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LIPSCHITZ STABILITY OF n-CUBIC FUNCTIONAL EQUATIONS AND APPLICATIONS

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Abstract. In this paper, we establish the stability of n-cubic functional equations in Lipschitz spaces and as a consequence we give the stability of cubic functional equations.

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1. INTRODUCTION

Lipschitz spaces have a rich and beautiful algebra structure and these spaces possess various universal properties. Some of open problems in this area are given in Chapter 7 of [25]. Much work has been done on the Banach space of Lipschitz functions (cf. [6]). The Lipschitz condition is one of the central concepts of some important subjects. The Lipschitz condition was used in the elementary theory of ordinary differential equations, real analysis, and metric geometry. There are also striking applications to topology. Every topological manifold outside dimension four admits a unique Lipschitz structure, while such a manifold may have no smooth or piecewise linear structures or it may have many. On a practical side, questions about Lipschitz functions arise in image processing and in the study of internet search engines. (cf. [10, 21]).

Let G be an Abelian group and V a vector space. We say that $S(V)$, a family of subsets of V, is linearly invariant if it is closed under the addition and scalar multiplication defined as usual sense and translation invariant, in the sense that $x+A \in S(V)$, for every $A \in S(V)$ and every $x \in V$ (see[3]). Note that $S(V)$ contains all singleton subsets of V. For instance, $CB(V)$ the family of all closed balls is a linearly invariant family in a normed vector space V. By $B(G^n, S(V))$ we denote the family of all functions $f : G^n \longrightarrow V$ such that $\mathtt{Im} f \subset A$ for some $A \in S(V)$, where G^n is the Cartesian product of G. This family is a vector space and contains all constant functions.

The problem of the stability of functional equations has been posed by Ulam in [24]. Hyers in [5] gave an affirmative partial answer for the stability of the linear functional equation for Banach spaces. For more detailed definitions of such terminology one can refer to [8, 9, 15, 18] and [16, 17, 19, 20] and references therein.

In Lipschitz spaces, Czerwik et al. [3] and Tabor [22, 23] studied the stability type problems for some functional equations. The author of the present paper proved the stability of quadratic, cubic and quartic functional equations in Lipschitz spaces (cf. [4, 11–14]). Czerwik et al. [3] considered the stability of the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1.1}$$

Jun and Kim [7] introduced the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x), \tag{1.2}$$

which are somewhat different from (1.1). They established the general solution and the Hyers-Ulam-Rassias stability problem of (1.2) for mapping from a real vector space to a Banach space. Bae and Park [1] proved the general solution and the stability of the following 2-variable quadratic functional equation

$$f(x+z,y+w)+f(x-z,y-w)=2f(x,y)+2f(z,w)$$

in complete normed spaces.

In this paper, we introduce the n-variable cubic functional equations as follows

$$\begin{aligned} 2f(x_1+y_1,\dots,x_n+y_n)+2f(x_1-y_1,\dots,x_n-y_n)+12f(x_1,\dots,x_n)& \\ =f(2x_1+y_1,\dots,2x_n+y_n)+f(2x_1-y_1,\dots,2x_n-y_n).& \end{aligned} \tag{1.3}$$

We say that a function $f: G^n \longrightarrow V$ is n-cubic if f satisfies (1.3). We verify the stability of the n-cubic functional equation in Lipschitz spaces and as a consequence we give the stability of the cubic functional equation.

2. APPROXIMATION WITH $\mathbf{d}$-LIPSCHITZ FUNCTIONS

Suppose that $f: G^n \longrightarrow V$ is a function. We say that f is an odd function if $f(-x_1,\dots,-x_n)=-f(x_1,\dots,x_n)$ for all $(x_1,\dots,x_n)\in G^n$. We consider the n-variable cubic difference as follows:

$$\begin{aligned} Sf(x_1,\dots,x_n;y_1,\dots,y_n) &:= 2f(x_1+y_1,\dots,x_n+y_n)+2f(x_1-y_1,\dots,x_n-y_n) \\ &+12f(x_1,\dots,x_n)-f(2x_1+y_1,\dots,2x_n+y_n)-f(2x_1-y_1,\dots,2x_n-y_n) \end{aligned}$$

for all $(x_1,\dots,x_n),(y_1,\dots,y_n)\in G^n$. A function f is n-cubic if

$$Sf(x_1,\dots,x_n;y_1,\dots,y_n)=0.$$

It is clear that if a function f is n-cubic, then $f(0,\dots,0)=0$. Let $S(V)$ be a family of subsets of V. It is easy to verify that $S(V)$ contains all singleton subsets of V. Following [2, 22] let $\mathbf{d}: G^n\times G^n \longrightarrow S(V)$ be a set-valued function such that

