Fluid Forces on a Circular Cylinder Moving Transversely in Cylindrical Confinement: Extension of the Fritz Model to Larger Amplitude Motions

Cédric Leblond, Jean-François Sigrist, Christian Laine, Bruno Auvity, Hassan Peerhossaini

FLUID FORCES ON A CIRCULAR CYLINDER MOVING TRANSVERSELY IN CYLINDRICAL CONFINEMENT: EXTENSION OF THE FRITZ MODEL TO LARGER AMPLITUDE MOTIONS

Cedric Leblond
Service Technique et Scientifique
DCN Propulsion
Indret - BP 30
44620 LA MONTAGNE, FRANCE
Email: cedric.leblond@dcn.fr

Jean Francois Sigrist
Christian Laine
Service Technique et Scientifique
DCN Propulsion
Indret - BP 30
44620 LA MONTAGNE, FRANCE

Bruno Auvity
Hassan Peerhossaini
Laboratoire de Thermocinetique
CNRS UMR 6607
Rue C. Pauc - BP 50609
44306 NANTES, FRANCE

ABSTRACT

This paper is related to the fluid forces prediction on a rapidly moving circular cylinder in cylindrical confinement. The Fritz model, which mainly assumes infinitesimal motions of the inner cylinder in an inviscid fluid, is one of the simplest model available in the scientific literature and is often used by design engineers in the nuclear industry.

In this paper, simple non-linear expressions of fluid forces are derived for the case of finite amplitude motions of the inner cylinder. Assuming a potential flow, advection term and geometrical deformations can be taken into account. The problem, formulated as a boundary-perturbation problem, is solved thanks to a regular expansion. The range of validity of the approximate analytical solution thus obtained is theoretically discussed. The results are also confronted to numerical simulations, which allows to emphasize some limits and advantages of the analytical approach.

NOMENCLATURE

$(x,y)_{(\mathbf{e_x},\mathbf{e_y})}$ Cartesian coordinates system.

$(r,\theta)_{(\mathbf{e_r},\mathbf{e_\theta})}$ Polar coordinates system.

ρ Fluid density.

$\mathcal{C}(t)$ Inner circular cylinder.

Ψ Parametric curbe of $\mathcal{C}(t)$.

R_1 Inner circular cylinder radius.

R_2 Outer circular cylinder radius.

α Cylinder radius ratio R_2/R_1.

$\mathbf{n}$ Exact outward normal to inner cylinder.

$\mathbf{n_0}$, $\mathbf{n_1}$, $\mathbf{n_2}$ Approximate outward normals to the inner cylinder respectively at leading order, first order and second order.

$e(t)$, $\dot{e}(t)$, $\ddot{e}(t)$ Displacement, velocity and acceleration imposed to the inner circular cylinder.

ξ Ratio between the maximum displacement of the inner cylinder e_{max} and the radial clearance $R_2 - R_1$, $\xi = e_{max}/(r_2 - R_1)$

r_c Exact inner circular position.

r_0, r_1, r_2 Approximate inner circular cylinder positions at leading order, first order and second order.

$\mathbf{u}$ Local fluid velocity.

$\mathbf{u_0}$, $\mathbf{u_1}$, $\mathbf{u_2}$ Approximate velocity at leading order and velocity rectification of first and second order.

p Local pressure.

p_0, p_1, p_2 Approximate pressure at leading order, and first and second order rectifications.

Φ Velocity potential.

Φ_0, Φ_1, Φ_2 Approximate velocity potentials at leading order, and first and second order rectifications.

ds Infinitesimal element of the curvilign abscissa of $\mathcal{C}(t)$.

$\mathcal{F}(t)$ Integrated force on the inner cylinder.

INTRODUCTION

When a moving body is submerged, it can experience strong forces induced by the surrounding fluid. Since the body motion modifies the fluid flow, and the fluid flow can modify the body motion, this is a non-linear fluid/structure interaction problem. Furthemore, the induced fluid forces are not only functions of the whole history of the solid motion, which is sometimes determined, but also of the ambiant perturbation level. Hence, it can be helpful to isolate physical phenomena in simple cases so as to identify their influences. Once it is done, models may be built and used to interpret real or numerical experiments. Moreover, if they are validated, they can avoid the use of a numerical code to solve the fluid domain in fluid/structure interaction problems.

This paper focuses on the fluid forces experienced by a rapidly moving circular cylinder in a annular fluid region. The motion is assumed radial, unidirectional and without rotation. It is related to a study whose aim is to predict impulsive fluid loads on naval components during a typical military shock. The simplest and most currently used model available in the scientific litterature related to this geometry is the Fritz one [1]. This model makes the following assumptions:

(i) the flow is two-dimensional,
(ii) the flow is incompressible,
(iii) the fluid is initially at rest,
(iv) the fluid is inviscid,
(v) the advection term $\mathbf{u}\cdot\nabla\mathbf{u}$ can be neglected,
(vi) the displacement imposed to the inner cylinder is very small compared to its radius : $e(t)/R_1 << 1$.

Assumptions (i) to (iv) are also made in this paper. (i) is valid if the length of the cylinders is much longer than their radius, if there is no axial flow and if the two-dimensional flow is stable - or the effects of three-dimensional instabilities are negligible in the forces compared to potential effects. Some cases where hypothesis (ii) is not allowed are discussed in a companion paper [2] and more generally in [3,4]. Hypothesis (iii) can be relaxed in this study to an initially irrotational flow. Assumption (iv) is only roughly valid if the inner cylinder displacements are sufficiently small so that no separation occurs and if the motions are rapid enough to produce boundary layers [5] whose thickness are much thinner than the radial clearance and more specifically for high number $\omega R_1^2(\alpha-1)^2/\nu$ [4, 6]. This paper puts the focus on the relaxation of (v) and (vi). Hence the advection term is taken into account and the fluid force is expected to be valid, as it will be seen later, when the relation $(e(t)/R_1)^3 << 1$ is satisfied, which is less restrictive than that of (vi). Another way of thinking large displacements effects can be found in [7, 8].

In the first section the problem is formulated as a boundary-perturbation problem [9] for the velocity potential and a regular expansion method used to solve it analytically is exposed. The resolution up to the second order is achieved in the second section where the local pressure, velocity and integrated forces are given. In the third section, these results are compared with those obtained from a CFD code [10] based on a finite-volume discretization on a moving mesh. Some conclusions are given in the last part.

