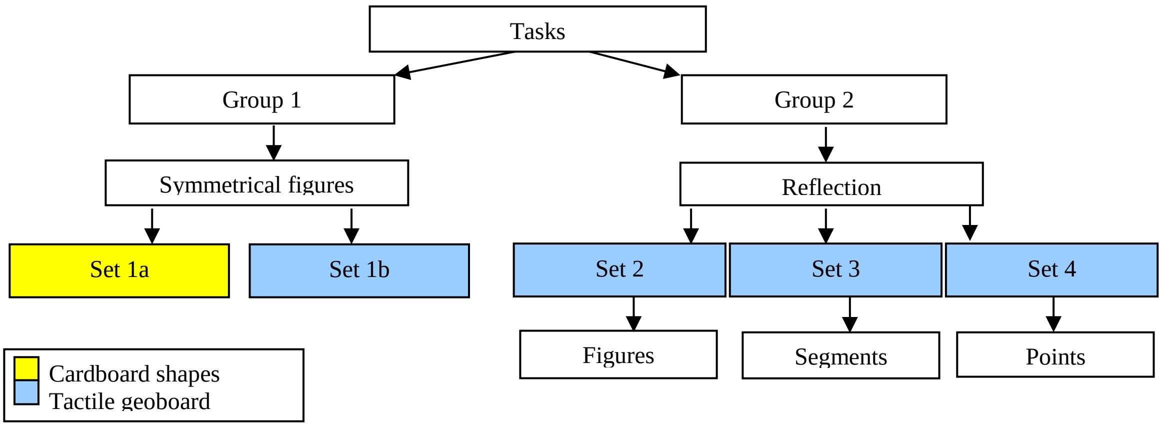


Figure 2: Task set

Seeing with hands

> *Knowledge has three entrances to reach our mind, and we keep one barricaded for the lack of signs* (Diderot, 1979, p. 89-90).

In the above quote, Diderot talks of the eyes (vision), the ears (audition) and the hands (touch) as the three sense organs which act as entrances through which the exterior world enters in contact with our minds. In fact, as Diderot also wrote in his 1749 letter, the information that the blind receive through the tips of their fingers makes it possible, when properly managed, for them to "imagine a solid as large as this terrestrial globe" (p.86). However the "embodied cognition perspective rejects the notion that behind perceptual-motor activity there is a 'mind', suffused with formal propositions and inferential rules" (Nemirovsky & Ferrara, 2009, p. 161), that commands our action either through some kind of homunculus as implied by Descartes or in the "separating in thought of the perceptible qualities of the body, either from one another, or from the body itself" as implied in Diderot's empiricist view (Diderot, 1749/1916; p. 88). The embodied cognition perspectives of today, as we have argued above, sit more easily with ideas of the phenomenological world proposed by Merleau-Ponty (1962/1945, p.xxii):

> "The phenomenological world is not pure being, but the sense which is revealed where the paths of my various experiences intersect, and also where my own and other people's intersect and engage each other like gears. It is thus inseparable from subjectivity and intersubjectivity, which find their unity when I either take up my past experiences in those of the present, or other people's in my own."

[4] That is, stimuli that activate different channels of the sensory system, impressing upon more than one of the sense organs.

Actually, Merleau-Ponty's position, here, enables us to extend the idea of simulation of by Barsalou (2008; 2009) – we might say that simulation involves the re-enactment of states previously experienced by ourselves in past happenings or the (re)-enactment of states observed in others as they participate in particular happening – simulation hence becomes a social as well as an individual process.

