



Elliptic methods for solving the linearized field equations of causal variational principles

Felix Finster¹ · Magdalena Lottner¹

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Abstract

The existence theory is developed for solutions of the inhomogeneous linearized field equations for causal variational principles. These equations are formulated weakly with an integral operator which is shown to be bounded and symmetric on a Hilbert space endowed with a suitably adapted weighted L^2 -scalar product. Guided by the procedure in the theory of linear elliptic partial differential equations, we use the spectral calculus to define Sobolev-type Hilbert spaces and invert the linearized field operator as an operator between such function spaces. The uniqueness of the resulting weak solutions is analyzed. Our constructions are illustrated in simple explicit examples. The connection to the causal action principle for static causal fermion systems is explained.

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✉ Felix Finster
finster@ur.de

Magdalena Lottner
magdalena.lottner@ur.de

¹ Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

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1 Introduction

Causal variational principles were introduced in [4] as a mathematical generalization of the causal action principle, being the analytical core of the physical theory of causal fermion systems (for the general context see the reviews [6, 9, 12], the textbooks [5, 18] or the website [22]). In general terms, given a manifold \mathcal{G} together with a non-negative function $\mathcal{L} : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{R}_0^+$, in a *causal variational principle* one minimizes the action \mathcal{S} given by

$$\mathcal{S}(\mu) = \int_{\mathcal{G}} d\mu(\mathbf{x}) \int_{\mathcal{G}} d\mu(\mathbf{y}) \mathcal{L}(\mathbf{x}, \mathbf{y})$$

under variations of the measure μ on \mathcal{G} , keeping the total volume $\mu(\mathcal{G})$ fixed (for the precise mathematical setup see Sect. 2.1 below). The support of the measure μ denoted by

$$N := \text{supp } \mu \subset \mathcal{G}$$

has the interpretation as the underlying *space* or *spacetime*. A minimizing measure satisfies corresponding *Euler–Lagrange (EL) equations* (for details see the preliminaries in Sect. 2.1 or [18, Chapter 7]). For the detailed analysis of minimizing measures, it is very useful to consider first variations of the measure μ which preserve the EL equations. Such a variation is described by a so-called *jet* $\mathfrak{v} = (b, v)$, which consists of a scalar function b and a vector field v [for details see (2.8) in Sect. 2.2 below] satisfying the *homogeneous linearized field equations*

$$\Delta \mathfrak{v} = 0,$$

where the operator Δ is defined by

$$\Delta \mathfrak{v}(\mathbf{x}) = \nabla \left(\int_N (\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}) \mathcal{L}(\mathbf{x}, \mathbf{y}) d\mu(\mathbf{y}) - \nabla_{\mathfrak{v}} \mathfrak{s} \right).$$

Here \mathfrak{s} is a positive parameter, and the jet derivative ∇ is a combination of multiplication and directional derivative (for details see again Sect. 2.2). For the mathematical analysis of the linearized field equations, it is preferable to include an inhomogeneity \mathfrak{w} ,

$$\Delta \mathfrak{v} = \mathfrak{w}. \tag{1.1}$$

In view of the formal similarity to the Poisson equation, we sometimes refer to the operator Δ as the *Laplacian*. Despite this analogy, one should keep in mind that the *inhomogeneous linearized field equations are not differential equations*, but instead they are nonlocal equations involving integrals of specific integral kernels. Nevertheless, it turns out that methods of partial differential equations (PDEs) can be used for their analysis. This has first been explored in [3], where methods of hyperbolic PDEs were used to prove under general assumptions that the Cauchy problem is well-posed. In the present paper, we explore how methods of the theory of *elliptic* PDEs are applicable to the analysis of the linearized field equations. The main application in mind are static linearized fields in static causal fermion systems. In analogy to the scalar wave equation, which for time-independent fields goes over to the Poisson equation, one might expect that the linearized field equations of the underlying causal variational principle might change from hyperbolic to elliptic type. It is the goal of this paper to show that this naive expectation is indeed correct. Moreover, we work out abstractly how elliptic estimates can be used for the analysis of the linearized field equations in the static setting. To this end, we realize the Laplacian as a bounded symmetric operator on a weighted L^2 -space. Using the spectral calculus, we define Sobolev-type Hilbert spaces and prove that the Laplacian, considered as an operator between such function spaces, has a well-defined inverse. Our abstract constructions are illustrated in various examples.

The paper is organized as follows. Section 2 provides the necessary preliminaries on causal variational principles and the linearized field equations. In Sect. 3, the bilinear form obtained by weak evaluation of the linearized field equations is written as $\langle\langle u, \Delta_N v \rangle\rangle$, where $\langle\langle \cdot, \cdot \rangle\rangle$ is a weighted L^2 -scalar product (with a suitable “adapted” weight function h constructed out of the Lagrangian), and Δ_N is a bounded symmetric operator on the resulting Hilbert space. In Sect. 4, we proceed by solving the inhomogeneous linearized field equations. In Sect. 5 we illustrate our constructions in simple explicit examples. In Sect. 6 we explain the connection to static linearized fields in static causal fermion systems. Finally, in “Appendix” the role of the scalar components of the jets is clarified.

2 Preliminaries

We now recall the basics on causal variational principles in the setting needed here. More details can be found in [13, 18]. We use a slightly different notation in order to get consistency with the causal variational principle in the static case to be introduced in Sect. 6.

2.1 Causal variational principles in the non-compact setting

We consider causal variational principles in the non-compact setting as first introduced in [13, Section 2]. Thus we let \mathcal{G} be a (possibly non-compact) smooth manifold of dimension $m \geq 1$ and μ a (positive) Borel measure on \mathcal{G} . Moreover, we are given a non-negative function $\mathcal{L} : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{R}_0^+$ (the *Lagrangian*) with the following properties:

- (i) \mathcal{L} is symmetric: $\mathcal{L}(\mathbf{x}, \mathbf{y}) = \mathcal{L}(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{G}$.
- (ii) \mathcal{L} is lower semi-continuous, i.e. for all sequences $\mathbf{x}_n \rightarrow \mathbf{x}$ and $\mathbf{y}_{n'} \rightarrow \mathbf{y}$,

$$\mathcal{L}(\mathbf{x}, \mathbf{y}) \leq \liminf_{n, n' \rightarrow \infty} \mathcal{L}(\mathbf{x}_n, \mathbf{y}_{n'}).$$

The *causal variational principle* is to minimize the action

$$\mathcal{S}(\mu) = \int_{\mathcal{G}} d\mu(\mathbf{x}) \int_{\mathcal{G}} d\mu(\mathbf{y}) \mathcal{L}(\mathbf{x}, \mathbf{y}) \tag{2.1}$$

under variations of the measure μ , keeping the total volume $\mu(\mathcal{G})$ fixed (*volume constraint*).

If the total volume $\mu(\mathcal{G})$ is finite, one minimizes the causal action within the class of all regular Borel measures with fixed total volume. If the total volume $\mu(\mathcal{G})$ is infinite, however, it is not obvious how to implement the volume constraint, making it necessary to proceed as follows. We make the following additional assumptions:

- (iii) The measure μ is *locally finite* (meaning that any $\mathbf{x} \in \mathcal{G}$ has an open neighborhood U with $\mu(U) < \infty$) and *regular* (meaning that the measure of a set can be recovered by approximation from inside with compact and from outside with open sets).
- (iv) The function $\mathcal{L}(\mathbf{x}, \cdot)$ is μ -integrable for all $\mathbf{x} \in \mathcal{G}$, giving a lower semi-continuous and bounded function on \mathcal{G} .

Given a regular Borel measure μ on \mathcal{G} , we vary over all regular Borel measures $\tilde{\mu}$ with

$$|\tilde{\mu} - \mu|(\mathcal{G}) < \infty \quad \text{and} \quad (\tilde{\mu} - \mu)(\mathcal{G}) = 0 \tag{2.2}$$

(where $|\cdot|$ denotes the total variation of a measure). For such variations, the difference of the actions is well-defined by

$$\begin{aligned} \mathcal{S}(\tilde{\mu}) - \mathcal{S}(\mu) &= \int_{\mathcal{G}} d(\tilde{\mu} - \mu)(\mathbf{x}) \int_{\mathcal{G}} d\mu(\mathbf{y}) \mathcal{L}(\mathbf{x}, \mathbf{y}) + \int_{\mathcal{G}} d\mu(\mathbf{x}) \int_{\mathcal{G}} d(\tilde{\mu} - \mu)(\mathbf{y}) \mathcal{L}(\mathbf{x}, \mathbf{y}) \\ &+ \int_{\mathcal{G}} d(\tilde{\mu} - \mu)(\mathbf{x}) \int_{\mathcal{G}} d(\tilde{\mu} - \mu)(\mathbf{y}) \mathcal{L}(\mathbf{x}, \mathbf{y}). \end{aligned} \tag{2.3}$$

The measure μ is said to be a *minimizer under variations of finite volume* if this difference is non-negative for all regular Borel measures satisfying (2.2). The existence theory for such minimizers is developed in [14]. Moreover, it is shown in [13, Lemma 2.3] that a minimizer satisfies the *Euler–Lagrange (EL) equations* which state that for a suitable value of the parameter $\varepsilon > 0$, the lower semi-continuous function $\ell : \mathcal{G} \rightarrow \mathbb{R}_0^+$ defined by

$$\ell(\mathbf{x}) := \int_{\mathcal{G}} \mathcal{L}(\mathbf{x}, \mathbf{y}) d\mu(\mathbf{y}) - \varepsilon \tag{2.4}$$

is minimal and vanishes on the support of μ ,

$$\ell|_{\text{supp } \mu} \equiv \inf_{\mathcal{G}} \ell = 0. \tag{2.5}$$

For further details we refer to [13, Section 2].

2.2 The restricted Euler–Lagrange equations and jet spaces

We denote the support of μ by N ,

$$N := \text{supp } \mu \subset \mathcal{G}.$$

The EL equations (2.5) are nonlocal in the sense that they make a statement on ℓ even for points $\mathbf{x} \in \mathcal{G}$ which are far away from N . It turns out that for the applications in this paper, it is preferable to evaluate the EL equations only locally in a neighborhood of N . This leads to the *restricted EL equations* introduced in [13, Section 4]. We here give a slightly less general

version of these equations which is sufficient for our purposes. In order to explain how the restricted EL equations come about, we begin with the simplified situation that the function ℓ is smooth. In this case, the minimality of ℓ implies that the derivative of ℓ vanishes on N , i.e.

$$\ell|_N \equiv 0 \quad \text{and} \quad D\ell|_N \equiv 0 \tag{2.6}$$

(where $D\ell(p) : T_p\mathcal{G} \rightarrow \mathbb{R}$ is the derivative). In order to combine these two equations in a compact form, it is convenient to consider a pair $u := (a, u)$ consisting of a real-valued function a on N and a vector field u on $T\mathcal{G}$ along N , and to denote the combination of multiplication and directional derivative by

$$\nabla_u \ell(\mathbf{x}) := a(\mathbf{x}) \ell(\mathbf{x}) + (D_u \ell)(\mathbf{x}). \tag{2.7}$$

Then the Eq. (2.6) imply that $\nabla_u \ell(\mathbf{x})$ vanishes for all $\mathbf{x} \in N$. The pair $u = (a, u)$ is referred to as a *jet*.

In the general lower-continuous setting, one must be careful because the directional derivative $D_u \ell$ in (2.7) need not exist. Our method for dealing with this issue is to restrict attention to vector fields for which the directional derivative is well-defined. Moreover, we must specify the regularity assumptions on a and u . To begin with, we always assume that a and u are *smooth* in the sense that they have a smooth extension to the manifold \mathcal{G} . Thus the jet u should be an element of the jet space

$$\mathfrak{J} := \{u = (a, u) \text{ with } a \in C^\infty(N, \mathbb{R}) \text{ and } u \in \Gamma(N, T\mathcal{G})\}, \tag{2.8}$$

where $C^\infty(N, \mathbb{R})$ and $\Gamma(N, T\mathcal{G})$ denote the space of real-valued functions and vector fields on N , respectively, which admit a smooth extension to \mathcal{G} . We always denote spaces of compactly supported jets with a subscript 0; for example,

$$\mathfrak{J}_0 := \{u \in \mathfrak{J} \mid \text{supp } u \text{ compact}\}. \tag{2.9}$$

Clearly, the fact that a jet u is smooth does not imply that the functions ℓ or \mathcal{L} are differentiable in the direction of u . This must be ensured by additional conditions which are satisfied by suitable subspaces of \mathfrak{J} which we now introduce. First, we let Γ^{diff} be those vector fields for which the directional derivative of the function ℓ exists,

$$\Gamma^{\text{diff}} = \{u \in C^\infty(N, T\mathcal{G}) \mid D_u \ell(\mathbf{x}) \text{ exists for all } \mathbf{x} \in N\}.$$

This gives rise to the jet space

$$\mathfrak{J}^{\text{diff}} := C^\infty(N, \mathbb{R}) \oplus \Gamma^{\text{diff}} \subset \mathfrak{J}. \tag{2.10}$$

For the jets in $\mathfrak{J}^{\text{diff}}$, the combination of multiplication and directional derivative in (2.7) is well-defined. We choose a linear subspace $\mathfrak{J}^{\text{test}} \subset \mathfrak{J}^{\text{diff}}$ with the properties that its scalar and vector components are both vector spaces,

$$\mathfrak{J}^{\text{test}} = C^{\text{test}}(N, \mathbb{R}) \oplus \Gamma^{\text{test}} \subseteq \mathfrak{J}^{\text{diff}},$$

and that the scalar component is nowhere trivial in the sense that¹

$$\text{for all } \mathbf{x} \in N \text{ there is } a \in C^{\text{test}}(N, \mathbb{R}) \text{ with } a(\mathbf{x}) \neq 0. \tag{2.11}$$

Then the *restricted EL equations* read (for details cf. [13, (eq. (4.10))])

$$\nabla_u \ell|_N = 0 \quad \text{for all } u \in \mathfrak{J}^{\text{test}}. \tag{2.12}$$

¹ This assumption is convenient, because then the restricted EL equations (2.12) imply that ℓ vanishes identically on M .

We remark that, in the literature, the restricted EL equations are sometimes also referred to as the *weak* EL equations. Here we prefer the notion “restricted” in order to avoid confusion with weak solutions of these equations as constructed in [3] (in the static setting, such weak solutions will be constructed in Sect. 4). The purpose of introducing $\mathfrak{J}^{\text{test}}$ is that it gives the freedom to restrict attention to the part of information in the EL equations which is relevant for the application in mind.

