Propiedades de algunos sistemas de polinomios ortogonales Sobolev en varias variables

Properties of some Sobolev orthogonal polynomial systems in several variables

Omar Salazar Morales

Universidad Nacional de Colombia
Facultad de Ciencias
Departamento de Matemáticas
Bogotá, Colombia

February, 2022

Propiedades de algunos sistemas de polinomios ortogonales Sobolev en varias variables

Properties of some Sobolev orthogonal polynomial systems in several variables

Omar Salazar Morales

Tesis presentada como requisito parcial para optar al título de
Doctor en Ciencias Matemáticas

A thesis submitted in fulfillment of the requirements for the Ph.D. degree in
Mathematical Sciences

Director/Advisor
Herbert Alonso Dueñas Ruiz, Ph.D.

Universidad Nacional de Colombia
Facultad de Ciencias
Departamento de Matemáticas
Bogotá, Colombia
February, 2022

This thesis was evaluated and accepted by the following reviewers

---

Teresa E. Pérez
IMAG - Instituto de Matemáticas y Departamento de Matemática Aplicada, Universidad de Granada, Spain
tperez@ugr.es

Cleonice F. Bracciali
Department of Mathematics, UNESP - Universidade Estadual Paulista, Brasil
cleonice.bracciali@unesp.br

Andrei Martinez-Finkelshtein
Baylor University, United States
A_Martinez-Finkelshtein@baylor.edu

Bogotá, Colombia
May, 2022

Acknowledgment

I would like to thank all the valuable support from my advisor, Herbert A. Dueñas[1], who greatly contributed to improve the ideas presented in this thesis. His comments and contributions during the mathematical seminars inside of our research group[2] clarified many aspects of my study on Sobolev polynomials in several variables.

I would like to thank to Miguel A. Piñar[3], for all his valuable ideas concerning polynomials on product domains during the *UN Encuentro de Matemáticas 2018 (UNEMAT 2018)* that was carried out in Bogotá, Colombia.

I would like to thank to Héctor E. Pijeira[4] for extending me an invitation to attend to the online session of the *Seminario Iberoamericano de Análisis Matemático y Matemática Aplicada (GAMMA)*[5] that was carried out on december, 2021.

Also, i would like to thank to Luis E. Garza[6] for all his valuable recommendations in preparing the dissertation of this thesis, during his visit to Bogotá on may, 2022.

Finally, i would like to thank to the *Universidad Nacional de Colombia*[7] for all the resources provided during all these four years (2017–2021), and thanks to all my friends and relatives.

Omar Salazar Morales
omsalazarmo@unal.edu.co
Bogotá, Colombia
May, 2022

[1]Departamento de Matemáticas. Universidad Nacional de Colombia, sede Bogotá. Contact e-mail: haduenasr@unal.edu.co

[2]Grupo de investigación en polinomios ortogonales y aplicaciones.

[3]Departamento de Matemática Aplicada. Universidad de Granada. Spain. Contact e-mail: mpinar@ugr.es

[4]Departamento de Matemáticas. Universidad Carlos III de Madrid. Spain. Contact e-mail: hpijeira@math.uc3m.es

[5]Website: http://pijeira.com/gamma/

[6]Universidad de Colima, México. Contact e-mail: garzaleg@gmail.com

[7]Website: https://unal.edu.co/

This thesis is dedicated to
my godfather,
my mother,
my wife,
my brothers and sisters,
my cats and dogs.

Abstract

In this work we study some algebraic and analytical properties of the orthogonal polynomials in d real variables $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ with respect to the continuous-discrete Sobolev inner product:

$$\langle f, g \rangle_S = c \int_\Omega \nabla^\kappa f(\mathbf{x}) \cdot \nabla^\kappa g(\mathbf{x}) W(\mathbf{x}) d\mathbf{x} + \sum_{i=0}^{\kappa-1} \lambda_i \nabla^i f(\mathbf{p}) \cdot \nabla^i g(\mathbf{p}),$$

where W is a non-negative weight function on the domain $\Omega \subseteq \mathbb{R}^d$; $\lambda_i > 0$ for $i = 0, 1, \ldots, \kappa - 1$, $\kappa \in \mathbb{N}$; $\mathbf{p} = (p_1, p_2, \ldots, p_d)$ is a given point in $\mathbb{R}^d$; $\nabla^i f$, $i = 0, 1, \ldots, \kappa$, is a column vector of size d^i which contains all the partial derivatives of order i of f; and c is the normalization constant of W:

$$c = \left(\int_\Omega W(\mathbf{x}) d\mathbf{x} \right)^{-1}.$$

We consider the Sobolev polynomials on different domains, namely: a product domain $\Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$, where $[a_i, b_i]$, $1 \leq i \leq d$, is an interval of the real line; the unit ball $\Omega = \mathbb{B}^d := \left\{ \mathbf{x} \in \mathbb{R}^d : x_1^2 + x_2^2 + \cdots + x_d^2 \leq 1 \right\}$; the simplex $\Omega = \mathbb{T}^d := \left\{ \mathbf{x} \in \mathbb{R}^d : x_1 \geq 0, \ldots, x_d \geq 0, x_1 + x_2 + \cdots + x_d \leq 1 \right\}$; and the cone $\Omega = \mathbb{V}_\vartheta^d := \left\{ \mathbf{x} \in \mathbb{R}^d : x_1^2 + x_2^2 + \cdots + x_{d-1}^2 \leq x_d^2, 0 \leq x_d \leq \vartheta \right\}$, $0 < \vartheta \leq \infty$. Our main results consist of an iterative method for constructing the polynomials with respect to $\langle \cdot, \cdot \rangle_S$, properties that involve the main (continuous) part of this inner product, a connection formula, and some results on partial differential equations. In order to illustrate our main ideas, at the end of this work we present some numerical examples in two variables. In addition, we discuss some open problems.

Key words: orthogonal polynomials, Sobolev polynomials, polynomials in several variables, inner products, Sobolev inner products, differential equations, partial differential equations.

Mathematics Subject Classification (2020): 33C45, 33C47, 33C50, 42C05

Resumen

En este trabajo estudiamos algunas propiedades algebraicas y analíticas de los polinomios ortogonales en d variables reales $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ con respecto al producto interno Sobolev continuo-discreto:

$$\langle f, g\rangle_S = c\int_\Omega \nabla^\kappa f(\mathbf{x}) \cdot \nabla^\kappa g(\mathbf{x}) W(\mathbf{x}) d\mathbf{x} + \sum_{i=0}^{\kappa-1} \lambda_i \nabla^i f(\mathbf{p}) \cdot \nabla^i g(\mathbf{p}),$$

donde W es una función de peso no negativa sobre el dominio $\Omega \subseteq \mathbb{R}^d$; $\lambda_i > 0$ para $i = 0, 1, \ldots, \kappa - 1$, $\kappa \in \mathbb{N}$; $\mathbf{p} = (p_1, p_2, \ldots, p_d)$ es un punto dado de $\mathbb{R}^d$; $\nabla^i f$, $i = 0, 1, \ldots, \kappa$, es un vector columna de tamaño d^i que contiene todas las derivadas parciales de orden i de f; y c es la constante de normalización de W:

$$c = \left(\int_\Omega W(\mathbf{x}) d\mathbf{x}\right)^{-1}.$$

Consideramos los polinomios Sobolev sobre diferentes dominios, a saber: un dominio producto $\Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$, donde $[a_i, b_i]$, $1 \le i \le d$, es un intervalo de la recta real; la bola unitaria $\Omega = \mathbb{B}^d := \left\{\mathbf{x} \in \mathbb{R}^d : x_1^2 + x_2^2 + \cdots + x_d^2 \le 1\right\}$; el simplex $\Omega = \mathbb{T}^d := \left\{\mathbf{x} \in \mathbb{R}^d : x_1 \ge 0, \ldots, x_d \ge 0, x_1 + x_2 + \cdots + x_d \le 1\right\}$; y el cono $\Omega = \mathbb{V}_\vartheta^d := \left\{\mathbf{x} \in \mathbb{R}^d : x_1^2 + x_2^2 + \cdots + x_{d-1}^2 \le x_d^2, 0 \le x_d \le \vartheta\right\}$, $0 < \vartheta \le \infty$. Nuestros principales resultados consisten en un método iterativo de construcción de los polinomios ortogonales con respecto a $\langle\cdot,\cdot\rangle_S$, propiedades que involucran su parte principal (continua), una fórmula de conexión, y algunos resultados sobre ecuaciones diferenciales parciales. Con el fin de ilustrar nuestras principales ideas, al final de este trabajo presentamos varios ejemplos numéricos en dos variables. Además, discutimos algunos problemas abiertos.

Palabras clave: polinomios ortogonales, polinomios Sobolev, polinomios en varias variables, productos internos, productos internos Sobolev, ecuaciones diferenciales, ecuaciones diferenciales parciales.

Contents

Introduction

The theory of *standard orthogonal polynomials* in one and several variables is well established and documented in several books, for example, Dunkl and Xu [44] and Szegö [111]. Properties like the three-term relation, the Favard's theorem, or the Christoffel-Darboux identity and their importance for the study of this kind of polynomials (for example, their zeros) are well known. Conversely, orthogonal polynomials with respect to inner products that involve derivatives are called *Sobolev orthogonal polynomials*. The non-standard character of these inner products makes their study more difficult, mainly because the three-term relation no longer holds. The lack of this tool has motivated the study of new tools and techniques in recent years. As a result, the theory of Sobolev polynomials is non-uniform and fragmented [86].

Sobolev orthogonal polynomials in one variable have been studied since the decade of the 60s when the first paper was published on this topic due to Althammer [7]. This first paper was motivated by an optimization problem which was proposed by Lewis [73] in the 40s. In the last 30 years a big number of publications have appeared. Some authors worked on properties like asymptotic behavior [11] and zeros [33, 34] of those polynomials. On the other hand, some applications have been considered, for example, electrostatic models [35, 40] and generalizations for higher-order derivatives [38, 95]. On the subject of differential equations, it is well-known that the classical orthogonal polynomials are eigenfunctions of a second-order differential equation. Orthogonal polynomials which are eigenfunctions of a fourth-order differential operator were classified by Krall [67] in the 40s, and some higher-order cases were studied by Koornwinder [65] and Krall [66] in the 80s. In two and several variables, a similar problem was considered by Fernández, Pérez, and Piñar [50] and Martínez and Piñar [87]. A classification of the so-called admissible equations in two variables was made by Krall and Sheffer [68] in the 60s. In the multivariate case, there are well-known results on second-order partial differential equations for which orthogonal polynomials are eigenfunctions [44]. These cases include polynomials on product domains, the unit ball, and the simplex. Most of these results in the theory of orthogonal polynomials have helped to find differential equations that are satisfied by families of Sobolev polynomials in two and several variables (see, for example, [72, 99, 113]). Therefore, a very interesting question is to find new differential operators for Sobolev polynomials derived from the existing ones. We remit the reader to a detailed survey on Sobolev orthogonal polynomials by Marcellán and Xu [86], and other references by Meijer [92] and Martínez-Finkelshtein [88, 89] who give the state of the art on this topic.

In contrast to one variable, the study of Sobolev orthogonal polynomials in several variables is most recent. The tools and techniques for studying this kind of polynomials are even fewer than in one variable. We refer some studies [23, 30, 32, 36, 94, 98, 99, 112, 113] on the unit ball $\mathbb{B}^d := \left\{\mathbf{x} \in \mathbb{R}^d : x_1^2 + x_2^2 + \cdots + x_d^2 \leq 1\right\}$, on the unit sphere $\mathbb{S}^{d-1} := \left\{\mathbf{x} \in \mathbb{R}^d : x_1^2 + x_2^2 + \cdots + x_d^2 = 1\right\}$, on the simplex [2, 114] $\mathbb{T}^d := \left\{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0, x_2 \geq 0, \ldots, x_d \geq 0, x_1 + x_2 + \cdots + x_d \leq 1\right\}$, and on a product domain $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$ [41, 49]. Most of the results were obtained in two variables where the inner products involved only first-order derivatives [18].

In the case of the ball and the sphere, several results provided an explicit basis for the spaces of orthogonal polynomials with respect to Sobolev inner products defined on $\mathbb{B}^d$ and $\mathbb{S}^{d-1}$, or they provided an approximation to functions on these domains. For example, Xu [112] constructed the orthogonal polynomials with respect to certain Sobolev inner product on $\mathbb{B}^d$ which introduced the Laplacian operator $\triangle$. This study was motivated by a paper due to Atkinson and Hansen [13], where the same inner product was found for two variables in the numerical solution of the Poisson equation $-\triangle u = f(\cdot, u)$. Xu [113], motivated by a problem related to dwell time for polishing tools in fabricating optical surfaces, constructed the orthogonal polynomials with respect to an inner product involving the gradient operator ∇ on the unit ball. Pérez, Piñar, and Xu [98] showed a similar work in this way. Piñar and Xu [99] studied a partial differential equation with Sobolev orthogonal polynomials as eigenfunctions involving the operators $\triangle$ and ∇. The approximation by polynomials on the sphere and the ball was studied by Dai and Xu [23], and asymptotic properties on $\mathbb{B}^d$ were studied in [30, 32, 36, 94].

We found just a few studies of Sobolev orthogonal polynomials on the simplex $\mathbb{T}^d$ of $\mathbb{R}^d$, and on a product domain $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$. Xu [114] considered approximation problems and orthogonality on the triangle $\mathbb{T}^2$. Aktaş and Xu [2] analyzed the orthogonal polynomials on the simplex with special attention to those on the triangle $\mathbb{T}^2$. Recently, Fernández, Marcellán, Pérez, Piñar, and Xu [49] studied the Sobolev orthogonal polynomials in two variables on the product domain $\Omega = [a_1, b_1] \times [a_2, b_2]$ with respect to the inner product:

$$\langle f, g \rangle_S = c \int_\Omega \nabla f(x, y) \cdot \nabla g(x, y) W(x, y) dx dy + \lambda f(p_1, p_2) g(p_1, p_2), \quad \lambda > 0. \quad (1)$$

Following a similar strategy, Dueñas, Pinzón-Cortés, and Salazar-Morales [41] replaced in (1) the gradient operator $\nabla = (\partial_x, \partial_y)^T$, where T represents the transpose operator, by the gradient of order two $\nabla^2 = (\partial_{xx}, \partial_{xy}, \partial_{yx}, \partial_{yy})^T$ and the corresponding Sobolev orthogonal polynomials were discussed.

In d real variables $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ our general study of the orthogonal polynomials with respect to the Sobolev inner product:

$$\langle f, g \rangle_S = c \int_\Omega \nabla^\kappa f(\mathbf{x}) \cdot \nabla^\kappa g(\mathbf{x}) W(\mathbf{x}) d\mathbf{x} + \sum_{i=0}^{\kappa-1} \lambda_i \nabla^i f(\mathbf{p}) \cdot \nabla^i g(\mathbf{p}), \quad \lambda_i > 0, \quad (2)$$

started in [42] on a product domain of the form $\Omega = [a_1, b_1] \times \cdots \times [a_d, b_d]$, where $[a_i, b_i]$ is an interval of the real line. As mentioned above, in [41, 49, 101] some particular studies for specific weights W appeared for the cases $d = 2$ and $\kappa = 1, 2, 3$. In the present work we study some algebraic and analytical properties of the orthogonal polynomials with respect to (2). We also consider other domains like the simplex $\Omega = \mathbb{T}^d$, the unit ball $\Omega = \mathbb{B}^d$, and the cone $\Omega = \mathbb{V}^d_\vartheta$. For some of these domains and specific weights we provide some results on partial differential equations. As far as we know, no other studies in several variables regarding this type of continuous-discrete inner products have been reported in literature.

In contrast, in one variable inner products of the form (2) were studied in [3, 6, 8, 47, 48, 57, 70, 71, 75, 97, 105–108, 114] to name just a few. Pérez and Piñar [97] proved that the monic generalized Laguerre polynomials, with arbitrary parameter $\alpha \in \mathbb{R}$, are orthogonal with respect to a Sobolev inner product which involved higher-order derivatives. These authors also observed that if $\alpha \in \{-1, -2, -3, \ldots\}$ then the inner product reduced to the continuous-discrete case studied by Kwon and Littlejohn [70], which had the same form as (2). Therefore, the paper [97] contains a generalization of the results in [70]. These generalized Laguerre polynomials with negative integer parameter were also studied in [47, 48] in spectral theory. Kwon and Littlejohn [71] studied some particular cases of (2) with first-order derivatives and classical weights, with additional attention to differential equations. Alfaro, Pérez, Piñar, and Rezola [6] presented a more general study of the orthogonal polynomials with respect to a bilinear form which had the same form as (2). These authors provided examples, which included the classical cases with negative parameters: Laguerre polynomials $\left\{L_n^{(-N)}(x)\right\}_{n\geq 0}$, and Jacobi polynomials $\left\{P_n^{(-N,\beta)}(x)\right\}_{n\geq 0}$ and $\left\{P_n^{(\alpha,-N)}(x)\right\}_{n\geq 0}$, with $\beta + N$ and $\alpha + N$ not being a negative integer and N being a natural number. Alfaro, Álvarez de Morales, and Rezola [3] and Álvarez de Morales, Pérez, and Piñar [8] studied the remainder cases of the Jacobi and Gegenbauer polynomials $\left\{C_n^{(-N+1/2)}(x)\right\}_{n\geq 0}$, with N a natural number. A similar study in a general setting, but only with first-order derivatives, was presented by Jung, Kwon, and Lee [57]. Li and Xu [75] and Xu [114] defined the generalized Jacobi polynomials and they proved that these polynomials are orthogonal with respect to an inner product of the form (2). These authors used some of theirs results in one variable for studying Sobolev polynomials in several variables on the unit ball and on the triangle. Sharapudinov [105–108] studied the Sobolev polynomials with respect to an inner product of the form (2) in the spectral theory for solving differential equations, with special attention to the Chebyshev, Legendre, and Gegenbauer cases.

This document is organized as follows. In Chapter 1 we present some basic background on standard orthogonal polynomials in one and several variables. In Chapter 2 we provide a state of the art on Sobolev polynomials in one and several variables. Our main results are presented in Chapter 3, namely, an iterative method for constructing the orthogonal polynomials with respect to (2), properties that involve the main (continuous) part of this inner product, a connection formula, and some results on partial differential equations. In order to illustrate our main ideas, in Chapter 4 we present some numerical examples in two variables. In addition, in Chapter 5 we state some open problems derived from the present work.

Chapter 1

Basic background

In this chapter, we introduce notation and basic background concerning standard orthogonal polynomials in one and several variables. The results in this chapter are well-known and they can be found in classical references by Chihara [21], Dunkl and Xu [44], and Szegö [111]. Complementary material is due to Abramowitz and Stegun [1], Dai and Xu [24], Duistermaat and Kolk [43], Saint Raymond [100], and Xu [115].

1.1 Notation

We use the usual symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ for the natural, integer, rational, real and complex numbers, respectively. We also denote by $\mathbb{N}_0$ the set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and by $\mathbb{R}_+$ and $\mathbb{R}_-$ the sets $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_- := \{x \in \mathbb{R} : x \leq 0\}$. All functions in this work are real valued. We use the symbols $\delta_{n,m}$ and $(x)_n$ for the Kronecker delta and the Pochhammer symbol, respectively, which are defined by:

$$\delta_{n,m} := \begin{cases} 1, & n = m, \\ 0, & n \neq m, \end{cases} \quad n, m \in \mathbb{Z},$$

$$(x)_n := \begin{cases} x(x+1)(x+2)\cdots(x+n-1), & n \geq 1, \\ 1, & n = 0, \end{cases} \quad x \in \mathbb{R}, \quad n \in \mathbb{N}_0.$$

For $n \in \mathbb{N}_0$, we denote by $n!$ the factorial of n which is given by $n! = (1)_n$. Also, we denote by $\Gamma(x)$ the gamma function which is defined by the integral:

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1} dt, \quad x > 0,$$

This function satisfies the well-known property $\Gamma(x+1) = x\Gamma(x)$, $x > 0$. And more generally, for $n \in \mathbb{N}_0$ and $x > 0$ the equation $\Gamma(x+n) = (x)_n \Gamma(x)$ holds. This last relation is used for extending the gamma function to the set $\mathbb{R}_- \setminus \{0, -1, -2, -3, \ldots\}$ by the equation $\Gamma(x) := \Gamma(x+n)/(x)_n$, $-n < x < -n+1$, $n \in \mathbb{N}$. Two well-known values of Γ are $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1) = 1$. See Abramowitz and Stegun [1, Chapter 6] for more properties.

Let $d \in \mathbb{N}$. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}_0^d$, is a d-tuple of non-negative integers α_i, we call α a multi-index for which $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_d$. We say $\alpha \leq \beta$, where α and β are both multi-indices, if $\alpha_i \leq \beta_i$ for all $i = 1, 2, \dots, d$. We denote by $\alpha!$ and $\delta_{\alpha,\beta}$ the symbols

$$\alpha! := \alpha_1!\alpha_2! \cdots \alpha_d!, \qquad \delta_{\alpha,\beta} := \delta_{\alpha_1,\beta_1}\delta_{\alpha_2,\beta_2} \cdots \delta_{\alpha_d,\beta_d},$$

and if $\beta \leq \alpha$ then

$$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2} \cdots \binom{\alpha_d}{\beta_d},$$

where

$$\binom{x}{n} := \frac{(x-n+1)_n}{n!}, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}_0.$$

If $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ then $(\mathbf{x})_\alpha$ denotes

$$(\mathbf{x})_\alpha := (x_1)_{\alpha_1}(x_2)_{\alpha_2} \cdots (x_d)_{\alpha_d}.$$

Associated with $\mathbf{x} \in \mathbb{R}^d$, for each i define by $\mathbf{x}_i$ a truncation of $\mathbf{x}$, namely,

$$\mathbf{x}_0 := 0, \quad \mathbf{x}_i = (x_1, x_2, \dots, x_i) \in \mathbb{R}^i, \quad 1 \leq i \leq d.$$

Notice that $\mathbf{x}_d = \mathbf{x}$. For two (row or column) vectors $\mathbf{x}$ and $\mathbf{y}$, we use the usual notation of $\mathbf{x} \cdot \mathbf{y}$ and $\|\mathbf{x}\|$ to denote the dot product $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^d x_i y_i$ and the Euclidean norm $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$.

If α is a multi-index, and if $\partial_i^{\alpha_i} := \partial^{\alpha_i}/\partial x_i^{\alpha_i}$ denotes the α_i-th partial derivative with respect to x_i for $1 \leq i \leq d$, then $\partial^\alpha := \partial_1^{\alpha_1}\partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}$ denotes a differential operator of order $|\alpha|$ where $\partial^{(0,0,\dots,0)}u := u$. For one variable, we use the usual symbols du/dx, d^2u/dx^2, d^3u/dx^3 (or u', u'', u''') for the first, second, third derivative with respect to x, respectively, and d^nu/dx^n (or $u^{(n)}$) for higher-order derivatives. For later use, we define the following differential operators:

$$\triangle_i := \sum_{j=1}^{i} \partial_j^2, \quad \nabla_i := \big(\partial_1, \partial_2, \dots, \partial_i\big)^T, \quad \big\langle \mathbf{x}_i^T, \nabla_i \big\rangle := \sum_{j=1}^{i} x_j \partial_j, \quad 1 \leq i \leq d,$$

where T is the transpose operator. When $i = d$, we drop the subscript and we write $\triangle := \triangle_d$, $\nabla := \nabla_d$ and $\big\langle \mathbf{x}^T, \nabla \big\rangle := \big\langle \mathbf{x}_d^T, \nabla_d \big\rangle$ instead[1]. The operators $\triangle$ and ∇ are known as *Laplacian* and *gradient*, respectively.

1.2 Orthogonal polynomials in one variable

Let $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$. We denote by x^n a monomial of degree n. A (real) polynomial p of degree n in one variable x is a finite linear combination of monomials of the form

[1]Some authors use the notation $\langle \mathbf{x}, \nabla \rangle$ when ∇ is defined to be a row vector.

$p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$, $c_n \neq 0$, where c_i, $0 \le i \le n$, is a real number. The number c_n is called the *leading coefficient*. If $c_n = 1$ the polynomial is said to be *monic*. Sometimes we use the notation $\deg p$ to denote the degree of p. Let Π denote the linear space of polynomials with real coefficients on the real line and, for $n = 0, 1, 2, \ldots$ let Π_n denote the linear subspace of polynomials of degree at most n. A basis of Π_n is the set $\{1, x, x^2, \ldots, x^n\}$, which is known as the *canonical basis*. Then $\dim \Pi_n = n + 1$.

Let $\langle \cdot, \cdot \rangle$ be a symmetric bilinear form[2] defined on Π. It is an inner product if $\langle p, p \rangle > 0$ for all non-zero polynomial $p \in \Pi$. A sequence of polynomials $\{p_n\}_{n\ge 0}$ is called an *orthogonal polynomial sequence* (OPS) with respect to $\langle \cdot, \cdot \rangle$ if:

1. $\deg p_n = n$,
2. $\langle p_n, p_m \rangle = 0$ if $n \neq m$, and
3. $\langle p_n, p_n \rangle \neq 0$, $n \ge 0$.

If $\{p_n\}_{n\ge 0}$ is an OPS for $\langle \cdot, \cdot \rangle$ and, in addition, we also have $\langle p_n, p_n \rangle = 1$, $n \ge 0$, then the sequence is said to be *orthonormal*.

Let

$$\mathbf{M}_n = \begin{pmatrix} \langle 1, 1 \rangle & \langle 1, x \rangle & \cdots & \langle 1, x^n \rangle \\ \langle x, 1 \rangle & \langle x, x \rangle & \cdots & \langle x, x^n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x^n, 1 \rangle & \langle x^n, x \rangle & \cdots & \langle x^n, x^n \rangle \end{pmatrix}, \quad n \ge 0.$$

If $\langle \cdot, \cdot \rangle$ is an inner product, then $\mathbf{M}_n$ is definite positive, that is, $\det \mathbf{M}_n > 0$ for every $n \in \mathbb{N}_0$. If $\det \mathbf{M}_n \neq 0$ for all $n \in \mathbb{N}_0$ then a sequence of monic orthogonal polynomials exists. In this case, the monic sequence is given by $p_0(x) = 1$ and

$$p_n(x) = \frac{1}{\det \mathbf{M}_{n-1}} \det \left(\begin{array}{cccc|c} & & & & \langle 1, x^n \rangle \\ & & & & \langle x, x^n \rangle \\ & \mathbf{M}_{n-1} & & & \vdots \\ & & & & \langle x^{n-1}, x^n \rangle \\ \hline 1 & x & \cdots & x^{n-1} & x^n \end{array}\right), \quad n \ge 1.$$

If the multiplication operator is a symmetric operator with respect to $\langle \cdot, \cdot \rangle$, that is,

$$\langle xp, q \rangle = \langle p, xq \rangle, \quad p, q \in \Pi, \tag{1.1}$$

then there exist constants b_n and $c_n \neq 0$ such that the monic OPS $\{p_n\}_{n\ge 0}$ satisfies the three-term recurrence relation:

$$x p_n(x) = p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \quad n \ge 0, \quad p_{-1}(x) := 0.$$

[2] In a more general setting, the theory of (real) standard orthogonal polynomials in one variable can be described in terms of a linear functional $\mathcal{L} : \Pi \mapsto \mathbb{R}$. In this work, we restrict our study to bilinear forms in order to get concrete results for particular domains and weight functions. A similar comment applies for orthogonal polynomials in several variables. See [21, 44, 111] for more details of the general theory.

On the other hand, $K_n(x,y)$, the n-th reproducing kernel associated with $\{p_n\}_{n\geq 0}$ is defined by:

$$K_n(x,y) = \sum_{k=0}^{n} \frac{p_k(x)p_k(y)}{\langle p_k, p_k\rangle}.$$

The three-term recurrence relation implies a closed formula for computing $K_n(x,y)$ in terms of p_{n+1} and p_n, the so-called Christoffel-Darboux identity:

$$K_n(x,y) = \sum_{k=0}^{n} \frac{p_k(x)p_k(y)}{\langle p_k, p_k\rangle} = \begin{cases} \dfrac{1}{\langle p_n, p_n\rangle}\dfrac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y}, & x \neq y, \\ \dfrac{p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x)}{\langle p_n, p_n\rangle}, & x = y. \end{cases}$$

Then, the three-term recurrence relation plays an important role in the study of standard orthogonal polynomials. Conversely, in the theory of Sobolev polynomials the condition (1.1), in general, is not satisfied. This fact makes the study of these non-standard polynomials more difficult, mainly because the three-term relation no longer holds. The lack of this tool has motivated the study of new techniques in recent years.

1.2.1 Classical orthogonal polynomials

These polynomials are associated with inner products that involve the following weight functions:

1. Hermite: $w(x) = e^{-x^2}$, $x \in (-\infty, \infty)$,
2. Laguerre: $w_a(x) = x^a e^{-x}$, $a > -1$, $x \in [0, \infty)$,
3. Jacobi: $w_{a,b}(x) = (1-x)^a(1+x)^b$, $a, b > -1$, $x \in [-1, 1]$.

In the last case, the well-known families of Legendre ($a = b = 0$), Tchebichef (first kind $a = b = -1/2$, second kind $a = b = 1/2$), and Gegenbauer ($a = b$) polynomials are renormalizations of the Jacobi polynomials for particular values of the parameters[3] a and b. There is another family which satisfies many properties of the classical orthogonal polynomials: the so-called *Bessel polynomials*. They are orthogonal with respect to a weight function defined on the complex plane. This case involves complex variable and integration on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Therefore, these polynomials are not considered in this work. See Chihara [21, Chapter 6] for more details.

There are several characterizations of classical orthogonal polynomials but we will present only the most basic facts concerning them. For example, the polynomials in each classical family are eigenfunctions of a second-order linear differential operator with polynomial coefficients, and also, they can be expressed by a Rodrigues' formula. For more properties see Szegö [111, Chapters 4 and 5] and Chihara [21, Chapter 5].

[3] In the references is usual to find α and β to denote the parameters of the Laguerre and Jacobi polynomials. In this work we reserve greek letters for denoting multi-indices.

Hermite polynomials The monic sequence $\{H_n(x)\}_{n\geq 0}$ of Hermite polynomials is orthogonal with respect to the inner product:

$$\langle H_n, H_m\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = \frac{n!}{2^n}\delta_{n,m}, \quad n,m = 0,1,2,3,\ldots,$$

which is normalized such that $\langle 1,1\rangle = 1$. This monic sequence satisfies the three-term recurrence relation:

$$H_{n+1}(x) = xH_n(x) - \frac{n}{2}H_{n-1}(x), \quad n\geq 1, \quad H_0(x) = 1, \quad H_1(x) = x.$$

Also, these polynomials satisfy the second-order linear differential equation:

$$y'' - 2xy' + 2ny = 0, \quad y = H_n(x), \tag{1.2}$$

that is, these polynomials are eigenfunctions of the second-order differential operator $\mathcal{H}$ given by:

$$\mathcal{H}y = -2ny, \quad \mathcal{H} := \frac{d^2}{dx^2} - 2x\frac{d}{dx}, \quad y = H_n(x). \tag{1.3}$$

For more properties and relations see Szegö [111, Section 5.5].

Laguerre polynomials The monic sequence $\left\{L_n^{(a)}(x)\right\}_{n\geq 0}$, $a > -1$, of Laguerre polynomials is orthogonal with respect to the inner product:

$$\left\langle L_n^{(a)}, L_m^{(a)}\right\rangle = \frac{1}{\Gamma(a+1)}\int_0^{\infty} L_n^{(a)}(x)L_m^{(a)}(x)x^a e^{-x}dx = n!(a+1)_n\delta_{n,m}, \quad n,m = 0,1,2,\ldots,$$

which is normalized such that $\langle 1,1\rangle = 1$. This monic sequence satisfies the three-term recurrence relation:

$$L_{n+1}^{(a)}(x) = (x-2n-a-1)L_n^{(a)}(x) - n(n+a)L_{n-1}^{(a)}(x), \quad n\geq 1,$$
$$L_0^{(a)}(x) = 1, \quad L_1^{(a)}(x) = x-a-1.$$

Also, these polynomials satisfy the second-order linear differential equation:

$$xy'' + (a+1-x)y' + ny = 0, \quad y = L_n^{(a)}(x), \tag{1.4}$$

that is, these polynomials are eigenfunctions of the second-order differential operator $\mathcal{L}_a$ given by:

$$\mathcal{L}_a y = -ny, \quad \mathcal{L}_a := x\frac{d^2}{dx^2} + (a+1-x)\frac{d}{dx}, \quad y = L_n^{(a)}(x). \tag{1.5}$$

For more properties and relations see Szegö [111, Section 5.1].

Jacobi polynomials The monic sequence $\left\{P_n^{(a,b)}(x)\right\}_{n\geq 0}$, $a, b > -1$, of Jacobi polynomials is orthogonal with respect to the inner product:

$$\begin{aligned}\left\langle P_n^{(a,b)}, P_m^{(a,b)}\right\rangle &= \frac{\Gamma(a+b+2)}{2^{a+b+1}\Gamma(a+1)\Gamma(b+1)} \int_{-1}^{1} P_n^{(a,b)}(x) P_m^{(a,b)}(x)(1-x)^a(1+x)^b dx \\ &= \frac{4^n (a+1)_n (b+1)_n}{(n+a+b+1)_n (a+b+2)_{2n}} \delta_{n,m}, \quad n, m = 0, 1, 2, 3, \dots,\end{aligned}$$

which is normalized such that $\langle 1, 1\rangle = 1$. This monic sequence satisfies the three-term recurrence relation:

$$\begin{aligned}P_{n+1}^{(a,b)}(x) &= \left(x + \frac{a^2-b^2}{(2n+a+b)(2n+a+b+2)}\right) P_n^{(a,b)}(x) \\ &- \frac{4n(a+n)(b+n)(a+b+n)}{(2n+a+b-1)(2n+a+b)^2(2n+a+b+1)} P_{n-1}^{(a,b)}(x), \quad n \geq 1, \\ &P_0^{(a,b)}(x) = 1, \quad P_1^{(a,b)}(x) = x + \frac{a-b}{a+b+2}.\end{aligned}$$

Also, these polynomials satisfy the second-order linear differential equation:

$$(1-x^2)y'' + [b-a-(a+b+2)x]y' + n(n+a+b+1)y = 0, \quad y = P_n^{(a,b)}(x), \tag{1.6}$$

that is, these polynomials are eigenfunctions of the second-order differential operator $\mathcal{J}_{a,b}$ given by:

$$\begin{aligned}\mathcal{J}_{a,b} y &= -n(n+a+b+1)y, \quad y = P_n^{(a,b)}(x), \\ \mathcal{J}_{a,b} &:= (1-x^2)\frac{d^2}{dx^2} + [b-a-(a+b+2)x]\frac{d}{dx}.\end{aligned} \tag{1.7}$$

For more properties and relations see Szegö [111, Chapter 4].

1.3 Orthogonal polynomials in several variables

Let $d \in \mathbb{N}$. If α is a multi-index and $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, we denote by $\mathbf{x}^\alpha$ the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$ which has total degree $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$. Similarly, $(\mathbf{x}-\mathbf{y})^\alpha$ denotes the shifted monomial $(x_1-y_1)^{\alpha_1}(x_2-y_2)^{\alpha_2}\cdots(x_d-y_d)^{\alpha_d}$. A polynomial P in d real variables $x_1, x_2, \dots, x_d$ is a finite linear combination of monomials in the form $P(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha$, where c_α is a real number. The total degree of P is defined as the highest degree of its monomials. We denote the linear space of polynomials in d variables by Π^d, and the subspace of polynomials of degree at most n by Π_n^d. When $d = 1$, we drop the superscript and we write Π and Π_n instead. A basis of Π_n^d is the set $\{\mathbf{x}^\alpha : |\alpha| \leq n\}$, which is known as the *canonical basis*. It is known [44] that:

$$\dim \Pi_n^d = \binom{n+d}{n}.$$

A polynomial $P \in \Pi_n^d$ is said to be monic if it is of the form $P(\mathbf{x}) = \mathbf{x}^\alpha + Q(\mathbf{x})$, with $Q \in \Pi_{n-1}^d$, $|\alpha| = n$.

A polynomial is called homogeneous if all its monomials have the same total degree. We denote by $\mathscr{P}_n^d$ the linear space of homogeneous polynomials of degree n in d variables, that is,

$$\mathscr{P}_n^d = \left\{ P \in \Pi_n^d : P(\mathbf{x}) = \sum_{|\alpha|=n} c_\alpha \mathbf{x}^\alpha \right\}.$$

A basis of $\mathscr{P}_n^d$ is the set $\{\mathbf{x}^\alpha : |\alpha| = n\}$, hence, $\dim \mathscr{P}_n^d = \#\left\{\alpha \in \mathbb{N}_0^d : |\alpha| = n\right\}$. It is known [44] that:

$$r_n^d := \dim \mathscr{P}_n^d = \binom{n+d-1}{n}.$$

Let $\langle \cdot, \cdot \rangle$ be a symmetric bilinear form defined on Π^d. It is an inner product if $\langle P, P \rangle > 0$ for all non-zero polynomial $P \in \Pi^d$. Two polynomials P and Q are said to be orthogonal to each other with respect to the bilinear form if $\langle P, Q \rangle = 0$. A polynomial P is called an orthogonal polynomial if it is orthogonal to all polynomials of lower degree, that is, if $\langle P, Q \rangle = 0$ for all $Q \in \Pi^d$ such that $\deg Q < \deg P$. We denote by $\mathscr{V}_n^d$ the linear space of orthogonal polynomials of degree exactly n with respect to $\langle \cdot, \cdot \rangle$, that is,

$$\mathscr{V}_n^d = \left\{ P \in \Pi_n^d : \langle P, Q \rangle = 0, \forall Q \in \Pi_{n-1}^d \right\}.$$

When $\langle \cdot, \cdot \rangle$ is defined in terms of a weight function W, we write $\mathscr{V}_n^d(W)$. It is known [44] that:

$$\dim \mathscr{V}_n^d = r_n^d = \binom{n+d-1}{n}.$$

Since $r_n^d = \#\left\{\alpha \in \mathbb{N}_0^d : |\alpha| = n\right\}$, it is natural to use a multi-index to index the elements of an orthogonal basis of $\mathscr{V}_n^d$. Let $\{P_\alpha^n : |\alpha| = n\}$ denote a basis of $\mathscr{V}_n^d$. If the elements of the basis are orthogonal to each other, that is, $\left\langle P_\alpha^n, P_\beta^n \right\rangle = 0$ whenever $\alpha \neq \beta$, we call the basis mutually orthogonal. A basis is said to be monic if each P_α^n is a monic polynomial. If, in addition, $\langle P_\alpha^n, P_\alpha^n \rangle = 1$, we call the basis orthonormal. It is common to write a basis $\{P_\alpha^n : |\alpha| = n\}$ of $\mathscr{V}_n^d$ as the column vector:

$$\mathbb{P}_n(\mathbf{x}) := \left(P^n_{\alpha^{(1)}}(\mathbf{x}), P^n_{\alpha^{(2)}}(\mathbf{x}), \ldots, P^n_{\alpha^{(r_n^d)}}(\mathbf{x}) \right)^T,$$

where $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(r_n^d)}$ is the arrangement of elements in $\left\{\alpha \in \mathbb{N}_0^d : |\alpha| = n\right\}$ according to the reverse lexicographical order[4]. We will say that $\{\mathbb{P}_n\}_{n\geq 0}$ is an *orthogonal polynomial system* (OPS). With this notation, the orthogonality of $\{\mathbb{P}_n\}_{n\geq 0}$

[4]Sometimes the lexicographical order is used (see [44, page 61]). In this work we use the *reverse* lexicographical order as usual in literature.

can be expressed as $\left\langle \mathbb{P}_n, \mathbb{P}_m^T \right\rangle = \mathbf{0}$ if $n \neq m$, and $\left\langle \mathbb{P}_n, \mathbb{P}_n^T \right\rangle = \mathbf{H}_n$, where $\mathbf{H}_n = \left(\left\langle P^n_{\alpha^{(i)}}, P^n_{\alpha^{(j)}} \right\rangle\right)_{i,j=1}^{r_n^d}$ is a symmetric non-singular matrix of size $r_n^d \times r_d^d$. In addition, $\mathbf{H}_n$ is positive definite if $\langle \cdot, \cdot \rangle$ is an inner product.

Let $\mathbb{X}_n := \left(\mathbf{x}^{\alpha^{(1)}}, \mathbf{x}^{\alpha^{(2)}}, \ldots, \mathbf{x}^{\alpha^{(r_n^d)}}\right)^T$ and let

$$\mathbf{M}_{n,d} = \begin{pmatrix} \left\langle \mathbb{X}_0, \mathbb{X}_0^T \right\rangle & \left\langle \mathbb{X}_0, \mathbb{X}_1^T \right\rangle & \cdots & \left\langle \mathbb{X}_0, \mathbb{X}_n^T \right\rangle \\ \left\langle \mathbb{X}_1, \mathbb{X}_0^T \right\rangle & \left\langle \mathbb{X}_1, \mathbb{X}_1^T \right\rangle & \cdots & \left\langle \mathbb{X}_1, \mathbb{X}_n^T \right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle \mathbb{X}_n, \mathbb{X}_0^T \right\rangle & \left\langle \mathbb{X}_n, \mathbb{X}_1^T \right\rangle & \cdots & \left\langle \mathbb{X}_n, \mathbb{X}_n^T \right\rangle \end{pmatrix}, \quad n \geq 0.$$

We call $\mathbf{M}_{n,d}$ a moment matrix, and its elements are $\left\langle \mathbf{x}^\alpha, \mathbf{x}^\beta \right\rangle$ for $|\alpha| \leq n$ and $|\beta| \leq n$. This matrix preserves several properties like in the univariate case. If $\langle \cdot, \cdot \rangle$ is an inner product, then $\mathbf{M}_{n,d}$ is definite positive, that is, $\det \mathbf{M}_{n,d} > 0$ for every $n \in \mathbb{N}_0$. If $\det \mathbf{M}_{n,d} \neq 0$ for all $n \in \mathbb{N}_0$ then a sequence of monic orthogonal polynomials in several variables exists. This monic sequence is given by $P^0_{\mathbf{0}}(\mathbf{x}) = 1$ and

$$P^n_\alpha(\mathbf{x}) = \frac{1}{\det \mathbf{M}_{n-1,d}} \det \left(\begin{array}{cccc|c} & & & & \langle \mathbb{X}_0, \mathbf{x}^\alpha \rangle \\ & & & & \langle \mathbb{X}_1, \mathbf{x}^\alpha \rangle \\ & \mathbf{M}_{n-1,d} & & & \vdots \\ & & & & \langle \mathbb{X}_{n-1}, \mathbf{x}^\alpha \rangle \\ \hline \mathbb{X}_0^T & \mathbb{X}_1^T & \cdots & \mathbb{X}_{n-1}^T & \mathbf{x}^\alpha \end{array}\right), \quad |\alpha| = n \geq 1.$$

As in the univariate case, if the multiplication operator is a symmetric operator with respect to $\langle \cdot, \cdot \rangle$, that is,

$$\langle x_i P, Q \rangle = \langle P, x_i Q \rangle, \quad P, Q \in \Pi^d, \quad 1 \leq i \leq d, \tag{1.8}$$

then there exist matrices $\mathbf{A}_{n,i}$, $\mathbf{B}_{n,i}$ and $\mathbf{C}_{n,i}$, of sizes $r_n^d \times r_{n+1}^d$, $r_n^d \times r_n^d$ and $r_n^d \times r_{n-1}^d$, respectively, such that $\{\mathbb{P}_n\}_{n \geq 0}$ satisfies the three-term relation:

$$x_i \mathbb{P}_n(\mathbf{x}) = \mathbf{A}_{n,i} \mathbb{P}_{n+1}(\mathbf{x}) + \mathbf{B}_{n,i} \mathbb{P}_n(\mathbf{x}) + \mathbf{C}_{n,i} \mathbb{P}_{n-1}(\mathbf{x}), \quad n \geq 0, \quad 1 \leq i \leq d, \quad \mathbb{P}_{-1} := 0.$$

The matrices $\mathbf{A}_{n,i}$, $\mathbf{B}_{n,i}$ and $\mathbf{C}_{n,i}$ have several additional properties that can be found in [44, Sections 3.3 and 3.5].

On the other hand, $K_n(\mathbf{x}, \mathbf{y})$, the n-th reproducing kernel associated with $\{\mathbb{P}_n\}_{n \geq 0}$ is defined by:

$$K_n(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{n} \mathbb{P}_k^T(\mathbf{x}) (\mathbf{H}_k)^{-1} \mathbb{P}_k(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$

It is known [44, Theorem 3.6.1] that $K_n(\cdot, \cdot)$ depends only on $\mathscr{V}_k^d$ rather than a particular basis of $\mathscr{V}_k^d$. Therefore, it is usual to work with an orthonormal basis of $\mathscr{V}_k^d$ for which $\mathbf{H}_k = \mathbf{I}_{r_k^d}$, the identity matrix of size $r_k^d \times r_k^d$. In this case $K_n(\cdot, \cdot)$ takes a simpler form. The three-term relation implies that the corresponding Christoffel-Darboux identity for several variables is:

$$K_n(\mathbf{x},\mathbf{y}) = \sum_{k=0}^{n} \mathbb{P}_k^T(\mathbf{x})(\mathbf{H}_k)^{-1}\mathbb{P}_k(\mathbf{y})$$
$$= \begin{cases} \dfrac{[\mathbf{A}_{n,i}\mathbb{P}_{n+1}(\mathbf{x})]^T(\mathbf{H}_n)^{-1}\mathbb{P}_n(\mathbf{y}) - \mathbb{P}_n^T(\mathbf{x})(\mathbf{H}_n)^{-1}[\mathbf{A}_{n,i}\mathbb{P}_{n+1}(\mathbf{y})]}{x_i - y_i}, & \mathbf{x}\neq\mathbf{y}, \\ \mathbb{P}_n^T(\mathbf{x})(\mathbf{H}_n)^{-1}[\mathbf{A}_{n,i}\partial_i\mathbb{P}_{n+1}(\mathbf{x})] - [\mathbf{A}_{n,i}\mathbb{P}_{n+1}(\mathbf{x})]^T(\mathbf{H}_n)^{-1}\partial_i\mathbb{P}_n(\mathbf{x}), & \mathbf{x}=\mathbf{y}. \end{cases}$$

Even though for each of the above formula the right-hand side seems to depend on i, the left-hand side shows that it does not.

The three-term relation is essential in understanding the structure of standard orthogonal polynomials in several variables. The lack of (1.8) (and consequently of the three-term relation) in the theory of Sobolev polynomials has motivated new tools and techniques for this type of non-standard polynomials.

1.3.1 Orthogonal polynomials on product domains

We consider the product domain:

$$\Omega := [a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_d,b_d],$$

where $[a_i,b_i]$, $i=1,2,\dots,d$, is an interval of $\mathbb{R}$, and where $|a_i|$ and $|b_i|$ can be infinity.

Let $w_i(x_i)$ be a non-negative weight function defined on the interval $[a_i,b_i]$, $i = 1,2,\dots,d$, . Let $\{p_n(w_i;x_i)\}_{n\geq 0}$, $1\leq i\leq d$ be a sequence of polynomials that are orthogonal with respect to w_i, that is,

$$\begin{aligned} \langle p_n,p_m\rangle_{w_i} &= c_i\int_{a_i}^{b_i} p_n(x_i)p_m(x_i)w_i(x_i)dx_i = h_n(w_i)\delta_{n,m}, \\ c_i &:= \left(\int_{a_i}^{b_i} w_i(x_i)dx_i\right)^{-1}, \end{aligned} \tag{1.9}$$

where $h_n(w_i)$ is the L^2 norm:

$$h_n(w_i) := \langle p_n,p_n\rangle_{w_i}, \tag{1.10}$$

and c_i is the normalization constant of w_i such that $\langle 1,1\rangle_{w_i}=1$.

Let W be the product weight function:

$$W(\mathbf{x}) = w_1(x_1)w_2(x_2)\cdots w_d(x_d), \quad \mathbf{x} = \big(x_1,x_2,\dots,x_d\big)\in\Omega, \tag{1.11}$$

and the inner product:

$$\langle f,g\rangle_W = c\int_\Omega f(\mathbf{x})g(\mathbf{x})W(\mathbf{x})d\mathbf{x}, \quad c = \left(\int_\Omega W(\mathbf{x})d\mathbf{x}\right)^{-1}, \tag{1.12}$$

where c is a normalization constant of W such that $\langle 1,1\rangle_W = 1$ and $d\mathbf{x} = dx_1\cdots dx_d$. We denote by $\mathscr{V}_n^d(W)$ the space of orthogonal polynomials of total degree n in d variables with respect to (1.12), and by $\|\cdot\|_W := \sqrt{\langle\cdot,\cdot\rangle_W}$ the norm induced by the inner product $\langle\cdot,\cdot\rangle_W$. Notice that if c_i is given in (1.9), then c in (1.12) can be written as $c = c_1c_2\cdots c_d$. The product structure implies that:

$$P_\alpha^n(W;\mathbf{x}) := p_{\alpha_1}(w_1;x_1)p_{\alpha_2}(w_2;x_2)\cdots p_{\alpha_d}(w_d;x_d), \quad \alpha\in\mathbb{N}_0^d, \quad |\alpha| = n, \quad (1.13)$$

is an orthogonal polynomial of degree $|\alpha| = n$ with respect to W on the product domain Ω. We have the following proposition for the polynomials in (1.13).

Proposition 1.1. *[44, Theorem 5.1.1] For $n = 0,1,2,\ldots$, the set $\{P_\alpha^n(W) : |\alpha| = n\}$ is a mutually orthogonal basis of $\mathscr{V}_n^d(W)$. More precisely,*

$$\left\langle P_\alpha^n, P_\beta^m\right\rangle_W = h_\alpha(W)\delta_{\alpha,\beta},$$

where, with $h_{\alpha_i}(w_i)$ given in (1.10),

$$h_\alpha(W) = h_{\alpha_1}(w_1)h_{\alpha_2}(w_2)\cdots h_{\alpha_d}(w_d). \quad (1.14)$$

From Proposition 1.1 it follows that if the polynomials $p_{\alpha_i}(w_i)$ are orthonormal ($h_{\alpha_i}(w_i) = 1$) with respect to w_i for each $i = 1,2,\ldots,d$ then the polynomials $P_\alpha^n(W)$ in (1.13) are also orthonormal with respect to W. In addition, if the polynomials $p_{\alpha_i}(w_i)$ are monic for each $i = 1,2,\ldots,d$ then the polynomials $P_\alpha^n(W)$ are also monic, that is, $P_\alpha^n(W)$ is of the form $P_\alpha^n(W;\mathbf{x}) = \mathbf{x}^\alpha + Q(\mathbf{x})$, with $Q\in\Pi_{n-1}^d$.

In a matrix form,

$$\left\langle \mathbb{P}_n, \mathbb{P}_m^T\right\rangle_W = \left(\left\langle P_{\alpha^{(i)}}^n, P_{\beta^{(j)}}^m\right\rangle_W\right)_{1\le i\le r_n^d, 1\le j\le r_m^d}$$

is matrix of size $r_n^d\times r_m^d$ such that $\left\langle \mathbb{P}_n, \mathbb{P}_m^T\right\rangle_W = \mathbf{0}$ if $n\neq m$ and $\left\langle \mathbb{P}_n, \mathbb{P}_n^T\right\rangle_W = \mathbf{H}_n^W$ where

$$\mathbf{H}_n^W = \operatorname{diag}\left(h_{\alpha^{(1)}}(W), h_{\alpha^{(2)}}(W),\ldots,h_{\alpha^{(r_n^d)}}(W)\right)$$

is a diagonal positive definite matrix. Since $\det\mathbf{H}_n^W = \prod_{i=1}^{r_n^d} h_{\alpha^{(i)}}(W) > 0$, it follows that $\mathbf{H}_n^W$ is a non-singular matrix.

Several examples of polynomials on a product domain can be obtained from well-known families of orthogonal polynomials on the real line. Next, we show some important examples obtained from the classical orthogonal polynomials. See [44, Sections 5.1.3 and 5.1.4] for more details.

1.3.1.1 Multiple Hermite polynomials

The multiple[5] Hermite polynomials are orthogonal on the product domain $\mathbb{R}^d$ with respect to the product weight function:

$$W^H(\mathbf{x}) = e^{-x_1^2}e^{-x_2^2}\cdots e^{-x_d^2} = e^{-\|\mathbf{x}\|^2}, \quad \mathbf{x}\in\mathbb{R}^d,$$

[5] The word *multiple* was extracted directly from Dunkl and Xu [44, pages 139–141].

that is, with respect to the inner product:

$$\langle f,g\rangle_{W^H}=c\int_{\mathbb{R}^d} f(\mathbf{x})g(\mathbf{x})e^{-\|\mathbf{x}\|^2}d\mathbf{x},\quad c=\left(\int_{\mathbb{R}^d} e^{-\|\mathbf{x}\|^2}d\mathbf{x}\right)^{-1}=\frac{1}{\pi^{d/2}}. \tag{1.15}$$

From Proposition 1.1, a monic mutually orthogonal basis of $\mathscr{V}_n^d(W^H)$, the space of orthogonal polynomials with respect to (1.15), is given by $\left\{P_\alpha^n(W^H):|\alpha|=n\right\}$, where

$$\begin{aligned} P_\alpha^n(W^H;\mathbf{x})&=H_{\alpha_1}(x_1)H_{\alpha_2}(x_2)\cdots H_{\alpha_d}(x_d),\quad |\alpha|=n,\\ h_\alpha(W^H)&=\|P_\alpha^n\|_{W^H}^2=\frac{\alpha!}{2^n}, \end{aligned} \tag{1.16}$$

and where $H_{\alpha_i}(x_i)$, $1\le i\le d$, is the monic Hermite polynomial of degree α_i. The polynomials in $\mathscr{V}_n^d(W^H)$ are eigenfunctions of a second-order differential operator $\mathcal{H}$:

$$\mathcal{H}P=-2nP,\quad P\in\mathscr{V}_n^d(W^H), \tag{1.17}$$

where

$$\mathcal{H}:=\triangle-2\left\langle \mathbf{x}^T,\nabla\right\rangle. \tag{1.18}$$

This fact follows from the product structure (1.16) and the differential equation (1.2) (or differential operator (1.3)) satisfied by the Hermite polynomials.

1.3.1.2 Multiple Laguerre polynomials

The multiple Laguerre polynomials, with parameter $\eta=(\eta_1,\eta_2,\dots,\eta_d)\in\mathbb{R}^d$, $\eta_i>-1$, $1\le i\le d$, are orthogonal on the product domain $\mathbb{R}_+^d$ with respect to the product weight function:

$$\begin{aligned} W_\eta^L(\mathbf{x})=x_1^{\eta_1}e^{-x_1}x_2^{\eta_2}e^{-x_2}\cdots x_d^{\eta_d}e^{-x_d}&=\mathbf{x}^\eta e^{-|\mathbf{x}|},\quad \mathbf{x}\in\mathbb{R}_+^d,\\ \eta_i>-1,\quad 1\le i\le d,&\quad |\mathbf{x}|=x_1+x_2+\cdots+x_d, \end{aligned}$$

that is, with respect to the inner product:

$$\begin{aligned} \langle f,g\rangle_{W_\eta^L}&=c_\eta\int_{\mathbb{R}_+^d} f(\mathbf{x})g(\mathbf{x})\mathbf{x}^\eta e^{-|\mathbf{x}|}d\mathbf{x},\\ c_\eta&=\left(\int_{\mathbb{R}_+^d}\mathbf{x}^\eta e^{-|\mathbf{x}|}d\mathbf{x}\right)^{-1}=\frac{1}{\prod_{i=1}^d\Gamma(\eta_i+1)}. \end{aligned} \tag{1.19}$$

From Proposition 1.1, a monic mutually orthogonal basis of $\mathscr{V}_n^d(W_\eta^L)$, the space of orthogonal polynomials with respect to (1.19), is given by $\left\{P_\alpha^n(W_\eta^L):|\alpha|=n\right\}$, where

$$\begin{aligned} P_\alpha^n(W_\eta^L;\mathbf{x})&=L_{\alpha_1}^{(\eta_1)}(x_1)L_{\alpha_2}^{(\eta_2)}(x_2)\cdots L_{\alpha_d}^{(\eta_d)}(x_d),\quad |\alpha|=n,\\ h_\alpha(W_\eta^L)&=\|P_\alpha^n\|_{W_\eta^L}^2=\alpha!(\eta+\mathbf{1})_\alpha,\quad \mathbf{1}=(1,1,\dots,1), \end{aligned} \tag{1.20}$$

and where $L_{\alpha_i}^{(\eta_i)}(x_i)$, $1 \le i \le d$, is the monic Laguerre polynomial of degree α_i and parameter η_i. The polynomials in $\mathscr{V}_n^d(W_\eta^L)$ are eigenfunctions of a second-order differential operator $\mathcal{L}_\eta$:

$$\mathcal{L}_\eta P = -nP, \quad P \in \mathscr{V}_n^d(W_\eta^L), \tag{1.21}$$

where

$$\mathcal{L}_\eta := \sum_{i=1}^d x_i \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^d (\eta_i + 1 - x_i)\frac{\partial}{\partial x_i}, \quad \eta_i > -1, \quad 1 \le i \le d. \tag{1.22}$$

This fact follows from the product structure (1.20) and the differential equation (1.4) (or differential operator (1.5)) satisfied by the Laguerre polynomials.

1.3.1.3 Multiple Jacobi polynomials

The multiple Jacobi polynomials, with parameters $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_d) \in \mathbb{R}^d$, $\eta = (\eta_1, \eta_2, \ldots, \eta_d) \in \mathbb{R}^d$, $\zeta_i, \eta_i > -1$, $1 \le i \le d$, are orthogonal on the product domain $[-1,1]^d$ with respect to the product weight function:

$$W_{\zeta,\eta}^J(\mathbf{x}) = \prod_{i=1}^d (1-x_i)^{\zeta_i}(1+x_i)^{\eta_i} = (\mathbf{1}-\mathbf{x})^\zeta(\mathbf{1}+\mathbf{x})^\eta,$$
$$\mathbf{x} \in [-1,1]^d, \quad \zeta_i, \eta_i > -1, \quad 1 \le i \le d,$$

that is, with respect to the inner product:

$$\begin{aligned} \langle f, g\rangle_{W_{\zeta,\eta}^J} &= c_{\zeta,\eta}\int_{[-1,1]^d} f(\mathbf{x})g(\mathbf{x})(\mathbf{1}-\mathbf{x})^\zeta(\mathbf{1}+\mathbf{x})^\eta d\mathbf{x}, \\ c_{\zeta,\eta} &= \left(\int_{[-1,1]^d}(\mathbf{1}-\mathbf{x})^\zeta(\mathbf{1}+\mathbf{x})^\eta d\mathbf{x}\right)^{-1} = \prod_{i=1}^d \frac{\Gamma(\zeta_i+\eta_i+2)}{2^{\zeta_i+\eta_i+1}\Gamma(\zeta_i+1)\Gamma(\eta_i+1)}. \end{aligned} \tag{1.23}$$

From Proposition 1.1, a monic mutually orthogonal basis of $\mathscr{V}_n^d(W_{\zeta,\eta}^J)$, the space of orthogonal polynomials with respect to (1.23), is given by $\left\{P_\alpha^n(W_{\zeta,\eta}^J) : |\alpha| = n\right\}$, where

$$\begin{aligned} P_\alpha^n(W_{\zeta,\eta}^J;\mathbf{x}) &= P_{\alpha_1}^{(\zeta_1,\eta_1)}(x_1)P_{\alpha_2}^{(\zeta_2,\eta_2)}(x_2)\cdots P_{\alpha_d}^{(\zeta_d,\eta_d)}(x_d), \quad |\alpha| = n, \\ h_\alpha(W_{\zeta,\eta}^J) &= \|P_\alpha^n\|_{W_{\zeta,\eta}^J}^2 = \frac{4^n(\zeta+\mathbf{1})_\alpha(\eta+\mathbf{1})_\alpha}{(\alpha+\zeta+\eta+\mathbf{1})_\alpha(\zeta+\eta+\mathbf{2})_{2\alpha}}, \quad \mathbf{2} = (2,2,\ldots,2), \end{aligned} \tag{1.24}$$

and where $P_{\alpha_i}^{(\zeta_i,\eta_i)}(x_i)$, $1 \le i \le d$, is the monic Jacobi polynomial of degree α_i and parameters ζ_i, η_i.

