Hyperelasticity with volumetric damage

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ABSTRACT: The present paper presents a simple framework to model continuous volumetric damage in elastomers. The formulation predicts phenomenologically the growth of microscopic cavities, and can be applied to both static and fatigue loading conditions. This first version of the approach cannot handle cavitation and is limited to small values of porosities. The derivation is based on the use of a simple scalar damage parameter, the irreversible volume change, and takes naturally into account the change in stiffness through the explicit dependence of the material parameters on the damage variable. The thermodynamic force which drives the volume change contains the hydrostatic stress and also a contribution due to stiffness evolution. As a first application, a damage compressible neo-Hookean constitutive equation is derived and a simple example is studied.

1 INTRODUCTION

Rubber-like materials are usually considered as incompressible. However, under multiaxial or fatigue loading conditions, cavitation and cavities growth take place, and lead to damage and finally to fracture (Farris 1968; Le Cam et al. 2004; Le Gorju 2007). Special experiments can be carried out to exhibit this behaviour as proposed by Gent and Thomas (1958), Gent and Wang (1991) or Legorju-Jago and Bathias (2002). For modelling, on the one hand the cavitation phenomenon under hydrostatic loading conditions is studied considering the stability conditions for the sudden growth of microscopic cavities in the incompressible bulk (see Ball (1982), Horgan and Abeyaratne (1986) for example). On the other hand, several phenomenological approaches have been proposed to predict the growth of pre-existing cavities; the corresponding models incorporate damage variables into compressible hyperelastic approaches (see Boyce and Arruda (2000) for a short review) to quantify the irreversible change of porosity (Andrieux et al. 1997; Dorfmann et al. 2002; Layouni et al. 2003; Li et al. 2007). These models can also be extended to cavitation by adapting the rate equation of the damage variable (Dorfmann 2003). Nevertheless, they are limited to small values of the porosity.

In the present paper, similarly to Andrieux et al. (1997), we propose a simple theoretical framework to model the compressibility induced by damage in hyperelastic materials. Our approach is phenomenological and is restricted to small values of porosity, such that the growing cavities do not interfere. The scalar damage variable is the irreversible volume change and its influence on the stiffness of the material is taken into account through the material parameters. The rate equation chosen here is not adapted to sudden volume change (cavitation) but only to continuous volume change (damage by continuous growth of cavities).

The derivation of the model is described in the next section, the emphasize being laid on the determination of the thermodynamic force which drives the volume change. Then, a very simple constitutive equation which generalizes the compressible neo-Hookean model is considered to illustrate the relevance of the method.

2 DERIVATION OF THE CONSTITUTIVE EQUATION

2.1 *General formulation*

2.1.1 *Kinematics*

It is considered here that under loading, the body exhibits an irreversible volume change due to what

can be called *a volumetric damage*. In the general framework of large strain, the material is supposed homogeneous, isotropic and hyperelastic. Moreover, we assume that it initially contains small flaws that can be considered as holes. To simplify the derivation, the RVE is assumed to deform as sketched in Figure 1; the deformation gradient $\mathbf{F}$ is separated into two parts:

- an irreversible volume change between the initial configuration (C_0) and the intermediate configuration (C_i). It is defined by the deformation gradient $J_c^{1/3}\mathbf{I}$, in which J_c is the ratio of irreversible volume change between the two configurations. In the figure, this volume change is illustrated by the irreversible growth of small holes; nevertheless, we do not consider their microscopic evolution: interaction, coalescence, …
- a classical elastic deformation, which gradient is denoted $\mathbf{f}$, between the intermediate configuration (C_i) and the deformed configuration (C).

The well-established multiplicative decomposition of the deformation gradient is adopted:

$$\mathbf{F} = \mathbf{f}\, J_c^{1/3}\mathbf{I} = J_c^{1/3}\mathbf{f}. \tag{1}$$

Indeed, the deformation process can be described by the two following variables: the observable strain (through $\mathbf{F}$) and the internal variable J_c which describes damage.

