Reaction-diffusion Model Describing the Morphogenesis of Urban Systems in the US

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Abstract: Urbanization is currently one of the greatest challenges facing mankind. In order to be able to anticipate the rapid changes in urban structures, models are in need, mapping the morphological development of these structures. In this paper we present an urban development model that is based on reaction diffusion equations and can be interpreted sociologically. We apply this model to the urban development of US-American cities and show that this very simple model can already map basic characteristics of urban development.

1 INTRODUCTION

More than every second person on earth lives in a city. According to UN estimates, this share will rise to almost two thirds by 2050 (United Nations, 2016). These rapid changes lead to major challenges for the infrastructure systems (energy, water, wastewater) of urban areas (Kraas et al., 2016). In order to develop adequat solution stragetegies to face this problem it is necessary to anticipate the development of cities. Therefore, for many years efforts have been made to understand, model and simulate the development of urban structures. Very common are statistically, cellular automata and agent based models. A comprehensive summary of these different approaches can be found in (Benenson and Torrens, 2004). The question is, what the central mechanisms of this structure formation are and how they can be simulated.

While in earlier studies it was repeatedly assumed that urban systems behave scale free, more recent studies have shown that urban systems do indeed contain typical variables, as shown for example in the work of (González-Val et al., 2015) for cities within countries, or of (Friesen et al., 2018) for slums within cities. Assuming that urban systems are free of scales, simulation models are also developed that lead to scale free systems. This can be seen for example in the development of models for fractal morphogenesis (Frankhauser, 1998). However, if urban systems have typical scales, it is advantageous to use models that take this circumstance into account.

Therefore, in this paper we use an analogy from another scientific discipline to model urban change. Structural changes can not only be identified in urban structures, but are also observed in other systems, for example in biology. These processes of morphogenesis were fundamentally described in a mathematical way in 1952 by Alan Turing, who was able to show how processes of pattern formation in biological systems can be described using relatively simple reaction-diffusion equations (Turing, 1952). He showed, that when certain conditions are satisfied, diffusion can lead to instability of the system.

In this paper we use a framework based on the work of Turing for describing the structural development of cities with a system of reaction-diffusion (RD) equations. Although RD equations have already been used to describe urban processes (Schweitzer and Steinbink, 2002), no stability analyses of the equations have been performed. An exeption is a recent paper of (Pelz et al., 2019) using instability effects to explain the formation of slums, a specific urban class. Also worth mentioning in this context is the work of Paul Krugman, who also relates the spatial distribution of industry to the work of Turing (Krugman, 1996). However, the equations presented in the last mentioned paper describe economic and not sociological aspects.

We take up this idea and ask the research question whether and to what extend it is possible to predict the structural development of cities with reaction-diffusion equations. For this purpose, we use Census data from cities in the USA from the years 2000 and 2010, showing particularly strong economic growth. After first recapitulating a stability analysis of the equations from literature, we interpret the equations

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sociologically and examine their suitability for describing structural changes in the cities. To do this, we conduct a simulation study varying four different parameters on two cities to find the parameter combinitation describing the structural change within the cities the best. We then simulate the structal change in a thrid city by using the best parameters found before and analyse the results.

In this paper we first describe the reaction-diffusion model, the sociological processes it describes, the data used and then our approach (sec. 2). We then present our results (sec. 3), discuss them (sec. 4) and finally summarize the work (sec. 5).

2 METHODS AND DATA

Our approach is as follows: First, we briefly present the mathematical principles of the RD equations. Building on this, we explain the sociological processes that can be mapped with these equations and discuss the assumptions we made within our setup.

2.1 Reaction-diffusion Model

The Reaction-Diffusion Model which shall be used for this simulation study is presented by Gray and Scott first in (Gray and Scott, 1983; Gray and Scott, 1984), which is why we shall refer to it as the "Reaction-Diffusion Model by Gray/Scott" or short the "Gray/Scott-model". The basic model as described in (Gray and Scott, 1983) can be expressed as follows:

$$\begin{aligned}\frac{\partial v}{\partial t} &= D_v \Delta v + R[uv^2 - (F+k)v] \\ \frac{\partial u}{\partial t} &= D_u \Delta u + R[-uv^2 + F(1-u)]\end{aligned} \tag{1}$$