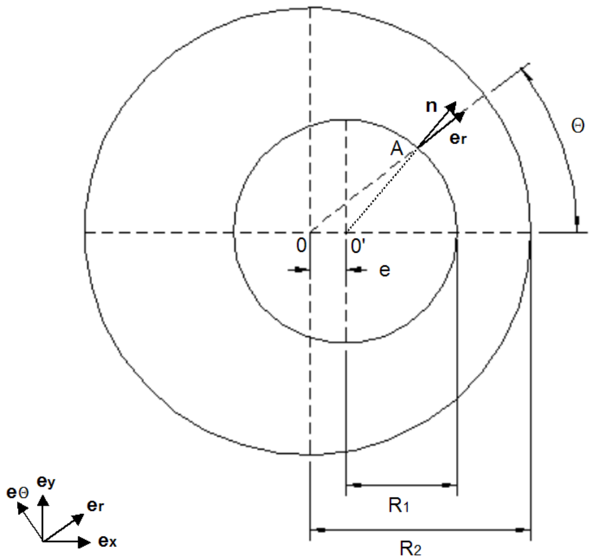


Figure 1. *THE GEOMETRICAL CONFIGURATION.*

PROBLEM FORMULATION

General equations

With assumptions (ii) and (iv) of the last section, the Navier-Stokes equations governing the fluid motion are reduced to:

$$\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\,\mathbf{u}=-\frac{1}{\rho}\nabla p \quad (1)$$

$$\nabla\cdot\mathbf{u}=0 \quad (2)$$

Since the fluid is inviscid and the density is constant, the Kelvin theorem[1] can be used and provides the classical consequence: an inviscid fluid initially irrotational remains irrotational at latter times:

$$\nabla\times\mathbf{u}=0 \quad (3)$$

Hence, there exists a function Φ, called the velocity potential, such that:

$$\mathbf{u}=\nabla\Phi \quad (4)$$

Introducing it in Eqn. (2) gives the equation satisfied by Φ in the fluid domain:

$$\nabla^2\Phi=0 \quad (5)$$

which is the Laplace equation. This derivation is classical and can be found in all fluid dynamics books (see [11] for example).

[1] the circulation on all closed material line is conserved [11].

Since the fluid is assumed inviscid, only the normal component of the velocity has to be conserved on the solid boundaries. For the geometry of interest in this paper (see Fig. 1), the boundary conditions in term of velocity potentiel are:

$$\nabla\Phi \cdot \mathbf{n} = 0 \quad \text{on the fixed outer cylinder} \tag{6}$$

$$\nabla\Phi \cdot \mathbf{n} = \dot{e}(t)\mathbf{e_x} \cdot \mathbf{n} \quad \text{on the moving inner cylinder} \tag{7}$$

In order to make this problem analytically tractable, the inner cylinder position and the unit outward normal $\mathbf{n}$ have to be expressed in explicit terms in Eqn. (7). This is done in the following subsection.

Geometrical considerations

By considering the triangle $OO'A$ in Fig. 1, it is straightforward to find that the inner circular cylinder position r_c satisfies the following relation:

$$R_1^2 = r_c^2 + e^2(t) - 2r_c e(t)\cos\theta \tag{8}$$

Since we are interested by motions of the inner cylinder much smaller than its radius, the physical solution of Eqn. (8) is:

$$r_c(\theta,t) = e(t)\cos\theta + R_1\sqrt{1 - \frac{e^2(t)}{R_1^2}\sin^2\theta} \tag{9}$$

This is the polar equation of the inner cylinder $\mathcal{C}(t)$. Expanding the square root in terms of series gives

$$\begin{aligned} r_c(\theta,t) = R_1 \Bigg(& 1 + \cos\theta\frac{e(t)}{R_1} \\ & + \sum_{n=1}^{\infty}(-1)^n(\sin\theta)^{2n}\frac{1}{n!}\prod_{k=0}^{n-1}\left(\frac{1}{2}-k\right)\left(\frac{e(t)}{R_1}\right)^{2n}\Bigg) \end{aligned} \tag{10}$$

which always converges for $|e(t)/R_1| < 1$. Thanks to this formula, we define approximate positions of the inner cylinder:

$$r_0(\theta) = R_1 \tag{11}$$

$$r_1(\theta) = R_1\left(1 + \cos\theta\frac{e(t)}{R_1}\right) \tag{12}$$

$$r_2(\theta) = R_1\left(1 + \cos\theta\frac{e(t)}{R_1} - \frac{1}{2}\sin^2\theta\left(\frac{e(t)}{R_1}\right)^2\right) \tag{13}$$

The boundary condition Eqn. (7) will be expressed thanks to this formula. r_0 is used in the Fritz model and is the leading order approximation of Eqn. (10). r_1 and r_2, respectivily the first and second order approximations of Eqn. (10), will be used to locate $\mathcal{C}(t)$ in the first order and second order models. It is also of interest to write in explict terms the unit outward normal $\mathbf{n}$ on the moving inner cylinder. We consider for this the parametric curve Ψ of $\mathcal{C}(t)$ which is defined by:

$$\mathcal{C}(t) : \theta \mapsto \Psi(\theta) = O + r_c(\theta,t)\mathbf{e_r}(\theta,t) \tag{14}$$

where O is the centre of the outer cylinder. The unit tangent $\mathbf{T}$ to $\mathcal{C}(t)$ at the position θ is given by:

$$\mathbf{T}(\theta) = \frac{\Psi'(\theta)}{\|\Psi'(\theta)\|} \tag{15}$$

where $\Psi'(\theta)$ and $\|\Psi'(\theta)\|$ can be written:

$$\Psi'(\theta) = r_c'(\theta)\mathbf{e_r} + r_c(\theta)\mathbf{e_\theta} \tag{16}$$

$$\|\Psi'(\theta)\| = \sqrt{r_c'^2(\theta) + r_c^2(\theta)} \tag{17}$$

and where the prime denotes derivative according to θ. The unit normal $\mathbf{n}$ which is orthogonal to $\mathbf{T}$ can then be evaluated thanks to Eqns. (11-16). Its truncation at the leading, first and second orders are respectively:

$$\mathbf{n_0}(\theta) = \frac{1}{\|\Psi'(\theta)\|}\left(R_1\cos\theta\,\mathbf{e_x} + R_1\sin\theta\,\mathbf{e_y}\right) \tag{18}$$

$$\begin{aligned} \mathbf{n_1}(\theta) = & \frac{1}{\|\Psi'(\theta)\|}\left(R_1\cos\theta + e(t)\cos 2\theta\right)\mathbf{e_x} \\ + & \frac{1}{\|\Psi'(\theta)\|}\left(R_1\sin\theta + e(t)\sin 2\theta\right)\mathbf{e_y} \end{aligned} \tag{19}$$

$$\begin{aligned} \mathbf{n_2}(\theta) = & \frac{1}{\|\Psi'(\theta)\|}\left(\left(R_1 - \frac{3}{8}\frac{e^2(t)}{R_1}\right)\cos\theta\right. \\ + & \left. e(t)\cos 2\theta + \frac{3}{8}\frac{e^2(t)}{R_1}\cos 3\theta\right)\mathbf{e_x} \\ + & \frac{1}{\|\Psi'(\theta)\|}\left(\left(R_1 - \frac{3}{8}\frac{e^2(t)}{R_1}\right)\sin\theta\right. \\ + & \left. e(t)\sin 2\theta + \frac{3}{8}\frac{e^2(t)}{R_1}\sin 3\theta\right)\mathbf{e_y} \end{aligned} \tag{20}$$

In order to evaluate the fluid forces on the inner cylinder, the knowledge of $\mathbf{n}ds$ is also required. ds is an infinitesimal element of the curviligne abscissa of $\mathcal{C}(t)$ and is given by the formula:

$$ds = \|\Psi'(\theta)\|\,d\theta \tag{21}$$

Hence the expression of $\mathbf{n}ds$ can be directly deduced from Eqns. (18,19,20).