With this perspective in mind, we now turn to the data we collected during our interviews with Lucas and Edson. We began each interview by asking the young men about some of the mathematical terms that would be used during the activities. We were interested in ascertaining possible *entrances* that might be opened for geometrical knowledge and this led us to enquire into their past experiences in respect to learning geometry. Both of them initially stressed the importance of touch in allowing them to conceive the shape of a given figure and went on to mention the necessity of further information (or experiences) involving the *other*. Lucas's response serves as an illustration:

> ***Res**: Ok, I would like you to tell me something about figures..*
> ***Lucas**: Figures for the visually impaired who have never seen, like me – I lost my sight aged two years and consider myself, in truth, never to have seen because I have no memory, visual memory – For the visually impaired who have never seen figure is a complicated thing. We have an idea of figure after we have touched the figure. Without touch it would be difficult to define a figure.*
> ***Res**: So, you know shapes through touch?*
> ***Lucas**: Yes. If I have had some kind of tactile access, yes. If not, no. We try and get an idea on the basis of what people say.*

Observing the paths of the hands of both the blind subjects during their explorations of the tactile tools and the objects presented by them (Figures 2 and 3), we could see how they moved their hands intentionally to capture the particularities of the shapes in question and hence constructed for themselves an image[5] of the objects involved in the activities. This intentionality shows the active quality of their touch and how images are made, not passively received. We noticed, in particular, that both subjects explored the geoboard and the figures displayed upon it in similar ways, moving their hands successively from the outside to the inside and back again, with an initial tendency to move their hands together, following symmetrical trajectories – something we had by no means anticipated when planning the tasks.

In their first haptic contact with an object, the blind (like the sighted) act according to the cultural system in which both themselves and the objects are inserted, hence the object presents itself in ways that are to some extent already impregnated with a conceptualisation constituted historically and culturally. The perceptions associated with the physical contact are processed in forms which draw from its connections to other cultural systems associated with the past experiences of each learner. In other words, more than constituting their own image for the object, the learner actively engages in a process of interpretation and association, which enables the connection of this object to others already known.

[5] Here we use the term "image" following Damásio (2005) and Oakley (2007)

Figure 2: Exploring the tool symmetrically

Figure 3: Exploring the figures symmetrically

Edson's solutions

Edson was not born blind. He lost the sight in his left eye at the age of four years. Even after thirteen surgical attempts to save the sight in his right eye, at the age of fifteen, after suffering a dislocation of the retina, he lost all but two percent of the vision in this eye too, leaving him with the capacity to distinguish only between light and dark and to identify the direction from which light is coming. During the time that he participated in this research project, he was a student in the third year of High School, studying at night in a public school in the state of São Paulo. For his mathematics lessons, he was included in the regular classroom, and although Braille translations of the exercises were given for these lessons, he received no extra additional support. During the day, Edson worked as a receptionist for the organisation ADEVA (Association of the visually impaired and friends). During the first interview, he talked about his visual memories. He explained that not only did he have visual memories, but that he used them to recognise shapes, now experienced through touch, that he had learnt about when he could still see. Edson's description of the process by which he recognized these shapes resonates with the claim that thinking is multimodal and that the states associated with past experiences of a category of knowledge are relived through simulation – even when the sensory modes in question are no longer physically accessible. Our interpretation is that it is through the simulation process that Edson was able make to present, through a temporal activation, cerebral processes created in association with past experiences.

Edson had not studied geometrical transformations before and the term axis of symmetry was unfamiliar to him at the start of the research – although he was familiar, at least nominally, with some other elements that were fundamental in the developing dialogues, including midpoint, perpendicular lines and angle bisector. The first tasks that he worked upon involved determining the axis of symmetry in geometrical figures, first by folding cardboard representations of polygons and then by positioning an elastic band on the geoboard. On the basis of the activities with cardboard figures, he recognised some properties associated with the axis of symmetry, for example "*it divides into two identical figures*" and some properties associated with symmetrical figures stating, "*a figure is symmetrical when the two sides are identical, that is, if I fold, the figure stays equal*".

To determine the axis of symmetry of the figures represented on the geoboard, Edson employed the same strategy that he had used to locate the axes of the cardboard figures – he simulated the act of folding (Figure 4), and used his imagination to validate that the two figures determined by the axis would coincide were this folding action physically possible.

Figure 4: Edson simulates the act of folding

Edson: *First, I used touch to get an idea of the drawing. Now I am using my head, as if I was folding it with my hands*.
Res: *You're imagining the figure*?
Edson: *Yes, visually. Here one* (at this point Edson makes a gesture to indicate folding around the horizontally place axis as shown in figure). *And here two* (uses a similar gesture to simulate a fold on the vertical axis). *It just has the two* (axes of symmetry).