Each polynomial in the basis $\left\{P_\alpha^n(W_{\zeta,\eta}^J) : |\alpha| = n\right\}$ satisfies the second-order partial differential equation:

$$\mathcal{J}_{\zeta,\eta}P_\alpha^n = -\sum_{i=1}^d \alpha_i(\alpha_i+\zeta_i+\eta_i+1)P_\alpha^n, \tag{1.25}$$

where

$$\mathcal{J}_{\zeta,\eta} := \sum_{i=1}^{d}(1-x_i^2)\frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{d}[\eta_i - \zeta_i - (\zeta_i + \eta_i + 2)x_i]\frac{\partial}{\partial x_i}, \quad \zeta_i, \eta_i > -1, \quad 1 \le i \le d. \tag{1.26}$$

This fact follows from (1.24) and the differential equation (1.6) (or differential operator (1.7)) satisfied by the Jacobi polynomials.

Remark 1.1. Notice that equations (1.17) and (1.21) are of the form $\mathcal{H}P = \lambda_n P$ and $\mathcal{L}_\eta P = \lambda_n P$, respectively, with $n = \deg P$, and where λ_n is a number, called an *eigenvalue*, which depends on n only. But this is not the case for the Multiple Jacobi polynomials in (1.25).

1.3.2 Orthogonal polynomials on the simplex

The simplex of $\mathbb{R}^d$ is the set:

$$\mathbb{T}^d := \left\{\mathbf{x} \in \mathbb{R}^d : x_1 \ge 0, x_2 \ge 0, \ldots, x_d \ge 0, 1 - |\mathbf{x}| \ge 0\right\}, \quad |\mathbf{x}| := x_1 + x_2 + \cdots + x_d.$$

Orthogonal polynomials on the simplex [44, Section 5.3] are orthogonal with respect to the weight function:

$$W_\gamma(\mathbf{x}) := x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_d^{\gamma_d}(1-|\mathbf{x}|)^{\gamma_{d+1}}, \quad \mathbf{x} \in \mathbb{T}^d, \quad \gamma_i > -1, \quad 1 \le i \le d+1, \tag{1.27}$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{d+1}) \in \mathbb{R}^{d+1}$ is such that $\gamma_i > -1$ for $i = 1, 2, \ldots, d+1$. That is, these polynomials are orthogonal with respect to the inner product:

$$\langle f, g \rangle_\gamma := c_\gamma \int_{\mathbb{T}^d} f(\mathbf{x}) g(\mathbf{x}) W_\gamma(\mathbf{x}) d\mathbf{x}, \tag{1.28}$$

$$c_\gamma := \left(\int_{\mathbb{T}^d} W_\gamma(\mathbf{x}) d\mathbf{x}\right)^{-1} = \frac{\Gamma(|\gamma| + d + 1)}{\prod_{i=1}^{d+1} \Gamma(\gamma_i + 1)}, \quad |\gamma| := \gamma_1 + \gamma_2 + \cdots + \gamma_{d+1}, \tag{1.29}$$

and where c_γ is the normalization constant of W_γ such that $\langle 1, 1\rangle_\gamma = 1$. We denote by $\mathscr{V}_n^d(W_\gamma)$ the space of orthogonal polynomials in d variables of degree n with respect to (1.28).

For $\alpha \in \mathbb{N}_0^d$ and $|\alpha| = n$, a monic orthogonal basis of $\mathscr{V}_n^d(W_\gamma)$ is given by the polynomials:

$$V_\alpha^n(W_\gamma; \mathbf{x}) = \sum_{\mathbf{0} \le \beta \le \alpha} (-1)^{n+|\beta|} \binom{\alpha}{\beta} \prod_{i=1}^{d} \frac{(\gamma_i + 1)_{\alpha_i}}{(\gamma_i + 1)_{\beta_i}} \frac{(|\gamma| + d)_{n+|\beta|}}{(|\gamma| + d)_{n+|\alpha|}} \mathbf{x}^\beta. \tag{1.30}$$

Proposition 1.2. *[44, Proposition 5.3.2] Let $V_\alpha^n(W_\gamma)$ be defined in* (1.30)*, then the set $\{V_\alpha^n(W_\gamma) : |\alpha| = n\}$ forms a monic orthogonal basis of $\mathscr{V}_n^d(W_\gamma)$.*

It is known [44] that the orthogonal polynomials $P \in \mathscr{V}_n^d(W_\gamma)$ with respect to W_γ are eigenfunctions of a second-order differential operator $\mathcal{T}_\gamma$, that is,

$$\mathcal{T}_\gamma P = -n(n+|\gamma|+d)P, \quad P \in \mathscr{V}_n^d(W_\gamma), \tag{1.31}$$

where

$$\mathcal{T}_\gamma := \sum_{i=1}^{d} x_i(1-x_i)\frac{\partial^2}{\partial x_i^2} - 2\sum_{1\le i<j\le d} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d}\left[(\gamma_i+1)-(|\gamma|+d+1)x_i\right]\frac{\partial}{\partial x_i}. \tag{1.32}$$

1.3.3 Spherical harmonics

The unit ball $\mathbb{B}^d$ and the unit sphere $\mathbb{S}^{d-1}$ are the sets:

$$\mathbb{B}^d := \left\{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \le 1\right\}, \quad \mathbb{S}^{d-1} := \left\{\xi \in \mathbb{R}^d : \|\xi\| = 1\right\}.$$

The volume of $\mathbb{B}^d$ and the surface area of $\mathbb{S}^{d-1}$, denoted by $\mathrm{vol}(\mathbb{B}^d)$ and ω_{d-1}, respectively, are given [44] by:

$$\mathrm{vol}(\mathbb{B}^d) = \frac{\omega_{d-1}}{d}, \quad \omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{1.33}$$

A harmonic polynomial P is a homogeneous polynomial that satisfies the Laplace equation $\triangle P = 0$ [24]. We denote by $\mathscr{H}_n^d$ the space of harmonic polynomials in d variables of total degree n, that is,

$$\mathscr{H}_n^d = \left\{P \in \mathscr{P}_n^d : \triangle P = 0\right\}.$$

It is known [24, 44] that:

$$a_n^d := \dim \mathscr{H}_n^d = \binom{n+d-1}{d-1} - \binom{n+d-3}{d-1}.$$

Spherical harmonics are the restriction of harmonic polynomials on the unit sphere $\mathbb{S}^{d-1}$. In spherical-polar coordinates $\mathbf{x} = r\xi$, $\mathbf{x} \in \mathbb{R}^d$, $r \ge 0$, $\xi \in \mathbb{S}^{d-1}$, we use the notation $Y(\mathbf{x})$ for harmonic polynomials and $Y(\xi)$ for spherical harmonics. Since $Y \in \mathscr{H}_n^d$ is homogeneous then $Y(\mathbf{x}) = r^n Y(\xi)$.

In spherical-polar coordinates $\mathbf{x} = r\xi$, $\mathbf{x} \in \mathbb{R}^d$, $r \ge 0$, $\xi \in \mathbb{S}^{d-1}$, the differential operators ∇ and $\triangle$ can be decomposed[6] as follows [24]:

$$\nabla = \frac{1}{r}\nabla_0 + \xi^T \frac{\partial}{\partial r}, \tag{1.34}$$

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\triangle_0. \tag{1.35}$$

[6] ξ is used in (1.34) instead of ξ^T when ∇ is defined to be a row vector.

The operators ∇_0 and $\triangle_0$ are the spherical parts of the gradient and the Laplacian, respectively. The operator $\triangle_0$ is the Laplace-Beltrami operator, and it has spherical harmonics as its eigenfunctions [24, Theorem 1.4.5], that is, for $Y \in \mathscr{H}_n^d$:

$$\triangle_0 Y(\xi) = -n(n+d-2)Y(\xi), \quad \xi \in \mathbb{S}^{d-1}. \tag{1.36}$$

Spherical harmonics of different degrees are orthogonal with respect to the inner product on the sphere [24, Theorem 1.1.2]:

$$\langle f, g\rangle_{\mathbb{S}^{d-1}} := \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} f(\xi)g(\xi)d\omega(\xi), \tag{1.37}$$

where $d\omega$ is the surface area measure. We use the notation $\{Y_\nu^n : 1 \le \nu \le a_n^d\}$ to denote an orthonormal basis for $\mathscr{H}_n^d$ with respect to (1.37), that is,

$$\frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} Y_\nu^n(\xi) Y_\eta^m(\xi) d\omega(\xi) = \delta_{n,m}\delta_{\nu,\eta}. \tag{1.38}$$

1.3.4 Orthogonal polynomials on the unit ball

The orthogonal polynomials on the unit ball $\mathbb{B}^d$ [44, Section 5.2] are orthogonal with respect to the weight function:

$$W_\mu(\mathbf{x}) := (1 - \|\mathbf{x}\|^2)^\mu, \quad \mathbf{x} \in \mathbb{B}^d, \quad \mu > -1, \tag{1.39}$$

that is, with respect to the inner product:

$$\langle f, g\rangle_\mu = c_\mu \int_{\mathbb{B}^d} f(\mathbf{x})g(\mathbf{x})W_\mu(\mathbf{x})d\mathbf{x}, \tag{1.40}$$

$$c_\mu := \left(\int_{\mathbb{B}^d} W_\mu(\mathbf{x})d\mathbf{x}\right)^{-1} = \frac{\Gamma(\mu + d/2 + 1)}{\pi^{d/2}\Gamma(\mu+1)}, \tag{1.41}$$

and where c_μ is a normalization constant such that $\langle 1, 1\rangle_\mu = 1$. We denote by $\mathscr{V}_n^d(W_\mu)$ the space of orthogonal polynomials in d variables of degree n with respect to (1.40).

A mutually orthogonal basis of $\mathscr{V}_n^d(W_\mu)$ is given in terms of the Jacobi polynomials[7] $P_n^{(a,b)}$ and harmonic polynomials.

Proposition 1.3. *[44, Proposition 5.2.1] For $n = 0, 1, 2, 3, \dots$ and $0 \le j \le n/2$, let $\{Y_\nu^{n-2j} : 1 \le \nu \le a_{n-2j}^d\}$ denote an orthonormal basis of $\mathscr{H}_{n-2j}^d$. The polynomials:*

$$P_{j,\nu}^n(W_\mu; \mathbf{x}) = P_j^{(\mu, n-2j+\frac{d-2}{2})}(2\|\mathbf{x}\|^2 - 1) Y_\nu^{n-2j}(\mathbf{x}), \tag{1.42}$$

form a mutually orthogonal basis of $\mathscr{V}_n^d(W_\mu)$. More precisely,

$$\langle P_{j,\nu}^n, P_{k,\eta}^m \rangle_\mu = h_{j,n}^\mu \delta_{n,m}\delta_{j,k}\delta_{\nu,\eta},$$

where $h_{j,n}^\mu$ is given by:

$$h_{j,n}^\mu = \frac{(\mu+1)_j (d/2)_{n-j} (n-j+\mu+d/2)}{j!(\mu+1+d/2)_{n-j}(n+\mu+d/2)}.$$

[7] Proposition 1.3 assumes that the Jacobi polynomials are non-monic. Therefore, $h_{j,n}^\mu$ was computed under this assumption.

It is known [44, 99] that the orthogonal polynomials $P \in \mathscr{V}_n^d(W_\mu)$ with respect to W_μ are eigenfunctions of a second-order differential operator $\mathcal{B}_\mu$, that is,

$$\mathcal{B}_\mu P = -n(n+2\mu+d)P, \quad \mu > -1, \quad P \in \mathscr{V}_n^d(W_\mu), \tag{1.43}$$

where

$$\mathcal{B}_\mu := \triangle - \left\langle \mathbf{x}^T, \nabla \right\rangle^2 - (2\mu+d)\left\langle \mathbf{x}^T, \nabla \right\rangle, \tag{1.44}$$

Another second-order differential operator $\mathcal{D}_\mu$ that appears in literature [44, Section 5.2] for which the polynomials $P \in \mathscr{V}_n^d(W_\mu)$ are eigenfunctions is:

$$\mathcal{D}_\mu P = -(n+d)(n+2\mu)P, \quad \mu > -1, \quad P \in \mathscr{V}_n^d(W_\mu), \tag{1.45}$$

where

$$\mathcal{D}_\mu := \triangle - \sum_{j=1}^{d} \frac{\partial}{\partial x_j} x_j \left[2\mu + \sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i} \right] \tag{1.46}$$

It is not difficult to show that (1.44) and (1.46) satisfy the relation:

$$\mathcal{B}_\mu = \mathcal{D}_\mu + 2d\mu\mathcal{I}, \quad \mathcal{I} \text{ is the identity operator.}$$

1.3.5 Orthogonal polynomials on a cone

Let us recall that for $\mathbf{x} \in \mathbb{R}^d$, the symbol $\mathbf{x}_i = (x_1, x_2, \ldots, x_i) \in \mathbb{R}^i$, $1 \le i \le d$, with $\mathbf{x}_0 := 0$, denotes a truncation of $\mathbf{x}$. The solid cone[8] (or simply the cone) of $\mathbb{R}^d$ is the set:

$$\mathbb{V}_\vartheta^d := \left\{ \mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}_{d-1}\| \le x_d, 0 \le x_d \le \vartheta \right\}, \quad 0 < \vartheta \le \infty.$$

If ϑ is finite then we have a bounded cone, otherwise we have an unbounded cone. In literature [115] we found the cases $\vartheta = 1$ and $\vartheta = \infty$ as we will show in the sequel.

Orthogonal polynomials on $\mathbb{V}_\vartheta^d$ are orthogonal with respect to the weight function:

$$W_{w,\mu}(\mathbf{x}) := (x_d^2 - \|\mathbf{x}_{d-1}\|^2)^\mu w(x_d), \quad \mu > -1, \quad \mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{V}_\vartheta^d, \tag{1.47}$$

where w is a non-negative weight function on the interval $0 \le x_d \le \vartheta$, that is, with respect to the inner product:

$$\langle f, g \rangle_{w,\mu} := c_{w,\mu} \int_{\mathbb{V}_\vartheta^d} f(\mathbf{x}) g(\mathbf{x}) W_{w,\mu}(\mathbf{x}) d\mathbf{x}, \tag{1.48}$$

$$c_{w,\mu} := \left(\int_{\mathbb{V}_\vartheta^d} W_{w,\mu}(\mathbf{x}) d\mathbf{x} \right)^{-1}, \tag{1.49}$$

[8] In [115] all the theory on the cone $\mathbb{V}^{d+1} := \left\{ (\mathbf{x}, t) \in \mathbb{R}^{d+1} : \|\mathbf{x}\| \le t, 0 \le t \le \vartheta \right\}$ was presented. In order to fit notation and theory to d variables, in this work we use the cone $\mathbb{V}^d$. It is just a matter of changing $d+1$ by d.

and where $c_{w,\mu}$ is the normalization constant of $W_{w,\mu}$ such that $\langle 1,1\rangle_{w,\mu} = 1$. We denote by $\mathcal{V}_n^d(W_{w,\mu})$ the space of orthogonal polynomials in d variables of degree n with respect to (1.48).

Xu [115] showed that under the change of variables $\mathbf{x}_{d-1} = x_d\mathbf{y}$, $\mathbf{y} \in \mathbb{B}^{d-1}$, we have $d\mathbf{x}_{d-1} = x_d^{d-1}d\mathbf{y}$, and therefore, the following integral formula on the cone holds:

$$\begin{aligned}\int_{\mathbb{V}_\vartheta^d} f(\mathbf{x})d\mathbf{x} &= \int_0^\vartheta \int_{\|\mathbf{x}_{d-1}\|\le x_d} f(\mathbf{x}_{d-1},x_d)d\mathbf{x}_{d-1}dx_d \\ &= \int_0^\vartheta x_d^{d-1}\int_{\mathbb{B}^{d-1}} f(x_d\mathbf{y},x_d)d\mathbf{y}dx_d.\end{aligned} \tag{1.50}$$

In particular, by (1.41), (1.49) and (1.50), the normalization constant $c_{w,\mu}$ is given by:

$$\begin{aligned}c_{w,\mu} &= \left(\int_0^\vartheta w(x_d)x_d^{2\mu+d-1}dx_d\right)^{-1}\left(\int_{\mathbb{B}^{d-1}}(1-\|\mathbf{y}\|^2)^\mu d\mathbf{y}\right)^{-1} \\ &= \frac{\Gamma\left(\mu+\frac{d-1}{2}+1\right)}{\pi^{\frac{d-1}{2}}\Gamma(\mu+1)}\left(\int_0^\vartheta w(x_d)x_d^{2\mu+d-1}dx_d\right)^{-1}.\end{aligned} \tag{1.51}$$

Particular examples presented by Xu [115, section 3] for the weight function w include Jacobi and Laguerre cases:

$$w_{a,b}(t) = t^a(1-t)^b, \quad a,b > -1, \quad 0 \le t \le 1 = \vartheta, \tag{1.52}$$

$$w_a(t) = t^a e^{-t}, \quad a > -1, \quad 0 \le t < \infty = \vartheta. \tag{1.53}$$

From (1.51), in these two cases the normalization constants are given in (1.55) and (1.59), respectively.

1.3.5.1 Jacobi polynomials on the bounded cone ($\vartheta = 1$)

The Jacobi polynomials on the cone are orthogonal on the set

$$\mathbb{V}_1^d = \left\{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}_{d-1}\| \le x_d, 0 \le x_d \le 1\right\}$$

with respect to the weight function:

$$W_{a,b,\mu}^J(\mathbf{x}) = (x_d^2 - \|\mathbf{x}_{d-1}\|^2)^\mu x_d^a(1-x_d)^b, \quad \mathbf{x} \in \mathbb{V}_1^d, \quad a,b,\mu > -1,$$

that is, with respect to the inner product:

$$\langle f,g\rangle_{W_{a,b,\mu}^J} = c_{a,b,\mu}\int_{\mathbb{V}_1^d} f(\mathbf{x})g(\mathbf{x})W_{a,b,\mu}^J(\mathbf{x})d\mathbf{x}, \tag{1.54}$$

$$c_{a,b,\mu} = \left(\int_{\mathbb{V}_1^d} W_{a,b,\mu}^J(\mathbf{x})d\mathbf{x}\right)^{-1} = \frac{\Gamma\left(\mu+\frac{d+1}{2}\right)\Gamma(a+b+2\mu+d+1)}{\pi^{\frac{d-1}{2}}\Gamma(\mu+1)\Gamma(a+2\mu+d)\Gamma(b+1)}. \tag{1.55}$$

An orthogonal basis of $\mathcal{V}_n^d(W_{a,b,\mu}^J)$, the space of orthogonal polynomials with respect to (1.54), is given in terms of Jacobi polynomials in one variable and an orthonormal basis in $d-1$ variables on the unit ball $\mathbb{B}^{d-1}$.

Proposition 1.4. *[115, Proposition 3.1] For* $m = 0, 1, 2, 3, \ldots$, *let*

$$\left\{ P_\alpha^m(W_\mu) : |\alpha| = m, \alpha \in \mathbb{N}_0^{d-1} \right\}$$

denote an orthonormal basis of $\mathscr{V}_m^{d-1}(W_\mu)$ *on the unit ball* $\mathbb{B}^{d-1}$. *Define*

$$Q_{m,\alpha}^n(W_{a,b,\mu}^J; \mathbf{x}) = P_{n-m}^{(2\mu+2m+a+d-1,b)}(1-2x_d)\, x_d^m\, P_\alpha^m\left(W_\mu; \frac{\mathbf{x}_{d-1}}{x_d}\right).$$

Then $\left\{ Q_{m,\alpha}^n(W_{a,b,\mu}^J) : |\alpha| = m, \alpha \in \mathbb{N}_0^{d-1}, 0 \le m \le n \right\}$ *is an orthogonal basis of the space* $\mathscr{V}_n^d(W_{a,b,\mu}^J)$.

There is an important case that arises when the parameter $a = 0$. It is known [115, Theorem 3.2] that the orthogonal polynomials $P \in \mathscr{V}_n^d(W_{0,b,\mu}^J)$ with respect to $W_{0,b,\mu}^J$ are eigenfunctions of a second-order partial differential operator $\mathcal{V}_{b,\mu}^J$, that is,

$$\mathcal{V}_{b,\mu}^J P = -n(n+2\mu+b+d)P, \quad b, \mu > -1, \quad P \in \mathscr{V}_n^d(W_{0,b,\mu}^J), \tag{1.56}$$

where[9]

$$\mathcal{V}_{b,\mu}^J := x_d(1-x_d)\frac{\partial^2}{\partial x_d^2} + 2(1-x_d)\left\langle \mathbf{x}_{d-1}^T, \nabla_{d-1}\right\rangle \frac{\partial}{\partial x_d} + x_d \triangle_{d-1} - \left\langle \mathbf{x}_{d-1}^T, \nabla_{d-1}\right\rangle^2$$
$$+ (2\mu+d)\frac{\partial}{\partial x_d} - (2\mu+b+d+1)\left\langle \mathbf{x}^T, \nabla\right\rangle + \left\langle \mathbf{x}_{d-1}^T, \nabla_{d-1}\right\rangle. \tag{1.57}$$

Remark 1.2. Accordingly with Xu [115, remark 3.1], when the parameter $a \neq 0$, the Jacobi polynomials $Q_{m,\alpha}^n$ on the cone with respect to $W_{a,b,\mu}^J$ also satisfy a differential equation, but the eigenvalues depend on both m and n. In this case, $\mathscr{V}_n^d(W_{a,b,\mu}^J)$ is not an eigenspace of such a differential operator.

1.3.5.2 Laguerre polynomials on the unbounded cone ($\vartheta = \infty$)

The Laguerre polynomials on the cone are orthogonal on the set

$$\mathbb{V}_\infty^d = \left\{ \mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}_{d-1}\| \le x_d, 0 \le x_d < \infty \right\}$$

with respect to the weight function:

$$W_{a,\mu}^L(\mathbf{x}) = (x_d^2 - \|\mathbf{x}_{d-1}\|^2)^\mu x_d^a\, e^{-x_d}, \quad \mathbf{x} \in \mathbb{V}_\infty^d, \quad a, \mu > -1,$$

that is, with respect to the inner product:

$$\langle f, g\rangle_{W_{a,\mu}^L} = c_{a,\mu} \int_{\mathbb{V}_\infty^d} f(\mathbf{x})g(\mathbf{x})W_{a,\mu}^L(\mathbf{x})d\mathbf{x}, \tag{1.58}$$

$$c_{a,\mu} = \left(\int_{\mathbb{V}_\infty^d} W_{a,\mu}^L(\mathbf{x})d\mathbf{x}\right)^{-1} = \frac{\Gamma\left(\mu + \frac{d+1}{2}\right)}{\pi^{\frac{d-1}{2}}\Gamma(\mu+1)\Gamma(a+2\mu+d)}. \tag{1.59}$$

An orthogonal basis of $\mathscr{V}_n^d(W_{a,\mu}^L)$, the space of orthogonal polynomials with respect to (1.58), is given in terms of Laguerre polynomials in one variable and an orthonormal basis in $d-1$ variables on the unit ball $\mathbb{B}^{d-1}$.

[9] The operator $\mathcal{V}_{b,\mu}^J$ was written for fitting to our notation. See [115, pages 12–13] for more details.

Proposition 1.5. *[115, Proposition 3.3] For $m = 0, 1, 2, 3, \ldots$, let*

$$\left\{ P_\alpha^m(W_\mu) : |\alpha| = m, \alpha \in \mathbb{N}_0^{d-1} \right\}$$

denote an orthonormal basis of $\mathscr{V}_m^{d-1}(W_\mu)$ on the unit ball $\mathbb{B}^{d-1}$. Define

$$L_{m,\alpha}^n(W_{a,\mu}^L; \mathbf{x}) = L_{n-m}^{(2\mu+2m+a+d-1)}(x_d)\, x_d^m\, P_\alpha^m \left(W_\mu; \frac{\mathbf{x}_{d-1}}{x_d} \right).$$

Then $\left\{ L_{m,\alpha}^n(W_{a,\mu}^L) : |\alpha| = m, \alpha \in \mathbb{N}_0^{d-1}, 0 \le m \le n \right\}$ is an orthogonal basis of $\mathscr{V}_n^d(W_{a,\mu}^L)$.

As in the Jacobi case, when the parameter $a = 0$ [115, Theorem 3.4] the orthogonal polynomials $P \in \mathscr{V}_n^d(W_{0,\mu}^L)$ with respect to $W_{0,\mu}^L$ are eigenfunctions of a second-order partial differential operator $\mathcal{V}_\mu^L$, that is,

$$\mathcal{V}_\mu^L P = -nP, \quad \mu > -1, \quad P \in \mathscr{V}_n^d(W_{0,\mu}^L), \tag{1.60}$$

where[10]

$$\mathcal{V}_\mu^L := x_d \triangle + 2 \left\langle \mathbf{x}_{d-1}^T, \nabla_{d-1} \right\rangle \frac{\partial}{\partial x_d} - \left\langle \mathbf{x}_{d-1}^T, \nabla_{d-1} \right\rangle + (2\mu + d - x_d) \frac{\partial}{\partial x_d}. \tag{1.61}$$

Remark 1.3. Accordingly with Xu [115, remark 3.2], when the parameter $a \neq 0$, the Laguerre polynomials $L_{m,\alpha}^n$ on the cone with respect to $W_{a,\mu}^L$ also satisfy a differential equation, but the eigenvalues depend on both m and n. In this case, $\mathscr{V}_n^d(W_{a,\mu}^L)$ is not an eigenspace of such a differential operator.

1.4 Taylor's formula in several variables

Let us recall that a function $u(\mathbf{x})$ is said to be of class $\mathscr{C}^\kappa$ if it has continuous derivatives up to order κ, and these derivatives do not depend on the order used to achieve the differentiations. The following classic result known as Taylor's formula can be found in [100, Theorem 1.1].

Theorem 1.1 (Taylor's formula)**.** *[100, theorem 1.1] Let u be a $\mathscr{C}^\kappa$ function defined on $\mathbb{R}^d$. Then for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$:*

$$u(\mathbf{x} + \mathbf{y}) = \sum_{|\beta| \le \kappa - 1} \frac{\mathbf{y}^\beta}{\beta!} \partial^\beta u(\mathbf{x}) + \sum_{|\beta| = \kappa} \frac{\mathbf{y}^\beta}{\beta!} \int_0^1 \kappa (1-t)^{\kappa-1} \partial^\beta u(\mathbf{x} + t\mathbf{y}) dt. \tag{1.62}$$

The first sum in (1.62) is the well-known Taylor polynomial of degree $\kappa - 1$ of u at $\mathbf{x} \in \mathbb{R}^d$, and the second sum is the remainder term.

For our purposes, let $\mathbf{p} \in \mathbb{R}^d$ be a fixed point in $\mathbb{R}^d$. If $P \in \Pi^d$ is a polynomial (a $\mathscr{C}^\infty$ function) in d variables, we denote by $\mathcal{T}^{\kappa-1}(P, \mathbf{p}; \mathbf{x})$ the Taylor polynomial of

[10] The operator $\mathcal{V}_\mu^L$ was written for fitting to our notation. See [115, page 15] for more details.

degree $\kappa - 1$ of P at $\mathbf{p}$, and by $\mathcal{R}_\kappa(P, \mathbf{p}; \mathbf{x})$ its remainder term. Then by (1.62), and by setting $\mathbf{x} = \mathbf{p}$ and $\mathbf{y} = \mathbf{x} - \mathbf{p}$, we have:

$$P(\mathbf{x}) = \mathcal{T}^{\kappa-1}(P, \mathbf{p}; \mathbf{x}) + \mathcal{R}_\kappa(P, \mathbf{p}; \mathbf{x}),$$

where

$$\mathcal{T}^{\kappa-1}(P, \mathbf{p}; \mathbf{x}) = \sum_{|\beta| \leq \kappa-1} \frac{\partial^\beta P(\mathbf{p})}{\beta!}(\mathbf{x} - \mathbf{p})^\beta, \tag{1.63}$$

$$\mathcal{R}_\kappa(P, \mathbf{p}; \mathbf{x}) = \sum_{|\beta| = \kappa} \frac{(\mathbf{x} - \mathbf{p})^\beta}{\beta!} \int_0^1 \kappa(1-t)^{\kappa-1} \partial^\beta P(\mathbf{p} + t(\mathbf{x} - \mathbf{p}))dt. \tag{1.64}$$

A well-known property of the Taylor polynomial (1.63) is that, at the point $\mathbf{p} \in \mathbb{R}^d$, it satisfies:

$$(\partial^\alpha P)(\mathbf{p}) = (\partial^\alpha \mathcal{T}^{\kappa-1}(P, \mathbf{p}))(\mathbf{p}), \quad |\alpha| \leq \kappa - 1,$$

which implies that

$$(\partial^\alpha \mathcal{R}_\kappa(P, \mathbf{p}))(\mathbf{p}) = 0, \quad |\alpha| \leq \kappa - 1.$$

Chapter 2

State of the art

Chapter 1 was devoted to the main topics for the so-called *standard orthogonal polynomials*. In this chapter we focus our attention to a type of non-standard polynomials, which are known in literature as *Sobolev orthogonal polynomials*. In contrast to the standard case, the theory of Sobolev polynomials is non-uniform and fragmented. This chapter is not comprehensive. Conversely, we only mention some well-known results in literature and we remit the reader to a detailed survey by Marcellán and Xu [86], and some other references by Meijer [92] and Martínez-Finkelshtein [88, 89] who give the state of the art on this topic.

2.1 Sobolev orthogonal polynomials in one variable

Sobolev orthogonal polynomials in one variable have been studied since the decade of the 60s when the first paper on this topic was published by Althammer [7]. This first paper was motivated by an optimization problem proposed by Lewis [73] in the 40s. The Lewis' problem consists in finding a polynomial P_n, of degree at most n, such that it minimizes:

$$\sum_{k=0}^{p} \int_a^b [f^{(k)}(x) - P_n^{(k)}(x)]^2 d\alpha_k(x),$$

where $\alpha_0(x), \alpha_1(x), \ldots, \alpha_p(x)$ are $p+1$ monotonic non-decreasing functions defined for $a \leq x \leq b$, and f is a function of class $\mathscr{C}^{p-1}$ over an interval $A \leq x \leq B$, where $A \leq a < b \leq B$, such that its p-th derivative $f^{(p)}(x)$ exists almost everywhere with respect to α_p and is such that the Lebesgue-Stieltjes integral,

$$\int_a^b [f^{(p)}(x)]^2 d\alpha_p(x),$$

exists. More than fifty years have passed and a big number of publications have appeared. In the next sections we present a survey of some references on Sobolev polynomials.

2.1.1 First publications on Sobolev polynomials

Initial studies showed that many properties of the standard polynomials were not preserved on Sobolev polynomials. Althammer [7], for example, considered the sequence $\{S_n(\lambda;x)\}_{n\geq 0}$ of Sobolev-Legendre orthogonal polynomials with respect to the inner product:

$$\langle f,g\rangle_S = \int_{-1}^{1} f(x)g(x)dx + \lambda\int_{-1}^{1} f'(x)g'(x)dx, \quad \lambda > 0. \tag{2.1}$$

This author gave an example in which dx is replaced in the second integral of (2.1) by $w(x)dx$, where $w(x) = 10$ for $-1 \leq x < 0$ and $w(x) = 1$ for $0 \leq x \leq 1$, and he observed that $S_2(\lambda;x) = K(x^2 + 27x/35 - 1/3)$, with K a real constant, for this new inner product has a zero at $x = -1.08$, which is outside[1] the interval $[-1,1]$. Schäfke [103] made important simplifications to the calculations presented by Althammer, and in particular, he observed that the normalization $S_n(\lambda;1) = 1$ simplified many results. Gröbner [51] also studied the Sobolev-Legendre polynomials on the interval $[0,1]$, and he found a generalized Rodrigues' formula for these polynomials. Cohen [22] studied the zeros of $S_n(\lambda;\cdot)$ and he proved that they interlace with those of the Legendre polynomial P_{n-1} if $\lambda \geq 2/n$, among other results. Brenner [19] also studied the Sobolev orthogonal polynomials with respect to the inner product:

$$\langle f,g\rangle_S = \int_{0}^{\infty} f(x)g(x)e^{-x}dx + \lambda\int_{0}^{\infty} f'(x)g'(x)e^{-x}dx, \quad \lambda > 0, \tag{2.2}$$

with similar results to those of Althammer.

Schäfke and Wolf [104] considered a family of inner products of the form:

$$\langle f,g\rangle_S = \sum_{j,k=0}^{\infty}\int_a^b f^{(j)}(x)g^{(k)}(x)v_{j,k}(x)w(x)dx, \tag{2.3}$$

where $w(x)$ and (a,b) are one of the three classical cases (Hermite, Laguerre or Jacobi), and where $v_{j,k}$ are polynomials that satisfy $v_{j,k} = v_{k,j}$, $j,k = 0,1,2,\ldots$, and other additional restrictions. With their study, Schäfke and Wolf showed eight classes of Sobolev orthogonal polynomials, which they called a *generalization of classical orthogonal polynomials*. In addition, through their study they extended known results to the Sobolev case. Recently, analytical and algebraic properties were studied for particular cases of (2.3). See for example [4, 11, 37–39, 84, 96].

After the Schäfke and Wolf's paper, the theory remained without significant contributions for about two decades until the *coherent pairs* appeared in a paper due to Iserles, Koch, Norsett, and Sanz-Serna [56] when they studied the polynomials with respect to the inner product:

$$\langle f,g\rangle_\lambda = \int_{-\infty}^{\infty} f(x)g(x)d\varphi(x) + \lambda\int_{-\infty}^{\infty} f'(x)g'(x)d\psi(x), \quad \lambda \geq 0, \tag{2.4}$$

[1] It is well-known [111, Theorem 3.3.1] that a standard orthogonal polynomial has all its zeros inside the interval of orthogonality.

where $d\varphi$ and $d\psi$ are two Borel measures. They showed, among several results and under certain conditions, that the orthogonal polynomials with respect to (2.4) can be expanded in terms of the orthogonal polynomials with respect to $d\varphi$. According with [56, Theorem 3], the pair $\{d\varphi, d\psi\}$ is *coherent* if there exist non-zero constants $a_1, a_2, a_3, \ldots$ such that:

$$P_n(d\psi; x) = a_{n+1}P'_{n+1}(d\varphi; x) - a_n P'_n(d\varphi; x), \quad n \geq 1, \tag{2.5}$$

where $P_n(d\varphi; \cdot)$ and $P_n(d\psi; \cdot)$ are the orthogonal polynomials with respect to $d\varphi$ and $d\psi$, respectively. If both measures $d\varphi$ and $d\psi$ are symmetric (that is, invariant under the transformation $x \mapsto -x$), then the pair $\{d\varphi, d\psi\}$ is *symmetrically coherent* [56, Theorem 4] if there exist non-zero constants $a_1, a_2, a_3, \ldots$ such that:

$$P_n(d\psi; x) = a_{n+1}P'_{n+1}(d\varphi; x) - a_{n-1}P'_{n-1}(d\varphi; x), \quad n \geq 2.$$

Inner products like (2.1) and (2.2), which involve derivatives, do not satisfy the symmetry property (1.1). This fact makes that the three-term recurrence relation no longer holds, and as a consequence, many properties (their zeros, for example) of the corresponding Sobolev polynomials become more difficult to study. Many techniques have been developed through the years to balance the lack of this tool as we will show in the sequel. In next sections we present some references that have appeared in the last thirty years.

2.1.2 Recent publications on Sobolev polynomials

After the notion of coherent pair appeared in 1991, this idea was incorporated by some authors to the study of some sequences of Sobolev orthogonal polynomials.

Meijer [90] derived general results for the zero distribution of $\{S_n(\lambda; x)\}_{n\geq 0}$, the sequence of orthogonal polynomials with respect to (2.4), when $\{d\varphi, d\psi\}$ is a coherent pair. Meijer showed that, for $n \geq 2$ and if λ is large enough, the polynomial $S_n(\lambda; \cdot)$ has n different, real zeros, and these zeros interlace with those of $P_{n-1}(d\varphi; \cdot)$ and $P_{n-1}(d\psi; \cdot)$. This author also studied the case when $\{d\varphi, d\psi\}$ is a symmetrically coherent pair. This last situation is more complicated, even leading to complex zeros of $S_n(\lambda; \cdot)$ when n is an even number.

The results in [90] were generalized by De Bruin and Meijer [26]. They showed that, under certain conditions, the polynomials $\{S_n(\lambda; x)\}_{n\geq 0}$ satisfy a 5-term recurrence relation. Marcellán, Pérez, and Piñar [78–80] deduced some properties concerning the localization and separation of the zeros of these polynomials in the Laguerre case ($d\varphi(x) = d\psi(x) = x^\alpha e^{-x}dx$, $\alpha > -1$) and Gegenbauer case ($d\varphi(x) = d\psi(x) = (1-x^2)^{\alpha-1/2}dx$, $\alpha > -1/2$), and a similar work was carried out by Kim, Kwon, Marcellán, and Yoon [59] in the Jacobi case ($d\varphi(x) = (1-x)^\alpha(1+x)^\beta dx$, $d\psi(x) = (1-x)^{\alpha+1}(1+x)^\beta dx$, $\alpha > -1$, $-1 < \beta \leq 0$).

Marcellán and Petronilho [81] extended the notion of coherent pair to linear functionals $\{\Phi, \Psi\}$. These authors found all the coherent pairs when one of the measures is classical. Meijer [93] proved that if $\{\Phi, \Psi\}$ is a coherent pair, then at least one of them has to be classical (Hermite, Laguerre, Jacobi, or Bessel). A similar result was derived for symmetrically coherent pairs.

The *generalized coherent pairs* appeared in a paper due to Kim, Kwon, Marcellán, and Yoon [58] when they solved an inverse problem concerning coherent pairs. According with [58], the pair $\{d\varphi, d\psi\}$ is called a generalized coherent pair if the following relation holds for all $n \geq 1$:

$$P_n(d\psi;x) + b_{n-1}P_{n-1}(d\psi;x) = \frac{P'_{n+1}(d\varphi;x)}{n+1} + a_n\frac{P'_n(d\varphi;x)}{n}, \quad n \geq 1,$$

which is a more general relation than (2.5). In this case $P_n(d\varphi;\cdot)$ and $P_n(d\psi;\cdot)$ are monic polynomials. These authors also extended the definition to linear functionals $\{\Phi, \Psi\}$. Alfaro, Marcellán, Peña, and Rezola [5] identified all the generalized coherent pairs when Φ is a classical linear functional. Delgado and Marcellán [31] made a complete identification of generalized coherent pairs, and they found that either Φ or Ψ must be a semiclassical linear functional. Berti, Bracciali, and Sri Ranga [16] and Berti and Sri Ranga [17] provided two examples of generalized coherent pairs in the Jacobi case:

$$\begin{aligned} d\varphi(x) &= (1-x)^\alpha(1+x)^\beta dx, \\ d\psi(x) &= \frac{x-\xi_0}{x-\xi_1}(1-x)^{\alpha+1}(1+x)^{\beta+1}dx + M\delta_{\xi_1}, \end{aligned} \quad \alpha,\beta > -1, \quad |\xi_0|,|\xi_1| \geq 1,$$

and Laguerre case:

$$d\varphi(x) = x^\alpha e^{-x}dx, \quad d\psi(x) = \frac{x-\xi_0}{x-\xi}x^{\alpha+1}e^{-x}dx + M\delta_\xi, \quad \alpha > -1, \quad |\xi_0| \geq 1, \quad \xi \leq 0,$$

where δ_ξ is the Dirac delta at ξ.

A further generalization of coherent pair is the so-called (M,K)-*coherent pair of order* (m,k), where M, K, m, k are non-negative integers. A pair of linear functionals $\{\Phi, \Psi\}$ is said to be a (M,K)-coherent pair of order (m,k) if $\{P_n(\Phi;x)\}_{n\geq 0}$ and $\{P_n(\Psi;x)\}_{n\geq 0}$, the monic sequences of orthogonal polynomials with respect to Φ and Ψ, respectively, satisfy a linear algebraic structure relation of the form:

$$\sum_{i=0}^{M} r_{i,n}P^{(m)}_{n-i+m}(\Phi;x) = \sum_{i=0}^{K} s_{i,n}P^{(k)}_{n-i+k}(\Psi;x), \tag{2.6}$$

where $r_{i,n}$ and $s_{i,n}$ are complex numbers satisfying some conditions. The relation (2.6) was studied by de Jesus and Petronilho [28] in the context of an inverse problem in the theory of orthogonal polynomials. Kwon, Lee, and Marcellán [69] considered the $(2,0)$-coherent pair of order $(1,0)$ when they studied the Sobolev orthogonal polynomials with respect to (2.4). These authors provided [69, section 5] an efficient way for computing the Sobolev-Fourier coefficients at the Fourier's series expansion. Delgado and Marcellán [31] characterized all the pairs of linear functionals $\{\Phi, \Psi\}$ that are $(1,1)$-coherent of order $(1,0)$. The (M,K)-coherence of order $(1,0)$ was studied by de Jesus and Petronilho [29] and the Sobolev orthogonal polynomials with respect to

(2.4) were considered. In this case, these authors also gave an an efficient algorithm for computing the coefficients at the Fourier's series expansion with respect to (2.4). The case $(M,0)$-coherent of order $(1,0)$ was studied by Marcellán, Martínez-Finkelshtein, and Moreno-Balcázar [77], where several examples of non-trivial measures were given by these authors. The more general case of (M,K)-coherence of order (m,k) was studied by de Jesus, Marcellán, Petronilho, and Pinzón-Cortés [27]. In particular, in their paper they studied the Sobolev orthogonal polynomials with respect to the inner product:

$$\langle f,g\rangle_\lambda = \int_{-\infty}^{\infty} f(x)g(x)d\varphi(x) + \lambda\int_{-\infty}^{\infty} f^{(m)}(x)g^{(m)}(x)d\psi(x), \quad \lambda>0, \tag{2.7}$$

where $\{d\varphi, d\psi\}$ is a (M,K)-coherent pair of measures of order $(m,0)$, $m\geq 1$. The inner product (2.7) was also studied by Marcellán and Pinzón-Cortés [82]. The notion of coherent pairs has been also extended to complex domains with the corresponding study of the associated Sobolev orthogonal polynomials. See, for example, Marcellán and Pinzón-Cortés [83]. At the time of writing this document, the research on coherent pairs still continues.

It is known that the classical polynomials (with parameters greater than minus one) are also Sobolev orthogonal polynomials with respect to a certain non-standard inner products [86, Section 6]. More generally, when their parameters are taken to be real numbers, it is also known that the Laguerre and Jacobi polynomials are orthogonal with respect to a Sobolev inner product. For example, Pérez and Piñar [97] proved that the *monic generalized Laguerre polynomials*:

$$L_n^{(\alpha)}(x) = (-1)^n n! \sum_{j=0}^{n} \frac{(\alpha+j+1)_{n-j}}{j!(n-j)!}(-x)^j, \quad \alpha\in\mathbb{R},$$

are orthogonal with respect to the Sobolev inner product:

$$\begin{aligned}\langle f,g\rangle_S^{(k,\alpha+k)} &= \int_0^{\infty} \mathbf{F}(x)\mathbf{M}(k)\mathbf{G}(x)^T x^{\alpha+k}e^{-x}dx,\\ k &= \max\{0,\lfloor -\alpha\rfloor\}, \quad \alpha\in\mathbb{R},\end{aligned} \tag{2.8}$$

where:

$$\mathbf{F}(x) = \left(f(x), f'(x), \ldots, f^{(k)}(x)\right), \quad \mathbf{G}(x) = \left(g(x), g'(x), \ldots, g^{(k)}(x)\right),$$

and $\mathbf{M}(k) = \left(m_{ij}(k)\right)_{i,j=0}^{k}$ is a positive definite matrix of size $(k+1)\times(k+1)$ whose entries are given by:

$$m_{ij}(k) = \sum_{l=0}^{\min\{i,j\}} (-1)^{i+j}\binom{k-l}{i-l}\binom{k-l}{j-l}, \quad 0\leq i,j\leq k,$$

and where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. These authors also observed that if $\alpha\in\{-1,-2,-3,\ldots\}$ then the inner product (2.8), taking integration by parts, reduces to:

$$\langle f,g\rangle_S^{(k,0)} = \int_0^\infty f^{(k)}(x)g^{(k)}(x)e^{-x}dx + \sum_{i=0}^{k-1}\sum_{j=0}^{i}\frac{m_{ij}(k)}{2}[f^{(i)}(0)g^{(j)}(0)+f^{(j)}(0)g^{(i)}(0)]. \quad (2.9)$$

This last inner product was also studied by Kwon and Littlejohn [70], and the particular case $k=1$ was also considered by the same authors in [71].

In addition, Alfaro, Pérez, Piñar, and Rezola [6] studied the Sobolev polynomials with respect to the bilinear form:

$$\langle f,g\rangle_S^{(N)} = \mathscr{L}(f^{(N)}g^{(N)}) + \mathbf{F}(c)\mathbf{A}\mathbf{G}(c)^T, \quad (2.10)$$

which is more general than (2.9), where $\mathscr{L}$ is a quasi-definite linear functional on Π, c is a real number, N is a positive integer number, $\mathbf{A}$ is a symmetric $N\times N$ real matrix such that each of its principal submatrices is regular, $\mathbf{F}(c) = \Big(f(c), f'(c), \ldots, f^{(N-1)}(c)\Big)$, and $\mathbf{G}(c) = \Big(g(c), g'(c), \ldots, g^{(N-1)}(c)\Big)$. These authors provided examples of orthogonal polynomials with respect to (2.10), with an adequate choice of c. These examples are the Laguerre polynomials $\Big\{L_n^{(-N)}(x)\Big\}_{n\geq 0}$ with $c=0$, Jacobi polynomials $\Big\{P_n^{(-N,\beta)}(x)\Big\}_{n\geq 0}$ with $c=1$ and $\beta+N$ not being a negative integer, and $\Big\{P_n^{(\alpha,-N)}(x)\Big\}_{n\geq 0}$ with $c=-1$ and $\alpha+N$ not being a negative integer.

In a similar way, the Sobolev orthogonality with respect an inner product like (2.10) was studied by Alfaro, Álvarez de Morales, and Rezola [3] for the remainder cases of the Jacobi polynomials, and by Álvarez de Morales, Pérez, and Piñar [8] for the Gegenbauer polynomials $\Big\{C_n^{(-N+1/2)}(x)\Big\}_{n\geq 0}$, with $N\geq 1$ being a non-negative integer. Jung, Kwon, and Lee [57] made a similar study for the orthogonal polynomials with respect to (2.10) in a general setting, but only for first-order derivatives.

Xu [113, section 2.3] used some of the results in [57] to deduce that the sequence $\{q_n(x)\}_{n\geq 0}$ of orthogonal polynomials with respect to the Sobolev inner product:

$$\langle f,g\rangle = 2^{2-d/2}\lambda\int_{-1}^{1} f'(x)g'(x)(1+x)^{d/2}dx + f(-1)g(-1), \quad \lambda>0, \quad (2.11)$$

where d is a non-negative integer, is defined by:

$$q_0(x)=1, \quad q_n(x) = \frac{2}{n+\frac{d-2}{2}}\left(P_n^{(-1,\frac{d-2}{2})}(x) - (-1)^n\frac{(d/2)_n}{n!}\right), \quad n\geq 1, \quad (2.12)$$

where $P_n^{(\alpha,\beta)}$ is the Jacobi Polynomial of degree n.

Pérez, Piñar, and Xu [98, Section 4] studied the Sobolev orthogonal polynomials with respect to the inner product:

$$\langle f,g\rangle_{\alpha,\beta} := \int_{-1}^{1} f(x)g(x)w_{\alpha,\beta}(x)dx +$$

$$2\lambda \int_{-1}^{1} \left(f, f'\right) \begin{pmatrix} \beta(2\beta-d+2) & (2\beta-d+2)(1+x) \\ (2\beta-d+2)(1+x) & 4(1+x)^2 \end{pmatrix} \begin{pmatrix} g \\ g' \end{pmatrix} w_{\alpha,\beta-1}(x)dx, \tag{2.13}$$

where $d \in \mathbb{N}$, $\lambda > 0$, $\alpha > -1$, $\beta > \max\{0, (d-2)/2\}$, and $w_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ is the Jacobi weight function. The restriction $\beta > \max\{0, (d-2)/2\}$ guarantees that the 2×2 matrix in (2.13) is positive definite, and consequently, (2.13) is indeed an inner product. This inner product appeared naturally when these authors studied a family of Sobolev orthogonal polynomials in several variables on the unit ball (see Theorem 2.13 below).

Sobolev polynomials also have been used in the spectral theory for solving differential equations. For example, Sharapudinov [105–108] considered the Sobolev orthogonal polynomials with respect to the inner product:

$$\langle f, g\rangle_S = \int_a^b f^{(r)}(x)g^{(r)}(x)w(x)dx + \sum_{i=0}^{r-1} f^{(i)}(a)g^{(i)}(a), \quad r \in \mathbb{N}, \tag{2.14}$$

where w is a weight function on $[a, b]$. This author studied the approximation properties of Fourier series with weights for the Haar functions and Jacobi polynomials (with special attention to the Chebyshev, Legendre and Gegenbauer cases), and he showed that the Fourier series and sums of orthogonal polynomials with respect to (2.14) are an efficient tool for the approximate solution of the Cauchy problem for ordinary differential equations.

The generalized Jacobi polynomials (with arbitrary parameters $\alpha, \beta \in \mathbb{R}$) also have been used in spectral methods. See, for example, [52, 74, 75, 109]. Li and Xu [75] and Xu [114] defined the generalized Jacobi polynomial by the equation:

$$J_n^{\alpha,\beta}(x) := \sum_{k=\iota_0}^{n} \frac{(k+\alpha+1)_{n-k}}{(n-k)!k!(n+\alpha+\beta+k+1)_{n-k}} \left(\frac{x-1}{2}\right)^k, \qquad \alpha, \beta \in \mathbb{R}, \quad n \in \mathbb{N}_0, \tag{2.15}$$

where $\iota_0 = \iota_0^{\alpha,\beta}(n) := -n-\alpha-\beta$ if $-n-\alpha-\beta \in \{1, 2, 3, \ldots, n\}$ and $\iota_0 = 0$ otherwise. This extends the definition of the Jacobi polynomials to all $\alpha, \beta \in \mathbb{R}$ avoiding the problem of a degree reduction. Indeed, if $\alpha, \beta > -1$ then $J_n^{\alpha,\beta}$ is a renormalization of the ordinary Jacobi polynomial $P_n^{(\alpha,\beta)}$ because $\iota_0^{\alpha,\beta}(n) = 0$ and the following relation holds [114, proposition 2.2]:

$$J_n^{\alpha,\beta}(x) = \frac{1}{(n+\alpha+\beta+1)_n} P_n^{(\alpha,\beta)}(x), \quad \text{if} \quad \iota_0^{\alpha,\beta}(n) = 0.$$

Many other identities [114, Section 2] for the ordinary Jacobi polynomials were extended to the generalized polynomials. From these generalized polynomials, Xu [114] defined the polynomials $\widehat{J}_n^{\alpha,-m}$, $\widetilde{J}_n^{\alpha,-m}$, $\widehat{J}_n^{-l,\beta}$, and $\widetilde{J}_n^{-l,\beta}$, with $n \in \mathbb{N}_0$, $m, l \in \mathbb{N}$,

$\alpha, \beta \in \mathbb{R}$, and $\alpha^\sharp := \max\{0, \lfloor -\alpha \rfloor\}$, by the following relations:

$$\widehat{J}_n^{\alpha,-m}(x) = \begin{cases} J_n^{\alpha,-m}(2x-1), & n \geq m+\alpha^\sharp, \\ J_n^{\alpha,-m}(2x-1) + \displaystyle\sum_{k=0}^{\min\{n,m\}-1} J_{n-k}^{\alpha+k,-m+k}(-1)\frac{x^k}{k!}, & n < m+\alpha^\sharp, \end{cases}$$

$$\widetilde{J}_n^{\alpha,-m}(x) = \begin{cases} (-1)^n J_n^{\alpha,-m}(1-2x), & n \geq m+\alpha^\sharp, \\ (-1)^n J_n^{\alpha,-m}(1-2x) - \displaystyle\sum_{k=0}^{n-1} (-1)^{n-k} J_{n-k}^{\alpha+k,-m+k}(-1)\frac{(x-1)^k}{k!}, & n < m+\alpha^\sharp, \end{cases}$$

$$\widehat{J}_n^{-l,\beta}(x) = \begin{cases} J_n^{-l,\beta}(2x-1), & n \geq l, \\ J_n^{-l,\beta}(2x-1) + \displaystyle\sum_{k=0}^{n-1} J_{n-k}^{-l+k,\beta+k}(1)\frac{(x-1)^k}{k!}, & n < l, \end{cases}$$

$$\widetilde{J}_n^{-l,\beta}(x) = \begin{cases} (-1)^n J_n^{-l,\beta}(1-2x), & n \geq l, \\ (-1)^n J_n^{-l,\beta}(1-2x) - \displaystyle\sum_{k=0}^{n-1} (-1)^{n-k} J_{n-k}^{-l+k,\beta+k}(1)\frac{x^k}{k!}, & n < l, \end{cases}$$

and he also defined the Sobolev inner products:

$$\langle f, g\rangle_{\alpha,-m} := \int_0^1 f^{(m)}(x)g^{(m)}(x)(1-x)^{\alpha+m}dx + \sum_{k=0}^{m-1}\lambda_k f^{(k)}(0)g^{(k)}(0), \tag{2.16}$$
$$\lambda_k > 0, \quad k = 0,1,\dots m-1, \quad \alpha+m > -1,$$

$$\langle f, g\rangle_{-l,\beta} := \int_0^1 f^{(l)}(x)g^{(l)}(x)x^{\beta+l}dx + \sum_{k=0}^{l-1}\lambda_k f^{(k)}(1)g^{(k)}(1), \tag{2.17}$$
$$\lambda_k > 0, \quad k = 0,1,\dots l-1, \quad \beta+l > -1.$$

Then, Xu [114, Propositions 3.1 to 3.4] proved the following results.

1. The sequence $\left\{\widehat{J}_n^{\alpha,-m}\right\}_{n\geq 0}$, with $m = 1,2,3,\dots$ and $\alpha+m > -1$, is orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_{\alpha,-m}$ defined in (2.16).

2. The sequence $\left\{\widetilde{J}_n^{\alpha,-m}\right\}_{n\geq 0}$, with $m = 1,2,3,\dots$ and $\alpha+m > -1$, is orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_{-m,\alpha}$ defined in (2.17).

3. The sequence $\left\{\widehat{J}_n^{-l,\beta}\right\}_{n\geq 0}$, with $l = 1,2,3,\dots$ and $\beta+l > -1$, is orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_{-l,\beta}$ defined in (2.17).

4. The sequence $\left\{\widetilde{J}_n^{-l,\beta}\right\}_{n\geq 0}$, with $l = 1,2,3,\dots$ and $\beta+l > -1$, is orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_{\beta,-l}$ defined in (2.16).

The so-called Sobolev-type inner products, for which the derivatives appear only on function evaluations on a finite discrete set, also have considered in literature. They have the form:

$$\langle f,g\rangle_S = \int_{\mathbb{R}} f(x)g(x)d\mu + \sum_{k=0}^{m}\int_{\mathbb{R}} f^{(k)}(x)g^{(k)}(x)d\mu_k,$$

where $d\mu$ is a positive Borel measure supported on an *infinite* subset of $\mathbb{R}$, and $d\mu_k$, $k=0,1,\dots,m$, are Borel measures supported on *finite* subsets of $\mathbb{R}$. Koekoek [62] studied the Laguerre case $d\mu = x^\alpha e^{-x}dx/\Gamma(\alpha+1)$, $x\in\mathbb{R}_+$, $\alpha>-1$, and $d\mu_k = M_k\delta_0$, $M_k\geq 0$, $k=0,1,2,\dots,m$, where δ_c is the Dirac delta measure supported at $c\in\mathbb{R}$. Bavinck and Meijer [14, 15] studied the Gegenbauer case $d\mu = \Gamma(2\alpha+2)(1-x^2)^\alpha dx/(2^{2\alpha+1}\Gamma^2(\alpha+1))$, $x\in[-1,1]$, $\alpha>-1$, and $d\mu_0 = M(\delta_{-1}+\delta_1)$, $d\mu_1 = N(\delta_{-1}+\delta_1)$, $M,N\geq 0$. The case $d\mu = w(x)dx$, where w is a weight function, $d\mu_k = 0$, $k=0,1,\dots,m-1$, $d\mu_m = \lambda^{-1}\delta_c$, $\lambda>0$, $c\in\mathbb{R}$, was studied by Marcellán and Ronveaux [85].

The well-known Favard's theorem for standard polynomials, which guarantees the orthogonality of a sequence of polynomials if it satisfies a three-term recurrence relation, was generalized by Durán [45]. A similar work in this direction is due to Evans, Littlejohn, Marcellán, Markett, and Ronveaux [46].

Koekoek [63] studied the sequence $\left\{P_n^{\alpha,\beta,N,M}(x)\right\}_{n\geq 0}$ of orthogonal polynomials with respect to the inner product:

$$\langle f,g\rangle = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1} f(x)g(x)(1-x)^\alpha(1+x)^\beta dx + \\ Mf(-1)g(-1) + Nf(1)g(1),$$

with $\alpha,\beta>-1$ and $M,N\geq 0$. This author found that $\left\{P_n^{\alpha,\beta,N,M}(x)\right\}_{n\geq 0}$ satisfied a differential equation of the form:

$$M\sum_{i=0}^{\infty} a_i(x)y^{(i)}(x) + N\sum_{i=0}^{\infty} b_i(x)y^{(i)}(x) + MN\sum_{i=0}^{\infty} c_i(x)y^{(i)}(x) + \\ (1-x^2)y''(x) + [\beta-\alpha-(\alpha+\beta+2)x]y'(x) + n(n+\alpha+\beta+1)y(x) = 0,$$

where $a_i(x), b_i(x), c_i(x)$ are polynomials, independent of n. Similarly, in [63], a study for the polynomials $\left\{L_n^{\alpha,N,M}(x)\right\}_{n\geq 0}$ with respect to the Sobolev inner product:

$$\langle f,g\rangle = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty f(x)g(x)x^\alpha e^{-x}dx + Mf(0)g(0) + Nf'(0)g'(0),$$

with $\alpha>-1$ and $M,N\geq 0$, was carried out. This author also found that the sequence $\left\{L_n^{\alpha,N,M}(x)\right\}_{n\geq 0}$ satisfied a differential equation of the form:

$$M\sum_{i=0}^{\infty} a_i(x)y^{(i)}(x) + N\sum_{i=0}^{\infty} b_i(x)y^{(i)}(x) + MN\sum_{i=0}^{\infty} c_i(x)y^{(i)}(x) +$$

$$xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0,$$

where $a_i(x), b_i(x), c_i(x)$ are again polynomials. Complementary results on these generalized Jacobi and Laguerre polynomials can be found in [60, 61, 64]. Some other results on differential equations for Sobolev-type orthogonal polynomials are due to Arvesú, Álvarez-Nodarse, Marcellán, and Pan [12], Dueñas and Garza [35], and Dueñas and Marcellán [39].