Resolution method

Equations (5,6,7) governing the velocity potential can be rewritten in polar coordinates to give:

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0 \tag{22}$$

in the fluid domain $(r,\theta) \in]r_c(\theta), R_2[\times[0,2\pi[$ and:

$$\frac{\partial\Phi}{\partial r}(R_2,\theta) = 0 \tag{23}$$

$$\nabla\Phi(r_c,\theta)\cdot\mathbf{n}(r_c,\theta) = \dot{e}(t)\mathbf{e_x}\cdot\mathbf{n}(r_c,\theta) \tag{24}$$

on the boundaries. Hence the system of equations to solve is a laplacian with Neumann boundary condition on the outer cylinder. A difficulty arises from the boundary condition on the inner cylinder since r_c and $\mathbf{n}$ are functions of θ and t. Since there is no differentiation with time in this system, t is only a parameter. The problem is tackled with a boundary-perturbation method [9] thanks to a regular expansion. Equation (24) is seen as the extreme boundary condition in the following family of boundary conditions:

$$\nabla\Phi\left(r_n(\theta),\theta\right)\cdot\mathbf{n}\left(r_n(\theta),\theta\right)=\dot{e}(t)\mathbf{e_x}\cdot\mathbf{n}\left(r_n(\theta),\theta\right) \tag{25}$$

where $r_n(\theta)$ takes the form:

$$r_n(\theta)=\sum_{p=0}^{n}A_p\,\varepsilon^p \tag{26}$$

and satisfies:

$$\lim_{n\mapsto\infty}r_n(\theta)=r_c(\theta). \tag{27}$$

The perturbation parameter ε is in our case $e(t)/R_1$ and the coefficients A_p can be identified by considering Eqn. (10). Performing a Taylor expansion of Eqn (24) about R_1 and using the decomposition Eqn. (26) allow to turn the original problem into an equivalent one. We can now divide the problem into a sequence of problems where we can separately find the functions Φ_0, Φ_1, Φ_2 ... in the desired solution:

$$\Phi=\sum_{n=0}^{\infty}\left(\frac{e(t)}{R_1}\right)^n\Phi_n \tag{28}$$

If the perturbation parameter is sufficiently small, the serie will converge rapidly and few terms will be sufficient to provide a good approximation of the solution. In this paper, only the main order Φ_0, first order Φ_1 and second order Φ_2 approximations are found. Hence the solution is expected to be valid in cases where $(e(t)/R_1)^3<<1$. Once the velocity potential is found, the velocity distribution is written thanks to Eqn. (4). In order to obtain the pressure in the fluid domain, Eqn. (1) is rewritten after elementary manipulations [11]:

$$\nabla p=-\rho\frac{\partial\mathbf{u}}{\partial t}-\rho\left[\nabla\left(\frac{\mathbf{u}^2}{2}\right)-\mathbf{u}\times(\nabla\times\mathbf{u})\right] \tag{29}$$

Taking into account Eqn. (3) and integrating the resulting formula in space coordinates gives the following expression for the pressure:

$$p(r,\theta)=-\rho\frac{\partial\Phi}{\partial t}(r,\theta)-\rho\frac{1}{2}\mathbf{u}^2(r,\theta)+C \tag{30}$$

where C is a constant available in the whole fluid domain and will be taken as null in the following. This formula is the classical generalized Bernoulli equation. Since the flow is supposed inviscid, integrated fluid forces on the moving inner circular cylinder are given by:

$$\mathcal{F}(t)=-\int_0^{2\pi}p(r(\theta))\overline{\overline{I}}\cdot\mathbf{n}(\theta)\,\|\Psi'(\theta)\|\,d\theta \tag{31}$$

where $\overline{\overline{I}}$ is the identity matrix. In the next section, the problem is analytically solved at the leading, first and second orders.

Approximate analytical solution

Leading-order resolution

In this model, boundary condition Eqn. (24) is expressed with the leading-order approximations of r_c and $\mathbf{n}$ given respectively by Eqn. (11) and Eqn. (18). In this case, the leading-order solution Φ_0 satisfies the following problem:

$$\frac{\partial^2\Phi_0}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi_0}{\partial\theta^2}=0 \tag{32}$$

for $(r,\theta)\in]R_1,R_2[\times[0,2\pi[$, with the simplified boundary conditions:

$$\frac{\partial\Phi_0}{\partial r}(R_2,\theta)=0 \tag{33}$$

$$\frac{\partial\Phi_0}{\partial r}(r_0,\theta)=\dot{e}(t)\cos\theta \tag{34}$$

for $\theta\in[0,2\pi[$. Hence, this problem consists in solving a Laplacian with Neumann boundary conditions in an annular geometry and has been solved by Fritz [1] for example. The main steps are repeated here for completeness. Since the problem is elliptic, we search a solution by the method of separation of variables in the form:

$$\Phi_0(r,\theta)=\Phi_r(r)\Phi_\theta(\theta) \tag{35}$$

Introducing it in Eqn. (32) and noting that Eqns. (32,33,34) are invariant under the transformations:

$$(r,\theta,\Phi_0)\mapsto(r,\theta+2\pi,\Phi_0) \tag{36}$$

$$(r,\theta,\Phi_0)\mapsto(r,-\theta,\Phi_0), \tag{37}$$

the solution has the form:

$$\Phi_0(r,\theta)=\sum_{n=1}^{\infty}\left(\left(A_nr^n+B_nr^{-n}\right)cos(n\theta)\right)+A_0\ln r+B_0 \tag{38}$$

The coefficients A_n and B_n are determined with the boundary conditions and the leading-order solution arises:

$$\Phi_0(r,\theta)=-\frac{1}{\alpha^2-1}\left(r+\frac{R_2^2}{r}\right)\dot{e}(t)\cos\theta+B_0 \tag{39}$$