We conclude this section by introducing spaces of jets with suitable differentiability properties. Before beginning, we point out that, here and throughout this paper, we use the following conventions for partial derivatives and jet derivatives:

- Partial and jet derivatives with an index $i \in \{1, 2\}$, as for example in (2.14), only act on the respective variable of the function \mathcal{L} . This implies, for example, that the derivatives commute,

$$\nabla_{1,v} \nabla_{1,u} \mathcal{L}(\mathbf{x}, \mathbf{y}) = \nabla_{1,u} \nabla_{1,v} \mathcal{L}(\mathbf{x}, \mathbf{y}).$$

- The partial or jet derivatives which do not carry an index act as partial derivatives on the corresponding argument of the Lagrangian. This implies, for example, that

$$\nabla_u \int_{\mathfrak{G}} \nabla_{1,v} \mathcal{L}(\mathbf{x}, \mathbf{y}) \, d\mu(\mathbf{y}) = \int_{\mathfrak{G}} \nabla_{1,u} \nabla_{1,v} \mathcal{L}(\mathbf{x}, \mathbf{y}) \, d\mu(\mathbf{y}). \tag{2.13}$$

Thus *jets are never differentiated*. We now introduce the spaces \mathfrak{J}^ℓ , where $\ell \in \mathbb{N}_0 \cup \{\infty\}$ can be thought of as the order of differentiability if the derivatives act simultaneously on both arguments of the Lagrangian:

Definition 2.1 For any $\ell \in \mathbb{N}_0 \cup \{\infty\}$, the jet space $\mathfrak{J}^\ell \subset \mathfrak{J}$ is defined as the vector space of test jets with the following properties:

- (i) For all $\mathbf{y} \in N$ and all \mathbf{x} in an open neighborhood of N , directional derivatives

$$(\nabla_{1,v_1} + \nabla_{2,v_1}) \cdots (\nabla_{1,v_p} + \nabla_{2,v_p}) \mathcal{L}(\mathbf{x}, \mathbf{y}) \tag{2.14}$$

(computed componentwise in charts around \mathbf{x} and \mathbf{y}) exist for all $p \in \{1, \dots, \ell\}$ and all $v_1, \dots, v_p \in \mathfrak{J}^\ell$.

- (ii) The functions in (2.14) are μ -integrable in the variable \mathbf{y} , giving rise to locally bounded functions in \mathbf{x} . More precisely, these functions are in the space

$$L^\infty_{\text{loc}}\left(N, L^1(N, d\mu(\mathbf{y})); d\mu(\mathbf{x})\right).$$

- (iii) Integrating the expression (2.14) in \mathbf{y} over N with respect to the measure μ , the resulting function (defined for all \mathbf{x} in an open neighborhood of N) is continuously differentiable in the direction of every jet $\mathbf{u} \in \mathfrak{J}^{\text{test}}$.

2.3 The linearized field equations

The EL equations (2.5) [and similarly the restricted EL equations (2.6)] are nonlinear equations because they involve the measure ρ in a twofold way: first, the measure comes up as the integration measure in (2.4), and second the function ℓ is evaluated on the support of this measure. Following the common procedure in mathematics and physics, one can simplify the problem by considering linear perturbations about a given solution. As an example in classical field theory, considering a family $(g_\tau)_{\tau \in [0,1]}$ of Lorentzian metrics which all satisfy the vacuum Einstein equations, the linearization $\partial_\tau g_\tau$ describes gravitational waves

propagating in the spacetime with metric g_τ . The analogous notion in the setting of causal fermion systems is a linearization of a family of measures $(\tilde{\mu}_\tau)_{\tau \in [0,1]}$ which all satisfy the restricted EL equations (2.12) (for fixed values of the parameters κ and \mathfrak{s}). It turns out to be fruitful to construct this family of measures by multiplying a given minimizing measure μ by a weight function f_τ and then “transporting” the resulting measure with a mapping F_τ . More precisely, we consider the ansatz

$$\tilde{\mu}_\tau = (F_\tau)_*(f_\tau \mu), \tag{2.15}$$

where $f_\tau \in C^\infty(N, \mathbb{R}^+)$ and $F_\tau \in C^\infty(N, \mathfrak{G})$ are smooth mappings, and $(F_\tau)_*\mu$ denotes the push-forward (defined for a subset $\Omega \subset \mathfrak{G}$ by $((F_\tau)_*\mu)(\Omega) = \mu(F_\tau^{-1}(\Omega))$); see for example [2, Section 3.6]).

The property of the family of measures $(\tilde{\mu}_\tau)_{\tau \in [0,1]}$ of the form (2.15) to satisfy the restricted EL equation for all τ means infinitesimally in τ that the jet \mathfrak{v} defined by

$$\mathfrak{v} = (b, v) := \frac{d}{d\tau}(f_\tau, F_\tau)|_{\tau=0} \tag{2.16}$$

satisfies the *linearized field equations*. We now recall the main step of the construction. Using the definition of the push-forward measure, we can write the restricted EL equations (2.12) for the measure $\tilde{\mu}_\tau$ as

$$\nabla_u \left(\int_N \mathcal{L}(F_\tau(\mathbf{x}), F_\tau(\mathbf{y})) f_\tau(\mathbf{y}) d\mu - \mathfrak{s} \right) = 0.$$

Since the function ℓ vanishes on the support, we may multiply by $f_\tau(\mathbf{x})$ to obtain

$$\nabla_u \left(\int_N f_\tau(\mathbf{x}) \mathcal{L}(F_\tau(\mathbf{x}), F_\tau(\mathbf{y})) f_\tau(\mathbf{y}) d\mu - f_\tau(\mathbf{x}) \mathfrak{s} \right) = 0. \tag{2.17}$$

At this point, the technical complication arise that one must specify the τ -dependence of the jet spaces, and moreover the last transformation makes it necessary to transform the jet spaces. Here we do not enter the details but refer instead to the rigorous derivation in [8, Section 3.3] or to the simplified presentation in the smooth setting in the textbook [18, Chapter 6]. Differentiating (2.17) with respect to τ gives the homogeneous linearized field equations

$$\langle u, \Delta v \rangle_N = 0 \quad \text{for all } u \in \mathfrak{J}^{\text{test}}, \tag{2.18}$$

where

$$\langle u, \Delta v \rangle(\mathbf{x}) := \nabla_u \left(\int_N (\nabla_{1,v} + \nabla_{2,v}) \mathcal{L}(\mathbf{x}, \mathbf{y}) d\mu(\mathbf{y}) - \nabla_v \mathfrak{s} \right). \tag{2.19}$$

We denote the vector space of all solutions of the linearized field equations by $\mathfrak{J}^{\text{lin}} \subset \mathfrak{J}^1$.

2.4 Positive functionals arising from second variations

Another ingredient to our constructions are positive functionals which arise in the analysis of second variations [7]. We now recall a few concepts and results.

Clearly, if μ is a *minimizing* measure, then second variations are non-negative. For our purposes, it again suffices to consider variations of the form (2.15), where for simplicity we assume that f_τ and F_τ are trivial outside a compact set. Under these assumptions, it is proven in [7, Theorem 1.1] that the positivity of second variations gives rise to the inequality

$$\int_N d\mu(\mathbf{x}) \int_N d\mu(\mathbf{y}) \nabla_{1,v} \nabla_{2,v} \mathcal{L}(\mathbf{x}, \mathbf{y}) + \int_N \nabla^2 \ell|_{\mathbf{x}}(v, v) d\mu(\mathbf{x}) \geq 0,$$

where $\text{jet } v \in \mathfrak{J}_0$ is again an infinitesimal generator of a variation of the form (2.16), which, however, does not need to respect the EL equations [and \mathfrak{J}_0 are the compactly supported jets (2.9)]. For our purposes, it is preferable to write this inequality as

$$\frac{1}{2} \int_N d\mu(\mathbf{x}) \int_N d\mu(\mathbf{y}) (\nabla_{1,v} + \nabla_{2,v})^2 \mathcal{L}(\mathbf{x}, \mathbf{y}) - \int_N (\nabla^2 \mathfrak{g})(v, v) d\mu \geq 0. \tag{2.20}$$

Then it is obvious that the integrals are well-defined if we assume that $u, v \in \mathfrak{J}^2$ (see Definition 2.1). Moreover, using (2.4) and (2.19), the inequality can be written in the compact form

$$\langle v, \Delta v \rangle_N \geq 0 \quad \text{for all } v \in \mathfrak{J}_0^2, \tag{2.21}$$

where we used the notation

$$\langle u, \Delta v \rangle_N := \int_N \langle u, \Delta v \rangle(\mathbf{x}) d\mu(\mathbf{x}). \tag{2.22}$$

In other words, the operator representing $\langle \cdot, \Delta \cdot \rangle_N$ is *positive semi-definite*. This positivity property is not an assumption, but it follows already from the structure of causal variational principles. It might come as a surprise, because the analogous inequality for the wave operator in Minkowski space is violated. Instead, this inequality holds (up to an irrelevant sign) for the Laplacian in the *Riemannian* setting. These facts are not a contradiction if one keeps in mind that the operator Δ has a structure which is very different from a differential operator. The basic reason why (2.21) holds is that, in the setting of causal variational principles, we consider minimizers. In contrast, the Dirichlet energy in the hyperbolic setting is unbounded from below, making it necessary to work instead with critical points. The fact that, in the theory of causal fermion systems, the dynamics in spacetime is described by minimizers (and not merely critical points) is a specific feature of the causal action principle. We refer the interested reader to the text books [5, 18].

3 The Laplacian as a bounded symmetric operator

3.1 The adapted weighted L^2 -scalar product

In preparation, we let $\Gamma_{\mathbf{x}}$ be the subspace of the tangent space spanned by the test jets,

$$\Gamma_{\mathbf{x}} := \{u(\mathbf{x}) \mid u \in \Gamma^{\text{test}}\} \subset T_{\mathbf{x}}\mathfrak{J}.$$

We introduce a Riemannian metric $g_{\mathbf{x}}$ on $\Gamma_{\mathbf{x}}$. This Riemannian metric also induces a pointwise scalar product on the jets. Namely, setting

$$\mathfrak{J}_{\mathbf{x}} := \mathbb{R} \oplus \Gamma_{\mathbf{x}},$$

we obtain the scalar product on $\mathfrak{J}_{\mathbf{x}}$

$$\langle \cdot, \cdot \rangle_{\mathbf{x}} : \mathfrak{J}_{\mathbf{x}} \times \mathfrak{J}_{\mathbf{x}} \rightarrow \mathbb{R}, \quad \langle v, \tilde{v} \rangle_{\mathbf{x}} := b(\mathbf{x}) \tilde{b}(\mathbf{x}) + g_{\mathbf{x}}(v(\mathbf{x}), \tilde{v}(\mathbf{x})), \tag{3.1}$$

were the scalar and vector components of v and \tilde{v} are denoted by $v = (v, b)$, $\tilde{v} = (\tilde{v}, \tilde{b})$. We denote the corresponding norm by $\|\cdot\|_{\mathbf{x}}$. We note for clarity that the choice of the Riemannian metric $g_{\mathbf{x}}$ involves a certain degree of freedom. This freedom can be used to our advantage in order to help to arrange the technical assumptions needed below (in particular that the norm $\|\bar{\nabla}^2 \mathcal{L}(x, y)\|$ in (3.4) be finite almost everywhere). We point out that the subsequent constructions may depend on the choice of $g_{\mathbf{x}}$.

In order to have the largest possible flexibility, we shall work with a subspace $\mathfrak{J}^{\text{vary}}$ of the compactly supported test jets,

$$\mathfrak{J}^{\text{vary}} \subset \mathfrak{J}^{\text{test}} \cap \mathfrak{J}_0^2, \tag{3.2}$$

which we can choose arbitrarily depending on the application in mind. In particular, the scalar component of $\mathfrak{J}^{\text{vary}}$ does not need to be nontrivial in the sense (2.11). Indeed, a case of particular interest is when the $\mathfrak{J}^{\text{vary}}$ has no scalar component, as will be explained in detail in ‘‘Appendix’’. We now consider the bilinear form [see (2.22)]

$$\langle \cdot, \Delta \cdot \rangle_N : \mathfrak{J}^{\text{vary}} \times \mathfrak{J}^{\text{vary}} \rightarrow \mathbb{R}. \tag{3.3}$$

This bilinear form is *positive semi-definite* according to (2.21). Using the abbreviation

$$\bar{\nabla}_{\mathfrak{v}} := (\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}),$$

we write (2.20) as

$$\langle \mathfrak{v}, \Delta \mathfrak{v} \rangle_N = \frac{1}{2} \int_N d\mu(\mathbf{x}) \int_N d\mu(\mathbf{y}) \bar{\nabla}_{\mathfrak{v}} \bar{\nabla}_{\mathfrak{v}} \mathcal{L}(\mathbf{x}, \mathbf{y}) - \int_N (\nabla^2 \mathfrak{s})(\mathfrak{v}, \mathfrak{v}) d\mu \geq 0$$

(and $(\nabla^2 \mathfrak{s})(\mathfrak{v}, \mathfrak{v}) = \nabla_{\mathfrak{v}} \nabla_{\mathfrak{v}} \mathfrak{s} = b(x)^2 \mathfrak{s}$). This shows in particular that the bilinear form $\langle \cdot, \Delta \cdot \rangle_N$ is *symmetric*. For all $\mathbf{x}, \mathbf{y} \in N$ we define

$$\|\bar{\nabla}^2 \mathcal{L}(\mathbf{x}, \mathbf{y})\| := \sup_{\mathbf{u}, \mathbf{v} \in \mathfrak{J}^{\text{vary}}} \frac{|\bar{\nabla}_{\mathbf{u}} \bar{\nabla}_{\mathbf{v}} \mathcal{L}(\mathbf{x}, \mathbf{y})|}{(\|\mathbf{u}(\mathbf{x})\|_{\mathbf{x}} + \|\mathbf{u}(\mathbf{y})\|_{\mathbf{y}})(\|\mathbf{v}(\mathbf{x})\|_{\mathbf{x}} + \|\mathbf{v}(\mathbf{y})\|_{\mathbf{y}})}, \tag{3.4}$$

where the supremum is taken only over those jets for which the denominator is non-zero. Furthermore, we assume that the Lagrangian is sufficiently regular such that this quantity is finite for almost all $\mathbf{x}, \mathbf{y} \in N$. Then for any $\mathfrak{v} = (v, b) \in \mathfrak{J}^{\text{vary}}$:

$$\begin{aligned} \|\bar{\nabla}_{\mathfrak{v}} \bar{\nabla}_{\mathfrak{v}} \mathcal{L}(\mathbf{x}, \mathbf{y})\| &\leq \|\bar{\nabla}^2 \mathcal{L}(\mathbf{x}, \mathbf{y})\| (\|\mathfrak{v}(\mathbf{x})\|_{\mathbf{x}} + \|\mathfrak{v}(\mathbf{y})\|_{\mathbf{y}})^2 \\ &\leq 2 \|\bar{\nabla}^2 \mathcal{L}(\mathbf{x}, \mathbf{y})\| (\|\mathfrak{v}(\mathbf{x})\|_{\mathbf{x}}^2 + \|\mathfrak{v}(\mathbf{y})\|_{\mathbf{y}}^2). \end{aligned}$$