2.1.2 *Constitutive equations*

First, the reversible deformation between configurations (C_i) and (C) is considered hyperelastic: it exists a strain energy function $w_{J_c}(\mathbf{f})$ per unit of volume in (C_i). It is highly important to note that this strain energy function depends on J_c in two ways: through $\mathbf{f}$, recalling Eq. (1), and through the material parameters which *explicitly* depend on the irreversible volume change, as notified by the subscript $\cdot_{J_c}$. Second, the deformation between (C_0) and (C_i) being totally irreversible, no elastic strain energy is involved. Then, the total strain energy of the material is simply the strain energy w_{J_c} written per unit of undeformed volume, i.e. unit of volume in (C_0):
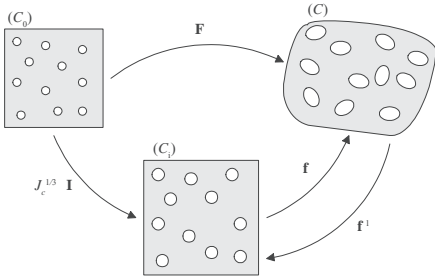

Figure 1. Deformation of the RVE.

$$W(\mathbf{F}, J_c) = J_c\, w_{J_c}(\mathbf{f}). \tag{2}$$

Once the strain energy function defined, one can easily derive the constitutive equations. Restricting the problem to a purely mechanical theory, i.e. ignoring thermal effects, the Clausius-Planck inequality is

$$\mathcal{D}_{\text{int}} = \mathbf{P} : \dot{\mathbf{F}} - \dot{W} \geq 0, \tag{3}$$

where $\mathcal{D}_{\text{int}}$ is the internal dissipation and $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor. Recalling that W depends on both the deformation gradient $\mathbf{F}$ and the internal variable J_c,

$$\mathcal{D}_{\text{int}} = \mathbf{P} : \dot{\mathbf{F}} - \left.\frac{\partial W}{\partial \mathbf{F}}\right|_{J_c} : \dot{\mathbf{F}} - \left.\frac{\partial W}{\partial J_c}\right|_{\mathbf{F}} \dot{J}_c \geq 0. \tag{4}$$

Following Coleman and Noll (1963), $\dot{\mathbf{F}}$ and $\dot{J}_c$ can be chosen arbitrarily and then the constitutive equation for $\mathbf{F}$, i.e. the stress-strain relationship, is

$$\mathbf{P} = \left.\frac{\partial W}{\partial \mathbf{F}}\right|_{J_c}, \tag{5}$$

and the internal dissipation reduces to

$$\mathcal{D}_{\text{int}} = G\dot{J}_c \geq 0 \quad \text{with} \quad G = -\left.\frac{\partial W}{\partial J_c}\right|_{\mathbf{F}} \tag{6}$$

where G is the thermodynamic force which drives the irreversible change in volume. Eq. $(6)_2$ is the constitutive equation for J_c.

Stress-strain relationship. One can now derive Eq. (5):

$$\mathbf{P} = \left.\frac{\partial W}{\partial \mathbf{F}}\right|_{J_c} = \frac{\partial J_c w_{J_c}(\mathbf{f})}{\partial \mathbf{F}} = J_c\, \frac{\partial w_{J_c}}{\partial \mathbf{f}} : \frac{\partial \mathbf{f}}{\partial \mathbf{F}}. \tag{7}$$

Introducing $\mathbf{p} = \partial w_{J_c}/\partial \mathbf{f}$ the first Piola-Kirchhoff stress tensor with respect to the intermediate configuration (C_i), and after some algebraic manipulations, the nominal stress reduces to (see for example Holzapfel (2000) for such derivation)

$$\mathbf{P} = J_c^{2/3}\, \frac{\partial w_{J_c}(\mathbf{f})}{\partial \mathbf{f}}. \tag{8}$$

Note that the derivation of $\mathbf{p}$ is straightforward: once the strain energy function w_{J_c} is chosen, the classical hyperelastic theory applies.