In these activities, Edson's doings (the folds) were associated with what Gallese e Lakoff (2005) have termed imaginative simulations and what Barsalou (2008) calls simple simulations, that is, he imagined manipulating the forms that were presented on the (unfoldable) geoboard and the figures that would result from this manipulation. Considering the stages of geometrical thinking proposed by Piaget and Garcia (1987), in these activities Edson was concerned only with reproducing two congruent parts on a single figure, which suggests his strategy was characteristic of the intrafigural level.

The next activities involved the reflection of geometrical shapes in relation to a given axis, with all the geometrical figures displayed using elastic on the geoboard. In his first attempt at the task presented in Figure 5, Edson constructed a triangle congruent to the given figure, but not inverted in relation to the axis. In an attempt to get him to focus on the inverted nature of the image, the researcher intervened, asking him about possible previous experiences with mirrors.

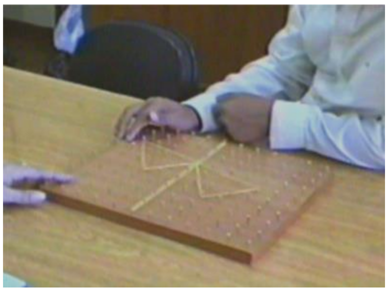


Figure 5: Reflecting a triangle

Res: *In your visual memory, do you have your image in a mirror?*
Edson: *Yes.*
Res: *And how did you see your image in a mirror?*
Edson: *Like how?*
Res: *... Pretend that I am your image in a mirror. I am in front of you. Raise your left hand. If I was to touch your hand, which hand would I have to raise?*
Edson: *You're right. I get it, it's inverted.*

In this exchange, the teacher attempted to call Edson's attention to a particular property associated with the reflection transformation by inviting him relive an experience associated with his visual past. In this case, the mental simulation was not a spontaneous act on his part, but motivated by a suggestion from the researcher. Nonetheless, the evoking of his previous experiences with mirrors still seemed to favour the use of previously stored memories in an imaginary simulation of an activity that was no longer physically possible for Edson. The intervention of the researcher in this case contributed to the reactivation of the multimodal representation structured during a past experience – that of seeing himself in a mirror, simulating the states, perceptive, motor and introspective, associated with this act. This helped Edson to (re)experience the inverted nature of images in a mirror.

As he concluded the series of reflection activities, Edson, making use of his visual memory (and experiences of the past) defined the process of reflecting as how "*to draw something in the head…to imagine it*" and he used expressions in which he re-used suggestions originally offered by the researcher "*Like you gave the example of the mirror*. *Looking at myself in the mirror. I created an image of me*". This way of experiencing reflection, however, emphasised mainly reproducing two parts of a whole, an intrafigural approach. The axis of symmetry continues for Edson to be an element which divides a whole into two congruent parts:

Res: *Tell me something now: what is the axis of symmetry for you?*
Edson: *It's what separates a figure into two or into one and its image.*

The next set of tasks involved the reflection of line segments. Edson's attempt to reflect a line segment placed parallel to the axis was somewhat surprising (Figure 6). He developed a strategy which involved positioning an elastic band on the endpoints of the segments and positioning it as a segment perpendicular to the axis of symmetry (Figure 7). By counting the number of pins between the given segments and the axis, he reproduced the same distance in the opposite semi-plane, correctly determining the symmetrical point. Using this point, he went on to construct the image segment, and verified his construction by checking for equal distances between the segments and the axis and congruence of the same.