We thus obtain the estimate

$$\begin{aligned} \langle \mathfrak{v}, \Delta \mathfrak{v} \rangle_N &\leq \int_N d\mu(\mathbf{x}) \int_N d\mu(\mathbf{y}) \|\bar{\nabla}^2 \mathcal{L}(\mathbf{x}, \mathbf{y})\| (\|\mathfrak{v}(\mathbf{x})\|_{\mathbf{x}}^2 + \|\mathfrak{v}(\mathbf{y})\|_{\mathbf{y}}^2) - \int_N b(\mathbf{x})^2 \mathfrak{s} d\mu(\mathbf{x}) \\ &= 2 \int_N d\mu(\mathbf{x}) \|\mathfrak{v}(\mathbf{x})\|_{\mathbf{x}}^2 \int_N d\mu(\mathbf{y}) \|\bar{\nabla}^2 \mathcal{L}(\mathbf{x}, \mathbf{y})\| - \int_N b(\mathbf{x})^2 \mathfrak{s} d\mu(\mathbf{x}). \end{aligned} \tag{3.5}$$

Definition 3.1 The *adapted weighted L^2 -scalar product* is defined by

$$\langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle := \int_N \langle \mathbf{u}(\mathbf{x}), \mathbf{v}(\mathbf{x}) \rangle_{\mathbf{x}} h(\mathbf{x}) d\mu(\mathbf{x}), \tag{3.6}$$

where h is the weight function

$$h(\mathbf{x}) := 1 + \int_N \|\bar{\nabla}^2 \mathcal{L}(\mathbf{x}, \mathbf{y})\| d\mu(\mathbf{y}). \tag{3.7}$$

The corresponding norm is denoted by $\|\cdot\|$.

We assume that this norm is finite for all jets in $\mathfrak{J}^{\text{vary}}$, thus defining a scalar product on $\mathfrak{J}^{\text{vary}}$,

$$\langle\langle \cdot, \cdot \rangle\rangle : \mathfrak{J}^{\text{vary}} \times \mathfrak{J}^{\text{vary}} \rightarrow \mathbb{R}. \tag{3.8}$$

This assumption is satisfied for example if we assume that the integral in (3.7) is a locally bounded function in \mathbf{x} (note that the jets in $\mathfrak{J}^{\text{vary}}$ are all compactly supported). If the function h is unbounded, one can still arrange in many situations that the scalar product (3.8) is finite by adapting $\mathfrak{J}^{\text{vary}}$ appropriately (for instance by working with compactly supported jets; see the example in Sect. 5.3). Taking the completion, we obtain the Hilbert space $(\mathfrak{h}, \langle \cdot | \cdot \rangle)$. By definition, the weighted norm is larger than the L^2 -norm, i.e.

$$\|u\|_{L^2(N, d\mu)} \leq \|u\| \quad \text{for all } u \in \mathfrak{J}^{\text{vary}},$$

giving rise to a natural embedding $\mathfrak{h} \hookrightarrow L^2(N, d\mu)$.

3.2 Spectral decomposition of the Laplacian

The estimate (3.5) shows that the bilinear form $\langle \cdot, \Delta \cdot \rangle_N$ is bounded with respect to the adapted weighted L^2 -scalar product introduced in Definition 3.1, i.e. there is a constant $C > 0$ such that

$$|\langle u, \Delta v \rangle_N| \stackrel{(*)}{\leq} \sqrt{\langle u, \Delta u \rangle_N} \sqrt{\langle v, \Delta v \rangle_N} \leq C \|u\| \|v\| \quad \text{for all } u, v \in \mathfrak{J}^{\text{vary}},$$

where in (*) we applied the Cauchy–Schwarz inequality to the positive semi-definite bilinear form $\langle \cdot, \Delta \cdot \rangle$ [see (2.21)]. The Fréchet–Riesz theorem yields a uniquely determined bounded and symmetric operator Δ_N with the property that

$$\langle u, \Delta v \rangle_N = \langle u, \Delta_N v \rangle \quad \text{for all } u, v \in \mathfrak{h}. \tag{3.9}$$

We let $\mathfrak{h}^{\mathbb{C}}$ be the complexification of the vector space \mathfrak{h} , i.e.

$$\mathfrak{h}^{\mathbb{C}} = \mathfrak{h} \oplus i\mathfrak{h}$$

with the obvious multiplication by complex numbers. Extending the bilinear form $\langle \cdot | \cdot \rangle$ to a sesquilinear form on $\mathfrak{h}^{\mathbb{C}}$ by

$$\langle\langle (u + iv) | (z + i\mathfrak{w}) \rangle\rangle := \langle u | z \rangle - i \langle v | z \rangle + i \langle u | \mathfrak{w} \rangle + \langle v | \mathfrak{w} \rangle,$$

we obtain the complex Hilbert space $(\mathfrak{h}^{\mathbb{C}}, \langle\langle \cdot | \cdot \rangle\rangle)$. The Laplacian Δ_N extends to a complex-linear bounded operator on $\mathfrak{h}^{\mathbb{C}}$ by

$$\Delta_N(u + iv) := \Delta_N u + i \Delta_N v.$$

The resulting operator is again symmetric and positive semi-definite, i.e.

$$\langle\langle u | \Delta_N v \rangle\rangle = \langle\langle \Delta_N u | v \rangle\rangle \quad \text{and} \quad \langle\langle u | \Delta_N u \rangle\rangle \geq 0 \quad \text{for all } u, v \in \mathfrak{h}^{\mathbb{C}}.$$

The spectral theorem for bounded symmetric operators on a complex Hilbert space yields the decomposition

$$\Delta_N = \int_{\sigma(\Delta_N)} \lambda \, dE_\lambda, \tag{3.10}$$

where dE_λ is a compactly supported projection-valued Borel measure on \mathbb{R}_0^+ (for details see for example [23, Section VII.3]). Let f be a real-valued (possibly unbounded) Borel function on $\sigma(\Delta_N) \subset \mathbb{R}_0^+$ which is finite almost everywhere. Using the spectral calculus for (possibly unbounded) Borel functions (for details see for example [23, Section VIII.3]), the operator

$$f(\Delta_N) = \int_{\sigma(\Delta_N)} f(\lambda) \, dE_\lambda \tag{3.11}$$

with domain

$$\mathcal{D}(f(\Delta_N)) := \left\{ u \in \mathfrak{h}^{\mathbb{C}} \mid \int_{\sigma(\Delta_N)} |f(\lambda)|^2 d\langle u | E_\lambda u \rangle < \infty \right\}$$

is selfadjoint. If $E_\lambda = 0$ for all points λ with $f(\lambda) \in \{\pm\infty\}$, then the domain is dense. In particular, this is the case if E has no point spectrum. We remark that, if the function f is bounded, then the operator $f(\Delta_N)$ is also bounded, and its domain is the whole Hilbert space.

Remark 3.2 We also have a spectral decomposition on the real Hilbert space \mathfrak{h} in the following sense. Starting from the spectral decomposition (3.10), we write the spectral measure in a block operator form for the real and imaginary parts,

$$dE_\lambda = \begin{pmatrix} dE_\lambda^{11} & dE_\lambda^{12} \\ dE_\lambda^{21} & dE_\lambda^{22} \end{pmatrix}. \tag{3.12}$$

Next, by construction of the complexification, the operator Δ_N has the block operator form

$$\Delta_N = \begin{pmatrix} \Delta_N^{\mathbb{R}} & 0 \\ 0 & \Delta_N^{\mathbb{R}} \end{pmatrix},$$

where for clarity $\Delta_N^{\mathbb{R}} : \mathfrak{h} \rightarrow \mathfrak{h}$ is the operator on the real Hilbert space. Using the functional calculus, we find that for every real-valued Borel function f , the operator $f(\Delta_N)$ is again block diagonal,

$$f(\Delta_N) = \begin{pmatrix} f(\Delta_N^{\mathbb{R}}) & 0 \\ 0 & f(\Delta_N^{\mathbb{R}}) \end{pmatrix}.$$

Comparing this equation with (3.12) and using that f is arbitrary, we conclude that

$$dE_\lambda^{11} = dE_\lambda^{22} \quad \text{and} \quad dE_\lambda^{12} = 0 = dE_\lambda^{21}.$$

Hence dE_λ^{11} is the desired spectral measure of the operator $\Delta_N^{\mathbb{R}}$.

4 Construction of weak solutions

4.1 The weak linearized field equations

Given a suitable jet \mathfrak{w} , we want to solve the inhomogeneous linearized field equations (1.1) in the weak formulation

$$\langle u, \Delta v \rangle_N = \langle u, \mathfrak{w} \rangle_N \quad \text{for all } u \in \mathfrak{J}^{\text{vary}}, \tag{4.1}$$

where we use the pointwise scalar product on the jets (3.1) in order to identify dual jets with jets. Using (3.9), we can rewrite the left side with the operator Δ_N ,

$$\langle\langle u, \Delta_N v \rangle\rangle = \langle u, \mathfrak{w} \rangle_N \quad \text{for all } u \in \mathfrak{J}^{\text{vary}}. \tag{4.2}$$

Using this formula in (4.1) has the disadvantage that we obtain different inner products on the left and on the right. This problem can be cured by absorbing one over the weight factor (3.7) into the inhomogeneity. Thus we write (4.2) equivalently as

$$\langle\langle u, \Delta_N v \rangle\rangle = \langle\langle u, \hat{\mathfrak{w}} \rangle\rangle \quad \text{for all } u \in \mathfrak{J}^{\text{vary}}, \tag{4.3}$$

where \hat{w} is the new inhomogeneity

$$\hat{w}(\mathbf{x}) := \frac{1}{h(\mathbf{x})} w(\mathbf{x}).$$

We refer to (4.3) as the *weak linearized field equations* formulated with the adapted weighted L^2 -scalar product.

4.2 Solutions in Sobolev-type Hilbert spaces

Formally, the weak linearized field equations (4.3) can be solved by setting $v = (\Delta_N)^{-1}u$. Our strategy for making mathematical sense of the inverse is to work in suitable Hilbert spaces constructed from the spectral calculus for the Laplacian (3.11). Since the Laplacian Δ_N is bounded, the interesting point is the behavior of the spectrum near zero. This leads us to consider negative powers of the spectral parameter.

Definition 4.1 Given $k \in \mathbb{N}_0$, the complex scalar product space $(V, \langle\langle \cdot | \cdot \rangle\rangle_{\mathfrak{h}^k})$ is defined by

$$V = \left\{ u \in \mathfrak{h}^{\mathbb{C}} \mid \int_{\sigma(\Delta_N)} \lambda^{-k} d\langle\langle u | E_\lambda u \rangle\rangle < \infty \right\} \tag{4.4}$$

$$\langle\langle u | v \rangle\rangle_{\mathfrak{h}^k} = \int_{\sigma(\Delta_N)} \lambda^{-k} d\langle\langle u | E_\lambda v \rangle\rangle. \tag{4.5}$$

Its Hilbert space completion is denoted by $(\mathfrak{h}^k, \langle\langle \cdot | \cdot \rangle\rangle_{\mathfrak{h}^k})$. We refer to the \mathfrak{h}^k as *Sobolev-type Hilbert spaces*.

Clearly, the Hilbert space \mathfrak{h}^0 coincides with $\mathfrak{h}^{\mathbb{C}}$. If $k \geq 1$, the condition for the integral in (4.4) to be finite implies in particular that u must be orthogonal to the kernel of Δ_N , i.e.

$$\langle\langle u | v \rangle\rangle = 0 \quad \text{for all } v \in \ker \Delta_N. \tag{4.6}$$

Moreover, for any $\varepsilon > 0$, the subspace $E_{(\varepsilon, \infty)}(\mathfrak{h}^{\mathbb{C}})$ is contained in \mathfrak{h}^k . This means that (4.5) poses a condition only for vectors whose spectral decomposition extends to the origin. The name *Sobolev-type Hilbert space* is motivated by the fact that the spectral decomposition with respect to Δ_N gives information on the regularity of the jets; this will be explained in Sect. 5.1 in a simple example.

We also note that, the higher k is chosen, the stronger the condition in (4.4) becomes. This shows that we have the sequence of inclusions

$$\mathfrak{h}^{\mathbb{C}} = \mathfrak{h}^0 \supset \mathfrak{h}^1 \supset \mathfrak{h}^2 \supset \dots$$

Moreover, the Laplacian Δ_N is a well-defined mapping

$$\Delta_N : \mathfrak{h}^k \rightarrow \mathfrak{h}^{k+2}.$$

Theorem 4.2 Let $\hat{w} \in \mathfrak{h}^{k+2}$ with $k \in \mathbb{N}_0$. Then the inhomogeneous linearized field equations

$$\Delta_N u = \hat{w}$$

have a unique weak solution $u \in \mathfrak{h}^k$ given by

$$u = \int_{\sigma(\Delta_N)} \lambda^{-1} dE_\lambda \hat{w}, \tag{4.7}$$

where the integral converges in \mathfrak{h}^k .

