**Algebraic, graphic and natural language registers to interrelate different worlds of mathematics: the case of function**

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ABSTRACT

We designed a set of activities on one real variable functions, based on various semiotic registers of representation (Duval, 2000), expecting to answer the following research question: "Can the individual concept image (Tall & Vinner, 1981) of function be enriched by the use of various functions and their associate ones, using verbal texts, algebraic laws and graphs?". Seven in-service Mathematics teachers from Brazil carried out these activities, during six two and a half hours sessions. In this paper, we focused on protocols and observers' written notes of one teacher from the group, which were analysed in the light of the theoretical framework of The Three Worlds of Mathematics (Tall, 2004a, 2004b). We found that this teacher had, in his concept image, notions related with only first and second-degree polynomial functions, and used mostly the embodied world to justify his answers. The activities seemed useful for him to broaden his concept image.

Key-words**:** Registers, function, concept image, embodiment, symbolism, formalism.

RESUMO

Elaboramos um conjunto de atividades sobre funções de uma variável real, baseadas em vários registros semióticos de representação (Duval, 2000), esperando responder a seguinte questão de pesquisa: "A imagem de conceito individual (Tall & Vinner, 1981) de função pode ser enriquecida por meio do uso de várias funções e de suas associadas, usando textos verbais, leis algébricas e gráficos?". Sete professores brasileiros de Matemática trabalharam com essas atividades, durante seis sessões de duas horas e meia cada uma. Neste artigo, focamo-nos nos protocolos e nas anotações feitas por observadores do trabalho de um professor desse grupo, que foram analisados à luz do quadro teórico dos Três Mundos da Matemática (Tall, 2004a, 2004b). Resultados mostram que este professor tinha, em sua imagem de conceito, noções relacionadas apenas com funções polinomiais de primeiro e segundo graus e que usou predominantemente características do mundo corporificado para justificar suas respostas. As atividades nos pareceram úteis para que esse professor ampliasse a sua imagem de conceito.

Palavras-chave: registros, função, imagem de conceito, corporificado, simbólico, formal

## Introduction

For several years, we worked, in weekly sessions, with seven Brazilian public schools mathematics teachers, discussing pedagogical and mathematical aspects of compulsory school contents. In order to provide a pedagogical tool that could help their students to cope with difficulties in algebraic methods to solve inequations as reported by many studies (for instance, De Souza & Campos, 2006; Tsamir & Bazzini, 2004), the group agreed to discuss a functional graphic approach, meaning to represent each side of the inequation as a function, to plot their graphs in the same coordinate system in order to compare them according to the sign of the inequation, and to find the final solution.

To present this approach, we started a discussion about real functions and their graphs, and those teachers expressed some of their ideas about the subject. Utterances of four of them, in particular, caught our attention:

> "I am scared to death of graphs!" (teacher T1).
> "When I start the subject 'function', I bring to the classroom the naval battle game" (teacher T2).
> "I can only understand the graph if the horizontal variable is t [meaning the time] if it's not, I cannot …" (teacher T3).
> "I always use a table of values, with x in one column and y in the other." (teacher T4).

Teacher T1's comment suggests that he does not discuss graphs in his classroom and, if he does, it might be "painful" for him. To change this situation, and in order to deal with the functional graphic approach, he needs to have a *better understanding of graphs.*

Teachers T2 and T4's utterances show emphasis in a point-to-point graph plot which may bring difficulties for students to grasp graphic ideas of function (Duval, 1993) and is inefficient for graph visualization (Duval, 1999) and consequently to graphically solve inequations. In this way, they need to *enlarge their repertoire of graphs of functions*, and learn to visualise the graph as a whole, because graphs are more general than functions' algebraic expressions, in the sense that there are graphs which are functions but do not have an algebraic expression related to it. In addition, we live in a world of images, graphic calculators and computers. With them, graphical images are spread out.

Considering Brazilian textbooks, the independent variable is almost always represented by *x*. So, we can suppose that Teacher T3 struggles in his work with graphs of functions in the classroom. We understand that T3 needs to *generalise the idea of variable* (Sierpinska, 1992, obstacle #16, p.55) for both the work with graphs and functional graphic approach for inequations.