The zeros of the Sobolev polynomials also have been studied in recent years. Contrary to the standard polynomials on the real line, for which their zeros are all real, simple, interlace, and they lie at the interior of the interval of orthogonality, in the Sobolev setting some of these properties are lost. On this subject, we could reference the works [4, 10, 12, 15, 22, 25, 26, 33, 34, 59, 84, 90, 91] to name just a few. And more recently, over the last ten years, the works by Dueñas and Garza [35], Huertas, Marcellán, and Rafaeli [55], and Molano-Molano [96].

2.2 Sobolev orthogonal polynomials in several variables

In contrast to one variable, the study of Sobolev orthogonal polynomials in several variables is recent. Most of the results were obtained in two variables and where the inner products introduced only first-order derivatives [18]. Next, we show a summary in regards to Sobolev orthogonal polynomials of several variables on different domains.

2.2.1 Sobolev-type orthogonal polynomials in several variables

Mello, Paschoa, Pérez, and Piñar [94] studied the Sobolev-type orthogonal polynomials with respect to the inner product:

$$\langle f, g \rangle_S = \langle f, g \rangle_G + \lambda \nabla f(\mathbf{p}) \cdot \nabla g(\mathbf{p}), \quad \lambda > 0, \tag{2.18}$$

where $\langle \cdot, \cdot \rangle_G$ is the inner product:

$$\langle f, g \rangle_G = \int_G f(\mathbf{x}) g(\mathbf{x}) d\mu(\mathbf{x}), \tag{2.19}$$

and where $G \subset \mathbb{R}^d$ is a domain having a non-empty interior, $d\mu$ is a positive measure defined on the domain G, and $\mathbf{p}$ is a given point in $\mathbb{R}^d$.

Let $\{P_\alpha^n : |\alpha| = n\}$ be an orthonormal basis of $\mathscr{V}_n^d(G)$, the space of orthogonal polynomials with respect to (2.19), and let $\mathbb{P}_n$ denote the column vector:

$$\mathbb{P}_n(\mathbf{x}) := \left(P^n_{\alpha^{(1)}}(\mathbf{x}), P^n_{\alpha^{(2)}}(\mathbf{x}), \cdots, P^n_{\alpha^{(r^d_n)}}(\mathbf{x}) \right)^T,$$

where $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(r^d_n)}$ is the arrangement of elements in $\left\{\alpha \in \mathbb{N}_0^d : |\alpha| = n\right\}$ according to the reverse lexicographical order. In addition, let $\nabla \mathbb{P}_n(\mathbf{x})$ denote the matrix of size $r_n^d \times d$ given by:

$$\nabla \mathbb{P}_n(\mathbf{x}) = \Big(\partial_1 \mathbb{P}_n(\mathbf{x}), \partial_2 \mathbb{P}_n(\mathbf{x}), \cdots, \partial_d \mathbb{P}_n(\mathbf{x})\Big)$$

and the kernel function of Π_n^d by:

$$K_n(\mathbf{x}, \mathbf{y}) = \sum_{j=0}^{n} \mathbb{P}_j^T(\mathbf{x})(\mathbf{H}_j)^{-1}\mathbb{P}_j(\mathbf{y}), \quad \mathbf{H}_j = \left\langle \mathbb{P}_j, \mathbb{P}_j^T \right\rangle_G,$$

for which $\mathbf{K}_n^{(1,0)}$, $\mathbf{K}_n^{(0,1)}$ and $\mathbf{K}_n^{(1,1)}$ are matrices of sizes $d \times 1$, $1 \times d$, and $d \times d$, respectively, given by:

$$\mathbf{K}_n^{(1,0)}(\mathbf{x}, \mathbf{y}) = \sum_{j=0}^{n} (\nabla \mathbb{P}_j(\mathbf{x}))^T (\mathbf{H}_j)^{-1}\mathbb{P}_j(\mathbf{y}), \quad \mathbf{K}_n^{(0,1)}(\mathbf{x}, \mathbf{y}) = \sum_{j=0}^{n} \mathbb{P}_j^T(\mathbf{x})(\mathbf{H}_j)^{-1}\nabla \mathbb{P}_j(\mathbf{y}),$$

$$\mathbf{K}_n^{(1,1)}(\mathbf{x}, \mathbf{y}) = \Big(\partial_{x_i}\partial_{y_j} K_n(\mathbf{x}, \mathbf{y})\Big)_{i,j=1}^{d}.$$

Then the following result was proved in [94].

Theorem 2.1. *[94, Theorem 3.1] Let $\{\mathbb{P}_n\}_{n\geq 0}$ be an orthonormal polynomial system associated with* (2.19)*. We define the polynomial system $\{\mathbb{Q}_n\}_{n\geq 0}$ by means of*

$$\begin{aligned} \mathbb{Q}_0(\mathbf{x}) &:= \mathbb{P}_0(\mathbf{x}), \\ \mathbb{Q}_n(\mathbf{x}) &:= \mathbb{P}_n(\mathbf{x}) - \lambda \nabla \mathbb{P}_n(\mathbf{p})[\mathbf{I}_d + \lambda \mathbf{K}_{n-1}^{(1,1)}(\mathbf{p}, \mathbf{p})]^{-1}\mathbf{K}_{n-1}^{(1,0)}(\mathbf{p}, \mathbf{x}), \quad n \geq 1. \end{aligned} \tag{2.20}$$

Then $\{\mathbb{Q}_n\}_{n\geq 0}$ is a sequence of orthogonal polynomials with respect to $\langle \cdot, \cdot \rangle_S$ defined in (2.18)*. Reciprocally, any sequence of orthogonal polynomials with respect to* (2.18) *can be expressed as in* (2.20)*.*

Additional results [94, Lemma 2.1, Proposition 3.2, Theorem 3.3] showed that $\mathbf{I}_d + \lambda \mathbf{K}_n^{(1,1)}(\mathbf{p}, \mathbf{p})$, $\lambda > 0$, $n \geq 0$, is a symmetric and non-singular matrix of size $d \times d$, and explicit expressions were given for $\mathbf{G}_n = \left\langle \mathbb{Q}_n, \mathbb{Q}_n^T \right\rangle_S$ and its inverse $\mathbf{G}_n^{-1}$, and for the kernel function $\widehat{K}_n(\mathbf{x}, \mathbf{y})$ associated with $\langle \cdot, \cdot \rangle_S$ which is given by:

$$\widehat{K}_n(\mathbf{x}, \mathbf{y}) = \sum_{j=0}^{n} \mathbb{Q}_j^T(\mathbf{x}) \mathbf{G}_j^{-1} \mathbb{Q}_j(\mathbf{y}).$$

As a generalization from the previous result, Dueñas, Garza, and Piñar [36] studied the Sobolev-type polynomials with respect to:

$$\langle f, g \rangle_S = \langle f, g \rangle_\sigma + M \nabla^{(j)} f(\xi)(\nabla^{(j)} g(\xi))^T, \quad \langle f, g \rangle_\sigma := \int_G f(\mathbf{x}) g(\mathbf{x}) d\sigma(\mathbf{x}). \tag{2.21}$$

where σ is a measure defined on the domain $G \subseteq \mathbb{R}^d$ with a non-empty interior, $\xi \in \mathbb{R}^d$, $\nabla^{(j)} f$ is the row vector which contains all the partial derivatives of order j of f, and $M \in \mathbb{R}_+$. The main result we found in [36] is the following.

Theorem 2.2. *[36, Theorem 1] Let $\{\mathbb{P}_n\}_{n\geq 0}$ be an orthonormal polynomial system (OPS) associated with the inner product $\langle\cdot,\cdot\rangle_\sigma$. Define a Sobolev-type inner product as in* (2.21)*. Then, if we denote by $\{\mathbb{Q}_n\}_{n\geq 0}$ its corresponding OPS, normalized in such a way that $\mathbb{Q}_n - \mathbb{P}_n$ is a r_n^d dimensional vector whose components are polynomials of total degree lower than n, we have*

$$\mathbb{Q}_n(\mathbf{x}) = \begin{cases} \mathbb{P}_n(\mathbf{x}), & n < j, \\ \mathbb{P}_n(\mathbf{x}) - M\nabla^{(j)}\mathbb{P}_n(\xi)(\mathbf{I}_{d^j} + M\mathbf{K}_{n-1}^{(j,j)}(\xi,\xi))^{-1}\mathbf{K}_{n-1}^{(j,0)}(\xi,\mathbf{x}), & n \geq j, \end{cases} \tag{2.22}$$

Conversely, if we define $\{\mathbb{Q}_n\}_{n\geq 0}$ as in (2.22)*, then they are an OPS with respect to* (2.21)*.*

In (2.22), $\nabla^{(j)}\mathbb{P}_n$ is a matrix of size $r_n^d \times d^j$ which contains the partial derivatives of order j of $\mathbb{P}_n$, $K_n(\mathbf{x},\mathbf{y})$ is the kernel polynomial of Π_n^d, $\mathbf{K}_n^{(j,j)}(\mathbf{x},\mathbf{y})$ is a matrix of size $d^j \times d^j$ with all the partial derivatives of order j of $K_n(\mathbf{x},\mathbf{y})$, and

$$\mathbf{K}_n^{(j,0)}(\mathbf{x},\mathbf{y}) = \sum_{j=0}^{n}(\nabla^{(j)}\mathbb{P}_j(\mathbf{x}))^T(\mathbf{H}_j)^{-1}\mathbb{P}_j(\mathbf{y}), \quad \mathbf{K}_n^{(0,j)}(\mathbf{x},\mathbf{y}) = \sum_{j=0}^{n}\mathbb{P}_j^T(\mathbf{x})(\mathbf{H}_j)^{-1}\nabla^{(j)}\mathbb{P}_j(\mathbf{y}).$$

2.2.2 Sobolev orthogonal polynomials on the unit ball

At this moment, Sobolev orthogonal polynomials on the unit ball $\mathbb{B}^d$ are the most studied polynomials in several variables [86].

Xu [112] considered the Sobolev orthogonal polynomials in d variables on the unit ball with respect to the inner product:

$$\langle f,g\rangle_\triangle = \frac{1}{4d^2\,\mathrm{vol}(\mathbb{B}^d)}\int_{\mathbb{B}^d}\triangle\left[(1-\|\mathbf{x}\|^2)f(\mathbf{x})\right]\triangle\left[(1-\|\mathbf{x}\|^2)g(\mathbf{x})\right]d\mathbf{x}, \tag{2.23}$$

where $\triangle$ is the Laplacian operator, and where $\mathrm{vol}(\mathbb{B}^d)$ is the volume of $\mathbb{B}^d$ given in (1.33). This work was motivated by a study due to Atkinson and Hansen [13], where they found the same inner product (2.23) for the case $d=2$, in the numerical solution of the Poisson equation $-\triangle u = f(\cdot,u)$. The main result in [112, Theorem 2.4] showed an explicit construction for a mutually orthogonal basis of $\mathscr{V}_n^d(\triangle)$, the space of orthogonal polynomials with respect to (2.23), in terms of the Jacobi polynomials $P_n^{(a,b)}(t)$, which are orthogonal on $[-1,1]$ with respect to the weight $(1-t)^a(1+t)^b$, and harmonic polynomials. That is,

Theorem 2.3. *[112, Theorem 2.4] A mutually orthogonal basis for $\mathscr{V}_n^d(\triangle)$ is given by:*

$$Q_{0,\nu}^n(\mathbf{x}) = Y_\nu^n(\mathbf{x}),$$
$$Q_{j,\nu}^n(\mathbf{x}) = (1-\|\mathbf{x}\|^2)P_{j-1}^{(2,n-2j+\frac{d-2}{2})}(2\|\mathbf{x}\|^2-1)Y_\nu^{n-2j}(\mathbf{x}), \quad 1\leq j\leq n/2,$$

where $\left\{Y_\nu^{n-2j} : 1\leq\nu\leq a_{n-2j}^d\right\}$ is an orthonormal basis of $\mathscr{H}_{n-2j}^d$. Furthermore,

$$\left\langle Q_{0,\nu}^n, Q_{0,\nu}^n\right\rangle_\triangle = \frac{2n+d}{d}, \quad \left\langle Q_{j,\nu}^n, Q_{j,\nu}^n\right\rangle_\triangle = \frac{8j^2(j+1)^2}{d(n+d/2)}.$$

Similarly, Xu [113] considered the Sobolev orthogonal polynomials on the unit ball with respect to the inner products:

$$\langle f,g\rangle_I = \frac{\lambda}{\omega_{d-1}}\int_{\mathbb{B}^d}\nabla f(\mathbf{x})\cdot\nabla g(\mathbf{x})d\mathbf{x} + \frac{1}{\omega_{d-1}}\int_{\mathbb{S}^{d-1}} f(\xi)g(\xi)d\omega(\xi), \tag{2.24}$$

$$\langle f,g\rangle_{\|} = \frac{\lambda}{\omega_{d-1}}\int_{\mathbb{B}^d}\nabla f(\mathbf{x})\cdot\nabla g(\mathbf{x})d\mathbf{x} + f(\mathbf{0})g(\mathbf{0}), \tag{2.25}$$

where $\lambda > 0$, ∇ is the gradient operator, and ω_{d-1} is the area of the unit sphere given in (1.33). Similar results [113, Theorem 2.3 and 2.6] showed an explicit construction for mutually orthogonal bases for $\mathscr{V}_n^d(I)$ and $\mathscr{V}_n^d(\|)$, the spaces of orthogonal polynomials with respect to (2.24) and (2.25), respectively.

Theorem 2.4. *[113, Theorem 2.3] A mutually orthogonal basis*

$$\left\{U_{j,\nu}^n : 0\le j\le n/2, 1\le\nu\le a_{n-2j}^d\right\}$$

for $\mathscr{V}_n^d(I)$ is given by:

$$U_{0,\nu}^n(\mathbf{x}) = Y_\nu^n(\mathbf{x}),$$
$$U_{j,\nu}^n(\mathbf{x}) = (1-\|\mathbf{x}\|^2)P_{j-1}^{(1,n-2j+\frac{d-2}{2})}(2\|\mathbf{x}\|^2-1)Y_\nu^{n-2j}(\mathbf{x}), \quad 1\le j\le n/2,$$

where $\left\{Y_\nu^{n-2j} : 1\le\nu\le a_{n-2j}^d\right\}$ is an orthonormal basis of $\mathscr{H}_{n-2j}^d$. Furthermore,

$$\left\langle U_{0,\nu}^n, U_{0,\nu}^n\right\rangle_I = n\lambda+1, \quad \left\langle U_{j,\nu}^n, U_{j,\nu}^n\right\rangle_I = \frac{2j^2}{n+\frac{d-2}{2}}\lambda.$$

Theorem 2.5. *[113, Theorem 2.6] A mutually orthogonal basis*

$$\left\{V_{j,\nu}^n : 0\le j\le n/2, 1\le\nu\le a_{n-2j}^d\right\}$$

for $\mathscr{V}_n^d(\|)$ is given by:

$$V_{j,\nu}^n(\mathbf{x}) = U_{j,\nu}^n(\mathbf{x}), \quad 0\le j\le \left\lfloor\frac{n-1}{2}\right\rfloor,$$
$$V_{n/2}^n(\mathbf{x}) = \frac{4}{n+d-2}\left(P_{n/2}^{(-1,\frac{d-2}{2})}(2\|\mathbf{x}\|^2-1) - (-1)^{n/2}\frac{(d/2)_{n/2}}{(n/2)!}\right),$$

where $V_{n/2}^n(\mathbf{x}) := V_{n/2,\nu}^n(\mathbf{x})$ holds only when n is even. Furthermore,

$$\left\langle V_{j,\nu}^n, V_{j,\nu}^n\right\rangle_{\|} = \left\langle U_{j,\nu}^n, U_{j,\nu}^n\right\rangle_I, \quad 0\le j\le\left\lfloor\frac{n-1}{2}\right\rfloor,$$
$$\left\langle V_{n/2}^n, V_{n/2}^n\right\rangle_{\|} = \frac{8\lambda}{n+\frac{d-2}{2}}.$$

Piñar and Xu [99] studied the second-order differential operator $\mathcal{B}_\mu$ given in (1.43) for $\mu = -1,-2,-3,-4,\ldots$. One result [99, Theorem 3.3] showed that the orthogonal polynomials with respect to $\langle\cdot,\cdot\rangle_I$ in (2.24) satisfy the equation (1.43) for $\mu=-1$, that is,

Theorem 2.6. *[99, Theorem 3.3] Elements of $\mathscr{V}_n^d(I)$ satisfy $\mathcal{B}_{-1}P = -n(n+d-2)P$. In particular, the eigenfunctions of the operator $\mathcal{B}_{-1}$ consist of a complete polynomial basis.*

For $\mu = -k$, $k = 2, 3, \ldots$ Piñar and Xu defined:

$$\mathscr{U}_n^d(W_{-k}) := \mathscr{H}_n^d \cup \left(\bigcup_{j=1}^{k-1} \left[\sum_{\nu=0}^{j} a_{j,\nu}^n (1-\|\mathbf{x}\|^2)^\nu \right] \mathscr{H}_{n-2j}^d \right) \cup (1-\|\mathbf{x}\|^2)^k \mathscr{V}_{n-2k}^d(W_k),$$

where, for $1 \le j \le k-1$,

$$a_{j,\nu}^n := \frac{(-1)^{j-\nu} j!(-k+1)_j (n-j-k+d/2)_\nu}{\nu!(j-\nu)!(-k+1)_\nu (n-j-k+d/2)_j}, \quad 0 \le \nu \le j,$$

and where $a_{j,\nu}^n$ is well-defined if $n-j-k+\nu+d/2 \neq 0$. Piñar and Xu [99, Theorem 3.4] showed the following result.

Theorem 2.7. *[99, Theorem 3.4] If $\mu = -k$ and $k = 2, 3, \ldots$, then the polynomials in $\mathscr{U}_n^d(W_{-k})$ satisfy equation (1.43); that is, $\mathcal{B}_{-k}P = -n(n-2k+d)P$ for $P \in \mathscr{U}_n^d(W_{-k})$. Furthermore,*

$$\dim \mathscr{U}_n^d = \dim \mathscr{P}_n^d, \quad \text{if } n-2k-1+d/2 \neq 0.$$

In particular, the operator $\mathcal{B}_{-k}$ has a complete polynomial basis of eigenfunctions if the dimension d is odd.

Piñar and Xu [99, Theorem 4.1] observed that for $\mu = -2$, the polynomials in

$$\mathscr{V}_n^d(W_{-2}) := \mathscr{H}_n^d \cup (1-\|\mathbf{x}\|^2)\mathscr{H}_{n-2}^d \cup (1-\|\mathbf{x}\|^2)^2 \mathscr{V}_{n-4}^d(W_2),$$

are orthogonal with respect to the inner product:

$$\langle f, g \rangle_{-2} = \frac{\lambda}{\omega_{d-1}} \int_{\mathbb{B}^d} \triangle f(\mathbf{x}) \triangle g(\mathbf{x}) d\mathbf{x} + \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} f(\mathbf{x}) g(\mathbf{x}) d\omega(\mathbf{x}), \quad \lambda > 0.$$

Theorem 2.8. *[99, Theorem 4.1] The elements in $\mathscr{V}_n^d(W_{-2})$ are orthogonal polynomials with respect to $\langle \cdot, \cdot \rangle_{-2}$. Moreover, they contain an orthonormal basis; in other words,*

$$\mathscr{V}_n^d(W_{-2}) = \mathscr{H}_n^d \oplus (1-\|\mathbf{x}\|^2)\mathscr{H}_{n-2}^d \oplus (1-\|\mathbf{x}\|^2)^2 \mathscr{V}_{n-4}^d(W_2).$$

With respect to the inner product:

$$\begin{aligned}\langle f, g \rangle_n^* = {} & \frac{\lambda_1}{\omega_{d-1}} \int_{\mathbb{B}^d} \triangle f(\mathbf{x}) \triangle g(\mathbf{x}) d\mathbf{x} + \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} f(\mathbf{x}) g(\mathbf{x}) d\omega(\mathbf{x}) \\ & + \frac{\lambda_2}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} \frac{d}{dr}[r^{n-4-d} f(\mathbf{x})] \frac{d}{dr}[r^{n-4-d} g(\mathbf{x})] d\omega(\mathbf{x}), \quad \lambda_1, \lambda_2 > 0,\end{aligned}$$

where d/dr is the normal derivative, another result is the following.

Theorem 2.9. *[99, Theorem 4.2] The elements of $\mathscr{V}_n^d(W_{-2})$ are orthogonal polynomials with respect to the inner product $\langle f,g\rangle_n^*$.*

Li and Xu [75] studied the polynomials in $\mathscr{V}_n^d(W_{-s})$, $s\in\mathbb{N}$, the space of Sobolev orthogonal polynomials with respect to the inner product:

$$\langle f,g\rangle_{-s} = \langle \nabla^s f, \nabla^s g\rangle_{\mathbb{B}^d} + \sum_{i=0}^{\lceil s/2\rceil-1} \lambda_i \left\langle \triangle^i f, \triangle^i g\right\rangle_{\mathbb{S}^{d-1}}, \quad s=1,2,3\ldots, \tag{2.26}$$

where $\lambda_i>0$ for $i=0,1,\ldots,\lceil s/2\rceil-1$,

$$\nabla^{2m} := \triangle^m, \quad \text{and} \quad \nabla^{2m+1} := \nabla\triangle^m, \quad m=1,2,3,\ldots,$$

and where $\langle\cdot,\cdot\rangle_{\mathbb{B}^d} := \langle\cdot,\cdot\rangle_0$, with[2] $\langle\cdot,\cdot\rangle_\mu$ given in (1.40), and where $\langle\cdot,\cdot\rangle_{\mathbb{S}^{d-1}}$ is given in (1.37). For $s=1$ the inner product (2.26) is essentially $\langle\cdot,\cdot\rangle_I$ in (2.24). The motivation for introducing this inner product was to study the orthogonal structure in the Sobolev space[3] $\mathscr{W}_p^s(\mathbb{B}^d)$. In fact, (2.26) is an inner product on $\mathscr{W}_2^s(\mathbb{B}^d)$ [75, Definition 3.1].

Li and Xu [75, Proposition 2.1, Definition A.2] extended the polynomials in Proposition 1.3 to the following definition:

$$P_{j,l}^{\mu,n}(\mathbf{x}) := (n-j+d/2)_j J_j^{\mu,n-2j+\frac{d-2}{2}}(2\|\mathbf{x}\|^2-1)Y_l^{n-2j}(\mathbf{x}),$$
$$\mu\in\mathbb{R}, \quad n\in\mathbb{N}_0, \quad 0\le j\le n/2, \quad 1\le l\le a_{n-2j}^d,$$

where $J_j^{\alpha,\beta}$, $\alpha,\beta\in\mathbb{R}$, is the generalized Jacobi polynomial (2.15) of degree j [75, (A.3)], and $\left\{Y_l^{n-2j} : 1\le l\le a_{n-2j}^d\right\}$ is an orthonormal basis for $\mathscr{H}_{n-2j}^d$. Then the set $\left\{P_{j,l}^{\mu,n}(\mathbf{x}) : 0\le j\le n/2, 1\le l\le a_{n-2j}^d\right\}$ is an orthogonal basis of $\mathscr{V}_n^d(W_\mu)$ whenever $\mu>-1$, and for $s=1,2,3,\ldots$ the polynomial $P_{j,l}^{-s,n}$ can be expressed as [75, Lemma 3.2]:

$$P_{j,l}^{-s,n}(\mathbf{x}) := \frac{(1-n-d/2)_j}{(-j)_s(1-n-d/2+2s)_{j-s}}(\|\mathbf{x}\|^2-1)^s P_{j-s,l}^{s,n-2s}(\mathbf{x}),$$
$$s\in\mathbb{N}, \quad n\in\mathbb{N}_0, \quad s\le j\le n/2, \quad 1\le l\le a_{n-2j}^d.$$

From these definitions, and for $s\in\mathbb{N}$, $n\in\mathbb{N}_0$, $0\le j\le n/2$, $1\le l\le a_{n-2j}^d$, $\mathbf{x}\in\mathbb{B}^d$, and $\xi\in\mathbb{S}^{d-1}$, define:

$$Q_{j,l}^{-s,n}(\mathbf{x}) = \begin{cases} P_{j,l}^{-s,n}(\mathbf{x}), & j\ge s,\\ P_{j,l}^{-s,n}(\mathbf{x}) - \displaystyle\sum_{k=0}^{\lceil s/2\rceil-1} \frac{\triangle^k P_{j,l}^{-s,n}(\xi)}{Y_l^{n-2j}(\xi)} Y_l^{n-2j,k}(\mathbf{x}), & \lceil s/2\rceil\le j\le s-1,\\ Y_l^{n-2j,j}(\mathbf{x}), & 0\le j\le\lceil s/2\rceil-1,\end{cases}$$

[2]In [75, Sections 2.1 and 2.2] the definition of $\langle f,g\rangle_\mu$ and $\langle f,g\rangle_{\mathbb{S}^{d-1}}$ assumes complex variable. Therefore, at their definitions, $\overline{g}$ appears instead of g.

[3]In [75] the Sobolev space $\mathscr{W}_p^s(W_\mu,\mathbb{B}^d)$ is defined to the space of functions whose derivatives up to the s-th order are all in $\mathscr{L}^p(W_\mu,\mathbb{B}^d)$, $1\le p\le\infty$. For $p=\infty$, $\mathscr{L}^p$ is replaced by the space $\mathscr{C}(\mathbb{B}^d)$ of continuous functions. The space $\mathscr{W}_p^s(\mathbb{B}^d)$ is defined to be $\mathscr{W}_p^s(W_\mu,\mathbb{B}^d)$ when $\mu=0$.

where for any $n, j \in \mathbb{N}_0$, $Y_l^{n,j}(\mathbf{x}) := 0$ if $j < 0$ and, if $j \geq 0$,

$$Y_l^{n,j}(\mathbf{x}) := \sum_{i=0}^{j} c_i^{n,j}(1-\|\mathbf{x}\|^2)^i Y_l^n(\mathbf{x}), \quad 1 \leq l \leq a_n^d,$$

and $c_i^{n,j}$, $0 \leq i \leq j$, is the unique solution of the system of linear equations:

$$4^k \sum_{i=k}^{j} (-i)_k(-k)_{i-k} \frac{(n+d/2)_k}{(n+d/2)_{i-k}} c_i^{n,j} = \delta_{k,j}, \quad 0 \leq k \leq j.$$

Among several results in [75], most of them on approximation theory on the unit sphere, we have the following concerning the inner product (2.26):

Theorem 2.10. *[75, Theorem 3.7] The polynomials in*

$$\left\{ Q_{j,l}^{-s,n}(\mathbf{x}) : 0 \leq j \leq n/2, 1 \leq l \leq a_{n-2j}^d \right\}$$

form an orthogonal basis of $\mathscr{V}_n^d(W_{-s})$. *More precisely,* $\left\langle Q_{j,l}^{-s,n}, Q_{i,k}^{-s,m} \right\rangle_{-s} = h_{j,l}^{-s}\delta_{n,m}\delta_{j,i}\delta_{l,k}$ *for* $\langle\cdot,\cdot\rangle_{-s}$ *defined in* (2.26), *where:*

$$h_{j,l}^{-s} = \begin{cases} 2^{2s-1}d(n+d/2-s)_s(n+d/2-s+1)_{s-1}, & j \geq \lceil s/2 \rceil, \\ d(n-2j)+\lambda_j, & j = (s-1)/2, \\ \lambda_j, & 0 \leq j \leq (s-1)/2. \end{cases}$$

Another Sobolev inner product on the unit ball was considered by Pérez, Piñar, and Xu [98, Definition 3.2]. It is given for $\mu > -1$ by:

$$\langle f, g \rangle_{\nabla, W_\mu} := \frac{\lambda}{\omega_{d-1}} \int_{\mathbb{B}^d} \nabla f(\mathbf{x}) \cdot \nabla g(\mathbf{x}) W_{\mu+1}(\mathbf{x}) d\mathbf{x} + \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} f(\xi) g(\xi) d\omega(\xi), \quad (2.27)$$

where W_μ is the weight function (1.39). The inner product $\langle\cdot,\cdot\rangle_I$ in (2.24) corresponds to the limiting case of (2.27) when $\mu \to -1$. A result by Pérez, Piñar, and Xu [98, Theorem 3.4] showed an explicit mutually orthogonal basis of $\mathscr{V}_n^d(\nabla, W_\mu)$, the space of orthogonal polynomials with respect to (2.27), that is,

Theorem 2.11. *[98, Theorem 3.4] For* $0 \leq j \leq n/2$, *let* $\left\{ Y_\nu^{n-2j} : 1 \leq \nu \leq a_{n-2j}^d \right\}$ *be an orthonormal basis of* $\mathscr{H}_{n-2j}^d$. *Define*

$$R_{0,\nu}^n(\mathbf{x}) = Y_\nu^n(\mathbf{x}),$$
$$R_{j,\nu}^n(\mathbf{x}) = \left[P_j^{(\mu, n-2j+\frac{d-2}{2})}(2\|\mathbf{x}\|^2-1) - P_j^{(\mu, n-2j+\frac{d-2}{2})}(1) \right] Y_\nu^{n-2j}(\mathbf{x}), \quad 1 \leq j \leq n/2,$$

Then $\left\{ R_{j,\nu}^n : 0 \leq j \leq n/2, 1 \leq \nu \leq a_{n-2j}^d \right\}$ *forms a mutually orthogonal basis of* $\mathscr{V}_n^d(\nabla, W_\mu)$. *Furthermore,*

$$\left\langle R_{0,\nu}^n, R_{0,\nu}^n \right\rangle_{\nabla, W_\mu} = \lambda n \frac{\Gamma(\mu+2)\Gamma(n+d/2)}{\Gamma(n+\mu+1+d/2)} + 1,$$

$$\begin{aligned}\left\langle R^n_{j,\nu}, R^n_{j,\nu}\right\rangle_{\nabla,W_\mu} &= \lambda(n(2j+\mu+1)-j(2j-d+2))\\ &\quad\times\frac{\Gamma(\mu+j+1)\Gamma(n-j+d/2)(n+\mu-j+d/2)}{j!\Gamma(n+\mu+1-j+d/2)(n+\mu+d/2)}\\ &\quad+\lambda(n-2j)\frac{\Gamma(\mu+2)\Gamma(n-2j+d/2)(\mu+1)_j^2}{j^2\Gamma(n-2j+\mu+d/2)}, \quad 1\le j\le n/2.\end{aligned}$$

Another result we found in Pérez, Piñar, and Xu [98, Theorem 3.5] is the following.

Theorem 2.12. *[98, Theorem 3.5] Let $\mu > -1$ and let $P^n_{j,\nu}$ be the mutually orthogonal polynomials in $\mathscr{V}^d_n(W_\mu)$, defined in (1.42). Then they are also mutually orthogonal with respect to the Sobolev inner product:*

$$\langle f, g\rangle = c_\mu\left[\int_{\mathbb{B}^d} f(\mathbf{x})g(\mathbf{x})W_\mu(\mathbf{x})d\mathbf{x} + \lambda\int_{\mathbb{B}^d}\nabla f(\mathbf{x})\cdot\nabla g(\mathbf{x})W_{\mu+1}(\mathbf{x})d\mathbf{x}\right], \quad (2.28)$$

where $\lambda > 0$ is a fixed constant.

Notice that the parameters of the weight functions in (2.28) are μ and $\mu+1$. According to Pérez, Piñar, and Xu [98], the orthogonal structure becomes far more complicated if we want the weight functions have the same parameter. The main result in [98, Theorem 5.2] showed a mutually orthogonal basis for $\mathscr{V}^d_n(\nabla, W_\mu, \mathbb{B}^d)$, the space of Sobolev orthogonal polynomials with respect to the inner product:

$$\langle f, g\rangle_{\nabla,W_\mu,\mathbb{B}^d} = c_\mu\left[\int_{\mathbb{B}^d} f(\mathbf{x})g(\mathbf{x})W_\mu(\mathbf{x})d\mathbf{x} + \lambda\int_{\mathbb{B}^d}\nabla f(\mathbf{x})\cdot\nabla g(\mathbf{x})W_{\mu}(\mathbf{x})d\mathbf{x}\right], \quad (2.29)$$

with $\lambda > 0$ and $\mu > -1$.

Theorem 2.13. *[98, Theorem 5.2] Let $\lambda > 0$. For $0 \le j \le n/2$, let $\beta_j := n-2j+\frac{d-2}{2}$ and let $q_k^{(\mu,\beta_j)}(t)$ be the k-th Sobolev orthogonal polynomial associated with the inner product $\langle\cdot,\cdot\rangle_{\mu,\beta_j}$ in (2.13). Let $\left\{Y^{n-2j}_\nu : 1\le\nu\le a^d_{n-2j}\right\}$ be an orthonormal basis of $\mathscr{H}^d_{n-2j}$. Define*

$$T^n_{j,\nu}(\mathbf{x}) := q_j^{(\mu,\beta_j)}(2\|\mathbf{x}\|^2-1)Y^{n-2j}_\nu(\mathbf{x}).$$

Then the set $\left\{T^n_{j,\nu} : 0\le j\le n/2, 1\le\nu\le a^d_{n-2j}\right\}$ is a mutually orthogonal basis of $\mathscr{V}^d_n(\nabla, W_\mu, \mathbb{B}^d)$. Moreover,

$$\left\langle T^n_{j,\nu}, T^n_{j,\nu}\right\rangle_{\nabla,W_\mu,\mathbb{B}^d} := \frac{\Gamma(\mu+1+d/2)}{\Gamma(\mu+1)\Gamma(d/2)2^{\beta_j+\mu}}\left\langle q_j^{(\mu,\beta_j)}, q_j^{(\mu,\beta_j)}\right\rangle_{\mu,\beta_j}.$$

Among several results on spherical harmonics, Pérez, Piñar, and Xu [98, Lemma 2.2] provided the following result, with ∇_0 the spherical part of the gradient (see equation (1.34)).

Lemma 2.1. *[98, Lemma 2.2] Let $\left\{Y^n_\nu : 1\le\nu\le a^d_n\right\}$ be an orthonormal basis of $\mathscr{H}^d_n$. Let $\mathbf{x} = r\xi$, with $r > 0$ and $\xi\in\mathbb{S}^{d-1}$. Then we have the following:*

1. $\xi \cdot \nabla_0 Y_\nu^n(\mathbf{x}) = 0$.
2. $\nabla Y_\nu^n(\mathbf{x}) \cdot \nabla Y_\eta^m(\mathbf{x}) = \frac{1}{r^2}\nabla_0 Y_\nu^n(\mathbf{x}) \cdot \nabla_0 Y_\eta^m(\mathbf{x}) + \frac{nm}{r^2} Y_\nu^n(\mathbf{x}) Y_\eta^m(\mathbf{x})$.
3. *For* $1 \le \nu \le a_n^d$ *and* $1 \le \eta \le a_m^d$, *the following relation holds:*

$$\frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} \nabla Y_\nu^n(\xi) \cdot \nabla Y_\eta^m(\xi) d\omega(\xi) = n(2n+d-2)\delta_{n,m}\delta_{\nu,\eta}. \tag{2.30}$$

Delgado, Pérez, and Piñar [32] studied the Sobolev-type orthogonal polynomials on the unit ball with respect to the inner product:

$$\langle f, g \rangle_S = \langle f, g \rangle_\mu + \lambda \sum_{k=0}^{N} \frac{\partial f(\mathbf{s}_k)}{\partial n} \frac{\partial g(\mathbf{s}_k)}{\partial n}, \quad \lambda > 0, \tag{2.31}$$

where $\langle \cdot, \cdot \rangle_\mu$ is the inner product (1.40) on the unit ball, $N \in \mathbb{N}$, $S = \{\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_N\}$ is a set of $N+1$ points on $\mathbb{S}^{d-1}$, and $\partial f/\partial n$ is the normal derivative on $\mathbb{S}^{d-1}$ given by:

$$\frac{\partial f(\mathbf{x})}{\partial n} = x_1 \frac{\partial f(\mathbf{x})}{\partial x_1} + x_2 \frac{\partial f(\mathbf{x})}{\partial x_2} + \cdots + x_d \frac{\partial f(\mathbf{x})}{\partial x_d}.$$

In addition, $\partial f(S)/\partial n$ denotes the row vector of dimension $N+1$:

$$\frac{\partial f(S)}{\partial n} := \left(\frac{\partial f(\mathbf{s}_0)}{\partial n}, \frac{\partial f(\mathbf{s}_1)}{\partial n}, \cdots, \frac{\partial f(\mathbf{s}_N)}{\partial n} \right).$$

If $K_n(\mathbf{x}, \mathbf{y})$, $n \ge 0$, is the kernel function of Π_n^d associated with the inner product (1.40) defined by:

$$K_n(\mathbf{x}, \mathbf{y}) = \sum_{m=0}^{n} \sum_{|\alpha|=m} P_\alpha^m(\mathbf{x}) P_\alpha^m(\mathbf{y}),$$

where $\{P_\alpha^n(\mathbf{x}) : |\alpha| = n\}_{n \ge 0}$ is an orthonormal polynomial system on the unit ball with respect to (1.40), and for $n \ge 1$,

$$K_n^{(1,0)}(\mathbf{x}, \mathbf{y}) = \sum_{m=0}^{n} \sum_{|\alpha|=m} \frac{\partial P_\alpha^m(\mathbf{x})}{\partial n} P_\alpha^m(\mathbf{y}), \tag{2.32}$$

$$K_n^{(1,1)}(\mathbf{x}, \mathbf{y}) = \sum_{m=0}^{n} \sum_{|\alpha|=m} \frac{\partial P_\alpha^m(\mathbf{x})}{\partial n} \frac{\partial P_\alpha^m(\mathbf{y})}{\partial n}, \tag{2.33}$$

$$K_n^{(1,0)}(S, \mathbf{y}) = \left(K_n^{(1,0)}(\mathbf{s}_0, \mathbf{y}), K_n^{(1,0)}(\mathbf{s}_1, \mathbf{y}), \cdots, K_n^{(1,0)}(\mathbf{s}_N, \mathbf{y}) \right), \tag{2.34}$$

$$\mathbf{K}_n^{(1,1)} := K_n^{(1,1)}(S, S) = \left(K_n^{(1,1)}(\mathbf{s}_i, \mathbf{s}_j) \right)_{i,j=0}^{N}, \tag{2.35}$$

where $\mathbf{K}_n^{(1,1)}$ is a symmetric matrix of size $(N+1) \times (N+1)$, then a result with respect to the inner product (2.31) is the following.

Theorem 2.14. *[32, Theorem 3.1] Let* $\{P_\alpha^n(\mathbf{x}) : |\alpha| = n\}_{n\geq 0}$ *denote an orthonormal polynomial system on the unit ball with respect to* (1.40)*, and let* $K_{n-1}^{(1,0)}(S, \mathbf{x})$ *and* $\mathbf{K}_n^{(1,1)}$ *be as in* (2.34) *and* (2.35)*, respectively. Define the polynomials* $\{Q_\alpha^n(\mathbf{x}) : |\alpha| = n\}_{n\geq 0}$ *by means of:*

$$
\begin{aligned}
&Q_{\mathbf{0}}^0(\mathbf{x}) = P_{\mathbf{0}}^0(\mathbf{x}),\\
&Q_\alpha^1(\mathbf{x}) = P_\alpha^1(\mathbf{x}), \quad |\alpha| = 1,\\
&Q_\alpha^n(\mathbf{x}) = P_\alpha^n(\mathbf{x}) - \lambda \frac{\partial P_\alpha^n(S)}{\partial n}[\mathbf{I} + \lambda \mathbf{K}_{n-1}^{(1,1)}]^{-1} K_{n-1}^{(1,0)}(S, \mathbf{x})^T, \quad n \geq 2, \quad |\alpha| = n,
\end{aligned}
$$

there the superscript T *denotes the transpose. Then* $\{Q_\alpha^n(\mathbf{x}) : |\alpha| = n\}_{n\geq 0}$ *is a sequence of polynomials satisfying the following weak orthogonality with respect to the Sobolev-type inner product* (2.31)*:*

$$\left\langle Q_\alpha^n, Q_\beta^m \right\rangle_S = h_{\alpha,\alpha}^n \delta_{n,m}, \quad n, m \geq 0, \quad h_{\alpha,\alpha}^n > 0.$$

Additional results [32, Lemmas 4.1 and 4.2, Corollary 4.3] give explicit expressions for computing the matrix $\mathbf{K}_n^{(1,1)}$ in terms of the Jacobi polynomials. These results on the matrix $\mathbf{K}_n^{(1,1)}$ allowed the authors deduce asymptotics for the Christoffel functions, which are the reciprocal of the kernel functions (see [32, Section 5]).

2.2.3 Sobolev orthogonal polynomials on the simplex

In two variables, Xu [114] studied in a extensive paper the approximation by polynomials on the triangle $\mathbb{T}^2 = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}$ in the Sobolev space $\mathscr{W}_2^r$, which consists of functions whose derivatives of up to r-th order have bounded L^2 norm. His work was motivated by the fact that with respect to the inner product:

$$\langle f, g\rangle_{\gamma_1,\gamma_2,\gamma_3} = \int_{\mathbb{T}^2} f(x, y)g(x, y)W_{\gamma_1,\gamma_2,\gamma_3}(x, y)dxdy, \tag{2.36}$$

with $W_{\gamma_1,\gamma_2,\gamma_3}(x, y) = x^{\gamma_1} y^{\gamma_2} (1 - x - y)^{\gamma_3}$, $\gamma_1, \gamma_2, \gamma_3 > -1$, a mutually orthogonal basis $\left\{P_{k,n}^{\gamma_1,\gamma_2,\gamma_3} : 0 \leq k \leq n\right\}$ of $\mathscr{V}_n^2(W_{\gamma_1,\gamma_2,\gamma_3})$, the space of orthogonal polynomials with respect to (2.36), is given by:

$$P_{k,n}^{\gamma_1,\gamma_2,\gamma_3}(x, y) := (x+y)^k P_k^{(\gamma_1,\gamma_2)} \left(\frac{y - x}{x + y}\right) P_{n-k}^{(2k+\gamma_1+\gamma_2+1,\gamma_3)}(1-2x-2y), \quad 0 \leq k \leq n,$$

where $P_n^{(\alpha,\beta)}$ denotes the Jacobi polynomial of degree n. Therefore, with respect to this basis, the best polynomial approximation (Hilbert spaces theory) for a function $f \in L^2(W_{\gamma_1,\gamma_2,\gamma_3})$ is given by its Fourier orthogonal expansion:

$$S_n^{\gamma_1,\gamma_2,\gamma_3} f := \sum_{m=0}^{n} \sum_{k=0}^{m} \widehat{f}_{k,m}^{\gamma_1,\gamma_2,\gamma_3} P_{k,m}^{\gamma_1,\gamma_2,\gamma_3}, \qquad \widehat{f}_{k,m}^{\gamma_1,\gamma_2,\gamma_3} = \frac{\left\langle f, P_{k,m}^{\gamma_1,\gamma_2,\gamma_3} \right\rangle_{\gamma_1,\gamma_2,\gamma_3}}{\left\langle P_{k,m}^{\gamma_1,\gamma_2,\gamma_3}, P_{k,m}^{\gamma_1,\gamma_2,\gamma_3} \right\rangle_{\gamma_1,\gamma_2,\gamma_3}}.$$

More precisely, the standard Hilbert space result shows that:

$$E_n(f)_{\gamma_1,\gamma_2,\gamma_3} := \inf_{P\in\Pi_n^2} \|f - P\|_{L^2(W_{\gamma_1,\gamma_2,\gamma_3})} = \|f - S_n^{\gamma_1,\gamma_2,\gamma_3} f\|_{L^2(W_{\gamma_1,\gamma_2,\gamma_3})}.$$

Xu motivated with finding similar results on the Sobolev space $\mathscr{W}_2^r$, in the first part of his work he extended $P_{k,n}^{\gamma_1,\gamma_2,\gamma_3}$ to negative parameters, by using the generalized Jacobi polynomials $J_n^{\alpha,\beta}$ defined in (2.15), to the following definitions:

$$J_{k,n}^{\gamma_1,\gamma_2,\gamma_3}(x,y) := (x+y)^k J_k^{\gamma_1,\gamma_2}\left(\frac{y-x}{x+y}\right) J_{n-k}^{2k+\gamma_1+\gamma_2+1,\gamma_3}(1-2x-2y), \quad 0 \le k \le n,$$
$$K_{k,n}^{\gamma_1,\gamma_2,\gamma_3}(x,y) := (1-y)^k J_k^{\gamma_3,\gamma_1}\left(\frac{2x}{1-y}-1\right) J_{n-k}^{2k+\gamma_1+\gamma_3+1,\gamma_2}(2y-1), \quad 0 \le k \le n,$$
$$L_{k,n}^{\gamma_1,\gamma_2,\gamma_3}(x,y) := (1-x)^k J_k^{\gamma_2,\gamma_3}\left(1-\frac{2y}{1-x}\right) J_{n-k}^{2k+\gamma_2+\gamma_3+1,\gamma_1}(2x-1), \quad 0 \le k \le n.$$

For $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ and $n = 0, 1, 2, \ldots$, let $\mathscr{V}_n^2(W_{\gamma_1,\gamma_2,\gamma_3})_J$ be the space spanned by $\left\{J_{k,n}^{\gamma_1,\gamma_2,\gamma_3} : 0 \le k \le n\right\}$. The spaces $\mathscr{V}_n^2(W_{\gamma_1,\gamma_2,\gamma_3})_K$ and $\mathscr{V}_n^2(W_{\gamma_1,\gamma_2,\gamma_3})_L$ are defined in a similar way.

Proposition 2.1. *[114, Proposition 4.2] If $\gamma_1, \gamma_2, \gamma_3 > -1$ or $-\gamma_1, -\gamma_2, -\gamma_3 \notin \mathbb{N}$ then $\mathscr{V}_n^2(W_{\gamma_1,\gamma_2,\gamma_3})_J = \mathscr{V}_n^2(W_{\gamma_1,\gamma_2,\gamma_3})_K = \mathscr{V}_n^2(W_{\gamma_1,\gamma_2,\gamma_3})_L =: \mathscr{V}_n^2(W_{\gamma_1,\gamma_2,\gamma_3})$, where $=:$ means we remove the subscript J, K, L when they are equal*[4]*.*

Xu proved that the subspace $\mathscr{V}_n^2(W_{\gamma_1,\gamma_2,-1})_J$ is the space of orthogonal polynomials for three different Sobolev inner products as shown in the propositions below.

For $\gamma_1, \gamma_2 > -1$, let the inner products:

$$\begin{aligned}
\langle f,g\rangle_{\gamma_1,\gamma_2,-1}^{J} &= \int_{\mathbb{T}^2} [x\partial_x f(x,y)\partial_x g(x,y) + y\partial_y f(x,y)\partial_y g(x,y)] W_{\gamma_1,\gamma_2,0}(x,y)\,dxdy \\
&\quad + \lambda \int_0^1 f(x,1-x)g(x,1-x)W_{\gamma_1,\gamma_2,0}(x,1-x)\,dx, \quad \lambda > 0, \\
\langle f,g\rangle_{\gamma_1,\gamma_2,-1}^{K} &= \int_{\mathbb{T}^2} x\partial_x f(x,y)\partial_x g(x,y) W_{\gamma_1,\gamma_2,0}(x,y)\,dxdy \\
&\quad + \lambda \int_0^1 f(x,1-x)g(x,1-x)W_{\gamma_1,\gamma_2,0}(x,1-x)\,dx, \quad \lambda > 0, \\
\langle f,g\rangle_{\gamma_1,\gamma_2,-1}^{L} &= \int_{\mathbb{T}^2} y\partial_y f(x,y)\partial_y g(x,y) W_{\gamma_1,\gamma_2,0}(x,y)\,dxdy \\
&\quad + \lambda \int_0^1 f(x,1-x)g(x,1-x)W_{\gamma_1,\gamma_2,0}(x,1-x)\,dx, \quad \lambda > 0.
\end{aligned}$$

Proposition 2.2. *[114, Proposition 5.1] The space $\mathscr{V}_n^2(W_{\gamma_1,\gamma_2,-1})_J$ consists of orthogonal polynomials of degree n with respect to $\langle\cdot,\cdot\rangle_{\gamma_1,\gamma_2,-1}^{J}$ and $\left\{J_{k,n}^{\gamma_1,\gamma_2,-1} : 0 \le k \le n\right\}$ is a mutually orthogonal basis of this space.*

Proposition 2.3. *[114, Proposition 5.2] The set $\left\{K_{k,n}^{\gamma_1,\gamma_2,-1} : 0 \le k \le n\right\}$ is a basis of $\mathscr{V}_n^2(W_{\gamma_1,\gamma_2,-1})_J$, and so is $\left\{L_{k,n}^{\gamma_1,\gamma_2,-1} : 0 \le k \le n\right\}$.*

[4] In [114] the symbol $\mathscr{V}_n^2(W_{\gamma_1,\gamma_2,\gamma_3})$ is used to denote the space of polynomials with respect to (2.36) and, at the same time, to denote the equality between the spaces $\mathscr{V}_n^2(W_{\gamma_1,\gamma_2,\gamma_3})_J$, $\mathscr{V}_n^2(W_{\gamma_1,\gamma_2,\gamma_3})_K$ and $\mathscr{V}_n^2(W_{\gamma_1,\gamma_2,\gamma_3})_L$. Such an equality also holds for some triplets $\gamma_1, \gamma_2, \gamma_3$ that contain negative integers, but does not hold for all such triplets.

Proposition 2.4. *[114, Proposition 5.3] The set* $\left\{K_{k,n}^{\gamma_1,\gamma_2,-1} : 0 \le k \le n\right\}$ *consists of a mutually orthogonal basis of* $\mathscr{V}_n^2(W_{\gamma_1,\gamma_2,-1})_J$ *under the inner product* $\langle\cdot,\cdot\rangle_{\gamma_1,\gamma_2,-1}^K$.

Proposition 2.5. *[114, Proposition 5.4] The set* $\left\{L_{k,n}^{\gamma_1,\gamma_2,-1} : 0 \le k \le n\right\}$ *consists of a mutually orthogonal basis of* $\mathscr{V}_n^2(W_{\gamma_1,\gamma_2,-1})_J$ *under the inner product* $\langle\cdot,\cdot\rangle_{\gamma_1,\gamma_2,-1}^L$.

He also defined suitable inner products for $\mathscr{W}_2^r$, and he formally extended $S_n^{\gamma_1,\gamma_2,\gamma_3} f$ to $\gamma_1, \gamma_2, \gamma_3$ being -1 or -2. The main results on approximation are given in [114, Theorem 1.2 and 1.3] where the extended polynomials $S_n^{-1,-1,-1} f$ and $S_n^{-2,-2,-2} f$ appear.

2.2.4 Sobolev orthogonal polynomials on a product domain

Recently Fernández, Marcellán, Pérez, Piñar, and Xu [49] studied the orthogonal polynomials in two variables with respect to the Sobolev inner product:

$$\langle f, g\rangle_S = c\int_\Omega \nabla f(x,y)\cdot\nabla g(x,y) W(x,y)\,dxdy + \lambda f(c_1,c_2)g(c_1,c_2), \qquad (2.37)$$

where $\nabla f = \left(\partial_x f, \partial_y f\right)^T$ is the gradient vector, (c_1, c_2) is a given point in $\mathbb{R}^2$, $\Omega := [a_1, b_1] \times [a_2, b_2]$ is a product domain, $W(x, y) = w_1(x)w_2(y)$ is a non-negative weight function which is obtained as a product of two weights in one variable, $c = 1/\int_\Omega W(x, y)dxdy$ is the normalization constant of W, and $\lambda > 0$. These authors proposed a strategy for constructing the Sobolev polynomials with respect to (2.37), which includes the definition of the product polynomials:

$$Q_k^n(x,y) = q_{n-k}(w_1;x)q_k(w_2;y), \quad 0\le k\le n, \quad n = 0,1,2,\ldots, \qquad (2.38)$$

where $q_n(w_i)$, $i = 1, 2$, is a monic polynomial of degree n in one variable defined by:

$$q_n(w_i;x) = p_n(w_i;x) + na_{n-1}(w_i)p_{n-1}(w_i;x) + nb_{n-1}(w_i)p_{n-2}(w_i;x), \quad n\ge 1,$$

which satisfies the property $q_n'(w_i) = np_{n-1}(w_i)$, and $\{p_n(w_i; x)\}_{n\ge0}$ is a sequence of *self-coherent* monic orthogonal polynomials with respect to the weight w_i, that is, this sequence satisfies also the relation:

$$p_n(w_i;x) = \frac{p_{n+1}'(w_i;x)}{n+1} + a_n(w_i)p_n'(w_i;x) + b_n(w_i)p_{n-1}'(w_i;x), \quad n\ge 1, \qquad (2.39)$$

where $a_n(w_i)$ and $b_n(w_i)$ are real constants. Marcellán, Branquinho, and Petronilho [76] proved that the only families of self-coherent polynomials on the real line are, up to a linear change of variable, Hermite, Laguerre and Jacobi.

Let $\langle\cdot,\cdot\rangle_\nabla$ denote the bilinear form:

$$\langle f, g\rangle_\nabla = c\int_\Omega \nabla f(x,y)\cdot\nabla g(x,y) W(x,y)\,dxdy. \qquad (2.40)$$

If $\mathscr{V}_n^2(S)$ and $\mathscr{V}_n^2(\nabla)$ denote the space of orthogonal polynomials of degree n with respect to (2.37) and (2.40), respectively, then the following result is given in [49].

Proposition 2.6. *[49, Proposition 2.3] For $n \geq 1$, let $\{S_k^n : 0 \leq k \leq n\}$ denote a monic orthogonal basis of $\mathscr{V}_n^2(\nabla)$. Then, the monic orthogonal basis $\{\mathcal{S}_k^n : 0 \leq k \leq n\}$ of $\mathscr{V}_n^2(S)$ is given by $\mathcal{S}_0^0(x,y) = 1$ and*

$$\mathcal{S}_k^n(x,y) = S_k^n(x,y) - S_k^n(c_1,c_2), \quad n \geq 1.$$

The previous proposition justified working with (2.40) only. In order to find a basis $\{S_k^n : 0 \leq k \leq n\}$ for the space $\mathscr{V}_n^2(\nabla)$, the authors denoted by $\mathbb{Q}_n$ the column vector $\mathbb{Q}_n = \Big(Q_0^n(x,y), Q_1^n(x,y), \ldots, Q_n^n(x,y)\Big)^T$, where $Q_k^n(x,y)$ is defined in (2.38), and by $\mathbb{S}_n = \Big(S_0^n(x,y), S_1^n(x,y), \ldots, S_n^n(x,y)\Big)^T$ the polynomials in the basis $\{S_k^n : 0 \leq k \leq n\}$. Then, they proved [49, theorem 2.5] that there exist matrices $\mathbf{A}_n$ and $\mathbf{B}_n$ such that:

$$\mathbb{Q}_n \stackrel{c}{=} \mathbb{S}_n + \mathbf{A}_{n-1}\mathbb{S}_{n-1} + \mathbf{B}_{n-2}\mathbb{S}_{n-2}, \tag{2.41}$$

where $\stackrel{c}{=}$ denotes a congruence relation on Π^2, where two polynomials P and Q are equal up to a generic constant c, that is, $P(x,y) \stackrel{c}{=} Q(x,y)$ if $P(x,y) - Q(x,y) = c$. Equation (2.41) provided an iterative method for computing the polynomials in $\mathbb{S}_n$ in the form $\mathbb{S}_n \stackrel{c}{=} \mathbb{Q}_n - \mathbf{A}_{n-1}\mathbb{S}_{n-1} - \mathbf{B}_{n-2}\mathbb{S}_{n-2}$. Then, two particular cases were discussed by these authors: Laguerre-Laguerre weight $W_{\alpha,\beta}(x,y) = x^\alpha e^{-x} y^\beta e^{-y}$, $\alpha, \beta > -1$, and Gegenbauer-Gegenbauer weight $W_{\alpha,\beta}(x,y) = (1-x^2)^{\alpha-1/2}(1-y^2)^{\beta-1/2}$, $\alpha, \beta > -1/2$.

Following a similar strategy, Dueñas, Pinzón-Cortés, and Salazar-Morales [41] studied the Sobolev orthogonal polynomials with respect to the bilinear form:

$$\langle f, g\rangle_S = c\int_\Omega \nabla^2 f(x,y) \cdot \nabla^2 g(x,y) W(x,y)dxdy + \lambda f(c_1,c_2)g(c_1,c_2), \tag{2.42}$$

where $\nabla^2 f = \Big(\partial_{xx} f, \partial_{xy} f, \partial_{yx} f, \partial_{yy} f\Big)^T$ is the gradient or order two, and where the remaining symbols in (2.42) have the same meaning that in (2.37). In this case, similar results like those in [49] were obtained.

Chapter 3

Main results

3.1 List of publications

The results in Chapter 3 and Chapter 4 were considered for publication at different journals. The following papers form a list of publications until February, 2022.

- Dueñas, Herbert A., Salazar-Morales, Omar, and Piñar, Miguel A. "Sobolev orthogonal polynomials of several variables on product domains". In: *Mediterr. J. Math.* 18.5 (2021). Article 227, pp. 1–21
- Salazar-Morales, Omar and Dueñas, Herbert A. "Laguerre-Gegenbauer-Sobolev orthogonal polynomials in two variables on product domains". In: *Rev. Colombiana Mat.* (2021). Accepted. To appear
- Salazar-Morales, Omar and Dueñas, Herbert A. "Partial differential equations for some families of Sobolev orthogonal polynomials". Submitted to journal. 2022

3.2 Introduction

Chapter 1 was devoted for a basic background on standard orthogonal polynomials in one and several variables. In Chapter 2 a state of the art on Sobolev polynomials was presented. In this chapter we study some algebraic and analytic properties of the Sobolev orthogonal polynomials in several variables with respect to the inner product (2) that involves higher-order derivatives.

In order to get a better understanding of this chapter, the results of our study were divided into sections that can be summarized as follows:

1. In a similar way that the gradient vector ∇f contains all the first-order partial derivatives of a function f, in Section 3.3.1 we introduce a column vector, denoted by $\nabla^\kappa f$, which contains all the partial derivatives of order $\kappa \in \mathbb{N}$ of f. In this same section, we present some of its properties and related results.

2. Our Sobolev inner product (2), denoted by $\langle\cdot,\cdot\rangle_S$, is presented in Section 3.3.2 and it is divided into a continuous (main) part and a discrete part. The continuous part is denoted by $\langle\cdot,\cdot\rangle_{\nabla^\kappa}$ and it includes a non-negative weight function W .

3. Some properties of $\langle\cdot,\cdot\rangle_{\nabla^\kappa}$ are developed in Section 3.3.3. In particular, we will show that $\langle\cdot,\cdot\rangle_{\nabla^\kappa}$ is a positive semi-definite bilinear form, and therefore, the orthogonal polynomials with respect to $\langle\cdot,\cdot\rangle_{\nabla^\kappa}$ can be determined up to a polynomial of degree at most $\kappa-1$. Even though this seems to be a drawback, in Section 3.3.4 we will show that the orthogonal polynomials with respect to $\langle\cdot,\cdot\rangle_S$ can be uniquely determined by means of the orthogonal polynomials with respect to $\langle\cdot,\cdot\rangle_{\nabla^\kappa}$ and through a connection formula. Some additional properties of this connection formula are presented in Section 3.3.5.

4. Since most of the work is reduced at studying the orthogonal polynomials with respect to $\langle\cdot,\cdot\rangle_{\nabla^\kappa}$, in Section 3.3.6 and Section 3.3.7 we present an iterative method for constructing the polynomials with respect to this bilinear form. This method requires explicit computation for the entries of some matrices that are involved. This is performed for particular weight functions in Section 3.4.1 to Section 3.4.4.