The corresponding velocity field $\mathbf{u_0}$ is found by putting the decomposition Eqn. (28) into Eqn. (4) and keeping only the leading-order term. Writing $\mathbf{u_0}=u_{r0}\mathbf{e_r}+u_{\theta0}\mathbf{e}_\theta$, it gives:

$$u_{r0}(r,\theta)=\frac{1}{\alpha^2-1}\left(\frac{R_2^2}{r^2}-1\right)\dot{e}(t)\cos\theta \tag{40}$$

$$u_{\theta0}(r,\theta)=\frac{1}{\alpha^2-1}\left(\frac{R_2^2}{r^2}+1\right)\dot{e}(t)\sin\theta \tag{41}$$

Introducing the decomposition Eqn. (28) in Eqn. (30) allows to express the leading-order pressure in the fluid domain:

$$p_0(r,\theta)=-\rho\frac{\partial}{\partial t}\Phi_0-\rho\frac{\mathbf{u_0}^2}{2} \tag{42}$$

With the above expressions for the potential and the velocity, the pressure is fully determined and can be written:

$$p_0(r,\theta) = \rho\ddot{e}(t)\frac{1}{\alpha^2-1}\left(\frac{R_2^2}{r}+r\right)\cos\theta - \rho\dot{e}^2(t)\frac{1}{(\alpha^2-1)^2}\left[\frac{1}{2}\left(\frac{R_2^4}{r^4}+1\right)-\frac{R_2^2}{r^2}\cos 2\theta\right] \tag{43}$$

At this order, the local pressure on the moving inner cylinder is evaluated at $r_c(\theta) = r_0$ (see Eqn. (11)) which gives:

$$p_0(R_1,\theta) = \rho\ddot{e}(t)R_1\frac{\alpha^2+1}{\alpha^2-1}\cos\theta - \rho\dot{e}^2(t)\frac{1}{(\alpha^2-1)^2}\left(\frac{1}{2}\left(\alpha^4+1\right)-\alpha^2\cos 2\theta\right) \tag{44}$$

This differs from the expression of Fritz [1] in which the second term of the right hand side in the above equation does not appear. Integrated forces at the leading order on the inner cylinder can be expressed thanks to Eqns. (31,18) and take the form:

$$\mathcal{F}_0(t) = -\int_0^{2\pi} p_0(R_1,\theta)R_1\cos\theta\, d\theta\, \mathbf{e_x} \tag{45}$$

With the pressure given in Eqn. (44), the fluid forces are fully determined:

$$\mathcal{F}_0(t) = -\rho\pi R_1^2\frac{\alpha^2+1}{\alpha^2-1}\ddot{e}(t)\,\mathbf{e_x} \tag{46}$$

This is exactly the expression given by the Fritz model. It can be inferred that even if the advection term modifies the local pressure on the moving cylinder, integrated forces are not influenced by it at the leading-order. In the following subsection, it will be shown that it nevertheless changes integrated forces at the next order.

First order resolution

At this order, the boundary condition on the moving inner circular cylinder Eqn. (24) is expressed at $r_c \approx r_1$ (given in Eqn. (12)) and the unit normal $\mathbf{n}$ is approximated by $\mathbf{n_1}$ (given in Eqn. (19)). Flow quantities are truncated at the first order. Using Taylor series expansions, the boundary condition on the moving cylinder becomes:

$$\frac{\partial\Phi}{\partial r}(R_1,\theta)+\frac{e(t)}{R_1}f(\Phi,R_1,\theta) = \dot{e}(t)\cos\theta+\frac{e(t)}{R_1}\dot{e}(t)\cos 2\theta \tag{47}$$

$$\text{where } f(\Phi,R_1,\theta) = \cos\theta\frac{\partial\Phi}{\partial r}(R_1,\theta)+R_1\cos\theta\frac{\partial^2\Phi}{\partial r^2}(R_1,\theta) + \frac{\sin\theta}{R_1}\frac{\partial\Phi}{\partial\theta}(R_1,\theta)$$

In accordance with the pertubation method, we search a function Φ_1 such as:

$$\Phi = \Phi_0+\frac{e(t)}{R_1}\Phi_1+O\left(\frac{e^2(t)}{R_1^2}\right) \tag{48}$$

Introducing this decomposition in Eqn. (47) and keeping in mind the problem solved by Φ_0 in the previous subsection, the boundary condition for Φ_1 at the inner cylinder is fully determined. After some manipulations it gives:

$$\frac{\partial\Phi_1}{\partial r}(R_1,\theta) = \frac{2\alpha^2}{\alpha^2-1}\dot{e}(t)\cos 2\theta \tag{49}$$

The equation governing Φ_1 in the fluid domain and the boundary condition on the outer cylinder are found by putting Eqn. (48) in Eqns. (22,23) which results in:

$$\frac{\partial^2\Phi_1}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi_1}{\partial\theta^2} = 0 \tag{50}$$

for $(r,\theta)\in]R_1,R_2[\times[0,2\pi[$ and:

$$\frac{\partial\Phi_1}{\partial r}(R_2,\theta) = 0 \tag{51}$$

on the outer cylinder. Hence, the problem to solve for Φ_1 is again a laplacian in an annular region with Neumann boundary conditions. The solution is found with exactly the same method as that used for the leading-order problem and can be written:

$$\Phi_1(r,\theta) = \frac{-\alpha^2}{(\alpha^2-1)(\alpha^4-1)}\dot{e}(t)\frac{R_2^2}{R_1}\left(\frac{r^2}{R_2^2}+\frac{R_2^2}{r^2}\right)\cos 2\theta \tag{52}$$

The corresponding first order velocity and pressure rectifications $\mathbf{u_1}$ and p_1 defined such that:

$$\mathbf{u} = \mathbf{u_0}+\frac{e(t)}{R_1}\mathbf{u_1}+O\left(\frac{e^2(t)}{R_1^2}\right) \tag{53}$$

$$p = p_0+\frac{e(t)}{R_1}p_1+O\left(\frac{e^2(t)}{R_1^2}\right) \tag{54}$$

are respectively found by introducing Eqn. (48) in Eqn. (4) and in Eqn. (30), which gives:

$$\mathbf{u_1} = \nabla\Phi_1 \tag{55}$$

$$p_1 = -\rho\frac{\partial}{\partial t}\Phi_1-\rho\mathbf{u_0}\cdot\mathbf{u_1} \tag{56}$$