All these teachers' difficulties with graphs point out to a need to discuss a global interpretation to functions' graphs in opposition to a point to point one before discussing the functional graphic approach. We understand that, to grasp important characteristics of a function, an individual, specially a teacher, has to know *what contributions graph and algebraic expression give to each other and relate these two ways of seen functions*.

In this way, we looked for answers to the following research question: *"Can the individual concept image* (Tall & Vinner, 1981) of function *be enriched by the use of various functions and their associate ones, using verbal texts, algebraic laws and graphs?"*.

By enriching the individual concept image in the context of functions, to our research study, we mean to *propitiate a better understanding of graphs*, to *enlarge the repertoire of graphs of functions* of the subjects of our study, to *help them visualising the graph as a whole* and *generalising the idea of variable*.

In order to do so, we designed a set of activities based in algebraic, graphical and natural language semiotic registers of representation (Duval, 1995, 2000), composed by questions in verbal text to propitiate connections between those registers, and to provoke teachers to find out those contributions and make those relations. When starting the analysis of the protocols, we found many characteristics of embodied world in these teachers work and this motivated us to analyse data using the theoretical framework of the Three Worlds of Mathematics (Tall, 2004a), looking for characteristics of embodied, symbolic and formal worlds in these teachers' concept image (Tall & Vinner, 1981) of function. We used this theoretical framework because we believe that using characteristics of only one of these worlds is not enough to make connections between graphs and algebraic expressions. More than that, it is essential to use characteristics of all three worlds.

**A theoretical framework**

Based on three different kinds of concepts in Mathematics, Tall (2004a, 2004b) proposed a categorisation of mathematical cognitive development into three different but interacting ways of thinking, which lead to three different worlds of Mathematics:

A conceptual-embodied world of perceptions, in which individuals make sense of properties of objects by observing and verbalizing them through descriptions and acting upon them.

A proceptual-symbolic world, in which mathematical entities are symbolized and the actions performed as procedures can be flexibly seen either as process or as concept. This flexibility is defined as a procept (an amalgam of process and concept) by Gray & Tall (1994) and is also related to the ability of understanding that different procepts give the same effect (Watson, 2002). Hence an individual can choose which procept is more suitable for a situation.

A formal-axiomatic world, in which mathematical structure is deduced from axioms, definitions and theorems by formal proof.

In this research study, we are implicitly using a concept of function, but not in an axiomatic way, which is not usual in the secondary school level. Hence, we do not expect the formal world to manifest itself in its whole, but we search for formal characteristics present in the study of this concept and in the teachers work.

As Tall (2004), we believe that an individual has to relate characteristics from all those worlds in order to have a broad concept image of a concept, which is defined as

> "… the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures." (Tall & Vinner, 1981, p. 152)

And all learning experiences of an individual are incorporated to his or her concept image.

In order to analyse teachers' protocols using this theoretical framework, it is important to make clear its relation with each kind of register we used and the concept of function.

**Functions' semiotic registers of representation and the three worlds of mathematics**

To develop our set of activities, we used algebraic, graphical and natural language semiotic registers of representation of functions, and we believe they allow different connections between characteristics of the Three Worlds of Mathematics. We explain ourselves.

**Algebraic registers:** we believe it mainly emphasise characteristics of the symbolic world because it is formed by mathematical symbols and it is possible to solve an inequation, for instance, merely by manipulating them.

For example, $f(x) = (x-2)^3$ is part of the symbolic world. An individual has to understand this symbolism as a whole and also each of its components in order to manipulate it and use the best procept for the situation at hand. For instance, the individual should understand that each value $f(x)$ is the dependent variable and is the result of the operations performed over x; especially in which order these operations have been performed. Also, it is important to know how to get another algebraic law with the same effect, such as $f(x) = x^3 - 6x^2 + 12x - 8$.