v and u are (dimensionless) concentrations of two chemicals V and U. Here V is the autocatalytic activator, whereas U represents the substrate in the activator-depleted substrate scheme. D_v and D_u are the diffusion constants of V and U. F and k are two more constants, which have an outstanding effect on the emergence of the outcoming patterns. We added the reaction rate R, which is set to one in the original papers, but is needed for proper dimensional analysis. As a first step Eq. 1 shall be transformed to a non-dimensional representation. Therefore we introduce the dimensonsless parameters

$$d := \frac{D_u}{D_v}; \gamma := \frac{RL^2}{D_v}; t^* := \frac{D_v t}{L^2}; x^* := \frac{x}{L} \tag{2}$$

with the time t and the typical length L and of the system. The non-dimensional reaction-diffusion equations are then:

$$\begin{aligned}\frac{\partial v}{\partial t^*} &= \Delta v + \gamma[uv^2 - (F+k)v] \\ \frac{\partial u}{\partial t^*} &= d\Delta u + \gamma[-uv^2 + F(1-u)]\end{aligned} \tag{3}$$

We use the common zero-flux boundary conditions $((n \cdot \nabla))$. For the stability analysis we follow the structure of the derivations by (Gray and Scott, 1984). When investigating the steady state (i.e. $\partial/(\partial t^*) = 0$), one can find three steady states. The first is the trivial solution and shall be called "red state" (index R) in accordance with (Gray and Scott, 1984): $(v_R, u_R) = (0, 1)$. Furthermore, one can derive the "blue state" (index "B")

$$(v_B, u_B) = \left(\frac{F}{2(F+k)}(1+\sqrt{p}), \frac{1}{2}(1-\sqrt{p})\right) \tag{4}$$

and the "intermediate state" (index "I")

$$(v_I, u_I) = \left(\frac{F}{2(F+k)}(1-\sqrt{p}), \frac{1}{2}(1+\sqrt{p})\right) \tag{5}$$

when the discriminator p is:

$$p := 1 - \frac{4(F+k)^2}{F} > 0. \tag{6}$$

A further analysis of the steady states shows, that the red state is always stable, the intermediate state is always unstable and the blue state can be stable, but does not have to be. Thus the blue state is the only steady state which can lead to diffusion driven instability, the so called Turing instability. The necessary and sufficient condition for Turing instability can be derived to:

$$2\sqrt{d \det A_B} < a_{11} d + a_{22} \tag{7}$$

with the Jacobian

$$A_B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{8}$$

Eq. 7 and Eq. 8 deliver the stability map (Fig. 1) of the Gray/Scott-model for different values of d. Every (F,k)-combination to the right of the Saddle-Node-Bifurcation curve leads to a system which only leads towards the red state. Every (F,k)-combiantion that is located between the Saddle-Node-Bifurcation and the respective instability curve for a given value of d leads towards Turing patterns, whereas every (F,k)-combination to the left of the instability curve can either lead to a stable blue state or a red state solution, depending on the initial conditions.

Further examination of the stability of the steady states leads to the implicit formula that is the necessary and sufficient condition for Turing instability in the Gray-Scott-model:

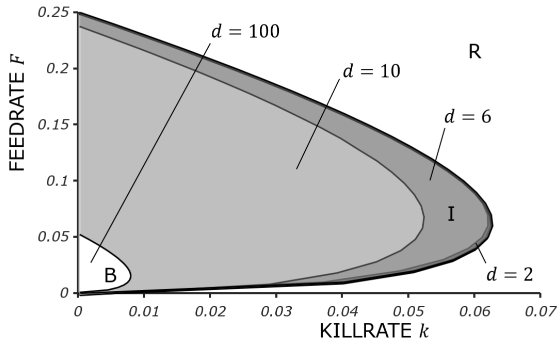

Figure 1: Stability Map of the Gray/Scott-model for different values of d. The region right to the curve is the red state (R). The grey regions are the Intermediate states according to the different diffusion numbers (I). The region left to the intermediate regions is the blue state (B).

$$2\sqrt{d\left[\left[(F+k)\left(\left(\frac{F}{2(F+k)}(1+\sqrt{p})\right)^2+F\right)\right]-\left[\left(\frac{F}{2(F+k)}(1+\sqrt{p})\right)^2(-2(F+k))\right]\right]} < (F+k)d-\left(\left(\frac{F}{2(F+k)}(1+\sqrt{p})\right)^2+F\right) \tag{9}$$

2.2 Methodology

A simulation study is chosen to test the capability of reaction diffusion equations to predict human movement patterns within cities. To reach the highest data availability, the USA was chosen to provide city data. One can get data down to the level of so called "Census Blocks", which vary in size but typically cover an area of about 10 000 m^2. At this level of resolution, the accessible data is limited to:

- Population count,
- Age and sex,
- Marital status and
- Ethnicity.