$$\begin{aligned} \mathbf{d}((x_1+a_1,\dots,x_n+a_n),(y_1+a_1,\dots,y_n+a_n)) &= \mathbf{d}((a_1+x_1,\dots,a_n+x_n),(a_1+y_1,\dots,a_n+y_n)) \\ &= \mathbf{d}((x_1,\dots,x_n),(y_1,\dots,y_n)) \end{aligned}$$

for all $(x_1,...,x_n),(y_1,...,y_n),(a_1,...,a_n) \in G^n$. A function $f : G^n \longrightarrow V$ is called **d**-Lipschitz if

$$f(x_1,...,x_n) - f(y_1,...,y_n) \in \mathbf{d}((x_1,...,x_n),(y_1,...,y_n))$$

for all $(x_1,...,x_n),(y_1,...,y_n) \in G^n$. Let (G^n,d) be a metric group and V a normed space. A function $\mathsf{mc}_f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a module of continuity of $f : G^n \longrightarrow V$ if for all $\varepsilon > 0$ and all $(x_1,...,x_n),(y_1,...,y_n) \in G^n$ the condition $d((x_1,...,x_n),(y_1,...,y_n)) \leq \varepsilon$ implies $||f(x_1,...,x_n) - f(y_1,...,y_n)|| \leq \mathsf{mc}_f(\varepsilon)$. A function $f : G^n \longrightarrow V$ is called Lipschitz function if it satisfies the condition

$$||f(x_1,...,x_n) - f(y_1,...,y_n)|| \leq Ld((x_1,...,x_n),(y_1,...,y_n)), \tag{2.1}$$

where $L > 0$ is a constant and $(x_1,...,x_n),(y_1,...,y_n) \in G^n$. Let $Lip(G^n,V)$ be the Lipschitz space consisting of all bounded Lipschitz functions with the norm

$$||f||_{Lip} := ||f||_\infty + \mathtt{P}(f),$$

where $||.||_\infty$ is the supremum norm and

$$\mathtt{P}(f) = \sup\Big\{ \frac{||f(x_1,...,x_n) - f(y_1,...,y_n)||}{d((x_1,...,x_n),(y_1,...,y_n))} : (x_1,...,x_n),(y_1,...,y_n) \in G^n, \\ (x_1,...,x_n) \neq (y_1,...,y_n) \Big\}.$$

Definition 1. We say that $B(G^n,S(V))$ admits a left invariant mean (briefly LIM), if the family $S(V)$ is linearly invariant and there exists a linear operator η from $B(G^n,S(V))$ to V such that

(i) if $\mathtt{Im} f \subset A$ for some $A \in S(V)$, then $\eta[f] \in A$,
(ii) if $f \in B(G^n,S(V))$ and $(a_1,...,a_n) \in G^n$, then $\eta[f^{a_1,...,a_n}] = \eta[f]$, where we define $f^{a_1,...,a_n}(x_1,...,x_n) = f(x_1 + a_1,...,x_n + a_n)$.

Definition 2. Consider an Abelian group $(G^n,+)$ with a metric d invariant under translation, i.e., satisfying the condition

$$\begin{aligned} &d((x_1 + a_1,...,x_n + a_n),(y_1 + a_1,y_n + a_n)) \\ &= d((a_1 + x_1,...,a_n + x_n),(a_1 + y_1,...,a_n + y_n)) \\ &= d((x_1,..,x_n),(y_1,...,y_n)) \end{aligned}$$

for all $(x_1,..,x_n),(y_1,...,y_n),(a_1,...,a_n) \in G^n$. We say that a metric ρ is a product metric in G^{2n} if it is an invariant metric and the following condition holds

$$\begin{aligned} &\rho((x_1,...,x_n;a_1,...,a_n),(y_1,...,y_n;a_1,...,a_n)) \\ &= \rho((a_1,...,a_n;x_1,...,x_n),(a_1,...,a_n;y_1,...,y_n)) \\ &= d((x_1,...,x_n),(y_1,...,y_n)) \end{aligned}$$

for all $(x_1,...,x_n;a_1,...,a_n),(y_1,...,y_n;a_1,...,a_n) \in G^{2n}$.

Theorem 1. *Let G be an Abelian group, V a vector space, and $f : G^n \longrightarrow V$ an odd function. If the family $B(G^n, S(V))$ admits LIM and $Sf(t_1, ..., t_n; \cdot, ..., \cdot) : G^n \longrightarrow V$ is **d**-Lipschitz for all $(t_1, ..., t_n) \in G^n$, then there exists an n-cubic function Γ such that $f - \Gamma$ is $\frac{1}{12}$**d**-Lipschitz.*

Proof. For every $(a_1, ..., a_n) \in G^n$ we define $F_{a_1,...,a_n} : G^n \longrightarrow V$ by

$$F_{a_1,...,a_n}(x_1, ..., x_n) := \frac{1}{12} f(2a_1 + x_1, ..., 2a_n + x_n) + \frac{1}{12} f(2a_1 - x_1, ..., 2a_n - x_n) - \frac{1}{6} f(a_1 + x_1, ..., a_n + x_n) - \frac{1}{6} f(a_1 - x_1, ..., a_n - x_n).$$