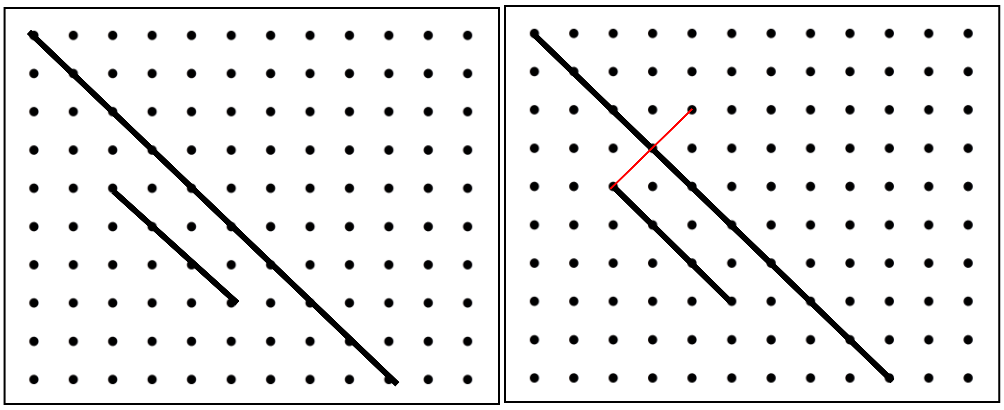


Figure 6: A new strategy

Figure 7: Constructing the image

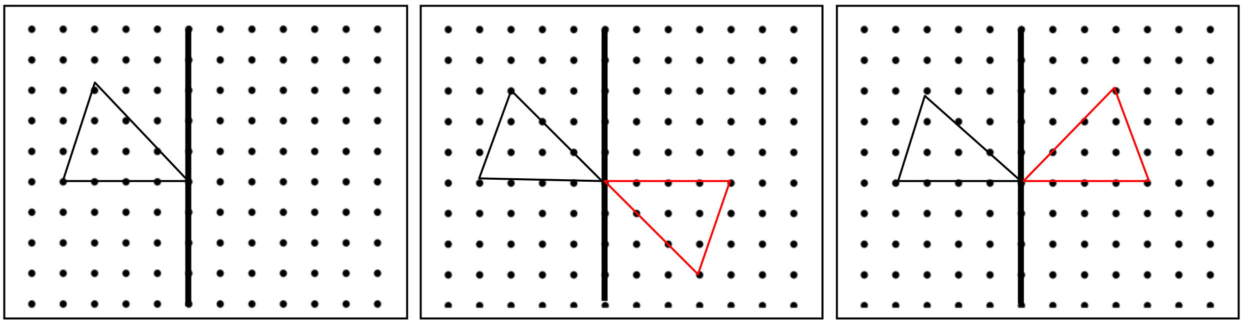
The work with line segments seems to have imposed a necessity to maintain the distances and angles, demanding the control of various elements – something that went beyond the visual memories Edson associated with mirrors. To orientate himself, Edson created a new external sign (using elastic bands to delineate a perpendicular), which helped him locate the symmetric point. To justify these new actions, however, Edson once more evoked the perceptive states he associated with looking in mirror: "*I used a bit of a crazy example. As if I was working with a mirror and I tried to measure the distance, the same distance*". We might say, paraphrasing Damásio (2005, p.236), that in evoking images of an object of his consciousness, it was not only aspects of its physical structure that were evoked, but also perceptive and motor states associated with the changes provoked by this object on the organism (Edson) when he learnt about it.

The strategy applied by Edson to deal with the line segments also suggests a change in geometrical perspective – from the intrafigural to the interfigural – the use of the external sign evidence of his attempts to relate figures in a structured mathematical space. Edson used this external sign in all the tasks which followed, including those which involved locating and constructing a missing axis of symmetry.

Lucas's solutions

Lucas was born with a congenital disease which led to a complete loss of sight by the time he was two years old. He had already completed High School within the mainstream school system when he participated in the research and he was familiar with a variety of geometrical objects and relations, although he told us he had never studied symmetry or geometrical transformations.

Lucas worked upon the same activities and in the same order as Edson. After working with the symmetrical cardboard figures, he described how the axis "*has the property of changing a figure into two figures of equal dimension*." This statement is characteristic of an intrafigural perspective – and in fact it may well be that the demands of the task itself privilege such a view: the students were given whole shapes and invited to explore properties internal to them. In the activities which involved reflecting polygons in a given axis, he declared to have constructed "*the same figure on the opposite side. They are equal. But everything is inverted".* He progressed with relative ease until he came to a task in which one of the vertices of the given figure (a triangle) was positioned upon the axis (Figure 8a).