Although this algebraic register of representation is part of the symbolic world, it can also have its roots in the other worlds. When a function is seen as a "machine", the input x is transformed in the output $= f(x)$, by means of the operations dictated by the machine and this idea provides an embodied way of dealing with the algebraic law of f. A function law can also be part of the embodied world when an individual uses a calculator to find out the order in which its operations are performed.

The formal world is present when the individual understands that $f(x)$ represents a function f with independent variable x, and f(x) is different from f, meaning that the "output" (f(x)) is the result of applying an algebraic law (f) to a value of x.

**Graphic registers:** it is possible to relate them to embodied and formal characteristics. Formal, because the graph needs to be defined and understood according to formal rules, such as $graph(f) := \{(a,b): a \in D(f) \; and \; b = f(a)\}$. Embodied, because we can use it as a way either to describe or to observe a function.

We believe all these characteristics are needed in order to transform an algebraic register into a graphic register. Specially the fact that $f(x) = (x-2)^3$ and $f(x) = x^3 - 6x^2 + 12x - 8$ are two procepts that give the same effect, but the former is much more useful to have a global and instantaneous visualization of the graph, when dealing with paper and pencil and taking $g(x) = x^3$ as the reference function. The graph itself is an embodied representation, from which it is possible to get an idea of how the function behaves and how the shape of the graph is in the interval that can be seen, for instance, in the computer screen.

However, the plotting of a graph is fundamentally based on the formal world because it is necessary to follow some formal conventions, such as taking two perpendicular axis (for functions with one real variable), representing the independent variable on the horizontal axis and the dependent one, on the vertical axis. A point in the Cartesian plane is found by predetermined formal laws, and the graph is the set of points described by $(x, f(x))$.

In addition, if a graph of a function like $f(x) = (x-2)^3$ is plotted based on the graph of its reference function $g(x) = x^3$, its behaviour is defined by the one of g, because the addition of the constant –2 to the independent variable causes a horizontal translation of two units to the right in the graph of g. Such understanding of what happens to the graph of a function (g) when adding a constant (–2) to the independent variable (x) involves characteristics of the symbolic and the formal worlds. Formal characteristics, because it is necessary the individual to realise that there is a composition of two functions in f, one is the addition of –2 in the independent variable, and the other is the function to the third. Bringing it to the symbolic world, we can write: $x \rightarrow x-2 \rightarrow (x-2)^3$.

**Natural language registers:** it is used by any individual to express him or herself, and it is useful to help making connections between any of the worlds. It can be used to express any characteristic of a function, be it in the embodied, symbolic or formal world, or related to algebraic or graphic registers. We believe that when converting from algebraic to graphic register, or vice-versa, there is, in between, a conversion to the natural language, that is either mental or verbal, because it is necessary to mentally or verbally analyse the registers and the process of converting one into another.

**The research**

In this paper, we use data from a research study regarding semiotic registers of representation in the case of functions, with a broader intention: to carry on the development of the theoretical framework of the Three Worlds of

Mathematics. Before us, Akkoç (2007) and Angelini (2010) used this framework as a tool to analyse data related to the concept of function. Akkoç (2007) found that previous experiences with functions in high school levels are not enough to help students to deal with sophisticated situations regarding functions in different domains rather than real numbers. Angelini (2010) concluded that 14-15 year-old students mainly used characteristics of embodied world in order to solve situations proposed with algebraic, graphic and natural language registers.

We believe that analysing our data in another light may bring a new view of how to treat difficulties in grasping the concept of function.

As we said, this research study was part of a research group working with seven mathematics teachers (teachers T1, T2, T3, T4 mentioned above, and three others, T5, T6 and T7). Four of these teachers participated in this group for five years (T1, T2, T3 and T4), and the others for two years by the time our data was collected. Six of them taught mathematics at high school level for at least four years (T1 to T6), and the other one was a Chemistry teacher for more than four years (T7). Two of the Mathematics teachers also used to teach Physics (T3 and T5). All of them worked in public schools and only one of them (T2) in private school as well. In our research study, we worked with these teachers a set of activities regarding the concept of function. The whole set of activities was carried out in six two and a half hours long sessions, in which the seven teachers worked alone or in pairs using the software Cabri-géomètre II. Teachers worked in three groups, and one observer for group accompanied their work, taking notes of their verbal discussions, doubts and decisions. At the end of each session, we collected teachers' written protocols and observers' notes and used them as data to be analysed.