This paper connects the natural science in form of reaction-diffusion-model by Gray-Scott with certain sociological phenomena (Figure 2). A broad literature analysis led to two phenomena that were regularly stated as typical for (fast) growing cities. These are segregation (S. Fowler, 2016; Farrell, 2016; Alba and Logan, 1993; England et al., 1988; Koch, 2011; Schulz et al., 2006) and gentrification (Timberlake and Johns-Wolfe, 2017; Hwang and Sampson, 2014; Janoschka et al., 2014; Lees, 2012; Smith, 2002; Smith, 2005). Segregation in this context means a separation of people and groups with similar social (religious, ethnical, class-specific, a.o.) backgrounds from groups with other social backgrounds to avoid reciprocal contact. The Gray/Scott-model shows similar behavior in being able to produce stripe and bubble patterns out of a simulated "mixed fluid bowl".

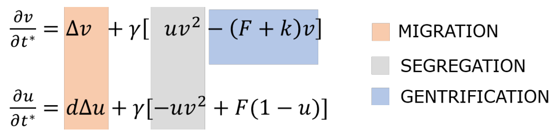

Figure 2: Sociological interpretation of the Gray/Scott Model.

The gentrification in contrast to the segregation means that a wealthier population is slowly squeezing the former – poorer – population out of their neighborhood or district. This squeeze-out is happening due to a valorization of these neighborhoods by renovations and redevelopment of houses, as well as a different business structure being attracted by a more solvent clientele, up to a point where the former population cannot or does not want to afford to live there anymore. Looking at the Gray/Scott-model, this can be incorporated if you see the richer population as the activator (V) and the poorer population as the inhibitor (U). The Gray/Scott-model contains the diminisher term $-(F+k)v$ and the refill term $F(1-u)$. Given above made assumption, the rich population is diminished more likely in areas where there is already a high concentration of rich people $v \approx 1$). At the same time some of the poorer population has to move out of their neighborhoods into close-by districts (indicated by the higher diffusion of the inhibitor $[d > 1]$) or further away (indicated by the refill term). This shows that wealth would be an excellent morphogene for the Gray/Scott-model. Unfortunately – as stated above – at the highest resolution, no wealth data is available. Therefore ethnicity data was chosen, as it shows the highest correlation to wealth of the accessible data (Shapiro et al., 2013; Tessmann and Friesen, 2017).

For the simulation study different cities have to be chosen. As megacities in developing countries are particularly interesting in their fast growth and fast changes in both population and infrastructure, cities were chosen that behave similarly to those megacities of developing countries. Four criteria can be identified, that characterize such cities: An already large number of inhabitants, a fast growth of population, a high inequality and strong economic growth.

Especially strong economic growth is seen to be a key factor in attracting rural population to migrate

Table 1: Variation of non-dimensional parameters in the simulation study.

Parameters	Values			
d	$d_- = 2$	$d_0 = 6$	$d_+ = 10$	$d_{++} = 100$
F	$F_{min} = 0$		$F_{max} = 0.25$	
k	$k_{min} = 0$		$k_{max} = 0.65$	
γ	$\gamma_- = \begin{cases} (588.31 + \frac{1961.7}{d})L^2 & \text{for } 2 \leq d \leq 10 \\ 280.82L^2 & \text{for } d = 100 \end{cases}$		$\gamma_+ = \begin{cases} (58831 + \frac{196170}{d})L^2 & \text{for } 2 \leq d \leq 10 \\ 28082L^2 & \text{for } d = 100 \end{cases}$	

towards those mega cities. Economic growth also has the potential to change city layouts when e.g. residential areas are replacing industrial zones due to the influx of inhabitants. Therefore, the US sample set was chosen based on economic growth, i.e. the GDP growth between the census years 2000 and 2010 of all Metropolitan Statistical Areas (MSA). The cities with the highest GDP growth rates were:

1. Midland/Odessa, TX
2. Victoria, TX
3. Bismarck, ND.

For these cities/metropolitan areas all available shapefiles and available population data was gathered from the respective sources at the highest available resolution. As simulation software a combination of FlexPDE for the calculation of the reaction-diffusion-equations and MATLAB for the preparation and evaluation of FlexPDE results was used. For the simulation various pre-assessments had to be performed to define the variation of the non-dimensional parameters. L is defined to be the square root of the city area in m^2 and the other parameters were defined like shown in Table 1.