a b c

Figure 8: Lucas's attempts to reflecting the given triangle

In his first attempt, Lucas extended two of the sides of the triangle (Figure 8b) leaving the third side on the image triangle parallel to its correspondent on the given. He felt the figure constructed and its relationship to the whole configuration, before, without offering any explanation, he remade the image as shown in Figure 8c. In order to follow his thinking, the researcher asked him about his first attempt:

> ***Lucas****: I was trying to reproduce this figure here, but I wasn't managing to fine the alignment. So, observing better, I could see that the base of the triangle (*the side horizontally represented on the geoboard*), which I hadn't noticed before. I was more concerned with the sides and not the base.*
> ***Res****: OK, this base in relation to the axis, how is its position? How did it help you, in a way that the sloping sides hadn't helped?*
> ***Lucas:*** *I imagined it dividing the axis of symmetry. Imagining another axis of symmetry to reproduce these sides here.*

In his second attempt, Lucas correctly inverted the given figure when he reproduced it on "the other side". To do this, he appears to have imagined a second axis, passing through the base of the given triangle as illustrated by the dotted line in Figure 9. Two points in particular stand out in the strategy employed by Lucus. First, his work at the intrafigural level – he considered the whole geoboard as a potentially symmetrical figure, on which could be traced two axes of symmetry. This seems to have helped him imagine what the required triangle would look like. Second, from the moment he began to imagine the additional axis of symmetry, the strategy changed to from intra to interfigural, with his actions and explanations suggesting that he imagined the sequence of reflections as shown in Figure 10. That is, Lucas, having indicated the imagined axis of symmetry, first traced out the reflection of the given figure in this new axis (the orange triangle), then reflected this in the original axis to obtain the green triangle, before offering the red triangle as his final answer.

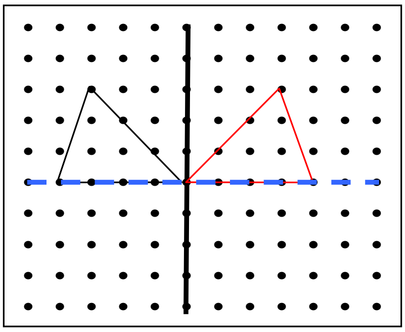


Figure 9: Imagining another axis

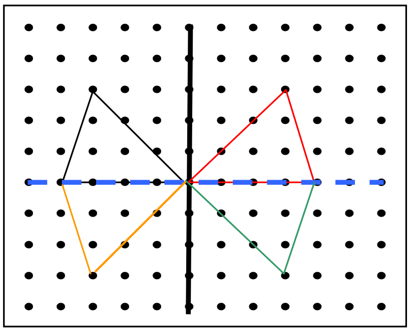
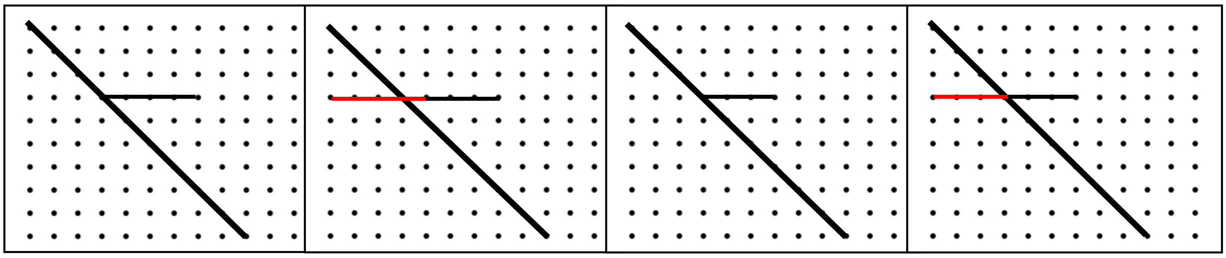