5. Finally, in Section 3.4.1 to Section 3.4.4 we consider additional properties (including partial differential equations) on each one of the following domains:

   - A product domain:

     $$\Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]\,,$$

     where $[a_i, b_i]$, $i = 1, 2, \ldots, d$, is an interval of $\mathbb{R}$.
   - The simplex:

     $$\Omega = \mathbb{T}^d := \left\{\mathbf{x}\in\mathbb{R}^d : x_1 \geq 0, x_2 \geq 0, \ldots, x_d \geq 0, 1-|\mathbf{x}| \geq 0\right\},$$

     where $|\mathbf{x}| := x_1 + x_2 + \cdots + x_d$.
   - The unit ball:

     $$\Omega = \mathbb{B}^d := \left\{\mathbf{x}\in\mathbb{R}^d : \|\mathbf{x}\| \leq 1\right\}.$$

   - The cone:

     $$\Omega = \mathbb{V}^d_\vartheta := \left\{\mathbf{x}\in\mathbb{R}^d : \|\mathbf{x}_{d-1}\| \leq x_d, 0 \leq x_d \leq \vartheta\right\}, \quad 0 < \vartheta \leq \infty,$$

     where $\mathbf{x}_{d-1} = (x_1, \ldots, x_{d-1})$.

3.3 General properties

The definition of our Sobolev inner product involves higher-order partial derivatives. In addition, many of its properties are independent of the domain Ω where its continuous part is defined. In this section we present some of these properties. We begin our study with the introduction of a column vector which contains all the partial derivatives of order κ of a function f.

3.3.1 Gradient of order κ

Let f be a real-valued function of d variables $x_1, x_2, \ldots, x_d$, and let $\nabla = \big(\partial_1, \partial_2, \ldots, \partial_d\big)^T$ be the gradient operator. We define recursively the gradient of order $\kappa \in \mathbb{N}$, which is denoted by $\nabla^\kappa f$, as the column vector:

$$\nabla^\kappa f := \begin{pmatrix} \partial_1(\nabla^{\kappa-1} f) \\ \partial_2(\nabla^{\kappa-1} f) \\ \vdots \\ \partial_d(\nabla^{\kappa-1} f) \end{pmatrix}, \quad \kappa \geq 1, \quad \text{where} \quad \nabla^0 f := f. \tag{3.1}$$

Notice that $\nabla^1 f := \nabla f$. It is not difficult to show that ∇^κ is a linear operator and $\nabla^\kappa f$ is a column vector of size d^κ which contains all the partial derivatives of order κ. Let us recall that if f has derivatives of all orders then the order of differentiation does not matter. This is our case because we work with polynomials. Therefore, $\partial_i(\nabla^\kappa f) = \nabla^\kappa(\partial_i f)$, $i = 1, 2, \ldots, d$. We will use this property in the sequel.

Similarly, let $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ and $\mathbf{y} = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d$. We define recursively the column vector $(\mathbf{x} - \mathbf{y})^{[\kappa]}$, $\kappa \in \mathbb{N}$, by:

$$(\mathbf{x} - \mathbf{y})^{[\kappa]} := \begin{pmatrix} (x_1 - y_1)(\mathbf{x} - \mathbf{y})^{[\kappa-1]} \\ (x_2 - y_2)(\mathbf{x} - \mathbf{y})^{[\kappa-1]} \\ \vdots \\ (x_d - y_d)(\mathbf{x} - \mathbf{y})^{[\kappa-1]} \end{pmatrix}, \quad \kappa \geq 1, \quad \text{where} \quad (\mathbf{x} - \mathbf{y})^{[0]} := 1. \tag{3.2}$$

Notice that $(\mathbf{x} - \mathbf{y})^{[1]} = \big(x_1 - y_1, x_2 - y_2, \ldots, x_d - y_d\big)^T$, and therefore, (3.2) can be written in terms of the Kronecker product [54, section 4.2], denoted by $\otimes$, in the form $(\mathbf{x} - \mathbf{y})^{[\kappa]} = (\mathbf{x} - \mathbf{y})^{[1]} \otimes (\mathbf{x} - \mathbf{y})^{[\kappa-1]}$. Let us observe that $(\mathbf{x} - \mathbf{y})^{[\kappa]}$ is a column vector of size d^κ which contains all the possible products of the differences $x_1 - y_1, x_2 - y_2, \ldots, x_d - y_d$.

Proposition 3.1. *Let $\kappa \geq 0$, and let f and g be real-valued functions of d variables with partial derivatives up to order κ. Then*

$$\nabla^\kappa f \cdot \nabla^\kappa g = \sum_{|\alpha| = \kappa} \binom{\kappa}{\alpha_1, \alpha_2, \ldots, \alpha_d} \partial^\alpha f \partial^\alpha g, \tag{3.3}$$

where

$$\binom{\kappa}{\alpha_1, \alpha_2, \ldots, \alpha_d} = \frac{\kappa!}{\alpha_1!\alpha_2!\cdots\alpha_d!} = \frac{\kappa!}{\alpha!}, \quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d = \kappa,$$

denotes the multinomial coefficient.

Proof. We use mathematical induction on κ. The result is immediate for $\kappa = 0$. Let us suppose that (3.3) holds for $\kappa - 1$, $\kappa \geq 1$. Now, we use the induction hypothesis on $\nabla^{\kappa-1}(\partial_i f) \cdot \nabla^{\kappa-1}(\partial_i g)$, $1 \leq i \leq d$, that is,

$$\nabla^{\kappa-1}(\partial_i f) \cdot \nabla^{\kappa-1}(\partial_i g) = \sum_{|\alpha|=\kappa-1} \binom{\kappa-1}{\alpha_1, \alpha_2, \ldots, \alpha_d} \partial^\alpha(\partial_i f)\partial^\alpha(\partial_i g).$$

Then, by its definition

$$\nabla^\kappa f \cdot \nabla^\kappa g = \sum_{i=1}^{d} \partial_i(\nabla^{\kappa-1} f) \cdot \partial_i(\nabla^{\kappa-1} g) = \sum_{i=1}^{d} \nabla^{\kappa-1}(\partial_i f) \cdot \nabla^{\kappa-1}(\partial_i g),$$

and the property:

$$\sum_{i=1}^{d} \binom{\kappa-1}{\alpha_1, \alpha_2, \ldots, \alpha_i - 1, \ldots \alpha_d} = \binom{\kappa}{\alpha_1, \alpha_2, \ldots, \alpha_d}, \quad \alpha_1 + \alpha_2 + \cdots + \alpha_d = \kappa,$$

of the multinomial coefficients follows the result. □

The proof of the following proposition is similar. Therefore, we omit it.

Proposition 3.2. *Let $\kappa \geq 0$, and let f be a real-valued function of d variables with partial derivatives up to order κ with respect to $\mathbf{x}$. Then*

$$(\mathbf{x} - \mathbf{y})^{[\kappa]} \cdot \nabla^\kappa f = \sum_{|\alpha|=\kappa} \binom{\kappa}{\alpha_1, \alpha_2, \ldots, \alpha_d} (\mathbf{x} - \mathbf{y})^\alpha \partial^\alpha f.$$

It is well-known [44] that if $P \in \mathscr{P}_n^d$ is homogeneous polynomial of degree n then its partial derivative $\partial^\theta P \in \mathscr{P}_{n-|\theta|}^d$, $\theta \in \mathbb{N}_0^d$, is also a homogeneous polynomial of degree $n - |\theta|$. That is, if $P(\mathbf{x}) = \sum_{|\alpha|=n} c_\alpha \mathbf{x}^\alpha$ then $\partial^\theta P(\mathbf{x}) = \sum_{|\alpha|=n} c_\alpha \partial^\theta \mathbf{x}^\alpha = \sum_{|\alpha|=n} c_\alpha (-1)^{|\theta|}(-\alpha)_\theta \mathbf{x}^{\alpha-\theta}$ is a linear combination of monomials of degree $|\alpha - \theta| = (\alpha_1 - \theta_1) + \cdots + (\alpha_d - \theta_d) = n - |\theta|$. Some other properties that include ∇^κ, when it is applied to polynomials, are generalizations of well-known properties of the gradient ∇. For example, if P is a homogeneous polynomial of degree n, it is known [44] the Euler's equation:

$$\sum_{i=1}^{d} x_i \partial_i P(\mathbf{x}) = nP(\mathbf{x}), \quad P \in \mathscr{P}_n^d. \tag{3.4}$$

Therefore, homogeneous polynomials are eigenfunctions of the differential operator $\sum_{i=1}^{d} x_i\partial_i$. With our notation, the equation (3.4) can be written in the form (with $\kappa = 1$ and $\mathbf{y} = \mathbf{0}$):

$$\mathbf{x}^{[1]} \cdot \nabla P(\mathbf{x}) = nP(\mathbf{x}), \quad P \in \mathscr{P}_n^d. \tag{3.5}$$

Now we present a generalization of (3.5) to higher-order derivatives.

Proposition 3.3. *Let $\kappa \geq 0$ and let $\mathbf{p} = (p_1, p_2, \ldots, p_d)$ be a given point in $\mathbb{R}^d$. Then, if $P \in \Pi_n^d$ is a polynomial of the form $P(\mathbf{x}) = \sum_{|\beta|=n} c_\beta(\mathbf{x} - \mathbf{p})^\beta$, where c_β is a real constant, then the following equation holds:*

$$(\mathbf{x} - \mathbf{p})^{[\kappa]} \cdot \nabla^\kappa P(\mathbf{x}) = (n - \kappa + 1)_\kappa P(\mathbf{x}). \tag{3.6}$$

In particular, if P is a homogeneous polynomial then we have the following generalized Euler's equation:

$$\mathbf{x}^{[\kappa]} \cdot \nabla^\kappa P(\mathbf{x}) = (n - \kappa + 1)_\kappa P(\mathbf{x}), \quad P \in \mathscr{P}_n^d. \tag{3.7}$$

Proof. First notice that if P is of the form $P(\mathbf{x}) = \sum_{|\beta|=n} c_\beta(\mathbf{x} - \mathbf{p})^\beta$, then $\partial_i P$, $1 \leq i \leq d$, has the same form, that is:

$$\partial_i P(\mathbf{x}) = \sum_{|\beta|=n} c_\beta \beta_i (x_1 - p_1)^{\beta_1} \cdots (x_i - p_i)^{\beta_i - 1} \cdots (x_d - p_d)^{\beta_d} \tag{3.8}$$

$$= \sum_{|\eta_i|=n-1} \widehat{c}_{\eta_i}(\mathbf{x} - \mathbf{p})^{\eta_i} \in \Pi_{n-1}^d, \quad \widehat{c}_{\eta_i} = c_\beta \beta_i, \quad \eta_i = (\beta_1, \ldots, \beta_i - 1, \ldots, \beta_d). \tag{3.9}$$

For $\kappa = 0$ the equation (3.6) is immediate. For $\kappa = 1$, by (3.8) we have that:

$$(\mathbf{x} - \mathbf{p})^{[1]} \cdot \nabla P(\mathbf{x}) = \sum_{i=1}^{d} (x_i - p_i)\partial_i P(\mathbf{x}) = nP(\mathbf{x}), \tag{3.10}$$

and (3.6) also holds in this case. Let us suppose that the proposition holds for $\kappa - 1$, $\kappa \geq 1$. By the induction hypothesis applied to $\partial_i P \in \Pi_{n-1}^d$ we have that:

$$(\mathbf{x} - \mathbf{p})^{[\kappa-1]} \cdot \nabla^{\kappa-1}\partial_i P(\mathbf{x}) = (n - \kappa + 1)_{\kappa-1}\partial_i P(\mathbf{x}), \quad 1 \leq i \leq d. \tag{3.11}$$

Then by (3.1), (3.2), (3.10) and (3.11):

$$\begin{aligned}(\mathbf{x} - \mathbf{p})^{[\kappa]} \cdot \nabla^\kappa P(\mathbf{x}) &= \sum_{i=1}^{d} (x_i - p_i)(\mathbf{x} - \mathbf{p})^{[\kappa-1]} \cdot \nabla^{\kappa-1}\partial_i P(\mathbf{x}) \\ &= (n - \kappa + 1)_{\kappa-1} \sum_{i=1}^{d} (x_i - p_i)\partial_i P(\mathbf{x}) = (n - \kappa + 1)_\kappa P(\mathbf{x}).\end{aligned}$$

Finally, equation (3.7) follows with $P \in \mathscr{P}_n^d$ and $\mathbf{p} = \mathbf{0}$. □

3.3.2 Sobolev inner product

Let $\kappa \in \mathbb{N}$ be a fixed number. We denote by $\mathscr{V}_n^d(S,W)$ the linear space of Sobolev orthogonal polynomials of total degree n in d variables with respect to the inner product:

$$\langle f,g\rangle_S = c\int_\Omega \nabla^\kappa f(\mathbf{x})\cdot\nabla^\kappa g(\mathbf{x})W(\mathbf{x})d\mathbf{x} + \sum_{i=0}^{\kappa-1}\lambda_i\nabla^i f(\mathbf{p})\cdot\nabla^i g(\mathbf{p}), \tag{3.12}$$

with $\Omega \subseteq \mathbb{R}^d$ being a domain having a non-empty interior, and where[1] $\lambda_i > 0$ for $i = 0,1,\ldots,\kappa-1$, W is a non-negative weight function on Ω, c is the normalization constant of W, that is,

$$c := \left(\int_\Omega W(\mathbf{x})d\mathbf{x}\right)^{-1},$$

and $\mathbf{p} = (p_1,p_2,\ldots,p_d)$ is a given point in $\mathbb{R}^d$, typically on the boundary of Ω. The sum in (3.12) is added to make the inner product well-defined on Π^d. For $\langle\cdot,\cdot\rangle_S$ we denote its continuous (main) part by:

$$\langle f,g\rangle_{\nabla^\kappa} := c\int_\Omega \nabla^\kappa f(\mathbf{x})\cdot\nabla^\kappa g(\mathbf{x})W(\mathbf{x})d\mathbf{x}, \tag{3.13}$$

where we observe that $\langle\cdot,\cdot\rangle_{\nabla^\kappa}$ can be defined for $\kappa = 0$ by $\langle\cdot,\cdot\rangle_{\nabla^0} := \langle\cdot,\cdot\rangle_W$, where $\langle\cdot,\cdot\rangle_W$ is the inner product:

$$\langle f,g\rangle_W := c\int_\Omega f(\mathbf{x})g(\mathbf{x})W(\mathbf{x})d\mathbf{x}. \tag{3.14}$$

We denote by $\|\cdot\|_W := \sqrt{\langle\cdot,\cdot\rangle_W}$ the norm induced by (3.14). In addition, we denote by $\mathscr{V}_n^d(W)$ the space of orthogonal polynomials with respect to (3.14), and by $\mathscr{V}_n^d(\nabla^\kappa,W)$ the linear space of orthogonal polynomials of total degree n with respect to (3.13). Then $\langle\cdot,\cdot\rangle_S$ can be written as:

$$\langle f,g\rangle_S = \langle f,g\rangle_{\nabla^\kappa} + \sum_{i=0}^{\kappa-1}\lambda_i\nabla^i f(\mathbf{p})\cdot\nabla^i g(\mathbf{p}).$$

We will give some properties of (3.13) in order to find a connection formula between polynomials in the spaces $\mathscr{V}_n^d(\nabla^\kappa,W)$ and $\mathscr{V}_n^d(S,W)$.

3.3.3 Some properties of $\langle\cdot,\cdot\rangle_{\nabla^\kappa}$

Observe that the recursive definition of $\nabla^\kappa f$ implies for $\kappa \geq 1$ that:

$$\langle f,g\rangle_{\nabla^\kappa} = c\int_\Omega\left(\sum_{i=1}^{d}\nabla^{\kappa-1}(\partial_i f)\cdot\nabla^{\kappa-1}(\partial_i g)\right)W(\mathbf{x})d\mathbf{x}$$

[1] If we choose $\lambda_0 = 1$, then we get the normalization $\langle 1,1\rangle_S = 1$ for this Sobolev inner product.

$$= \sum_{i=1}^{d} c \int_{\Omega} \nabla^{\kappa-1}(\partial_i f) \cdot \nabla^{\kappa-1}(\partial_i g) W(\mathbf{x}) d\mathbf{x} = \sum_{i=1}^{d} \langle \partial_i f, \partial_i g \rangle_{\nabla^{\kappa-1}},$$

and more generally we have the following proposition.

Proposition 3.4. *Let $\kappa \geq 0$. Then $\langle \cdot, \cdot \rangle_{\nabla^\kappa}$ is a symmetric bilinear form which is positive semidefinite and it is related to the inner product $\langle \cdot, \cdot \rangle_W$ by:*

$$\langle f, g \rangle_{\nabla^\kappa} = \sum_{|\alpha|=\kappa} \binom{\kappa}{\alpha_1, \alpha_2, \ldots, \alpha_d} \langle \partial^\alpha f, \partial^\alpha g \rangle_W, \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d. \quad (3.15)$$

Proof. That $\langle \cdot, \cdot \rangle_{\nabla^\kappa}$ is a symmetric bilinear form which is positive semidefinite follows from its definition. Equation (3.15) follows by (3.13), (3.14) and (3.3). □

The bilinear form $\langle \cdot, \cdot \rangle_{\nabla^\kappa}$ is not an inner product on Π^d, except for $\kappa = 0$. To see this, if $\kappa \geq 1$ and $P \in \Pi_0^d$, $P \neq 0$, then $\langle P, P \rangle_{\nabla^\kappa} = 0$ but $P \neq 0$. As a consequence the orthogonal polynomials with respect to $\langle \cdot, \cdot \rangle_{\nabla^\kappa}$ can be determined up to a polynomial of degree at most $\kappa - 1$ as we will show in the sequel.

For $\kappa \geq 1$ we adopt the following notation for two polynomials P and Q of d variables that are equal up to a polynomial of degree at most $\kappa - 1$:

$$P \overset{\kappa-1}{=} Q \quad \text{if} \quad P - Q \in \Pi_{\kappa-1}^d.$$

The relation $\overset{\kappa-1}{=}$ is a congruence relation[2] on Π^d. We denote by $\|\cdot\|_{\nabla^\kappa}$ the seminorm $\|\cdot\|_{\nabla^\kappa} := \sqrt{\langle \cdot, \cdot \rangle_{\nabla^\kappa}}$ induced by $\langle \cdot, \cdot \rangle_{\nabla^\kappa}$. Next we show a characterization of $\overset{\kappa-1}{=}$ in terms of ∇^κ, $\langle \cdot, \cdot \rangle_{\nabla^\kappa}$ and $\|\cdot\|_{\nabla^\kappa}$.

Proposition 3.5. *Let $P, Q \in \Pi^d$ and $\kappa \geq 1$. The following statements are equivalent.*

1. $P \overset{\kappa-1}{=} Q$,
2. $\langle P, R \rangle_{\nabla^\kappa} = \langle Q, R \rangle_{\nabla^\kappa}$ *for all* $R \in \Pi^d$,
3. $\|P - Q\|_{\nabla^\kappa} = 0$,
4. $\nabla^\kappa P = \nabla^\kappa Q$.

Proof. If $P \overset{\kappa-1}{=} Q$ then $P - Q \in \Pi_{\kappa-1}^d$. Then $\nabla^\kappa(P - Q) = \mathbf{0}$ and as a consequence $\langle P - Q, R \rangle_{\nabla^\kappa} = 0$, that is, $\langle P, R \rangle_{\nabla^\kappa} = \langle Q, R \rangle_{\nabla^\kappa}$ for all $R \in \Pi^d$.

If $\langle P, R \rangle_{\nabla^\kappa} = \langle Q, R \rangle_{\nabla^\kappa}$ for all $R \in \Pi^d$ then, in particular, for $R = P - Q \in \Pi^d$ we have:

$$0 = \langle P, R \rangle_{\nabla^\kappa} - \langle Q, R \rangle_{\nabla^\kappa} = \langle P - Q, R \rangle_{\nabla^\kappa} = \langle P - Q, P - Q \rangle_{\nabla^\kappa} = \|P - Q\|_{\nabla^\kappa}^2.$$

[2] A *congruence relation* is an equivalence relation (reflexivity, symmetry, transitivity) which satisfies the *compatibility property* [20, Definition 5.1] with the operations of the linear space Π^d: if $P, Q, R, S \in \Pi^d$ then $P \overset{\kappa-1}{=} Q$ implies $(aP) \overset{\kappa-1}{=} (aQ)$, $a \in \mathbb{R}$, and $P \overset{\kappa-1}{=} Q$ and $R \overset{\kappa-1}{=} S$ imply $(P + R) \overset{\kappa-1}{=} (Q + S)$.

If $\|P-Q\|_{\nabla^\kappa}=0$ then by (3.15):

$$0=\|P-Q\|^2_{\nabla^\kappa}=\langle P-Q,P-Q\rangle_{\nabla^\kappa}=\sum_{|\alpha|=\kappa}\binom{\kappa}{\alpha_1,\ldots,\alpha_d}\|\partial^\alpha(P-Q)\|^2_W,$$

which implies that $\partial^\alpha P=\partial^\alpha Q$ for all $\alpha\in\mathbb{N}_0^d$ such that $|\alpha|=\kappa$, that is, $\nabla^\kappa P=\nabla^\kappa Q$.

Finally, if $\nabla^\kappa P=\nabla^\kappa Q$ then $\nabla^\kappa(P-Q)=\mathbf{0}$. Therefore, $P-Q$ is a polynomial of degree at most $\kappa-1$, that is, $P\overset{\kappa-1}{=}Q$. □

Let $[P]$ denote the equivalence class that contains $P\in\Pi^d$ due to the congruence relation $\overset{\kappa-1}{=}$. The equivalence class that contains the zero polynomial is exactly the subspace $\Pi^d_{\kappa-1}$, that is:

$$[0]=\left\{P\in\Pi^d : P\overset{\kappa-1}{=}0\right\}=\Pi^d_{\kappa-1}.$$

Let us observe that if P is any polynomial in $\Pi^d_{\kappa-1}$ (that is, $\nabla^\kappa P=\mathbf{0}$) and if S is a polynomial in $\mathscr{V}^d_n(\nabla^\kappa,W)$ (that is, $\langle S,Q\rangle_{\nabla^\kappa}=0$ for all $Q\in\Pi^d_{n-1}$) then $S+P$ is also in $\mathscr{V}^d_n(\nabla^\kappa,W)$ because of the equality $\langle S+P,Q\rangle_{\nabla^\kappa}=\langle S,Q\rangle_{\nabla^\kappa}=0$ for all $Q\in\Pi^d_{n-1}$. Then the polynomials in $\mathscr{V}^d_n(\nabla^\kappa,W)$ are determined up to a polynomial of degree at most $\kappa-1$. This remark is important and it seems to be a drawback but, as we will see later, the polynomials in the space $\mathscr{V}^d_n(S,W)$ do not depend on the representative we choose of each equivalence class.

3.3.4 Connection formula between polynomials in the spaces $\mathscr{V}^d_n(\nabla^\kappa,W)$ and $\mathscr{V}^d_n(S,W)$

Let us recall (see [43, pp. 51] and Section 1.4) that the Taylor polynomial $\mathcal{T}^{\kappa-1}(P,\mathbf{p};\mathbf{x})$ of total degree $\kappa-1$ in d variables of $P\in\Pi^d$ at $\mathbf{p}=(p_1,p_2,\ldots,p_d)\in\mathbb{R}^d$ is given by:

$$\mathcal{T}^{\kappa-1}(P,\mathbf{p};\mathbf{x})=\sum_{|\beta|\le\kappa-1}\frac{(\partial^\beta P)(\mathbf{p})}{\beta!}(\mathbf{x}-\mathbf{p})^\beta \tag{3.16}$$

$$=\sum_{i=0}^{\kappa-1}\frac{1}{i!}(\nabla^i P)(\mathbf{p})\cdot(\mathbf{x}-\mathbf{p})^{[i]}, \tag{3.17}$$

and the corresponding remainder term (and its integral form) in the Taylor's formula is:

$$\mathcal{R}_\kappa(P,\mathbf{p};\mathbf{x})=P(\mathbf{x})-\mathcal{T}^{\kappa-1}(P,\mathbf{p};\mathbf{x}) \tag{3.18}$$

$$=\sum_{|\beta|=\kappa}\frac{(\mathbf{x}-\mathbf{p})^\beta}{\beta!}\int_0^1\kappa(1-t)^{\kappa-1}(\partial^\beta P)(\mathbf{p}+t(\mathbf{x}-\mathbf{p}))dt \tag{3.19}$$

$$=\int_0^1\frac{(1-t)^{\kappa-1}}{(\kappa-1)!}(\mathbf{x}-\mathbf{p})^{[\kappa]}\cdot(\nabla^\kappa P)(\mathbf{p}+t(\mathbf{x}-\mathbf{p}))dt, \tag{3.20}$$

where the expressions (3.17) and (3.20) follow from Proposition 3.2. Notice that, because of the linearity of ∂^β, if $P, Q \in \Pi^d$ and $a, b \in \mathbb{R}$ then:

$$\mathcal{T}^{\kappa-1}(aP + bQ, \mathbf{p}; \mathbf{x}) = a\mathcal{T}^{\kappa-1}(P, \mathbf{p}; \mathbf{x}) + b\mathcal{T}^{\kappa-1}(Q, \mathbf{p}; \mathbf{x}),$$
$$\mathcal{R}_\kappa(aP + bQ, \mathbf{p}; \mathbf{x}) = a\mathcal{R}_\kappa(P, \mathbf{p}; \mathbf{x}) + b\mathcal{R}_\kappa(Q, \mathbf{p}; \mathbf{x}),$$

that is,

$$\Pi^d \mapsto \Pi^d : P \mapsto \mathcal{T}^{\kappa-1}(P, \mathbf{p}), \tag{3.21}$$
$$\Pi^d \mapsto \Pi^d : P \mapsto \mathcal{R}_\kappa(P, \mathbf{p}), \tag{3.22}$$

are linear operators. A well-known property of the Taylor polynomial (3.16) is that, at the point $\mathbf{p} \in \mathbb{R}^d$, it satisfies:

$$(\partial^\theta P)(\mathbf{p}) = (\partial^\theta \mathcal{T}^{\kappa-1}(P, \mathbf{p}))(\mathbf{p}), \quad |\theta| \le \kappa - 1,$$

which implies that:

$$(\partial^\theta \mathcal{R}_\kappa(P, \mathbf{p}))(\mathbf{p}) = 0, \quad |\theta| \le \kappa - 1,$$

or the latter in a vector form:

$$(\nabla^i \mathcal{R}_\kappa(P, \mathbf{p}))(\mathbf{p}) = \mathbf{0}, \quad 0 \le i \le \kappa - 1.$$

Therefore, if $P, Q \in \Pi^d$, and since $\nabla^\kappa \mathcal{T}^{\kappa-1}(P, \mathbf{p}; \mathbf{x}) = \mathbf{0}$, then:

$$\langle \mathcal{R}_\kappa(P, \mathbf{p}), Q \rangle_S = \langle \mathcal{R}_\kappa(P, \mathbf{p}), Q \rangle_{\nabla^\kappa} + \sum_{i=0}^{\kappa-1} \lambda_i \underbrace{(\nabla^i \mathcal{R}_\kappa(P, \mathbf{p}))(\mathbf{p})}_{=\mathbf{0}} \cdot (\nabla^i Q)(\mathbf{p}) = \langle P, Q \rangle_{\nabla^\kappa}. \tag{3.23}$$

From the last equation, in particular, if $P \in \mathscr{V}_n^d(\nabla^\kappa, W)$, $Q \in \Pi_{n-1}^d$, $n \ge \kappa$, then by (3.18), $\mathcal{R}_\kappa(P, \mathbf{p})$ is a polynomial of degree n for which $\langle \mathcal{R}_\kappa(P, \mathbf{p}), Q \rangle_S = 0$ for all $Q \in \Pi_{n-1}^d$, that is, $\mathcal{R}_\kappa(P, \mathbf{p}) \in \mathscr{V}_n^d(S, W)$. This shows that the linear operator:

$$\mathscr{R}_{\kappa,\mathbf{p}} : \mathscr{V}_n^d(\nabla^\kappa, W) \mapsto \mathscr{V}_n^d(S, W) : P \mapsto \mathcal{R}_\kappa(P, \mathbf{p}), \quad n \ge \kappa,$$

is well-defined. Based on these properties, we have the following theorem.

Theorem 3.1. *Let $\{S_\alpha^n : |\alpha| = n\}$ denote a monic orthogonal basis of $\mathscr{V}_n^d(\nabla^\kappa, W)$. Then, a monic orthogonal basis $\{\mathcal{S}_\alpha^n : |\alpha| = n\}$ of $\mathscr{V}_n^d(S, W)$ is given by:*

$$\mathcal{S}_\alpha^n(\mathbf{x}) = (\mathbf{x} - \mathbf{p})^\alpha, \quad 0 \le |\alpha| = n < \kappa, \tag{3.24}$$
$$\mathcal{S}_\alpha^n(\mathbf{x}) = S_\alpha^n(\mathbf{x}) - \mathcal{T}^{\kappa-1}(S_\alpha^n, \mathbf{p}; \mathbf{x}), \quad |\alpha| = n \ge \kappa, \tag{3.25}$$

where $(\mathbf{x} - \mathbf{p})^\alpha$ denotes the shifted monomial $(x_1 - p_1)^{\alpha_1}(x_2 - p_2)^{\alpha_2} \cdots (x_d - p_d)^{\alpha_d}$ and $\mathcal{T}^{\kappa-1}(S_\alpha^n, \mathbf{p}; \mathbf{x})$ denotes the Taylor polynomial of total degree $\kappa - 1$ in d variables of S_α^n at $\mathbf{p} = (p_1, p_2, \ldots, p_d)$.

Proof. It is not difficult to see that (3.24) and (3.25) are monic of degree exactly n, that is, $\mathcal{S}_\alpha^n$ is of the form $\mathcal{S}_\alpha^n(\mathbf{x}) = \mathbf{x}^\alpha + R_\alpha(\mathbf{x})$, $|\alpha| = n$, $R_\alpha \in \Pi_{n-1}^d$, if S_α^n is monic. Now we prove the orthogonality of $\mathcal{S}_\alpha^n$ with respect to $\langle\cdot,\cdot\rangle_S$. We consider two cases.

Case $0 \le |\alpha| = n < \kappa$ in (3.24): If $n = 0$ we have nothing to prove. Let $n \ge 1$ and $Q \in \Pi_{n-1}^d$. If $n < i \le \kappa$ then $\nabla^i \mathcal{S}_\alpha^n = \mathbf{0}$, and also $\nabla^n Q = \mathbf{0}$ and $\nabla^0 \mathcal{S}_\alpha^n(\mathbf{p}) := \mathcal{S}_\alpha^n(\mathbf{p}) = 0$. Therefore $\langle \mathcal{S}_\alpha^n, Q\rangle_S$ in (3.12) reduces to:

$$\langle \mathcal{S}_\alpha^n, Q\rangle_S = \sum_{i=1}^{n-1} \lambda_i \nabla^i \mathcal{S}_\alpha^n(\mathbf{p}) \cdot \nabla^i Q(\mathbf{p}).$$

Now for $1 \le i \le n-1$ let us observe that each entry in the vector $\nabla^i \mathcal{S}_\alpha^n = \nabla^i (\mathbf{x}-\mathbf{p})^\alpha$ is a polynomial of total degree $n - i > 0$, therefore each entry in $\nabla^i \mathcal{S}_\alpha^n$ has at least one factor of the form $(x_j - p_j)$ for some $j \in \{1, 2, \ldots, d\}$, where we get $\nabla^i \mathcal{S}_\alpha^n(\mathbf{p}) = \mathbf{0}$. Consequently, $\langle \mathcal{S}_\alpha^n, Q\rangle_S = 0$ for all $Q \in \Pi_{n-1}^d$.

Case $|\alpha| = n \ge \kappa$ in (3.25): By hypothesis $\langle S_\alpha^n, Q\rangle_{\nabla^\kappa} = 0$ for all $Q \in \Pi_{n-1}^d$. Since $(\nabla^i S_\alpha^n)(\mathbf{p}) = (\nabla^i \mathcal{T}^{\kappa-1}(S_\alpha^n, \mathbf{p}))(\mathbf{p})$ for all $i = 0, 1, \ldots, \kappa-1$, and also $\nabla^\kappa \mathcal{T}^{\kappa-1}(S_\alpha^n, \mathbf{p}) = \mathbf{0}$ then:

$$\begin{aligned}\langle \mathcal{S}_\alpha^n, Q\rangle_S &= \left\langle S_\alpha^n - \mathcal{T}^{\kappa-1}(S_\alpha^n, \mathbf{p}), Q\right\rangle_{\nabla^\kappa} + \sum_{i=0}^{\kappa-1} \lambda_i \underbrace{\nabla^i (S_\alpha^n - \mathcal{T}^{\kappa-1}(S_\alpha^n, \mathbf{p}))(\mathbf{p})}_{=\mathbf{0}} \cdot \nabla^i Q(\mathbf{p}) \\ &= \left\langle S_\alpha^n - \mathcal{T}^{\kappa-1}(S_\alpha^n, \mathbf{p}), Q\right\rangle_{\nabla^\kappa} = \langle S_\alpha^n, Q\rangle_{\nabla^\kappa} = 0,\end{aligned}$$

for all $Q \in \Pi_{n-1}^d$. □

Remark 3.1. Notice that if $P \in \Pi_{\kappa-1}^d$ then $\mathcal{T}^{\kappa-1}(P, \mathbf{p}; \mathbf{x}) = P(\mathbf{x})$ for all $\mathbf{x}$. As a consequence, if $S \in \Pi^d$ is such that $S \overset{\kappa-1}{=} S_\alpha^n$, that is, $S(\mathbf{x}) = S_\alpha^n(\mathbf{x}) + P(\mathbf{x})$ where $P \in \Pi_{\kappa-1}^d$, then:

$$\begin{aligned} S(\mathbf{x}) - \mathcal{T}^{\kappa-1}(S, \mathbf{p}; \mathbf{x}) &= (S_\alpha^n + P)(\mathbf{x}) - \mathcal{T}^{\kappa-1}(S_\alpha^n + P, \mathbf{p}; \mathbf{x}) \\ &= S_\alpha^n(\mathbf{x}) + P(\mathbf{x}) - \mathcal{T}^{\kappa-1}(S_\alpha^n, \mathbf{p}; \mathbf{x}) - \mathcal{T}^{\kappa-1}(P, \mathbf{p}; \mathbf{x}) \\ &= S_\alpha^n(\mathbf{x}) - \mathcal{T}^{\kappa-1}(S_\alpha^n, \mathbf{p}; \mathbf{x}). \end{aligned}$$

Therefore, the polynomial $\mathcal{S}_\alpha^n$ in (3.25) does not depend on the representative we choose of each equivalence class due to the congruence relation $\overset{\kappa-1}{=}$.

Theorem 3.1 shows it is only necessary to work with the bilinear form $\langle\cdot,\cdot\rangle_{\nabla^\kappa}$ and on the set $\Pi^d \backslash \Pi_{\kappa-1}^d$ (polynomials of degree at least κ). Notice that the polynomials in $\mathscr{V}_n^d(\nabla^\kappa, W)$ are determined up to a polynomial of degree at most $\kappa-1$, but according to Theorem 3.1 and the previous remark, this issue does not affect the polynomials in $\mathscr{V}_n^d(S, W)$.

3.3.5 Some consequences of the connection formula

In this section we present some consequences of (3.25) in the case $n \ge \kappa$. Let us observe that for any polynomial in several variables $P \in \Pi^d$, the Taylor polynomial of P at $\mathbf{p}$ satisfies the following relation:

$$\partial^\theta \mathcal{T}^{k-1}(P, \mathbf{p}; \mathbf{x}) = \mathcal{T}^{k-|\theta|-1}(\partial^\theta P, \mathbf{p}; \mathbf{x}), \quad \theta \in \mathbb{N}_0^d, \quad |\theta| \le \kappa - 1. \tag{3.26}$$

Equation (3.26) can be proved directly from (3.16) and it implies, by (3.18), that:

$$\partial^\theta \mathcal{R}_\kappa(P,\mathbf{p};\mathbf{x}) = \mathcal{R}_{\kappa-|\theta|}(\partial^\theta P,\mathbf{p};\mathbf{x}), \quad |\theta| \le \kappa-1, \tag{3.27}$$

$$\partial^\theta \mathcal{R}_\kappa(P,\mathbf{p};\mathbf{x}) = \partial^\theta P(\mathbf{x}), \quad |\theta| > \kappa-1. \tag{3.28}$$

Then, for the monic orthogonal basis $\{\mathcal{S}^n_\alpha : |\alpha| = n\}$ of $\mathscr{V}^d_n(S,W)$ given in Theorem 3.1 we have that the partial derivatives of $\mathcal{S}^n_\alpha$ are given for the case $|\alpha| = n \ge \kappa$ by:

$$\partial^\theta \mathcal{S}^n_\alpha(\mathbf{x}) = \partial^\theta S^n_\alpha(\mathbf{x}) - \mathcal{T}^{k-|\theta|-1}(\partial^\theta S^n_\alpha,\mathbf{p};\mathbf{x}) = \mathcal{R}_{\kappa-|\theta|}(\partial^\theta S^n_\alpha,\mathbf{p};\mathbf{x}), \quad |\alpha| = n \ge k > |\theta|,$$
$$\partial^\theta \mathcal{S}^n_\alpha(\mathbf{x}) = \partial^\theta S^n_\alpha(\mathbf{x}), \quad |\alpha| = n \ge k, \quad |\theta| \ge k.$$

These two last equations and the Taylor's formula prove the following result.

Proposition 3.6. *Let $P \in \mathscr{V}^d_n(S,W)$. Then, the partial derivative $\partial^\theta P$, $\theta \in \mathbb{N}^d_0$, satisfies the equation:*

$$\mathcal{R}_{\kappa-|\theta|}(\partial^\theta P,\mathbf{p}) = \partial^\theta P, \quad n \ge \kappa > |\theta|.$$

And moreover, $(\partial^\theta P)(\mathbf{p}) = 0$, $n \ge \kappa > |\theta|$, or the latter in a vector form:

$$(\nabla^i P)(\mathbf{p}) = \mathbf{0}, \quad n \ge \kappa > i. \tag{3.29}$$

In particular, for $\theta = (0,0,\dots,0)$, the polynomial P satisfies the equation:

$$\mathcal{R}_\kappa(P,\mathbf{p}) = P, \quad n \ge \kappa. \tag{3.30}$$

Proof. Let $n \ge \kappa$ and let $\{\mathcal{S}^n_\alpha : |\alpha| = n\}$ and $\{S^n_\alpha : |\alpha| = n\}$ be monic orthogonal bases of $\mathscr{V}^d_n(S,W)$ and $\mathscr{V}^d_n(\nabla^\kappa,W)$, respectively, as in Theorem 3.1. Let us observe that $\mathcal{S}^n_\alpha$, given in (3.25), is $\mathcal{S}^n_\alpha = \mathcal{R}_\kappa(S^n_\alpha,\mathbf{p})$. Then, by (3.27) and (3.19), we have that:

$$\partial^\theta \mathcal{S}^n_\alpha(\mathbf{x}) = \partial^\theta \mathcal{R}_\kappa(S^n_\alpha,\mathbf{p};\mathbf{x}) = \mathcal{R}_{\kappa-|\theta|}(\partial^\theta S^n_\alpha,\mathbf{p};\mathbf{x}) =$$
$$\sum_{|\eta|=\kappa-|\theta|} \frac{(\mathbf{x}-\mathbf{p})^\eta}{\eta!} \int_0^1 (\kappa-|\theta|)(1-t)^{\kappa-|\theta|-1}(\partial^{\eta+\theta} S^n_\alpha)(\mathbf{p}+t(\mathbf{x}-\mathbf{p}))dt, \quad |\theta| < \kappa. \tag{3.31}$$

Since the multi-index $\eta+\theta$ in (3.31) satisfies $|\eta+\theta| = |\eta| + |\theta| = \kappa$, we have again by (3.25) that $\partial^{\eta+\theta}\mathcal{S}^n_\alpha = \partial^{\eta+\theta} S^n_\alpha$. Therefore, the right-hand side of (3.31) is equal to $\mathcal{R}_{\kappa-|\theta|}(\partial^\theta \mathcal{S}^n_\alpha,\mathbf{p})$, that is, we have the equality $\partial^\theta \mathcal{S}^n_\alpha = \mathcal{R}_{\kappa-|\theta|}(\partial^\theta \mathcal{S}^n_\alpha,\mathbf{p})$. Now, $(\partial^\theta \mathcal{S}^n_\alpha)(\mathbf{p}) = 0$ because the remainder term $\mathcal{R}_{\kappa-|\theta|}(\partial^\theta \mathcal{S}^n_\alpha,\mathbf{p};\mathbf{x})$ vanishes at $\mathbf{x} = \mathbf{p}$. The result follows by the linearity of $\mathcal{R}_{\kappa-|\theta|}(\cdot,\mathbf{p})$ on its first argument and because $P \in \mathscr{V}^d_n(S,W)$ is a linear combination of the polynomials in the basis $\{\mathcal{S}^n_\alpha : |\alpha| = n\}$. ☐

Notice that (3.29) implies that if $P \in \mathscr{V}^d_n(S,W)$, $n \ge \kappa$, then

$$0 = \langle P,Q\rangle_S = \langle P,Q\rangle_{\nabla^\kappa} + \sum_{i=0}^{\kappa-1} \lambda_i \underbrace{(\nabla^i P)(\mathbf{p})}_{=\mathbf{0}} \cdot (\nabla^i Q)(\mathbf{p}), \quad Q \in \Pi^d_{n-1},$$

that is, $P \in \mathscr{V}_n^d(\nabla^\kappa, W)$. Then, by (3.30), the linear operator $\mathscr{R}_{\kappa,\mathbf{p}}$ is surjective. It is not difficult to prove that $\ker \mathscr{R}_{\kappa,\mathbf{p}} = \Pi_{\kappa-1}^d$. Therefore, by the first isomorphism theorem [20, theorem 6.12], there is an isomorphism $\mathscr{S}_{\kappa,\mathbf{p}}$ from the linear quotient space $\mathscr{V}_n^d(\nabla^\kappa, W)/\Pi_{\kappa-1}^d$ to the linear space $\mathscr{V}_n^d(S, W)$ defined by $\mathscr{R}_{\kappa,\mathbf{p}} = \mathscr{S}_{\kappa,\mathbf{p}} \circ \mathscr{N}$, where $\mathscr{N}$ is the natural mapping defined by $\mathscr{N}(P) = [P]$, $P \in \mathscr{V}_n^d(\nabla^\kappa, W)$, such that the following diagram commutes.

$$\begin{array}{ccc} \mathscr{V}_n^d(\nabla^\kappa, W) & \xrightarrow{\mathscr{R}_{\kappa,\mathbf{p}}} & \mathscr{V}_n^d(S, W) \\ \big\downarrow{\scriptstyle \mathscr{N}} & \nearrow{\scriptstyle \mathscr{S}_{\kappa,\mathbf{p}}} & \\ \mathscr{V}_n^d(\nabla^\kappa, W)/\Pi_{\kappa-1}^d & & \end{array}$$

Recall that on the linear space $\mathscr{V}_n^d(\nabla^\kappa, W)/\Pi_{\kappa-1}^d$ the operations $[P]+[Q] := [P+Q]$ and $a[P] := [aP]$, $a \in \mathbb{R}$, $P, Q \in \mathscr{V}_n^d(\nabla^\kappa, W)$, are well-defined. On the quotient space $\mathscr{V}_n^d(\nabla^\kappa, W)/\Pi_{\kappa-1}^d$ there is a well-defined inner product (that is, it does not depend on the representative we choose of each equivalence class), induced by the bilinear form $\langle \cdot, \cdot \rangle_{\nabla^\kappa}$, defined by

$$\langle [P], [Q] \rangle_{\mathscr{V}_n^d(\nabla^\kappa, W)/\Pi_{\kappa-1}^d} := \langle P, Q \rangle_{\nabla^\kappa}, \quad P, Q \in \mathscr{V}_n^d(\nabla^\kappa, W).$$

Moreover, (3.23) implies that

$$\langle [P], [Q] \rangle_{\mathscr{V}_n^d(\nabla^\kappa, W)/\Pi_{\kappa-1}^d} = \langle \mathscr{S}_{\kappa,\mathbf{p}}([P]), \mathscr{S}_{\kappa,\mathbf{p}}([Q]) \rangle_S, \quad P, Q \in \mathscr{V}_n^d(\nabla^\kappa, W),$$

that is, $\mathscr{V}_n^d(\nabla^\kappa, W)/\Pi_{\kappa-1}^d$ and $\mathscr{V}_n^d(S, W)$, $n \geq \kappa$, are isomorphic inner product spaces.

Remark 3.2. The multinomial theorem [1, Section 24.1.2] in d variables:

$$(x_1 + x_2 + \cdots + x_d)^\kappa = \sum_{|\theta|=\kappa} \binom{\kappa}{\theta_1, \theta_2, \ldots, \theta_d} \mathbf{x}^\theta, \quad \mathbf{x}^\theta = x_1^{\theta_1} x_2^{\theta_2} \cdots x_d^{\theta_d}, \tag{3.32}$$

proves the factored form of the differential operator:

$$(\partial_1 + \partial_2 + \cdots + \partial_d)^\kappa = \sum_{|\theta|=\kappa} \binom{\kappa}{\theta_1, \theta_2, \ldots, \theta_d} \partial^\theta, \quad \partial^\theta = \partial_1^{\theta_1} \partial_2^{\theta_2} \cdots \partial_d^{\theta_d}, \tag{3.33}$$

which appears in Proposition 3.7.

Proposition 3.7. *Let $n \geq \kappa$, and let $P \in \mathscr{V}_n^d(S, W)$ or let $P \in \mathscr{V}_n^d(\nabla^\kappa, W)$. Then:*

$$(\partial_1 + \partial_2 + \cdots + \partial_d)^\kappa P \in \mathscr{V}_{n-\kappa}^d(W). \tag{3.34}$$

Proof. Notice that for any monomial $\mathbf{x}^\beta$ we have:

$$\partial^\theta \mathbf{x}^\beta = \begin{cases} \dfrac{\beta!}{(\beta-\theta)!} \mathbf{x}^{\beta-\theta}, & \theta \leq \beta, \\ 0, & \theta \nleq \beta, \end{cases} \qquad \theta, \beta \in \mathbb{N}_0^d,$$

and for a fixed multi-index $\theta \in \mathbb{N}_0^d$ we have the equality of sets:

$$\left\{\beta \in \mathbb{N}_0^d : |\beta| \leq n-1, \theta \leq \beta\right\} = \left\{\eta + \theta \in \mathbb{N}_0^d : |\eta| \leq n - |\theta| - 1\right\}.$$

If $P \in \mathscr{V}_n^d(\nabla^\kappa, W)$, and since $\left\{\dfrac{\mathbf{x}^\beta}{\beta!} : |\beta| \leq n-1\right\}$ is a basis of Π_{n-1}^d, then by (3.15) and the orthogonality of P with respect to $\langle\cdot,\cdot\rangle_{\nabla^\kappa}$, we have that:

$$\begin{aligned}
0 &= \left\langle P, \frac{\mathbf{x}^\beta}{\beta!}\right\rangle_{\nabla^\kappa} = \sum_{|\theta|=\kappa} \binom{\kappa}{\theta_1,\ldots,\theta_d} \left\langle \partial^\theta P, \partial^\theta\left(\frac{\mathbf{x}^\beta}{\beta!}\right)\right\rangle_W \\
&= \sum_{|\theta|=\kappa} \binom{\kappa}{\theta_1,\ldots,\theta_d} \left\langle \partial^\theta P, \underbrace{\frac{\mathbf{x}^{\beta-\theta}}{(\beta-\theta)!}}_{|\beta|\leq n-1,\, \theta\leq\beta} \right\rangle_W = \sum_{|\theta|=\kappa} \binom{\kappa}{\theta_1,\ldots,\theta_d} \left\langle \partial^\theta P, \underbrace{\frac{\mathbf{x}^{\eta}}{\eta!}}_{|\eta|\leq n-\kappa-1} \right\rangle_W \\
&= \left\langle \sum_{|\theta|=\kappa} \binom{\kappa}{\theta_1,\ldots,\theta_d} \partial^\theta P, \frac{\mathbf{x}^\eta}{\eta!}\right\rangle_W,
\end{aligned}$$

where we conclude that $(\partial_1+\partial_2+\cdots+\partial_d)^\kappa P = \sum_{|\theta|=\kappa} \binom{\kappa}{\theta_1,\ldots,\theta_d} \partial^\theta P$ is a polynomial of degree $n-\kappa$ that is orthogonal to all polynomials in $\Pi_{n-\kappa-1}^d$ with respect to $\langle\cdot,\cdot\rangle_W$, that is, we have (3.34). If $P \in \mathscr{V}_n^d(S, W)$ then by (3.29) we have that:

$$0 = \left\langle P, \frac{\mathbf{x}^\beta}{\beta!}\right\rangle_S = \left\langle P, \frac{\mathbf{x}^\beta}{\beta!}\right\rangle_{\nabla^\kappa} + \sum_{i=0}^{\kappa-1} \lambda_i \underbrace{(\nabla^i P)(\mathbf{p})}_{=\mathbf{0}} \cdot \left(\nabla^i \mathbf{x}^\beta/\beta!\right)(\mathbf{p}), \quad |\beta| \leq n-1.$$

We follow the same steps as above and again we conclude (3.34). □

Notice that if the polynomials in the space $\mathscr{V}_n^d(W)$ are eigenfunctions of some differential operator $\mathcal{L}$, that is,

$$\mathcal{L}P = \lambda_n P, \quad P \in \mathscr{V}_n^d(W), \tag{3.35}$$

where λ_n is an eigenvalue which depends on the degree n only, then Proposition 3.7 implies that the polynomials in the space $\mathscr{V}_n^d(S, W)$ (or $\mathscr{V}_n^d(\nabla^\kappa, W)$) satisfy a partial differential equation of the form:

$$[\mathcal{L} - \lambda_{n-\kappa}\mathcal{I}](\partial_1 + \partial_2 + \cdots + \partial_d)^\kappa P = 0, \quad (\mathcal{I} \text{ is the identity operator}).$$

Equations (1.17), (1.21), (1.31), (1.43), (1.45), (1.56), and (1.60) are examples of the more general equation (3.35). Therefore, for those equations we have some corollaries in Section 3.4.1 to Section 3.4.4.

3.3.6 Construction of a basis for the space $\mathscr{V}_n^d(\nabla^\kappa, W)$

Theorem 3.1 shows that we only need to know a basis for the space $\mathscr{V}_n^d(\nabla^\kappa, W)$ for $n \geq \kappa$. In this section we present an iterative method for constructing such a basis.

As in Theorem 3.1, let us denote by $\{S_\alpha^n : |\alpha| = n\}$ and $\{\mathcal{S}_\alpha^n : |\alpha| = n\}$ monic orthogonal bases of $\mathscr{V}_n^d(\nabla^\kappa, W)$ and $\mathscr{V}_n^d(S, W)$, respectively. We know that $\dim \mathscr{V}_n^d(\nabla^\kappa, W) = \dim \mathscr{V}_n^d(S, W) = r_n^d$. The elements of these two bases can be arranged in a vector form. We denote by $\mathbb{S}_n$ and $\mathfrak{S}_n$ the column vectors:

$$\mathbb{S}_n(\mathbf{x}) = \left(S_{\alpha^{(1)}}^n(\mathbf{x}), S_{\alpha^{(2)}}^n(\mathbf{x}), \ldots, S_{\alpha^{(r_n^d)}}^n(\mathbf{x})\right)^T,$$
$$\mathfrak{S}_n(\mathbf{x}) = \left(\mathcal{S}_{\alpha^{(1)}}^n(\mathbf{x}), \mathcal{S}_{\alpha^{(2)}}^n(\mathbf{x}), \ldots, \mathcal{S}_{\alpha^{(r_n^d)}}^n(\mathbf{x})\right)^T,$$

where $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(r_n^d)}$ is the arrangement of elements in $\left\{\alpha \in \mathbb{N}_0^d : |\alpha| = n\right\}$ according to the reverse lexicographical order. Then, by Theorem 3.1 we have that:

$$\mathfrak{S}_n(\mathbf{x}) = \left((\mathbf{x}-\mathbf{p})^{\alpha^{(1)}}, (\mathbf{x}-\mathbf{p})^{\alpha^{(2)}}, \ldots, (\mathbf{x}-\mathbf{p})^{\alpha^{(r_n^d)}}\right)^T, \quad 0 \leq n < \kappa,$$
$$\mathfrak{S}_n(\mathbf{x}) = \mathbb{S}_n(\mathbf{x}) - \mathcal{T}^{\kappa-1}(\mathbb{S}_n, \mathbf{p}; \mathbf{x}), \quad n \geq \kappa,$$

where $\mathcal{T}^{\kappa-1}(\mathbb{S}_n, \mathbf{p})$ denotes the column vector:

$$\mathcal{T}^{\kappa-1}(\mathbb{S}_n, \mathbf{p}) = \left(\mathcal{T}^{\kappa-1}(S_{\alpha^{(1)}}^n, \mathbf{p}), \mathcal{T}^{\kappa-1}(S_{\alpha^{(2)}}^n, \mathbf{p}), \ldots, \mathcal{T}^{\kappa-1}(S_{\alpha^{(r_n^d)}}^n, \mathbf{p})\right)^T.$$

In addition, with this notation $\left\langle \mathbb{S}_n, \mathbb{S}_m^T \right\rangle_{\nabla^\kappa}$ and $\left\langle \mathfrak{S}_n, \mathfrak{S}_m^T \right\rangle_S$ are both matrices of size $r_n^d \times r_m^d$ such that

$$\left\langle \mathbb{S}_n, \mathbb{S}_m^T \right\rangle_{\nabla^\kappa} = \left(\left\langle S_{\alpha^{(i)}}^n, S_{\beta^{(j)}}^m \right\rangle_{\nabla^\kappa}\right)_{1\leq i\leq r_n^d, 1\leq j\leq r_m^d} = \begin{cases} \mathbf{0}, & n \neq m, \\ \mathbf{H}_n^{\nabla^\kappa}, & n = m, \end{cases}$$
$$\left\langle \mathfrak{S}_n, \mathfrak{S}_m^T \right\rangle_S = \left(\left\langle \mathcal{S}_{\alpha^{(i)}}^n, \mathcal{S}_{\beta^{(j)}}^m \right\rangle_S\right)_{1\leq i\leq r_n^d, 1\leq j\leq r_m^d} = \begin{cases} \mathbf{0}, & n \neq m, \\ \mathbf{H}_n^S, & n = m, \end{cases}$$

where $\mathbf{H}_n^{\nabla^\kappa}$ and $\mathbf{H}_n^S$ are both symmetric matrices.

Remark 3.3. Even though the polynomials in the space $\mathscr{V}_n^d(\nabla^\kappa, W)$ are determined up to a polynomial of degree $\kappa - 1$, notice that the matrix $\left\langle \mathbb{S}_n, \mathbb{S}_m^T \right\rangle_{\nabla^\kappa}$ is well-defined. If $P_{\alpha^{(i)}}^n, Q_{\beta^{(j)}}^m \in \Pi^d$ are such that $S_{\alpha^{(i)}}^n \overset{\kappa-1}{=} P_{\alpha^{(i)}}^n$ and $S_{\beta^{(j)}}^m \overset{\kappa-1}{=} Q_{\beta^{(j)}}^m$, that is, $S_{\alpha^{(i)}}^n(\mathbf{x}) = P_{\alpha^{(i)}}^n(\mathbf{x}) + Q(\mathbf{x})$ and $S_{\beta^{(j)}}^m(\mathbf{x}) = Q_{\beta^{(j)}}^m(\mathbf{x}) + R(\mathbf{x})$, with $Q, R \in \Pi_{\kappa-1}^d$, then $\left\langle S_{\alpha^{(i)}}^n, S_{\beta^{(j)}}^m \right\rangle_{\nabla^\kappa} = \left\langle P_{\alpha^{(i)}}^n, Q_{\beta^{(j)}}^m \right\rangle_{\nabla^\kappa}$, because $\nabla^\kappa Q = \nabla^\kappa R = \mathbf{0}$. Then, each entry in $\left\langle \mathbb{S}_n, \mathbb{S}_m^T \right\rangle_{\nabla^\kappa}$ does not depend on the representative we choose of each equivalence class. In particular, $\mathbf{H}_n^{\nabla^\kappa}$ is well-defined.

Let us observe that $\mathbf{H}_n^{\nabla^\kappa}$ and $\mathbf{H}_n^S$ are given by:

$$\mathbf{H}_n^{\nabla^\kappa} := \left\langle \mathbb{S}_n, \mathbb{S}_n^T \right\rangle_{\nabla^\kappa} = \left(\left\langle S_{\alpha^{(i)}}^n, S_{\alpha^{(j)}}^n \right\rangle_{\nabla^\kappa}\right)_{i,j=1}^{r_n^d}, \tag{3.36}$$

$$\mathbf{H}_n^S := \left\langle \mathfrak{S}_n, \mathfrak{S}_n^T \right\rangle_S = \left(\left\langle \mathcal{S}_{\alpha^{(i)}}^n, \mathcal{S}_{\alpha^{(j)}}^n \right\rangle_S \right)_{i,j=1}^{r_n^d}, \tag{3.37}$$

that is, (3.36) and (3.37) are Gram matrices [53, pp. 407]. Therefore, they are always positive semi-definite. To see this in the case of $\mathbf{H}_n^{\nabla^\kappa}$, let $\mathbf{a} = \left(a_1, a_2, \dots, a_{r_n^d}\right)^T$ be a non-null column vector in $\mathbb{R}^{r_n^d}$, then

$$\begin{aligned} \mathbf{a}^T \mathbf{H}_n^{\nabla^\kappa} \mathbf{a} = \sum_{i=1}^{r_n^d} \sum_{j=1}^{r_n^d} a_i a_j \left\langle S_{\alpha^{(i)}}^n, S_{\alpha^{(j)}}^n \right\rangle_{\nabla^\kappa} = \sum_{i=1}^{r_n^d} \sum_{j=1}^{r_n^d} \left\langle a_i S_{\alpha^{(i)}}^n, a_j S_{\alpha^{(j)}}^n \right\rangle_{\nabla^\kappa} = \\ \left\langle \sum_{i=1}^{r_n^d} a_i S_{\alpha^{(i)}}^n, \sum_{j=1}^{r_n^d} a_j S_{\alpha^{(j)}}^n \right\rangle_{\nabla^\kappa} = \left\| \sum_{i=1}^{r_n^d} a_i S_{\alpha^{(i)}}^n \right\|_{\nabla^\kappa}^2 \geq 0. \end{aligned} \tag{3.38}$$

In fact, since $\langle \cdot, \cdot \rangle_S$ is an inner product then $\mathbf{H}_n^S$ is positive definite, but this is not always the case for $\mathbf{H}_n^{\nabla^\kappa}$ as we will show in the following proposition.

Proposition 3.8. $\mathbf{H}_n^{\nabla^\kappa} = \left\langle \mathbb{S}_n, \mathbb{S}_n^T \right\rangle_{\nabla^\kappa}$ *is positive definite if, and only if,* $n \geq \kappa$.

Proof. If $n < \kappa$ then every polynomial in the set $\{S_\alpha^n : |\alpha| = n\}$ has degree at most $\kappa - 1$. Then, $\nabla^\kappa S_\alpha^n = \mathbf{0}$ and as a consequence $\langle S_\alpha^n, Q \rangle_{\nabla^\kappa} = 0$ for all $Q \in \Pi^d$. Therefore, $\mathbf{H}_n^{\nabla^\kappa} = \left\langle \mathbb{S}_n, \mathbb{S}_n^T \right\rangle_{\nabla^\kappa} = \left(\left\langle S_{\alpha^{(i)}}^n, S_{\alpha^{(j)}}^n \right\rangle_{\nabla^\kappa} \right)_{i,j=1}^{r_n^d} = \mathbf{0}$ and $\mathbf{H}_n^{\nabla^\kappa}$ is not positive definite.