Thermodynamic force for change in volume. The thermodynamic force which drives the irreversible change in volume is

$$G = -\left.\frac{\partial W}{\partial J_c}\right|_{\mathbf{F}} = -w_{J_c} - J_c\left.\frac{\partial w_{J_c}(\mathbf{f})}{\partial J_c}\right|_{\mathbf{F}}. \tag{9}$$

The second right-hand side term can be calculated as follow

$$\left.\frac{\partial w_{J_c}(\mathbf{f})}{\partial J_c}\right|_{\mathbf{F}} = \left.\frac{\partial w_{J_c}}{\partial J_c}\right|_{\text{expl}} + \frac{\partial w_{J_c}}{\partial \mathbf{f}} : \frac{\partial \mathbf{f}}{\partial J_c}, \tag{10}$$

where the subscript $\cdot_{\text{expl}}$ denotes the explicit differentiation with respect to J_c, which involves the differentiation of the material parameters. Considering again the intermediate engineering stress tensor $\mathbf{p}$, we have

$$\frac{\partial w_{J_c}}{\partial \mathbf{f}} : \frac{\partial \mathbf{f}}{\partial J_c} = \mathbf{p} : \left(-\frac{1}{3}J_c^{-4/3}\mathbf{F}\right). \tag{11}$$

So, invoking Eqs. (1) and (8), the thermodynamic force is

$$G = -J_c\left.\frac{\partial w_{J_c}}{\partial J_c}\right|_{\text{expl}} - \left(w_{J_c} - \frac{1}{3}\mathbf{p} : \mathbf{f}\right). \tag{12}$$

This equation can be transformed by introducing the Eshelby stress tensor $\Sigma = W\mathbf{I} - \mathbf{F}^T\mathbf{P}$:

$$G = -J_c\left.\frac{\partial w_{J_c}}{\partial J_c}\right|_{\text{expl}} - J_c^{-1}\Sigma : \frac{\mathbf{I}}{3}. \tag{13}$$

Moreover, considering the velocity gradient associated with the irreversible volume change $\mathbf{L}_c$, which is simply $J_c^{-1}\dot{J}_c\,\mathbf{I}/3$ here, and its symmetric part, the rate of deformation tensor $\mathbf{D}_c$ (equal to $\mathbf{L}_c$ in the present case), the internal dissipation can be written as

$$\mathcal{D}_{\text{int}} = -J_c\left.\frac{\partial w_{J_c}}{\partial J_c}\right|_{\text{expl}}\dot{J}_c - \Sigma : \mathbf{D}_c, \tag{14}$$

and the thermodynamic force is then

$$G = -J_c\left.\frac{\partial w_{J_c}}{\partial J_c}\right|_{\text{expl}} - J_c^{-1}\text{tr}\left(\frac{\Sigma}{3}\right). \tag{15}$$

Finally, this expression can be simplified by rewriting the Eshelby stress tensor as proposed by Verron and Andriyana (2008): noting $(\mathbf{N}_i)_{i=1,2,3}$ the principal strain directions in the reference configuration and recalling that the Eshelby stress tensor is symmetric for isotropic elastic materials, Σ becomes

$$\Sigma = \sum_{i=1}^{3}(W - \lambda_i^2 S_i)\mathbf{N}_i \otimes \mathbf{N}_i, \tag{16}$$

where $(\lambda_i)_{i=1,2,3}$ are the principal stretch ratio and $(S_i)_{i=1,2,3}$ the eigenvalues of the second Piola-Kirchhoff stress tensor. Introducing the relationship between these principal stresses and the principal Cauchy stresses

$$\sigma_i = J^{-1}\lambda_i^2 S_i \quad i = 1,2,3 \tag{17}$$

where $J = \det \mathbf{F}$ characterizes the total change in volume between (C_0) and C, i.e. reversible and irreversible, the Eshelby stress tensor can be written as