Proof First of all, we know from (4.6) that $\hat{w} \in (\ker \Delta_N)^\perp$. Thus $E_{\{0\}}\hat{w} = 0$. As a consequence, the following integral exists and is finite,

$$\int_{\sigma(\Delta_N)} \lambda^{-k} d\langle u | E_\lambda u \rangle = \int_{\sigma(\Delta_N)} \lambda^{-k-2} d\langle \hat{w} | E_\lambda \hat{w} \rangle = \|\hat{w}\|_{\mathfrak{h}^{k+2}}^2 < \infty.$$

As a consequence, the integral (4.7) converges in the Hilbert space \mathfrak{h}^k . Moreover, it follows from the spectral calculus that

$$\Delta_N u = \Delta_N \int_{\sigma(\Delta_N)} \lambda^{-1} dE_\lambda \hat{w} = \int_{\sigma(\Delta_N)} \lambda \lambda^{-1} dE_\lambda \hat{w} = \hat{w}.$$

This concludes the proof. □

Finally, it is convenient to introduce the function spaces

$$\mathfrak{W}^k := \{h(\mathbf{x}) \hat{w}(\mathbf{x}) \mid \hat{w} \in \mathfrak{h}^k\}.$$

Then the mapping

$$\Delta^{-1} : \mathfrak{W}^{k+2} \rightarrow \mathfrak{h}^k, \quad w \mapsto v := \int_{\sigma(\Delta_N)} \lambda^{-1} dE_\lambda \frac{w}{h}$$

gives the desired weak solution of the inhomogeneous linearized field equations (1.1).

5 A few simple explicit examples

In this section we illustrate the previous abstract constructions and results by a few simple examples. These examples are chosen specifically in such a way that a minimizing measure can be given in closed form, making it possible to analyze the system explicitly.

5.1 A one-dimensional Gaussian

We let $\mathfrak{G} = \mathbb{R}$ and choose the Lagrangian as the Gaussian

$$\mathcal{L}(\mathbf{x}, \mathbf{y}) = \frac{1}{\sqrt{\pi}} e^{-(\mathbf{x}-\mathbf{y})^2}. \tag{5.1}$$

Lemma 5.1 *The Lebesgue measure*

$$d\mu = d\mathbf{x}$$

is a minimizer of the causal action principle for the Lagrangian (5.1) in the class of variations of finite volume [see (2.3) and (2.2)]. It is the unique minimizer within this class of variations.

Proof Writing the difference of the actions as in (2.3), we can carry out the integrals over μ using that the Gaussian is normalized,

$$\int_{\mathfrak{G}} \mathcal{L}(\mathbf{x}, \mathbf{y}) d\mu(\mathbf{y}) = 1.$$

We thus obtain

$$\begin{aligned} \mathcal{S}(\mu) - \mathcal{S}(\tilde{\mu}) &= 2 \int_N d(\mu - \tilde{\mu})(\mathbf{x}) + \int_N d(\mu - \tilde{\mu})(\mathbf{x}) \int_N d(\mu - \tilde{\mu})(\mathbf{y}) \mathcal{L}(\mathbf{x}, \mathbf{y}) \\ &= \int_N d(\mu - \tilde{\mu})(\mathbf{x}) \int_N d(\mu - \tilde{\mu})(\mathbf{y}) \mathcal{L}(\mathbf{x}, \mathbf{y}), \end{aligned}$$

where in the last line we used the volume constraint (2.2). In order to show that the last double integral is positive, we take the Fourier transform and use that the Fourier transform of a Gaussian is again a Gaussian. More precisely,

$$\int_N e^{ip\mathbf{x}} \mathcal{L}(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} = e^{-\frac{p^2}{4}} =: f(p).$$

Moreover, the estimate

$$\left| \int_N e^{ip\mathbf{x}} d(\mu - \tilde{\mu})(\mathbf{x}) \right| \leq |\tilde{\mu} - \mu|(\mathcal{G}) < \infty$$

shows that the Fourier transform of the signed measure $\tilde{\mu} - \mu$ is a bounded function $g \in L^\infty(\mathbb{R})$. Approximating this function in $L^2(\mathbb{R})$, we can apply Plancherel’s theorem and use the fact that convolution in position space corresponds to multiplication in momentum space. We thus obtain

$$\begin{aligned} & \int_N d(\mu - \tilde{\mu})(\mathbf{x}) \int_N d(\mu - \tilde{\mu})(\mathbf{y}) \mathcal{L}(\mathbf{x}, \mathbf{y}) \\ &= \int_N (\mathcal{F}^{-1}(fg))(\mathbf{x}) \, d(\mu - \tilde{\mu})(\mathbf{x}) = \int_{-\infty}^\infty \overline{g(p)} e^{-\frac{p^2}{4}} g(p) \, dp \geq 0, \end{aligned} \tag{5.2}$$

and the inequality is strict unless $\tilde{\mu} = \mu$. This concludes the proof. □

The EL equations read

$$\int_{\mathcal{G}} \mathcal{L}(\mathbf{x}, \mathbf{y}) \, d\mu(\mathbf{y}) = 1 \quad \text{for all } \mathbf{x} \in \mathbb{R}. \tag{5.3}$$

We now specify the jet spaces. Since the Lagrangian is smooth, it is obvious that

$$\mathfrak{J}^{\text{diff}} = \mathfrak{J} = C^\infty(\mathbb{R}) \oplus C^\infty(\mathbb{R})$$

(where we identify a vector field $a(\mathbf{x}) \partial_{\mathbf{x}}$ on \mathbb{R} with the function $a(\mathbf{x})$). The choice of $\mathfrak{J}^{\text{test}}$ is less obvious. In order to ensure that the conditions in Definition 2.1 are satisfied, we restrict attention to functions which are bounded together with all their derivatives, denoted by

$$C_b^\infty := \{f \in C^\infty(\mathbb{R}) \mid f^{(n)} \in L^\infty(\mathbb{R}) \text{ for all } n \in \mathbb{N}_0\}.$$

Now different choices are possible. Our first choice is to consider jets whose scalar components are compactly supported,

$$\mathfrak{J}^{\text{test}} = C_0^\infty(\mathbb{R}) \oplus C_b^\infty(\mathbb{R}).$$

The linearized field equations (2.18) reduce to the scalar equation

$$\int_N (\nabla_{1,v} + \nabla_{2,v}) \mathcal{L}(\mathbf{x}, \mathbf{y}) \, d\mu(\mathbf{y}) - \nabla_v 1 = 0 \quad \text{for all } \mathbf{x} \in \mathbb{R},$$

because if this equation holds, then the \mathbf{x} -derivative of the left side is also zero. Using the EL equations (5.3), the linearized field equations simplify to

$$\int_N \nabla_{2,v} \mathcal{L}(\mathbf{x}, \mathbf{y}) \, d\mu(\mathbf{y}) = 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}.$$

A specific class of solutions can be given explicitly. Indeed, choosing

$$\mathbf{u} = (a, A) \quad \text{with } a \in C_0^\infty(\mathbb{R}) \quad \text{and } A(\mathbf{x}) := \int_0^{\mathbf{x}} a(t) \, dt \in C_b^\infty(\mathbb{R}), \tag{5.4}$$

integration by parts yields

$$\int_N \nabla_{2,u} \mathcal{L}(\mathbf{x}, \mathbf{y}) \, d\mu(\mathbf{y}) = \int_N (A'(\mathbf{y}) + A(\mathbf{y}) \partial_{\mathbf{y}}) \mathcal{L}(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} = 0. \tag{5.5}$$

These linearized solutions are referred to as *inner solutions*, as introduced in a more general context in [10]. Inner solutions can be regarded as infinitesimal generators of transformations of N which leave the measure μ unchanged. Therefore, inner solutions do not change the causal fermion system, but merely describe symmetry transformations of the measure. With this in mind, inner solutions are not of interest by themselves. But they can be used in order to simplify the form of the jet spaces. We shall come back to these inner solutions in the general setting in ‘‘Appendix’’, where we will show that, by adding suitable inner solutions, one can arrange that the test jets have vanishing scalar components. In our example, this can be arranged by the transformation

$$\mathfrak{v} = (b, v) \mapsto \tilde{\mathfrak{v}} := \mathfrak{v} + \mathfrak{u} \quad \text{with} \quad \mathfrak{u} = (-b, -B),$$

where B is an indefinite integral of b .

In our example, we can also use the inner solutions alternatively in order to eliminate the vector component of the test jets. To this end, it is preferable to choose the space of test jets as

$$\mathfrak{J}^{\text{test}} = C_b^\infty(\mathbb{R}) \oplus C_b^\infty(\mathbb{R}). \tag{5.6}$$

Now the vector component disappears under the transformation

$$\mathfrak{v} = (b, v) \mapsto \tilde{\mathfrak{v}} := \mathfrak{v} + \mathfrak{u} \quad \text{with} \quad \mathfrak{u} = (-v', -v) \in \mathfrak{J}^{\text{test}}.$$

Therefore, it remains to consider the scalar components of jets. For technical simplicity, we restrict attention to compactly supported functions. Thus we choose the jet space $\mathfrak{J}^{\text{vary}}$ in (3.2) as

$$\mathfrak{J}^{\text{vary}} = C_0^\infty(\mathbb{R}) \oplus \{0\}.$$

Then the Laplacian reduces to the integral operator with kernel $\mathcal{L}(\mathbf{x}, \mathbf{y})$,

$$(\Delta(b, 0))(\mathbf{x}) = \int_{\mathcal{G}} \mathcal{L}(\mathbf{x}, \mathbf{y}) \, b(\mathbf{y}) \, d\mathbf{y}.$$

The bilinear form (3.3) takes the form

$$\langle \cdot, \Delta \cdot \rangle_N : \mathfrak{J}^{\text{vary}} \times \mathfrak{J}^{\text{vary}} \rightarrow \mathbb{R}, \quad \langle a, \Delta b \rangle_N = \int_{\mathcal{G}} d\mu(\mathbf{x}) \int_{\mathcal{G}} d\mu(\mathbf{y}) \mathcal{L}(\mathbf{x}, \mathbf{y}) \, a(\mathbf{x}) \, b(\mathbf{y}).$$

Moreover, $\|\bar{\nabla}^2 \mathcal{L}(\mathbf{x}, \mathbf{y})\| = \mathcal{L}(\mathbf{x}, \mathbf{y})$, so that the function h in (3.7) is constant,

$$h(\mathbf{x}) := 1 + \int_N \mathcal{L}(\mathbf{x}, \mathbf{y}) \, d\mu(\mathbf{y}) = 2.$$

Therefore, the adapted weighted scalar product $\langle\langle \cdot, \cdot \rangle\rangle$ is twice the L^2 -scalar product,

$$\langle\langle a, b \rangle\rangle := 2 \int_N ab \, d\mu,$$

making it possible to identify \mathfrak{h} with $L^2(N, d\mu)$. Consequently, the Laplacian Δ_N defined by (3.9) simply is the convolution with the Gaussian,

$$\Delta_N : \mathfrak{h} \rightarrow \mathfrak{h}, \quad (\Delta_N a)(\mathbf{x}) = \frac{1}{2} \int_N \mathcal{L}(\mathbf{x}, \mathbf{y}) \, a(\mathbf{y}) \, d\mu(\mathbf{y}).$$

The spectral decomposition of this operator is obtained by Fourier transformation,

$$\Delta_N a = \mathcal{F}^{-1} \left(\frac{1}{2} e^{-\frac{p^2}{4}} (\mathcal{F}a)(p) \right).$$

Thus, in momentum space, the operator Δ_N is a multiplication operator. Therefore, its spectral measure is obtained by multiplying with the characteristic function of the level sets,

$$E_\Omega = \mathcal{F}^{-1} \chi_{g^{-1}(\Omega)} \mathcal{F} \quad \text{with} \quad g(p) := \frac{1}{2} e^{-\frac{p^2}{4}}.$$

In particular, one sees that the high momenta (i.e. large $|p|$) correspond to the spectrum near zero. Therefore, the negative powers of the spectral parameter in (4.5) capture the high-frequency behavior of the jets. This is quite similar to the usual Sobolev norms on $W^{2,k}(\mathbb{R})$, as becomes clear when writing them in momentum space as

$$\|u\|_{W^{2,k}(\mathbb{R})}^2 = \int_{-\infty}^{\infty} (1 + |p|^2 + \dots + |p|^{2k}) |\hat{u}(p)|^2 dp.$$

This is why we refer to the \mathfrak{h}^k as *Sobolev-type Hilbert spaces*.

Before going on, we remark that the above methods works more generally if $\mathcal{L}(\mathbf{x}, \mathbf{y})$ is chosen as a function of $\mathbf{x} - \mathbf{y}$ which has the properties that it is non-negative and that its Fourier transform is strictly positive (or non-negative, in which case the minimizing measure may not be unique). In order to give a simple example, choosing the Lagrangian as the one-dimensional Yukawa potential,

$$\mathcal{L}(\mathbf{x}, \mathbf{y}) = g(\mathbf{x} - \mathbf{y}) := \frac{e^{-|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} \geq 0,$$

its Fourier transform is computed by

$$\hat{g}(k) = \frac{4\pi}{1 + |k|^2},$$

which, as desired, is strictly positive. Therefore, the Lebesgue measure $d\mu = d\mathbf{x}$ is again the unique minimizer of the causal action within the class of variations of finite volume.

5.2 A minimizing measure supported on a hyperplane

In the previous example, the support of the minimizing measure was the whole space \mathcal{G} . In most examples motivated from the physical applications, however, the minimizing measure will be supported on a low-dimensional subset of \mathcal{G} (see for instance the minimizers with singular support in [1, 17]). We now give a simple example where the minimizing measure is supported on a hyperplane of \mathcal{G} . We let $\mathcal{G} = \mathbb{R}^2$ and choose the Lagrangian as

$$\mathcal{L}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}') = \frac{1}{\sqrt{\pi}} e^{-(\mathbf{x}-\mathbf{x}')^2} (1 + \mathbf{y}^2)(1 + \mathbf{y}'^2), \tag{5.7}$$

where $(\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}') \in \mathcal{G}$.

Lemma 5.2 *The measure*

$$d\mu = d\mathbf{x} \times \delta_{\mathbf{y}} \tag{5.8}$$

(where $\delta_{\mathbf{y}}$ is the Dirac measure) is the unique minimizer of the causal action principle for the Lagrangian (5.7) under variations of finite volume [see (2.3) and (2.2)].