Conversely, let us suppose $n \geq \kappa$. Since the polynomial $\sum_{i=1}^{r_n^d} a_i S_{\alpha^{(i)}}^n = \mathbf{a}^T \mathbb{S}_n$ in (3.38) has degree n, $\mathbf{a} \in \mathbb{R}^{r_n^d}$, $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{a}^T \mathbb{S}_n \notin \Pi_{\kappa-1}^d = [0(\mathbf{x})]$, where $0(\mathbf{x})$ is the zero polynomial, that is, $\mathbf{a}^T \mathbb{S}_n \underset{\kappa-1}{\neq} 0(\mathbf{x})$. By Proposition 3.5 we have that:

$$\left\| \sum_{i=1}^{r_n^d} a_i S_{\alpha^{(i)}}^n(\mathbf{x}) - 0(\mathbf{x}) \right\|_{\nabla^\kappa} = \left\| \sum_{i=1}^{r_n^d} a_i S_{\alpha^{(i)}}^n \right\|_{\nabla^\kappa} \neq 0.$$

Therefore, from (3.38) we have that $\mathbf{H}_n^{\nabla^\kappa}$ is positive definite. □

Corollary 3.1. $\mathbf{H}_n^{\nabla^\kappa} = \left\langle \mathbb{S}_n, \mathbb{S}_n^T \right\rangle_{\nabla^\kappa}$ *is non-singular if, and only if,* $n \geq \kappa$.

Proof. Let us denote by $\lambda(\mathbf{H}_n^{\nabla^\kappa})$ an eigenvalue of $\mathbf{H}_n^{\nabla^\kappa}$. In the proof of Proposition 3.8 we showed that if $n < \kappa$ then $\mathbf{H}_n^{\nabla^\kappa} = \mathbf{0}$ and, therefore, $\mathbf{H}_n^{\nabla^\kappa}$ is singular.

Conversely, if $n \geq \kappa$ then $\mathbf{H}_n^{\nabla^\kappa}$ is positive definite by Proposition 3.8, and therefore $\lambda_i(\mathbf{H}_n^{\nabla^\kappa}) > 0$ for all $i = 1, 2, \dots, r_n^d$. As a consequence $\det(\mathbf{H}_n^{\nabla^\kappa}) = \prod_{i=1}^{r_n^d} \lambda_i(\mathbf{H}_n^{\nabla^\kappa}) > 0$ and we conclude that $\mathbf{H}_n^{\nabla^\kappa}$ is non-singular. □

The following proposition shows a relation between the matrices $\mathbf{H}_n^S$ and $\mathbf{H}_n^{\nabla^\kappa}$.

Proposition 3.9. *Let* $\lambda_i > 0$, $i = 0, 1, \dots, \kappa - 1$ *be the positive constants in the Sobolev inner product* (3.12)*, and let* $\mathbf{H}_n^{\nabla^\kappa}$ *and* $\mathbf{H}_n^S$ *be defined in* (3.36) *and* (3.37)*, respectively. Then:*

$$\mathbf{H}_n^S = \lambda_n n! \operatorname{diag}\left(\alpha^{(1)}!, \alpha^{(2)}!, \dots, \alpha^{(r_n^d)}!\right), \quad |\alpha^{(i)}| = n, \quad 1 \leq i \leq r_n^d, \quad 0 \leq n < \kappa,$$

$$\mathbf{H}_n^S = \mathbf{H}_n^{\nabla^\kappa}, \quad n \geq \kappa.$$

Proof. Case $0 \leq |\alpha| = n < \kappa$ in (3.24): Since $\nabla^i \mathcal{S}_\alpha^n = \nabla^i (\mathbf{x}-\mathbf{p})^\alpha = \mathbf{0}$ if $n < i \leq \kappa$, then by (3.3) and for all $Q \in \Pi^d$ the inner product $\langle \mathcal{S}_\alpha^n, Q\rangle_S$ in (3.12) reduces to:

$$\langle \mathcal{S}_\alpha^n, Q\rangle_S = \sum_{i=0}^{n} \lambda_i \nabla^i \mathcal{S}_\alpha^n(\mathbf{p}) \cdot \nabla^i Q(\mathbf{p}) = \sum_{i=0}^{n} \lambda_i \sum_{|\theta|=i} \binom{i}{\theta_1, \dots, \theta_d} \partial^\theta \mathcal{S}_\alpha^n(\mathbf{p}) \partial^\theta Q(\mathbf{p}). \quad (3.39)$$

But $\partial^\theta \mathcal{S}_\alpha^n(\mathbf{x}) = \partial^\theta (\mathbf{x}-\mathbf{p})^\alpha = \prod_{i=1}^d (\alpha_i - \theta_i + 1)_{\theta_i} (x_i - p_i)^{\alpha_i - \theta_i}$, where we get $\partial^\theta \mathcal{S}_\alpha^n(\mathbf{p}) = \alpha! \delta_{\alpha,\theta}$. Therefore, (3.39) reduces even more to:

$$\langle \mathcal{S}_\alpha^n, Q\rangle_S = \lambda_n \binom{n}{\alpha_1, \alpha_2, \dots, \alpha_d} \alpha! \partial^\alpha Q(\mathbf{p}) = \lambda_n n! \partial^\alpha Q(\mathbf{p}),$$

and, in particular, the entries of $\mathbf{H}_n^S$ are given by:

$$\left\langle \mathcal{S}_{\alpha^{(i)}}^n, \mathcal{S}_{\alpha^{(j)}}^n \right\rangle_S = \lambda_n n! \partial^{\alpha^{(i)}} \mathcal{S}_{\alpha^{(j)}}^n(\mathbf{p}) = \lambda_n n! \alpha^{(i)}! \delta_{\alpha^{(i)},\alpha^{(j)}}, \quad 1 \leq i, j \leq r_n^d.$$

Case $|\alpha| = n \geq \kappa$ in (3.25): Since $(\nabla^i S_\alpha^n)(\mathbf{p}) = (\nabla^i \mathcal{T}^{\kappa-1}(S_\alpha^n, \mathbf{p}))(\mathbf{p})$ for all $i = 0, 1, \dots, \kappa-1$, and also $\nabla^\kappa \mathcal{T}^{\kappa-1}(S_\alpha^n, \mathbf{p}) = \mathbf{0}$ then for all $Q \in \Pi^d$:

$$\begin{aligned}\langle \mathcal{S}_\alpha^n, Q\rangle_S &= \left\langle S_\alpha^n - \mathcal{T}^{\kappa-1}(S_\alpha^n, \mathbf{p}), Q\right\rangle_{\nabla^\kappa} + \sum_{i=0}^{\kappa-1} \lambda_i \underbrace{\nabla^i (S_\alpha^n - \mathcal{T}^{\kappa-1}(S_\alpha^n, \mathbf{p}))(\mathbf{p})}_{=\mathbf{0}} \cdot \nabla^i Q(\mathbf{p}) \\ &= \left\langle S_\alpha^n - \mathcal{T}^{\kappa-1}(S_\alpha^n, \mathbf{p}), Q\right\rangle_{\nabla^\kappa} = \langle S_\alpha^n, Q\rangle_{\nabla^\kappa},\end{aligned}$$

and, in particular, the entries of $\mathbf{H}_n^S$ are given by:

$$\left\langle \mathcal{S}_{\alpha^{(i)}}^n, \mathcal{S}_{\alpha^{(j)}}^n \right\rangle_S = \left\langle S_{\alpha^{(i)}}^n, S_{\alpha^{(j)}}^n \right\rangle_{\nabla^\kappa}, \quad 1 \leq i, j \leq r_n^d. \qquad \square$$

In view of Proposition 3.9, $\mathbf{H}_n^S$ can be computed in a closed form for $0 \leq n < \kappa$. In Proposition 3.11 we will show an iterative method for computing the matrix $\mathbf{H}_n^{\nabla^\kappa}$ (and therefore $\mathbf{H}_n^S$) for $n \geq \kappa$. In addition, $\mathbf{H}_n^S$ inherits all the properties of $\mathbf{H}_n^{\nabla^\kappa}$ for $n \geq \kappa$. In particular, we can confirm that for all $n \geq 0$:

1. $\mathbf{H}_n^S$ is positive definite. This follows from Proposition 3.8 and Proposition 3.9.

2. $\mathbf{H}_n^S$ is non-singular. This follows from Corollary 3.1 and Proposition 3.9.

3. If $\mathbf{H}_n^{\nabla^\kappa}$ is a diagonal matrix then $\mathbf{H}_n^S$ so is. This implies that if $\{S_\alpha^n : |\alpha| = n\}$ is a mutually orthogonal basis of $\mathscr{V}_n^d(\nabla^\kappa, W)$ then $\{\mathcal{S}_\alpha^n : |\alpha| = n\}$ is a mutually orthogonal basis of $\mathscr{V}_n^d(S, W)$.

In order to construct a monic orthogonal basis $\{S_\alpha^n : |\alpha| = n\}$ for the space $\mathscr{V}_n^d(\nabla^\kappa, W)$, we expand each S_α^n in terms of well-known polynomials Q_β^m of degree m, $|\beta| = m$, $0 \leq m \leq n$, in the form:

$$S_\alpha^n(\mathbf{x}) = \sum_{m=0}^{n} \sum_{|\beta|=m} c_{\alpha,\beta} Q_\beta^m(\mathbf{x}), \quad (3.40)$$

and then we determine the coefficients $c_{\alpha,\beta}$ of such an expansion by orthogonality. Since each polynomial in the space $\mathscr{V}_n^d(\nabla^\kappa, W)$ is determined up to a polynomial of degree $\kappa-1$, the equality in (3.40) must be replaced by the relation $\stackrel{\kappa-1}{=}$. The choice of Q_β^m clearly matters, mainly because in our construction method we need to compute explicitly some matrices where the polynomials Q_β^m are involved (see Proposition 3.10, Proposition 3.11, and Section 3.3.7). In addition, many computations involve higher-order derivatives of Q_β^m. Then, our choice criteria for Q_β^m depend on the domain Ω (product domain, ball, simplex, or cone) where the $\langle\cdot,\cdot\rangle_{\nabla^\kappa}$ is defined and the simplification of several computations.

Definition 3.1. *Let $P, Q \in \Pi_n^d$, with $n = \deg P = \deg Q$. We say P and Q have the same leading coefficient if $P - Q \in \Pi_{n-1}^d$*

Let $Q_\alpha^n \in \Pi_n^d$, $|\alpha| = n$, be a polynomial that has the same leading coefficient than S_α^n. Notice that our assumption that S_α^n is a monic polynomial leads to that Q_α^n is also monic. We denote by $\mathbb{Q}_n$ the column vector:

$$\mathbb{Q}_n = \left(Q_{\alpha^{(1)}}^n(\mathbf{x}), Q_{\alpha^{(2)}}^n(\mathbf{x}), \ldots, Q_{\alpha^{(r_n^d)}}^n(\mathbf{x})\right)^T, \tag{3.41}$$

where $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(r_n^d)}$ is the arrangement of elements in $\left\{\alpha \in \mathbb{N}_0^d : |\alpha| = n\right\}$ according to the reverse lexicographical order. We have the following proposition that relates the sequences $\{\mathbb{S}_n\}_{n\geq 0}$ and $\{\mathbb{Q}_n\}_{n\geq 0}$.

Proposition 3.10. *There exist real matrices $\mathbf{A}_{n,i}$ of size $r_n^d \times r_i^d$, $\kappa \leq i \leq n-1$, such that:*

$$\mathbb{Q}_n \stackrel{\kappa-1}{=} \mathbb{S}_n, \quad n \leq \kappa, \quad \textit{and} \quad \mathbb{Q}_n \stackrel{\kappa-1}{=} \mathbb{S}_n + \sum_{i=\kappa}^{n-1} \mathbf{A}_{n,i}\mathbb{S}_i, \quad n > \kappa.$$

Proof. If $n \leq \kappa$ then $Q_\alpha^n - S_\alpha^n \in \Pi_{n-1}^d$ is a polynomial of degree at most $\kappa - 1$ and therefore $\nabla^\kappa(Q_\alpha^n - S_\alpha^n) = \mathbf{0}$, that is, $\nabla^\kappa Q_\alpha^n = \nabla^\kappa S_\alpha^n$. By Proposition 3.5 we have that $Q_\alpha^n \stackrel{\kappa-1}{=} S_\alpha^n$ and as a consequence:

$$\mathbb{Q}_n = \left(Q_{\alpha^{(1)}}^n, Q_{\alpha^{(2)}}^n, \ldots, Q_{\alpha^{(r_n^d)}}^n\right)^T \stackrel{\kappa-1}{=} \left(S_{\alpha^{(1)}}^n, S_{\alpha^{(2)}}^n, \ldots, S_{\alpha^{(r_n^d)}}^n\right)^T = \mathbb{S}_n.$$

Now, let us suppose $n > \kappa$. If $\mathbb{Q}_n = \sum_{j=0}^n \mathbf{A}_{n,j}\mathbb{S}_j$ then we have:

$$\left\langle \mathbb{Q}_n, \mathbb{S}_i^T \right\rangle_{\nabla^\kappa} = \sum_{j=0}^n \mathbf{A}_{n,j} \left\langle \mathbb{S}_j, \mathbb{S}_i^T \right\rangle_{\nabla^\kappa} = \mathbf{A}_{n,i} \left\langle \mathbb{S}_i, \mathbb{S}_i^T \right\rangle_{\nabla^\kappa} = \mathbf{A}_{n,i}\mathbf{H}_i^{\nabla^\kappa}, \quad 0 \leq i \leq n.$$

By Corollary 3.1, $\mathbf{H}_i^{\nabla^\kappa}$ is non-singular if, and only if, $i \geq \kappa$. Therefore:

$$\mathbf{A}_{n,i} = \left\langle \mathbb{Q}_n, \mathbb{S}_i^T \right\rangle_{\nabla^\kappa} (\mathbf{H}_i^{\nabla^\kappa})^{-1}, \quad \kappa \leq i \leq n,$$

and $\mathbb{Q}_n - \sum_{j=\kappa}^n \left\langle \mathbb{Q}_n, \mathbb{S}_j^T \right\rangle_{\nabla^\kappa} (\mathbf{H}_j^{\nabla^\kappa})^{-1}\mathbb{S}_j = \sum_{j=0}^{\kappa-1} \mathbf{A}_{n,j}\mathbb{S}_j$ is a column vector whose entries are polynomials of degree at most $\kappa - 1$, that is,

$$\mathbb{Q}_n \stackrel{\kappa-1}{=} \sum_{j=\kappa}^n \left\langle \mathbb{Q}_n, \mathbb{S}_j^T \right\rangle_{\nabla^\kappa} (\mathbf{H}_j^{\nabla^\kappa})^{-1}\mathbb{S}_j.$$

Now, since each entry in the column vector $\mathbb{Q}_n - \mathbb{S}_n$ is a polynomial of degree at most $n-1$ we have that $\left\langle \mathbb{Q}_n - \mathbb{S}_n, \mathbb{S}_n^T \right\rangle_{\nabla^\kappa} = \mathbf{0}$, that is, $\left\langle \mathbb{Q}_n, \mathbb{S}_n^T \right\rangle_{\nabla^\kappa} = \left\langle \mathbb{S}_n, \mathbb{S}_n^T \right\rangle_{\nabla^\kappa} = \mathbf{H}_n^{\nabla^\kappa}$. Therefore, $\mathbf{A}_{n,n} = \left\langle \mathbb{Q}_n, \mathbb{S}_n^T \right\rangle_{\nabla^\kappa} (\mathbf{H}_n^{\nabla^\kappa})^{-1} = \mathbf{H}_n^{\nabla^\kappa} (\mathbf{H}_n^{\nabla^\kappa})^{-1} = \mathbf{I}_n$, and we have the result. □

As a consequence of Proposition 3.10, for $n > \kappa$ the polynomials in $\mathbb{S}_n$ can be found recursively, up to a vector of polynomials of degree at most $\kappa - 1$, in terms of $\mathbb{S}_\kappa, \mathbb{S}_{\kappa+1}, \ldots, \mathbb{S}_{n-1}$ of lower degrees by means of the relation:

$$\mathbb{S}_n \overset{\kappa-1}{=} \mathbb{Q}_n - \sum_{i=\kappa}^{n-1} \mathbf{A}_{n,i} \mathbb{S}_i, \quad n > \kappa, \quad \text{with} \quad \mathbb{S}_\kappa \overset{\kappa-1}{=} \mathbb{Q}_\kappa. \tag{3.42}$$

In addition, Proposition 3.10 shows that $\{\mathbb{S}_n\}_{n\geq 0}$ can be expressed, up to a vector of polynomials of degree at most $\kappa - 1$, in terms of $\{\mathbb{Q}_n\}_{n\geq 0}$ as follows:

1. $\mathbb{S}_n \overset{\kappa-1}{=} \mathbb{Q}_n$, $n \leq \kappa$,
2. $\mathbb{S}_{\kappa+1} \overset{\kappa-1}{=} \mathbb{Q}_{\kappa+1} - \mathbf{A}_{\kappa+1,\kappa} \mathbb{Q}_\kappa$,
3. $\mathbb{S}_{\kappa+2} \overset{\kappa-1}{=} \mathbb{Q}_{\kappa+2} - \mathbf{A}_{\kappa+2,\kappa+1}(\mathbb{Q}_{\kappa+1} - \mathbf{A}_{\kappa+1,\kappa}\mathbb{Q}_\kappa) - \mathbf{A}_{\kappa+2,\kappa}\mathbb{Q}_\kappa$,
4. etc.

Then, all we need is to compute the matrices $\mathbf{A}_{n,i}$ that appear in Proposition 3.10. Since we cannot calculate directly the $n - \kappa$ matrices $\mathbf{A}_{n,i} = \left\langle \mathbb{Q}_n, \mathbb{S}_i^T \right\rangle_{\nabla^\kappa} (\mathbf{H}_i^{\nabla^\kappa})^{-1}$, $\kappa \leq i \leq n-1$, because we do not know explicitly the polynomials $\mathbb{S}_n$, we must proceed inductively in the sequel.

We define the matrix $\mathbf{B}_{n,i} := \left\langle \mathbb{Q}_n, \mathbb{S}_i^T \right\rangle_{\nabla^\kappa}$, $\kappa \leq i \leq n-1$, of size $r_n^d \times r_i^d$ such that we can write $\mathbf{A}_{n,i}$ in the form $\mathbf{A}_{n,i} = \mathbf{B}_{n,i}(\mathbf{H}_i^{\nabla^\kappa})^{-1}$. By Proposition 3.10 we have that:

$$\mathbf{H}_\kappa^{\nabla^\kappa} = \left\langle \mathbb{S}_\kappa, \mathbb{S}_\kappa^T \right\rangle_{\nabla^\kappa} = \left\langle \mathbb{Q}_\kappa, \mathbb{Q}_\kappa^T \right\rangle_{\nabla^\kappa}, \quad \mathbf{B}_{n,\kappa} = \left\langle \mathbb{Q}_n, \mathbb{S}_\kappa^T \right\rangle_{\nabla^\kappa} = \left\langle \mathbb{Q}_n, \mathbb{Q}_\kappa^T \right\rangle_{\nabla^\kappa}.$$

Therefore, for $i = \kappa$ we have that:

$$\mathbf{A}_{n,\kappa} = \mathbf{B}_{n,\kappa}(\mathbf{H}_\kappa^{\nabla^\kappa})^{-1} = \left\langle \mathbb{Q}_n, \mathbb{Q}_\kappa^T \right\rangle_{\nabla^\kappa} (\left\langle \mathbb{Q}_\kappa, \mathbb{Q}_\kappa^T \right\rangle_{\nabla^\kappa})^{-1}.$$

We need to find $\mathbf{A}_{n,i}$ only for $\kappa < i \leq n-1$. By Proposition 3.10, the orthogonality of $\{\mathbb{S}_n\}_{n\geq 0}$ with respect to $\langle \cdot, \cdot \rangle_{\nabla^\kappa}$, and Corollary 3.1 we have that:

$$\begin{aligned}
\mathbf{H}_i^{\nabla^\kappa} = \left\langle \mathbb{S}_i, \mathbb{S}_i^T \right\rangle_{\nabla^\kappa} &= \left\langle \mathbb{Q}_i - \sum_{j=\kappa}^{i-1} \mathbf{A}_{i,j}\mathbb{S}_j, (\mathbb{Q}_i - \sum_{l=\kappa}^{i-1} \mathbf{A}_{i,l}\mathbb{S}_l)^T \right\rangle_{\nabla^\kappa} \\
&= \left\langle \mathbb{Q}_i, \mathbb{Q}_i^T \right\rangle_{\nabla^\kappa} - \sum_{j=\kappa}^{i-1} \mathbf{A}_{i,j} \left\langle \mathbb{S}_j, \mathbb{Q}_i^T \right\rangle_{\nabla^\kappa} - \sum_{l=\kappa}^{i-1} \left\langle \mathbb{Q}_i, \mathbb{S}_l^T \right\rangle_{\nabla^\kappa} \mathbf{A}_{i,l}^T \\
&+ \sum_{j=\kappa}^{i-1} \sum_{l=\kappa}^{i-1} \mathbf{A}_{i,j} \left\langle \mathbb{S}_j, \mathbb{S}_l^T \right\rangle_{\nabla^\kappa} \mathbf{A}_{i,l}^T
\end{aligned}$$

$$
\begin{aligned}
&= \left\langle \mathbb{Q}_i, \mathbb{Q}_i^T \right\rangle_{\nabla^\kappa} - \sum_{j=\kappa}^{i-1} \mathbf{A}_{i,j} \mathbf{H}_j^{\nabla^\kappa} \mathbf{A}_{i,j}^T - \sum_{l=\kappa}^{i-1} \mathbf{A}_{i,l} \mathbf{H}_l^{\nabla^\kappa} \mathbf{A}_{i,l}^T + \sum_{l=\kappa}^{i-1} \mathbf{A}_{i,l} \mathbf{H}_l^{\nabla^\kappa} \mathbf{A}_{i,l}^T \\
&= \left\langle \mathbb{Q}_i, \mathbb{Q}_i^T \right\rangle_{\nabla^\kappa} - \sum_{j=\kappa}^{i-1} \mathbf{B}_{i,j} (\mathbf{H}_j^{\nabla^\kappa})^{-1} \mathbf{B}_{i,j}^T, \quad \kappa < i \leq n-1,
\end{aligned}
$$

and also we have:

$$
\begin{aligned}
\mathbf{B}_{n,i} &= \left\langle \mathbb{Q}_n, \mathbb{S}_i^T \right\rangle_{\nabla^\kappa} = \left\langle \mathbb{Q}_n, (\mathbb{Q}_i - \sum_{j=\kappa}^{i-1} \mathbf{A}_{i,j} \mathbb{S}_j)^T \right\rangle_{\nabla^\kappa} \\
&= \left\langle \mathbb{Q}_n, \mathbb{Q}_i^T \right\rangle_{\nabla^\kappa} - \sum_{j=\kappa}^{i-1} \left\langle \mathbb{Q}_n, \mathbb{S}_j^T \right\rangle_{\nabla^\kappa} \mathbf{A}_{i,j}^T = \left\langle \mathbb{Q}_n, \mathbb{Q}_i^T \right\rangle_{\nabla^\kappa} - \sum_{j=\kappa}^{i-1} \mathbf{A}_{n,j} \mathbf{H}_j^{\nabla^\kappa} \mathbf{A}_{i,j}^T \\
&= \left\langle \mathbb{Q}_n, \mathbb{Q}_i^T \right\rangle_{\nabla^\kappa} - \sum_{j=\kappa}^{i-1} \mathbf{B}_{n,j} (\mathbf{H}_j^{\nabla^\kappa})^{-1} \mathbf{B}_{i,j}^T, \quad \kappa < i \leq n-1.
\end{aligned}
$$

Therefore, we have proved the following proposition.

Proposition 3.11. *Let $n > \kappa$. The $n-\kappa$ real matrices $\mathbf{A}_{n,i}$, $\kappa \leq i \leq n-1$, are given by $\mathbf{A}_{n,i} = \mathbf{B}_{n,i}(\mathbf{H}_i^{\nabla^\kappa})^{-1}$, where $\mathbf{B}_{n,i}$ and $\mathbf{H}_i^{\nabla^\kappa}$, of size $r_n^d \times r_i^d$ and $r_i^d \times r_i^d$ respectively, satisfy the recursive relations:*

$$
\mathbf{B}_{n,i} = \begin{cases} \left\langle \mathbb{Q}_n, \mathbb{Q}_\kappa^T \right\rangle_{\nabla^\kappa}, & i = \kappa, \\ \left\langle \mathbb{Q}_n, \mathbb{Q}_i^T \right\rangle_{\nabla^\kappa} - \sum_{j=\kappa}^{i-1} \mathbf{B}_{n,j} (\mathbf{H}_j^{\nabla^\kappa})^{-1} \mathbf{B}_{i,j}^T, & i > \kappa, \end{cases}
$$

$$
\mathbf{H}_i^{\nabla^\kappa} = \begin{cases} \left\langle \mathbb{Q}_\kappa, \mathbb{Q}_\kappa^T \right\rangle_{\nabla^\kappa}, & i = \kappa, \\ \left\langle \mathbb{Q}_i, \mathbb{Q}_i^T \right\rangle_{\nabla^\kappa} - \sum_{j=\kappa}^{i-1} \mathbf{B}_{i,j} (\mathbf{H}_j^{\nabla^\kappa})^{-1} \mathbf{B}_{i,j}^T, & i > \kappa. \end{cases}
$$

In order to find recursively the polynomials $\{\mathbb{S}_n\}_{n\geq 0}$ by means of (3.42), Proposition 3.11 shows us it is necessary to know explicitly the $n-\kappa$ rectangular matrices $\left\langle \mathbb{Q}_n, \mathbb{Q}_i^T \right\rangle_{\nabla^\kappa}$ of size $r_n^d \times r_i^d$, $\kappa \leq i \leq n-1$, and also the $n-\kappa$ square matrices $\left\langle \mathbb{Q}_i, \mathbb{Q}_i^T \right\rangle_{\nabla^\kappa}$ of size $r_i^d \times r_i^d$, $\kappa \leq i \leq n-1$. In the next subsection we present some considerations for computing these matrices. In Section 3.4.1 to Section 3.4.4 we present computations for particular domains Ω (product domain, the simplex, the unit ball and the cone). Chapter 4 presents some examples in two variables on different domains. In addition, in [42, Section 4] there is an example in three variables on a product domain.

3.3.7 Some considerations for computing $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^\kappa}$

As mentioned above, our recursive method (Proposition 3.10 and Proposition 3.11) for computing a monic orthogonal basis for the space $\mathscr{V}_n^d(\nabla^\kappa, W)$ requires the explicit computation of the matrix $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^\kappa}$, $n, m \geq \kappa$, of size $r_n^d \times r_m^d$, given by:

$$
\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^\kappa} = \left(\left\langle Q_{\alpha(i)}^n, Q_{\beta(j)}^m \right\rangle_{\nabla^\kappa} \right)_{1\leq i\leq r_n^d, 1\leq j\leq r_m^d}, \tag{3.43}
$$

where $Q_\alpha^n(\mathbf{x}) = \mathbf{x}^\alpha + R_\alpha(\mathbf{x})$, $|\alpha| = n$, $R_\alpha \in \Pi_{n-1}^d$, that is, Q_α^n is monic, and $\mathbb{Q}_n$ is the column vector defined in (3.41).

Remark 3.4. Notice that the matrix $\langle \mathbb{Q}_n, \mathbb{Q}_m^T \rangle_{\nabla^\kappa}$ is well-defined. If $R_{\alpha^{(i)}}^n, T_{\beta^{(j)}}^m \in \Pi^d$ are such that $Q_{\alpha^{(i)}}^n \overset{\kappa-1}{=} R_{\alpha^{(i)}}^n$ and $Q_{\beta^{(j)}}^m \overset{\kappa-1}{=} T_{\beta^{(j)}}^m$, that is, $Q_{\alpha^{(i)}}^n(\mathbf{x}) = R_{\alpha^{(i)}}^n(\mathbf{x}) + U(\mathbf{x})$ and $Q_{\beta^{(j)}}^m(\mathbf{x}) = T_{\beta^{(j)}}^m(\mathbf{x}) + V(\mathbf{x})$, with $U, V \in \Pi_{\kappa-1}^d$, then $\left\langle Q_{\alpha^{(i)}}^n, Q_{\beta^{(j)}}^m \right\rangle_{\nabla^\kappa} = \left\langle R_{\alpha^{(i)}}^n, T_{\beta^{(j)}}^m \right\rangle_{\nabla^\kappa}$, because $\nabla^\kappa U = \nabla^\kappa V = \mathbf{0}$. Then, each entry in $\langle \mathbb{Q}_n, \mathbb{Q}_m^T \rangle_{\nabla^\kappa}$ does not depend on the representative we choose of each equivalence class.

From equation (3.15), each entry of the matrix $\langle \mathbb{Q}_n, \mathbb{Q}_m^T \rangle_{\nabla^\kappa}$ can be computed in terms of the inner product $\langle \cdot, \cdot \rangle_W$ and partial derivatives $\partial^\theta = \partial_1^{\theta_1} \partial_2^{\theta_2} \cdots \partial_d^{\theta_d}$ of order $|\theta| = \kappa$ by:

$$\left\langle Q_{\alpha^{(i)}}^n, Q_{\beta^{(j)}}^m \right\rangle_{\nabla^\kappa} = \sum_{|\theta|=\kappa} \binom{\kappa}{\theta_1, \theta_2, \ldots, \theta_d} \left\langle \partial^\theta Q_{\alpha^{(i)}}^n, \partial^\theta Q_{\beta^{(j)}}^m \right\rangle_W. \tag{3.44}$$

Therefore, the computation of (3.44) depends significantly on the weight function W (see Remark 3.5), the domain Ω, and a suitable choice of Q_α^n. An obvious choice is the basis of orthogonal polynomials with respect to $\langle \cdot, \cdot \rangle_W$. This basis, however, is not a good choice because we need to work with higher-order derivatives of the basis elements. Our choice of Q_α^n depends basically on the following criteria:

1. that Q_α^n is monic, and
2. that a reduction of a big amount of calculations for computing (3.44) is desirable, depending on the weight function W and the domain Ω.

Then, in order to get additional results we will work with specific weight functions in the next sections.

Remark 3.5. If Q_α^n is expanded in terms of the canonical basis:

$$Q_\alpha^n(\mathbf{x}) = \sum_{|\phi| \le n} c_{\alpha,\phi} \mathbf{x}^\phi, \quad c_{\alpha,\phi} \in \mathbb{R}, \quad c_{\alpha,\phi} = \delta_{\alpha,\phi} \quad \text{if} \quad |\alpha| = |\phi| = n, \tag{3.45}$$

then we have:

$$\partial^\theta Q_\alpha^n(\mathbf{x}) = \sum_{\kappa \le |\phi| \le n} c_{\alpha,\phi} \partial^\theta \mathbf{x}^\phi = \sum_{\kappa \le |\phi| \le n} c_{\alpha,\phi} (-1)^\kappa (-\phi)_\theta \mathbf{x}^{\phi-\theta}, \quad |\theta| = \kappa, \tag{3.46}$$

where by properties of the Pochhammer symbol:

$$(-1)^\kappa (-\phi)_\theta = (-1)^{|\theta|} \prod_{i=1}^d (-\phi_i)_{\theta_i} = \prod_{i=1}^d (\phi_i - \theta_i + 1)_{\theta_i}.$$

Therefore, (3.44) reduces to:

$$\left\langle Q^n_{\alpha^{(i)}}, Q^m_{\beta^{(j)}} \right\rangle_{\nabla^\kappa} =$$
$$\sum_{|\theta|=\kappa} \sum_{\kappa\le|\phi|\le n} \sum_{\kappa\le|\varphi|\le m} \binom{\kappa}{\theta_1,\ldots,\theta_d} c_{\alpha^{(i)},\phi} c_{\beta^{(j)},\varphi} (-\phi)_\theta (-\varphi)_\theta \left\langle \mathbf{x}^{\phi-\theta}, \mathbf{x}^{\varphi-\theta} \right\rangle_W . \quad (3.47)$$

Equation (3.47) shows that the entries of the matrix $\langle \mathbb{Q}_n, \mathbb{Q}_m^T \rangle_{\nabla^\kappa}$ depend on the moments $\langle \mathbf{x}^{\phi-\theta}, \mathbf{x}^{\varphi-\theta} \rangle_W$ of the weight function W. In addition, notice that the expression (3.47) can be reduced even more if we choose Q^n_α as a monomial, that is, $Q^n_\alpha(\mathbf{x}) = \mathbf{x}^\alpha$, $|\alpha| = n$. In this last case, $\mathbb{Q}_n$ is defined to be the column vector:

$$\mathbb{Q}_n = \mathbb{X}_n = \left(\mathbf{x}^{\alpha^{(1)}}, \mathbf{x}^{\alpha^{(2)}}, \ldots, \mathbf{x}^{\alpha^{(r^d_n)}}\right)^T, \quad (3.48)$$

where $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(r^d_n)}$ is the arrangement of the elements in $\{\alpha \in \mathbb{N}_0^d : |\alpha| = n\}$ according to the reverse lexicographical order, and (3.47) can be simplified even more to:

$$\left\langle Q^n_{\alpha^{(i)}}, Q^m_{\beta^{(j)}} \right\rangle_{\nabla^\kappa} = \sum_{|\theta|=\kappa} \binom{\kappa}{\theta_1,\theta_2,\ldots,\theta_d} (-\alpha^{(i)})_\theta (-\beta^{(j)})_\theta \left\langle \mathbf{x}^{\alpha^{(i)}-\theta}, \mathbf{x}^{\beta^{(j)}-\theta} \right\rangle_W . \quad (3.49)$$

A direct comparison of (3.47) and (3.49) shows that choosing Q^n_α as a monomial then we get a considerable reduction of calculations.

3.4 Computing $\langle \mathbb{Q}_n, \mathbb{Q}_m^T \rangle_{\nabla^\kappa}$ on different domains and other results on partial differential equations

3.4.1 Product domains

For this section we remit the reader to the results from Section 1.3.1 on the space $\mathscr{V}^d_n(W)$ of standard orthogonal polynomials on the product domain:

$$\Omega := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d], \quad (3.50)$$

where $[a_i, b_i]$, $i = 1, 2, \ldots, d$, is an interval of $\mathbb{R}$ ($|a_i|$ and $|b_i|$ can be infinite), with respect to the product weight function:

$$W(\mathbf{x}) = w_1(x_1) w_2(x_2) \cdots w_d(x_d), \quad \mathbf{x} = (x_1, x_2, \ldots, x_d) \in \Omega, \quad (3.51)$$

and where $w_i(x_i)$ is a non-negative weight function on $[a_i, b_i]$.

The results in this section are mainly devoted in considering two subjects:

1. The problem of computing the matrix $\langle \mathbb{Q}_n, \mathbb{Q}_m^T \rangle_{\nabla^\kappa}$, defined in (3.43), on the product domain (3.50). Because of the properties of (3.50) and (3.51) this problem will be approached in two different ways:

   (a) Considering the moments of the product weight function W (see Section 3.4.1.1).

(b) Considering classical weight functions (Jacobi, Hermite, Laguerre) on each interval $[a_i, b_i]$, $i = 1, 2, \dots, d$ (see Section 3.4.1.2). This second case was motivated by the paper [49] as a generalization to several variables and higher-order derivatives. Our results in this case were published in [42].

2. To present some results on partial differential equations for Sobolev orthogonal polynomials on specific product domains (see Section 3.4.1.3).

3.4.1.1 Computing $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^\kappa}$ by means of the moments of the weight function

As mentioned in Remark 3.5, the entries of the matrix $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^\kappa}$, $n, m \geq \kappa$, can be computed in a simplified form by means of (3.49) if we choose $Q_\alpha^n(W)$ as the monomial $Q_\alpha^n(W; \mathbf{x}) = \mathbf{x}^\alpha$, $|\alpha| = n$. We need to compute the moments $\left\langle \mathbf{x}^{\alpha^{(i)}-\theta}, \mathbf{x}^{\beta^{(j)}-\theta} \right\rangle_W$ of the product weight function (3.51). Let $\mathbb{Q}_n$ denote the column vector defined in (3.48). Then, we have the following proposition.

Proposition 3.12. *Let $n, m \geq \kappa$ and let $\mathbb{Q}_n$ be defined in* (3.48)*. Then, each entry of the matrix $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^\kappa}$ of size $r_n^d \times r_m^d$, which is defined in* (3.43)*, can be computed on the product domain* (3.50) *by:*

$$\left\langle Q_{\alpha^{(i)}}^n(W), Q_{\beta^{(j)}}^m(W) \right\rangle_{\nabla^\kappa} =$$
$$\sum_{|\theta|=\kappa} \binom{\kappa}{\theta_1, \theta_2, \dots, \theta_d} \prod_{l=1}^d (-\alpha_l^{(i)})_{\theta_l} (-\beta_l^{(j)})_{\theta_l} \left\langle x_l^{\alpha_l^{(i)}-\theta_l}, x_l^{\beta_l^{(j)}-\theta_l} \right\rangle_{w_l},$$
$$|\theta| = \kappa, \quad |\alpha^{(i)}| = n, \quad |\beta^{(j)}| = m, \quad 1 \leq i \leq r_n^d, \quad 1 \leq j \leq r_m^d.$$

Proof. Since W is a product of non-negative weight functions w_l, $1 \leq l \leq d$, the product structure implies by (1.11) and (1.12) that the moments of (3.51) are given by:

$$\left\langle \mathbf{x}^{\alpha^{(i)}-\theta}, \mathbf{x}^{\beta^{(j)}-\theta} \right\rangle_W = \prod_{l=1}^d \left\langle x_l^{\alpha_l^{(i)}-\theta_l}, x_l^{\beta_l^{(j)}-\theta_l} \right\rangle_{w_l},$$

where $\langle \cdot, \cdot \rangle_{w_l}$, $1 \leq l \leq d$, is the inner product (1.9). The result follows from (3.49). □

In view of Proposition 3.12, the calculation of the moments $\left\langle x_l^{\alpha_l^{(i)}-\theta_l}, x_l^{\beta_l^{(j)}-\theta_l} \right\rangle_{w_l}$, $1 \leq l \leq d$, needs the explicit knowledge of the weight function w_l. In Chapter 4 we present some numerical examples in two variables with specific weights.

3.4.1.2 Computing $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^\kappa}$ with a weight function that is a product of classical weights

The results that we present in this section were motivated by Fernández, Marcellán, Pérez, Piñar, and Xu [49] as a generalization to several variables and higher-order derivatives from their results obtained in two wariables and first-order derivatives.

First, we construct the sequence of monic polynomials $\{\mathbb{Q}_n\}_{n\geq 0}$ defined in (3.41) by means of monic sequences in one variable of self-coherent polynomials, that is, we suppose that the weight function (3.51) is a product of classical weights in one variable (Jacobi, Hermite, Laguerre). Then, we compute $\langle \mathbb{Q}_n, \mathbb{Q}_m^T \rangle_{\nabla^\kappa}$ using this construction.

Let $\{p_n(w;x)\}_{n\geq 0}$ be a sequence of monic orthogonal polynomials in one variable with respect to the weight function w. A weight function w defined on the real line is called self-coherent if its monic orthogonal polynomials $p_n(w)$ satisfy the relation [49, pp. 205]:

$$p_n(w;x) = \frac{p'_{n+1}(w;x)}{n+1} + a_n(w)p'_n(w;x) + b_n(w)p'_{n-1}(w;x), \tag{3.52}$$

where $a_n(w)$ and $b_n(w)$ are constants. The self-coherent orthogonal polynomials are essentially, up to a linear change of variable, the classical orthogonal polynomials (Jacobi, Laguerre and Hermite) [49, 76]. Equation (3.52) is a well-known structure relation of classic orthogonal polynomials [9, Theorem 3.3.2]. We present a generalization of (3.52) to higher-order derivatives.

Proposition 3.13. *Let $l \in \mathbb{N}$ and let $\{p_n(w;x)\}_{n\geq 0}$ be a sequence of monic orthogonal polynomials which satisfies* (3.52)*. Then $p_n(w)$ satisfies the relation:*

$$p_n(w;x) = \sum_{i=-l}^{l} \gamma_i^{n,l}(w) p_{n+i}^{(l)}(w;x), \tag{3.53}$$

where $\gamma_i^{n,l}(w)$, $-l \leq i \leq l$, are constants such that they can be found recursively for $l \geq 2$ by:

$$\gamma_i^{n,l}(w) = \begin{cases} \gamma_{-l+1}^{n,l-1}(w) b_{n-l+1}(w), & i=-l, \\ \gamma_{-l+2}^{n,l-1}(w) b_{n-l+2}(w) + \gamma_{-l+1}^{n,l-1}(w) a_{n-l+1}(w), & i=-l+1, \\ \dfrac{\gamma_{i-1}^{n,l-1}(w)}{n+i} + \gamma_i^{n,l-1}(w) a_{n+i}(w) + \gamma_{i+1}^{n,l-1}(w) b_{n+i+1}(w), & -l+2 \leq i \leq l-2, \\ \dfrac{\gamma_{l-2}^{n,l-1}(w)}{n+l-1} + \gamma_{l-1}^{n,l-1}(w) a_{n+l-1}(w), & i=l-1, \\ \dfrac{\gamma_{l-1}^{n,l-1}(w)}{n+l}, & i=l, \end{cases} \tag{3.54}$$

and where the initial iterations are:

$$\gamma_{-1}^{n,1}(w) := b_n(w), \quad \gamma_0^{n,1}(w) := a_n(w), \quad \gamma_1^{n,1}(w) := \frac{1}{n+1}.$$

Proof. We use mathematical induction on l. If $l=1$ then (3.53) reduces to:

$$p_n(w;x) = \gamma_{-1}^{n,1}(w)p'_{n-1}(w;x) + \gamma_0^{n,1}(w)p'_n(w;x) + \gamma_1^{n,1}(w)p'_{n+1}(w;x),$$

which coincides with (3.52) if we choose $\gamma_{-1}^{n,1}(w) := b_n(w)$, $\gamma_0^{n,1}(w) := a_n(w)$ and $\gamma_1^{n,1}(w) := 1/(n+1)$. Let us suppose that (3.53) is valid for $l-1$, $l \geq 2$, that is,

$$p_n(w;x) = \sum_{i=-(l-1)}^{l-1} \gamma_i^{n,l-1}(w) p_{n+i}^{(l-1)}(w;x).$$

Then:

$$\begin{aligned}
&p_n(w;x) = \\
&\sum_{i=-(l-1)}^{l-1} \gamma_i^{n,l-1}(w) \left[\frac{p'_{n+i+1}(w;x)}{n+i+1} + a_{n+i}(w) p'_{n+i}(w;x) + b_{n+i}(w) p'_{n+i-1}(w;x) \right]^{(l-1)} \\
&= \underbrace{\gamma_{-l+1}^{n,l-1}(w) b_{n-l+1}(w)}_{=\gamma_i^{n,l}(w),\ \text{if } i=-l} p_{n-l}^{(l)}(w;x) \\
&+ \underbrace{\left(\gamma_{-l+2}^{n,l-1}(w) b_{n-l+2}(w) + \gamma_{-l+1}^{n,l-1}(w) a_{n-l+1}(w)\right)}_{=\gamma_i^{n,l}(w),\ \text{if } i=-l+1} p_{n-l+1}^{(l)}(w;x) \\
&+ \sum_{i=-(l-2)}^{l-2} \underbrace{\left(\frac{\gamma_{i-1}^{n,l-1}(w)}{n+i} + \gamma_i^{n,l-1}(w) a_{n+i}(w) + \gamma_{i+1}^{n,l-1}(w) b_{n+i+1}(w) \right)}_{=\gamma_i^{n,l}(w),\ \text{if } -(l-2)\leq i \leq l-2} p_{n+i}^{(l)}(w;x) \\
&+ \underbrace{\left(\frac{\gamma_{l-2}^{n,l-1}(w)}{n+l-1} + \gamma_{l-1}^{n,l-1}(w) a_{n+l-1}(w) \right)}_{=\gamma_i^{n,l}(w),\ \text{if } i=l-1} p_{n+l-1}^{(l)}(w;x) + \underbrace{\frac{\gamma_{l-1}^{n,l-1}(w)}{n+l}}_{=\gamma_i^{n,l}(w),\ \text{if } i=l} p_{n+l}^{(l)}(w;x).
\end{aligned}$$

This completes the proof. □

As a result of (3.54) we get recursively $\gamma_l^{n,l}(w) = 1/(n+1)_l$. Now we define a sequence of polynomials $\{q_n(w;x)\}_{n\geq 0}$, where $q_n(w)$ is a monic polynomial of degree n such that its κ-th derivative satisfies:

$$q_n^{(\kappa)}(w;x) := (n-\kappa+1)_\kappa p_{n-\kappa}(w;x), \tag{3.55}$$

where $\{p_n(w;x)\}_{n\geq 0}$ is a self-coherent monic sequence of orthogonal polynomials with respect to w. Notice that $\{q_n(w;x)\}_{n\geq 0}$ is not unique. The polynomial $q_n(w)$ and its higher-order derivatives $q'_n(w), q''_n(w), \ldots, q_n^{(\kappa-1)}(w)$ can be obtained if we use Proposition 3.13 on $p_{n-\kappa}(w)$ in (3.55). By setting $l = \kappa - j$, $0 \leq j \leq \kappa - 1$, we have that:

$$p_{n-\kappa}(w;x) = \sum_{i=-(\kappa-j)}^{\kappa-j} \gamma_i^{n-\kappa,\kappa-j}(w) p_{n-\kappa+i}^{(\kappa-j)}(w;x),$$

and (3.55) reduces to:

$$q_n^{(\kappa)}(w;x) = \left[(n-\kappa+1)_\kappa \sum_{i=-(\kappa-j)}^{\kappa-j} \gamma_i^{n-\kappa,\kappa-j}(w) p_{n-\kappa+i}(w;x) \right]^{(\kappa-j)}. \tag{3.56}$$

Then, it is natural to define the j-th derivative of $q_n(w)$, $0 \le j \le \kappa - 1$, by the term in parentheses of (3.56), that is:

$$q_n^{(j)}(w;x) := (n-\kappa+1)_\kappa \sum_{i=-(\kappa-j)}^{\kappa-j} \gamma_i^{n-\kappa,\kappa-j}(w) p_{n-\kappa+i}(w;x), \quad 0 \le j \le \kappa - 1. \quad (3.57)$$

Notice that, in particular for $j = 0$, we have the definition for $q_n(w)$:

$$q_n(w;x) := (n-\kappa+1)_\kappa \sum_{i=-\kappa}^{\kappa} \gamma_i^{n-\kappa,\kappa}(w) p_{n-\kappa+i}(w;x). \quad (3.58)$$

Let us observe that $q_n(w)$ is a linear combination of $p_n(w), p_{n-1}(w), \ldots, p_{n-2\kappa}(w)$, and as a consequence, the leading coefficient of $q_n(w)$ is $(n-\kappa+1)_\kappa \gamma_\kappa^{n-\kappa,\kappa}(w) = 1$, that is, $q_n(w)$ is monic.

If $\{q_n(w_i;x_i)\}_{n\ge 0}$, $1 \le i \le d$, denotes the sequence of monic polynomials defined in (3.58), we define the product polynomial $Q_\alpha^n(W)$ by:

$$Q_\alpha^n(W;\mathbf{x}) := q_{\alpha_1}(w_1;x_1) q_{\alpha_2}(w_2;x_2) \cdots q_{\alpha_d}(w_d;x_d), \quad \alpha \in \mathbb{N}_0^d, \quad |\alpha| = n, \quad (3.59)$$

which is a monic polynomial of total degree $|\alpha| = n$, that is, it is of the form $Q_\alpha^n(W;\mathbf{x}) = \mathbf{x}^\alpha + R_\alpha(\mathbf{x})$, where $R_\alpha \in \Pi_{n-1}^d$. We denote by $\mathbb{Q}_n$ the column vector defined in (3.41).

Let $\theta = (\theta_1, \theta_2, \ldots, \theta_d) \in \mathbb{N}_0^d$ a multi-index such that $|\theta| = \kappa$. The derivative of order κ of $Q_\alpha^n(W)$ defined in (3.59) is given by:

$$\begin{aligned}
\partial^\theta Q_\alpha^n(W;\mathbf{x}) &= \prod_{j=1}^d \partial_j^{\theta_j} q_{\alpha_j}(w_j;x_j) \\
&= \prod_{j=1}^d \left[(\alpha_j - \kappa + 1)_\kappa \sum_{i=-(\kappa-\theta_j)}^{\kappa-\theta_j} \gamma_i^{\alpha_j-\kappa,\kappa-\theta_j}(w_j) p_{\alpha_j-\kappa+i}(w_j;x_j) \right] \\
&= \sum_{n-(2d-1)\kappa \le r \le n-\kappa} \Gamma_\nu^{\theta,\alpha} P_\nu^r(W;\mathbf{x}), \quad |\alpha| = n, \quad |\theta| = \kappa, \quad |\nu| = r,
\end{aligned}$$

where $\Gamma_\nu^{\theta,\alpha}$ are constants that are obtained by developing the product of sums from the last expression, and which are given in terms of the constants $\gamma_i^{n,l}(w)$ defined in (3.54), and where $P_\nu^r(W)$ is the monic product polynomial defined in (1.13) orthogonal with respect to W. We can see that $\partial^\theta Q_\alpha^n(W)$ is a linear combination of the product polynomials (1.13) with total degrees that range from $(\alpha_1 - 2\kappa + \theta_1) + (\alpha_2 - 2\kappa + \theta_2) + \cdots + (\alpha_d - 2\kappa + \theta_d) = |\alpha| + |\theta| - 2d\kappa = n - (2d-1)\kappa$ to $(\alpha_1 - \theta_1) + (\alpha_2 - \theta_2) + \cdots + (\alpha_d - \theta_d) = |\alpha| - |\theta| = n - \kappa$.

Then the entries of the matrix $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^\kappa}$, $n, m \ge \kappa$, defined in (3.43) can be computed by:

$$\left\langle Q_{\alpha^{(i)}}^n(W), Q_{\beta^{(j)}}^m(W) \right\rangle_{\nabla^\kappa} = \sum_{|\theta|=\kappa} \binom{\kappa}{\theta_1, \theta_2, \ldots, \theta_d} \left\langle \partial^\theta Q_{\alpha^{(i)}}^n(W), \partial^\theta Q_{\beta^{(j)}}^m(W) \right\rangle_W$$

$$= \sum_{|\theta|=\kappa} \sum_{n-(2d-1)\kappa \le r \le n-\kappa} \sum_{m-(2d-1)\kappa \le s \le m-\kappa} \binom{\kappa}{\theta_1, \theta_2, \ldots, \theta_d} \Gamma_\nu^{\theta,\alpha^{(i)}} \Gamma_\sigma^{\theta,\beta^{(j)}} \langle P_\nu^r, P_\sigma^s \rangle_W$$
$$= \sum_{|\theta|=\kappa} \sum_{n-(2d-1)\kappa \le r \le n-\kappa} \sum_{m-(2d-1)\kappa \le s \le m-\kappa} \binom{\kappa}{\theta_1, \theta_2, \ldots, \theta_d} \Gamma_\nu^{\theta,\alpha^{(i)}} \Gamma_\sigma^{\theta,\beta^{(j)}} h_\nu(W) \delta_{\nu,\sigma},$$
$$|\theta| = \kappa, \quad |\alpha^{(i)}| = n, \quad |\beta^{(j)}| = m, \quad |\nu| = r, \quad |\sigma| = s.$$

Then $\langle \mathbb{Q}_n, \mathbb{Q}_m^T \rangle_{\nabla^\kappa}$ is a matrix that can be computed in a closed form in terms of constants that depend on the orthogonal polynomials with respect to the inner product $\langle \cdot, \cdot \rangle_W$, that is, each entry of this matrix is given in terms of the constants in (1.14) and (3.54).

For $\kappa = 1, 2, 3$ and $d = 2, 3$ detailed numerical examples were given in literature [41, 42, 49, 101] for particular weight functions. In Section 4.2 we present some examples for the Hermite-Laguerre and Laguerre-Gegenbauer product weight functions in two variables. In [42, Section 4] there is an example in three variables for the Hermite-Hermite-Laguerre weight function.

3.4.1.3 Partial differential equations for Sobolev polynomials on some product domains

As mentioned in Section 1.3.1, the orthogonal polynomials with respect to the product weight function (3.51) are eigenfunctions of a second-order differential operator for particular weights. In this subsection we consider two particular cases.

Multiple Sobolev-Hermite polynomials The space $\mathscr{V}_n^d(W^H)$ of multiple Hermite polynomials on the product domain $\Omega = \mathbb{R}^d$ and orthogonal with respect to the product weight function:

$$W^H(\mathbf{x}) = e^{-x_1^2} e^{-x_2^2} \cdots e^{-x_d^2} = e^{-\|\mathbf{x}\|^2}, \quad \mathbf{x} \in \mathbb{R}^d, \tag{3.60}$$

was discussed in Section 1.3.1.1. The polynomials in $\mathscr{V}_n^d(W^H)$ satisfy the partial differential equation (1.17).

Let us denote by $\mathscr{V}_n^d(S, W^H)$ and $\mathscr{V}_n^d(\nabla^\kappa, W^H)$ the spaces of Sobolev orthogonal polynomials with respect to (3.12) and (3.13), respectively, where it continuous part $\langle \cdot, \cdot \rangle_{\nabla^\kappa}$ is defined on $\Omega = \mathbb{R}^d$ and the weight is W^H. Then, we have the following corollary from Proposition 3.7 concerning partial differential equations for the polynomials in the spaces $\mathscr{V}_n^d(S, W^H)$ and $\mathscr{V}_n^d(\nabla^\kappa, W^H)$.

Corollary 3.2. *Let $P \in \mathscr{V}_n^d(S, W^H)$ or $P \in \mathscr{V}_n^d(\nabla^\kappa, W^H)$. Then P satisfies the partial differential equation:*

$$[\mathcal{H} + 2(n-\kappa)\mathcal{I}] (\partial_1 + \partial_2 + \cdots + \partial_d)^\kappa P = 0, \tag{3.61}$$

where $\mathcal{H}$ is the differential operator (1.18) *and $\mathcal{I}$ is the identity operator.*

Proof. If $n < \kappa$ then (3.61) is immediate because $(\partial_1+\partial_2+\cdots+\partial_d)^\kappa P = 0$. Let us suppose $n \geq \kappa$. If $P \in \mathscr{V}_n^d(S, W^H)$ or $P \in \mathscr{V}_n^d(\nabla^\kappa, W^H)$, we know from Proposition 3.7 that:

$$(\partial_1 + \partial_2 + \cdots + \partial_d)^\kappa P \in \mathscr{V}_{n-\kappa}^d(W^H), \quad n \geq \kappa.$$

The result follows from (1.17). □

Multiple Sobolev-Laguerre polynomials The space $\mathscr{V}_n^d(W_\eta^L)$ of multiple Laguerre polynomials on the product domain $\Omega = \mathbb{R}_+^d$ and orthogonal with respect to the product weight function:

$$W_\eta^L(\mathbf{x}) = x_1^{\eta_1} e^{-x_1} x_2^{\eta_2} e^{-x_2} \cdots x_d^{\eta_d} e^{-x_d} = \mathbf{x}^\eta e^{-|\mathbf{x}|}, \quad \mathbf{x} \in \mathbb{R}_+^d,$$
$$\eta_i > -1, \quad 1 \leq i \leq d, \quad |\mathbf{x}| = x_1 + x_2 + \cdots + x_d, \quad (3.62)$$

was discussed in Section 1.3.1.2. The polynomials in $\mathscr{V}_n^d(W_\eta^L)$ satisfy the partial differential equation (1.21).

Let us denote by $\mathscr{V}_n^d(S, W_\eta^L)$ and $\mathscr{V}_n^d(\nabla^\kappa, W_\eta^L)$ the spaces of Sobolev orthogonal polynomials with respect to (3.12) and (3.13), respectively, where it continuous part $\langle \cdot, \cdot \rangle_{\nabla^\kappa}$ is defined on $\Omega = \mathbb{R}_+^d$ and the weight is W_η^L. Then, we have the following corollary from Proposition 3.7 concerning partial differential equations for the polynomials in the spaces $\mathscr{V}_n^d(S, W_\eta^L)$ and $\mathscr{V}_n^d(\nabla^\kappa, W_\eta^L)$.

Corollary 3.3. *Let $P \in \mathscr{V}_n^d(S, W_\eta^L)$ or $P \in \mathscr{V}_n^d(\nabla^\kappa, W_\eta^L)$. Then P satisfies the partial differential equation:*

$$[\mathcal{L}_\eta + (n-\kappa)\mathcal{I}]\,(\partial_1 + \partial_2 + \cdots + \partial_d)^\kappa P = 0, \quad (3.63)$$

where $\mathcal{L}_\eta$ is the differential operator (1.22) *and $\mathcal{I}$ is the identity operator.*

Proof. Similar to Corollary 3.2. □

3.4.2 The simplex

We remit the reader to the results from Section 1.3.2 on the space $\mathscr{V}_n^d(W_\gamma)$ of standard orthogonal polynomials on the simplex:

$$\mathbb{T}^d := \left\{ \mathbf{x} \in \mathbb{R}^d : x_1 \geq 0, x_2 \geq 0, \ldots, x_d \geq 0, 1 - |\mathbf{x}| \geq 0 \right\}, \quad |\mathbf{x}| := x_1 + x_2 + \cdots + x_d,$$

with respect to the weight function:

$$W_\gamma(\mathbf{x}) := x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_d^{\gamma_d} (1 - |\mathbf{x}|)^{\gamma_{d+1}}, \quad \mathbf{x} \in \mathbb{T}^d, \quad \gamma_i > -1, \quad 1 \leq i \leq d+1, \quad (3.64)$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{d+1}) \in \mathbb{R}^{d+1}$ is such that $\gamma_i > -1$ for $i = 1, 2, \ldots, d+1$, and $|\gamma| := \gamma_1 + \gamma_2 + \cdots + \gamma_{d+1}$.

The results in this section are mainly devoted in considering two subjects:

1. The problem of computing the matrix $\langle \mathbb{Q}_n, \mathbb{Q}_m^T \rangle_{\nabla^\kappa}$, defined in (3.43), on the simplex $\mathbb{T}^d$. Because of the properties of (3.64) this problem will be approached considering the moments of the weight function W_γ (see Section 3.4.2.1).

2. To present some results on partial differential equations for Sobolev orthogonal polynomials on $\mathbb{T}^d$ (see Section 3.4.2.2).

3.4.2.1 Computing $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^\kappa}$ by means of the moments of the weight function

Notice that the weight function W_γ, defined in (3.64), is closed under products, that is, if $\gamma_a, \gamma_b \in \mathbb{R}^{d+1}$ then:

$$W_{\gamma_a}(\mathbf{x}) W_{\gamma_b}(\mathbf{x}) = W_{\gamma_a+\gamma_b}(\mathbf{x}).$$

In particular, for any monomial $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$, with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d$, we have that $(\alpha, 0) \in \mathbb{N}_0^{d+1}$, and therefore $\mathbf{x}^\alpha W_\gamma(\mathbf{x}) = W_{(\alpha,0)+\gamma}(\mathbf{x})$, that is:

$$\mathbf{x}^\alpha W_\gamma(\mathbf{x}) = x_1^{\alpha_1+\gamma_1} x_2^{\alpha_2+\gamma_2} \cdots x_d^{\alpha_d+\gamma_d} (1 - |\mathbf{x}|)^{\gamma_{d+1}}, \quad \gamma_i > -1, \quad 1 \le i \le d+1.$$

This property allows us to compute the matrix $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^\kappa}$ in a simplified form by means of the moments of W_γ.