Figure 10: Completing the reflections

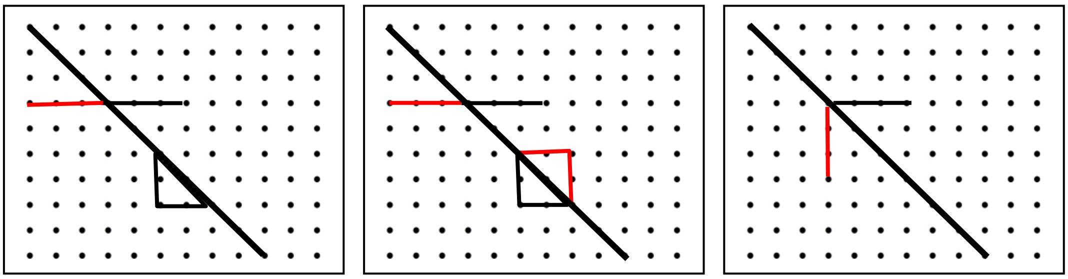
Following this task, Lucas began to work on the reflection of line segments. Faced with a horizontally orientated segment crossing a sloping axis, Lucas once again extended the line segment, seemingly not taking into account the angle that it made with the axis. As he added the segment shown in Figure 11b, he was clearly not satisfied with one of the properties (congruency) and declared "*Here, I am going to have to improvise because if I start from the same point it won´t have the same measure*". The strategy of extending the horizontally orientated segment may have been motivated by the Lucas's actions during the previous task – when it had led to success. Whatever, at this point the property of congruency still seems to be very salient to Lucas.



a b c d

Figure 11: A task involving a horizontal line segment

The researcher reduced the length of the original line segment (Figure 11c) and Lucas adjusted his image, satisfied with his result (Figure 11d). With the aim of helping Lucas to see that there was something wrong with his response, the researcher positioned two more line segments on the geoboard (Figura 12a).



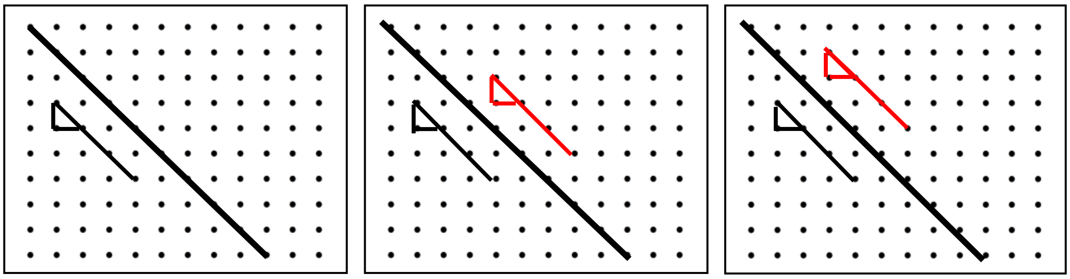
a b c

Figure 12: Adding two more line segments

This time Lucas had no difficulty in immediately constructing the correct image, as shown in Figure 12b and the following discussion ensued.

Res: *Analyse the first figure that you made in relation to the axis of symmetry.*
Lucas: *The angle that the figure that I made forms at the axis of symmetry is not the same as yours, at the axis of symmetry the angle is not the same,* (He repositions the elastic until he is happy the angles are the same). *I think it is better now* (Figure 12c).

The intervention made by the researcher seemed to enable Lucas to make connections between experiences from the recent past, when the object under study was a triangle touching the axis, and the current situation involving the reflection of line segments. By connecting these experiences, the aim was that he would be able to formulate a general strategy (an action of a prospective character) that could be applied to any geometrical object. Lucas's actions and words evince an integration of perspectives generated in past perceptive experiences which are coming to compose his arsenal of multimodal resources, usable in the actual moment of the experiment and, potentially, in future actions. However, when Lucas was challenged with a new figure for reflection (Figure 13a), he had to cope with a figure which did not share any points in common with the axis of symmetry. He perceived this difference immediately, declaring that "*the reference is not on the axis of symmetry*". Until this task, he had chosen as his initial point of reference, a point on the axis, and it was from this point that he had constructed the image – seemingly treating the given figure, axis and image as a whole, an intrafigural strategy.



a **b** **c**

Figure 13: A figure that is not touching the axis of symmetry

This time he quickly perceived that it would be necessary to select another point of reference – and perhaps his use of the word "reference" indicates that in fact hes activities were characteristics by a more interfigural flavour, as it seems he is, and may already have been, focused on transformations between parts, rather than decompositions of a whole.