Note that this measure is supported on the \mathbf{x} -axis,

$$N := \text{supp } \mu = \mathbb{R} \times \{0\}.$$

Proof of Lemma 5.2 Let $\tilde{\mu}$ be a regular Borel measure on \mathcal{G} satisfying (2.2). Then the difference of actions (2.3) is computed by

$$\mathcal{S}(\tilde{\mu}) - \mathcal{S}(\mu) = \frac{2}{\sqrt{\pi}} \int_{\mathcal{G}} d(\tilde{\mu} - \mu)(\mathbf{x}, \mathbf{y}) \int_N d\mathbf{x}' e^{-(\mathbf{x}-\mathbf{x}')^2} (1 + \mathbf{y}^2) \tag{5.9}$$

$$+ \frac{1}{\sqrt{\pi}} \int_{\mathcal{G}} d(\tilde{\mu} - \mu)(\mathbf{x}, \mathbf{y}) \int_{\mathcal{G}} d(\tilde{\mu} - \mu)(\mathbf{x}', \mathbf{y}') e^{-(\mathbf{x}-\mathbf{x}')^2} (1 + \mathbf{y}^2)(1 + \mathbf{y}'^2). \tag{5.10}$$

Using that the negative part of the measure $\tilde{\mu} - \mu$ is supported on the \mathbf{x} -axis, the first term (5.9) can be estimated by

$$\begin{aligned} & \frac{2}{\sqrt{\pi}} \int_{\mathcal{G}} d(\tilde{\mu} - \mu)(\mathbf{x}, \mathbf{y}) \int_N d\mathbf{x}' e^{-(\mathbf{x}-\mathbf{x}')^2} (1 + \mathbf{y}^2) \\ & \stackrel{(*)}{\geq} \frac{2}{\sqrt{\pi}} \int_{\mathcal{G}} d(\tilde{\mu} - \mu)(\mathbf{x}, \mathbf{y}) \int_N d\mathbf{x}' e^{-(\mathbf{x}-\mathbf{x}')^2} = \int_{\mathcal{G}} d(\tilde{\mu} - \mu)(\mathbf{x}, \mathbf{y}) = 0, \end{aligned}$$

where in the last step we used the volume constraint. The second term (5.10), on the other hand, can be rewritten as

$$\frac{1}{\sqrt{\pi}} \int_{\mathcal{G}} d\rho(\mathbf{x}, \mathbf{y}) \int_{\mathcal{G}} d\rho(\mathbf{x}', \mathbf{y}') e^{-(\mathbf{x}-\mathbf{x}')^2}$$

with the signed measure ρ defined by

$$d\rho(\mathbf{x}, \mathbf{y}) := (1 + \mathbf{y}^2) d(\tilde{\mu} - \mu)(\mathbf{x}, \mathbf{y}).$$

Now we can proceed as in the proof of Lemma 5.1 and use that the Fourier transform of the integral kernel is strictly positive. For the uniqueness statement one uses that the inequality in (*) is strict unless $\tilde{\mu}$ is supported on the \mathbf{x} -axis. Then one can argue as in the proof of Lemma 5.1.

For the minimizing measure (5.8), the function ℓ takes the form

$$\ell(\mathbf{x}, \mathbf{y}) = \int_{\mathcal{G}} \mathcal{L}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}') d\mu(\mathbf{x}', \mathbf{y}') - 1 = \mathbf{y}^2,$$

showing that the EL equations (2.5) are indeed satisfied. We now specify the jet spaces. Since the Lagrangian is smooth, it is obvious that

$$\mathfrak{J}^{\text{diff}} = \mathfrak{J} = C^\infty(\mathbb{R}) \oplus C^\infty(\mathbb{R}, \mathbb{R}^2), \tag{5.11}$$

where $C^\infty(\mathbb{R}, \mathbb{R}^2)$ should be regarded as the space of two-dimensional vector fields along the \mathbf{x} -axis. Similar as explained after (5.6), we want to use the inner solutions for simplifying the vector components of the jets. To this end, in analogy to (5.6) we choose

$$\mathfrak{J}^{\text{test}} = C_b^\infty(\mathbb{R}) \oplus C_b^\infty(\mathbb{R}, \mathbb{R}^2). \tag{5.12}$$

The linearized field equations (2.18) read

$$\nabla_{\mathbf{u}} \left(\int_{-\infty}^{\infty} (\nabla_{1,\mathbf{v}} + \nabla_{2,\mathbf{v}}) e^{-(\mathbf{x}-\mathbf{x}')^2} (1 + \mathbf{y}^2)(1 + \mathbf{y}'^2) d\mathbf{x}' - \nabla_{\mathbf{v}} \sqrt{\pi} \right) \Big|_{\mathbf{y}=\mathbf{y}'=0} = 0. \tag{5.13}$$

Now the inner solutions are generated by the vector fields tangential to the \mathbf{x} -axis. More precisely, in analogy to (5.4), we consider the jet

$$\mathbf{v} = (b, (B, 0)) \quad \text{with } b \in C_0^\infty \quad \text{and } B(\mathbf{x}) := \int_{-\infty}^{\mathbf{x}} b(t) dt \in C_b^\infty(\mathbb{R}). \tag{5.14}$$

Substituting this jet into (5.13), the linearized field equations simplify to

$$\nabla_{\mathbf{u}} \left((1 + \mathbf{y}^2) \int_{-\infty}^{\infty} (\nabla_{1,\mathbf{v}} + \nabla_{2,\mathbf{v}}) e^{-(\mathbf{x}-\mathbf{x}')^2} d\mathbf{x}' - \nabla_{\mathbf{v}} \sqrt{\pi} \right) \Big|_{\mathbf{y}=\mathbf{y}'=0} = 0.$$

The second component of the vector field u yields a \mathbf{y} -derivative, giving rise to a factor $2\mathbf{y}$, which vanishes at $\mathbf{y} = 0$. Therefore, it suffices to test with a vector field u which is tangential to the \mathbf{x} -axis. Now we are back in the example of the one-dimensional Gaussian. Integrating by parts as in (5.5) one sees that the jet \mathbf{v} indeed satisfies the linearized field equations.

By suitably subtracting inner solutions, we can compensate the tangential components of the jets. This leads us to choose

$$\mathfrak{J}^{\text{vary}} = C_0^\infty(\mathbb{R}) \oplus (\{0\} \oplus C_0^\infty(\mathbb{R})). \tag{5.15}$$

Then the Laplacian simplifies as follows,

$$\begin{aligned} \langle \mathbf{u}, \Delta \mathbf{v} \rangle(\mathbf{x}) &= \frac{1}{\sqrt{\pi}} \nabla_{\mathbf{u}} \left(\int_{-\infty}^{\infty} (\nabla_{1,\mathbf{v}} + \nabla_{2,\mathbf{v}}) e^{-(\mathbf{x}-\mathbf{x}')^2} (1 + \mathbf{y}^2)(1 + \mathbf{y}'^2) d\mathbf{x}' - \nabla_{\mathbf{v}} \sqrt{\pi} \right) \Big|_{\mathbf{y}=\mathbf{y}'=0} \\ &= \frac{2}{\sqrt{\pi}} u(\mathbf{x}) v(\mathbf{x}) \int_{-\infty}^{\infty} e^{-(\mathbf{x}-\mathbf{x}')^2} d\mathbf{x}' + \frac{1}{\sqrt{\pi}} a(\mathbf{x}) \int_{-\infty}^{\infty} e^{-(\mathbf{x}-\mathbf{x}')^2} b(\mathbf{x}') d\mathbf{x}' \\ &\quad + a(\mathbf{x}) \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} b(\mathbf{x}) e^{-(\mathbf{x}-\mathbf{x}')^2} d\mathbf{x}' - b(\mathbf{x}) \right) \\ &= 2 u(\mathbf{x}) v(\mathbf{x}) + \frac{1}{\sqrt{\pi}} a(\mathbf{x}) \int_{-\infty}^{\infty} e^{-(\mathbf{x}-\mathbf{x}')^2} b(\mathbf{x}') d\mathbf{x}', \end{aligned}$$

where $\mathbf{u} = (a, (0, u))$ and $\mathbf{v} = (b, (0, v))$. Hence the inhomogeneous linearized field equations (1.1) with $\mathfrak{w} = (e, w)$ give rise to separate equations for the scalar and vector components,

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\mathbf{x}-\mathbf{x}')^2} b(\mathbf{x}') d\mathbf{x}' = e(\mathbf{x}), \quad v(\mathbf{x}) = \frac{w(\mathbf{x})}{2}. \tag{5.16}$$

The bilinear form (3.3) reduces to

$$\langle \mathbf{u}, \Delta \mathbf{v} \rangle_N = 2 \int_{-\infty}^{\infty} u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\mathbf{x} \int_{-\infty}^{\infty} d\mathbf{x}' e^{-(\mathbf{x}-\mathbf{x}')^2} a(\mathbf{x}) b(\mathbf{x}').$$

Moreover,

$$\begin{aligned} \overline{\nabla}_{\mathbf{u}} \overline{\nabla}_{\mathbf{v}} \mathcal{L}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}') &= \frac{1}{\sqrt{\pi}} (a(\mathbf{x}) + a(\mathbf{x}'))(b(\mathbf{x}) + b(\mathbf{x}')) e^{-(\mathbf{x}-\mathbf{x}')^2} \\ &\quad + \frac{2}{\sqrt{\pi}} (u(\mathbf{x}) v(\mathbf{x}) + u(\mathbf{x}') v(\mathbf{x}')) e^{-(\mathbf{x}-\mathbf{x}')^2}. \end{aligned}$$

As a consequence, $\overline{\nabla}^2 \mathcal{L}$ is estimated from above and below by the Gaussian, i.e.

$$\frac{1}{c} e^{-(\mathbf{x}-\mathbf{x}')^2} \leq \|\overline{\nabla}^2 \mathcal{L}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}')\| \leq c e^{-(\mathbf{x}-\mathbf{x}')^2}$$

for a suitable numerical constant $c > 0$. As a consequence, the weight function (3.7) is bounded,

$$h(\mathbf{x}, 0) \leq 1 + c \sqrt{\pi}.$$

Hence the adapted scalar product agrees, up to an irrelevant constant, with the L^2 -scalar product. This reflects the fact that the Laplacian is already bounded with respect to the L^2 -scalar product, making it unnecessary to introduce the adapted scalar product. \square

5.3 An adapted L^2 -scalar product with non-trivial weight

In the previous example, the weight function $h(\mathbf{x})$ in (3.7) was constant by symmetry. Moreover, the linearized field equations for the vector component were local [see the second equation in (5.16)]. We now modify this example in order to make it more interesting in these respects. We again let $\mathcal{G} = \mathbb{R}^2$ and choose the Lagrangian as

$$\mathcal{L}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}') = \frac{1}{\sqrt{\pi}} e^{-(\mathbf{x}-\mathbf{x}')^2} e^{2\mathbf{y}\mathbf{y}'} (1 + \mathbf{x}^2\mathbf{y}^2)(1 + \mathbf{x}'^2\mathbf{y}'^2).$$

In order to show that the measure (5.8) is again a minimizer of the causal action principle under variations of finite volume [see (2.3) and (2.2)], one argues as follows. Similar as in the proof of Lemma 5.2], the difference of actions has a contribution linear in $\tilde{\mu} - \mu$ [see (5.9)] and a contribution quadratic in $\tilde{\mu} - \mu$ [see (5.10)]. Using again that the negative part of the measure $\tilde{\mu} - \mu$ is supported on the \mathbf{x} -axis, the linear term is estimated by

$$\begin{aligned} & \int_{\mathcal{G}} d(\tilde{\mu} - \mu)(\mathbf{x}, \mathbf{y}) \int_{\mathcal{G}} d\mu(\mathbf{x}', \mathbf{y}') \mathcal{L}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}') \\ &= \frac{1}{\sqrt{\pi}} \int_{\mathcal{G}} d(\tilde{\mu} - \mu)(\mathbf{x}, \mathbf{y}) \int_{\mathcal{N}} d\mathbf{x}' e^{-(\mathbf{x}-\mathbf{x}')^2} (1 + \mathbf{x}^2\mathbf{y}^2) \\ &\geq \frac{1}{\sqrt{\pi}} \int_{\mathcal{G}} d(\tilde{\mu} - \mu)(\mathbf{x}, \mathbf{y}) \int_{\mathcal{N}} d\mathbf{x}' e^{-(\mathbf{x}-\mathbf{x}')^2} = \int_{\mathcal{G}} d(\tilde{\mu} - \mu)(\mathbf{x}, \mathbf{y}) = 0. \end{aligned}$$

The quadratic term in $\tilde{\mu} - \mu$, on the other hand, can be rewritten as

$$\begin{aligned} & \int_{\mathcal{G}} d(\tilde{\mu} - \mu)(\mathbf{x}, \mathbf{y}) \int_{\mathcal{G}} d(\tilde{\mu} - \mu)(\mathbf{x}', \mathbf{y}') \mathcal{L}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}') \\ &= \frac{1}{\sqrt{\pi}} \int_{\mathcal{G}} d\rho(\mathbf{x}, \mathbf{y}) \int_{\mathcal{G}} d\rho(\mathbf{x}', \mathbf{y}') e^{-(\mathbf{x}-\mathbf{x}')^2} e^{-(\mathbf{y}-\mathbf{y}')^2}, \end{aligned}$$

where ρ is the signed measure

$$d\rho(\mathbf{x}, \mathbf{y}) = (1 + \mathbf{x}^2\mathbf{y}^2) e^{\mathbf{y}^2} d(\tilde{\mu} - \mu)(\mathbf{x}, \mathbf{y}).$$

Now we can proceed again as in the proof of Lemma 5.1 and use that the Fourier transform of the integral kernel is strictly positive.