It is important to note that using a computer environment and software Cabri-géomètre II is not foci for analysis in this paper. Firstly because the teachers were used to it, since they have used it in many opportunities during the five years of their work with us. Secondly, because in these activities, the aim is to work with global visualization and not point by point graph plotting. So, the computer was used to get the graphs more easily and faster than plotting them by hand, avoiding need for the teachers to look for specific values and to make a table of these.

We had a two and a half hour session once a week with these teachers, in a total of six sessions. They worked in groups of two or three members each, with one silent observer by the side of each group. Each group used the computer with Cabri-Géomètre, and they were given an activity sheet to work with.

The set of activities is composed by three ones (one activity for two sessions), with three main aims: to raise the use of graphic registers, its treatment and conversions between graphic, algebraic and natural language registers; to stimulate global visualization of functions graphs in a way in which it is possible to see the relation between independent and dependent variables, and not as a point-to-point approach or a table of values, that were already familiar to many of those teachers; and to broaden the teachers' set of known functions and, consequently, their concept image.

We chose these aims by taking into account that, in general, in Brazilian Mathematics text-books (at least the most common ones) the emphasis is on exploring algebraic register treatment, on stimulating construction of tables of values (with four or five points) from the algebraic expression and on presenting linear and quadratic functions only (De Souza, 2008).

The set of activities was designed in order to emphasise three registers of representation: algebraic, graphic and natural language. The algebraic register was chosen because it is the most usual in Brazilian Mathematics text-books in detriment of graphic and natural language registers; the graphic one, because of those teachers' difficulties in dealing with them, as long as the importance of this register to help individuals to understand functions; and the natural language because we believe it is necessary to interpret any kind of register.

In each activity, the teacher used a Cabri-géomètre II menu to easily obtain graphs of some "reference" functions and their "associate" ones. The chosen reference functions were $f(x) = x^2$, $h(x) = \frac{1}{x}$, $g(x) = x^3$, $p(x) = \sqrt{x}$ and $n(x) = cos(x)$, $m(x) = sin(x)$, and the associate ones were $f(x) + k$, $f(x + k)$, $h(x) + k$, $h(x + k)$, $|g(x)|$, $g(|x|)$, $|p(x)|$, $p(|x|)$, $|n(x)|$, $n(|x|)$, $k \cdot m(x)$ and $m(k \cdot x)$. In the activity sheet, the teacher was asked, in natural language, to plot the graphs of the reference function and the associate ones, using at least one or two values for the constant k, even rational values – in fact, he could have used as many values as he wished –, and to present domain, range and possible graph's symmetry of both reference and associate functions. We also asked for algebraic expressions of the associate ones. For instance, Activity 2 is presented on Figure 1. For each item in the activities, teachers were asked to write down the answers and their explanations of how they responded to the question.

**Activity 2**

In order to visualise the graph of each function, open Cabri-Géomètre II files sqr.fig, cub.fig, cos.fig

**I.** $p(x) = \sqrt{x}$ **II.** $g(x) = x^3$ **III.** $n(x) = cos(x)$

For each function I, II and III:

(1) draw on the paper the graph of the function; write down its domain and its range;

(2) does the graph have symmetry? If so, which one?

(3) taking the absolute value of the function (using the new tool in macro-constructions menu):

(a) What are the algebraic expression, domain and range of the associate function?

(b) What is the relationship between the domains of the function and the associate one?

(c) What is the relationship between the graphs of the function and the associate one?

(d) Does the graph of the associate function have symmetry? If so, which one?

(4) Taking the absolute value of x (using the new tool in macro-constructions menu):

(a) What is the algebraic expression, domain and range of the associate function?

(b) What is the relationship between the domains of the function and the associate one?