His first attempt to reflect the given figure is shown in Figure 13b. As he begins to explain to the researcher how he constructed this image, he continues his tactile explorations and his facial expression makes it clear that he is not yet satisfied. In his second attempt (Figure 13c), the longer line segment (the flag-pole) is correctly located but the triangle (flag) is not. As he re-explores his image once again, he makes a connection with the incorrect positioning of the triangle and his previous error on the task presented in Figure 11a.

> *Lucas: It isn't obeying the alignment* (refering to the incorrectly positioned triangle). *Ah, I have found the x in question. I made the same mistake again.*
> *Res: Which?*
> *Lucas: The segment. Once again I took into consideration only the axis of symmetry and not the direction of the angle. This angle is going outwards* (points to the triangle flag), *I put it inwards.*
> *Res: Yes.*
> *Lucas: Now I think I have got it* (Figure 14).

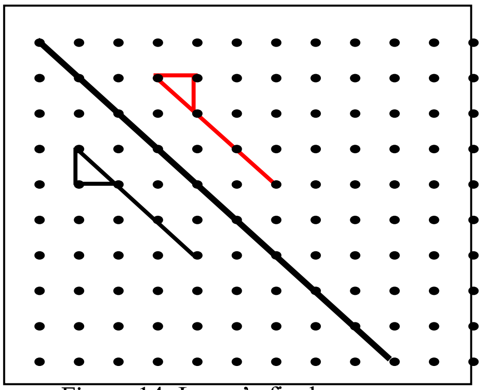


Figure 14: Lucas's final response

As Lucas progressed through the tasks, he became increasingly aware of the necessity to control the angles between the elements as well as shapes and measures. The distance used by Lucas at the end is not a (intrafigural) measure intrinsic to the figures involved but one which takes into account relationships between figures in the space in which they are contained. In particular, Lucas showed during the final task that he was considering the equidistance between the given figure, the axis of symmetry and the image figure, establishing comparisons between the points represented on the geoboard in the two semi-planes determined by the axis of symmetry. He also inverted the figure image – not locally – but with reference to the angle it needed to make with the axis of symmetry, a strong indication of interfigural thinking.

Some considerations

The research reported in this paper centred upon the domain of Geometry, an area usually associated with visual experiences. In relation to this domain, Piaget and Garcia (1987) have suggested that both the historical development of Geometry and the psychogenesis of geometrical structures can be characterized by three phases: intrafigural, interfigural and transfigural. We used this classification as a starting point from which to interpret how geometrical transformations are perceived when the entrances to the mind do not include the eyes.

In the first set of tasks (explorations of symmetrical figures), strategies characteristic of the intrafigural level predominated, although it should be said that their emergence depended as much on the structure of the tasks themselves and the interventions of the researcher as on any supposed developmental level on the part of the students. We suggest that symmetry began to become associated with the deconstructing of a single figure into two congruent parts separated by an axis of symmetry. And it was the activity of folding, either realized physically or in an imaginary simulation that structured the meanings that were expressed in the activities of this session. It is important to stress that it was a rather specific kind of folding that was involved, as it is not always that we fold to create congruent parts. This seemed to be quite clear to the students and we note that in a discussion with Edson, which occurred after the second session of activities, he made a connection between folding clothes, and, in particular, his bedcover, that could be associated with symmetry. He suggested that the fold is made "*point to point, it has an axis of symmetry … actually possibly various axes of symmetry*". Of course, there are a number of limitations in associating folding with reflection – it emphasises the intrafigural aspects of symmetry rather than the interfigural aspects of reflection, in that the relationship of dependence between the original points and their images is not highlighted. But one point was clear, as the students became more confident in articulating the mathematical properties of symmetrical shapes, in the multimodal images associated with the mathematical object that were becoming part of their memories, they were not wiping out any connections with the physical in favour of some kind of disembodied symbolic representation. Knowing about symmetry did not transcend feeling it.