The function ℓ takes the form

$$\begin{aligned} \ell(\mathbf{x}, \mathbf{y}) &= \int_{\mathcal{G}} \mathcal{L}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}') d\mu(\mathbf{x}', \mathbf{y}') - 1 \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\mathbf{x}-\mathbf{x}')^2} (1 + \mathbf{x}^2\mathbf{y}^2) d\mathbf{x}' - 1 = \mathbf{x}^2\mathbf{y}^2. \end{aligned}$$

We again choose the jet spaces $\mathfrak{J}^{\text{diff}}$, $\mathfrak{J}^{\text{test}}$ and $\mathfrak{J}^{\text{vary}}$ according to (5.11), (5.12) and (5.15). Then

$$\begin{aligned} & (u, \Delta v)(\mathbf{x}) \\ &= \nabla_u \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\nabla_{1,v} + \nabla_{2,v}) e^{-(\mathbf{x}-\mathbf{x}')^2} \right. \\ & \quad \left. \times e^{2\mathbf{y}\mathbf{y}'} (1 + \mathbf{x}^2 \mathbf{y}^2) (1 + \mathbf{x}'^2 \mathbf{y}'^2) d\mathbf{x}' - \nabla_v 1 \right) \Big|_{\mathbf{y}=\mathbf{y}'=0} \\ &= \frac{2}{\sqrt{\pi}} u(\mathbf{x}) v(\mathbf{x}) \int_{-\infty}^{\infty} e^{-(\mathbf{x}-\mathbf{x}')^2} \mathbf{x}^2 d\mathbf{x}' \\ & \quad - \frac{2}{\sqrt{\pi}} u(\mathbf{x}) \int_{-\infty}^{\infty} e^{-(\mathbf{x}-\mathbf{x}')^2} v(\mathbf{x}') d\mathbf{x}' + \frac{1}{\sqrt{\pi}} a(\mathbf{x}) \int_{-\infty}^{\infty} e^{-(\mathbf{x}-\mathbf{x}')^2} b(\mathbf{x}') d\mathbf{x}' \\ &= 2 u(\mathbf{x}) v(\mathbf{x}) \mathbf{x}^2 \\ & \quad - \frac{2}{\sqrt{\pi}} u(\mathbf{x}) \int_{-\infty}^{\infty} e^{-(\mathbf{x}-\mathbf{x}')^2} v(\mathbf{x}') d\mathbf{x}' + \frac{1}{\sqrt{\pi}} a(\mathbf{x}) \int_{-\infty}^{\infty} e^{-(\mathbf{x}-\mathbf{x}')^2} b(\mathbf{x}') d\mathbf{x}'. \end{aligned}$$

Thus, as desired, the linearized field equations are nonlocal also for the vector component.

Next, for any $(\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}') \in \mathbb{R} \times \{0\}$

$$\begin{aligned} \bar{\nabla}_u \bar{\nabla}_v \mathcal{L}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}') &= \frac{1}{\sqrt{\pi}} (a(\mathbf{x}) + a(\mathbf{x}')) (b(\mathbf{x}) + b(\mathbf{x}')) e^{-(\mathbf{x}-\mathbf{x}')^2} \\ & \quad + \frac{2}{\sqrt{\pi}} (u(\mathbf{x}) v(\mathbf{x}) \mathbf{x}^2 + u(\mathbf{x}') v(\mathbf{x}') \mathbf{x}'^2) e^{-(\mathbf{x}-\mathbf{x}')^2}. \end{aligned}$$

Hence

$$\| \bar{\nabla}_u \bar{\nabla}_v \mathcal{L}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}') \| \simeq (1 + \mathbf{x}^2 + \mathbf{x}'^2) e^{-(\mathbf{x}-\mathbf{x}')^2}$$

(here we make use of the fact that the jets in $\mathfrak{J}^{\text{vary}}$ can be chosen independently at the points \mathbf{x} and \mathbf{x}'). We conclude that the weight function (3.7) in the adapted weighted L^2 -scalar product takes the form

$$h(\mathbf{x}) = 1 + \int_N \| \bar{\nabla}^2 \mathcal{L}(\mathbf{x}, \mathbf{y}) \| d\mu(\mathbf{y}) \simeq 1 + \mathbf{x}^2.$$

Thus the adapted weighted L^2 -scalar product is *not* equivalent to the standard L^2 -scalar product. This example explains why the adaptation of the weight is needed in order to realize the Laplacian as a *bounded* symmetric operator on a Hilbert space.

5.4 A non-homogeneous minimizing measure

In the previous examples, the minimizing measure μ was translation invariant in the direction of the \mathbf{x} -axis. We now give a general procedure for constructing examples of causal variational principles where the minimizing measure has no translational symmetry. In order to work in a concrete example, our starting point is again the one-dimensional Gaussian (5.1). But the method can be adapted to other kernels in a straightforward way. In view of these generalizations, we begin with the following abstract result.

Lemma 5.3 *Let ρ be a measure on the m -dimensional manifold \mathcal{G} whose support is the whole manifold,*

$$\text{supp } \rho = \mathcal{G}.$$

Moreover, let $\mathcal{L}(\mathbf{x}, \mathbf{y}) \in (C^0 \cap L^\infty)(\mathcal{G} \times \mathcal{G}, \mathbb{R}_0^+)$ be a symmetric, non-negative, continuous and bounded kernel on $\mathcal{G} \times \mathcal{G}$. Next, let $h \in C^0(\mathcal{G}, \mathbb{R}^+)$ be a strictly positive, continuous function on \mathcal{G} . Assume that:

- (i) $\int_{\mathcal{G}} \mathcal{L}(\mathbf{x}, \mathbf{y}) h(\mathbf{y}) d\rho(\mathbf{y}) = 1$ for all $\mathbf{x} \in \mathcal{G}$.
- (ii) For all compactly supported bounded functions with zero mean,

$$g \in L_0^\infty(\mathcal{G}, \mathbb{R}^+) \text{ and } \int_{\mathcal{G}} g d\rho = 0,$$

the following inequality holds,

$$\int_{\mathcal{G}} d\rho(\mathbf{x}) \int_{\mathcal{G}} d\rho(\mathbf{y}) \mathcal{L}(\mathbf{x}, \mathbf{y}) g(\mathbf{x}) g(\mathbf{y}) \geq 0. \tag{5.17}$$

Then the measure $d\mu := h d\rho$ is a minimizer of the causal action principle under variations of finite volume [see (2.3) and (2.2)]. If the inequality (5.17) is strict for any non-zero g , then the minimizing measure is unique within the class of variations

$$d\tilde{\mu}_\tau = d\mu + \tau g d\rho. \tag{5.18}$$

Proof We begin with variations of the form (5.18), which we write equivalently as

$$d\tilde{\mu}_\tau = (h + \tau g) d\rho. \tag{5.19}$$

Note that the function h is continuous and strictly positive. Moreover, the function g is bounded and compactly supported. This implies that the function $h + \tau g$ is non-negative for sufficiently small $|\tau|$. Furthermore, using that g has mean zero, we conclude that (5.19) is an admissible variation of finite volume (2.2). Moreover, the difference of the actions (2.3) is well-defined and computed by

$$\begin{aligned} & \mathcal{S}(\tilde{\rho}_\tau) - \mathcal{S}(\mu) \\ &= 2\tau \int_{\mathcal{G}} d\mu(\mathbf{x}) g(\mathbf{x}) \int_{\mathcal{G}} d\mu(\mathbf{y}) h(\mathbf{y}) \mathcal{L}(\mathbf{x}, \mathbf{y}) + \tau^2 \int_{\mathcal{G}} d\mu(\mathbf{x}) \int_{\mathcal{G}} d\mu(\mathbf{x}) \mathcal{L}(\mathbf{x}, \mathbf{y}) g(\mathbf{x}) g(\mathbf{y}) \\ &\geq 2\tau \int_{\mathcal{G}} g(\mathbf{y}) d\mu(\mathbf{y}) = 0, \end{aligned}$$

where in the second step we used the above assumptions (i) and (ii). The last step follows from the fact that g has mean zero.

We conclude that the measure μ is a minimizer under variations of the form (5.19). In order to treat a general variation of finite volume (2.2), we use the following approximation argument. Our task is to show that

$$\int_{\mathcal{G}} d(\tilde{\mu} - \mu)(\mathbf{x}) \int_{\mathcal{G}} d(\tilde{\mu} - \mu) \mathcal{L}(\mathbf{x}, \mathbf{y}) \geq 0. \tag{5.20}$$

Exhausting \mathcal{G} by compact sets and using that the Lagrangian is bounded and that $|\tilde{\mu} - \mu|$ is finite, it suffices to consider the case that $\tilde{\mu} - \mu$ is compactly supported,

$$\text{supp}(\tilde{\mu} - \mu) \subset K \Subset \mathcal{G}.$$

We choose a partition of unity $(\eta_\ell)_{\ell \in \mathbb{N}}$ of \mathcal{G} which is subordinate to the atlas of \mathcal{G} . Then each measure $\tilde{\mu}_\ell := \eta_\ell(\tilde{\mu} - \mu)$ is compactly supported in the domain of a chart. Thus we can identify it with a compactly supported measure on \mathbb{R}^m , which for ease in notation we again denote by $\tilde{\mu}_\ell$. Given $k \in \mathbb{N}$, we decompose \mathbb{R}^m into cells of size $1/k$,

$$\mathbb{R}^m = \bigcup_{\vec{k} \in (\mathbb{Z}/k)^m} C_{\vec{k}} \quad \text{with} \quad C_{\vec{k}} := \vec{k} + \left[0, \frac{1}{k}\right)^m$$

and define functions $g_k^{(\ell)}$ by

$$g_k^{(\ell)}(\mathbf{x}) = \sum_{\vec{k} \in (\mathbb{Z}/k)^m} \frac{\tilde{\mu}_\ell(C_{\vec{k}})}{\rho(C_{\vec{k}})} \chi_{C_{\vec{k}}}(\mathbf{x}) \tag{5.21}$$

(where χ is again the characteristic function). Note that in (5.21) only a finite number of summands are non-zero because the measure $\tilde{\mu}_\ell$ is compactly supported. Moreover, each summand is finite because the functions $g_k^{(\ell)}$ are bounded since the measure $\rho(C_{\vec{k}}) > 0$; here we use that the support of ρ is all of \mathcal{G} . Hence $g_k^{(\ell)} \in L_0^\infty(\mathbb{R}^m)$, and a straightforward estimate shows that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^m} \phi(\mathbf{x}) g_k^{(\ell)}(\mathbf{x}) d\rho(\mathbf{x}) = \int_{\mathbb{R}^m} \phi(\mathbf{x}) d\tilde{\mu}_\ell(\mathbf{x}) \quad \text{for all} \quad \phi \in C_0^0(\mathbb{R}^m).$$

In other words, the measures $g_k^{(\ell)} d\rho$ converge in the weak*-topology to the measure $d\tilde{\mu}_\ell$. Moreover, one verifies immediately that the approximation preserves the total volume and the bound for the total variation, i.e.

$$\int_{\mathbb{R}^m} g_k^{(\ell)} d\rho = \tilde{\mu}_\ell(\mathbb{R}^m) \quad \text{and} \quad \int_{\mathbb{R}^m} |g_k^{(\ell)}| d\rho \leq |\tilde{\mu}_\ell|.$$

Finally, we introduce the function g_k by

$$g_k(\mathbf{x}) = \sum_{\ell=1}^\infty g_k^{(\ell)}(\mathbf{x}).$$

Here the sum is finite, because only a finite number of charts intersect K . Moreover, using that the measure $\tilde{\mu}$ has finite total variation, one sees from (5.21) that the functions g_k^ℓ are all bounded. Moreover, the measures $g_k d\rho$ all have finite total variation. Furthermore, they converge on K to the signed measure $\tilde{\mu} - \mu$. Using that the Lagrangian is continuous, it follows that

$$\begin{aligned} 0 &\leq \int_N d\mu(\mathbf{x}) \int_N d\mu(\mathbf{y}) \mathcal{L}(\mathbf{x}, \mathbf{y}) g_k(\mathbf{x}) g_k(\mathbf{y}) \\ &\xrightarrow{k \rightarrow \infty} \int_N d(\tilde{\mu} - \mu)(\mathbf{x}) \int_N d(\tilde{\mu} - \mu)(\mathbf{y}) \mathcal{L}(\mathbf{x}, \mathbf{y}), \end{aligned}$$

concluding the proof. □

Our goal is to apply this lemma to kernels of the form

$$\mathcal{L}(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) e^{-(\mathbf{x}-\mathbf{y})^2} f(\mathbf{y}) \tag{5.22}$$

with $\mathbf{x}, \mathbf{y} \in \mathbb{R}$ and a strictly positive function f , which for convenience we again choose as a Gaussian,

$$f(\mathbf{x}) = e^{\alpha \mathbf{x}^2} \quad \text{with} \quad \alpha \in \mathbb{R}. \tag{5.23}$$

In order for this Lagrangian to be bounded, we choose $\alpha < 1$. This kernel has the property (ii) with respect to the Lebesgue-measure $d\rho = d\mathbf{x}$ because for all non-trivial $g \in L_0^\infty(\mathcal{G}, \mathbb{R}^+)$,

$$\int_{\mathcal{G}} d\rho(\mathbf{x}) \int_{\mathcal{G}} d\rho(\mathbf{y}) \mathcal{L}(\mathbf{x}, \mathbf{y}) g(\mathbf{x}) g(\mathbf{y}) = \int_{\mathcal{G}} d\mathbf{x} \int_{\mathcal{G}} d\mathbf{y} e^{-(\mathbf{x}-\mathbf{y})^2} (fg)(\mathbf{x}) (fg)(\mathbf{y}) > 0,$$

where the last step is proved exactly as in the example of the Gaussian [see (5.2)] In order to arrange (i), for h we make an ansatz again with a Gaussian,

$$h(\mathbf{x}) = c e^{\beta \mathbf{x}^2}. \tag{5.24}$$

Then

$$\begin{aligned} \int_{\mathcal{G}} \mathcal{L}(\mathbf{x}, \mathbf{y}) h(\mathbf{y}) d\rho(\mathbf{y}) &= c \int_{-\infty}^{\infty} e^{\alpha \mathbf{x}^2} e^{-(\mathbf{x}-\mathbf{y})^2} e^{(\alpha+\beta)\mathbf{y}^2} d\mathbf{y} \\ &= c \exp\left(\alpha \mathbf{x}^2 - \mathbf{x}^2 - \frac{\mathbf{x}^2}{\alpha + \beta - 1}\right) \int_{-\infty}^{\infty} \exp\left\{(\alpha + \beta - 1) \left(\mathbf{y} + \frac{\mathbf{x}}{\alpha + \beta - 1}\right)^2\right\} d\mathbf{y} \\ &= c \sqrt{\frac{\pi}{1 - \alpha - \beta}} \exp\left(\alpha \mathbf{x}^2 - \mathbf{x}^2 - \frac{\mathbf{x}^2}{\alpha + \beta - 1}\right). \end{aligned}$$

In order to arrange that this function is constant one, we choose

$$c = \sqrt{\frac{1 - \alpha - \beta}{\pi}} \quad \text{and} \quad \beta = -\frac{\alpha(2 - \alpha)}{1 - \alpha}. \tag{5.25}$$

In order for the above Gaussian integral to converge, we need to ensure that $1 - \alpha - \beta > 0$. In view of the formula

$$1 - \alpha - \beta = \frac{1}{1 - \alpha},$$

this inequality holds because we chose $\alpha < 1$. Our finding is summarized as follows.