Fix $(a_1, ..., a_n) \in G^n$. We see that

$$\begin{aligned} F_{a_1,...,a_n}(x_1, ..., x_n) &= \frac{1}{12} f(2a_1 + x_1, ..., 2a_n + x_n) + \frac{1}{12} f(2a_1 - x_1, ..., 2a_n - x_n) \\ &\quad - \frac{1}{6} f(a_1 + x_1, ..., a_n + x_n) - \frac{1}{6} f(a_1 - x_1, ..., a_n - x_n) \\ &= f(0, ..., 0) + \frac{1}{12} f(2a_1 + x_1, ..., 2a_n + x_n) \\ &\quad + \frac{1}{12} f(2a_1 - x_1, ..., 2a_n - x_n) \\ &\quad - \frac{1}{6} f(a_1 + x_1, ..., a_n + x_n) - \frac{1}{6} f(a_1 - x_1, ..., a_n - x_n) \\ &\quad - f(a_1, ..., a_n) + f(a_1, ..., a_n) - f(0, ..., 0) \\ &= \frac{1}{12} Sf(0, ..., 0; x_1, ..., x_n) - \frac{1}{12} Sf(a_1, ..., a_n; x_1, ..., x_n) \\ &\quad + f(a_1, ..., a_n) - f(0, ..., 0). \end{aligned}$$

From the fact that $Sf(t_1, ...t_n; \cdot, ..., \cdot)$ is **d**-Lipschitz and $S(V)$ is translation invariant, we detect that $\mathtt{Im} F_{a_1,...,a_n} \subset A$ for some $A \in S(V)$ and hence $F_{a_1,...,a_n} \in B(G^n, S(V))$. By assumption the family $B(G^n, S(V))$ admits LIM and so there exists a linear operator $\eta : B(G^n, S(V)) \longrightarrow V$ such that

(i) $\eta[F_{a_1,...,a_n}] \in A$ for some $A \in S(V)$,
(ii) if for $(y_1, ..., y_n) \in G^n$, $F^{y_1,...,y_n}_{a_1,...,a_n} : G^n \longrightarrow V$ is defined by

$$F^{y_1,...,y_n}_{a_1,...,a_n}(t_1, ..., t_n) := F_{a_1,...,a_n}(t_1 + y_1, ..., t_n + y_n)$$

for every $(t_1, ..., t_n) \in G^n$, then $F^{y_1,...,y_n}_{a_1,...,a_n} \in B(G^n, S(V))$ and

$$\eta[F_{a_1,...,a_n}] = \eta[F^{y_1,...,y_n}_{a_1,...,a_n}].$$

Since $Sf(t_1, ..., t_n; \cdot, ..., \cdot)$ is a **d**-Lipschitz function,

$$\frac{1}{12} Sf(t_1, ..., t_n; x_1, ..., x_n) - \frac{1}{12} Sf(t_1, ..., t_n; y_1, ..., y_n) \in \frac{1}{12} \mathbf{d}((x_1, ..., x_n), (y_1, ..., y_n))$$

for all $(x_1,...,x_n),(y_1,...,y_n) \in G^n$ and so we observe that

$$\mathtt{Im}(\frac{1}{12}Sf(\cdot,...,\cdot;x_1,...,x_n)-\frac{1}{12}Sf(\cdot,...,\cdot;y_1,...,y_n)) \subseteq \frac{1}{12}\mathbf{d}((x_1,...,x_n),(y_1,...,y_n)).$$

From property (i) of η it becomes that

$$\eta[\frac{1}{12}Sf(\cdot,...,\cdot;x_1,...,x_n)-\frac{1}{12}Sf(\cdot,...,\cdot;y_1,...,y_n)] \in \frac{1}{12}\mathbf{d}((x_1,...,x_n),(y_1,...,y_n)) \tag{2.2}$$

for all $(x_1,...,x_n),(y_1,...,y_n) \in G^n$. Define the function $\Gamma : G^n \longrightarrow V$ by $\Gamma(x_1,...,x_n) := \eta[F_{x_1,...,x_n}]$ for $(x_1,...,x_n) \in G^n$. Note that $B(G^n,S(V))$ contains constant functions and hence using property (i) of η we see that that if $f : G^n \longrightarrow V$ is constant, i.e., $f(x_1,...,x_n) = c$ for $(x_1,...,x_n) \in G^n$, where $c \in V$, then $\eta[f] = c$. Fix $(x_1,...,x_n) \in G^n$ and define the constant function $C_{x_1,...,x_n} : G^n \longrightarrow V$ by $C_{x_1,...,x_n}(\cdot,...,\cdot) := f(x_1,...,x_n)$. One has