Hence, in explicit terms, the velocity in radial coordinates and the pressure take the form:

$$u_{r1}(r,\theta) = \frac{2\alpha^2}{(\alpha^2-1)(\alpha^4-1)}\left(\frac{R_2^4}{R_1 r^3}-\frac{r}{R_1}\right)\dot{e}(t)\cos 2\theta \tag{57}$$

$$u_{\theta 1}(r,\theta) = \frac{2\alpha^2}{(\alpha^2-1)(\alpha^4-1)}\left(\frac{R_2^4}{R_1 r^3}+\frac{r}{R_1}\right)\dot{e}(t)\sin 2\theta \tag{58}$$

$$p_1(r,\theta) = \frac{\alpha^2}{(\alpha^2-1)(\alpha^4-1)}\frac{R_2^2}{R_1}\left(\frac{r^2}{R_2^2}+\frac{R_2^2}{r^2}\right)\rho\ddot{e}(t)\cos 2\theta + \frac{2\alpha^2}{(\alpha^2-1)^2(\alpha^4-1)}\rho\dot{e}^2(t) \times\left[\left(\frac{R_2^2}{R_1 r}+\frac{R_2^4}{R_1 r^3}\right)\cos 3\theta-\left(\frac{R_2^6}{R_1 r^5}+\frac{r}{R_1}\right)\cos\theta\right] \tag{59}$$

The local pressure on the moving inner cylinder can be found by performing a Taylor expansion of $p(r_1(\theta))$ about R_1, inserting

Eqn. (54) in the resulting decomposition and keeping terms of order one. It gives in function of p_0 and p_1 and their derivatives:

$$p(r_1(\theta)) = p_0(R_1,\theta) + \frac{e(t)}{R_1}\left(p_1(R_1)+R_1\cos\theta\frac{\partial p_0}{\partial r}(R_1,\theta)\right)+O\left(\frac{e^2(t)}{R_1^2}\right) \quad (60)$$

which can be explicitly written thanks to Eqns. (44,59):

$$\begin{aligned} p(r_1(\theta)) &= \rho\ddot{e}(t)R_1\frac{\alpha^2+1}{\alpha^2-1}\cos\theta \\ &- \rho\dot{e}^2(t)\frac{1}{(\alpha^2-1)^2}\left(\frac{1}{2}\left(\alpha^4+1\right)-\alpha^2\cos2\theta\right) \\ &+ \frac{e(t)}{R_1}\left\{\rho\ddot{e}(t)R_1\left[-\frac{1}{2}+\right.\right. \\ &\left.\left.\left(\frac{\alpha^2\left(\alpha^4+1\right)}{(\alpha^2-1)(\alpha^4-1)}-\frac{1}{2}\right)\cos2\theta\right]\right\} \\ &+ \frac{e(t)}{R_1}\left(\rho\dot{e}^2(t)\frac{\alpha^2\left(\alpha^2\left(\alpha^2+2\right)+1\right)}{(\alpha^2-1)^2(\alpha^4-1)}(\cos(3\theta)-\cos\theta)\right) \end{aligned} \quad (61)$$

Integrating the above formula on the moving cylinder with Eqns. (31,19) gives the following global fluid forces:

$$\begin{aligned} \mathcal{F}(t) &= -\rho\pi R_1^2\frac{\alpha^2+1}{\alpha^2-1}\ddot{e}(t)\,\mathbf{e_x} \\ &+ \rho\pi e(t)\dot{e}^2(t)\frac{2\alpha^2\left(\alpha^2+1\right)}{(\alpha^2-1)^2(\alpha^4-1)}\mathbf{e_x}+O\left(\frac{e(t)}{R_1}\right)^2 \end{aligned} \quad (62)$$

A new term has appeared in the right hand side of the above formula. It can be inferred that the advection term which does not influence the fluid force at the leading-order (see Eqn. (46)), modify the global force at the first order. In the following subsection, the governing equations are solved at the second order.

Second order resolution

The boundary condition on the moving cylinder Eqn. (24) is expressed at $r_c \approx r_2$ (given in Eqn. (13)) with the approximate unit normal $\mathbf{n_2}$ (see Eqn. (20)). Flow quantities are truncated at the second order, neglecting the terms of order $(e(t)/R_1)^3$ and higher. Then the following decomposition:

$$\Phi = \Phi_0+\frac{e(t)}{R_1}\Phi_1+\frac{e^2(t)}{R_1^2}\Phi_2+O\left(\frac{e^3(t)}{R_1^3}\right) \quad (63)$$

is introduced in the boundary condition. Lastly, the resulting formula is expanded in Taylor series about R_1. This provides the boundary condition for the second order rectification potentiel Φ_2, expressed with the known function Φ_0, Φ_1 and their derivatives. After some manipulations, the boundary conditions for Φ_2 can be explicitly written as:

$$\begin{aligned} \frac{\partial\Phi_2}{\partial r}(R_1,\theta) &= \frac{2\alpha^2}{(\alpha^2-1)(\alpha^4-1)}\dot{e}(t)\cos\theta \\ &+ \frac{3\alpha^2\left(\alpha^4+1\right)}{(\alpha^2-1)(\alpha^4-1)}\dot{e}(t)\cos3\theta \end{aligned} \quad (64)$$

Equations satisfied by Φ_2 in the fluid domain and at the outer cylinder are found with the same method as that used in the previous subsection and consist once again of a laplacian with Neumann boundary conditions:

$$\frac{\partial^2\Phi_2}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi_2}{\partial\theta^2}=0 \quad (65)$$

for $(r,\theta)\in]R_1,R_2[\times[0,2\pi[$ and:

$$\frac{\partial\Phi_2}{\partial r}(R_2,\theta)=0 \quad (66)$$

So the problem for Φ_2 consists in solving Eqns. (65,66,64) which have exactly the same form as the problems for Φ_0 and Φ_1. It gives with the same method:

$$\begin{aligned} \Phi_2(r,\theta) &= -\dot{e}(t)\frac{2\alpha^2}{(\alpha^2-1)^2(\alpha^4-1)}\left(r+\frac{R_2^2}{r}\right)\cos\theta \\ &-\dot{e}(t)\frac{\alpha^2\left(\alpha^4+1\right)}{(\alpha^2-1)(\alpha^4-1)(\alpha^6-1)}\frac{1}{R_1^2}\left(r^3+\frac{R_2^6}{r^3}\right)\cos3\theta \end{aligned} \quad (67)$$

The corresponding second order velocity and pressure rectifications $\mathbf{u_2}$ and p_2 defined such that:

$$\mathbf{u} = \mathbf{u_0}+\frac{e(t)}{R_1}\mathbf{u_1}+\frac{e^2(t)}{R_1^2}\mathbf{u_2}+O\left(\frac{e^3(t)}{R_1^3}\right) \quad (68)$$

$$p = p_0+\frac{e(t)}{R_1}p_1+\frac{e^2(t)}{R_1^2}p_2+O\left(\frac{e^3(t)}{R_1^3}\right) \quad (69)$$

are found by introducing Eqn. (63) in Eqn. (4) and in Eqn. (30) which results in:

$$\mathbf{u_2} = \nabla\Phi_2 \quad (70)$$

$$p_2 = -\rho\frac{\partial}{\partial t}\Phi_2-\frac{\rho}{2}\left(\mathbf{u_1}^2+2\mathbf{u_0}\cdot\mathbf{u_2}\right) \quad (71)$$