Proposition 5.4 *For any $\alpha < 1$, we let f and h be the Gaussians (5.23) and (5.24) with c and β according to (5.25). Then the measure $d\mu = h d\mathbf{x}$ is a minimizer of the causal action corresponding to the Lagrangian (5.22) within the class of variations of finite volume. It is the unique minimizer within the class of variations (5.18).*

As a concrete example, we consider the well-known *Mehler kernel* (see for example [19, Section 1.5])

$$E(\mathbf{x}, \mathbf{y}) = \frac{1}{\sqrt{1 - \rho^2}} \exp\left(-\frac{\rho^2(\mathbf{x}^2 + \mathbf{y}^2) - 2\rho\mathbf{xy}}{(1 - \rho^2)}\right)$$

with $\rho > 0$. Rescaling \mathbf{x} and \mathbf{y} according to

$$\mathbf{x}, \mathbf{y} \rightarrow \sqrt{\frac{1 - \rho^2}{\rho}} \mathbf{x}, \sqrt{\frac{1 - \rho^2}{\rho}} \mathbf{y},$$

the Mehler kernel becomes

$$E(\mathbf{x}, \mathbf{y}) = \frac{1}{\sqrt{1 - \rho^2}} \exp\left(-\rho(\mathbf{x}^2 + \mathbf{y}^2) - 2\mathbf{xy}\right).$$

This kernel is of the desired form (5.22) if we choose

$$\alpha = 1 - \rho < 1, \quad \beta = \frac{\rho^2 - 1}{\rho}.$$

We finally remark that this non-homogeneous example can be used as the starting point for the construction of higher-dimensional examples with minimizing measures supported on lower-dimensional subsets, exactly as explained for the Gaussian in Sect. 5.2.

6 Example: static causal fermion systems

In view of the physical applications, the most important example of a causal variational principle is the causal action principle for causal fermion systems. In this context, the methods developed in the present paper should apply to *static* systems. This can be understood in general terms as follows. As worked out in detail in [3], the linearized field equations can be analyzed with energy methods inspired from the theory of hyperbolic PDEs. If one considers time-independent solutions, hyperbolic equations (like the wave equation) give rise to corresponding elliptic equations (like the Poisson equation). This suggests that, in the static situation, the linearized field equations should be analyzed with methods from elliptic PDEs. As already mentioned in the introduction, this analogy was our starting point for developing the methods in this paper.

Before introducing the static setting, we explain why the methods developed in the present paper do *not* apply to the causal action principle in the *time-dependent setting*. A simple explanation is that elliptic methods are not suitable for solving hyperbolic equations. On a more technical level, the reasons are more involved. Apart from regularity assumptions and technicalities, the main restriction for our methods to apply is that the second variation of the Lagrangian (3.4) should be finite and integrable, in the sense that the weight function (3.7) be finite almost everywhere. These conditions are harder to fulfill in the time-dependent setting, because the norm in (3.4) may be singular on the light cone. Another problem is related to the kernel of the Laplacian Δ_N . A hyperbolic equation (like the scalar wave equation) typically has a large kernel (namely, all homogeneous equations like plane scalar waves). In our treatment with Sobolev-like function spaces, we always assume that the inhomogeneity is orthogonal to this kernel, and the constructed solution is also orthogonal to this kernel. Such a treatment, even if it applies mathematically, does not seem useful if the kernel of Δ_N is too large. This is why we do not expect the methods in this paper to be helpful for analyzing time-dependent situations. Instead, it is preferable to use the hyperbolic methods developed in [3].

Static causal fermion systems were first considered in [16]. We here present a somewhat simpler setting where the constraints are built in right from the beginning.

6.1 Causal fermion systems with fixed local trace

We begin with the general definition of a causal fermion system with fixed local trace.

Definition 6.1 Given a separable complex Hilbert space \mathcal{H} with scalar product $\langle \cdot | \cdot \rangle_{\mathcal{H}}$ and a parameter $n \in \mathbb{N}$ (the *spin dimension*), we let $\mathcal{F} \subset L(\mathcal{H})$ be the set of all symmetric operators A on \mathcal{H} of finite rank which have trace one,

$$\mathrm{tr} A = 1, \tag{6.1}$$

and which (counting multiplicities) have at most n positive and at most n negative eigenvalues. On \mathcal{F} we are given a positive measure ρ (defined on a σ -algebra of subsets of \mathcal{F}). We refer to $(\mathcal{H}, \mathcal{F}, \rho)$ as a *causal fermion system with fixed local trace*.

On \mathcal{F} we consider the topology induced by the operator norm

$$\|A\| := \sup \{ \|Au\|_{\mathcal{H}} \text{ with } \|u\|_{\mathcal{H}} = 1 \}.$$

Spacetime M is defined to be the support of this measure,

$$M := \text{supp } \rho \subset \mathcal{F}.$$

It is a topological space (again with the topology induced by the operator norm). The fact that the spacetime points are operators gives rise to many additional structures which are *inherent* in the sense that they only use information already encoded in the causal fermion system. A detailed treatment can be found in [5, Section 1.1].

6.2 The reduced causal action principle

In order to single out the physically admissible causal fermion systems, one must formulate physical equations. To this end, we impose that the measure ρ should be a minimizer of the causal action principle, which we now introduce. For any $x, y \in \mathcal{F}$, the product xy is an operator of rank at most $2n$. However, in general it is no longer symmetric because $(xy)^* = yx$, and this is different from xy unless x and y commute. As a consequence, the eigenvalues of the operator xy are in general complex. We denote the nontrivial eigenvalues counting algebraic multiplicities by $\lambda_1^{xy}, \dots, \lambda_{2n}^{xy} \in \mathbb{C}$ (more specifically, denoting the rank of xy by $k \leq 2n$, we choose $\lambda_1^{xy}, \dots, \lambda_k^{xy}$ as all the non-zero eigenvalues and set $\lambda_{k+1}^{xy}, \dots, \lambda_{2n}^{xy} = 0$). Given a parameter $\kappa > 0$ (which will be kept fixed throughout this paper), we introduce the κ -Lagrangian and the causal action by

$$\begin{aligned} \kappa\text{-Lagrangian : } \mathcal{L}(x, y) &= \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 + \kappa \left(\sum_{j=1}^{2n} |\lambda_j^{xy}| \right)^2 \\ \text{causal action : } \mathcal{S}(\rho) &= \iint_{\mathcal{F} \times \mathcal{F}} \mathcal{L}(x, y) d\rho(x) d\rho(y). \end{aligned}$$

The *reduced causal action principle* is to minimize \mathcal{S} by varying the measure ρ under the

$$\text{volume constraint } \rho(\mathcal{F}) = \text{const.}$$

This variational principle is obtained from the general causal action principle as introduced in [5, §1.1.1] as follows. Using that minimizing measures are supported on operators of constant trace (see [5, Proposition 1.4.1]), we may fix the trace of the operators and leave out the trace constraint. Moreover, by rescaling all the operators according to $x \rightarrow \lambda x$ with $\lambda \in \mathbb{R}$, we may assume without loss of generality that this trace is equal to one (6.1). Next, the κ -Lagrangian arises when treating the so-called boundedness constraint with a Lagrange multiplier term. Here we slightly simplified the setting by combining this Lagrange multiplier term with the Lagrangian.

6.3 Static causal fermion systems

We now specialize our setting to the static case. Adapting the causal action principle to static causal fermion systems and imposing a regularity condition, we will then get into the setting of causal variational principles (Sects. 6.4 and 6.5).

Definition 6.2 Let $(\mathcal{U}_t)_{t \in \mathbb{R}}$ be a strongly continuous one-parameter group of unitary transformations on the Hilbert space \mathcal{H} (i.e. $s\text{-}\lim_{t' \rightarrow t} \mathcal{U}_{t'} = \mathcal{U}_t$ and $\mathcal{U}_t \mathcal{U}_{t'} = \mathcal{U}_{t+t'}$). The causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ is *static with respect to* $(\mathcal{U}_t)_{t \in \mathbb{R}}$ if it has the following properties:

- (i) Space-time $M := \text{supp } \rho \subset \mathcal{F}$ is a topological product,

$$M = \mathbb{R} \times N \quad \text{or} \quad M = S^1 \times N. \tag{6.2}$$

We write a space-time point $x \in M$ as $x = (t, \mathbf{x})$ with $t \in \mathbb{R}$ and $\mathbf{x} \in N$, where in the case $M = S^1 \times N$ we identify the circle with the unit interval via

$$S^1 \simeq \mathbb{R} \text{ mod } \mathbb{Z}.$$

- (ii) The one-parameter group $(\mathcal{U}_t)_{t \in \mathbb{R}}$ leaves the measure ρ invariant, i.e.

$$\rho(\mathcal{U}_t \Omega \mathcal{U}_t^{-1}) = \rho(\Omega) \quad \text{for all } \rho\text{-measurable } \Omega \subset \mathcal{F}.$$

Moreover,

$$\mathcal{U}_{t'}(t, \mathbf{x}) \mathcal{U}_{t'}^{-1} = (t + t', \mathbf{x})$$

(where in the case $M = S^1 \times N$ the sum $t + t'$ is taken modulo \mathbb{Z}).

The two cases in (6.2) are referred to as spacetimes of *finite* and *infinite lifetime*, respectively. Here we can treat both cases together.

Given a static causal fermion system, we also consider the set of operators

$$N := \{(0, \mathbf{x})\} \subset \mathcal{F}.$$

The measure ρ induces a measure μ on N defined by

$$\mu(\Omega) := \rho([0, 1] \times \Omega).$$

The fact that the causal fermion system is static implies that $\rho([t_1, t_2] \times \Omega) = (t_2 - t_1) \mu(\Omega)$, valid for all $t_1 < t_2$. This can be expressed more conveniently as

$$d\rho = dt d\mu.$$

6.4 The causal action principle in the static setting

The causal action principle can be formulated in a straightforward manner for static causal fermion systems. The only point to keep in mind is that, when considering families of measures, these measures should all be static with respect to the same group $(\mathcal{U}_t)_{t \in \mathbb{R}}$ of unitary operators (see Definition 6.2). In order to make this point clear, right from the beginning we choose a group of unitary operators $(\mathcal{U}_t)_{t \in \mathbb{R}}$ on \mathcal{H} . We denote the equivalence classes of \mathcal{F} under the action of the one-parameter group by

$$\mathcal{G} := \mathcal{F}/\mathbb{R} := \{\mathcal{U}_t x \mathcal{U}_t^{-1} \mid x \in \mathcal{F}, t \in \mathbb{R}\}.$$

We denote the elements of \mathcal{F}/\mathbb{R} just as the spatial points by \mathbf{x} and \mathbf{y} . Next, we define the following functions:

$$\text{static } \kappa\text{-Lagrangian } \mathcal{L}(\mathbf{x}, \mathbf{y}) := \int_I \mathcal{L}((0, \mathbf{x}), (t, \mathbf{y})) dt, \tag{6.3}$$

where I is a generalized interval chosen in infinite lifetime as $I = \mathbb{R}$, and in finite lifetime as $I = [0, 1)$. Note that, for ease in notation, we use the same symbol for the static as for the original Lagrangian. But they can be distinguished by their arguments, because the static Lagrangian depends on spatial points in \mathcal{G} whereas the original Lagrangian is defined on space-time operators in \mathcal{F} . For a measure ρ which is static with respect to $(\mathcal{U}_t)_{t \in \mathbb{R}}$, we introduce the

$$\text{static causal action } \mathcal{S}(\mu) = \int_{\mathcal{F}/\mathbb{R}} d\mu(\mathbf{x}) \int_{\mathcal{F}/\mathbb{R}} d\mu(\mathbf{y}) \mathcal{L}(\mathbf{x}, \mathbf{y}).$$

The *static causal action principle* is to minimize \mathcal{S} by varying the measure μ within the class of regular Borel measures on \mathcal{F}/\mathbb{R} under the

$$\text{volume constraint } \mu(\mathcal{F}/\mathbb{R}) = \text{const.}$$

6.5 The regular setting as a causal variational principle

We now explain how to get to the setting of causal variational principles introduced in Sect. 2.1. In order to give the set of operators a manifold structure, we assume that ρ is *regular* in the sense that all operators in its support have exactly n positive and exactly n negative eigenvalues. This leads us to introduce the set \mathcal{F}^{reg} as the set of all linear operators F on \mathcal{H} with the following properties:

- (i) F is selfadjoint, has finite rank and (counting multiplicities) has exactly n positive and n negative eigenvalues.
- (ii) The trace is constant, i.e. $\text{tr}(F) = c > 0$ (with c independent of $F \in \mathcal{F}^{\text{reg}}$).

At this point, one must distinguish the cases that \mathcal{H} is finite- or infinite-dimensional. In the *infinite-dimensional setting*, the set \mathcal{F}^{reg} is an infinite-dimensional Banach manifold (for details see [15]). In order to get into the setting of causal variational principles with a locally compact manifold \mathcal{G} , one must restrict attention to a finite-dimensional submanifold of \mathcal{F}^{reg} . Clearly, this submanifold must contain the supports of both measures ρ and all its considered variations, and the unitary group $(\mathcal{U}_t)_{t \in \mathbb{R}}$ must map the submanifold to itself. Moreover, the vector fields of the jets needed for the analysis must all be tangential to this submanifold. Then we can simply choose \mathcal{G} as the equivalence classes of this submanifold under the action of the group $(\mathcal{U}_t)_{t \in \mathbb{R}}$. One also needs to verify that the resulting static κ -Lagrangian (6.3) has the properties (i)–(iv) in Sect. 2.1.

The *finite-dimensional setting* is considerably easier. In this case, the set \mathcal{F}^{reg} has a smooth manifold structure (see the concept of a flag manifold in [20] or the detailed construction in [11, Section 3]). Assuming that the action of the group $(\mathcal{U}_t)_{t \in \mathbb{R}}$ on \mathcal{F}^{reg} is proper and has no fixed points, the quotient

$$\mathcal{G} := \mathcal{F}^{\text{reg}}/\mathbb{R}$$

is again a manifold. In this way, we get into the setting of causal variational principles as introduced in Sect. 2.1. The resulting static κ -Lagrangian indeed has the properties (i)–(iv) in Sect. 2.1 (for details see [16, Section 3.3]).