(c) What is the relationship between the graphs of the function and the associate one?

(d) Does the graph of the associate function have symmetry? If so, which one?

**Figure 1:** Activity 2 of our research study

For this paper, we present results obtained by analysing the protocols and observers' notes from one of those teachers – Teacher T2 –, a mathematics teacher participating in the research group for at least five years. We chose data from his work to be analysed because he was the most long-standing member of the group; he worked in both public and private schools; and he provided answers to almost all the questions proposed in the activities.

We analysed T2's protocols, searching for characteristics of the Three Worlds of Mathematics (Tall, 2004a) and observing if and how this teacher interrelated characteristics from all three worlds – embodied, symbolic and formal – when answering our questions and performing treatments or conversions requested in the tasks. We believe that, if he could relate those characteristics in the way we suggested above, he would be able to understand and work with global visualisation of graphs; hence, with functions; so, the teacher would have his concept image broader and richer than before.

**Data and results**

In our analysis of the data, we could find that, when calculating values for $f(x)$ using specific values for x, T2 behaves consistently with evaluation algebra (Tall & Thomas, 2001), in which an algebraic expression is evaluated given a value for x. And, in our opinion, evaluation algebra has characteristics of embodied world. Looking at T2's utterances when dealing with Activity 2 for $g(x) = x^3$:

> "If I take –1 to the third, it is –1, and its absolute value is 1. Then, it will be a parabola."

it seems that his behaviour is less sophisticated than evaluation algebra supposes, since, when dealing with the graph of the associate function $|g(x)| = |x^3|$, he uses two points on the graph to decide what kind of function it is.

So, he realises that (–1, 1) and (1, 1) belong to the graph of g and looking at the curve of the graph, he says it is a parabola. In our analysis, it seems that he is mainly considering the shape of the graph, an embodied characteristic, to describe what he sees, disregarding the formal and symbolic characteristics of the algebraic law.

Later on, in a task involving the function $n(x) = cos(x)$, he realizes that this function's graph also has "curves" and he knows it is not a parabola because of his previous experiences with trigonometric functions in a technical course. He goes back to the task involving the function $|g(x)| = |x^3|$, and realises that the graph of g could not be a parabola just because it is curved as a parabola. At this moment, he says

> "When I saw the graph [of function n], I said it was a second degree function. If I'm not seeing the function, then I cannot say that the degree is two. It could be a cubic one".

In this utterance, he is referring to the algebraic law as "the function", showing that he may be confusing the algebraic law (symbolic world) with the concept itself (formal world). Also, it seems that he realised that using solely the graph of a function (embodied world) to analyse the function itself may deceive who is interpreting it, and the algebraic law has to be analysed as well. In this example, the graph, as an embodiment, was taken as one of the ways he used to validate his conclusions. When he understands that it is important to compare both algebraic and graphic registers, he also reinforces our opinion that a teacher has to explore different worlds in order to propitiate a broaden understanding of function to his/her students.

In another task, regarding the graph of $h(x) + \frac{1}{2} = \frac{1}{x} + \frac{1}{2}$, having at hand the graph of $h(x) = \frac{1}{x}$, T2 did not use his experience with the associate function $f(x) + 2 = x^2 + 2$, in which he saw the vertical translation of a function's graph when adding a constant to the dependent variable. He added $\frac{1}{x} + \frac{1}{2}$, resulting in $\frac{2+x}{2x}$, probably in an attempt to get a more familiar expression. When he reached it, he, again, thought of giving values to x, but he changed his mind realising that treating the expression did not help him in plotting the graph. And he said

> "no, let's think of the previous task: when we add, it [the graph] goes up"

and makes an embodied gesture to show that he believed the branches would move apart, in opposite ways, following the quadrants bisector.

He may have understood that, in this case, using $\frac{1}{x} + \frac{1}{2}$ instead of $\frac{2+x}{2x}$ is better to plot the graph. In our opinion, this shows that using two or more procepts that give the same effect, is important in mathematics classrooms because it helps the subjects to decide which one is more useful for the situation at hand.