As they moved on to tasks which involved reflection, both students changed the multimodal resources they bought to bear on the activity. For Edson, who had visual memories associated with mirrors, it was by structuring his interactions around these past experiences that he became aware of some of the mathematical proprieties inherent in the activities he was working on. Once again, we suggest that this led to embodied forms of conceptual knowledge. It was he himself who suggested that although he could no longer see he could still make sense by imagining visual experiences from the past. These memories, coordinated with the tactile explorations and the ongoing discussions that accompanied them, came to form the basis of the meanings for the transformation reflection that were expressed during the research session. The researcher also played an important role in motivation the simulations behind Edson's meaning-making activity, guiding the associations between past and present experiences in ways intended to encourage Edson to build appropriate mathematical models for future activities.

In Edson's case, the researcher had the option of appealing to a visual past. For Lucas this was not possible and the term mirrors did not feature in his thinking about the activities at all. Perhaps because of this, he really did need to identify the properties invariant across all the tasks, determining by "touch that disposition required between the parts and the whole to enable it to be called beautiful", as Diderot (1916/1749, p.70) also described the blind man's view of symmetry. The ways that Lucas used his hands in order to seek and then reproduce these invariants seem more characteristic of the interfigural level than the intrafigural – he continually moved his hands to search for the relative positions of the geometrical points that constituted the required objects in a mathematically structurable space. Piaget and Garcia (1987) argued that the intrafigural and interfigural represent the first two of three hierarchically organised epistemological phases through which mathematical ideas develop. That is, their view is that all learners necessarily pass through a stage of intrafigural analyses before reaching the interfigural stage. The data we obtained from working with these two students suggests, however, that the perspective that comes to be adopted depends on the available means for mediating the ideas as well as on the demands of the tasks themselves and that it would be a mistake to expect those who do not see with their eyes to necessarily follow the same learning trajectories as those who do. Indeed, it was not even the case that Lucas, who had no visual memories, followed the same trajectory as Edson, who did.

From the point of view of the embodied cognition approaches introduced earlier in the paper, this should come as no surprise. For authors such as Gallese and Lakoff (2005, p.2), imagining, perceiving and doing are embodied acts, structured through our bodies' lived-in experiences of the world. Comparing the interactions which occurred during the episodes analysed in this paper, it is clear that at least some of the multimodal resources reactived were distinct for each of the participants and that they were products of their intrapersonal planes. For Edson, images in mirrors are memories of a sensory modality formulated during the period in which his eyes still functioned as tools to see with. These memories made up part of the repertoire of resources he could use in his imaginings and hence in his doings, by re-enacting perceptive states experienced bodily in the past. This was not the case for Lucas.

For him, the multimodal images were formulated and re-activated along the development of the empirical procedures of the interviews. Inserted into an emerging cultural system, Lucas came to associate the perceptions motivated by his fist haptic contact with the objects represented on the geoboard with other aspects of the cultural systems involving learning situations that had characterised his past. In other words, although he was surrounded by cardboard figures, pins, a wooden board and plenty of elastics, it was the interaction with an*other* (the researcher in the case) which bought these resources to life, as it were, endowing them with specific functionalities and meanings of the Merleau-Ponty's phenomenological world.

Before finishing, we make one last point in reference to the hand movements that both students used as the employed them to intentionally see. We noted earlier a tendency in both of them when they first explored a given tactile scenario to move their hands so that they followed almost symmetrical paths. Was this a simple coincidence or does it suggest that symmetry should be seen as more that what Diderot described as "perhaps a matter of pure convention", associated with particular kinds of spatial displays – after all, as one of the students Edson, jokily commented at the end of the second session, our bodies can be considered as roughly symmetrical, with an axis passing through our noses. We had not anticipated the use of symmetrical hand movements in the design of the tasks—since they had been developed on the basis of research into sighted learners' understandings of symmetry and reflection (Kuchemann, 1988; Grenier & Laborde, 1988; Healy, 2002). This stresses how our tendency to design learning scenarios for the blind relying exclusively on what we know about the learning trajectories of sighted might not offer them the best opportunities for mathematics learning. Moreover, by concentrating more specifically on the how hands can be used to conceive mathematical objects, we are beginning to recognise how very intimate the relationship between bodily groundings and mathematical abstractions is.

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