7 Conclusion and outlook

In the present paper we showed that linearized fields of causal variational principles can be analyzed with functional analytic tools in suitable function spaces. This opens the door to the analysis of corresponding nonlinear equations, in particular of the restricted EL equations (2.12). For the explicit analysis one can proceed perturbatively as worked out in general in [8]. But one can also use and adapt methods of nonlinear elliptic partial differential equations. In particular, it seems a promising strategy to analyze nonlinear equations with fixed point methods and suitable a-priori estimates.

Appendix: The kernel of the Laplacian

The kernel of the Laplacian Δ_N plays an important role for the solvability of the linearized field equations (4.3). Namely, since the operator Δ_N is symmetric, its image is the orthogonal complement of its kernel. Therefore, one sees immediately from (4.3) that the weak linearized field equations do not admit solutions unless the inhomogeneity \hat{w} is in the orthogonal complement of the kernel of Δ_N . This is reflected in our general existence results by the fact that the vectors in the weighted Hilbert spaces \mathfrak{h}^k for $k > 0$ are in the orthogonal complement of the kernel of the operator Δ_N [see (4.6)]. With this in mind, it is an important task to analyze this kernel.

In this appendix we show that, under additional smoothness assumptions, there is indeed an explicit class of jets which are in the kernel. We also show how these jets can be treated when constructing inhomogeneous solutions. We need to make the following assumptions.

Definition A.1 The support $N := \text{supp } \mu$ has a *smooth manifold structure* if the following conditions hold:

- (i) N is a k -dimensional smooth, oriented and connected submanifold of \mathcal{F} . Equipped with a smooth atlas, we also denote it by N .
- (ii) In a chart (x, U) of N , the measure μ is absolutely continuous with respect to the Lebesgue measure with a smooth, strictly positive weight function,

$$d\mu = h(x) d^k x \quad \text{with } h \in C^\infty(N, \mathbb{R}^+).$$

Let $v \in \Gamma(N, TN)$ be a vector field. Then its *divergence* $\text{div } v \in C^\infty(N, \mathbb{R})$ may be defined by the relation

$$\int_N \text{div } v \eta(x) d\mu = - \int_N D_v \eta(x) d\mu(x),$$

to be satisfied by all test functions $\eta \in C_0^\infty(N, \mathbb{R})$. In a local chart (x, U) , the divergence is computed by

$$\text{div } v = \frac{1}{h} \partial_j (h v^j)$$

(where, following the Einstein summation convention, we sum over $j = 1, \dots, k$).

When integrating by parts using the Gauß divergence theorem, we need to make sure that we do not get boundary values at infinity. To this end, it is convenient to choose the Riemannian metric g_x introduced before (3.1) to be compatible with the smooth manifold structure in the following sense.

Definition A.2 The Riemannian metric g on Γ_x is *adapted at infinity* if there is a sequence $(\eta_n)_{n \in \mathbb{N}}$ of compactly supported functions, $\eta_n \in C_0^\infty(N, \mathbb{R})$, with the following properties:

- (i) The functions η_n are non-negative, monotone increasing and exhaust N in the sense that for any compact set $K \subset N$ there is N with $\eta_n|_K \equiv 1$ for all $n \geq N$.
- (ii) The derivatives tend uniformly to zero, i.e.

$$\lim_{n \rightarrow \infty} \sup_{x \in N} \|D\eta_n\|_x = 0,$$

where $\|\cdot\|_x$ is the norm on $T_x N \subset \Gamma_x$ induced by the Riemannian metric g .

This definition poses an implicit condition on the behavior of the Riemannian metric g at infinity. A typical example is that N has one asymptotic end in the sense that there is a compact subset $K \subset N$ such that $N \setminus K$ is diffeomorphic to \mathbb{R}^k minus a closed ball. In this case, one can choose the Riemannian metric g outside K as the pull-back of the Euclidean metric, and extend it smoothly to N . Then one can choose the η_n as the pull-back of cutoff functions in \mathbb{R}^k which fall off for example in annular regions $B_{2n} \setminus B_n$.

Definition A.3 An *inner solution* is a jet \mathfrak{v} of the form

$$\mathfrak{v} = (\operatorname{div} v, v) \quad \text{with } v \in \Gamma(N, TN).$$

We make the following regularity and decay assumptions:

- (i) The vector field v can be extended to a vector field $\tilde{v} \in \Gamma(U, T\mathcal{F})$ defined in a neighborhood U of N such that the directional derivative $(D_{1, \tilde{v}} + D_{2, \tilde{v}})\mathcal{L}(x, y)$ exists for all $x \in U$ and $y \in N$ and is integrable in y , i.e.

$$\int_N \left| (D_{1, \tilde{v}} + D_{2, \tilde{v}})\mathcal{L}(x, y) \right| d\mu(y) < \infty \quad \text{for all } x \in U.$$

Moreover, the directional derivative $D_{\tilde{v}}\ell(x)$ exists for all $x \in U$ and is continuous in U .

- (ii) The integral

$$\int_N \mathcal{L}(x, y) \|\mathfrak{v}(y)\|_y d\mu(y)$$

is finite and bounded locally uniformly in a neighborhood of N (where $\|\cdot\|_y$ is again the norm corresponding to the scalar product (3.1) and the Riemannian metric is adapted at infinity according to Definition A.2).

- (iii) For any test jet $\mathfrak{u} \in \mathfrak{J}_\mu^{\text{test}}$, the directional derivative $D_v \mathfrak{u}$ (computed in the same charts used for computing the higher derivatives in Definition 2.1) is again in $\mathfrak{J}_\mu^{\text{test}}$.

The vector space of all inner solutions is denoted by $\mathfrak{J}_\mu^{\text{in}}$.

Note that (i) implies that every inner solution is in $\mathfrak{J}_\mu^1 \cap \mathfrak{J}_\mu^{\text{diff}}$ [see (2.10) and Definition 2.1]. The name “inner solution” is justified by the following lemma:

Lemma A.4 *Let μ be a critical measure [meaning that the EL equations (2.5) hold]. Then every inner solution $\mathfrak{v} \in \mathfrak{J}_\mu^{\text{in}}$ is a solution of the linearized field equations, i.e.*

$$\langle \mathfrak{u}, \Delta \mathfrak{v} \rangle|_N = 0 \quad \text{for all } \mathfrak{u} \in \mathfrak{J}_\mu^{\text{test}}.$$

The proof is given in [10, Lemma 3.3].

We next show that for any function a on N one can find an inner solution whose scalar component coincides with a . If N were compact, the analogous statement would be the infinitesimal version of Moser’s theorem (see for example [21, Section XVIII, §2]). Here we give a detailed proof if N is non-compact, based on [21, Theorem 1.2 in Section XVIII].

Proposition A.5 *Assume that N has a smooth manifold structure and that the Riemannian metric g on Γ_x is adapted at infinity (see Definitions A.1 and A.2). Then to any $a \in C^\infty(N)$ we can find a vector field $u \in \Gamma(N, TN)$ with $\operatorname{div} u = a$.*

Proof We again consider the functions η_n in Definition A.2. By construction, these functions η_n are monotone increasing and exhaust N in the sense that for any compact set $K \subset N$ there is N with $\eta_n|_K \equiv 1$ for all $n \geq N$. Next, we introduce the non-negative functions

$$\phi_n := \begin{cases} \eta_1 & \text{if } n = 1 \\ n - n + 1 & \text{if } n > 1. \end{cases}$$

Possibly by leaving out some of these functions we can arrange that none of the ϕ_n is identically equal to zero. Then all ϕ_n are in $C_0^\infty(N, \mathbb{R})$ and form a locally finite partition of unity. Moreover, the sets N_n defined by

$$N_n := \begin{cases} \operatorname{supp} \phi_1 & \text{if } n = 1 \\ \operatorname{supp} \phi_n \cup \operatorname{supp} \phi_{n-1} & \text{if } n > 1. \end{cases}$$

form a locally finite covering of N .

We adapt the proof of [16, Lemma 2.7] to our setting. We express the measure μ using a volume form $\psi \in \Lambda^k(N)$, i.e.:

$$\mu(U) = \int_U \psi \quad \text{for all compact } U \subseteq N.$$

Similarly, we can also find a volume form $\omega \in \Lambda^k(N)$ representing $a\mu$:

$$\int_U a(x) d\mu(x) = \int_U \omega, \quad \text{for all compact } U \subseteq N.$$

Now choose $c_1 \in \mathbb{R}$ such that:

$$0 = \int_N (\phi_1 \omega - c_1 \phi_1 \psi) = \int_{N_1} (\phi_1 \omega - c_1 \phi_1 \psi).$$

Then, in view of [21, XVIII, Theorem 1.2] applied to N_1 (which, due to our assumptions, is a connected oriented manifold without boundary), there exists a compactly supported $(k - 1)$ -form $\eta_1 \in \Lambda_0^{(k-1)}(N_1)$ with

$$(\phi_1 \omega - c_1 \phi_1 \psi)|_{N_1} = d\eta_1.$$

Clearly, we can extend η_1 by zero to N .

Now for any $n \in \mathbb{N}$, inductively choose $c_{n+1} \in \mathbb{R}$ such that

$$\int_N (\phi_{n+1} \omega + c_n \phi_n \psi - c_{n+1} \phi_{n+1} \psi) = \int_{N_{n+1}} (\phi_{n+1} \omega + c_n \phi_n \psi - c_{n+1} \phi_{n+1} \psi) = 0.$$

Now [21, XVIII, Theorem 1.2] again provides a $(k - 1)$ -form $\eta_{n+1} \in \Lambda_0^{(k-1)}(N_{n+1})$ such that

$$\phi_{n+1} \omega + c_n \phi_n \psi - c_{n+1} \phi_{n+1} \psi = d\eta_{n+1},$$

where we again extended η_{n+1} by zero to N . For ease in notation, from now on by η_n we always mean the smooth extensions to N . Then each η_n has compact support in N_n .

As $(N_n)_{n \in \mathbb{N}}$ is locally finite, the series $\eta := \sum_{n=1}^\infty \eta_n$ converges and can be computed to be

$$\begin{aligned} d\eta &= \sum_{n=1}^\infty d\eta_n = \phi_1 \omega - c_1 \phi_1 \psi + \sum_{n=1}^\infty (\phi_{n+1} \omega + c_n \phi_n \psi - c_{n+1} \phi_{n+1} \psi) \\ &= -c_1 \phi_1 \psi + \omega \sum_{n=1}^\infty \phi_n + \sum_{n=1}^\infty (c_n \phi_n \psi - c_{n+1} \phi_{n+1} \psi). \end{aligned}$$

The last sum is telescopic, giving $c_1 \phi_1 \psi$, which cancels the first summand in the last line. Furthermore, as $(\phi_n)_{n \in \mathbb{N}}$ is a partition of unity, the second summand in the last line reduces to ω . We conclude that

$$d\eta = \omega.$$

It remains to find a vector field $u \in \Gamma(N, TN)$ such that $\text{div } u d\mu = d\eta$. This can be done just as described in [16, proof of Lemma 2.7]: By a conformal transformation of the metric g we arrange that the corresponding volume form coincides with μ . Then we set

$$u^\alpha := \hat{g}^{\alpha\beta} (\star \eta)_\beta$$

with the Hodge star $\star : \Lambda^{(k-1)}(N) \rightarrow \Lambda^1(N)$. Then u has the desired property (up to a sign depending on the dimension of N). □

Let us discuss the significance of the inner solutions. As already mentioned in Sect. 5.1 in a simple example, inner solutions can be regarded as infinitesimal generators of transformations of N which leave the measure μ unchanged. Similar to gauge transformations, they describe infinitesimal symmetries of the system. Using these symmetry transformations, one can simplify the form of the jet spaces. For example, under suitable assumptions we can remove the scalar components of linearized solutions, as we now explain. Let $\mathbf{v} = (b, v) \in \mathfrak{J}^{\text{vary}}$. Then, according to (3.2), the scalar component b has compact support. We choose $a = -b$. Applying Proposition A.5, there is a smooth vector field u with $\text{div } u = a$. If a has mean zero,

$$\int_N a(\mathbf{x}) d\mu(\mathbf{x}) = 0,$$

then u can be chosen to be also compactly supported (as one sees immediately from the above proof or alternatively in [21, XVIII, Theorem 1.2]). Otherwise, the vector field u typically decays at infinity, but is not compactly supported. As a simple example, in the case of one asymptotic end introduced after Definition A.2, the vector field hu is divergence-free with respect to the Euclidean metric. Applying the Gauss divergence theorem, the flux of this vector field through a sphere of radius r does not depend on r , making it possible to arrange the decay rate

$$(hu)(\mathbf{x}) \lesssim \frac{1}{\|\mathbf{x}\|^{k-1}}$$

(where $\|\cdot\|$ is the Euclidean norm). As a consequence, the integral in (3.6) is finite if $k \geq 3$ and $d\mu$ goes over asymptotically to the Lebesgue measure. More generally, here we assume for simplicity that the resulting jet $\mathbf{u} := (a, u)$ has a finite adapted norm, $\|\mathbf{u}\| < \infty$. Then we can redefine $\mathfrak{J}^{\text{vary}}$ as all the jets $\mathbf{v} + \mathbf{u}$. In contrast to (3.2), the vector components of

the resulting jets are no longer compactly supported, but the adapted norm is still finite, making sure that our existence proof still goes through. As a result of this procedure, the scalar components of all linearized solutions vanish. As a further advantage, the kernel of the operator Δ_N has become smaller.

We finally point out that, even after the above procedure, in typical examples the Laplacian Δ_N still has a non-trivial kernel. For example, every compactly supported divergence-free vector field v gives rise to an inner solution $(0, v)$ with vanishing scalar component, which lies in the kernel of the Laplacian. In order to mod out the corresponding infinitesimal symmetry transformations in the construction of inhomogeneous linearized field equation, we set up the Hilbert space formulation in such a way that the inhomogeneity \hat{w} must be in the orthogonal complement of the kernel of Δ_N . Likewise, the constructed solution is orthogonal to the kernel. In this way, we can solve the linearized field equations modulo vectors in the kernel of the Laplacian.

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