As discussed in Remark 3.5, we choose $Q_\alpha^n(W_\gamma)$ to be the monomial $Q_\alpha^n(W_\gamma; \mathbf{x}) = \mathbf{x}^\alpha$, $|\alpha| = n$. We denote by $\mathbb{Q}_n$ the column vector (3.48). Therefore, we can compute the entries of the matrix $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^\kappa}$, $n, m \ge \kappa$, in a simplified form by means of (3.49) and the moments $\left\langle \mathbf{x}^{\alpha^{(i)}-\theta}, \mathbf{x}^{\beta^{(j)}-\theta} \right\rangle_\gamma$ of the weight (3.64).

Proposition 3.14. *Let $n, m \ge \kappa$ and let $\mathbb{Q}_n$ be defined in* (3.48)*. Then, each entry of the matrix $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^\kappa}$ of size $r_n^d \times r_m^d$, which is defined in* (3.43)*, can be computed on the simplex $\mathbb{T}^d$ by:*

$$\left\langle Q_{\alpha^{(i)}}^n(W_\gamma), Q_{\beta^{(j)}}^m(W_\gamma) \right\rangle_{\nabla^\kappa} =$$
$$\sum_{|\theta|=\kappa} \binom{\kappa}{\theta_1, \theta_2, \ldots, \theta_d} \frac{\prod_{l=1}^{d} (-\alpha_l^{(i)})_{\theta_l} (-\beta_l^{(j)})_{\theta_l} (\gamma_l + 1)_{\alpha_l^{(i)}+\beta_l^{(j)}-2\theta_l}}{(|\gamma| + d + 1)_{n+m-2\kappa}},$$
$$|\theta| = \kappa, \quad |\alpha^{(i)}| = n, \quad |\beta^{(j)}| = m, \quad 1 \le i \le r_n^d, \quad 1 \le j \le r_m^d.$$

Proof. Notice that by (1.28) and (1.29) we have:

$$\left\langle \mathbf{x}^{\alpha^{(i)}-\theta}, \mathbf{x}^{\beta^{(j)}-\theta} \right\rangle_\gamma = c_\gamma \int_{\mathbb{T}^d} W_{(\alpha^{(i)}+\beta^{(j)}-2\theta,0)+\gamma}(\mathbf{x}) d\mathbf{x}$$
$$= \frac{\Gamma(|\gamma| + d + 1)}{\prod_{l=1}^{d+1} \Gamma(\gamma_l + 1)} \frac{\prod_{l=1}^{d} \Gamma(\alpha_l^{(i)} + \beta_l^{(j)} - 2\theta_l + \gamma_l + 1) \Gamma(\gamma_{d+1} + 1)}{\Gamma(|\alpha^{(i)}| + |\beta^{(j)}| - 2|\theta| + |\gamma| + d + 1)}.$$

Therefore, by (3.49), with $|\alpha^{(i)}| = n$, $|\beta^{(j)}| = m$, $|\theta| = \kappa$, and further simplification we have the result. □

3.4.2.2 Partial differential equations for Sobolev polynomials on the simplex

As mentioned in Section 1.3.2, the orthogonal polynomials in the space $\mathscr{V}_n^d(W_\gamma)$ are eigenfunctions of a second-order differential operator, that is, they satisfy the partial differential equation (1.31).

Let us denote by $\mathscr{V}_n^d(S, W_\gamma)$ and $\mathscr{V}_n^d(\nabla^\kappa, W_\gamma)$ the spaces of Sobolev orthogonal polynomials with respect to (3.12) and (3.13), respectively, where it continuous part $\langle \cdot, \cdot \rangle_{\nabla^\kappa}$ is defined on $\Omega = \mathbb{T}^d$ and the weight is W_γ. Then, we have the following corollary that is a consequence of Proposition 3.7 concerning partial differential equations for the polynomials in the spaces $\mathscr{V}_n^d(S, W_\gamma)$ and $\mathscr{V}_n^d(\nabla^\kappa, W_\gamma)$.

Corollary 3.4. *Let $P \in \mathscr{V}_n^d(S, W_\gamma)$ or $P \in \mathscr{V}_n^d(\nabla^\kappa, W_\gamma)$. Then P satisfies the partial differential equation:*

$$[\mathcal{T}_\gamma + (n-\kappa)(n-\kappa+|\gamma|+d)\mathcal{I}]\,(\partial_1 + \partial_2 + \cdots + \partial_d)^\kappa P = 0, \tag{3.65}$$

with $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{d+1}) \in \mathbb{R}^{d+1}$, $\gamma_i > -1$, $1 \le i \le d+1$, $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_{d+1}$, and where $\mathcal{T}_\gamma$ is the differential operator (1.32) *and $\mathcal{I}$ is the identity operator.*

Proof. Similar to Corollary 3.2. □

3.4.3 The unit ball

We remit the reader to the results from Section 1.3.4 on the space $\mathscr{V}_n^d(W_\mu)$ of standard orthogonal polynomials on the unit ball:

$$\mathbb{B}^d := \left\{ \mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \le 1 \right\},$$

with respect to the weight function:

$$W_\mu(\mathbf{x}) := (1 - \|\mathbf{x}\|^2)^\mu, \quad \mathbf{x} \in \mathbb{B}^d, \quad \mu > -1. \tag{3.66}$$

The results in this section are mainly devoted in considering two subjects:

1. The problem of computing the matrix $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^\kappa}$, defined in (3.43), on the unit ball $\mathbb{B}^d$. Because of the properties of (3.66) this problem will be approached considering the moments of the weight function W_μ (see Section 3.4.3.1).

2. To present some results on partial differential equations for Sobolev orthogonal polynomials on $\mathbb{B}^d$ (see Section 3.4.3.2).

In addition, in Section 3.4.3.3 we present some miscellaneous results on the sphere $\mathbb{S}^{d-1}$.

3.4.3.1 Computing $\langle \mathbb{Q}_n, \mathbb{Q}_m^T \rangle_{\nabla^\kappa}$ by means of the moments of the weight function

Let us recall that for $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{x}_i = (x_1, x_2, \ldots, x_i) \in \mathbb{R}^i$, $1 \le i \le d$, with $\mathbf{x}_0 := 0$, denotes a truncation of $\mathbf{x}$. On the unit ball, there is an integral formula that relates the integral on the unit ball $\mathbb{B}^d$ with an integral on $\mathbb{B}^{d-1}$. This formula is given by [44, page 143]:

$$\int_{\mathbb{B}^d} f(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{B}^{d-1}} \int_{-1}^{1} f\left(\mathbf{x}_{d-1}, y\sqrt{1-\|\mathbf{x}_{d-1}\|^2}\right) \sqrt{1-\|\mathbf{x}_{d-1}\|^2} dy d\mathbf{x}_{d-1}, \quad (3.67)$$

that follows from the change of variable $x_d = y\sqrt{1-\|\mathbf{x}_{d-1}\|^2}$, $-1 \le y \le 1$. Using this formula repeatedly, by reducing the dimension d by 1 at each step, it is not difficult to show that:

$$\int_{\mathbb{B}^d} \mathbf{x}^\alpha W_\mu(\mathbf{x}) d\mathbf{x} = \frac{2^{-d}\Gamma(\mu+1)}{\Gamma\left(\dfrac{|\alpha|+d}{2} + \mu + 1\right)} \prod_{i=1}^{d} (1+(-1)^{\alpha_i}) \Gamma\left(\frac{\alpha_i+1}{2}\right), \quad (3.68)$$
$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d, \quad \mu > -1.$$

Notice that if any entry of α is odd, then (3.68) is zero.

As discussed in Remark 3.5, we choose $Q_\alpha^n(W_\mu)$ to be the monomial $Q_\alpha^n(W_\mu; \mathbf{x}) = \mathbf{x}^\alpha$, $|\alpha| = n$. We denote by $\mathbb{Q}_n$ the column vector (3.48). Then, we can compute the entries of the matrix $\langle \mathbb{Q}_n, \mathbb{Q}_m^T \rangle_{\nabla^\kappa}$ in a simplified form by means of (3.49) and the moments $\left\langle \mathbf{x}^{\alpha^{(i)}-\theta}, \mathbf{x}^{\beta^{(j)}-\theta} \right\rangle_\mu$ of the weight (3.66). Now, we use (3.68) on the following result.

Proposition 3.15. *Let $n, m \ge \kappa$ and let $\mathbb{Q}_n$ be defined in* (3.48)*. Then, each entry of the matrix $\langle \mathbb{Q}_n, \mathbb{Q}_m^T \rangle_{\nabla^\kappa}$ of size $r_n^d \times r_m^d$, which is defined in* (3.43)*, can be computed on the unit ball $\mathbb{B}^d$ by:*

$$\left\langle Q_{\alpha^{(i)}}^n(W_\mu), Q_{\beta^{(j)}}^m(W_\mu) \right\rangle_{\nabla^\kappa} = \sum_{|\theta|=\kappa} \binom{\kappa}{\theta_1, \ldots, \theta_d} \frac{\prod_{l=1}^{d} (-\alpha_l^{(i)})_{\theta_l} (-\beta_l^{(j)})_{\theta_l} \left(\frac{1}{2}\right)_{\frac{\alpha_l^{(i)}+\beta_l^{(j)}}{2} - \theta_l}}{(\mu + d/2 + 1)_{\frac{n+m}{2} - \kappa}},$$
$$|\theta| = \kappa, \quad |\alpha^{(i)}| = n, \quad |\beta^{(j)}| = m, \quad 1 \le i \le r_n^d, \quad 1 \le j \le r_m^d,$$

if $\alpha_l^{(i)} + \beta_l^{(j)}$ is even for all $l = 1, 2, \ldots, d$, and $\left\langle Q_{\alpha^{(i)}}^n(W_\mu), Q_{\beta^{(j)}}^m(W_\mu) \right\rangle_{\nabla^\kappa} = 0$ otherwise.

Proof. By (1.40), (1.41), and (3.68) we have that:

$$\left\langle \mathbf{x}^{\alpha^{(i)}-\theta}, \mathbf{x}^{\beta^{(j)}-\theta} \right\rangle_\mu = c_\mu \int_{\mathbb{B}^d} \mathbf{x}^{\alpha^{(i)}+\beta^{(j)}-2\theta} W_\mu(\mathbf{x}) d\mathbf{x}$$

$$= \frac{\Gamma(\mu + d/2 + 1)}{\pi^{d/2}\Gamma(\mu+1)} \frac{2^{-d}\Gamma(\mu+1)}{\Gamma\left(\frac{|\alpha^{(i)}| + |\beta^{(j)}| - 2|\theta| + d}{2} + \mu + 1\right)}$$
$$\times \prod_{l=1}^{d} (1 + (-1)^{\alpha_l^{(i)} + \beta_l^{(j)} - 2\theta_l}) \Gamma\left(\frac{\alpha_l^{(i)} + \beta_l^{(j)} - 2\theta_l + 1}{2}\right).$$

Therefore, by (3.49), with $|\alpha^{(i)}| = n$, $|\beta^{(j)}| = m$, $|\theta| = \kappa$, and further simplification we have the result. □

3.4.3.2 Partial differential equations for Sobolev polynomials on the unit ball

As mentioned in Section 1.3.4, the orthogonal polynomials in the space $\mathscr{V}_n^d(W_\mu)$ are eigenfunctions of a second-order differential operator, that is, they satisfy the partial differential equations (1.43) and (1.45).

Let us denote by $\mathscr{V}_n^d(S, W_\mu)$ and $\mathscr{V}_n^d(\nabla^\kappa, W_\mu)$ the spaces of Sobolev orthogonal polynomials with respect to (3.12) and (3.13), respectively, where it continuous part $\langle \cdot, \cdot \rangle_{\nabla^\kappa}$ is defined on $\Omega = \mathbb{B}^d$ and the weight is W_μ. Then, we have the following corollary from Proposition 3.7 concerning partial differential equations for the polynomials in the spaces $\mathscr{V}_n^d(S, W_\mu)$ and $\mathscr{V}_n^d(\nabla^\kappa, W_\mu)$.

Corollary 3.5. *Let $P \in \mathscr{V}_n^d(S, W_\mu)$ or $P \in \mathscr{V}_n^d(\nabla^\kappa, W_\mu)$. Then P satisfies the partial differential equations:*

$$[\mathcal{B}_\mu + (n-\kappa)(n-\kappa+2\mu+d)\mathcal{I}]\, (\partial_1 + \partial_2 + \cdots + \partial_d)^\kappa P = 0, \tag{3.69}$$
$$[\mathcal{D}_\mu + (n-\kappa+d)(n-\kappa+2\mu)\mathcal{I}]\, (\partial_1 + \partial_2 + \cdots + \partial_d)^\kappa P = 0, \tag{3.70}$$

with $\mu > -1$, and where $\mathcal{B}_\mu$ and $\mathcal{D}_\mu$ are the differential operators (1.44) *and* (1.46), *respectively, and $\mathcal{I}$ is the identity operator.*

Proof. Similar to Corollary 3.2. □

Notice that (3.69) and (3.70) are essentially the same because of the relation $\mathcal{B}_\mu = \mathcal{D}_\mu + 2d\mu\mathcal{I}$.

3.4.3.3 Some miscellaneous results on the sphere

For this section, we remit the reader to Section 1.3.3 for the basic background on harmonic polynomials and spherical harmonics.

The harmonic polynomials in the space $\mathscr{H}_n^d$ have several properties with respect to higher-order derivatives, and some of them can be expressed in terms of the gradient ∇^κ of order κ. For example, if $Y \in \mathscr{H}_n^d$ then from Proposition 3.3, equation (3.7), we know that:

$$\mathbf{x}^{[\kappa]} \cdot \nabla^\kappa Y(\mathbf{x}) = (n-\kappa+1)_\kappa Y(\mathbf{x}), \quad Y \in \mathscr{H}_n^d, \quad \kappa \in \mathbb{N}_0,$$

and this property is a consequence from the fact that Y is homogeneous. Equation (1.36) can be also extended to higher-order derivatives. We have some other properties of the spherical harmonics.

Lemma 3.1. *If* $Y \in \mathscr{H}_n^d$ *then* $\partial^\theta Y \in \mathscr{H}_{n-|\theta|}^d$, $\theta \in \mathbb{N}_0^d$, *and*

$$\triangle_0(\partial^\theta Y(\xi)) = -(n-|\theta|)(n-|\theta|+d-2)\partial^\theta Y(\xi), \quad \xi \in \mathbb{S}^{d-1}. \tag{3.71}$$

Proof. Since $Y \in \mathscr{H}_n^d$ is homogeneous, we know that $\partial^\theta Y$ is also homogeneous of degree $n-|\theta|$. In addition,

$$0 = \partial^\theta(\triangle Y) = \partial^\theta\left(\sum_{i=1}^d \partial_i^2 Y\right) = \sum_{i=1}^d \partial^\theta(\partial_i^2 Y) = \sum_{i=1}^d \partial_i^2(\partial^\theta Y) = \triangle(\partial^\theta Y).$$

This proves that $\partial^\theta Y \in \mathscr{H}_{n-|\theta|}^d$. Since $\partial^\theta Y$ is homogeneous, $\partial^\theta Y(\mathbf{x}) = r^{n-|\theta|}\partial^\theta Y(\xi)$, $\mathbf{x} = r\xi$, $r \geq 0$, $\xi \in \mathbb{S}^{d-1}$. We have by (1.35) that:

$$\begin{aligned} 0 = \triangle(\partial^\theta Y(\mathbf{x})) = \triangle(r^{n-|\theta|}\partial^\theta Y(\xi)) &= (n-|\theta|)(n-|\theta|-1)r^{n-|\theta|-2}\partial^\theta Y(\xi) + \\ &(d-1)(n-|\theta|)r^{n-|\theta|-2}\partial^\theta Y(\xi) + r^{n-|\theta|-2}\triangle_0(\partial^\theta Y(\xi)), \end{aligned}$$

which is, when restricted to the sphere, equation (3.71). □

Next, we prove a generalization to higher-order derivatives of Lemma 2.1. The following proof uses (3.71) and the Green's identity [24, Proposition 1.8.7] on the sphere:

$$\int_{\mathbb{S}^{d-1}} \nabla_0 f(\xi)\cdot\nabla_0 g(\xi) d\omega(\xi) = -\int_{\mathbb{S}^{d-1}} \triangle_0 f(\xi) g(\xi) d\omega(\xi). \tag{3.72}$$

Proposition 3.16. *Let* $\left\{Y_\nu^n : 1 \leq \nu \leq a_n^d\right\}$ *be an orthonormal basis of* $\mathscr{H}_n^d$. *Let* $\mathbf{x} = r\xi$, *with* $r > 0$, $\xi \in \mathbb{S}^{d-1}$, *and* $\theta \in \mathbb{N}_0^d$. *Then we have the following:*

$$\xi \cdot \nabla_0 \partial^\theta Y_\nu^n(\mathbf{x}) = 0, \tag{3.73}$$

$$\begin{aligned} \nabla\partial^\theta Y_\nu^n(\mathbf{x})\cdot\nabla\partial^\theta Y_\eta^m(\mathbf{x}) &= \frac{1}{r^2}\nabla_0\partial^\theta Y_\nu^n(\mathbf{x})\cdot\nabla_0\partial^\theta Y_\eta^m(\mathbf{x}) \\ &+ \frac{(n-|\theta|)(m-|\theta|)}{r^2}\partial^\theta Y_\nu^n(\mathbf{x})\partial^\theta Y_\eta^m(\mathbf{x}), \end{aligned} \tag{3.74}$$

and for $1 \leq \nu \leq a_n^d$, $1 \leq \eta \leq a_m^d$, $\kappa \in \mathbb{N}_0$, *the following relation holds:*

$$\frac{1}{\omega_{d-1}}\int_{\mathbb{S}^{d-1}} \nabla^\kappa Y_\nu^n(\xi)\cdot\nabla^\kappa Y_\eta^m(\xi) d\omega(\xi) = 2^\kappa(n-\kappa+1)_\kappa(n-\kappa+d/2)_\kappa\delta_{n,m}\delta_{\nu,\eta}. \tag{3.75}$$

Proof. Since $\partial^\theta Y_\nu^n$ is homogeneous of degree $n-|\theta|$, $\partial^\theta Y_\nu^n(\mathbf{x}) = r^{n-|\theta|}\partial^\theta Y_\nu^n(\xi)$. Then:

$$\frac{\partial}{\partial r}(\partial^\theta Y_\nu^n(\mathbf{x})) = (n-|\theta|)r^{n-|\theta|-1}\partial^\theta Y_\nu^n(\xi) = \frac{n-|\theta|}{r}\partial^\theta Y_\nu^n(\mathbf{x}).$$

By Euler's equation for homogeneous polynomials:

$$\mathbf{x}\cdot\nabla\partial^\theta Y_\nu^n(\mathbf{x}) = (n-|\theta|)\partial^\theta Y_\nu^n(\mathbf{x}).$$

Therefore, by (1.34) we have:

$$\xi \cdot \nabla_0 \partial^\theta Y_\nu^n(\mathbf{x}) = \mathbf{x} \cdot \nabla \partial^\theta Y_\nu^n(\mathbf{x}) - r\left(\frac{n-|\theta|}{r}\right)\partial^\theta Y_\nu^n(\mathbf{x}) =$$
$$(n-|\theta|)\partial^\theta Y_\nu^n(\mathbf{x}) - (n-|\theta|)\partial^\theta Y_\nu^n(\mathbf{x}) = 0.$$

Equation (3.74) follows from (1.34), (3.73), and a straightforward computation. Now we prove (3.75) by using induction on κ. The case $\kappa = 0$ is (1.38) (our hypothesis). The case $\kappa = 1$ is (2.30), which was proven in [98, Lema 2.2]. Let us suppose that (3.75) holds for κ. Using (3.3) and the definition of ∇^κ we have:

$$\nabla^{\kappa+1} Y_\nu^n(\mathbf{x}) \cdot \nabla^{\kappa+1} Y_\eta^m(\mathbf{x}) = \sum_{i=1}^d \partial_i \nabla^\kappa Y_\nu^n(\mathbf{x}) \cdot \partial_i \nabla^\kappa Y_\eta^m(\mathbf{x}) =$$
$$\sum_{i=1}^d \nabla^\kappa \partial_i Y_\nu^n(\mathbf{x}) \cdot \nabla^\kappa \partial_i Y_\eta^m(\mathbf{x}) = \sum_{i=1}^d \sum_{|\theta|=\kappa} \binom{\kappa}{\theta_1, \theta_2, \ldots, \theta_d} \partial^\theta \partial_i Y_\nu^n(\mathbf{x}) \partial^\theta \partial_i Y_\eta^m(\mathbf{x})$$
$$= \sum_{|\theta|=\kappa} \binom{\kappa}{\theta_1, \theta_2, \ldots, \theta_d} \nabla \partial^\theta Y_\nu^n(\mathbf{x}) \cdot \nabla \partial^\theta Y_\eta^m(\mathbf{x}). \quad (3.76)$$

By (3.71), (3.72), and (3.74), we have:

$$\int_{\mathbb{S}^{d-1}} \nabla \partial^\theta Y_\nu^n(\xi) \cdot \nabla \partial^\theta Y_\eta^m(\xi) d\omega(\xi) = \int_{\mathbb{S}^{d-1}} \nabla_0 \partial^\theta Y_\nu^n(\xi) \cdot \nabla_0 \partial^\theta Y_\eta^m(\xi) d\omega(\xi) +$$
$$(n-|\theta|)(m-|\theta|) \int_{\mathbb{S}^{d-1}} \partial^\theta Y_\nu^n(\xi) \partial^\theta Y_\eta^m(\xi) d\omega(\xi) = -\int_{\mathbb{S}^{d-1}} \triangle_0\, \partial^\theta Y_\nu^n(\xi) \partial^\theta Y_\eta^m(\xi) d\omega(\xi) +$$
$$(n-|\theta|)(m-|\theta|) \int_{\mathbb{S}^{d-1}} \partial^\theta Y_\nu^n(\xi) \partial^\theta Y_\eta^m(\xi) d\omega(\xi) =$$
$$(n-|\theta|)(n+m-2|\theta|+d-2) \int_{\mathbb{S}^{d-1}} \partial^\theta Y_\nu^n(\xi) \partial^\theta Y_\eta^m(\xi) d\omega(\xi). \quad (3.77)$$

Putting together (3.76) and (3.77), and again by (3.3) and the induction hypothesis:

$$\int_{\mathbb{S}^{d-1}} \nabla^{\kappa+1} Y_\nu^n(\xi) \cdot \nabla^{\kappa+1} Y_\eta^m(\xi) d\omega(\xi) =$$
$$(n-\kappa)(n+m-2\kappa+d-2) \sum_{|\theta|=\kappa} \binom{\kappa}{\theta_1, \theta_2, \ldots, \theta_d} \int_{\mathbb{S}^{d-1}} \partial^\theta Y_\nu^n(\xi) \partial^\theta Y_\eta^m(\xi) d\omega(\xi) =$$
$$(n-\kappa)(n+m-2\kappa+d-2) \int_{\mathbb{S}^{d-1}} \nabla^\kappa Y_\nu^n(\xi) \cdot \nabla^\kappa Y_\eta^m(\xi) d\omega(\xi) =$$
$$\omega_{d-1} 2^{\kappa+1} (n-\kappa)_{k+1} (n-\kappa-1+d/2)_{\kappa+1} \delta_{n,m} \delta_{\nu,\eta}.$$

This completes the proof. □

3.4.4 The cone

We remit the reader to the results from Section 1.3.5 on the space $\mathscr{V}_n^d(W_{w,\mu})$ of standard orthogonal polynomials on the cone:

$$\mathbb{V}_\vartheta^d := \left\{ \mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}_{d-1}\| \le x_d, 0 \le x_d \le \vartheta \right\}, \quad 0 < \vartheta \le \infty, \quad \mathbf{x}_{d-1} = (x_1, x_2, \ldots x_{d-1}),$$

with respect to the weight function:

$$W_{w,\mu}(\mathbf{x}) = (x_d^2 - \|\mathbf{x}_{d-1}\|^2)^\mu w(x_d), \quad \mu > -1, \quad \mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{V}_\vartheta^d, \tag{3.78}$$

where w is a non-negative weight function on the interval $0 \le x_d \le \vartheta$.

The results in this section are mainly devoted in considering two subjects:

1. The problem of computing the matrix $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^\kappa}$, defined in (3.43), on the cone $\mathbb{V}_\vartheta^d$. Because of the properties of (3.78) this problem will be approached considering the moments of the weight function $W_{w,\mu}$ (see Section 3.4.4.1).

2. To present some results on partial differential equations for Sobolev orthogonal polynomials on $\mathbb{V}_\vartheta^d$ (see Section 3.4.4.2).

3.4.4.1 Computing $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^\kappa}$ by means of the moments of the weight function

Using the integral formula (1.50) on the cone $\mathbb{V}_\vartheta^d$ it is not difficult to show that the following equation holds:

$$\int_{\mathbb{V}_\vartheta^d} \mathbf{x}^\alpha W_{w,\mu}(\mathbf{x}) d\mathbf{x} = \left(\int_0^\vartheta w(x_d) x_d^{|\alpha|+2\mu+d-1} dx_d \right) \left(\int_{\mathbb{B}^{d-1}} y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_{d-1}^{\alpha_{d-1}} W_\mu(\mathbf{y}) d\mathbf{y} \right),$$
$$\mathbf{x} \in \mathbb{V}_\vartheta^d, \quad \mathbf{y} \in \mathbb{B}^{d-1}, \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d, \quad \mu > -1, \quad 0 < \vartheta \le \infty, \tag{3.79}$$

where $W_{w,\mu}$ is the weight function (3.78) on the cone and W_μ is the weight function (3.66) on the ball. The integral on $\mathbb{B}^{d-1}$ in (3.79) can be computed by means of (3.68).

As discussed in Remark 3.5, we choose $Q_\alpha^n(W_{w,\mu})$ to be the monomial $Q_\alpha^n(W_{w,\mu}; \mathbf{x}) = \mathbf{x}^\alpha$, $|\alpha| = n$. Then, we can compute the entries of the matrix $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^\kappa}$ in a simplified form by means of (3.49) and the moments $\left\langle \mathbf{x}^{\alpha^{(i)}-\theta}, \mathbf{x}^{\beta^{(j)}-\theta} \right\rangle_{w,\mu}$ of the weight (3.78). We denote by $\mathbb{Q}_n$ the column vector (3.48).

Proposition 3.17. *Let $n, m \ge \kappa$ and let $\mathbb{Q}_n$ be defined in* (3.48)*. Then, each entry of the matrix $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^\kappa}$ of size $r_n^d \times r_m^d$, which is defined in* (3.43)*, can be computed on the cone $\mathbb{V}_\vartheta^d$ by:*

$$\left\langle Q_{\alpha^{(i)}}^n(W_{w,\mu}), Q_{\beta^{(j)}}^m(W_{w,\mu}) \right\rangle_{\nabla^\kappa} =$$
$$\sum_{|\theta|=\kappa} \binom{\kappa}{\theta_1, \theta_2, \ldots, \theta_d} \frac{(-\alpha^{(i)})_\theta (-\beta^{(j)})_\theta \varpi_{w,n,m,\kappa,\mu}}{\left(\mu + \frac{d+1}{2}\right)_{\frac{n+m-\alpha_d^{(i)}-\beta_d^{(j)}}{2}-\kappa+\theta_d}} \prod_{l=1}^{d-1} \left(\frac{1}{2}\right)_{\frac{\alpha_l^{(i)}+\beta_l^{(j)}}{2}-\theta_l},$$

$$|\theta| = \kappa, \quad |\alpha^{(i)}| = n, \quad |\beta^{(j)}| = m, \quad 1 \le i \le r_n^d, \quad 1 \le j \le r_m^d,$$

if $\alpha_l^{(i)} + \beta_l^{(j)}$ *is even for all* $l = 1, 2, \dots, d-1$, *where* $\varpi_{w,n,m,\kappa,\mu}$ *is:*

$$\varpi_{w,n,m,\kappa,\mu} := \frac{\displaystyle\int_0^\vartheta w(t) t^{n+m-2\kappa+2\mu+d-1} dt}{\displaystyle\int_0^\vartheta w(t) t^{2\mu+d-1} dt}, \quad \mu > -1, \quad 0 < \vartheta \le \infty, \tag{3.80}$$

and $\left\langle Q^n_{\alpha^{(i)}}(W_{w,\mu}), Q^m_{\beta^{(j)}}(W_{w,\mu}) \right\rangle_{\nabla^\kappa} = 0$ *otherwise.*

Proof. By (1.48), (1.51), (3.79), and (3.68) we have that:

$$\begin{aligned}
\left\langle \mathbf{x}^{\alpha^{(i)}-\theta}, \mathbf{x}^{\beta^{(j)}-\theta} \right\rangle_{w,\mu} &= c_{w,\mu} \int_{\mathbb{V}^d_\vartheta} \mathbf{x}^{\alpha^{(i)}+\beta^{(j)}-2\theta} W_{w,\mu}(\mathbf{x}) d\mathbf{x} \\
&= \frac{\Gamma\left(\mu + \frac{d+1}{2}\right) \int_0^\vartheta w(x_d) x_d^{|\alpha^{(i)}|+|\beta^{(j)}|-2|\theta|+2\mu+d-1} dx_d}{\pi^{\frac{d-1}{2}} \Gamma(\mu+1) \int_0^\vartheta w(x_d) x_d^{2\mu+d-1} dx_d} \\
&\times \frac{2^{-d+1}\Gamma(\mu+1)}{\Gamma\left(\frac{1}{2}\sum_{l=1}^{d-1}(\alpha_l^{(i)} + \beta_l^{(j)} - 2\theta_l) + \mu + \frac{d+1}{2}\right)} \\
&\times \prod_{l=1}^{d-1} (1 + (-1)^{\alpha_l^{(i)}+\beta_l^{(j)}-2\theta_l}) \Gamma\left(\frac{\alpha_l^{(i)} + \beta_l^{(j)} - 2\theta_l + 1}{2}\right).
\end{aligned}$$

Therefore, by (3.49), with $|\alpha^{(i)}| = n$, $|\beta^{(j)}| = m$, and $|\theta| = \kappa$, and further simplification we have the result. □

Notice that, in particular, for the Jacobi and Laguerre cases with weights (1.52) and (1.53), respectively, the constant $\varpi_{w,n,m,\kappa,\mu}$ in (3.80) is given for each case by:

1. Jacobi ($\vartheta = 1$):

$$\begin{aligned}
\varpi_{w_{a,b},n,m,\kappa,\mu} &:= \varpi_{a,b,n,m,\kappa,\mu} \\
&= \frac{(a+2\mu+d)_{n+m-2\kappa}}{(a+b+2\mu+d+1)_{n+m-2\kappa}}, \quad n, m \ge \kappa, \quad a, b, \mu > -1.
\end{aligned} \tag{3.81}$$

2. Laguerre ($\vartheta = \infty$):

$$\varpi_{w_a,n,m,\kappa,\mu} := \varpi_{a,n,m,\kappa,\mu} = (a+2\mu+d)_{n+m-2\kappa}, \quad n, m \ge \kappa, \quad a, \mu > -1. \tag{3.82}$$

3.4.4.2 Partial differential equations for some Sobolev polynomials on the cone

In Section 1.3.5 we showed that the orthogonal polynomials with respect to the weight (3.78) are eigenfunctions of a second-order differential operator for particular cases of w. In this subsection we consider the Jacobi ($\vartheta = 1$) and Laguerre ($\vartheta = \infty$) cases.

Sobolev-Jacobi polynomials on the bounded cone The space $\mathscr{V}_n^d(W_{a,b,\mu}^J)$ of Jacobi polynomials on the cone $\Omega = \mathbb{V}_1^d$ and orthogonal with respect to the weight function:

$$W_{a,b,\mu}^J(\mathbf{x}) = (x_d^2 - \|\mathbf{x}_{d-1}\|^2)^\mu x_d^a (1-x_d)^b, \quad \mathbf{x} \in \mathbb{V}_1^d, \quad a, b, \mu > -1, \tag{3.83}$$

was discussed in Section 1.3.5.1. When the parameter $a = 0$ in (3.83), the polynomials in $\mathscr{V}_n^d(W_{0,b,\mu}^J)$ satisfy the partial differential equation (1.56).

Let us denote by $\mathscr{V}_n^d(S, W_{a,b,\mu}^J)$ and $\mathscr{V}_n^d(\nabla^\kappa, W_{a,b,\mu}^J)$ the spaces of Sobolev orthogonal polynomials with respect to (3.12) and (3.13), respectively, where it continuous part $\langle \cdot, \cdot \rangle_{\nabla^\kappa}$ is defined on $\Omega = \mathbb{V}_1^d$ and the weight is $W_{a,b,\mu}^J$. Then, we have the following corollary from Proposition 3.7 concerning partial differential equations for the polynomials in the spaces $\mathscr{V}_n^d(S, W_{0,b,\mu}^J)$ and $\mathscr{V}_n^d(\nabla^\kappa, W_{0,b,\mu}^J)$ (special case $a = 0$).

Corollary 3.6. *Let $P \in \mathscr{V}_n^d(S, W_{0,b,\mu}^J)$ or $P \in \mathscr{V}_n^d(\nabla^\kappa, W_{0,b,\mu}^J)$. Then P satisfies the partial differential equation:*

$$\left[\mathcal{V}_{b,\mu}^J + (n-\kappa)(n-\kappa+2\mu+b+d)\mathcal{I}\right](\partial_1 + \partial_2 + \cdots + \partial_d)^\kappa P = 0, \tag{3.84}$$

with $b, \mu > -1$, and where $\mathcal{V}_{b,\mu}^J$ is the differential operator (1.57) *and $\mathcal{I}$ is the identity operator.*

Proof. Similar to Corollary 3.2. □

Sobolev-Laguerre polynomials on the unbounded cone The space $\mathscr{V}_n^d(W_{a,\mu}^L)$ of Laguerre polynomials on the cone $\Omega = \mathbb{V}_\infty^d$ and orthogonal with respect to the weight function:

$$W_{a,\mu}^L(\mathbf{x}) = (x_d^2 - \|\mathbf{x}_{d-1}\|^2)^\mu x_d^a e^{-x_d}, \quad \mathbf{x} \in \mathbb{V}_\infty^d, \quad a, \mu > -1, \tag{3.85}$$

was discussed in Section 1.3.5.2. When the parameter $a = 0$ in (3.85), the polynomials in $\mathscr{V}_n^d(W_{0,\mu}^L)$ satisfy the partial differential equation (1.60).

Let us denote by $\mathscr{V}_n^d(S, W_{a,\mu}^L)$ and $\mathscr{V}_n^d(\nabla^\kappa, W_{a,\mu}^L)$ the spaces of Sobolev orthogonal polynomials with respect to (3.12) and (3.13), respectively, where it continuous part $\langle \cdot, \cdot \rangle_{\nabla^\kappa}$ is defined on $\Omega = \mathbb{V}_\infty^d$ and the weight is $W_{a,\mu}^L$. Similarly, we have the following corollary concerning partial differential equations for the polynomials in the spaces $\mathscr{V}_n^d(S, W_{0,\mu}^L)$ and $\mathscr{V}_n^d(\nabla^\kappa, W_{0,\mu}^L)$ (special case $a = 0$).

Corollary 3.7. *Let $P \in \mathscr{V}_n^d(S, W_{0,\mu}^L)$ or $P \in \mathscr{V}_n^d(\nabla^\kappa, W_{0,\mu}^L)$. Then P satisfies the partial differential equation:*

$$\left[\mathcal{V}_\mu^L + (n-\kappa)\mathcal{I}\right](\partial_1 + \partial_2 + \cdots + \partial_d)^\kappa P = 0, \tag{3.86}$$

with $\mu > -1$, and where $\mathcal{V}_\mu^L$ is the differential operator (1.61) *and $\mathcal{I}$ is the identity operator.*

Proof. Similar to Corollary 3.2. □

Chapter 4

Some numerical examples

Chapter 3 was devoted to study some algebraic and analytic properties of the Sobolev orthogonal polynomials in several variables with respect to the inner product (3.12). In order to provide a better understanding of the theory presented in Chapter 3, in this chapter we present some numerical examples in two variables[1]. Each example was constructed independently, then the reader can study each one separately without referencing to any other section in this chapter. Some results in the following sections are corollaries in two variables of the more general results given in Chapter 3. Therefore, we present those corollaries without a proof.

Note 4.1. For the numerical evaluations of the matrices in the following examples, we used MATLAB® software, version 8.0.0.783 (R2012b). Therefore, some fractions at the entries of some matrices are only approximations to the real values.

4.1 Preliminaries and notation for the examples

For later use, we need additional notation for polynomials in two variables $(x, y) \in \mathbb{R}^2$. Let us denote by ∇f, $\nabla^2 f$, and $\nabla^3 f$ the column vectors:

$$\begin{aligned} \nabla f &= \big(\partial_1 f, \ \ \partial_2 f\big)^T, \quad \nabla^2 f = \Big(\partial_1^2 f, \ \ \partial_1\partial_2 f, \ \ \partial_2\partial_1 f, \ \ \partial_2^2 f\Big)^T, \\ \nabla^3 f &= \Big(\partial_1^3 f, \ \ \partial_1^2\partial_2 f, \ \ \partial_1\partial_2\partial_1 f, \ \ \partial_1\partial_2^2 f, \ \ \partial_2\partial_1^2 f, \ \ \partial_2\partial_1\partial_2 f, \ \ \partial_2^2\partial_1 f, \ \ \partial_2^3 f\Big)^T, \end{aligned} \tag{4.1}$$

with $\partial_1 := \partial/\partial x$, $\partial_2 := \partial/\partial y$, $\partial_i^2 := \partial_i\partial_i$, $\partial_i^3 := \partial_i\partial_i\partial_i$, $i = 1, 2$. Let us recall that the Taylor polynomials of first degree $\mathcal{T}^1$ and second degree $\mathcal{T}^2$ in two variables of $P \in \Pi^2$ at the point $\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2$ are given by:

$$\begin{aligned} \mathcal{T}^1(P, \mathbf{p}; x, y) &= P(\mathbf{p}) + \partial_1 P(\mathbf{p})(x - p_1) + \partial_2 P(\mathbf{p})(y - p_2), \\ \mathcal{T}^2(P, \mathbf{p}; x, y) &= P(\mathbf{p}) + \partial_1 P(\mathbf{p})(x - p_1) + \partial_2 P(\mathbf{p})(y - p_2) \\ &\quad + \frac{1}{2}\left[\partial_1^2 P(\mathbf{p})(x - p_1)^2 + 2\partial_1\partial_2 P(\mathbf{p})(x - p_1)(y - p_2) + \partial_2^2 P(\mathbf{p})(y - p_2)^2\right]. \end{aligned}$$

[1] In [42, Section 4] we provided one example of orthogonal polynomials in three variables that is not presented in this chapter.

In addition, for two polynomials $P, Q \in \Pi^2$ that are equal up to a polynomial of first degree we write:

$$P \stackrel{1}{=} Q \quad \text{if} \quad P - Q \in \Pi_1^2, \tag{4.2}$$

and if P and Q are equal up to a polynomial of second degree we write:

$$P \stackrel{2}{=} Q \quad \text{if} \quad P - Q \in \Pi_2^2. \tag{4.3}$$

The relations $\stackrel{1}{=}$ and $\stackrel{2}{=}$ are congruence relations on the space Π^2.

Note 4.2. In the following examples we restrict ourselves to polynomials of lower degrees. Mainly because polynomials of higher degrees have too many monomials that cannot be depicted here.

4.2 Sobolev orthogonal polynomials on a product domain

4.2.1 Hermite-Laguerre product weight

In this subsection we use the results from Section 3.3 and Section 3.4.1.2. For the Hermite-Laguerre case, we consider the product domain

$$(-\infty, \infty) \times [0, \infty).$$

We construct the Sobolev orthogonal polynomials in two variables with respect to the inner product:

$$\langle f, g \rangle_S = c_a \int_0^\infty \int_{-\infty}^\infty \nabla^2 f(x,y) \cdot \nabla^2 g(x,y) W_a(x,y) dx dy + \\ \lambda_1 \nabla f(p_1, p_2) \cdot \nabla g(p_1, p_2) + \lambda_0 f(p_1, p_2) g(p_1, p_2), \tag{4.4}$$

where $\mathbf{p} = (p_1, p_2)$ is a given point in $\mathbb{R}^2$, $\lambda_0, \lambda_1 > 0$, ∇f and $\nabla^2 f$ are given in (4.1), W_a is the Hermite-Laguerre product weight:

$$W_a(x,y) = e^{-x^2} y^a e^{-y}, \quad a > -1, \quad (x,y) \in (-\infty, \infty) \times [0, \infty),$$

c_a is the normalization constant:

$$c_a := \left(\int_0^\infty \int_{-\infty}^\infty W_a(x,y) dx dy \right)^{-1} = \frac{1}{\Gamma(a+1)\sqrt{\pi}},$$

and the main part of (4.4) is denoted by:

$$\langle f, g \rangle_{\nabla^2} = c_a \int_0^\infty \int_{-\infty}^\infty \nabla^2 f(x,y) \cdot \nabla^2 g(x,y) W_a(x,y) dx dy. \tag{4.5}$$

We denote by $\mathscr{V}_n^2(S, W_a)$ and $\mathscr{V}_n^2(\nabla^2, W_a)$ the spaces of orthogonal polynomials of degree n with respect to (4.4) and (4.5), respectively. The following corollary is a consequence of Theorem 3.1 and Proposition 3.9 for $\kappa = d = 2$.

Corollary 4.1. *Let* $\left\{S_j^n : 0 \le j \le n\right\}$ *denote a monic orthogonal basis of* $\mathscr{V}_n^2(\nabla^2, W_a)$. *Then, a monic orthogonal basis* $\left\{\mathcal{S}_j^n : 0 \le j \le n\right\}$ *of* $\mathscr{V}_n^2(S, W_a)$ *is given by:*

$$\begin{aligned}
&\mathcal{S}_0^0(x,y) = 1,\\
&\mathcal{S}_0^1(x,y) = x - p_1, \quad \mathcal{S}_1^1(x,y) = y - p_2,\\
&\mathcal{S}_j^n(x,y) = S_j^n(x,y) - \mathcal{T}^1(S_j^n, \mathbf{p}; x, y), \quad n \ge 2,
\end{aligned}$$

where $\mathcal{T}^1(S_j^n, \mathbf{p})$ *is the Taylor polynomial of first degree of* S_j^n *at* $\mathbf{p} = (p_1, p_2)$, *and where*

$$\begin{aligned}
&\left\langle \mathcal{S}_0^0, \mathcal{S}_0^0 \right\rangle_S = \lambda_0,\\
&\left\langle \mathcal{S}_0^1, \mathcal{S}_0^1 \right\rangle_S = \left\langle \mathcal{S}_1^1, \mathcal{S}_1^1 \right\rangle_S = \lambda_1,\\
&\left\langle \mathcal{S}_j^n, \mathcal{S}_j^n \right\rangle_S = \left\langle S_j^n, S_j^n \right\rangle_{\nabla^2}, \quad 0 \le j \le n, \quad n \ge 2.
\end{aligned}$$

Then, we need only to find a monic orthogonal basis $\left\{S_j^n : 0 \le j \le n\right\}$ of $\mathscr{V}_n^2(\nabla^2, W_a)$ for $n \ge 2$. Let us denote by $\mathbb{S}_n$ the column vector of size $n+1$:

$$\mathbb{S}_n = \Big(S_0^n(x,y), \quad S_1^n(x,y), \quad \ldots, \quad S_n^n(x,y)\Big)^T.$$

For this construction, we consider the monic sequences of Hermite $\{H_n(x)\}_{n\ge 0}$ and Laguerre $\left\{L_n^{(a)}(y)\right\}_{n\ge 0}$, $a > -1$, orthogonal polynomials (see [111, Chapter 5]) which are, respectively, orthogonal with respect to the weight functions:

$$u(x) = e^{-x^2}, \quad x \in (-\infty, \infty), \quad w_a(y) = y^a e^{-y}, \quad a > -1, \quad y \in [0, \infty),$$

that is,

$$\begin{aligned}
\langle H_n, H_m \rangle &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} H_n(x) H_m(x) u(x) dx = h_n(u) \delta_{n,m},\\
\left\langle L_n^{(a)}, L_m^{(a)} \right\rangle &= \frac{1}{\Gamma(a+1)} \int_0^{\infty} L_n^{(a)}(y) L_m^{(a)}(y) w_a(y) dy = h_n(w_a) \delta_{n,m},
\end{aligned}$$

and where their L^2 norms are, respectively:

$$h_n(u) = \langle H_n, H_n \rangle = \frac{n!}{2^n}, \quad h_n(w_a) = \left\langle L_n^{(a)}, L_n^{(a)} \right\rangle = n!(a+1)_n. \tag{4.6}$$

By convention, we take $H_n = L_n^{(a)} = 0$ for $n < 0$, and consequently $h_n(u) = h_n(w_a) = 0$ for $n < 0$. In addition, the Hermite and Laguerre (monic) sequences are self-coherent with the relations [111, (5.5.10), (5.1.13), (5.1.14)]:

$$H_n(x) = \frac{1}{n+1} H'_{n+1}(x), \quad n \ge 0, \tag{4.7}$$

$$L_n^{(a)}(y) = \frac{1}{n+1} (L_{n+1}^{(a)})'(y) + (L_n^{(a)})'(y), \quad n \ge 0. \tag{4.8}$$

From the monic sequences $\{H_n(x)\}_{n\geq 0}$ and $\left\{L_n^{(a)}(y)\right\}_{n\geq 0}$ we define the product polynomial P_j^n in two variables:

$$P_j^n(x,y) = H_{n-j}(x)L_j^{(a)}(y), \quad 0 \leq j \leq n, \quad n \geq 0. \tag{4.9}$$

According to Proposition 1.1, the set $\left\{P_j^n : 0 \leq j \leq n\right\}$ is a monic mutually orthogonal basis for the space $\mathscr{V}_n^2(W_a)$ of orthogonal polynomials of degree n with respect to the inner product:

$$\langle f, g\rangle_{W_a} = c_a \int_0^{\infty}\int_{-\infty}^{\infty} f(x,y)g(x,y)W_a(x,y)dxdy,$$

that is,

$$\left\langle P_j^n, P_k^m\right\rangle_{W_a} = h_j^n \delta_{n,m}\delta_{j,k}, \tag{4.10}$$

where, from (4.6), we get that the L^2 norm of P_j^n is:

$$h_j^n = \|P_j^n\|_{W_a}^2 = h_{n-j}(u)h_j(w_a) = \frac{(n-j)!j!(a+1)_j}{2^{n-j}}, \quad 0 \leq j \leq n, \quad n \geq 0. \tag{4.11}$$

From differential equations (1.2) and (1.4), and the polynomial (4.9), it is not difficult to prove that the polynomials in the space $\mathscr{V}_n^2(W_a)$ satisfy the partial differential equation[2]:

$$\frac{1}{2}\frac{\partial^2 P}{\partial x^2} + y\frac{\partial^2 P}{\partial y^2} - x\frac{\partial P}{\partial x} + (a+1-y)\frac{\partial P}{\partial y} = -nP, \quad P \in \mathscr{V}_n^2(W_a), \quad a > -1. \tag{4.12}$$

From Proposition 3.7, we know that if $P \in \mathscr{V}_n^2(S, W_a)$ or $P \in \mathscr{V}_n^2(\nabla^2, W_a)$ then:

$$(\partial_1 + \partial_2)^2 P := \frac{\partial^2 P}{\partial x^2} + 2\frac{\partial^2 P}{\partial x \partial y} + \frac{\partial^2 P}{\partial y^2} \in \mathscr{V}_{n-2}^2(W_a). \tag{4.13}$$

Putting (4.12) and (4.13) together, then they prove the following result.

Proposition 4.1. *Let $P \in \mathscr{V}_n^2(S, W_a)$ or $P \in \mathscr{V}_n^2(\nabla^2, W_a)$, $a > -1$. Then P satisfies the fourth-order partial differential equation:*

$$\left[\frac{1}{2}\frac{\partial^2}{\partial x^2} + y\frac{\partial^2}{\partial y^2} - x\frac{\partial}{\partial x} + (a+1-y)\frac{\partial}{\partial y} + (n-2)\mathcal{I}\right]\left[\frac{\partial^2}{\partial x^2} + 2\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}\right] P = 0,$$

where $\mathcal{I}$ is the identity operator.

In the following proposition we use the results from Section 3.4.1.2 because W_a is a product of classical weights.

[2] In (4.12) the factor $1/2$ appears at $\partial^2/\partial x^2$ because we assume e^{-x^2} to be the weight function for the Hermite polynomials. Suetin [110, page 40, equations (25)–(28)] works with the weight $e^{-x^2/2}$.

Proposition 4.2. *The monic sequences* $\{q_n(u;x)\}_{n\geq 0}$ *and* $\{q_n(w_a;y)\}_{n\geq 0}$ *defined in* (3.58)*, and their derivatives up to second-order, are given for the Hermite and Laguerre cases by:*

$$\begin{aligned} q_n''(u;x) &= n(n-1)H_{n-2}(x), \quad n \geq 0, \\ q_n'(u;x) &= nH_{n-1}(x), \quad n \geq 0, \\ q_n(u;x) &= H_n(x), \quad n \geq 0, \end{aligned}$$

and

$$\begin{aligned} q_n''(w_a;y) &= n(n-1)L_{n-2}^{(a)}(y), \quad n \geq 0, \\ q_n'(w_a;y) &= nL_{n-1}^{(a)}(y) + n(n-1)L_{n-2}^{(a)}(y), \quad n \geq 0, \\ q_n(w_a;y) &= L_n^{(a)}(y) + 2nL_{n-1}^{(a)}(y) + n(n-1)L_{n-2}^{(a)}(y), \quad n \geq 0. \end{aligned}$$

Proof. By comparing coefficients in (3.52), (4.7) and (4.8) we have for each case:

$$a_n(u) = b_n(u) = 0, \quad a_n(w_a) = 1, \quad b_n(w_a) = 0.$$

Then, from Proposition 3.13 we have the following constants:

$$\begin{aligned} &\gamma_{-1}^{n,1}(u) = \gamma_0^{n,1}(u) = 0, \quad \gamma_1^{n,1}(u) = \frac{1}{n+1}, \\ &\gamma_{-2}^{n,2}(u) = \gamma_{-1}^{n,2}(u) = \gamma_0^{n,2}(u) = \gamma_1^{n,2}(u) = 0, \quad \gamma_2^{n,2}(u) = \frac{1}{(n+1)(n+2)}, \end{aligned}$$

and also for the Laguerre case:

$$\begin{aligned} &\gamma_{-1}^{n,1}(w_a) = 0, \quad \gamma_0^{n,1}(w_a) = 1, \quad \gamma_1^{n,1}(w_a) = \frac{1}{n+1}, \quad \gamma_{-2}^{n,2}(w_a) = \gamma_{-1}^{n,2}(w_a) = 0, \\ &\gamma_0^{n,2}(w_a) = 1, \quad \gamma_1^{n,2}(w_a) = \frac{2}{n+1}, \quad \gamma_2^{n,2}(w_a) = \frac{1}{(n+1)(n+2)}. \end{aligned}$$

The result follows from (3.55) and (3.57) with $\kappa = 2$. □

From the monic sequences $\{q_n(u;x)\}_{n\geq 0}$ and $\{q_n(w_a;y)\}_{n\geq 0}$ we define the product polynomial Q_j^n in two variables:

$$Q_j^n(x,y) = q_{n-j}(u;x)q_j(w_a;y), \quad 0 \leq j \leq n, \quad n \geq 0, \tag{4.14}$$

where we denote by $\mathbb{Q}_n$ the column vector of size $n+1$:

$$\mathbb{Q}_n = \Big(Q_0^n(x,y), \quad Q_1^n(x,y), \quad \ldots, \quad Q_n^n(x,y)\Big)^T.$$

As a consequence, from Proposition 4.2 we have the following three results with respect to the matrix $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^2}$.

Proposition 4.3. *The second-order partial derivatives* $\partial_1^2 Q_j^n$, $\partial_1\partial_2 Q_j^n$ *and* $\partial_2^2 Q_j^n$ *of the polynomial* Q_j^n, $0 \le j \le n$, $n \ge 0$, *are given by:*

$$\begin{aligned}
\partial_1^2 Q_j^n(x,y) &= (n-j)(n-j-1)[P_j^{n-2}(x,y) + 2jP_{j-1}^{n-3}(x,y) + j(j-1)P_{j-2}^{n-4}(x,y)],\\
\partial_1\partial_2 Q_j^n(x,y) &= j(n-j)[P_{j-1}^{n-2}(x,y) + (j-1)P_{j-2}^{n-3}(x,y)],\\
\partial_2^2 Q_j^n(x,y) &= j(j-1)P_{j-2}^{n-2}(x,y).
\end{aligned}$$

Proof. From (4.9), (4.14) and Proposition 4.2 we have that:

$$\begin{aligned}
&\partial_1^2 Q_j^n(x,y) = q''_{n-j}(u;x)q_j(w_a;y) = \\
&(n-j)(n-j-1)H_{n-j-2}(x)[L_j^{(a)}(y) + 2jL_{j-1}^{(a)}(y) + j(j-1)L_{j-2}^{(a)}(y)] = \\
&\qquad (n-j)(n-j-1)[P_j^{n-2}(x,y) + 2jP_{j-1}^{n-3}(x,y) + j(j-1)P_{j-2}^{n-4}(x,y)].
\end{aligned}$$

The expressions for $\partial_1\partial_2 Q_j^n$ and $\partial_2^2 Q_j^n$ follow similarly. □

Proposition 4.4. $\langle Q_j^n, Q_k^m\rangle_{\nabla^2}$, $0 \le j \le n$, $0 \le k \le m$, $n,m \ge 0$, *is given by:*

$$\begin{aligned}
\langle Q_j^n, Q_k^m\rangle_{\nabla^2} = A_j^n\delta_{n,m-2}\delta_{j,k-2} + B_j^n\delta_{n,m-1}\delta_{j,k-1} + C_j^n\delta_{n,m}\delta_{j,k}&\\
+ B_{j-1}^{n-1}\delta_{n,m+1}\delta_{j,k+1} + A_{j-2}^{n-2}\delta_{n,m+2}\delta_{j,k+2}&,
\end{aligned}$$

where,

$$\begin{aligned}
A_j^n &= (j+1)(j+2)(n-j)^2(n-j-1)^2h_j^{n-2},\\
B_j^n &= 2(j+1)(n-j)^2(n-j-1)^2h_j^{n-2} + 2j^2(j+1)(n-j)^2h_{j-1}^{n-2}\\
&\quad + 2j^2(j+1)(n-j)^2(n-j-1)^2h_{j-1}^{n-3},\\
C_j^n &= (n-j)^2(n-j-1)^2h_j^{n-2} + 2j^2(n-j)^2h_{j-1}^{n-2} + j^2(j-1)^2h_{j-2}^{n-2}\\
&\quad + 4j^2(n-j)^2(n-j-1)^2h_{j-1}^{n-3} + 2j^2(j-1)^2(n-j)^2h_{j-2}^{n-3}\\
&\quad + j^2(j-1)^2(n-j)^2(n-j-1)^2h_{j-2}^{n-4},
\end{aligned}$$

and where h_j^n *is given in* (4.11).