So they can be written explicitly:

$$u_{r2}(r,\theta) = \dot{e}(t)\frac{2\alpha^2}{(\alpha^2-1)^2(\alpha^4-1)}\left(\frac{R_2^2}{r^2}-1\right)\cos\theta \quad (72)$$
$$+\ \dot{e}(t)\frac{3\alpha^2(\alpha^4+1)}{(\alpha^2-1)(\alpha^4-1)(\alpha^6-1)}\frac{1}{R_1^2}\left(\frac{R_2^6}{r^4}-r^2\right)\cos 3\theta$$

$$u_{\theta 2}(r,\theta) = \dot{e}(t)\frac{2\alpha^2}{(\alpha^2-1)^2(\alpha^4-1)}\left(\frac{R_2^2}{r^2}+1\right)\sin\theta \quad (73)$$
$$+\ \dot{e}(t)\frac{3\alpha^2(\alpha^4+1)}{(\alpha^2-1)(\alpha^4-1)(\alpha^6-1)}\frac{1}{R_1^2}\left(\frac{R_2^6}{r^4}+r^2\right)\sin 3\theta$$

$$p_2(r,\theta) = \rho\ddot{e}(t)R_1\frac{\alpha^2}{(\alpha^2-1)(\alpha^4-1)}A(r,\theta) \quad (74)$$
$$-\ \rho\dot{e}^2(t)\frac{\alpha^2}{(\alpha^2-1)^2(\alpha^4-1)}B(r,\theta)$$

where $A(r,\theta)$ and $B(r,\theta)$ are given by:

$$A(r,\theta) = \frac{2}{\alpha^2-1}\left(\frac{r}{R_1}+\frac{R_2^2}{R_1 r}\right)\cos\theta + \frac{\alpha^4+1}{\alpha^6-1}\left(\frac{r^3}{R_1^3}+\frac{R_2^6}{R_1^3 r^3}\right)\cos 3\theta$$

$$B(r,\theta) = \frac{2\alpha^2}{\alpha^4-1}\left(\frac{R_2^8}{R_1^2 r^6}+\frac{r^2}{R_1^2}-\frac{2R_2^4}{R_1^2 r^2}\cos 4\theta\right) + \frac{2}{\alpha^2-1}\left(\frac{R_2^4}{r^4}+1-2\frac{R_2^2}{r^2}\cos 2\theta\right) + \frac{3(\alpha^4+1)}{\alpha^6-1}\left(\frac{R_2^8}{R_1^2 r^6}+\frac{r^2}{R_1^2}-\left(\alpha^2+\frac{R_6^2}{R_1^2 r^4}\right)\cos 6\theta\right)$$

The local pressure until the second order, on the moving inner cylinder, can then be written by performing Taylor series expansions of the pressure $p(r_2(\theta))$ about R_1, which results in:

$$p(r_2(\theta)) = p_0(R_1,\theta)+\frac{e(t)}{R_1}\left(p_1(R_1,\theta)+R_1\cos\theta\frac{\partial p_0}{\partial r}(R_1,\theta)\right)$$
$$+\ \frac{e^2(t)}{R_1^2}\left(p_2(R_1,\theta)+R_1\cos\theta\frac{\partial p_1}{\partial r}(R_1,\theta) \quad (75)\right.$$
$$\left.-\ \frac{R_1}{2}\sin^2\theta\frac{\partial p_0}{\partial r}(R_1,\theta)+\frac{R_1^2}{2}\cos^2\theta\frac{\partial^2 p_0}{\partial r^2}(R_1,\theta)\right)$$

The functions of the right hand side are all known, so the local pressure until the second order is fully determined. Expressing explicitly each term of the above formula and integrating the resulting equation with Eqns. (31,20) give the integrated forces up to the second order:

$$\mathcal{F}(t) = -\rho\pi R_1^2\ddot{e}(t)\frac{\alpha^2+1}{\alpha^2-1}\mathbf{e_x} \quad (76)$$
$$+\ \rho\pi e(t)\dot{e}^2(t)\frac{2\alpha^2(\alpha^2+1)}{(\alpha^2-1)^2(\alpha^4-1)}\mathbf{e_x}$$
$$-\ \rho\pi e^2(t)\ddot{e}(t)\frac{4\alpha^4}{(\alpha^2-1)^2(\alpha^4-1)}\mathbf{e_x}+O\left(\frac{e(t)}{R_1}\right)^3$$

A new term appears. It can be seen as a non linear rectification of the added mass coefficient. Validity and limits of this fluid forces expression are compared with numerical simulation results in the next section.

Comparaison of the results with numerical simulations

Comparaisons of the analytical results are performed with a CFD code [10] based on a second-order finite volume discretization scheme. The Navier-Stokes equations are written in their general conservative form [12] with an arbitrary lagrangian eulerian formulation [13]. Hence moving boundaries can be taken into account. The PISO algorithm [14] is used to handle the coupling between pressure and velocity. The analytical model will be only tested on its ability to take advection term and geometrical deformation effects into account. Introduction of the fluid viscosity is the topic of a work currently in progress and will be the subject of a future paper. So it will not be considered here.

In order to compare the results issued from numerical simulations with the simple models developped in this paper, a sinusoidal motion of period T is imposed on the inner cylinder. Since the fluid is assumed inviscid, there is no history effect [4] and this motion can be considered without loss of generality. Three confinements are investigated: $\alpha = 1.1$, $\alpha = 1.5$ and $\alpha = 2$. For each confinement, nine cases are computed (from $\xi = 0,1$ to $\xi = 0,9$) so as to investigate the influence of large displacements in regards to the radial clearance. For each case, the numerical results have been checked to be independent of mesh refinements.