While d and γ are simulated as stated in Table 1, F and d are simulated with a higher resolution. In total 11 F steps of 0.025 between $F_{\min}$ and $F_{\max}$ and 14 k steps of 0.005 between $k_{\min}$ and $k_{\max}$ are simulated. This results in $4_d * 11_F * 14_k * 2_\gamma = 1232$ simulations per city.

The values u and v are calculated by determining the number of inhabitants N per block. $u := N_u/N$ at a given location then corresponds to the percentage of all non-white inhabitants N_u relative to the total number of inhabitants N in that block. Accordingly, v is then defined as $v := N_v/N$, where N_v is the number of white residents.

Defining the nondimensional simulation time $t*_{\max}$ is crucial to the simulation experiment as it needs to be equivalent to the 10 years of change that the census data represent. As it is – per definition – not independent of γ, two nondimensional simulation times – one per γ simulated – were defined based on pre-simulations to: $t^*_{\max,\mathrm{Sim}\gamma-} = 5*10^{-6}$ and $t^*_{\max,\mathrm{Sim}\gamma+} = 5*10^{-8}$. Furthermore a quality measure

$$QF = \sum_{i=1}^{B_{\max}} |AC_{2010,i} - AC_{Sim,i}| \tag{10}$$

has to be defined to evaluate the results of the simulation experiments, with $AC_{2010,i}$ the activator concentration at point i in 2010. $B_{\max}$ is the number of discrete simulation points used for the respective simulated city. The smaller the QF value is, the better is the result of the simulation. In the experiment Midland/Odessa, TX and Bismarck, ND shall be simulated to find the best factor combination and identify main and interdependency effects of the individual factors. Then, to test a linear dependency, Victoria, TX, the city with the 2nd fastest GDP growth is tested with the interpolated factor combination also.

2.3 Assumptions

For the simulations, some further general, simplifying assumptions are made in order to obtain a simplified city model, which can then provide insights into the suitability of the reaction-diffusion equations for predictions of city morphology by means of the simulation. First, it should be assumed that the cities to be simulated represent homogeneous levels, i.e. the entire urban area can in principle be developed within its boundaries. So undevelopable geographies (e.g. rivers or steep mountains) are neglected as well as deliberately undeveloped areas such as roads, parking lots, parks and other means of transport (e.g. trams, etc.). Besides that, only "concentrations" of people are considered. This means that ultimately no concrete statements can be made about the total number of people living in a block, but only the proportion of a certain ethnic group (or, if a lower resolution was chosen, the proportion of rich/poor population). However, since the core of the model is an interplay between two social groups, this should be regarded as a reasonable simplification.

3 RESULTS

In this section we present the results of our study. First, we present the two cities we trained on (Midland/Odessa, TX and Bismarck, ND). We show the effects, the different parameters have on the quality function and interpret the outcoming simulations qualitivly. Then we present the result for the city (Victoria, TX) we gained using the best parameters from the two cities simulated before.

3.1 Midland

The minimal QF value is 105,881 at a combination of $(\gamma, d, F, k) = (-, 10, 0.075, 0.005)$. The maximal QF value is 188,349 is the result of (-, 2, 0.25, 0.065). Furthermore, the main and interdependency effects (derived from a regression model) on the QF value are shown in Table 2. Here γ,d ,F and k are normed, so that $\gamma^*, d^*, F^*, k^* \in [-1, 1]$. The regression model based on Table 2, can be used to estimate the effects of the parameters γ, d, F and k on the QF value.

If minimized within the set boundaries, the regression model leads to an optimized factor combination of $(\gamma^*, d^*, F^*, k^*) = (-, +, -, -)$, which means that small γ, F and k and large d values seem to lead to better results for Midland/Odessa, TX. The graphical results of the simulation based on the factor combinations is shown in Figure 3.

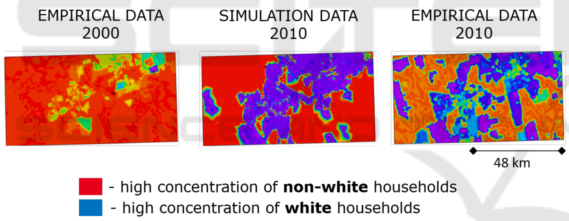

Figure 3: Representation of the empirical and simulation data for Midland/Odessa. The simulation was conducted with $(\gamma, d, F, k) = (-, 10, 0.007, 0.005)$.