Proof. From Proposition 3.4 for $\kappa = d = 2$ we have:

$$\langle Q_j^n, Q_k^m\rangle_{\nabla^2} = \langle \partial_1^2 Q_j^n, \partial_1^2 Q_k^m\rangle_{W_a} + 2\langle \partial_1\partial_2 Q_j^n, \partial_1\partial_2 Q_k^m\rangle_{W_a} + \langle \partial_2^2 Q_j^n, \partial_2^2 Q_k^m\rangle_{W_a},$$

where, for example, from (4.10) and Proposition 4.3:

$$\begin{aligned}
&\langle \partial_1^2 Q_j^n, \partial_1^2 Q_k^m\rangle_{W_a} = (n-j)^2(n-j-1)^2[h_j^{n-2}\delta_{n,m}\delta_{j,k} + \\
&2jh_{j-1}^{n-3}\delta_{n,m+1}\delta_{j,k+1} + j(j-1)h_{j-2}^{n-4}\delta_{n,m+2}\delta_{j,k+2} + 2(j+1)h_j^{n-2}\delta_{n,m-1}\delta_{j,k-1} + \\
&4j^2h_{j-1}^{n-3}\delta_{n,m}\delta_{j,k} + 2j(j-1)^2h_{j-2}^{n-4}\delta_{n,m+1}\delta_{j,k+1} + (j+1)(j+2)h_j^{n-2}\delta_{n,m-2}\delta_{j,k-2} + \\
&\qquad 2j^2(j+1)h_{j-1}^{n-3}\delta_{n,m-1}\delta_{j,k-1} + j^2(j-1)^2h_{j-2}^{n-4}\delta_{n,m}\delta_{j,k}],
\end{aligned}$$

and similarly for the other two expressions. We have the result by adding and simplifying. □

Proposition 4.5. *Let A_j^n, B_j^n, and C_j^n, $0 \le j \le n$, $n \ge 0$, be given in Proposition 4.4. Then, the matrices $\langle \mathbb{Q}_n, \mathbb{Q}_n^T \rangle_{\nabla^2}$, $\langle \mathbb{Q}_{n+1}, \mathbb{Q}_n^T \rangle_{\nabla^2}$ and $\langle \mathbb{Q}_{n+2}, \mathbb{Q}_n^T \rangle_{\nabla^2}$, of sizes $(n+1) \times (n+1)$, $(n+2) \times (n+1)$ and $(n+3) \times (n+1)$, respectively, are diagonal matrices of the form:*

$$\langle \mathbb{Q}_n, \mathbb{Q}_n^T \rangle_{\nabla^2} = \operatorname{diag}\left(C_0^n, \quad C_1^n, \quad \ldots, \quad C_n^n \right),$$

$$\langle \mathbb{Q}_{n+1}, \mathbb{Q}_n^T \rangle_{\nabla^2} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ B_0^n & 0 & \cdots & 0 \\ 0 & B_1^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n^n \end{pmatrix}, \quad \langle \mathbb{Q}_{n+2}, \mathbb{Q}_n^T \rangle_{\nabla^2} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ A_0^n & 0 & \cdots & 0 \\ 0 & A_1^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n^n \end{pmatrix}.$$

Proof. Let us recall that $\langle \mathbb{Q}_n, \mathbb{Q}_m^T \rangle_{\nabla^2} = \left(\langle Q_j^n, Q_k^m \rangle_{\nabla^2} \right)_{0 \le j \le n, 0 \le k \le m}$ is a matrix of size $(n+1) \times (m+1)$. By Proposition 4.4 we have that $\langle Q_j^n, Q_k^m \rangle_{\nabla^2} = 0$ with the exception of $m = n, m = n \pm 1, m = n \pm 2$. Therefore, for $m = n$, $m = n-1$, and $m = n-2$ we have:

$$\begin{aligned} \langle Q_j^m, Q_k^m \rangle_{\nabla^2} &= C_k^m \delta_{j,k}, \quad 0 \le j \le m, \quad 0 \le k \le m, \\ \langle Q_j^{m+1}, Q_k^m \rangle_{\nabla^2} &= B_k^m \delta_{j,k+1}, \quad 0 \le j \le m+1, \quad 0 \le k \le m, \\ \langle Q_j^{m+2}, Q_k^m \rangle_{\nabla^2} &= A_k^m \delta_{j,k+2}, \quad 0 \le j \le m+2, \quad 0 \le k \le m. \end{aligned}$$ □

Example 4.1 (Numerical, see Note 4.1). Let $a = 1$ for the Laguerre polynomials and let $\mathbf{p} = (p_1, p_2) = (1, 0)$ in the inner product (4.4). From the monic sequences of Hermite $\{H_n(x)\}_{n \ge 0}$ and Laguerre $\left\{ L_n^{(a)}(x) \right\}_{n \ge 0}$ polynomials [111, Chapter 5] we have by Proposition 4.2 that $\{q_n(u; x)\}_{n \ge 0}$ and $\{q_n(w_a; y)\}_{n \ge 0}$ are given for $0 \le n \le 5$ by:

$$q_0(u;x) = 1, \quad q_1(u;x) = x, \quad q_2(u;x) = x^2 - \frac{1}{2}, \quad q_3(u;x) = x^3 - \frac{3}{2}x,$$
$$q_4(u;x) = x^4 - 3x^2 + \frac{3}{4}, \quad q_5(u;x) = x^5 - 5x^3 + \frac{15}{4}x,$$

and

$$q_0(w_a;y) = 1, \quad q_1(w_a;y) = y, \quad q_2(w_a;y) = y^2 - 2y, \quad q_3(w_a;y) = y^3 - 6y^2 + 6y,$$
$$q_4(w_a;y) = y^4 - 12y^3 + 36y^2 - 24y, \quad q_5(w_a;y) = y^5 - 20y^4 + 120y^3 - 240y^2 + 120y.$$

Therefore, from (4.14), for $0 \le n \le 5$:

$$\mathbb{Q}_0 = 1, \quad \mathbb{Q}_1 = \left(x, \quad y\right)^T, \quad \mathbb{Q}_2 = \left(x^2 - \tfrac{1}{2}, \quad xy, \quad y^2 - 2y\right)^T,$$
$$\mathbb{Q}_3 = \left(x^3 - \tfrac{3}{2}x, \quad x^2 y - \tfrac{1}{2}y, \quad xy^2 - 2xy, \quad y^3 - 6y^2 + 6y\right)^T,$$

$$\mathbb{Q}_4 = \begin{pmatrix} x^4 - 3x^2 + \frac{3}{4} \\ x^3y - \frac{3}{2}xy \\ x^2y^2 - 2x^2y - \frac{1}{2}y^2 + y \\ xy^3 - 6xy^2 + 6xy \\ y^4 - 12y^3 + 36y^2 - 24y \end{pmatrix}, \quad \mathbb{Q}_5 = \begin{pmatrix} x^5 - 5x^3 + \frac{15}{4}x \\ x^4y - 3x^2y + \frac{3}{4}y \\ x^3y^2 - 2x^3y - \frac{3}{2}xy^2 + 3xy \\ x^2y^3 - 6x^2y^2 + 6x^2y - \frac{1}{2}y^3 + 3y^2 - 3y \\ xy^4 - 12xy^3 + 36xy^2 - 24xy \\ y^5 - 20y^4 + 120y^3 - 240y^2 + 120y \end{pmatrix}.$$

From Proposition 3.11 and Proposition 4.5, with $\kappa = 2$, we have:

1. First iteration: $\mathbf{B}_{n,2} = \mathbf{0}$ for $n \geq 5$ and

$$\mathbf{B}_{3,2} = \left\langle \mathbb{Q}_3, \mathbb{Q}_2^T \right\rangle_{\nabla^2} = \begin{pmatrix} 0 & 0 & 0 \\ 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}_{4,2} = \left\langle \mathbb{Q}_4, \mathbb{Q}_2^T \right\rangle_{\nabla^2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{H}_2^{\nabla^2} = \left\langle \mathbb{Q}_2, \mathbb{Q}_2^T \right\rangle_{\nabla^2} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

2. Second iteration: $\mathbf{B}_{n,3} = \mathbf{0}$ for $n \geq 6$ and

$$\mathbf{B}_{4,3} = \left\langle \mathbb{Q}_4, \mathbb{Q}_3^T \right\rangle_{\nabla^2} - \mathbf{B}_{4,2} \left(\mathbf{H}_2^{\nabla^2}\right)^{-1} \mathbf{B}_{3,2}^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 36 & 0 & 0 & 0 \\ 0 & 40 & 0 & 0 \\ 0 & 0 & 48 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{B}_{5,3} = \left\langle \mathbb{Q}_5, \mathbb{Q}_3^T \right\rangle_{\nabla^2} - \mathbf{B}_{5,2} \left(\mathbf{H}_2^{\nabla^2}\right)^{-1} \mathbf{B}_{3,2}^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 36 & 0 & 0 & 0 \\ 0 & 48 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{H}_3^{\nabla^2} = \left\langle \mathbb{Q}_3, \mathbb{Q}_3^T \right\rangle_{\nabla^2} - \mathbf{B}_{3,2} \left(\mathbf{H}_2^{\nabla^2}\right)^{-1} \mathbf{B}_{3,2}^T = \begin{pmatrix} 18 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 18 & 0 \\ 0 & 0 & 0 & 72 \end{pmatrix}.$$

3. Third iteration: $\mathbf{B}_{n,4} = \mathbf{0}$ for $n \geq 7$ and

$$\mathbf{B}_{5,4} = \left\langle \mathbb{Q}_5, \mathbb{Q}_4^T \right\rangle_{\nabla^2} - \sum_{j=2}^{3} \mathbf{B}_{5,j} \left(\mathbf{H}_j^{\nabla^2}\right)^{-1} \mathbf{B}_{4,j}^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 144 & 0 & 0 & 0 & 0 \\ 0 & 162 & 0 & 0 & 0 \\ 0 & 0 & 416 & 0 & 0 \\ 0 & 0 & 0 & 864 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{B}_{6,4} = \left\langle \mathbb{Q}_6, \mathbb{Q}_4^T \right\rangle_{\nabla^2} - \sum_{j=2}^{3} \mathbf{B}_{6,j} \left(\mathbf{H}_j^{\nabla^2}\right)^{-1} \mathbf{B}_{4,j}^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 144 & 0 & 0 & 0 & 0 \\ 0 & 216 & 0 & 0 & 0 \\ 0 & 0 & 576 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{H}_4^{\nabla^2} = \left\langle \mathbb{Q}_4, \mathbb{Q}_4^T \right\rangle_{\nabla^2} - \sum_{j=2}^{3} \mathbf{B}_{4,j} \left(\mathbf{H}_j^{\nabla^2}\right)^{-1} \mathbf{B}_{4,j}^T = \begin{pmatrix} 72 & 0 & 0 & 0 & 0 \\ 0 & 45 & 0 & 0 & 0 \\ 0 & 0 & \frac{278}{3} & 0 & 0 \\ 0 & 0 & 0 & 268 & 0 \\ 0 & 0 & 0 & 0 & 1728 \end{pmatrix}.$$

Therefore, from Proposition 3.11 we have:

$$\mathbf{A}_{n,2} = \mathbf{B}_{n,2} \left(\mathbf{H}_2^{\nabla^2}\right)^{-1} = \mathbf{0}, \quad n \geq 5, \quad \mathbf{A}_{n,3} = \mathbf{B}_{n,3} \left(\mathbf{H}_3^{\nabla^2}\right)^{-1} = \mathbf{0}, \quad n \geq 6,$$
$$\mathbf{A}_{n,4} = \mathbf{B}_{n,4} \left(\mathbf{H}_4^{\nabla^2}\right)^{-1} = \mathbf{0}, \quad n \geq 7,$$

and

$$\mathbf{A}_{3,2} = \mathbf{B}_{3,2} \left(\mathbf{H}_2^{\nabla^2}\right)^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_{4,2} = \mathbf{B}_{4,2} \left(\mathbf{H}_2^{\nabla^2}\right)^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{A}_{4,3} = \mathbf{B}_{4,3} \left(\mathbf{H}_3^{\nabla^2}\right)^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & \frac{10}{3} & 0 & 0 \\ 0 & 0 & \frac{8}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{A}_{5,4} = \mathbf{B}_{5,4} \left(\mathbf{H}_4^{\nabla^2}\right)^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \frac{18}{5} & 0 & 0 & 0 \\ 0 & 0 & \frac{624}{139} & 0 & 0 \\ 0 & 0 & 0 & \frac{216}{67} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{A}_{5,3} = \mathbf{B}_{5,3} \left(\mathbf{H}_3^{\nabla^2}\right)^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_{6,4} = \mathbf{B}_{6,4} \left(\mathbf{H}_4^{\nabla^2}\right)^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \frac{24}{5} & 0 & 0 & 0 \\ 0 & 0 & \frac{864}{139} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $\overset{1}{=}$ denotes the congruence relation (4.2) on Π^2 then, from Proposition 3.10, we have for $0 \leq n \leq 5$:

$\mathbb{S}_0 \overset{1}{=} \mathbb{Q}_0 = 1,$

$$\mathbb{S}_1 \overset{1}{=} \mathbb{Q}_1 = \left(x, \quad y\right)^T,$$
$$\mathbb{S}_2 \overset{1}{=} \mathbb{Q}_2 = \left(x^2 - \tfrac{1}{2}, \quad xy, \quad y^2 - 2y\right)^T,$$
$$\mathbb{S}_3 \overset{1}{=} \mathbb{Q}_3 - \mathbf{A}_{3,2}\mathbb{S}_2 \overset{1}{=} \left(x^3 - \tfrac{3}{2}x, \quad x^2y - 2x^2 - \tfrac{1}{2}y + 1, \quad xy^2 - 4xy, \quad y^3 - 6y^2 + 6y\right)^T,$$
$$\mathbb{S}_4 \overset{1}{=} \mathbb{Q}_4 - \mathbf{A}_{4,3}\mathbb{S}_3 - \mathbf{A}_{4,2}\mathbb{S}_2 \overset{1}{=} \begin{pmatrix} x^4 - 3x^2 + \frac{3}{4} \\ x^3y - 2x^3 - \frac{3}{2}xy + 3x \\ x^2y^2 - \frac{16}{3}x^2y + \frac{14}{3}x^2 - \frac{1}{2}y^2 + \frac{8}{3}y - \frac{7}{3} \\ xy^3 - \frac{26}{3}xy^2 + \frac{50}{3}xy \\ y^4 - 12y^3 + 36y^2 - 24y \end{pmatrix},$$
$$\mathbb{S}_5 \overset{1}{=} \mathbb{Q}_5 - \mathbf{A}_{5,4}\mathbb{S}_4 - \mathbf{A}_{5,3}\mathbb{S}_3 - \mathbf{A}_{5,2}\mathbb{S}_2$$
$$\overset{1}{=} \begin{pmatrix} x^5 - 5x^3 + \frac{15}{4}x \\ x^4y - 2x^4 - 3x^2y + 6x^2 + \frac{3}{4}y - \frac{3}{2} \\ x^3y^2 - \frac{28}{5}x^3y + \frac{26}{5}x^3 - \frac{3}{2}xy^2 + \frac{42}{5}xy - \frac{39}{5}x \\ x^2y^3 - \frac{1458}{139}x^2y^2 + \frac{3606}{139}x^2y - \frac{1}{2}y^3 - \frac{1800}{139}x^2 + \frac{729}{139}y^2 - \frac{1803}{139}y + \frac{900}{139} \\ xy^4 - \frac{1020}{67}xy^3 + \frac{4284}{67}xy^2 - \frac{5208}{67}xy \\ y^5 - 20y^4 + 120y^3 - 240y^2 + 120y \end{pmatrix}.$$

Finally, let $\mathbf{p} = (p_1, p_2) = (1, 0)$. The Taylor polynomials of first degree at $\mathbf{p}$ of $\mathbb{S}_2, \mathbb{S}_3, \mathbb{S}_4$ and $\mathbb{S}_5$ are given by:

$$\mathcal{T}^1(\mathbb{S}_2, \mathbf{p}; x, y) = \left(2x - \tfrac{3}{2}, \quad y, \quad -2y\right)^T,$$
$$\mathcal{T}^1(\mathbb{S}_3, \mathbf{p}; x, y) = \left(\tfrac{3}{2}x - 2, \quad -4x + \tfrac{1}{2}y + 3, \quad -4y, \quad 6y\right)^T,$$
$$\mathcal{T}^1(\mathbb{S}_4, \mathbf{p}; x, y) = \left(-2x + \tfrac{3}{4}, \quad -3x - \tfrac{1}{2}y + 4, \quad \tfrac{28}{3}x - \tfrac{8}{3}y - 7, \quad \tfrac{50}{3}y, \quad -24y\right)^T,$$
$$\mathcal{T}^1(\mathbb{S}_5, \mathbf{p}; x, y) = \begin{pmatrix} -\frac{25}{4}x + 6 \\ 4x - \frac{5}{4}y - \frac{3}{2} \\ \frac{39}{5}x + \frac{14}{5}y - \frac{52}{5} \\ -\frac{3600}{139}x + \frac{1803}{139}y + \frac{2700}{139} \\ -\frac{5208}{67}y \\ 120y \end{pmatrix}.$$

Then, from Corollary 4.1 we have a basis $\left\{\mathcal{S}_j^n : 0 \le j \le n\right\}$ for the space $\mathscr{V}_n^2(S, W_a)$ (with $a = 1$), together with their Sobolev L^2 norms, for $0 \le n \le 5$ given by:

- For the space $\mathscr{V}_0^2(S, W_a)$:

$$\mathcal{S}_0^0(x, y) = 1.$$

- For the space $\mathscr{V}_1^2(S, W_a)$:

$$\mathcal{S}_0^1(x,y) = x - 1,$$
$$\mathcal{S}_1^1(x,y) = y.$$

- For the space $\mathscr{V}_2^2(S, W_a)$:

$$\mathcal{S}_0^2(x,y) = x^2 - 2x + 1,$$
$$\mathcal{S}_1^2(x,y) = xy - y,$$
$$\mathcal{S}_2^2(x,y) = y^2.$$

- For the space $\mathscr{V}_3^2(S, W_a)$:

$$\mathcal{S}_0^3(x,y) = x^3 - 3x + 2,$$
$$\mathcal{S}_1^3(x,y) = x^2y - 2x^2 + 4x - y - 2,$$
$$\mathcal{S}_2^3(x,y) = xy^2 - 4xy + 4y,$$
$$\mathcal{S}_3^3(x,y) = y^3 - 6y^2.$$

- For the space $\mathscr{V}_4^2(S, W_a)$:

$$\mathcal{S}_0^4(x,y) = x^4 - 3x^2 + 2x,$$
$$\mathcal{S}_1^4(x,y) = x^3y - 2x^3 - \frac{3}{2}xy + 6x + \frac{1}{2}y - 4,$$
$$\mathcal{S}_2^4(x,y) = x^2y^2 - \frac{16}{3}x^2y + \frac{14}{3}x^2 - \frac{1}{2}y^2 - \frac{28}{3}x + \frac{16}{3}y + \frac{14}{3},$$
$$\mathcal{S}_3^4(x,y) = xy^3 - \frac{26}{3}xy^2 + \frac{50}{3}xy - \frac{50}{3}y,$$
$$\mathcal{S}_4^4(x,y) = y^4 - 12y^3 + 36y^2.$$

- For the space $\mathscr{V}_5^2(S, W_a)$:

$$\mathcal{S}_0^5(x,y) = x^5 - 5x^3 + 10x - 6,$$
$$\mathcal{S}_1^5(x,y) = x^4y - 2x^4 - 3x^2y + 6x^2 - 4x + 2y,$$
$$\mathcal{S}_2^5(x,y) = x^3y^2 - \frac{28}{5}x^3y + \frac{26}{5}x^3 - \frac{3}{2}xy^2 + \frac{42}{5}xy - \frac{78}{5}x - \frac{14}{5}y + \frac{52}{5},$$
$$\mathcal{S}_3^5(x,y) = x^2y^3 - \frac{1458}{139}x^2y^2 + \frac{3606}{139}x^2y - \frac{1}{2}y^3 - \frac{1800}{139}x^2 + \frac{729}{139}y^2 + \frac{3600}{139}x - \frac{3606}{139}y - \frac{1800}{139},$$
$$\mathcal{S}_4^5(x,y) = xy^4 - \frac{1020}{67}xy^3 + \frac{4284}{67}xy^2 - \frac{5208}{67}xy + \frac{5208}{67}y,$$
$$\mathcal{S}_5^5(x,y) = y^5 - 20y^4 + 120y^3 - 240y^2.$$

This completes our numerical example with the Hermite-Laguerre product weight.

4.2.2 Laguerre-Gegenbauer product weight

In this subsection we use the results from Section 3.3 and Section 3.4.1.2. For the Laguerre-Gegenbauer case, we consider the product domain:

$$[0,\infty)\times[-1,1].$$

We present all the calculations that are needed for constructing the Sobolev orthogonal polynomials of total degree n in two variables with respect to the Sobolev inner product:

$$\langle f,g\rangle_S = c_{a,b}\int_{-1}^{1}\int_{0}^{\infty}\nabla^3 f(x,y)\cdot\nabla^3 g(x,y)W_{a,b}(x,y)dxdy+\lambda_2\nabla^2 f(p_1,p_2)\cdot\nabla^2 g(p_1,p_2)$$
$$+\lambda_1\nabla f(p_1,p_2)\cdot\nabla g(p_1,p_2)+\lambda_0 f(p_1,p_2)g(p_1,p_2), \quad (4.15)$$

where (p_1,p_2) is a point in $\mathbb{R}^2$, $\lambda_0,\lambda_1,\lambda_2>0$, ∇f, $\nabla^2 f$ and $\nabla^3 f$ are given in (4.1), $W_{a,b}$ is the Laguerre-Gegenbauer product weight:

$$W_{a,b}(x,y)=x^a e^{-x}(1-y^2)^{b-1/2},\quad a>-1,\quad b>-1/2,$$
$$(x,y)\in[0,\infty)\times[-1,1], \quad (4.16)$$

$c_{a,b}$ is the normalization constant:

$$c_{a,b}=\left(\int_{-1}^{1}\int_{0}^{\infty}W_{a,b}(x,y)dxdy\right)^{-1}=\frac{\Gamma(b+1)}{\Gamma(a+1)\Gamma(1/2)\Gamma(b+1/2)}, \quad (4.17)$$

and the main part of (4.15) is denoted by:

$$\langle f,g\rangle_{\nabla^3}=c_{a,b}\int_{-1}^{1}\int_{0}^{\infty}\nabla^3 f(x,y)\cdot\nabla^3 g(x,y)W_{a,b}(x,y)dxdy. \quad (4.18)$$

We denote by $\mathscr{V}_n^2(S,W_{a,b})$ and $\mathscr{V}_n^2(\nabla^3,W_{a,b})$ the spaces of orthogonal polynomials of degree n with respect to (4.15) and (4.18), respectively, for which $\dim\mathscr{V}_n^2(S,W_{a,b})=\dim\mathscr{V}_n^2(\nabla^3,W_{a,b})=n+1$. The following corollary is a consequence of Theorem 3.1 and Proposition 3.9 for $\kappa=3$ and $d=2$.

Corollary 4.2. *Let $\left\{S_j^n:0\le j\le n\right\}$ denote a monic orthogonal basis of $\mathscr{V}_n^2(\nabla^3,W_{a,b})$. Then, a monic orthogonal basis $\left\{\mathcal{S}_j^n:0\le j\le n\right\}$ of $\mathscr{V}_n^2(S,W_{a,b})$ is given by:*

$$\mathcal{S}_0^0(x,y)=1,$$
$$\mathcal{S}_0^1(x,y)=x-p_1,\ \mathcal{S}_1^1(x,y)=y-p_2,$$
$$\mathcal{S}_0^2(x,y)=(x-p_1)^2,\ \mathcal{S}_1^2(x,y)=(x-p_1)(y-p_2),\ \mathcal{S}_2^2(x,y)=(y-p_2)^2,$$
$$\mathcal{S}_j^n(x,y)=S_j^n(x,y)-\mathcal{T}^2(S_j^n,\mathbf{p};x,y),\quad 0\le j\le n,\quad n\ge 3,$$

where $\mathcal{T}^2(S_j^n,\mathbf{p};x,y)$ denotes the Taylor polynomial of second degree in two variables of S_j^n at $\mathbf{p}=(p_1,p_2)$, and where

$$\left\langle\mathcal{S}_0^0,\mathcal{S}_0^0\right\rangle_S=\lambda_0,$$

$$\langle \mathcal{S}_0^1, \mathcal{S}_0^1 \rangle_S = \langle \mathcal{S}_1^1, \mathcal{S}_1^1 \rangle_S = \lambda_1,$$
$$\langle \mathcal{S}_0^2, \mathcal{S}_0^2 \rangle_S = 4\lambda_2, \quad \langle \mathcal{S}_1^2, \mathcal{S}_1^2 \rangle_S = 2\lambda_2, \quad \langle \mathcal{S}_2^2, \mathcal{S}_2^2 \rangle_S = 4\lambda_2,$$
$$\langle \mathcal{S}_j^n, \mathcal{S}_j^n \rangle_S = \langle S_j^n, S_j^n \rangle_{\nabla^3}, \quad 0 \leq j \leq n, \quad n \geq 3.$$

Then, we need to construct a monic orthogonal basis $\{S_j^n : 0 \leq j \leq n\}$ for the space $\mathscr{V}_n^2(\nabla^3, W_{a,b})$, where each S_j^n is of the form $S_j^n(x,y) = x^{n-j}y^j + R(x,y)$, with $R \in \Pi_{n-1}^2$. We denote by $\mathbb{S}_n$ the column vector:

$$\mathbb{S}_n = \Big(S_0^n(x,y), \quad S_1^n(x,y), \quad \ldots, \quad S_n^n(x,y) \Big)^T.$$

For this construction, let us consider the monic sequences of Laguerre $\left\{ L_n^{(a)}(x) \right\}_{n\geq 0}$, $a > -1$, and Gegenbauer[3] $\{C_n^b(y)\}_{n\geq 0}$, $b > -1/2$, $b \neq 0$, orthogonal polynomials [111, Chapter 4 and 5]. These two sequences are orthogonal with respect to the inner products:

$$\langle L_n^{(a)}, L_m^{(b)} \rangle_{w_a} = \frac{1}{\Gamma(a+1)} \int_0^\infty L_n^{(a)}(x) L_m^{(a)}(x) w_a(x) dx = h_n(w_a)\delta_{n,m},$$
$$\langle C_n^b, C_m^b \rangle_{u_b} = \frac{\Gamma(b+1)}{\Gamma(1/2)\Gamma(b+1/2)} \int_{-1}^1 C_n^b(y) C_m^b(y) u_b(y) dy = h_n(u_b)\delta_{n,m},$$

where w_a and u_b are the Laguerre and Gegenbauer weight functions:

$$w_a(x) = x^a e^{-x}, \quad x \in [0,\infty), \quad a > -1,$$
$$u_b(y) = (1-y^2)^{b-1/2}, \quad y \in [-1,1], \quad b > -1/2,$$

and their L^2 norms are given, respectively, by:

$$h_n(w_a) = \langle L_n^{(a)}, L_n^{(a)} \rangle_{w_a} = n!(a+1)_n, \quad h_n(u_b) = \langle C_n^b, C_n^b \rangle_{u_b} = \frac{n!(2b)_n}{4^n (b)_n (b+1)_n}.$$

By convention, we take $L_n^{(a)} = C_n^b = 0$ if $n < 0$, and consequently, $h_n(w_a) = h_n(u_b) = 0$ if $n < 0$. The monic sequences $\left\{ L_n^{(a)}(x) \right\}_{n\geq 0}$ and $\{C_n^b(y)\}_{n\geq 0}$ are self-coherent [111, (4.7.29), (5.1.13)–(5.1.14)] with the relations:

$$L_n^{(a)}(x) = \frac{1}{n+1} \frac{d}{dx} L_{n+1}^{(a)}(x) + \frac{d}{dx} L_n^{(a)}(x), \quad n \geq 0, \tag{4.19}$$
$$C_n^b(y) = \frac{1}{n+1} \frac{d}{dy} C_{n+1}^b(y) + b_n(u_b) \frac{d}{dy} C_{n-1}^b(y), \quad n \geq 1, \tag{4.20}$$

where

$$b_n(u_b) = -\frac{n}{4(n+b)(n+b-1)}, \quad n \geq 1. \tag{4.21}$$

[3]For $b = 0$ the Gegenbauer polynomials vanish identically for $n \geq 1$. This case must be treated separately. See [111, page 80] for more details.

From the monic sequences $\left\{L_n^{(a)}(x)\right\}_{n\geq 0}$ and $\left\{C_n^b(y)\right\}_{n\geq 0}$ we define the monic product polynomials of total degree n in two variables:

$$P_j^n(x,y) = L_{n-j}^{(a)}(x)C_j^b(y), \quad 0 \leq j \leq n, \quad n \geq 0, \tag{4.22}$$

which are mutually orthogonal with respect to the inner product:

$$\langle f, g\rangle_{W_{a,b}} = c_{a,b}\int_{-1}^{1}\int_{0}^{\infty} f(x,y)g(x,y)W_{a,b}(x,y)dxdy, \tag{4.23}$$

where $W_{a,b}$ and $c_{a,b}$ are given in (4.16) and (4.17), respectively. That is,

$$\left\langle P_j^n, P_l^m\right\rangle_{W_{a,b}} = h_j^n \delta_{n,m}\delta_{j,l}, \tag{4.24}$$

where the L^2 norm is:

$$h_j^n = \left\langle P_j^n, P_j^n\right\rangle_{W_{a,b}} = h_{n-j}(w_a)h_j(u_b) = \frac{j!(n-j)!(2b)_j(a+1)_{n-j}}{4^j(b)_j(b+1)_j}, \quad 0 \leq j \leq n. \tag{4.25}$$

The set $\left\{P_j^n : 0 \leq j \leq n\right\}$ forms a mutually orthogonal basis [44, proposition 2.2.1] for the space $\mathscr{V}_n^2(W_{a,b})$ of orthogonal polynomials with respect to (4.23).

Note 4.3. The Laguerre-Gegenbauer polynomials (4.22) also satisfy a partial differential equation, but the eigenvalues depend on both n and j. In this case, $\mathscr{V}_n^2(W_{a,b})$ is not an eigenspace of such a differential operator. See [110, page 41] for more details.

Now, we use the results from Section 3.4.1.2 because $W_{a,b}$ is a product of classical weights. From (3.57) and (3.58) with $\kappa = 3$, the monic sequence of polynomials $\{q_n(w;x)\}_{n\geq 0}$ is defined by:

$$\begin{aligned} q_n(w;x) &= (n-2)_3[\gamma_3^{n-3,3}(w)p_n(w;x) + \gamma_2^{n-3,3}(w)p_{n-1}(w;x) \\ &+ \gamma_1^{n-3,3}(w)p_{n-2}(w;x) + \gamma_0^{n-3,3}(w)p_{n-3}(w;x) + \gamma_{-1}^{n-3,3}(w)p_{n-4}(w;x) \\ &+ \gamma_{-2}^{n-3,3}(w)p_{n-5}(w;x) + \gamma_{-3}^{n-3,3}(w)p_{n-6}(w;x)], \end{aligned} \tag{4.26}$$

where $\{p_n(w;x)\}_{n\geq 0}$ is a self-coherent monic sequence of orthogonal polynomials with respect to w, and where $q_n'(w)$, $q_n''(w)$ and $q_n'''(w)$ are given by:

$$\begin{aligned} q_n'(w;x) &= (n-2)_3[\gamma_2^{n-3,2}(w)p_{n-1}(w;x) + \gamma_1^{n-3,2}(w)p_{n-2}(w;x) \\ &+ \gamma_0^{n-3,2}(w)p_{n-3}(w;x) + \gamma_{-1}^{n-3,2}(w)p_{n-4}(w;x) + \gamma_{-2}^{n-3,2}(w)p_{n-5}(w;x)], \end{aligned} \tag{4.27}$$

$$\begin{aligned} q_n''(w;x) &= (n-1)_2 p_{n-2}(w;x) + (n-2)_3 a_{n-3}(w)p_{n-3}(w;x) \\ &+ (n-2)_3 b_{n-3}(w)p_{n-4}(w;x), \end{aligned} \tag{4.28}$$

$$q_n'''(w;x) = (n-2)_3 p_{n-3}(w;x). \tag{4.29}$$

From (3.54) we have the following constants in (4.26) and (4.27) in terms of $a_n(w)$ and $b_n(w)$:

$$
\begin{aligned}
\gamma_3^{n,3}(w) &= \frac{1}{(n+1)_3}, \quad \gamma_2^{n,3}(w) = \frac{a_{n+2}(w)+a_{n+1}(w)+a_n(w)}{(n+1)_2},\\
\gamma_1^{n,3}(w) &= \frac{a_{n+1}^2(w)+a_{n+1}(w)a_n(w)+a_n^2(w)}{n+1} + \frac{b_{n+2}(w)}{(n+1)_2} + \frac{b_{n+1}(w)}{(n+1)^2} + \frac{b_n(w)}{(n)_2},\\
\gamma_0^{n,3}(w) &= (a_{n+1}(w)+2a_n(w))\frac{b_{n+1}(w)}{n+1} + a_n^3(w) + (2a_n(w)+a_{n-1}(w))\frac{b_n(w)}{n},\\
\gamma_{-1}^{n,3}(w) &= nb_n(w)\gamma_1^{n-1,3}(w), \quad \gamma_{-2}^{n,3}(w) = (n-1)_2 b_n(w)b_{n-1}(w)\gamma_2^{n-2,3}(w),\\
\gamma_{-3}^{n,3}(w) &= b_n(w)b_{n-1}(w)b_{n-2}(w), \quad \gamma_2^{n,2}(w) = \frac{1}{(n+1)_2},\\
\gamma_1^{n,2}(w) &= \frac{a_{n+1}(w)+a_n(w)}{n+1}, \quad \gamma_0^{n,2}(w) = a_n^2(w) + \frac{b_{n+1}(w)}{n+1} + \frac{b_n(w)}{n},\\
\gamma_{-1}^{n,2}(w) &= (a_n(w)+a_{n-1}(w))b_n(w), \quad \gamma_{-2}^{n,2}(w) = b_n(w)b_{n-1}(w).
\end{aligned}
$$

In particular, the Laguerre and Gegenbauer (monic) orthogonal polynomials are self-coherent with the relations (4.19) and (4.20), respectively. From (4.19) and (4.20) we get the following constants:

$$a_n(w_a) = 1, \qquad b_n(w_a) = 0, \tag{4.30}$$

$$a_n(u_b) = 0, \qquad b_n(u_b) = -\frac{n}{4(n+b)(n+b-1)}. \tag{4.31}$$

Therefore, from the previous discussion we have proved the following two propositions for the Laguerre and Gegenbauer cases.

Proposition 4.6. *The monic sequence of polynomials* $\{q_n(w_a;x)\}_{n\geq 0}$, $a>-1$, *which is defined in* (3.58), *and its derivatives up to third-order, are given in the Laguerre case by:*

$$
\begin{aligned}
q_n(w_a;x) &= L_n^{(a)}(x) + 3nL_{n-1}^{(a)}(x) + 3(n-1)_2 L_{n-2}^{(a)}(x) + (n-2)_3 L_{n-3}^{(a)}(x),\\
q_n'(w_a;x) &= nL_{n-1}^{(a)}(x) + 2(n-1)_2 L_{n-2}^{(a)}(x) + (n-2)_3 L_{n-3}^{(a)}(x),\\
q_n''(w_a;x) &= (n-1)_2 L_{n-2}^{(a)}(x) + (n-2)_3 L_{n-3}^{(a)}(x),\\
q_n'''(w_a;x) &= (n-2)_3 L_{n-3}^{(a)}(x).
\end{aligned}
$$

Proposition 4.7. *The monic sequence of polynomials* $\{q_n(u_b;y)\}_{n\geq 0}$, $b>-1/2$, *which is defined in* (3.58), *and its derivatives up to third-order, are given in the Gegenbauer case by:*

$$
\begin{aligned}
q_n(u_b;y) = C_n^b(y) &- \left[\frac{3(n-1)_2(n+b-3)_2}{4(n+b-4)_4}\right] C_{n-2}^b(y) + \left[\frac{3(n-3)_4}{16(n+b-5)_4}\right] C_{n-4}^b(y)\\
&- \left[\frac{(n-5)_6}{64(n+b-6)_4(n+b-5)_2}\right] C_{n-6}^b(y),
\end{aligned}
$$

$$
\begin{aligned}
q_n'(u_b;y) &= nC_{n-1}^b(y) - \left[\frac{(n-2)_3(n+b-3)}{2(n+b-4)_3}\right] C_{n-3}^b(y) \\
&\quad + \left[\frac{(n-4)_5}{16(n+b-4)(n+b-5)_3}\right] C_{n-5}^b(y), \\
q_n''(u_b;y) &= (n-1)_2 C_{n-2}^b(y) - \left[\frac{(n-3)_4}{4(n+b-4)_2}\right] C_{n-4}^b(y), \\
q_n'''(u_b;y) &= (n-2)_3 C_{n-3}^b(y).
\end{aligned}
$$

If $\{q_n(w_a;x)\}_{n\geq 0}$ and $\{q_n(u_b;y)\}_{n\geq 0}$ denote the monic sequences in Proposition 4.6 and Proposition 4.7, respectively, we define the product polynomial Q_j^n in two variables by:

$$
Q_j^n(x,y) = q_{n-j}(w_a;x)q_j(u_b;y), \quad 0 \leq j \leq n, \quad n \geq 0, \tag{4.32}
$$

which is a monic polynomial of total degree n. We denote by $\mathbb{Q}_n$ the column vector:

$$
\mathbb{Q}_n = \Big(Q_0^n(x,y), \quad Q_1^n(x,y), \quad \ldots, \quad Q_n^n(x,y)\Big)^T.
$$

We have the following proposition concerning to Q_j^n.

Proposition 4.8. *The third-order partial derivatives $\partial_1^3 Q_j^n$, $\partial_1^2\partial_2 Q_j^n$, $\partial_1\partial_2^2 Q_j^n$, $\partial_2^3 Q_j^n$ of the polynomial Q_j^n, $0 \leq j \leq n$, $n \geq 0$, in the Laguerre-Gegenbauer case are given by:*

$$
\begin{aligned}
\partial_1^3 Q_j^n(x,y) &= (n-j-2)_3 P_j^{n-3}(x,y) \\
&\quad - \left[\frac{3(n-j-2)_3(j-1)_2(j+b-3)_2}{4(j+b-4)_4}\right] P_{j-2}^{n-5}(x,y) \\
&\quad + \left[\frac{3(n-j-2)_3(j-3)_4}{16(j+b-5)_4}\right] P_{j-4}^{n-7}(x,y) \\
&\quad - \left[\frac{(n-j-2)_3(j-5)_6}{64(j+b-6)_4(j+b-5)_2}\right] P_{j-6}^{n-9}(x,y), \\
\partial_1^2\partial_2 Q_j^n(x,y) &= j(n-j-1)_2 P_{j-1}^{n-3}(x,y) + j(n-j-2)_3 P_{j-1}^{n-4}(x,y) \\
&\quad - \left[\frac{(n-j-1)_2(j-2)_3(j+b-3)}{2(j+b-4)_3}\right] P_{j-3}^{n-5}(x,y) \\
&\quad - \left[\frac{(n-j-2)_3(j-2)_3(j+b-3)}{2(j+b-4)_3}\right] P_{j-3}^{n-6}(x,y) \\
&\quad + \left[\frac{(n-j-1)_2(j-4)_5}{16(j+b-4)(j+b-5)_3}\right] P_{j-5}^{n-7}(x,y) \\
&\quad + \left[\frac{(n-j-2)_3(j-4)_5}{16(j+b-4)(j+b-5)_3}\right] P_{j-5}^{n-8}(x,y), \\
\partial_1\partial_2^2 Q_j^n(x,y) &= (n-j)(j-1)_2 P_{j-2}^{n-3}(x,y) + 2(n-j-1)_2(j-1)_2 P_{j-2}^{n-4}(x,y) \\
&\quad + (n-j-2)_3(j-1)_2 P_{j-2}^{n-5}(x,y) - \left[\frac{(n-j)(j-3)_4}{4(j+b-4)_2}\right] P_{j-4}^{n-5}(x,y)
\end{aligned}
$$

$$- \left[\frac{(n-j-1)_2(j-3)_4}{2(j+b-4)_2}\right] P_{j-4}^{n-6}(x,y) - \left[\frac{(n-j-2)_3(j-3)_4}{4(j+b-4)_2}\right] P_{j-4}^{n-7}(x,y),$$

$$\partial_2^3 Q_j^n(x,y) = (j-2)_3 P_{j-3}^{n-3}(x,y) + 3(n-j)(j-2)_3 P_{j-3}^{n-4}(x,y) + 3(n-j-1)_2(j-2)_3 P_{j-3}^{n-5}(x,y) + (n-j-2)_3(j-2)_3 P_{j-3}^{n-6}(x,y),$$

where P_j^n is given in (4.22).

Proof. From (4.32) we have that $\partial_1^3 Q_j^n = q_{n-j}'''(w_a;x) q_j(u_b;y)$. Then, we use Proposition 4.6 and Proposition 4.7 and we express the result in terms of P_j^n in (4.22). The other expressions follow similarly. □

Proposition 3.11 shows that it is necessary to know explicitly the $n-3$ rectangular matrices $\langle \mathbb{Q}_n, \mathbb{Q}_i^T \rangle_{\nabla^3}$ of size $(n+1)\times(i+1)$, $3 \le i \le n-1$, and also the $n-3$ square matrices $\langle \mathbb{Q}_i, \mathbb{Q}_i^T \rangle_{\nabla^3}$ of size $(i+1)\times(i+1)$, $3 \le i \le n-1$. For this purpose, it is necessary to know explicitly their entries $\langle Q_j^n, Q_l^m \rangle_{\nabla^3}$ for $0 \le j \le n$, $0 \le l \le m$, $n, m \ge 0$. These are our next two propositions.

Proposition 4.9. *Let $a > -1$, $b > -1/2$ and let h_j^n be given in (4.25). Then $\langle Q_j^n, Q_l^m \rangle_{\nabla^3}$, $0 \le j \le n$, $0 \le l \le m$, $n, m \ge 0$, is given by:*

$$\begin{aligned}
\langle Q_j^n, Q_l^m \rangle_{\nabla^3} &= \omega_{n+6,j+6}^{a,b}\delta_{j,l-6}\delta_{n,m-6} + \psi_{n+5,j+4}^{a,b}\delta_{j,l-4}\delta_{n,m-5} + \chi_{n+4,j+4}^{a,b}\delta_{j,l-4}\delta_{n,m-4} \\
&+ \phi_{n+4,j+2}^{a,b}\delta_{j,l-2}\delta_{n,m-4} + \tau_{n+3,j+4}^{a,b}\delta_{j,l-4}\delta_{n,m-3} + \sigma_{n+3,j+2}^{a,b}\delta_{j,l-2}\delta_{n,m-3} + \rho_{n+3,j}^{a,b}\delta_{j,l}\delta_{n,m-3} \\
&+ \pi_{n+2,j+2}^{a,b}\delta_{j,l-2}\delta_{n,m-2} + \xi_{n+2,j}^{a,b}\delta_{j,l}\delta_{n,m-2} + \nu_{n+1,j+2}^{a,b}\delta_{j,l-2}\delta_{n,m-1} + \mu_{n+1,j}^{a,b}\delta_{j,l}\delta_{n,m-1} \\
&+ \kappa_{n,j+2}^{a,b}\delta_{j,l-2}\delta_{n,m} + \theta_{n,j}^{a,b}\delta_{j,l}\delta_{n,m} + \kappa_{n,j}^{a,b}\delta_{j,l+2}\delta_{n,m} + \mu_{n,j}^{a,b}\delta_{j,l}\delta_{n,m+1} + \nu_{n,j}^{a,b}\delta_{j,l+2}\delta_{n,m+1} \\
&+ \xi_{n,j}^{a,b}\delta_{j,l}\delta_{n,m+2} + \pi_{n,j}^{a,b}\delta_{j,l+2}\delta_{n,m+2} + \rho_{n,j}^{a,b}\delta_{j,l}\delta_{n,m+3} + \sigma_{n,j}^{a,b}\delta_{j,l+2}\delta_{n,m+3} + \tau_{n,j}^{a,b}\delta_{j,l+4}\delta_{n,m+3} \\
&+ \phi_{n,j}^{a,b}\delta_{j,l+2}\delta_{n,m+4} + \chi_{n,j}^{a,b}\delta_{j,l+4}\delta_{n,m+4} + \psi_{n,j}^{a,b}\delta_{j,l+4}\delta_{n,m+5} + \omega_{n,j}^{a,b}\delta_{j,l+6}\delta_{n,m+6},
\end{aligned}$$

where:

$$\begin{aligned}
\theta_{n,j}^{a,b} &= (n-j-2)_3^2 h_j^{n-3} + \left[\frac{9(n-j-2)_3^2(j-1)_2^2(j+b-3)_2^2}{16(j+b-4)_4^2}\right] h_{j-2}^{n-5} \\
&+ \left[\frac{9(n-j-2)_3^2(j-3)_4^2}{256(j+b-5)_4^2}\right] h_{j-4}^{n-7} + \left[\frac{(n-j-2)_3^2(j-5)_6^2}{4096(j+b-6)_4^2(j+b-5)_2^2}\right] h_{j-6}^{n-9} \\
&+ 3j^2(n-j-1)_2^2 h_{j-1}^{n-3} + 3j^2(n-j-2)_3^2 h_{j-1}^{n-4} \\
&+ \left[\frac{3(n-j-1)_2^2(j-2)_3^2(j+b-3)^2}{4(j+b-4)_3^2}\right] h_{j-3}^{n-5} \\
&+ \left[\frac{3(n-j-2)_3^2(j-2)_3^2(j+b-3)^2}{4(j+b-4)_3^2}\right] h_{j-3}^{n-6} + \left[\frac{3(n-j-1)_2^2(j-4)_5^2}{256(j+b-4)^2(j+b-5)_3^2}\right] h_{j-5}^{n-7} \\
&+ \left[\frac{3(n-j-2)_3^2(j-4)_5^2}{256(j+b-4)^2(j+b-5)_3^2}\right] h_{j-5}^{n-8} + 3(n-j)^2(j-1)_2^2 h_{j-2}^{n-3} \\
&+ 12(n-j-1)_2^2(j-1)_2^2 h_{j-2}^{n-4} + 3(n-j-2)_3^2(j-1)_2^2 h_{j-2}^{n-5} \\
&+ \left[\frac{3(n-j)^2(j-3)_4^2}{16(j+b-4)_2^2}\right] h_{j-4}^{n-5} + \left[\frac{3(n-j-1)_2^2(j-3)_4^2}{4(j+b-4)_2^2}\right] h_{j-4}^{n-6}
\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{3(n-j-2)_3^2(j-3)_4^2}{16(j+b-4)_2^2}\right]h_{j-4}^{n-7}+(j-2)_3^2h_{j-3}^{n-3}+9(n-j)^2(j-2)_3^2h_{j-3}^{n-4}\\
&+9(n-j-1)_2^2(j-2)_3^2h_{j-3}^{n-5}+(n-j-2)_3^2(j-2)_3^2h_{j-3}^{n-6},\\
\kappa_{n,j}^{a,b}&=-\left[\frac{3(n-j)^2(n-j+1)_2(j-3)_2^2(j-1)_2}{4(j+b-4)_2}\right]h_{j-4}^{n-5},\\
\mu_{n,j}^{a,b}&=3j^2(n-j-2)_2^2(n-j)h_{j-1}^{n-4}+\left[\frac{3(n-j-2)_2^2(n-j)(j-2)_3^2(j+b-3)^2}{4(j+b-4)_3^2}\right]h_{j-3}^{n-6}\\
&+\left[\frac{3(n-j-2)_2^2(n-j)(j-4)_5^2}{256(j+b-4)^2(j+b-5)_3^2}\right]h_{j-5}^{n-8}+6(n-j-1)^2(n-j)(j-1)_2^2h_{j-2}^{n-4}\\
&+6(n-j-2)_2^2(n-j)(j-1)_2^2h_{j-2}^{n-5}+\left[\frac{3(n-j-1)^2(n-j)(j-3)_4^2}{8(j+b-4)_2^2}\right]h_{j-4}^{n-6}\\
&+\left[\frac{3(n-j-2)_2^2(n-j)(j-3)_4^2}{8(j+b-4)_2^2}\right]h_{j-4}^{n-7}+3(n-j)(j-2)_3^2h_{j-3}^{n-4}\\
&+9(n-j-1)^2(n-j)(j-2)_3^2h_{j-3}^{n-5}+3(n-j-2)_2^2(n-j)(j-2)_3^2h_{j-3}^{n-6},\\
\nu_{n,j}^{a,b}&=-\left[\frac{3(n-j-1)_2^2(n-j+1)(j-2)^2(j-1)_2(j+b-3)}{2(j+b-4)_3}\right]h_{j-3}^{n-5}\\
&-\left[\frac{3(n-j-1)_2^2(n-j+1)(j-4)_3^2(j-1)_2}{32(j+b-4)^2(j+b-6)_4}\right]h_{j-5}^{n-7}\\
&-\left[\frac{3(n-j)^2(n-j+1)(j-3)_2^2(j-1)_2}{2(j+b-4)_2}\right]h_{j-4}^{n-5}\\
&-\left[\frac{3(n-j-1)_2^2(n-j+1)(j-3)_2^2(j-1)_2}{2(j+b-4)_2}\right]h_{j-4}^{n-6},\\
\xi_{n,j}^{a,b}&=3(n-j-2)^2(n-j-1)_2(j-1)_2^2h_{j-2}^{n-5}\\
&+\left[\frac{3(n-j-2)^2(n-j-1)_2(j-3)_4^2}{16(j+b-4)_2^2}\right]h_{j-4}^{n-7}\\
&+3(n-j-1)_2(j-2)_3^2h_{j-3}^{n-5}+3(n-j-2)^2(n-j-1)_2(j-2)_3^2h_{j-3}^{n-6},\\
\pi_{n,j}^{a,b}&=-\left[\frac{3(n-j-2)_3^2(j-1)_2(j+b-3)_2}{4(j+b-4)_4}\right]h_{j-2}^{n-5}\\
&-\left[\frac{9(n-j-2)_3^2(j-3)_2^2(j-1)_2}{64(j+b-3)_2(j+b-6)_4}\right]h_{j-4}^{n-7}\\
&-\left[\frac{3(n-j-2)_3^2(j-5)_4^2(j-1)_2}{1024(j+b-6)_4(j+b-5)_2(j+b-7)_4}\right]h_{j-6}^{n-9}\\
&-\left[\frac{3(n-j-1)_2^2(j-2)^2(j-1)_2(j+b-3)}{2(j+b-4)_3}\right]h_{j-3}^{n-5}\\
&-\left[\frac{3(n-j-2)_3^2(j-2)^2(j-1)_2(j+b-3)}{2(j+b-4)_3}\right]h_{j-3}^{n-6}\\
&-\left[\frac{3(n-j-1)_2^2(j-4)_3^2(j-1)_2}{32(j+b-4)^2(j+b-6)_4}\right]h_{j-5}^{n-7}-\left[\frac{3(n-j-2)_3^2(j-4)_3^2(j-1)_2}{32(j+b-4)^2(j+b-6)_4}\right]h_{j-5}^{n-8}
\end{aligned}
$$

$$
\begin{aligned}
&- \left[\frac{3(n-j)^2(j-3)_2^2(j-1)_2}{4(j+b-4)_2}\right] h_{j-4}^{n-5} \\
&- \left[\frac{3(n-j-1)_2^2(j-3)_2^2(j-1)_2}{(j+b-4)_2}\right] h_{j-4}^{n-6} - \left[\frac{3(n-j-2)_3^2(j-3)_2^2(j-1)_2}{4(j+b-4)_2}\right] h_{j-4}^{n-7},
\end{aligned}
$$

$$
\rho_{n,j}^{a,b} = (n-j-2)_3(j-2)_3^2 h_{j-3}^{n-6},
$$

$$
\begin{aligned}
\sigma_{n,j}^{a,b} = &- \left[\frac{3(n-j-2)_2^2(n-j)(j-2)^2(j-1)_2(j+b-3)}{2(j+b-4)_3}\right] h_{j-3}^{n-6} \\
&- \left[\frac{3(n-j-2)_2^2(n-j)(j-4)_3^2(j-1)_2}{32(j+b-4)^2(j+b-6)_4}\right] h_{j-5}^{n-8} \\
&- \left[\frac{3(n-j-1)^2(n-j)(j-3)_2^2(j-1)_2}{2(j+b-4)_2}\right] h_{j-4}^{n-6} \\
&- \left[\frac{3(n-j-2)_2^2(n-j)(j-3)_2^2(j-1)_2}{2(j+b-4)_2}\right] h_{j-4}^{n-7},
\end{aligned}
$$

$$
\tau_{n,j}^{a,b} = \left[\frac{3(n-j-1)_2^2(n-j+1)(j-4)^2(j-3)_4}{16(j+b-4)(j+b-5)_3}\right] h_{j-5}^{n-7},
$$

$$
\phi_{n,j}^{a,b} = -\left[\frac{3(n-j-2)^2(n-j-1)_2(j-3)_2^2(j-1)_2}{4(j+b-4)_2}\right] h_{j-4}^{n-7},
$$

$$
\begin{aligned}
\chi_{n,j}^{a,b} = &\left[\frac{3(n-j-2)_3^2(j-3)_4}{16(j+b-5)_4}\right] h_{j-4}^{n-7} + \left[\frac{3(n-j-2)_3^2(j-5)_2^2(j-3)_4(j+b-7)_2}{256(j+b-6)_4(j+b-5)_2(j+b-8)_4}\right] h_{j-6}^{n-9} \\
&+ \left[\frac{3(n-j-1)_2^2(j-4)^2(j-3)_4}{16(j+b-4)(j+b-5)_3}\right] h_{j-5}^{n-7} + \left[\frac{3(n-j-2)_3^2(j-4)^2(j-3)_4}{16(j+b-4)(j+b-5)_3}\right] h_{j-5}^{n-8},
\end{aligned}
$$

$$
\psi_{n,j}^{a,b} = \left[\frac{3(n-j-2)_2^2(n-j)(j-4)^2(j-3)_4}{16(j+b-4)(j+b-5)_3}\right] h_{j-5}^{n-8},
$$

$$
\omega_{n,j}^{a,b} = -\left[\frac{(n-j-2)_3^2(j-5)_6}{64(j+b-6)_4(j+b-5)_2}\right] h_{j-6}^{n-9}.
$$

Proof. From (3.15), with $\kappa = 3$ and $d = 2$, we have that:

$$
\begin{aligned}
\left\langle Q_j^n, Q_l^m \right\rangle_{\nabla^3} = \left\langle \partial_1^3 Q_j^n, \partial_1^3 Q_l^m \right\rangle_{W_{a,b}} + 3\left\langle \partial_1^2\partial_2 Q_j^n, \partial_1^2\partial_2 Q_l^m \right\rangle_{W_{a,b}} + \\
3\left\langle \partial_1\partial_2^2 Q_j^n, \partial_1\partial_2^2 Q_l^m \right\rangle_{W_{a,b}} + \left\langle \partial_2^3 Q_j^n, \partial_2^3 Q_l^m \right\rangle_{W_{a,b}}. \quad (4.33)
\end{aligned}
$$

Then, by the linearity of $\langle \cdot, \cdot \rangle_{W_{a,b}}$, Proposition 4.8, (4.24) and (4.25), we compute each term in (4.33). We have the result by adding and simplifying. □

Proposition 4.10. *Let $3 \le i \le n$ and let $\theta_{n,j}^{a,b}$, $\kappa_{n,j}^{a,b}$, $\mu_{n,j}^{a,b}$, $\nu_{n,j}^{a,b}$, $\xi_{n,j}^{a,b}$, $\pi_{n,j}^{a,b}$, $\rho_{n,j}^{a,b}$, $\sigma_{n,j}^{a,b}$, $\tau_{n,j}^{a,b}$, $\phi_{n,j}^{a,b}$, $\chi_{n,j}^{a,b}$, $\psi_{n,j}^{a,b}$ and $\omega_{n,j}^{a,b}$ be given in Proposition 4.9, with $a > -1$ and $b > -1/2$. Then $\left\langle \mathbb{Q}_n, \mathbb{Q}_i^T \right\rangle_{\nabla^3} = \mathbf{0}$ for $3 \le i \le n-7$, and $\left\langle \mathbb{Q}_n, \mathbb{Q}_i^T \right\rangle_{\nabla^3}$ is given for $n-6 \le i \le n$ by:*

1. *$\left\langle \mathbb{Q}_n, \mathbb{Q}_n^T \right\rangle_{\nabla^3} = \left(a_{j,l}^{a,b}\right)_{0 \le j,l \le n}$, $n \ge 3$, is a symmetric tridiagonal matrix of size*

$(n+1)\times(n+1)$ where its entries are given by:

$$a_{j,l}^{a,b} = \begin{cases} \theta_{n,j}^{a,b}, & l=j, \quad 0\le j\le n, \\ \kappa_{n,j+2}^{a,b}, & l=j+2, \quad 0\le j\le n-2, \\ \kappa_{n,j}^{a,b}, & l=j-2, \quad 2\le j\le n, \\ 0, & \textit{otherwise}. \end{cases} \tag{4.34}$$

2. $\left\langle \mathbb{Q}_n, \mathbb{Q}_{n-1}^T \right\rangle_{\nabla^3} = \left(b_{j,l}^{a,b} \right)_{0\le j\le n, 0\le l\le n-1}$, *$n\ge 4$, is a bidiagonal matrix of size $(n+1)\times n$ where its entries are given by:*

$$b_{j,l}^{a,b} = \begin{cases} \mu_{n,j}^{a,b}, & l=j, \quad 0\le j\le n-1, \\ \nu_{n,j}^{a,b}, & l=j-2, \quad 2\le j\le n, \\ 0, & \textit{otherwise}. \end{cases} \tag{4.35}$$

3. $\left\langle \mathbb{Q}_n, \mathbb{Q}_{n-2}^T \right\rangle_{\nabla^3} = \left(c_{j,l}^{a,b} \right)_{0\le j\le n, 0\le l\le n-2}$, *$n\ge 5$, is a bidiagonal matrix of size $(n+1)\times(n-1)$ where its entries are given by:*

$$c_{j,l}^{a,b} = \begin{cases} \xi_{n,j}^{a,b}, & l=j, \quad 0\le j\le n-2, \\ \pi_{n,j}^{a,b}, & l=j-2, \quad 2\le j\le n, \\ 0, & \textit{otherwise}. \end{cases} \tag{4.36}$$

4. $\left\langle \mathbb{Q}_n, \mathbb{Q}_{n-3}^T \right\rangle_{\nabla^3} = \left(d_{j,l}^{a,b} \right)_{0\le j\le n, 0\le l\le n-3}$, *$n\ge 6$, is a tridiagonal matrix of size $(n+1)\times(n-2)$ where its entries are given by:*

$$d_{j,l}^{a,b} = \begin{cases} \rho_{n,j}^{a,b}, & l=j, \quad 0\le j\le n-3, \\ \sigma_{n,j}^{a,b}, & l=j-2, \quad 2\le j\le n-1, \\ \tau_{n,j}^{a,b}, & l=j-4, \quad 4\le j\le n, \\ 0, & \textit{otherwise}. \end{cases} \tag{4.37}$$

5. $\left\langle \mathbb{Q}_n, \mathbb{Q}_{n-4}^T \right\rangle_{\nabla^3} = \left(e_{j,l}^{a,b} \right)_{0\le j\le n, 0\le l\le n-4}$, *$n\ge 7$, is a bidiagonal matrix of size $(n+1)\times(n-3)$ where its entries are given by:*

$$e_{j,l}^{a,b} = \begin{cases} \phi_{n,j}^{a,b}, & l=j-2, \quad 2\le j\le n-2, \\ \chi_{n,j}^{a,b}, & l=j-4, \quad 4\le j\le n, \\ 0, & \textit{otherwise}. \end{cases} \tag{4.38}$$

6. $\left\langle \mathbb{Q}_n, \mathbb{Q}_{n-5}^T \right\rangle_{\nabla^3} = \left(f_{j,l}^{a,b} \right)_{0\le j\le n, 0\le l\le n-5}$, *$n\ge 8$, is a diagonal matrix of size $(n+1)\times(n-4)$ where its entries are given by:*

$$f_{j,l}^{a,b} = \begin{cases} \psi_{n,j}^{a,b}, & l=j-4, \quad 4\le j\le n-1, \\ 0, & \textit{otherwise}. \end{cases} \tag{4.39}$$

7. $\left\langle \mathbb{Q}_n, \mathbb{Q}_{n-6}^T \right\rangle_{\nabla^3} = \left(g_{j,l}^{a,b} \right)_{0 \le j \le n, 0 \le l \le n-6}$, $n \ge 9$, *is a diagonal matrix of size* $(n+1) \times (n-5)$ *where its entries are given by:*

$$g_{j,l}^{a,b} = \begin{cases} \omega_{n,j}^{a,b}, & l = j-6, \quad 6 \le j \le n, \\ 0, & \text{otherwise}. \end{cases} \tag{4.40}$$

Proof. Let us recall that $\left\langle \mathbb{Q}_n, \mathbb{Q}_i^T \right\rangle_{\nabla^3} = \left(\left\langle Q_j^n, Q_l^i \right\rangle_{\nabla^3} \right)_{0 \le j \le n, 0 \le l \le i}$ is a matrix of size $(n+1) \times (i+1)$. By Proposition 4.9 we have that $\left\langle Q_j^n, Q_l^i \right\rangle_{\nabla^3} = 0$ for $3 \le i \le n-7$. By Proposition 4.9 we have for $n-6 \le i \le n$ that:

$$\begin{aligned}
\left\langle Q_j^n, Q_l^{n-6} \right\rangle_{\nabla^3} &= \omega_{n,j}^{a,b} \delta_{j,l+6}, \quad 0 \le j \le n, \ 0 \le l \le n-6, \ n \ge 9, \\
\left\langle Q_j^n, Q_l^{n-5} \right\rangle_{\nabla^3} &= \psi_{n,j}^{a,b} \delta_{j,l+4}, \quad 0 \le j \le n, \ 0 \le l \le n-5, \ n \ge 8, \\
\left\langle Q_j^n, Q_l^{n-4} \right\rangle_{\nabla^3} &= \phi_{n,j}^{a,b} \delta_{j,l+2} + \chi_{n,j}^{a,b} \delta_{j,l+4}, \quad 0 \le j \le n, \ 0 \le l \le n-4, \ n \ge 7, \\
\left\langle Q_j^n, Q_l^{n-3} \right\rangle_{\nabla^3} &= \rho_{n,j}^{a,b} \delta_{j,l} + \sigma_{n,j}^{a,b} \delta_{j,l+2} + \tau_{n,j}^{a,b} \delta_{j,l+4}, \quad 0 \le j \le n, \ 0 \le l \le n-3, \ n \ge 6, \\
\left\langle Q_j^n, Q_l^{n-2} \right\rangle_{\nabla^3} &= \xi_{n,j}^{a,b} \delta_{j,l} + \pi_{n,j}^{a,b} \delta_{j,l+2}, \quad 0 \le j \le n, \ 0 \le l \le n-2, \ n \ge 5, \\
\left\langle Q_j^n, Q_l^{n-1} \right\rangle_{\nabla^3} &= \mu_{n,j}^{a,b} \delta_{j,l} + \nu_{n,j}^{a,b} \delta_{j,l+2}, \quad 0 \le j \le n, \ 0 \le l \le n-1, \ n \ge 4, \\
\left\langle Q_j^n, Q_l^{n} \right\rangle_{\nabla^3} &= \kappa_{n,j+2}^{a,b} \delta_{j,l-2} + \theta_{n,j}^{a,b} \delta_{j,l} + \kappa_{n,j}^{a,b} \delta_{j,l+2}, \quad 0 \le j \le n, \ 0 \le l \le n, \ n \ge 3.
\end{aligned}$$

By setting $l = j \pm 2$, $l = j$, $l = j-4$, and $l = j-6$ we have the result. □