The maximum integrated forces are compared to those predicted by the Fritz model (which corresponds to the leading-order formula Eqn. (46)) and the second order model (given in Eqn. (76)). The results are summarized in Fig. 2. As expected, the models are all the more valid as ξ is small, i.e. as the inner cylinder displacement is small compared to the radial clearance. Furthemore, at a given ξ, they are more accurate for small values of α. It is also an expected result since the perturbation parameter $\varepsilon = e(t)/R_1$ which have been used to construct the models is all the smaller as α tends to unity. Of course, this argument is only true as long as the inviscid hypothesis holds, i.e. as long as the distance between the boundary layers of the inner and outer cylinder is large enough $\sqrt{\nu/\omega} << R_2 - R_1$ and more specifically $\omega R_1^2(\alpha-1)^2/\nu >> 1$ [6]. These figures also show that the second order model gives better prediction than the Fritz one. The differences between the numerical code and the second or-

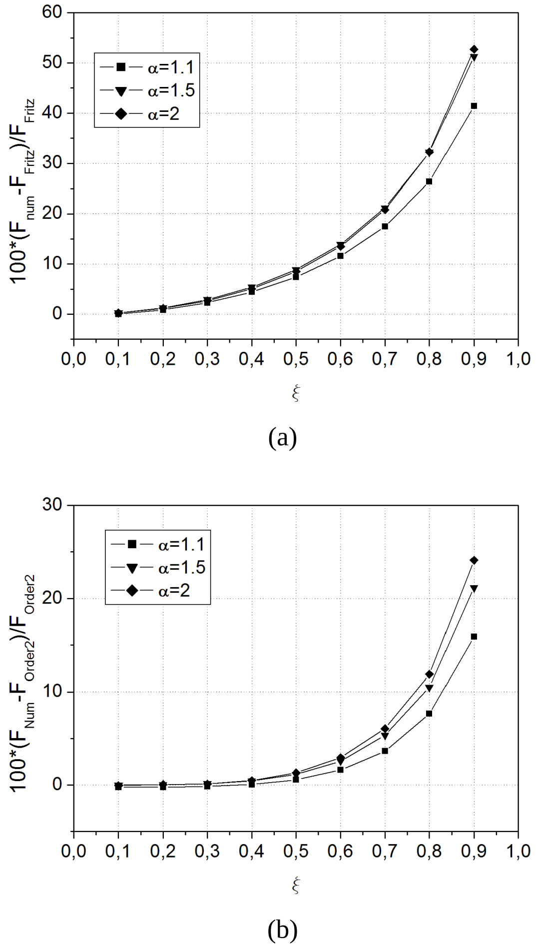


Figure 2. *COMPARISONS BETWEEN THE MAXIMUM FORCES FOUND BY NUMERICAL SIMULATIONS WITH THE FRITZ MODEL (a) AND WITH THE SECOND ORDER MODEL (b).*

der model are all under 3% until $\xi = 0.6$ whereas they are at more than 11% for the Fritz model at the same ξ. For the highest ξ achieved in this paper ($\xi = 0.9$), the differences are less than 25% for the second order model, whereas they are more than 40% for the Fritz model.

In order to gain some insight into the integrated fluid forces, it is fruitful to display their time history on a whole period. We will specialy consider the case $\alpha = 2$ for illustration but the same phenomena occur for $\alpha = 1.5$ and $\alpha = 1.1$. The results for different ξ (0.1, 0.6 and 0.9) are shown in Fig. 3. For small amplitudes of the inner cylinder ($\xi = 0.1$), the fluid forces predicted by the Fritz model, the second order model and the numerical simulation are the same at each time (see Fig. 3(a)). Increasing ξ, both the second order model and the numerical simulation predict bigger maximum forces than the Fritz model, as it was already men-

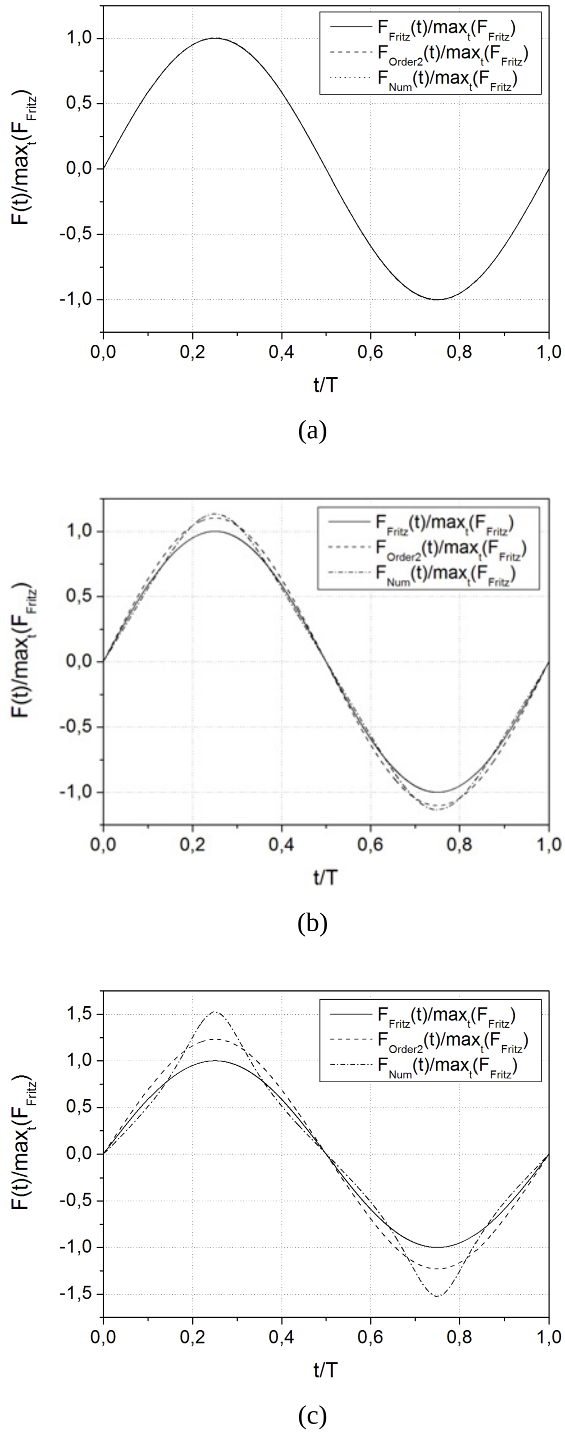
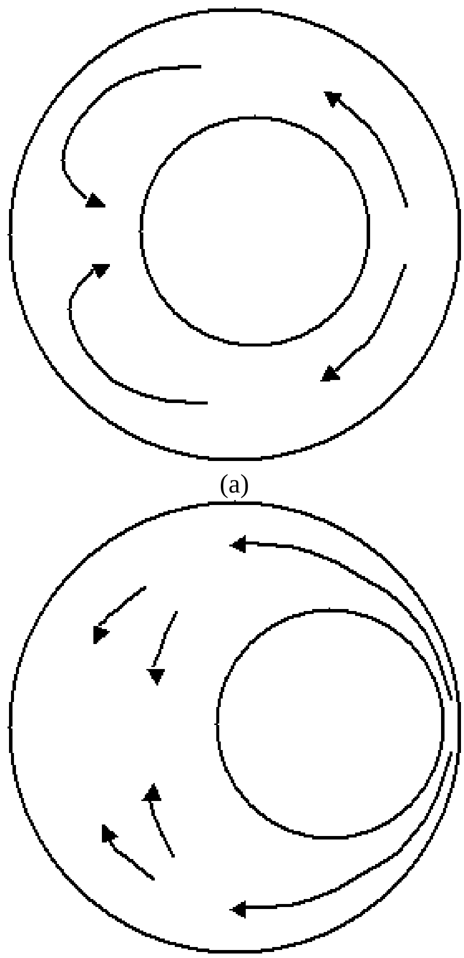