Looking at Midland/Odessa's graphical results, it can be seen that there have been significant demographic changes between 2000 and 2010. In the west and southeast of the simulation area completely new structures were created. These fundamental structural changes could be mapped by the simulation model, even if larger deviations can still be detected.

3.2 Bismarck

The minimal QF value is 154,584 at a combination of $(\gamma, d, F, k) = (+, 100, 0, 0)$. The maximal QF value is 287,979 is the result of (-, 2, 0, 0.05). Furthermore, the main and interdependency effects (derived from a regression model) on the QF value are as presented in Table 2. Once again γ, d, F and k are normed, so that $\gamma^*, d^*, F^*, k^* \in [-1, 1]$. The regression model based on Table 2, can be used to estimate the effects of the parameters γ, d, F and k on the QF value.

If minimized within the set boundaries, the regression model leads to an optimized factor combination of $(\gamma^*, d^*, F^*, k^*) = (+, +, -, -)$, which means that small F and k and large γ and d values seem to lead to better results for Bismarck, ND.

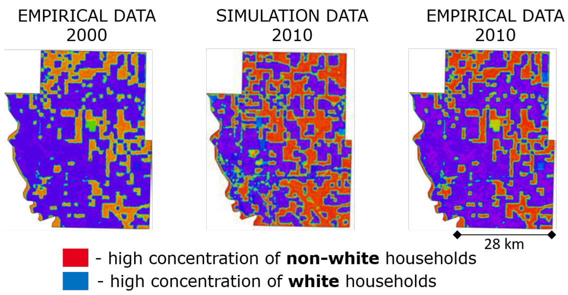

Figure 4: Representation of the empirical and simulation data for Bismarck. The simulation was conducted with $(\gamma, d, F, k) = (+, 100, 0, 0)$.

The so far presented results lead to the various effects on the QF factor for the two simulated cities presented in Table 2.

If the results just described are summarized, the best combinations found can be entered in the stability map (Figure 1). This Figure 5 shows that the best results are obtained with small values of k and F, as well as large values for d. The only difference is γ.

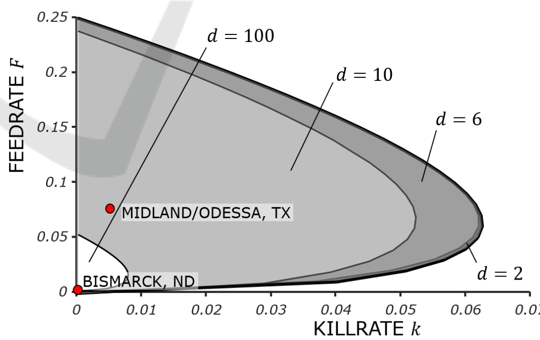

Figure 5: Stability Map of the Gray/Scott-model with the best factor combination for both cities investigated (Midland/Odessa, TX and Bismarck, ND).

3.3 Victoria

As described above, to test the linear dependency of the factor effects, Victoria, TX, which had the 2nd strongest GDP growth of all US cities, shall be tested with an altered simulation scenario. On the one hand, the "regular" test plan described in the methodology

Table 2: Effects.

	Midland/Odessa					
Main effect	γ^*	d^*	F^*	k^*		
	-8726	-6680	21317	13909		
Interdependency effect 1st O.	$\gamma^* d^*$	$\gamma^* F^*$	$\gamma^* k^*$	$D^* F^*$	$D^* k^*$	$F^* k^*$
	5853	-6729	-7823	-3600	-4554	732
Inter-dependency effect 2nd & 3rd O.	$\gamma^* d^* F^*$	$\gamma^* d^* k^*$	$\gamma^* F^* k^*$	$d^* F^* k^*$	$\gamma^* d^* F^* k^*$	
	-2318	455	1336	392	343	
	Bismarck					
Main effect	γ^*	d^*	F^*	k^*		
	-35605	-24505	8684	19681		
Interdependency effect 1st O.	$\gamma^* d^*$	$\gamma^* F^*$	$\gamma^* k^*$	$D^* F^*$	$D^* k^*$	$F^* k^*$
	8377	5952	-6703	2056	-4127	3903
Inter-dependency effect 2nd & 3rd O.	$\gamma^* d^* F^*$	$\gamma^* d^* k^*$	$\gamma^* F^* k^*$	$d^* F^* k^*$	$\gamma^* d^* F^* k^*$	
	-2362	-1393	-4388	-2107	1983	

Table 3: Overview to factor combinations, that promise low QF values for Midland/Odessa, TX and Bismarck, ND. The "best factor combination" is the one that delivered the lowest QF for the respective city. The factor combinations in the last column are based on a GRG (generalized reduced gradient method) minimization of the respective regression model when respecting all constraints.