$$\Sigma = \sum_{i=1}^{3}(W - J\sigma_i)\mathbf{N}_i \otimes \mathbf{N}_i, \tag{18}$$

and

$$\text{tr}\left(\frac{\Sigma}{3}\right) = W - J\sigma_s \tag{19}$$

where σ_s is the spherical part of the Cauchy stress tensor, i.e. the hydrostatic stress. So, the thermodynamic force reduces to

$$G = -\left.\frac{\partial W}{\partial J_c}\right|_{\text{expl}} + \frac{J}{J_c}\sigma_s. \tag{20}$$

2.1.3 Evolution equation

To close the general formulation of the model, we should precise the rate equation for the internal variable J_c. Such equation can be written as:

$$\dot{J}_c = f(G, \mathbf{F}) \tag{21}$$

where the function f must be positive to ensure the positivity of the dissipation. In order to simplify the model, we simply choose a very simple "damage-like" evolution equation under the following form:

$$\dot{J}_c = \begin{cases} kG & \text{if } G < G_{\max} \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

In this equation, the parameter k is a positive real scalar value which depends on the material and $G_{\max}$ stands for the maximum value of the thermodynamic force G previously endured by the material.

2.2 A particular model

Once the general theory derived, particular models can be proposed by specifying the strain energy density w_{J_c}. We consider the case where the elastic deformation (from (C_i) to (C)) is compressible and we adopt one of the simplest strain energy densities for compressible hyperelastic materials: the generalization of the incompressible neo-Hookean

model proposed by Simo and Pister (1984) and adopted later by both Ehlers and Elipper (1998) and Bischoff et al. (2001). So the complete strain energy density is

$$W(\mathbf{F}, J_c) = J_c \underbrace{[C(i_1 - 3 - \ln i_3) + D(\ln i_3)^2]}_{w_{J_c}(\mathbf{f})}$$

$$= CJ_c \left(J_c^{-2/3} I_1 - 3 + 2\ln \frac{J_c}{J} \right) \qquad (23)$$

$$+ 4DJ_c \left(\ln \frac{J_C}{J} \right)^2,$$

where i_1 and i_3 are the invariants of the right Cauchy-Green elastic strain tensor $\mathbf{f}^T \mathbf{f}$, i.e. its trace and determinant respectively. Note that for the deformation gradient $\mathbf{f}$, the spherical-deviatoric split is not considered. In this equation the material parameter C is twice the shear modulus, and the material parameter D is proportional to the compressibility modulus.

Two cases are now considered.

- The first one for which there is no stiffness decrease due to damage, i.e. material parameters are constant, $C = C_0$ and $D = D_0$. In that case, the engineering stress tensor $\mathbf{P}$ and the thermodynamic force G are respectively

$$\mathbf{P} = 2C_0 J_c^{1/3} \mathbf{F} - \left(2C_0 + 8D_0 \ln \frac{J_c}{J} \right) J_c \mathbf{F}^{-T} \qquad (24)$$

and

$$G = -C_0 \left(J_c^{-2/3} \frac{I_1}{3} - 1 \right)$$
$$+ \ln \frac{J_c}{J} \left(-2C_0 - 8D_0 - 4D_0 \ln \frac{J_c}{J} \right). \qquad (25)$$

- The second one for which the stiffness decreases, i.e. C and D depends on J_c. To specify the dependence of these parameters on J_c, we invoke the statistical theory of rubber elasticity which states that

$$C = \frac{1}{2} n k T, \qquad (26)$$

with n being the number of active polymer chains per unit of volume, k the Boltzmann constant and T the temperature. Considering the mass conservation equation between configurations (C_0) and (C_i),

$$n_0 dV_0 m_{\text{chain}} = n_i dV_i m_{\text{chain}} \qquad (27)$$

where n_0 and n_i are the chain densities per unit of volume in (C_0) and (C_i), dV_0 and dV_i the infinitesimal volumes in (C_0) and (C_i), and m_{chain} is the mean mass of the chains, and recalling that $dV_i = J_c dV_0$, one can state that