$$\begin{aligned}(f(x_1,...,x_n)&-\Gamma(x_1,...,x_n))-(f(y_1,...,y_n)-\Gamma(y_1,...,y_n))\\ &=(\eta[C_{x_1,...,x_n}]-\eta[F_{x_1,...,x_n}])-(\eta[C_{y_1,...,y_n}]-\eta[F_{y_1,...,y_n}])\\ &=\eta[C_{x_1,...,x_n}-F_{x_1,...,x_n}]-\eta[C_{y_1,...,y_n}-F_{y_1,...,y_n}]\\ &=\eta[\frac{1}{12}Sf(\cdot,...,\cdot;x_1,...,x_n)-\frac{1}{12}Sf(\cdot,...,\cdot;y_1,...,y_n)]\end{aligned} \tag{2.3}$$

for all $(x_1,...,x_n),(y_1,...,y_n) \in G^n$. From (2.2) and (2.3) it follows that

$$\begin{aligned}(f(x_1,...,x_n)-\Gamma(x_1,...,x_n))&-(f(y_1,...,y_n)-\Gamma(y_1,...,y_n))\\ &\in \frac{1}{12}\mathbf{d}((x_1,...,x_n),(y_1,...,y_n))\end{aligned}$$

for all $(x_1,...,x_n),(y_1,...,y_n) \in G^n$. This means that $f-\Gamma$ is a $\frac{1}{12}\mathbf{d}$-Lipschitz function. By applying property (ii) of η, we see that

$$\begin{aligned}&2\Gamma(y_1+z_1,...,y_n+z_n)+2\Gamma(y_1-z_1,...,y_n-z_n)+12\Gamma(y_1,...,y_n)\\ &=2\eta[F_{y_1+z_1,...,y_n+z_n}(x_1,...,x_n)]+2\eta[F_{y_1-z_1,...,y_n-z_n}(x_1,...,x_n)]\\ &\quad+12\eta[F_{y_1,...,y_n}(x_1,...,x_n)]\\ &=\eta[F^{2y_1,...,2y_n}_{y_1+z_1,...,y_n+z_n}(x_1,...,x_n)]+\eta[F^{-2y_1,...,-2y_n}_{y_1+z_1,...,y_n+z_n}(x_1,...,x_n)]\\ &\quad+\eta[F^{2y_1,...,2y_n}_{y_1-z_1,...,y_n-z_n}(x_1,...,x_n)]+\eta[F^{-2y_1,...,-2y_n}_{y_1-z_1,...,y_n-z_n}(x_1,...,x_n)]\\ &\quad+2\eta[F^{y_1+z_1,...,y_n+z_n}_{y_1,...,y_n}(x_1,...,x_n)]+2\eta[F^{y_1-z_1,...,y_n-z_n}_{y_1,...,y_n}(x_1,...,x_n)]\\ &\quad+2\eta[F^{-y_1-z_1,...,-y_n-z_n}_{y_1,...,y_n}(x_1,...,x_n)]\\ &\quad+2\eta[F^{-y_1+z_1,...,-y_1+z_1}_{y_1,...,y_n}(x_1,...,x_n)]+2\eta[F^{z_1,...,z_n}_{y_1,...,y_n}(x_1,...,x_n)]\\ &\quad+2\eta[F^{-z_1,...,-z_n}_{y_1,...,y_n}(x_1,...,x_n)].\end{aligned} \tag{2.4}$$

On the other hand,

$$\eta[F^{2y_1,\dots,2y_n}_{y_1+z_1,\dots,y_n+z_n}(x_1,\dots,x_n)]$$
$$=\eta[\frac{1}{12}f(4y_1+2z_1+x_1,\dots,4y_n+2z_n+x_n)+\frac{1}{12}f(2z_1-x_1,\dots,2z_n-x_n)$$
$$-\frac{1}{6}f(3y_1+x_1+z_1,\dots,3y_n+x_n+z_n)-\frac{1}{6}f(z_1-x_1-y_1,\dots,z_n-x_n-y_n)],\quad (2.5)$$

$$\eta[F^{-2y_1,\dots,-2y_n}_{y_1+z_1,\dots,y_n+z_n}(x_1,\dots,x_n)]$$
$$=\eta[\frac{1}{12}f(2z_1+x_1,\dots,2z_n+x_n)+\frac{1}{12}f(4y_1+2z_1-x_1,\dots,4y_n+2z_n-x_n)$$
$$-\frac{1}{6}f(x_1-y_1+z_1,\dots,x_n-y_n+z_n)-\frac{1}{6}f(3y_1+z_1-x_1,\dots,3y_n+z_n-x_n)],\quad (2.6)$$

$$\eta[F^{2y_1,\dots,2y_n}_{y_1-z_1,\dots,y_n-z_n}(x_1,\dots,x_n)]$$
$$=\eta[\frac{1}{12}f(4y_1-2z_1+x_1,\dots,4y_n-2z_n+x_n)+\frac{1}{12}f(-2z_1-x_1,\dots,-2z_n-x_n)$$
$$-\frac{1}{6}f(3y_1+x_1-z_1,\dots,3y_n+x_n-z_n)-\frac{1}{6}f(-z_1-x_1-y_1,\dots,-z_n-x_n-y_n)],$$
$$(2.7)$$