	Best factor combination				Main Effects				with interdependency effects			
	γ^*	d^*	F^*	k^*	γ^*	d^*	F^*	k^*	γ^*	d^*	F^*	k^*
Midland/ Odessa	-	+	-	-	+	+	-	-	-	+	-	-
Bismarck	+	+	-	-	+	+	0	-	+	+	-	-

and used for Midland/Odessa, TX and Bismarck, ND, was simplified to test less combinations of F and k. These were now only varied in three steps each. On the other hand, an additional factor combination was tested, that lies – with regards to the factor combination – in the middle of the optimized solutions found for Midland/Odessa, TX and Bismarck, ND, which are displayed in the table above.

As the results for the two cities are consistent for d, F and k, but not for $\gamma, \gamma_0 = 10\gamma_-$ was defined as the middle between γ_- and γ_+. Consequently, if the factor effects were linear dependent across the tested cities, one would expect the best factor combination for Victoria, TX to be at $(\gamma^*, d^*, F^*, k^*) = (0, +, -, -)$. The results for the Victoria, TX simulation can be seen in Figure 6.

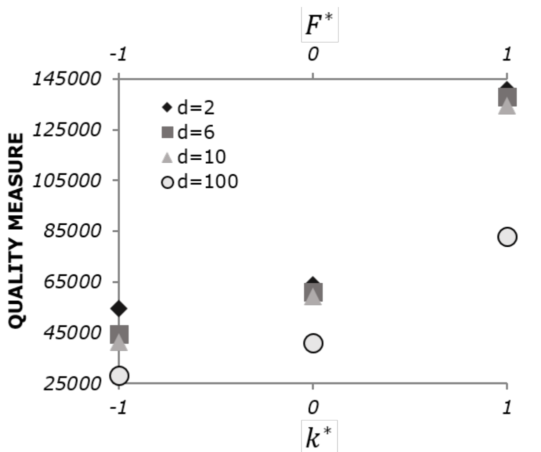

Figure 6: Simulation for Victoria, TX with $\gamma_0 = 10 * \gamma_-$ for variations of the factor d. F and k are changed equally from small $(-)$ to medium (0) to large $(+)$. $(-)$ is $F = k = 0$, (0) is $F = 0.125$ and $k = 0.03$ and $(+)$ is $F = 0.25$ and $k = 0.065$.

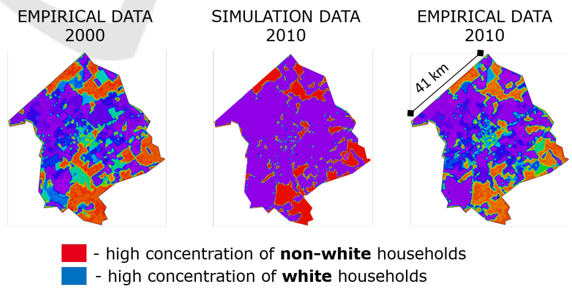

Figure 7: Representation of the empirical and simulation data for Victoria. The simulation was conducted with $(\gamma, d, F, k) = (10\gamma_-, 100, 0, 0)$.

The best QF value for Victoria, TX was found with the factor combination that was indicated by the linearity test, i.e. with small values for F and k, a large value for d and the middle value of γ, γ_0.

It is interesting to note that both the formation of a kind of coridor, in the east of the area under consideration, and the splitting off of a small "island" with a high concentration of non-white population at the northern edge of the area was represented by the model in the simulation and can also be seen in the empirical data (Figure 7).

4 DISCUSSION

An analysis of the above results shows that small values for F and k lead to the best quality values.