$$C(J_c) = C_0 / J_c. \qquad (28)$$

The second parameter D is not directly related to the chain density; nevertheless, we consider the same relationship with the initial compressibility modulus

$$D(J_c) = D_0 / J_c \qquad (29)$$

In that case, the engineering stress tensor $\mathbf{P}$ and the thermodynamic force G are respectively

$$\mathbf{P} = 2C_0 J_c^{-2/3} \mathbf{F} - \left(2C_0 + 8D_0 \ln \frac{J_c}{J} \right) \mathbf{F}^{-T} \qquad (30)$$

and

$$G = 2C_0 \left(J_c^{-5/3} \frac{I_1}{3} - \frac{1}{J_c} \right) - 8 \frac{D_0}{J_c} \ln \frac{J_c}{J}. \qquad (31)$$

3 FIRST RESULTS

In order to illustrate the previous theory, we consider a very simple problem: a sample is first subjected to cyclic triaxial loading conditions, then it is uniaxially stretched. More precisely, the loading conditions are

- Four equi-triaxial cycles under prescribed stretch ratio λ: from $\lambda = 1$ to $\lambda = 1.25$, then from $\lambda = 1$ to $\lambda = 1.5$, then from $\lambda = 1$ to $\lambda = 1.75$, and finally from $\lambda = 1$ to $\lambda = 2$. The deformation gradient and the engineering stress tensor are respectively

$$\mathbf{F} = \lambda(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3), \qquad (32)$$

$$\mathbf{P} = P(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3). \qquad (33)$$

- One uniaxial loading path under prescribed stretch ratio from $\lambda = 1$ to $\lambda = 3$ with

$$\mathbf{F} = \lambda \mathbf{e}_1 \otimes \mathbf{e}_1 + \mu(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3), \qquad (34)$$

$$\mathbf{P} = P\mathbf{e}_1 \otimes \mathbf{e}_1. \qquad (35)$$

The following values of the material parameters are adopted: $C_0 = 1$ MPa, $D_0 = 10$ MPa and $k = 0.01$.

First, for both models, we present the evolution of J_c as a function of the loading time and the stress-strain response during the triaxial loading phase: in Figures 2 and 3 for the first model without change in material parameters (Eqs. (24–25)),
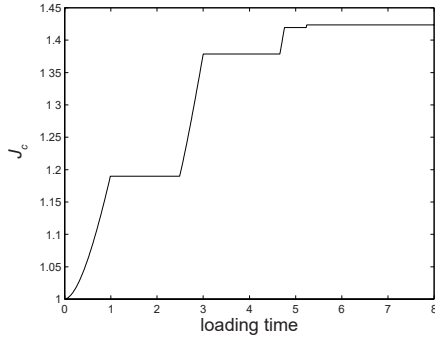
Figure 2. Model without change in material parameters: evolution of the volumetric damage during the four triaxial cycles.
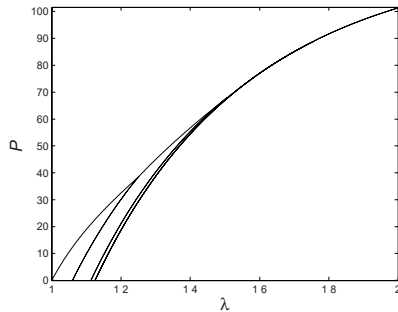

Figure 4. Model with change in material parameters: evolution of the volumetric damage during the four triaxial cycles.

Figure 3. Model without change in material parameters: stress-strain response during the four triaxial cycles.