$$\eta[F^{-2y_1,\dots,-2y_n}_{y_1-z_1,\dots,y_n-z_n}(x_1,\dots,x_n)]$$
$$=\eta[\frac{1}{12}f(-2z_1+x_1,\dots,-2z_n+x_n)+\frac{1}{12}f(4y_1-2z_1-x_1,\dots,4y_n-2z_n-x_n)$$
$$-\frac{1}{6}f(-z_1+x_1-y_1,\dots,-z_n+x_n-y_n)-\frac{1}{6}f(3y_1-z_1-x_1,\dots,3y_n-z_n-x_n)].$$
$$(2.8)$$

We can also find that

$$2\eta[F^{y_1+z_1,\dots,y_n+z_n}_{y_1,\dots,y_n}(x_1,\dots,x_n)]$$
$$=\eta[\frac{1}{6}f(3y_1+x_1+z_1,\dots,3y_n+x_n+z_n)+\frac{1}{6}f(y_1-x_1-z_1,\dots,y_n-x_n-z_n)$$
$$-\frac{1}{3}f(2y_1+x_1+z_1,\dots,2y_n+x_n+z_n)-\frac{1}{3}f(-x_1-z_1,\dots,-x_n-z_n)],\quad (2.9)$$

$$2\eta[F^{y_1-z_1,\dots,y_n-z_n}_{y_1,\dots,y_n}(x_1,\dots,x_n)]$$
$$=\eta[\frac{1}{6}f(3y_1+x_1-z_1,\dots,3y_n+x_n-z_n)+\frac{1}{6}f(y_1-x_1+z_1,\dots,y_n-x_n+z_n)$$
$$-\frac{1}{3}f(2y_1+x_1-z_1,\dots,2y_n+x_n-z_n)-\frac{1}{3}f(-x_1+z_1,\dots-x_n+z_n,)],\quad (2.10)$$

$$2\eta[F^{-y_1-z_1,\dots,-y_n+z_n}_{y_1,\dots,y_n}(x_1,\dots,x_n)]$$

$$
\begin{aligned}
&= \eta[\frac{1}{6}f(x_1+y_1-z_1,\ldots,x_n+y_n-z_n)+\frac{1}{6}f(3y_1-x_1+z_1,\ldots,3y_n-x_n+z_n)\\
&\quad -\frac{1}{3}f(x_1-z_1,\ldots,x_n-z_n)-\frac{1}{3}f(2y_1-x_1+z_1,\ldots,2y_n-x_n+z_n)], \qquad (2.11)
\end{aligned}
$$

$$
\begin{aligned}
&2\eta[F_{y_1,\ldots,y_n}^{-y_1+z_1,\ldots,-y_n+z_n}(x_1,\ldots,x_n)]\\
&= \eta[\frac{1}{6}f(x_1+y_1+z_1,\ldots,x_n+y_n+z_n)+\frac{1}{6}f(3y_1-x_1-z_1,\ldots,3y_n-x_n-z_n)\\
&\quad -\frac{1}{3}f(x_1+z_1,\ldots,x_n+z_n)-\frac{1}{3}f(2y_1-x_1-z_1,\ldots,2y_n-x_n-z_n)], \qquad (2.12)
\end{aligned}
$$

$$
\begin{aligned}
&2\eta[F_{y_1,\ldots,y_n}^{z_1,\ldots,z_n}(x_1,\ldots,x_n)]\\
&= \eta[\frac{1}{6}f(2y_1+x_1+z_1,\ldots,2y_n+x_n+z_n)+\frac{1}{6}f(2y_1-x_1-z_1,\ldots,2y_n-x_n-z_n)\\
&\quad -\frac{1}{3}f(x_1+y_1+z_1,\ldots,x_n+y_n+z_n)-\frac{1}{3}f(y_1-x_1-z_1,\ldots,y_n-x_n-z_n)], \qquad (2.13)
\end{aligned}
$$

$$
\begin{aligned}
&2\eta[F_{y_1,\ldots,y_n}^{-z_1,\ldots,-z_n}(x_1,\ldots,x_n)]\\
&= \eta[\frac{1}{6}f(2y_1+x_1-z_1,\ldots,2y_n+x_n-z_n)+\frac{1}{6}f(2y_1-x_1+z_1,\ldots,2y_n-x_n+z_n)\\
&\quad -\frac{1}{3}f(x_1+y_1-z_1,\ldots,x_n+y_n-z_n)-\frac{1}{3}f(y_1-x_1+z_1,\ldots,y_n-x_n+z_n)]. \qquad (2.14)
\end{aligned}
$$