T2's attempt to describe the translation of the graph of h using an embodied approach was not successful because he moves his hands in a quadrant bisector direction, instead of showing a vertical movement. In this case, his embodied idea was not consistent with formal world. The use of characteristics related to the embodied world, in this case, helped him to elaborate a first analysis of the graph he was trying to obtain, while his symbolic approach of adding terms was not helpful at all.

In the next task, looking at the graph of $n(x) = cos(x)$, T2 says

> "If I look to the graph and it cuts once the x-axis, it has one root, if it cuts twice, it has two roots. If it cuts three times, it has three roots, but I cannot say what its degree is because I don't know what happens with greater values of x".

Still looking at the same graph, he asks the researcher

> "If I cut the graph here [showing one part of the graph that cuts the horizontal axis six times], can I say its degree is six?"

These utterances show that he thought n could be a function "with a degree", meaning a polynomial function, although he never used this expression while referring to a function "with degree". When he saw the graph cutting the horizontal axis more than twice, he decided it was an embodiment for a function which was not a straight line or a parabola. Therefore, he believes that this graph must be related to the algebraic law of a polynomial function, which is part of the symbolic world. In this case, the relations he has made between the two worlds are detached from the formal world, and the lack of understanding guided him to choose the polynomial function as an explanation.

In general, T2 uses the name given to the reference function to the associate one. For example, if the reference function is $g(x) = x^3$, he names the associate $|g(x)|$ as $g(x) = |x^3|$. T2 also experienced difficulties in writing the algebraic law of some associate functions. For instance, to add k=2 to the independent variable in $f(x) = x^2$, T2 writes $f(x+2) = (x+2)^2$ in his protocol. However, as he is not sure about what he has done, he continues in his protocol to deal with the algebraic law. Analysing his work, it is possible to notice that he uses $x = -2$ in the expression $f(x+2) = (x+2)^2$, and concludes that $f(0) = 0$. In his mind, if this was true, (0, 0) would belong to the graph of the function, but he knows it does not because he is looking at the graph of $g(x) = (x+2)^2$ in the computer screen. In this way, he concludes that the right expression must be $f(x+2) = x^2$ because he uses $x = -2$ in both sides of the expression, obtaining $f(0) = 4$, and he can see that (0, 4) belongs to the graph he is dealing with. His work with the algebraic law of the associate function shows a lack of interrelation between symbolic and formal worlds, since he is using just one point of the graph to choose an algebraic expression he believes to be correct. In this case, only the embodied world is being used to validate a characteristic that is part of the symbolic world, apparently with no relation to the formal world. Regarding the formal world, he did not understand that zero is the image of –2 in function $g(x) = (x+2)^2$, since the taken value for x is –2, and not zero! Meaning that the point he was considering was (–2, 0), and not (0, 0). Apparently this mistake happened because he did not know how to algebraically represent g as an associate function of f, using characteristics of symbolic world.

Considering this analysis of T2's work, it seems that, when T2 is making any validation for his work, he uses embodied characteristics, even to validate characteristics from symbolic and formal worlds, and, at first, he associated all functions to affine and quadratic functions only, which could have made it more difficult for him to work those activities. This means that his concept image of function had embodied characteristics even of affine and quadratic functions. However, working with other functions helped him to broad his repertoire of functions, as now he knows some other functions and their behaviour, although it seems that he is still using only characteristics from embodied world.

**Conclusions**

In this paper, we have analysed the work of teacher T2 in dealing with algebraic, graphic and natural language registers of several functions, in an attempt to raise characteristics of embodied, symbolic and formal worlds of mathematics, and interrelations made by this teacher between mathematical worlds and these different registers, aiming to see if his concept image was enriched somehow by the proposed activities.

In our analysis, we concluded that T2 never searched for any kind of validation in the formal world when he tried to answer the questions in which he was asked to associate two registers. In fact, he never used any characteristic from formal or symbolic worlds for validation, apart from giving a value for x and finding f(x) in order to evaluate the algebraic law.