Figure 3. *DIMENSIONLESS TIME HISTORY FLUID FORCES FOR* $\alpha = 2$ *IN CASES* $\xi = 0.1$ *(a),* $\xi = 0.6$ *(b) and* $\xi = 0.9$ *(c).*

tioned. However a net difference can be seen: the second order model is for each time bigger than the Fritz model, whereas numerical simulations predict smaller forces during parts of the oscillation. Furthemore the time history contains two inflection points in each semi-period $]0,T/2[$, $]T/2,T[$, which are not predicted by neither the Fritz model nor the second order one. This behaviour is all the more pronounced as ξ tends to 1 (as shown in Fig. 3(b)(c)) and has been already described in [15]. The following physical interpretation is proposed. Once the inner cylinder is subjected to an imposed motion, the fluid in front of it is sent back and a significant part finally push it (see Fig. 4(a)). The resulting force is then lower (see Fig. 3(c) at $t \approx 0,12$) that the one obtained with the Fritz and second order models, which are not able to reproduce properly this coupled advection/geometrical deformation effect. When the inner cylinder approaches more closely the outer one, this phenomenon is relaxed since the fluid amount sent back is lower and distributed on a larger area (see Fig. 4(b)). Moreover, the fluid particules in the squeeze film are all the more accelerated as the cylinders are closed, which results in a force increase (see Fig. 3(c) at $t \approx 0,25$) as in the case of a body falling to a wall [16, 17].

Conclusions

Extensions of the Fritz model are performed by taking advection terms and the geometrical deformations induced by the inner circular cylinder movement into account. Approximated analytical solutions are found with a regular expansion performed until the second order on boundary-perturbation problem. At the leading-order, the advection term influences the local pressure on the moving inner cylinder, but not the integrated force, which can help to explain why the Fritz model, which consists in a pure added mass term, is accurate for small amplitude motions. At the first order, the advection term influences the fluid force and no term in the form $(e(t)/R_1)\ddot{e}(t)$ is found. At the second order the geometrical deformation gives rise to a non linear modification of the added mass and no term in the form $(e^2(t)/R_1^2)\dot{e}^2(t)$ appears. The resulting fluid forces are then compared to numerical simulation predictions performed with a code able to take into account moving fluid domains. The second order model is shown to be more efficient than the Fritz model, specialy for high geometrical deformation. Nevertheless these two models are not able to reproduce strongly non linear potential effects found with the numerical simulations for high inner cylinder displacements. For ξ lower than $0,6$, the differences between the Fritz and second order models are well below the incertainties on the input data in impulsive load analysis and these models are sufficient to have an idea about the forces level. However for higher ξ, these effects could influence the dynamical behaviour in fluid-structure interaction problems, and could prevent engineers from performing accurate prediction in case of shock or seismic loading. Hence further investigations need to be done so as to make this phenomenon more understandable. Moreover, future extensions of the presented work would include viscous effects in order to characterize the damping term in the fluid forces.



(a)

(b)

Figure 4. *COUPLED ADVECTION/GEOMETRICAL DEFORMATION EFFECTS FOR SMALL (a) AND HIGH (b) AMPLITUDE MOTIONS*

REFERENCES

[1] Fritz, R., 1972. “The effects of liquids on the dynamics motion of immersed solids”. *Trans. ASME, J. Eng. Ind.*, pp. 163–171.

[2] Leblond, C., Sigrist, J.-F., Laine, C., Auvity, B., and Peerhossaini, H., 2006. “A review of fluid forces induced by a circular cylinder oscillating at low amplitude and high frequency in cylindrical confinement”. In ASME, Pressure Vessel and Piping Conference, PVP2006-ICPVT-11, Vancouver.

[3] Landau, L., and Lifschitz, E., 1959. *Fluid Mechanics*. Pergamon Press, Oxford.

[4] DeLangre, E., 2001. *Fluides et Solides*. Les éditions de

l'Ecole Polytechnique.
[5] Schlichting, H., 1979. *Boundary Layer Theory,* $7^{th} Edition$. McGraw-Hill.
[6] Axisa, F., 2001. *Modélisation des systèmes mécaniques. Interactions fluide/structure*. Hermès.
[7] Lu, Y., and Rogers, R., 1995. "Instantaneous squeeze film force between a heat exchanger tube with arbitrary tube motion and a support plate". *J. Fluids Struct.,* **9**, pp. 835–860.
[8] Zhou, T., and Rogers, R., 1997. "Simulation of two-dimensional squeeze film and solid contact forces acting on a heat exchanger tube". *J. Sound. Vib.,* **203**, pp. 621–639.
[9] Dyke, M. V., 1964. *Perturbation Methods in Fluid Mechanics*. Academic Press, New York and London.
[10] Star-CD, 2004. *Methodology, User Guide*. CD Adapco.
[11] Batchelor, G. K., 1997. *An Introduction to Fluid Dynamics*. Cambridge University Press, Cambridge.
[12] Warsi, Z., 1980. "Conservation form of the navier-stokes equations in general non steady coordinates.". *AIAA Journal.,* **19**, pp. 240–242.
[13] Sarrate, J., Huerta, A., and Donea, J., 1998. *Arbitrary Lagrangian Eulerian Formulation for Fluid Multi-Rigid Bodies Interaction Problems.* Computational Mechanics, New trends and Application.
[14] Issa, R., 1985. "Solution of the implicitly discretised fluid flow equation by operator splitting.". *J. Comput. Phys.,* **62**, pp. 40–65.
[15] Sigrist, J.-F., Laine, C., Lemoine, D., and Peseux, B., 2003. "Choice and limits of a linear fluid model for the numerical study in dynamic fluid structure interaction problem". In Proceedings of ASME, Vol. **454** of *Pressure Vessel and Piping, Cleveland*, pp. 87–93.
[16] Wang, Q. X., 2004. "Interaction of two circular cylinders in inviscid fluid.". *Phys. Fluids.,* **16**, pp. 4412–4425.
[17] Joseph, G. G., Zenit, R., Hunt, M. L., and Rosenwinkel, A. M., 2001. "Particle-wall collisions in a viscous fluid". *J. Fluid Mech.,* **433**, pp. 329–346.