By summing Eqs. (2.5)–(2.14) and using (2.4) we conclude that

$$
\begin{aligned}
&2\Gamma(y_1+z_1,\ldots,y_n+z_n)+2\Gamma(y_1-z_1,\ldots,y_n-z_n)+12\Gamma(y_1,\ldots,y_n)\\
&= \eta\Big[\frac{1}{12}f(4y_1+2z_1+x_1,\ldots,4y_n+2z_n+x_n)\\
&\quad +\frac{1}{12}f(4y_1+2z_1-x_1,\ldots,4y_n+2z_n-x_n)\\
&\quad -\frac{1}{6}f(2y_1+x_1+z_1,\ldots,2y_n+x_n+z_n)-\frac{1}{6}f(2y_1+z_1-x_1,\ldots,2y_n+z_n-x_n)\Big]\\
&\quad +\eta\Big[\frac{1}{12}f(4y_1-2z_1+x_1,\ldots,4y_n-2z_n+x_n)\\
&\quad +\frac{1}{12}f(4y_1-2z_1-x_1,\ldots,4y_n-2z_n-x_n)\\
&\quad -\frac{1}{6}f(2y_1+x_1-z_1,\ldots,2y_n+x_n-z_n)-\frac{1}{6}f(2y_1-z_1-x_1,\ldots,2y_n-z_n-x_n)\Big]\\
&= \eta[F_{2y_1+z_1,\ldots,2y_n+z_n}(x_1,\ldots,x_n)]+\eta[F_{2y_1-z_1,\ldots,2y_n-z_n}(x_1,\ldots,x_n)]\\
&= \Gamma(2y_1+z_1,\ldots,2y_1+z_1)+\Gamma(2y_1-z_1,\ldots,2y_n-z_n).
\end{aligned}
$$

This entails that $S\Gamma(y_1,\ldots,y_n;z_1,\ldots,z_n)=0$ and so Γ is n-cubic. □

By $\Delta(G)$ we denote the diagonal set on G, i.e.,

$$\Delta(G) := \{(x,\dots,x) \in G^n : x \in G\}$$

and throughout the paper we define $F := f_{|_{\Delta(G)}}$.

Corollary 1. *Under the hypotheses of Theorem 1, there exists a cubic function $C : G \longrightarrow V$ such that $F - C$ is $\frac{1}{12}\mathbf{d}$-Lipschitz.*

Proof. Define the function $\gamma : G^n \longrightarrow V$ by $\gamma(x_1,\dots,x_n) := \eta[F_{x_1,\dots,x_n}]$ for $(x_1,\dots,x_n) \in G^n$. Then, γ is n-cubic as in the proof of Theorem 1. Let $C : G \longrightarrow V$ be a function defined by $C(x) := \gamma(x,\dots,x)$. We have

$$F - C = f_{|_{\Delta(G)}} - \gamma_{|_{\Delta(G)}} = (f-\gamma)_{|_{\Delta(G)}}.$$

The function $f-\gamma$ is $\frac{1}{12}\mathbf{d}$-Lipschitz and so is $F-C$. The following equality now entails that C is cubic:

$$\begin{aligned} 2C(x+y)+2C(x-y)+12C(x) &= 2\gamma(x+y,\dots,x+y) \\ &\quad + 2\gamma(x-y,\dots,x-y) + 12\gamma(x,\dots,x) \\ &= \gamma(2x+y,\dots,2x+y)+\gamma(2x-y,\dots,2x-y) \\ &= C(2x+y)+C(2x-y). \end{aligned}$$

□

Lemma 1. *Under the hypotheses of Theorem 1, if $\mathtt{Im}Sf \subset A$ for some $A \in S(V)$, then $\mathtt{Im}(f-\Gamma) \subset \frac{1}{12}A$.*

Proof. It is clear that

$$\mathtt{Im}(\frac{1}{12}Sf(x_1,\dots,x_n;\cdot,\dots,\cdot)) \subset \mathtt{Im}(\frac{1}{12}Sf) \subset \frac{1}{12}A$$

and so $\frac{1}{12}Sf(x_1,\dots,x_n;\cdot,\dots,\cdot) \in B(G^n,S(V))$ for all $(x_1,\dots,x_n) \in G^n$. Thus, property (i) of η implies

$$f(x_1,\dots,x_n) - \Gamma(x_1,\dots,x_n) = \eta[\frac{1}{12}Sf(x_1,\dots,x_n;\cdot,\dots,\cdot)] \in \frac{1}{12}A$$

for all $(x_1,\dots,x_n) \in G^n$. Therefore, $\mathtt{Im}(f-\Gamma) \subset \frac{1}{12}A$. □

Let $CB(V)$ be a family of closed balls of V.