Figure 5. Model with change in material parameters: stress-strain response during the four triaxial cycles.

and in Figures 4 and 5 for the second model with change in material parameters (Eqs. (30–31)). The comparison of the volumetric damage evolution in Figs. 2 and 4 exhibits that ignoring the induced change in material parameters leads to a limit in the irreversible change in volume (see the two final constant steps in Fig. 2), whereas if this change is taken into account the volumetric damage evolves continuously during triaxial cyclic loading. This mechanical response can be also observed with the stress-strain responses in Figs. 3 and 5: for the former model a unique curve is reached for about $\lambda = 1.5$ and unloading parts of the response become closer and closer as depicted in Fig. 3. For the latter model, Fig. 5 shows that the material stiffness evolves in a regular manner and the resulting cyclic response is similar to the one encountered with damage-like constitutive equations, see for example Chagnon et al. (2004).

Finally, we examine the mechanical response obtained during the final uniaxial extension in Figures 6 and 7. For both models, J_c does not change under loading (the curves are not shown here). Nevertheless, the first model exhibits a "strange" behaviour: the stiffness of the material
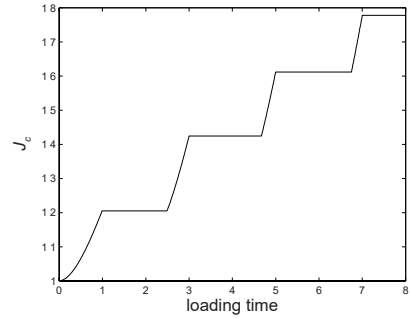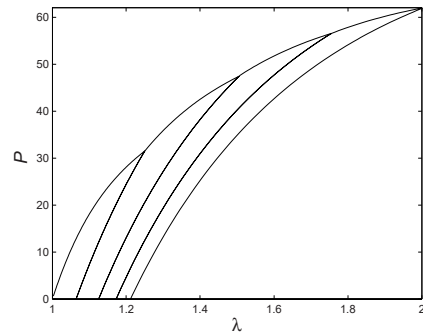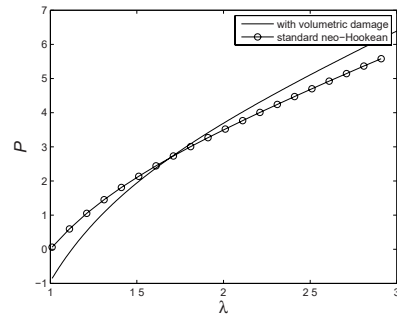

Figure 6. Model without change in material parameters: uniaxial stress-strain response.

is revealed always greater than its initial stiffness (before damage), and the undamaged and damaged curves intersect which means that for stretch ratios greater than 1.75 the stress is greater in the damaged material than in the undamaged one; this behaviour is physically irrelevant. For the second model, the stiffness of the damaged material is always lower than the initial stiffness and the curves never intersect.
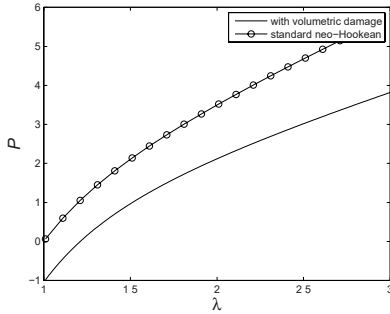
Figure 7. Model with change in material parameters: uniaxial stress-strain response.

4 CONCLUSION

In this paper, a new method has been proposed to consider irreversible growth of cavities in rubber-like materials. The general framework consists in introducing a scalar damage variable J_c which represents the irreversible volume change at a material point; then, the constitutive equations are derived with in the framework of Thermodynamics or Irreversible Processes. The originality of the present approach consists in taking into account the stiffness change induced by damage through the explicit dependence of the material parameters on the damage variable. Moreover, considering the statistical rubber elasticity theory, the extension of the neo-Hookean model has been derived and the dependence of the shear modulus on J_c has been easily established. Finally, a first proof of the model relevance has been proposed by considering a very simple loading history.

Further work will be carried out first to extend the model to cavitation through the choice of a more complex rate equation for J_c and second to implement the model in the finite element context.

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