For F, a slight tendency towards small values can be observed, which in terms of the reaction-diffusion equations means that the inhibitor or substrate U - i.e. in the sociological analogy the non-white population - is "filled up" at a low rate. This means that few non-white population "react" into the system, which can be interpreted in such a way that there is little influx of non-white population groups from outside the system boundaries under consideration. The factor k stands for the difference between the filling rate for the inhibitor/substrate and the reduction of the activator. If k is zero, the activator in the system decreases at the same rate as the inhibitor/substrate is replenished. The factor d, which represents the ratio of the diffusion coefficient of the inhibitor or substrate U to that of the activator V, gives a similar picture to the factor γ. For most of the cities investigated, it seems advantageous to use a large value for d for the simulation of urban morphology development. This means that the diffusion coefficient of the inhibitor or substrate - i.e. in the sociological analogy used here the diffusion coefficient of the non-white population - is significantly higher than the coefficient of the white population. This means that in relation to the real cities, the non-white population must have a higher migration rate in order to better reflect real developments. This means that non-white households, i.e. households that tend to be poorer, have to move more often. In principle, this is very much in line with the sociological phenomena of segregation and gentrification, since poorer households are displaced from their previous residential locations by gentrification and segregation and therefore have to move more frequently.

This interpretation is to be understood qualitatively and should be investigated in further studies with extended data material.

The simulations carried out depict a total of 10 years of urban development, as the best data availability in the USA is available for the census years 2000 and 2010. The population data from the year 2000 serve as starting values for the simulations. The boundary conditions correspond to the usual zero flux conditions for the Gray-Scott model, i.e. there is no flow across system boundaries. In later investigations it is therefore necessary to check more precisely whether the selected boundary conditions have a significant influence on the quality of the results and to what extent a different simulation time can have a positive or negative influence on the results. The evaluation of the result quality of a simulation was carried out using a so-called quality function, which forms the absolute difference between the reference solutions from the census data of 2010 and the simulation result at each simulation grid point. These simulation grid points depend on the individual geometry of the city under investigation, but always have a distance of 0.001. The results of the simulation study suggest that the investigated factor combinations are only applicable to a very limited extent for the prediction of urban structures. Thus, the initial values in for out of six cities are a better prediction for the reference solution than the result of the simulation with the best factor combination in each case. On the other hand, in two cases, for Midland/Odessa, TX and Victoria, TX, the simulation result of the best factor combination is a much better estimate of the reference solution than the corresponding initial values. It is noticeable that the changes from 2000 to 2010 in Midland/Odessa, TX are particularly large compared to the other cities investigated. This may suggest that the reaction-diffusion equations are better suited to approximate large changes than to depict the smallest change in detail. This in turn would mean that a lower data resolution than that available in the USA would suffice. A more detailed mapping of city structures may be possible with the reaction-diffusion equations, but then a local optimization of the factors is very likely necessary. At present, however, it remains questionable whether the Gray/Scott equations can be used to map the patterns.

Other quality values should also be examined in further investigations, since the quality measure presented here is only an integral quantity and does not say anything about which areas in the area under consideration the deviations between simulation and empiricism are large and which they are small.

Finally, it remains to be seen that good factor combinations can actually be derived to a limited extent from other, similar cities. This was investigated for two cities. To what extent future developments can be estimated by a historical determination of the best factor combination has to be examined in further investigations.

5 CONCLUSIONS

In this paper we have investigated a Gray/Scott reaction-diffusion equation model to simulate the development of economically strong cities in the USA. We transferred the model known from chemistry and biology to the development of urban systems. We show that reaction-diffusion equations can be interpreted sociologically and we describe fundamental structural changes in a third city by training the equations on two cities.

A more detailed mapping of urban structures may be possible with the reaction-diffusion equations, but then a local optimization of the factors will be necessary. At the moment, however, it remains questionable whether the patterns can be mapped with the Gray-Scott equations investigated here. Finally, it remains to be seen that good factor combinations can actually be derived to a limited extent from other, similar cities. This was investigated for two cities. To what extent future developments can be estimated by a historical determination of the best factor combination is to be examined in further investigations.

Finally, it should be noted that the approach presented here is not intended to show that simple mathematical equations can fully describe complex phenomena such as urbanisation. Rather, it addresses the question of whether (i) there are fundamental processes that dominate urbanization, while other processes may be neglected. On the other hand, (ii) it is examined whether these processes can be mapped with simple equations.

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REFERENCES

Alba, R. D. and Logan, J. R. (1993). Minority proximity to whites in suburbs: An individual-level analysis of segregation. *American journal of sociology*, 98(6):1388–1427.