On the other hand, using characteristics from embodied world has made him realise that the graph of $|g(x)| = |x^3|$ is not a parabola, and that the shape of the graph can be deceitful. Also the quantity of roots of a function is not related to its "degree", meaning that he realised that not all functions are polynomials.

In this case, although he has not used characteristics of formal world, he ended up being aware of their existence when he perceived that "what you see is not what you get", meaning that what is shown on the screen is not the whole function. It is necessary to analyse other characteristics, for instance the algebraic law, in order to understand all properties of a function, being it in any register.

When trying to work with symbolic world (for example, by manipulating $\frac{1}{x} + \frac{1}{2}$), he realised that there are situations in which one procept is better than the other to work with. We perceived that this idea of reducing an addition of fractions to the same denominator is a very strong procedure that is almost coercive for this teacher. In this way, we believe that it was a gain that he understood it is not always necessary to do it.

Still regarding characteristics of symbolic world, we wish to emphasise the example in which T2 tries to find the algebraic expression for the associate function $f(x+2)$ when the reference one is $f(x) = x^2$. As his knowledge of symbolic world and its associations with formal world seems to be lacking, he uses the correct expression, tries $x = -2$, and does not realise that the result is the point (–2, 0), instead of (0, 0), which he uses to modify the right expression to obtain $f(x+2) = x^2$.

Although he had many problems with symbolic and formal worlds, and even with embodied one, the use of different kinds of functions, as long as their graphs and algebraic laws, during this set of activities, was useful for T2 to broaden his concept image of function, one of our main goals. He now understands that there are other kinds of functions different from polynomial ones, it is necessary to relate graph and algebraic law of a function, and such relation made him realise that what is in the computer screen is not the function as a whole, which were also our goals. In addition, he understood that a graph with a curve may not be a parabola, which is important for him as a teacher.

Readdressing our research question, we concluded that this teacher have enriched his concept image in some ways, since, after working with the activities of this research study, he has a *better understanding of graphs* and he *enlarged his own repertoire of graphs of functions*. However, these activities were not enough for the complete enrichment of his concept image, as he still does not *visualise graph as a whole* and does not *generalise the idea of variable*. Considering this, we say that T2 does not make a complete journey through Three Worlds of Mathematics regarding the concept of function.

From the analysis of T2's work with these activities, we suggest that, in teacher education courses, it should be given more emphasis to characteristics of the formal world at least when discussing compulsory school Mathematics contents. For example, teachers should be able to understand formal characteristics present in different representations of functions, at least algebraic laws and graphs. Also, these formal characteristics should be related to embodied and symbolic worlds, so they can all be linked together, and all this must be done with a broad range of functions, not only first and second degrees polynomial ones. If characteristics from the three worlds are discussed in only two specific examples such these, we believe it will be difficult for the teacher to understand relations among them in other kinds of functions.

More attention should be given to both graphs and algebraic laws, discussing the embodied modifications that occur from graphs of reference functions to associate ones, and the symbolic manipulation from the law of the reference functions to the associate ones.

In this paper, we presented results from a single teacher, but our research study raised similar results from the group of seven mathematics teachers. Even so, we consider that this kind of research needs to be more comprehensive, including more teachers and also students at the end of high school or at the beginning of university levels. An interesting attempt should be a teacher who interrelate different mathematical worlds teaching a group of students, in order to find out whether this would incentive students to make such interrelations as well.

Our research study shows an attempt to use the theoretical framework of the Three Worlds of Mathematics with the idea of concept image. It is one of the few studies to do so. Considering Akkoç (2005), Angelini (2010), Bonomi (1999) and Sierpinska (1992) regarding difficulties of learning the concept of function, all of them state that students have difficulties in learning functions and in using it in a broader context. Apparently, these students did not have an approach to the study of functions that interrelates different worlds of Mathematics. Our research seems to reveal that it is by considering such approach that a rich concept image of function may be created, in a way to help students to have a broad repertoire of functions, not being confined to affine and quadratic functions, and to understand characteristics of all mathematical worlds, instead of just the embodied one.

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