Theorem 2. *Let $(G^n,+,d,\rho)$ be a product metric, V a normed space, and $f : G^n \longrightarrow E$ an odd function. If $B(G^n,CB(V))$ admits LIM and $Sf \in Lip(G^n \times G^n, E)$, then there exists an n-cubic function Γ such that*

$$||f-\Gamma||_{Lip} \le \frac{1}{12}||Sf||_{Lip}.$$

Proof. Define the set-valued function $\mathbf{d} : G \times G \longrightarrow CB(V)$ by

$$\mathbf{d}((x_1,\ldots,x_n),(y_1,\ldots,y_n)) := \inf_{d((x_1,\ldots,x_n),(y_1,\ldots,y_n))\leq\delta} \mathsf{mc}_{Sf}(\delta)B(0,1),$$

where $B(0,1)$ is the closed unit ball with center at zero. Let $\mathsf{mc}_{Qf} : \mathbb{R}^+ \to \mathbb{R}^+$ be the module of continuity of $Sf : G^n \times G^n \to E$ with the product metric ρ on $G^n \times G^n$. Then,

$$\begin{aligned}&||Sf(t_1,\ldots,t_n;x_1,\ldots,x_n) - Sf(t_1,\ldots,t_n;y_1,\ldots,y_n)||\\ &\qquad \leq \inf_{\rho((t_1,\ldots,t_n;x_1,\ldots,x_n),(t_1,\ldots,t_n;y_1,\ldots,y_n))\leq\delta} \mathsf{mc}_{Sf}(\delta) = \inf_{d((x_1,\ldots,x_n),(y_1,\ldots,y_n))\leq\delta} \mathsf{mc}_{Sf}(\delta)\end{aligned}$$

for all $(t_1,\ldots,t_n),(x_1,\ldots,x_n),(y_1,\ldots,y_n) \in G^n$. This inequality entails that $Sf(t_1,\ldots,t_n;\cdot,\ldots,\cdot)$ is $\mathbf{d}$-Lipschitz and so Theorem 1 implies that there exists an n-cubic function Γ such that $f-\Gamma$ is $\frac{1}{12}\mathbf{d}$-Lipschitz. Hence,

$$\begin{aligned}&||(f(x_1,\ldots,x_n) - \Gamma(x_1,\ldots,x_n)) - (f(y_1,\ldots,y_n) - \Gamma(y_1,\ldots,y_n))||\\ &\qquad\qquad \leq \inf_{d((x_1,\ldots,x_n),(y_1,\ldots,y_n))\leq\delta} \frac{1}{12}\mathsf{mc}_{Sf}(\delta),\end{aligned}$$

which shows that $\mathsf{mc}_{f-\Gamma} = \frac{1}{12}\mathsf{mc}_{Sf}$. Since $\mathtt{Im}Sf \subseteq ||Sf||_\infty B(0,1)$, Lemma 1 implies that

$$\mathtt{Im}(f-\Gamma) \subset \frac{1}{12}||Sf||_\infty B(0,1),$$

which is equivalent to

$$||f-\Gamma||_\infty \leq \frac{1}{12}||Sf||_\infty. \tag{2.15}$$

We may also prove that $\mathsf{mc}_{Sf} = \mathtt{P}(Sf)$ and so $\mathtt{P}(f-\Gamma) \leq \frac{1}{12}\mathtt{P}(Sf)$. The inequality (2.15) now ensures

$$||f-\Gamma||_{Lip} = ||f-\Gamma||_\infty + \mathtt{P}(f-\Gamma) \leq \frac{1}{12}||Sf||_\infty + \frac{1}{12}\mathtt{P}(Sf) = \frac{1}{12}||Sf||_{Lip}.$$

□

Corollary 2. *Under the hypotheses of Theorem 2, there exists a cubic function $C : G \longrightarrow V$ such that*

$$||F-C||_{Lip} \leq \frac{1}{12}||Sf||_{Lip}.$$

Proof. It follows from Theorem 2 that there exists an n-cubic function $\gamma : G^n \longrightarrow V$ such that

$$||f-\gamma||_{Lip} \leq \frac{1}{12}||Sf||_{Lip}.$$

Define $C : G \longrightarrow V$ by $C(x) := \gamma(x,\ldots,x)$. Corollary 1 entails that C is cubic and

$$F - C = (f-\gamma)|_{\Delta(G)}.$$

On the other hand, the function $f-\gamma$ is $\frac{1}{12}\mathbf{d}$-Lipschitz and so is $F-C$. The following inequality now ensures the result:

$$||F-C||_{Lip} \leq ||f-\gamma||_{Lip} \leq \frac{1}{12}||Sf||_{Lip}.$$

□

REFERENCES

[1] J.-H. Bae and W.-G. Park, "A functional equation originating from quadratic forms." *J. Math. Anal. Appl.*, vol. 326, pp. 1142–1148, 2007, doi: 10.1016/J.JMAA.2006.03.023.

[2] S. Czerwik, *Functional Equations and Inequalities in Several Variables.* River Edge, NJ: World Scientific, 2002.