Benenson, I. and Torrens, P. M. (2004). *Geosimulation: automata-based modelling of urban phenomena*. John Wiley & Sons, Hoboken, NJ.

England, P., Farkas, G., Kilbourne, B. S., and Dou, T. (1988). Explaining occupational sex segregation and wages: Findings from a model with fixed effects. *American Sociological Review*, pages 544–558.

Farrell, C. R. (2016). Immigrant suburbanisation and the shifting geographic structure of metropolitan segregation in the united states. *Urban Studies*, 53(1):57–76.

Frankhauser, P. (1998). Fractal geometry of urban patterns and their morphogenesis. *Discrete Dynamics in Nature and Society*, 2(2):127–145.

Friesen, J., Taubenböck, H., Wurm, M., and Pelz, P. F. (2018). The similar size of slums. *Habitat International*, 73:79–88.

González-Val, R., Ramos, A., Sanz-Gracia, F., and Vera-Cabello, M. (2015). Size distributions for all cities: Which one is best? *Papers in Regional Science*, 94(1):177–196.

Gray, P. and Scott, S. (1983). Autocatalytic reactions in the isothermal, continuous stirred tank reactor: isolas and other forms of multistability. *Chemical Engineering Science*, 38(1):29–43.

Gray, P. and Scott, S. (1984). Autocatalytic reactions in the isothermal, continuous stirred tank reactor: Oscillations and instabilities in the system a+ 2b→ 3b; b→ c. *Chemical Engineering Science*, 39(6):1087–1097.

Hwang, J. and Sampson, R. J. (2014). Divergent pathways of gentrification: Racial inequality and the social order of renewal in chicago neighborhoods. *American Sociological Review*, 79(4):726–751.

Janoschka, M., Sequera, J., and Salinas, L. (2014). Gentrification in s pain and l atin a merica—a critical dialogue. *International journal of urban and regional research*, 38(4):1234–1265.

Koch, A. (2011). Soziale segregation aus der modellierungs-und simulationsperspektive. *IÖR Schriften*, page 25.

Kraas, F., Leggewie, C., Lemke, P., Matthies, E., Messner, D., Nakicenovic, N., Schellnhuber, H. J., Schlacke, S., Schneidewind, U., and Brandi, C. (2016). *Humanity on the move: Unlocking the transformative power of cities*. WBGU-German Advisory Council on Global Change.

Krugman, P. R. (1996). *The self-organizing economy*, volume 122. Blackwell Oxford.

Lees, L. (2012). The geography of gentrification: Thinking through comparative urbanism. *Progress in Human Geography*, 36(2):155–171.

Pelz, P. F., Friesen, J., and Hartig, J. (2019). Similar size of slums caused by a turing-instability of migration behaviour. *Physical Review E*.

S. Fowler, C. (2016). Segregation as a multiscalar phenomenon and its implications for neighborhood-scale research: the case of south seattle 1990–2010. *Urban geography*, 37(1):1–25.

Schulz, A. J., Gravlee, C. C., Williams, D. R., Israel, B. A., Mentz, G., and Rowe, Z. (2006). Discrimination, symptoms of depression, and self-rated health among african american women in detroit: results from a longitudinal analysis. *American journal of public health*, 96(7):1265–1270.

Schweitzer, F. and Steinbink, J. (2002). Analysis and Computer Simulation of Urban Cluster Distributions. In *Fundamental Principles of Urban Growth*, pages 142–157. Müller + Busmann, Wuppertal.

Shapiro, T., Meschede, T., and Osoro, S. (2013). The roots of the widening racial wealth gap: Explaining the black-white economic divide. *Research and policy brief*.

Smith, N. (2002). New globalism, new urbanism: gentrification as global urban strategy. *Antipode*, 34(3):427–450.

Smith, N. (2005). *The new urban frontier: Gentrification and the revanchist city*. Routledge.

Tessmann, R. and Friesen, J. (2017). Study of Reaction-Diffusion Equations as structural forming mechanisms in urban morphologies in the USA. Research Project, TU Darmstadt.

Timberlake, J. M. and Johns-Wolfe, E. (2017). Neighborhood ethnoracial composition and gentrification in chicago and new york, 1980 to 2010. *Urban Affairs Review*, 53(2):236–272.

Turing, A. M. (1952). The Chemical Basis of Morphogenesis. *Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences*, 237(641):37–72.

United Nations (2016). *Urbanization and development: emerging futures. World cities report.* United Nations Publicatio, New York. OCLC: 953977423.