[3] S. Czerwik and K. Dlutek, "Stability of the quadratic functional equation in Lipschitz spaces." *J. Math. Anal. Appl.*, vol. 293, pp. 79–88, 2004, doi: 10.1016/j.jmaa.2003.12.034.

[4] A. Ebadian, N. Ghobadipour, I. Nikoufar, and M. E. Eshaghi Gordji, "Approximation of the cubic functional equations in Lipschitz spaces." *Anal. Theory Appl.*, vol. 30, pp. 354–362, 2014, doi: 10.4208/ata.2014.v30.n4.2.

[5] D. H. Hyers, "On the stability of the linear functional equation." *Proc. Natl. Acad. Sci.*, vol. 27, pp. 222–224, 1941, doi: 10.1073/pnas.27.4.222.

[6] J. A. Johnson, "Lipschitz spaces." *Pacific J. Math.*, vol. 51, pp. 177–186, 1974.

[7] K.-W. Jun and H.-M. Kim, "The generalized Hyers-Ulam-Rassias stability of a cubic functional equation." *J. Math. Anal. Appl.*, vol. 274, pp. 867–878, 2002, doi: 10.1016/S0022-247X(02)00415-8.

[8] S. Jung and M. Th. Rassias, "A linear functional equation of third order associated to the Fibonacci numbers." *Abstr. Appl. Anal.*, vol. 2014, Article ID 137466, 2014, doi: 10.1155/2014/137468.

[9] Y.-H. Lee, S. Jung, and M. Th. Rassias, "On an *n*-dimensional mixed type additive and quadratic functional equation." *Appl. Math. Comput.*, vol. 228, pp. 13–16, 2014, doi: 10.1016/j.amc.2013.11.091.

[10] J. Luukkainen and J. Vaisala, "Elements of Lipschitz topology." *Ann. Acad. Sci. Fenn. Ser. A I Math.*, vol. 3, pp. 85–122, 2014, doi: 10.5186/AASFM.1977.0315.

[11] I. Nikoufar, "Lipschitz approximation of the *n*-quadratic functional equations." *Mathematica (Cluj).*, vol. 57, pp. 67–74, 2015.

[12] I. Nikoufar, "Quartic functional equations in Lipschitz spaces." *Rend. Circ. Mat. Palermo.*, vol. 64, pp. 171–176, 2015, doi: 10.1007/s12215-014-0187-1.

[13] I. Nikoufar, "Erratum to "Quartic functional equations in Lipschitz spaces"." *Rend. Circ. Mat. Palermo.*, vol. 65, pp. 345–350, 2016, doi: 10.1007/s12215-015-0222-x.

[14] I. Nikoufar, "Approximate tri-quadratic functional equations via Lipschitz conditions." *Math. Bohem.*, vol. 142, pp. 337–344, 2017, doi: 10.21136/MB.2017.0028-16.

[15] I. Nikoufar and Th. M. Rassias, "Approximately algebraic tensor products." *Miskolc Math. Notes*, vol. 16, pp. 361–368, 2015, doi: 10.18514/MMN.2015.1260.

[16] C. Park and Th. M. Rassias, "Hyers-Ulam stability of a generalized Apollonius type quadratic mapping." *J. Math. Anal. Appl.*, vol. 322, no. 1, pp. 371–381, 2006, doi: 10.1016/J.JMAA.2005.09.027.

[17] M. Th. Rassias, "Solution of a functional equation problem of Steven Butler." *Octogon Math. Mag.*, vol. 12, pp. 152–153, 2004.

[18] Th. M. Rassias, *Functional Equations and Inequalities.* New York: Kluwer Academic Publishers, 2000.

[19] B. V. Senthil Kumar and H. Dutta, "Non-Archimedean stability of a generalized reciprocal-quadratic functional equation in several variables by direct and fixed point methods." *Filomat*, vol. 32, pp. 3199–3209, 2018, doi: 10.2298/FIL1809199.

[20] B. V. Senthil Kumar and H. Dutta, "Approximation of multiplicative inverse undecic and duodecic functional equations." *Math. Methods Appl. Sci.*, vol. 42, pp. 1073–1081, 2019, doi: 10.1002/mma.5413.

[21] L. Siebenmann and D. Sullivan, *On complexes that are Lipschitz manifolds, In Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977).* New York: Academic Press, 1979.

[22] J. Tabor, "Lipschitz stability of the Cauchy and Jensen equations." *Results Math.*, vol. 32, pp. 133–144, 1997, doi: 10.1007/BF03322533.

[23] J. Tabor, "Superstability of the Cauchy, Jensen and isometry equations." *Results Math.*, vol. 35, pp. 355–379, 1999, doi: 10.1007/BF03322824.

[24] S. M. Ulam, *Collection of the Mathematical Problems.* New York: Interscience Publ., 1940.

[25] N. Weaver, *Lipschitz Algebras.* River Edge, NJ: World Scientific Publishing Co., 1999.

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