Example 4.2 (Numerical, see Note 4.1). Let $a = 0$ and $b = 1/2$ the parameters for the Laguerre-Gegenbauer weight function $W_{a,b}$, and let $\mathbf{p} = (0, 1)$. Let us observe that from Proposition 4.10 the matrices $\left\langle \mathbb{Q}_3, \mathbb{Q}_3^T \right\rangle_{\nabla^3}$, $\left\langle \mathbb{Q}_4, \mathbb{Q}_4^T \right\rangle_{\nabla^3}$, $\left\langle \mathbb{Q}_4, \mathbb{Q}_3^T \right\rangle_{\nabla^3}$, $\left\langle \mathbb{Q}_5, \mathbb{Q}_3^T \right\rangle_{\nabla^3}$, and $\left\langle \mathbb{Q}_5, \mathbb{Q}_4^T \right\rangle_{\nabla^3}$ are given by:

$$\left\langle \mathbb{Q}_3, \mathbb{Q}_3^T \right\rangle_{\nabla^3} = \begin{pmatrix} 36 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix}, \quad \left\langle \mathbb{Q}_4, \mathbb{Q}_4^T \right\rangle_{\nabla^3} = \begin{pmatrix} 576 & 0 & 0 & 0 & 0 \\ 0 & 228 & 0 & 0 & 0 \\ 0 & 0 & 256 & 0 & 0 \\ 0 & 0 & 0 & 396 & 0 \\ 0 & 0 & 0 & 0 & 192 \end{pmatrix},$$

$$\left\langle \mathbb{Q}_4, \mathbb{Q}_3^T \right\rangle_{\nabla^3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 36 & 0 & 0 \\ 0 & 0 & 48 & 0 \\ 0 & 0 & 0 & 108 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \left\langle \mathbb{Q}_5, \mathbb{Q}_3^T \right\rangle_{\nabla^3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 24 & 0 & 72 & 0 \\ 0 & 48 & 0 & 216 \\ 0 & 0 & -48 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\left\langle \mathbb{Q}_5, \mathbb{Q}_4^T \right\rangle_{\nabla^3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 432 & 0 & 0 & 0 \\ 0 & 0 & 624 & 0 & 0 \\ 0 & 144 & 0 & 1008 & 0 \\ 0 & 0 & -192 & 0 & 576 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From Proposition 3.11 we have recursively the following matrices:

$$\mathbf{H}_3^{\nabla^3} = \left\langle \mathbb{Q}_3, \mathbb{Q}_3^T \right\rangle_{\nabla^3} = \begin{pmatrix} 36 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix}, \quad \mathbf{B}_{4,3} = \left\langle \mathbb{Q}_4, \mathbb{Q}_3^T \right\rangle_{\nabla^3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 36 & 0 & 0 \\ 0 & 0 & 48 & 0 \\ 0 & 0 & 0 & 108 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{H}_4^{\nabla^3} = \left\langle \mathbb{Q}_4, \mathbb{Q}_4^T \right\rangle_{\nabla^3} - \mathbf{B}_{4,3}(\mathbf{H}_3^{\nabla^3})^{-1}\mathbf{B}_{4,3}^T = \begin{pmatrix} 576 & 0 & 0 & 0 & 0 \\ 0 & 120 & 0 & 0 & 0 \\ 0 & 0 & 64 & 0 & 0 \\ 0 & 0 & 0 & 72 & 0 \\ 0 & 0 & 0 & 0 & 192 \end{pmatrix},$$

$$\mathbf{B}_{5,3} = \left\langle \mathbb{Q}_5, \mathbb{Q}_3^T \right\rangle_{\nabla^3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 24 & 0 & 72 & 0 \\ 0 & 48 & 0 & 216 \\ 0 & 0 & -48 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{B}_{5,4} = \left\langle \mathbb{Q}_5, \mathbb{Q}_4^T \right\rangle_{\nabla^3} - \mathbf{B}_{5,3}(\mathbf{H}_3^{\nabla^3})^{-1}\mathbf{B}_{4,3}^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 432 & 0 & 0 & 0 \\ 0 & 0 & 336 & 0 & 0 \\ 0 & 0 & 0 & 360 & 0 \\ 0 & 0 & 0 & 0 & 576 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the matrices $\mathbf{A}_{4,3}$, $\mathbf{A}_{5,3}$ and $\mathbf{A}_{5,4}$ are given by:

$$\mathbf{A}_{4,3} = \mathbf{B}_{4,3}(\mathbf{H}_3^{\nabla^3})^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_{5,3} = \mathbf{B}_{5,3}(\mathbf{H}_3^{\nabla^3})^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 6 & 0 \\ 0 & 4 & 0 & 6 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{A}_{5,4} = \mathbf{B}_{5,4}(\mathbf{H}_4^{\nabla^3})^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{18}{5} & 0 & 0 & 0 \\ 0 & 0 & \frac{21}{4} & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From Proposition 4.6, Proposition 4.7 and (4.32) the column vectors $\mathbb{Q}_n$ for $n =$

0, 1, 2, 3, 4, 5 are give by:

$$\mathbb{Q}_0 = 1, \quad \mathbb{Q}_1 = \begin{pmatrix} x+2 \\ y \end{pmatrix}, \quad \mathbb{Q}_2 = \begin{pmatrix} x^2+2x+2 \\ xy+2y \\ y^2+\frac{1}{3} \end{pmatrix}, \quad \mathbb{Q}_3 = \begin{pmatrix} x^3 \\ x^2y+2xy+2y \\ xy^2+\frac{1}{3}x+2y^2+\frac{2}{3} \\ y^3+3y \end{pmatrix},$$

$$\mathbb{Q}_4 = \begin{pmatrix} x^4-4x^3 \\ x^3y \\ x^2y^2+\frac{1}{3}x^2+2xy^2+\frac{2}{3}x+2y^2+\frac{2}{3} \\ xy^3+6y+3xy+2y^3 \\ y^4-6y^2-3 \end{pmatrix},$$

$$\mathbb{Q}_5 = \begin{pmatrix} x^5-10x^4+20x^3 \\ x^4y-4x^3y \\ x^3y^2+\frac{1}{3}x^3 \\ x^2y^3+3x^2y+2xy^3+6xy+2y^3+6y \\ xy^4-6xy^2-3x-12y^2+2y^4-6 \\ y^5-\frac{10}{3}y^3+5y \end{pmatrix}.$$

If $\overset{2}{=}$ denotes the congruence relation (4.3), then by (3.42) we get recursively the polynomials $\mathbb{S}_n$ for $n = 0, 1, 2, 3, 4, 5$:

$$\mathbb{S}_0 \overset{2}{=} \mathbb{Q}_0 = 1, \quad \mathbb{S}_1 \overset{2}{=} \mathbb{Q}_1 = \begin{pmatrix} x+2 \\ y \end{pmatrix}, \quad \mathbb{S}_2 \overset{2}{=} \mathbb{Q}_2 = \begin{pmatrix} x^2+2x+2 \\ xy+2y \\ y^2+\frac{1}{3} \end{pmatrix},$$

$$\mathbb{S}_3 \overset{2}{=} \mathbb{Q}_3 = \begin{pmatrix} x^3 \\ x^2y+2xy+2y \\ xy^2+\frac{1}{3}x+2y^2+\frac{2}{3} \\ y^3+3y \end{pmatrix},$$

$$\mathbb{S}_4 \overset{2}{=} \mathbb{Q}_4 - \mathbf{A}_{4,3}\mathbb{S}_3 \overset{2}{=} \begin{pmatrix} x^4-4x^3 \\ x^3y-3x^2y-6xy-6y \\ x^2y^2+\frac{1}{3}x^2-2xy^2-\frac{2}{3}x-6y^2-2 \\ xy^3+3xy-3y-y^3 \\ y^4-6y^2-3 \end{pmatrix},$$

$$\mathbb{S}_5 \stackrel{2}{=} \mathbb{Q}_5 - \mathbf{A}_{5,4}\mathbb{S}_4 - \mathbf{A}_{5,3}\mathbb{S}_3 \stackrel{2}{=} \begin{pmatrix} x^5 - 10x^4 + 20x^3 \\ x^4y - \frac{38}{5}x^3y + \frac{54}{5}x^2y + \frac{108}{5}xy + \frac{108}{5}y \\ x^3y^2 - \frac{1}{3}x^3 - \frac{21}{4}x^2y^2 - \frac{7}{4}x^2 + \frac{9}{2}xy^2 + \frac{3}{2}x + \frac{39}{2}y^2 + \frac{13}{2} \\ x^2y^3 - x^2y - 3xy^3 - 17xy + y^3 - 5y \\ xy^4 - 2xy^2 - \frac{5}{3}x + 14y^2 - y^4 + \frac{17}{3} \\ y^5 - \frac{10}{3}y^3 + 5y \end{pmatrix}.$$

Finally, let $\mathbf{p} = (0, 1)$. Then, the Taylor polynomials of second degree at $\mathbf{p}$ for each entry in $\mathbb{S}_n$, for $n = 3, 4, 5$, are given by:

$$\mathcal{T}^2(\mathbb{S}_3, \mathbf{p}) = \begin{pmatrix} 0 \\ x^2 + 2xy + 2y \\ 2y^2 + 2xy - \frac{2}{3}x + \frac{2}{3} \\ 3y^2 + 1 \end{pmatrix}, \quad \mathcal{T}^2(\mathbb{S}_4, \mathbf{p}) = \begin{pmatrix} 0 \\ -3x^2 - 6xy - 6y \\ \frac{4}{3}x^2 - 4xy + \frac{4}{3}x - 6y^2 - 2 \\ -3y^2 + 6xy - 2x - 1 \\ -8y \end{pmatrix},$$

$$\mathcal{T}^2(\mathbb{S}_5, \mathbf{p}) = \begin{pmatrix} 0 \\ \frac{54}{5}x^2 + \frac{108}{5}xy + \frac{108}{5}y \\ -7x^2 + 9xy - 3x + \frac{39}{2}y^2 + \frac{13}{2} \\ 6x - 8y - 26xy + 3y^2 + 1 \\ 8y^2 + 8y - \frac{8}{3}x + \frac{8}{3} \\ \frac{8}{3} \end{pmatrix}.$$

With the previous polynomials, and by Corollary 4.2, we have a monic orthogonal basis $\left\{\mathcal{S}_j^n : 0 \le j \le n\right\}$ for the space $\mathscr{V}_n^2(S, W_{a,b})$ for $n = 0, 1, 2, 3, 4, 5$, with $a = 0$, $b = 1/2$ and $\mathbf{p} = (0, 1)$:

- For the space $\mathscr{V}_0^2(S, W_{a,b})$:

  $\mathcal{S}_0^0(x, y) = 1.$

- For the space $\mathscr{V}_1^2(S, W_{a,b})$:

  $\mathcal{S}_0^1(x, y) = x,$
  $\mathcal{S}_1^1(x, y) = y - 1.$

- For the space $\mathscr{V}_2^2(S, W_{a,b})$:

  $\mathcal{S}_0^2(x, y) = x^2,$
  $\mathcal{S}_1^2(x, y) = xy - x,$
  $\mathcal{S}_2^2(x, y) = y^2 - 2y + 1.$

- For the space $\mathscr{V}_3^2(S, W_{a,b})$:

$$\begin{aligned}
\mathcal{S}_0^3(x,y) &= x^3,\\
\mathcal{S}_1^3(x,y) &= x^2y - x^2,\\
\mathcal{S}_2^3(x,y) &= xy^2 - 2xy + x,\\
\mathcal{S}_3^3(x,y) &= y^3 - 3y^2 + 3y - 1.
\end{aligned}$$

- For the space $\mathscr{V}_4^2(S, W_{a,b})$:

$$\begin{aligned}
\mathcal{S}_0^4(x,y) &= x^4 - 4x^3,\\
\mathcal{S}_1^4(x,y) &= x^3y - 3x^2y + 3x^2,\\
\mathcal{S}_2^4(x,y) &= x^2y^2 - x^2 - 2xy^2 + 4xy - 2x,\\
\mathcal{S}_3^4(x,y) &= xy^3 + 2x - 3y - 3xy + 3y^2 - y^3 + 1,\\
\mathcal{S}_4^4(x,y) &= y^4 - 6y^2 + 8y - 3.
\end{aligned}$$

- For the space $\mathscr{V}_5^2(S, W_{a,b})$:

$$\begin{aligned}
\mathcal{S}_0^5(x,y) &= x^5 - 10x^4 + 20x^3,\\
\mathcal{S}_1^5(x,y) &= x^4y + \frac{54}{5}x^2y - \frac{38}{5}x^3y - \frac{54}{5}x^2,\\
\mathcal{S}_2^5(x,y) &= x^3y^2 - \frac{1}{3}x^3 - \frac{21}{4}x^2y^2 + \frac{21}{4}x^2 + \frac{9}{2}xy^2 - 9xy + \frac{9}{2}x,\\
\mathcal{S}_3^5(x,y) &= x^2y^3 - x^2y - 3xy^3 + 9xy - 6x + y^3 - 3y^2 + 3y - 1,\\
\mathcal{S}_4^5(x,y) &= xy^4 + x - 8y - 2xy^2 + 6y^2 - y^4 + 3,\\
\mathcal{S}_5^5(x,y) &= y^5 - \frac{10}{3}y^3 + 5y - \frac{8}{3}.
\end{aligned}$$

This completes our example with the Laguerre-Gegenbauer product weight.

4.3 Sobolev orthogonal polynomials on the triangle

The triangle of $\mathbb{R}^2$ is the set:

$$\mathbb{T}^2 := \left\{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leq 1\right\}.$$

This is a particular example in two variables of the simplex $\mathbb{T}^d$. In this section we construct the Sobolev orthogonal polynomials in two variables with respect to the inner product:

$$\begin{aligned}
\langle f, g\rangle_S = c_\gamma \int_{\mathbb{T}^2} \nabla^2 f(x,y) \cdot \nabla^2 g(x,y) W_\gamma(x,y)\,dxdy\\
+ \lambda_1 \nabla f(p_1,p_2) \cdot \nabla g(p_1,p_2) + \lambda_0 f(p_1,p_2) g(p_1,p_2), \quad (4.41)
\end{aligned}$$

where (p_1, p_2) is a given point in $\mathbb{R}^2$, $\lambda_0, \lambda_1 > 0$, ∇f and $\nabla^2 f$ are given in (4.1), W_γ is the weight function on the triangle:

$$W_\gamma(x,y) = x^{\gamma_1} y^{\gamma_2} (1-x-y)^{\gamma_3}, \quad \gamma_1, \gamma_2, \gamma_3 > -1, \quad (x,y) \in \mathbb{T}^2, \tag{4.42}$$

$\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3$ is such that $\gamma_1, \gamma_2, \gamma_3 > -1$, c_γ is the normalization constant:

$$c_\gamma := \left(\int_{\mathbb{T}^2} W_\gamma(x,y) dx dy \right)^{-1} = \frac{\Gamma(\gamma_1 + \gamma_2 + \gamma_3 + 3)}{\Gamma(\gamma_1 + 1)\Gamma(\gamma_2 + 1)\Gamma(\gamma_3 + 1)},$$

and the main part of (4.41) is denoted by:

$$\langle f, g \rangle_{\nabla^2} = c_\gamma \int_{\mathbb{T}^2} \nabla^2 f(x,y) \cdot \nabla^2 g(x,y) W_\gamma(x,y) dx dy. \tag{4.43}$$

We denote by $\mathscr{V}_n^2(S, W_\gamma)$ and $\mathscr{V}_n^2(\nabla^2, W_\gamma)$ the spaces of orthogonal polynomials of degree n with respect to (4.41) and (4.43), respectively, where $\dim \mathscr{V}_n^2(S, W_\gamma) = \dim \mathscr{V}_n^2(\nabla^2, W_\gamma) = n+1$.

From (1.31)–(1.32), the polynomials in the space $\mathscr{V}_n^2(W_\gamma)$ with respect to the inner product:

$$\langle f, g \rangle_{W_\gamma} = c_\gamma \int_{\mathbb{T}^2} f(x,y) g(x,y) W_\gamma(x,y) dx dy, \tag{4.44}$$

satisfy the partial differential equation:

$$\begin{aligned} & x(1-x)\frac{\partial^2 P}{\partial x^2} - 2xy\frac{\partial^2 P}{\partial x \partial y} + y(1-y)\frac{\partial^2 P}{\partial y^2} + \\ & [(\gamma_1+1) - (|\gamma|+3)x]\frac{\partial P}{\partial x} + [(\gamma_2+1) - (|\gamma|+3)y]\frac{\partial P}{\partial y} = -n(n+|\gamma|+2)P, \\ & \qquad P \in \mathscr{V}_n^2(W_\gamma), \quad |\gamma| = \gamma_1 + \gamma_2 + \gamma_3. \end{aligned} \tag{4.45}$$

From Proposition 3.7, we know that if $P \in \mathscr{V}_n^2(S, W_\gamma)$ or $P \in \mathscr{V}_n^2(\nabla^2, W_\gamma)$ then:

$$(\partial_1 + \partial_2)^2 P := \frac{\partial^2 P}{\partial x^2} + 2\frac{\partial^2 P}{\partial x \partial y} + \frac{\partial^2 P}{\partial y^2} \in \mathscr{V}_{n-2}^2(W_\gamma). \tag{4.46}$$

Putting (4.45) and (4.46) together, then they prove the following result.

Proposition 4.11. *Let $P \in \mathscr{V}_n^2(S, W_\gamma)$ or $P \in \mathscr{V}_n^2(\nabla^2, W_\gamma)$, with $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, $\gamma_1, \gamma_2, \gamma_3 > -1$, $|\gamma| = \gamma_1 + \gamma_2 + \gamma_3$. Then P satisfies the fourth-order partial differential equation:*

$$\begin{aligned} & \Bigg[x(1-x)\frac{\partial^2}{\partial x^2} - 2xy\frac{\partial^2}{\partial x \partial y} + y(1-y)\frac{\partial^2}{\partial y^2} + [(\gamma_1+1) - (|\gamma|+3)x]\frac{\partial}{\partial x} + \\ & [(\gamma_2+1) - (|\gamma|+3)y]\frac{\partial}{\partial y} + (n-2)(n+|\gamma|)\mathcal{I} \Bigg] \left[\frac{\partial^2}{\partial x^2} + 2\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \right] P = 0, \end{aligned}$$

where $\mathcal{I}$ is the identity operator.

The following corollary is a consequence of Theorem 3.1 and Proposition 3.9 for $\kappa = d = 2$.

Corollary 4.3. *Let* $\left\{S_j^n : 0 \le j \le n\right\}$ *denote a monic orthogonal basis of* $\mathscr{V}_n^2(\nabla^2, W_\gamma)$. *Then, a monic orthogonal basis* $\left\{\mathcal{S}_j^n : 0 \le j \le n\right\}$ *of* $\mathscr{V}_n^2(S, W_\gamma)$ *is given by:*

$$\begin{aligned}
&\mathcal{S}_0^0(x,y) = 1,\\
&\mathcal{S}_0^1(x,y) = x - p_1, \quad \mathcal{S}_1^1(x,y) = y - p_2,\\
&\mathcal{S}_j^n(x,y) = S_j^n(x,y) - \mathcal{T}^1(S_j^n, \mathbf{p}; x, y), \quad n \ge 2.
\end{aligned}$$

where $\mathcal{T}^1(S_j^n, \mathbf{p})$ *is the Taylor polynomial of first degree of* S_j^n *at* $\mathbf{p} = (p_1, p_2)$, *and where*

$$\begin{aligned}
&\left\langle \mathcal{S}_0^0, \mathcal{S}_0^0 \right\rangle_S = \lambda_0,\\
&\left\langle \mathcal{S}_0^1, \mathcal{S}_0^1 \right\rangle_S = \left\langle \mathcal{S}_1^1, \mathcal{S}_1^1 \right\rangle_S = \lambda_1,\\
&\left\langle \mathcal{S}_j^n, \mathcal{S}_j^n \right\rangle_S = \left\langle S_j^n, S_j^n \right\rangle_{\nabla^2}, \quad 0 \le j \le n, \quad n \ge 2.
\end{aligned}$$

Then, we need only to find a monic orthogonal basis $\left\{S_j^n : 0 \le j \le n\right\}$ for the space $\mathscr{V}_n^2(\nabla^2, W_\gamma)$ for $n \ge 2$. Let us arrange the elements of this basis in a vector form. We denote by $\mathbb{S}_n$ the column vector of size $n+1$:

$$\mathbb{S}_n = \Big(S_0^n(x,y), \quad S_1^n(x,y), \quad \dots, \quad S_n^n(x,y)\Big)^T,$$

and by $\mathbb{Q}_n$ the column vector of size $n+1$:

$$\mathbb{Q}_n = \Big(x^n, \quad x^{n-1}y, \quad x^{n-2}y^2, \quad \dots, \quad y^n\Big)^T. \tag{4.47}$$

As we mentioned in Section 3.4.2.1, in order to simplify the computation of the matrix $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^2}$ we only need that each entry of $\mathbb{Q}_n$ to be defined as a monomial. Then, we have the following corollary that is a consequence of Proposition 3.14 with $\kappa = d = 2$.

Corollary 4.4. *Let* $n, m \ge 2$ *and let* $\mathbb{Q}_n$ *be defined in* (4.47). *Then, each entry of the matrix* $\langle \mathbb{Q}_n, \mathbb{Q}_m^T \rangle_{\nabla^2} = \Big(\langle x^{n-i}y^i, x^{m-j}y^j \rangle_{\nabla^2}\Big)_{0 \le i \le n, 0 \le j \le m}$ *of size* $(n+1) \times (m+1)$ *can be computed by:*

$$\left\langle x^{n-i}y^i, x^{m-j}y^j \right\rangle_{\nabla^2} = \frac{A_{i,j}^{n,m,\gamma 1,\gamma 2} + 2B_{i,j}^{n,m,\gamma 1,\gamma 2} + C_{i,j}^{n,m,\gamma 1,\gamma 2}}{(\gamma_1 + \gamma_2 + \gamma_3 + 3)_{n+m-4}},$$
$$\gamma_1, \gamma_2, \gamma_3 > -1, \quad 0 \le i \le n, \quad 0 \le j \le m,$$

where $A_{i,j}^{n,m,\gamma 1,\gamma 2}$, $B_{i,j}^{n,m,\gamma 1,\gamma 2}$, *and* $C_{i,j}^{n,m,\gamma 1,\gamma 2}$ *are:*

$$A_{i,j}^{n,m,\gamma 1,\gamma 2} = \begin{cases} (i-1)_2 (j-1)_2 (\gamma_1+1)_{n+m-i-j} (\gamma_2+1)_{i+j-4}, & 2 \le i \le n, \quad 2 \le j \le m,\\ 0, & \textit{otherwise}, \end{cases}$$

$$B_{i,j}^{n,m,\gamma1,\gamma2} = \begin{cases} ij(n-i)(m-j)(\gamma_1+1)_{n+m-i-j-2}(\gamma_2+1)_{i+j-2}, & 1 \le i \le n-1, \\ & 1 \le j \le m-1, \\ 0, & \textit{otherwise}, \end{cases}$$

$$C_{i,j}^{n,m,\gamma1,\gamma2} = \begin{cases} (n-i-1)_2(m-j-1)_2(\gamma_1+1)_{n+m-i-j-4}(\gamma_2+1)_{i+j}, & 0 \le i \le n-2, \\ & 0 \le j \le m-2, \\ 0, & \textit{otherwise}. \end{cases}$$

Then, on the triangle $\mathbb{T}^2$ the polynomials in the column vector $\mathbb{S}_n$ can be computed recursively by means of the relations in Proposition 3.10, Proposition 3.11 and Corollary 4.4. In order to illustrate the main ideas we will show a numerical example in the sequel.

Example 4.3 (Numerical, see Note 4.1). Let $\gamma = (\gamma_1, \gamma_2, \gamma_3) = (-19/20, -19/20, 3/5)$ the parameters[4] for the weight function W_γ, which is defined in (4.42), and let $\mathbf{p} = (p_1, p_2) = (1/2, 1/2)$ in the inner product (4.41). From (4.47) we have for $0 \le n \le 4$:

$$\mathbb{Q}_0 = 1, \quad \mathbb{Q}_1 = \begin{pmatrix} x, & y \end{pmatrix}^T, \quad \mathbb{Q}_2 = \begin{pmatrix} x^2, & xy, & y^2 \end{pmatrix}^T,$$
$$\mathbb{Q}_3 = \begin{pmatrix} x^3, & x^2y, & xy^2, & y^3 \end{pmatrix}^T, \quad \mathbb{Q}_4 = \begin{pmatrix} x^4, & x^3y, & x^2y^2, & xy^3, & y^4 \end{pmatrix}^T.$$

From Proposition 3.11 and Corollary 4.4, with $\kappa = 2$, we have the following iterations:

1. First iteration:

$$\mathbf{B}_{3,2} = \left\langle \mathbb{Q}_3, \mathbb{Q}_2^T \right\rangle_{\nabla^2} = \begin{pmatrix} \frac{6}{17} & 0 & 0 \\ \frac{2}{17} & \frac{2}{17} & 0 \\ 0 & \frac{2}{17} & \frac{2}{17} \\ 0 & 0 & \frac{6}{17} \end{pmatrix}, \quad \mathbf{B}_{4,2} = \left\langle \mathbb{Q}_4, \mathbb{Q}_2^T \right\rangle_{\nabla^2} = \begin{pmatrix} \frac{14}{51} & 0 & 0 \\ \frac{1}{153} & \frac{7}{102} & 0 \\ \frac{7}{153} & \frac{2}{459} & \frac{7}{153} \\ 0 & \frac{7}{102} & \frac{1}{153} \\ 0 & 0 & \frac{14}{51} \end{pmatrix},$$

$$\mathbf{H}_2^{\nabla^2} = \left\langle \mathbb{Q}_2, \mathbb{Q}_2^T \right\rangle_{\nabla^2} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

2. Second iteration:

$$\mathbf{B}_{4,3} = \left\langle \mathbb{Q}_4, \mathbb{Q}_3^T \right\rangle_{\nabla^2} - \mathbf{B}_{4,2} \left(\mathbf{H}_2^{\nabla^2}\right)^{-1} \mathbf{B}_{3,2}^T = \begin{pmatrix} \frac{4620}{10693} & -\frac{140}{32079} & 0 & 0 \\ \frac{160}{32079} & \frac{7090}{96237} & -\frac{70}{32079} & 0 \\ -\frac{70}{32079} & \frac{7570}{288711} & \frac{7570}{288711} & -\frac{70}{32079} \\ 0 & -\frac{70}{32079} & \frac{7090}{96237} & \frac{160}{32079} \\ 0 & 0 & -\frac{140}{32079} & \frac{4620}{10693} \end{pmatrix},$$

[4] The values for γ were chosen to get reduced fractions at the entries of the matrix $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^2}$. See Corollary 4.4.

$$\mathbf{H}_3^{\nabla^2} = \left\langle \mathbb{Q}_3, \mathbb{Q}_3^T \right\rangle_{\nabla^2} - \mathbf{B}_{3,2} \left(\mathbf{H}_2^{\nabla^2}\right)^{-1} \mathbf{B}_{3,2}^T = \begin{pmatrix} \frac{110}{289} & -\frac{10}{2601} & 0 & 0 \\ -\frac{10}{2601} & \frac{110}{867} & -\frac{20}{7803} & 0 \\ 0 & -\frac{20}{7803} & \frac{110}{867} & -\frac{10}{2601} \\ 0 & 0 & -\frac{10}{2601} & \frac{110}{289} \end{pmatrix}.$$

Therefore, from Proposition 3.11, we have:

$$\mathbf{A}_{3,2} = \mathbf{B}_{3,2} \left(\mathbf{H}_2^{\nabla^2}\right)^{-1} = \begin{pmatrix} \frac{3}{34} & 0 & 0 \\ \frac{1}{34} & \frac{1}{17} & 0 \\ 0 & \frac{1}{17} & \frac{1}{34} \\ 0 & 0 & \frac{3}{34} \end{pmatrix}, \quad \mathbf{A}_{4,2} = \mathbf{B}_{4,2} \left(\mathbf{H}_2^{\nabla^2}\right)^{-1} = \begin{pmatrix} \frac{7}{102} & 0 & 0 \\ \frac{1}{612} & \frac{7}{204} & 0 \\ \frac{7}{612} & \frac{1}{459} & \frac{7}{612} \\ 0 & \frac{7}{204} & \frac{1}{612} \\ 0 & 0 & \frac{7}{102} \end{pmatrix},$$

$$\mathbf{A}_{4,3} = \mathbf{B}_{4,3} \left(\mathbf{H}_3^{\nabla^2}\right)^{-1} = \begin{pmatrix} \frac{42}{37} & 0 & 0 & 0 \\ \frac{172}{9065} & \frac{5268}{9065} & -\frac{99}{18130} & -\frac{1}{18130} \\ -\frac{2}{555} & \frac{39}{185} & \frac{39}{185} & -\frac{2}{555} \\ -\frac{1}{18130} & -\frac{99}{18130} & \frac{5268}{9065} & \frac{172}{9065} \\ 0 & 0 & 0 & \frac{42}{37} \end{pmatrix}.$$

If $\stackrel{1}{=}$ denotes the congruence relation (4.2) then, from Proposition 3.10, we have for $0 \leq n \leq 4$ that:

$$\mathbb{S}_0 \stackrel{1}{=} \mathbb{Q}_0 = 1,$$

$$\mathbb{S}_1 \stackrel{1}{=} \mathbb{Q}_1 = \begin{pmatrix} x, & y \end{pmatrix}^T,$$

$$\mathbb{S}_2 \stackrel{1}{=} \mathbb{Q}_2 = \left(x^2, \quad xy, \quad y^2\right)^T,$$

$$\begin{aligned} \mathbb{S}_3 &\stackrel{1}{=} \mathbb{Q}_3 - \mathbf{A}_{3,2}\mathbb{S}_2 \\ &\stackrel{1}{=} \left(x^3 - \tfrac{3}{34}x^2, \quad x^2y - \tfrac{1}{17}xy - \tfrac{1}{34}x^2, \quad xy^2 - \tfrac{1}{17}xy - \tfrac{1}{34}y^2, \quad y^3 - \tfrac{3}{34}y^2\right)^T, \end{aligned}$$

$$\begin{aligned} \mathbb{S}_4 &\stackrel{1}{=} \mathbb{Q}_4 - \mathbf{A}_{4,3}\mathbb{S}_3 - \mathbf{A}_{4,2}\mathbb{S}_2 \\ &\stackrel{1}{=} \begin{pmatrix} x^4 - \frac{42}{37}x^3 + \frac{7}{222}x^2 \\ x^3y - \frac{172}{9065}x^3 - \frac{5268}{9065}x^2y + \frac{99}{18130}xy^2 + \frac{1}{18130}y^3 + \frac{5591}{326340}x^2 - \frac{1}{2220}xy - \frac{3}{18130}y^2 \\ x^2y^2 + \frac{2}{555}x^3 - \frac{39}{185}x^2y - \frac{39}{185}xy^2 + \frac{2}{555}y^3 - \frac{1}{180}x^2 + \frac{113}{4995}xy - \frac{1}{180}y^2 \\ xy^3 + \frac{1}{18130}x^3 + \frac{99}{18130}x^2y - \frac{5268}{9065}xy^2 - \frac{172}{9065}y^3 - \frac{3}{18130}x^2 - \frac{1}{2220}xy + \frac{5591}{326340}y^2 \\ y^4 - \frac{42}{37}y^3 + \frac{7}{222}y^2 \end{pmatrix}. \end{aligned}$$

Finally, let $\mathbf{p} = (p_1, p_2) = (1/2, 1/2)$. The Taylor polynomials of first degree of $\mathbb{S}_2$, $\mathbb{S}_3$ and $\mathbb{S}_4$ at $\mathbf{p}$ are given by:

$$\mathcal{T}^1(\mathbb{S}_2, \mathbf{p}) = \left(x - \tfrac{1}{4}, \quad \tfrac{1}{2}x + \tfrac{1}{2}y - \tfrac{1}{4}, \quad y - \tfrac{1}{4}\right)^T,$$

$$\mathcal{T}^1(\mathbb{S}_3,\mathbf{p}) = \left(\tfrac{45}{68}x-\tfrac{31}{136},\quad \tfrac{15}{34}x+\tfrac{15}{68}y-\tfrac{31}{136},\quad \tfrac{15}{68}x+\tfrac{15}{34}y-\tfrac{31}{136},\quad \tfrac{45}{68}y-\tfrac{31}{136}\right)^T,$$

$$\mathcal{T}^1(\mathbb{S}_4,\mathbf{p}) = \begin{pmatrix} -\tfrac{71}{222}x+\tfrac{157}{1776} \\ \tfrac{11549}{130536}x-\tfrac{779}{43512}y-\tfrac{229}{5328} \\ \tfrac{401}{3996}x+\tfrac{401}{3996}y-\tfrac{1387}{15984} \\ -\tfrac{779}{43512}x+\tfrac{11549}{130536}y-\tfrac{229}{5328} \\ -\tfrac{71}{222}y+\tfrac{157}{1776} \end{pmatrix}.$$

Then from Corollary 4.3, the following polynomials are a monic orthogonal basis $\left\{\mathcal{S}_j^n : 0 \le j \le n\right\}$ for the space $\mathscr{V}_n^2(S, W_\gamma)$ for $0 \le n \le 4$:

- For the space $\mathscr{V}_0^2(S, W_\gamma)$:

$$\mathcal{S}_0^0(x,y) = 1.$$

- For the space $\mathscr{V}_1^2(S, W_\gamma)$:

$$\mathcal{S}_0^1(x,y) = x - \frac{1}{2},$$
$$\mathcal{S}_1^1(x,y) = y - \frac{1}{2}.$$

- For the space $\mathscr{V}_2^2(S, W_\gamma)$:

$$\mathcal{S}_0^2(x,y) = x^2 - x + \frac{1}{4},$$
$$\mathcal{S}_1^2(x,y) = xy - \frac{1}{2}x - \frac{1}{2}y + \frac{1}{4},$$
$$\mathcal{S}_2^2(x,y) = y^2 - y + \frac{1}{4}.$$

- For the space $\mathscr{V}_3^2(S, W_\gamma)$:

$$\mathcal{S}_0^3(x,y) = x^3 - \frac{3}{34}x^2 - \frac{45}{68}x + \frac{31}{136},$$
$$\mathcal{S}_1^3(x,y) = x^2y - \frac{1}{17}xy - \frac{1}{34}x^2 - \frac{15}{34}x - \frac{15}{68}y + \frac{31}{136},$$
$$\mathcal{S}_2^3(x,y) = xy^2 - \frac{1}{17}xy - \frac{1}{34}y^2 - \frac{15}{68}x - \frac{15}{34}y + \frac{31}{136},$$
$$\mathcal{S}_3^3(x,y) = y^3 - \frac{3}{34}y^2 - \frac{45}{68}y + \frac{31}{136}.$$

- For the space $\mathscr{V}_4^2(S, W_\gamma)$:

$$\mathcal{S}_0^4(x,y) = x^4 - \frac{42}{37}x^3 + \frac{7}{222}x^2 + \frac{71}{222}x - \frac{157}{1776},$$

$$
\begin{aligned}
\mathcal{S}_1^4(x,y) &= x^3y - \frac{172}{9065}x^3 - \frac{5268}{9065}x^2y + \frac{99}{18130}xy^2 + \frac{1}{18130}y^3 + \frac{5591}{326340}x^2 \\
&\quad - \frac{1}{2220}xy - \frac{3}{18130}y^2 - \frac{11549}{130536}x + \frac{779}{43512}y + \frac{229}{5328}, \\
\mathcal{S}_2^4(x,y) &= x^2y^2 + \frac{2}{555}x^3 - \frac{39}{185}x^2y - \frac{39}{185}xy^2 + \frac{2}{555}y^3 - \frac{1}{180}x^2 + \frac{113}{4995}xy \\
&\quad - \frac{1}{180}y^2 - \frac{401}{3996}x - \frac{401}{3996}y + \frac{1387}{15984}, \\
\mathcal{S}_3^4(x,y) &= xy^3 + \frac{1}{18130}x^3 + \frac{99}{18130}x^2y - \frac{5268}{9065}xy^2 - \frac{172}{9065}y^3 - \frac{3}{18130}x^2 \\
&\quad - \frac{1}{2220}xy + \frac{5591}{326340}y^2 + \frac{779}{43512}x - \frac{11549}{130536}y + \frac{229}{5328}, \\
\mathcal{S}_4^4(x,y) &= y^4 - \frac{42}{37}y^3 + \frac{7}{222}y^2 + \frac{71}{222}y - \frac{157}{1776}.
\end{aligned}
$$

This completes our numerical example on the triangle $\mathbb{T}^2$.

4.4 Sobolev orthogonal polynomials on the disk

The disk of $\mathbb{R}^2$ is the set:

$$\mathbb{B}^2 := \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\}.$$

This is a particular example of the ball $\mathbb{B}^d$ in two variables. In this section we construct the Sobolev orthogonal polynomials in two variables with respect to the inner product:

$$\langle f,g\rangle_S = c_\mu \int_{\mathbb{B}^2} \nabla^3 f(x,y)\cdot\nabla^3 g(x,y)W_\mu(x,y)dxdy + \lambda_2\nabla^2 f(p_1,p_2)\cdot\nabla^2 g(p_1,p_2) + \lambda_1\nabla f(p_1,p_2)\cdot\nabla g(p_1,p_2) + \lambda_0 f(p_1,p_2)g(p_1,p_2), \quad (4.48)$$

where (p_1,p_2) is a given point in $\mathbb{R}^2$, $\lambda_0,\lambda_1,\lambda_2>0$, ∇f, $\nabla^2 f$ and $\nabla^3 f$ are given in (4.1), W_μ is the weight function on the disk:

$$W_\mu(x,y) = (1-x^2-y^2)^\mu, \quad \mu > -1, \quad (x,y)\in\mathbb{B}^2, \quad (4.49)$$

c_μ is the normalization constant (see (1.41)):

$$c_\mu := \left(\int_{\mathbb{B}^2} W_\mu(x,y)dxdy\right)^{-1} = \frac{\mu+1}{\pi},$$

and the main part of (4.48) is denoted by:

$$\langle f,g\rangle_{\nabla^3} = c_\mu \int_{\mathbb{B}^2} \nabla^3 f(x,y)\cdot\nabla^3 g(x,y)W_\mu(x,y)dxdy. \quad (4.50)$$

We denote by $\mathscr{V}_n^2(S,W_\mu)$ and $\mathscr{V}_n^2(\nabla^3,W_\mu)$ the spaces of orthogonal polynomials of degree n with respect to (4.48) and (4.50), respectively, where $\dim\mathscr{V}_n^2(S,W_\mu) = \dim\mathscr{V}_n^2(\nabla^3,W_\mu) = n+1$.

From (1.43)–(1.44), the polynomials in the space $\mathscr{V}_n^2(W_\mu)$ with respect to the inner product:

$$\langle f,g\rangle_{W_\mu} = c_\mu \int_{\mathbb{B}^2} f(x,y)g(x,y)W_\mu(x,y)dxdy, \tag{4.51}$$

satisfy the partial differential equation:

$$(1-x^2)\frac{\partial^2 P}{\partial x^2} - 2xy\frac{\partial^2 P}{\partial x\partial y} + (1-y^2)\frac{\partial^2 P}{\partial y^2} - (2\mu+3)x\frac{\partial P}{\partial x} - (2\mu+3)y\frac{\partial P}{\partial y} = \\ -n(n+2\mu+2)P, \quad P\in\mathscr{V}_n^2(W_\mu), \quad \mu>-1. \tag{4.52}$$

From Proposition 3.7, we know that if $P\in\mathscr{V}_n^2(S,W_\mu)$ or $P\in\mathscr{V}_n^2(\nabla^3,W_\mu)$ then:

$$(\partial_1+\partial_2)^3P := \frac{\partial^3 P}{\partial x^3} + 3\frac{\partial^3 P}{\partial x^2\partial y} + 3\frac{\partial^3 P}{\partial x\partial y^2} + \frac{\partial^3 P}{\partial y^3} \in \mathscr{V}_{n-3}^2(W_\mu). \tag{4.53}$$

Putting (4.52) and (4.53) together, then they prove the following result.

Proposition 4.12. *Let $P\in\mathscr{V}_n^2(S,W_\mu)$ or $P\in\mathscr{V}_n^2(\nabla^3,W_\mu)$, with $\mu>-1$. Then P satisfies the fifth-order partial differential equation:*

$$\Big[(1-x^2)\frac{\partial^2}{\partial x^2} - 2xy\frac{\partial^2}{\partial x\partial y} + (1-y^2)\frac{\partial^2}{\partial y^2} - (2\mu+3)x\frac{\partial}{\partial x} \\ -(2\mu+3)y\frac{\partial}{\partial y} + (n-3)(n+2\mu-1)\mathcal{I}\Big]\left[\frac{\partial^3}{\partial x^3} + 3\frac{\partial^3}{\partial x^2\partial y} + 3\frac{\partial^3}{\partial x\partial y^2} + \frac{\partial^3}{\partial y^3}\right]P = 0,$$

where $\mathcal{I}$ is the identity operator.

The following corollary is a consequence of Theorem 3.1 and Proposition 3.9 for $\kappa=3$ and $d=2$.

Corollary 4.5. *Let $\{S_j^n : 0\le j\le n\}$ denote a monic orthogonal basis of $\mathscr{V}_n^2(\nabla^3,W_\mu)$. Then, a monic orthogonal basis $\{\mathcal{S}_j^n : 0\le j\le n\}$ of $\mathscr{V}_n^2(S,W_\mu)$ is given by:*

$$\begin{aligned}
&\mathcal{S}_0^0(x,y) = 1,\\
&\mathcal{S}_0^1(x,y) = x-p_1, \quad \mathcal{S}_1^1(x,y) = y-p_2,\\
&\mathcal{S}_0^2(x,y) = (x-p_1)^2, \quad \mathcal{S}_1^2(x,y) = (x-p_1)(y-p_2), \quad \mathcal{S}_2^2(x,y) = (y-p_2)^2,\\
&\mathcal{S}_j^n(x,y) = S_j^n(x,y) - \mathcal{T}^2(S_j^n,\mathbf{p};x,y), \quad n\ge 3.
\end{aligned}$$

where $\mathcal{T}^2(S_j^n,\mathbf{p})$ is the Taylor polynomial of second degree of S_j^n at $\mathbf{p}=(p_1,p_2)$, and where

$$\begin{aligned}
&\langle\mathcal{S}_0^0,\mathcal{S}_0^0\rangle_S = \lambda_0,\\
&\langle\mathcal{S}_0^1,\mathcal{S}_0^1\rangle_S = \langle\mathcal{S}_1^1,\mathcal{S}_1^1\rangle_S = \lambda_1,\\
&\langle\mathcal{S}_0^2,\mathcal{S}_0^2\rangle_S = 4\lambda_2, \quad \langle\mathcal{S}_1^2,\mathcal{S}_1^2\rangle_S = 2\lambda_2, \quad \langle\mathcal{S}_2^2,\mathcal{S}_2^2\rangle_S = 4\lambda_2,\\
&\langle\mathcal{S}_j^n,\mathcal{S}_j^n\rangle_S = \langle S_j^n,S_j^n\rangle_{\nabla^3}, \quad 0\le j\le n, \quad n\ge 3.
\end{aligned}$$

.

Let us arrange the elements of the basis $\left\{S_j^n : 0 \le j \le n\right\}$ of $\mathscr{V}_n^2(\nabla^3, W_\mu)$ in a vector form. We denote by $\mathbb{S}_n$ the column vector of size $n+1$:

$$\mathbb{S}_n = \left(S_0^n(x,y), \quad S_1^n(x,y), \quad \ldots, \quad S_n^n(x,y)\right)^T,$$

and by $\mathbb{Q}_n$ the column vector of size $n+1$:

$$\mathbb{Q}_n = \left(x^n, \quad x^{n-1}y, \quad x^{n-2}y^2, \quad \ldots, \quad y^n\right)^T. \tag{4.54}$$

As we mentioned in Section 3.4.3.1, in order to simplify the computation of the matrix $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^3}$ we only need that each entry of $\mathbb{Q}_n$ to be defined as a monomial. The following corollary is a consequence of Proposition 3.15 with $\kappa = 3$ and $d = 2$.

Corollary 4.6. *Let $n, m \ge 3$, $\mu > -1$, and let $\mathbb{Q}_n$ be defined in* (4.54)*. Then, each entry of the matrix $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^3} = \left(\langle x^{n-i}y^i, x^{m-j}y^j\rangle_{\nabla^3}\right)_{0\le i\le n, 0\le j\le m}$ of size $(n+1)\times(m+1)$ can be computed for $0 \le i \le n$ and $0 \le j \le m$ by:*

$$\langle x^{n-i}y^i, x^{m-j}y^j\rangle_{\nabla^3} = \begin{cases} \dfrac{A_{i,j}^{n,m} + 3B_{i,j}^{n,m} + 3C_{i,j}^{n,m} + D_{i,j}^{n,m}}{(\mu+2)_{\frac{n+m}{2}-3}}, & n+m-i-j \text{ and } i+j \text{ are even}, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$A_{i,j}^{n,m} = \begin{cases} (-i)_3(-j)_3 \left(\frac{1}{2}\right)_{\frac{n+m-i-j}{2}} \left(\frac{1}{2}\right)_{\frac{i+j}{2}-3}, & 3 \le i \le n, \quad 3 \le j \le m, \\ 0, & \text{otherwise}, \end{cases}$$

$$B_{i,j}^{n,m} = \begin{cases} (n-i)(m-j)(-i)_2(-j)_2 \left(\frac{1}{2}\right)_{\frac{n+m-i-j}{2}-1} \left(\frac{1}{2}\right)_{\frac{i+j}{2}-2}, & 2 \le i < n, \quad 2 \le j < m, \\ 0, & \text{otherwise}, \end{cases}$$

$$C_{i,j}^{n,m} = \begin{cases} ij(i-n)_2(j-m)_2 \left(\frac{1}{2}\right)_{\frac{n+m-i-j}{2}-2} \left(\frac{1}{2}\right)_{\frac{i+j}{2}-1}, & 1 \le i < n-1, \quad 1 \le j < m-1, \\ 0, & \text{otherwise}, \end{cases}$$

$$D_{i,j}^{n,m} = \begin{cases} (i-n)_3(j-m)_3 \left(\frac{1}{2}\right)_{\frac{n+m-i-j}{2}-3} \left(\frac{1}{2}\right)_{\frac{i+j}{2}}, & 0 \le i < n-2, \quad 0 \le j < m-2, \\ 0, & \text{otherwise}. \end{cases}$$

On the disk $\mathbb{B}^2$ the polynomials in $\mathbb{S}_n$ can be computed recursively by means of the relations in Proposition 3.10, Proposition 3.11 and Corollary 4.6. Next we show a numerical example.

Example 4.4 (Numerical, see Note 4.1). Let $\mu = -1/2$ the parameter for the weight function W_μ on the disk, which is defined in (4.49), and let $\mathbf{p} = (p_1, p_2) = (1, 0)$. From (4.54) we have for $0 \le n \le 5$:

$$\mathbb{Q}_0 = 1, \quad \mathbb{Q}_1 = \left(x, \quad y\right)^T, \quad \mathbb{Q}_2 = \left(x^2, \quad xy, \quad y^2\right)^T, \quad \mathbb{Q}_3 = \left(x^3, \quad x^2y, \quad xy^2, \quad y^3\right)^T,$$
$$\mathbb{Q}_4 = \left(x^4, \quad x^3y, \quad x^2y^2, \quad xy^3, \quad y^4\right)^T, \quad \mathbb{Q}_5 = \left(x^5, \quad x^4y, \quad x^3y^2, \quad x^2y^3, \quad xy^4, \quad y^5\right)^T.$$

From Proposition 3.11 and Corollary 4.6, with $\kappa = 3$, we have the following iterations:

1. First iteration:

$$\mathbf{B}_{4,3} = \left\langle \mathbb{Q}_4, \mathbb{Q}_3^T \right\rangle_{\nabla^3} = \mathbf{0}, \quad \mathbf{B}_{5,3} = \left\langle \mathbb{Q}_5, \mathbb{Q}_3^T \right\rangle_{\nabla^3} = \begin{pmatrix} 120 & 0 & 0 & 0 \\ 0 & 24 & 0 & 0 \\ 12 & 0 & 12 & 0 \\ 0 & 12 & 0 & 12 \\ 0 & 0 & 24 & 0 \\ 0 & 0 & 0 & 120 \end{pmatrix},$$

$$\mathbf{H}_3^{\nabla^3} = \left\langle \mathbb{Q}_3, \mathbb{Q}_3^T \right\rangle_{\nabla^3} = \begin{pmatrix} 36 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix}.$$

2. Second iteration:

$$\mathbf{B}_{5,4} = \left\langle \mathbb{Q}_5, \mathbb{Q}_4^T \right\rangle_{\nabla^3} - \mathbf{B}_{5,3} \left(\mathbf{H}_3^{\nabla^3}\right)^{-1} \mathbf{B}_{4,3}^T = \mathbf{0},$$

$$\mathbf{H}_4^{\nabla^3} = \left\langle \mathbb{Q}_4, \mathbb{Q}_4^T \right\rangle_{\nabla^3} - \mathbf{B}_{4,3} \left(\mathbf{H}_3^{\nabla^3}\right)^{-1} \mathbf{B}_{4,3}^T = \begin{pmatrix} 192 & 0 & 0 & 0 & 0 \\ 0 & 48 & 0 & 0 & 0 \\ 0 & 0 & 32 & 0 & 0 \\ 0 & 0 & 0 & 48 & 0 \\ 0 & 0 & 0 & 0 & 192 \end{pmatrix}.$$

Therefore, from Proposition 3.11, we have:

$$\mathbf{A}_{4,3} = \mathbf{B}_{4,3} \left(\mathbf{H}_3^{\nabla^3}\right)^{-1} = \mathbf{0}, \quad \mathbf{A}_{5,3} = \mathbf{B}_{5,3} \left(\mathbf{H}_3^{\nabla^3}\right)^{-1} = \begin{pmatrix} \frac{10}{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \frac{1}{3} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \frac{10}{3} \end{pmatrix},$$

$$\mathbf{A}_{5,4} = \mathbf{B}_{5,4} \left(\mathbf{H}_4^{\nabla^3}\right)^{-1} = \mathbf{0}.$$

If $\stackrel{2}{=}$ denotes the congruence relation (4.3) then, from Proposition 3.10, we have for $0 \le n \le 5$ that:

$$\mathbb{S}_0 \stackrel{2}{=} \mathbb{Q}_0 = 1,$$

$$\mathbb{S}_1 \stackrel{2}{=} \mathbb{Q}_1 = \begin{pmatrix} x, & y \end{pmatrix}^T,$$

$$\mathbb{S}_2 \stackrel{2}{=} \mathbb{Q}_2 = \begin{pmatrix} x^2, & xy, & y^2 \end{pmatrix}^T,$$

$$\mathbb{S}_3 \stackrel{2}{=} \mathbb{Q}_3 = \begin{pmatrix} x^3, & x^2y, & xy^2, & y^3 \end{pmatrix}^T,$$

$$\mathbb{S}_4 \stackrel{2}{=} \mathbb{Q}_4 - \mathbf{A}_{4,3}\mathbb{S}_3 \stackrel{2}{=} \begin{pmatrix} x^4, & x^3y, & x^2y^2, & xy^3, & y^4 \end{pmatrix}^T,$$

$$\mathbb{S}_5 \stackrel{2}{=} \mathbb{Q}_5 - \mathbf{A}_{5,4}\mathbb{S}_4 - \mathbf{A}_{5,3}\mathbb{S}_3 \stackrel{2}{=} \begin{pmatrix} x^5 - \frac{10}{3}x^3 \\ x^4y - 2x^2y \\ x^3y^2 - \frac{1}{3}x^3 - xy^2 \\ x^2y^3 - x^2y - \frac{1}{3}y^3 \\ xy^4 - 2xy^2 \\ y^5 - \frac{10}{3}y^3 \end{pmatrix}.$$

Finally, let $\mathbf{p} = (p_1, p_2) = (1, 0)$. The Taylor polynomials of second degree at $\mathbf{p}$ of $\mathbb{S}_3, \mathbb{S}_4$ and $\mathbb{S}_5$ are given by:

$$\mathcal{T}^2(\mathbb{S}_3, \mathbf{p}; x, y) = \Big(3x^2 - 3x + 1, \quad 2xy - y, \quad y^2, \quad 0\Big)^T,$$

$$\mathcal{T}^2(\mathbb{S}_4, \mathbf{p}; x, y) = \Big(6x^2 - 8x + 3, \quad 3xy - 2y, \quad y^2, \quad 0, \quad 0\Big)^T,$$

$$\mathcal{T}^2(\mathbb{S}_5, \mathbf{p}; x, y) = \begin{pmatrix} -5x + \frac{8}{3} \\ -y \\ -x^2 + x - \frac{1}{3} \\ -2xy + y \\ -2y^2 \\ 0 \end{pmatrix}.$$

Then, from Corollary 4.5, the following polynomials are a monic orthogonal basis $\{\mathcal{S}_j^n : 0 \le j \le n\}$ for the space $\mathscr{V}_n^2(S, W_\mu)$, with $\mu = -1/2$, on the disk for $0 \le n \le 5$:

- For the space $\mathscr{V}_0^2(S, W_\mu)$:

  $\mathcal{S}_0^0(x, y) = 1.$

- For the space $\mathscr{V}_1^2(S, W_\mu)$:

  $\mathcal{S}_0^1(x, y) = x - 1,$
  $\mathcal{S}_1^1(x, y) = y.$

- For the space $\mathscr{V}_2^2(S, W_\mu)$:

  $\mathcal{S}_0^2(x, y) = x^2 - 2x + 1,$
  $\mathcal{S}_1^2(x, y) = xy - y,$
  $\mathcal{S}_2^2(x, y) = y^2.$

- For the space $\mathscr{V}_3^2(S, W_\mu)$:

  $\mathcal{S}_0^3(x, y) = x^3 - 3x^2 + 3x - 1,$
  $\mathcal{S}_1^3(x, y) = x^2y - 2xy + y,$
  $\mathcal{S}_2^3(x, y) = xy^2 - y^2,$
  $\mathcal{S}_3^3(x, y) = y^3.$

- For the space $\mathscr{V}_4^2(S, W_\mu)$:

$$\begin{aligned}
\mathcal{S}_0^4(x,y) &= x^4 - 6x^2 + 8x - 3,\\
\mathcal{S}_1^4(x,y) &= x^3y - 3xy + 2y,\\
\mathcal{S}_2^4(x,y) &= x^2y^2 - y^2,\\
\mathcal{S}_3^4(x,y) &= xy^3,\\
\mathcal{S}_4^4(x,y) &= y^4.
\end{aligned}$$

- For the space $\mathscr{V}_5^2(S, W_\mu)$:

$$\begin{aligned}
\mathcal{S}_0^5(x,y) &= x^5 - \frac{10}{3}x^3 + 5x - \frac{8}{3},\\
\mathcal{S}_1^5(x,y) &= x^4y - 2x^2y + y,\\
\mathcal{S}_2^5(x,y) &= x^3y^2 - \frac{1}{3}x^3 - xy^2 + x^2 - x + \frac{1}{3},\\
\mathcal{S}_3^5(x,y) &= x^2y^3 - x^2y - \frac{1}{3}y^3 + 2xy - y,\\
\mathcal{S}_4^5(x,y) &= xy^4 - 2xy^2 + 2y^2,\\
\mathcal{S}_5^5(x,y) &= y^5 - \frac{10}{3}y^3.
\end{aligned}$$

This completes our numerical example on the disk $\mathbb{B}^2$.

4.5 Sobolev orthogonal polynomials on the cone

4.5.1 Jacobi weight function

The bounded cone of $\mathbb{R}^2$ is the set:

$$\mathbb{V}_1^2 := \left\{(x,y) \in \mathbb{R}^2 : |x| \le y, 0 \le y \le 1\right\}.$$

This is a particular example of the bounded cone $\mathbb{V}_1^d$ in two variables. In this section we construct the Sobolev orthogonal polynomials in two variables with respect to the inner product:

$$\langle f, g\rangle_S = c_{a,b,\mu} \int_{\mathbb{V}_1^2} \nabla^2 f(x,y) \cdot \nabla^2 g(x,y) W^J_{a,b,\mu}(x,y)dxdy \\ + \lambda_1 \nabla f(p_1,p_2) \cdot \nabla g(p_1,p_2) + \lambda_0 f(p_1,p_2) g(p_1,p_2), \quad (4.55)$$

where (p_1, p_2) is a given point in $\mathbb{R}^2$, $\lambda_0, \lambda_1 > 0$, ∇f and $\nabla^2 f$ are given in (4.1), $W^J_{a,b,\mu}$ is the weight function on the cone $\mathbb{V}_1^2$:

$$W^J_{a,b,\mu}(x,y) = y^a(1-y)^b(y^2-x^2)^\mu, \quad a,b,\mu > -1, \quad (x,y) \in \mathbb{V}_1^2, \quad (4.56)$$

$c_{a,b,\mu}$ is the normalization constant (see (1.55)):

$$c_{a,b,\mu} := \left(\int_{\mathbb{V}_1^2} W_{a,b,\mu}^J(x,y)dxdy \right)^{-1} = \frac{\Gamma(\mu+3/2)\Gamma(a+b+2\mu+3)}{\sqrt{\pi}\Gamma(\mu+1)\Gamma(a+2\mu+2)\Gamma(b+1)},$$

and the main part of (4.55) is denoted by:

$$\langle f,g\rangle_{\nabla^2} = c_{a,b,\mu} \int_{\mathbb{V}_1^2} \nabla^2 f(x,y) \cdot \nabla^2 g(x,y) W_{a,b,\mu}^J(x,y)dxdy. \tag{4.57}$$

We denote by $\mathscr{V}_n^2(S, W_{a,b,\mu}^J)$ and $\mathscr{V}_n^2(\nabla^2, W_{a,b,\mu}^J)$ the spaces of orthogonal polynomials of degree n with respect to (4.55) and (4.57), respectively, where $\dim \mathscr{V}_n^2(S, W_{a,b,\mu}^J) = \dim \mathscr{V}_n^2(\nabla^2, W_{a,b,\mu}^J) = n+1$. Similarly, we denote by $\mathscr{V}_n^2(W_{a,b,\mu}^J)$ the orthogonal polynomials with respect to:

$$\langle f,g\rangle_{W_{a,b,\mu}^J} = c_{a,b,\mu} \int_{\mathbb{V}_1^2} f(x,y)g(x,y) W_{a,b,\mu}^J(x,y)dxdy. \tag{4.58}$$

From (1.56)–(1.57), the polynomials in the space $\mathscr{V}_n^2(W_{0,b,\mu}^J)$, when the parameter $a=0$, satisfy the partial differential equation:

$$\begin{aligned} &y(1-y)\frac{\partial^2 P}{\partial y^2} + 2x(1-y)\frac{\partial^2 P}{\partial x \partial y} + (y-x^2)\frac{\partial^2 P}{\partial x^2} + (2\mu+2)\frac{\partial P}{\partial y} \\ &- (2\mu+b+3)\left[x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y}\right] = -n(n+2\mu+b+2)P, \\ &\qquad\qquad P \in \mathscr{V}_n^2(W_{0,b,\mu}^J), \quad b,\mu > -1. \end{aligned} \tag{4.59}$$

From Proposition 3.7, we know that if $P \in \mathscr{V}_n^2(S, W_{a,b,\mu}^J)$ or $P \in \mathscr{V}_n^2(\nabla^2, W_{a,b,\mu}^J)$ then:

$$(\partial_1+\partial_2)^2 P := \frac{\partial^2 P}{\partial x^2} + 2\frac{\partial^2 P}{\partial x \partial y} + \frac{\partial^2 P}{\partial y^2} \in \mathscr{V}_{n-2}^2(W_{a,b,\mu}^J). \tag{4.60}$$

Putting (4.59) and (4.60) together, then they prove the following result.

Proposition 4.13. *Let $P \in \mathscr{V}_n^2(S, W_{0,b,\mu}^J)$ or $P \in \mathscr{V}_n^2(\nabla^2, W_{0,b,\mu}^J)$, with $b,\mu > -1$. Then P satisfies the fourth-order partial differential equation:*

$$\begin{aligned} &\Bigg[y(1-y)\frac{\partial^2}{\partial y^2} + 2x(1-y)\frac{\partial^2}{\partial x \partial y} + (y-x^2)\frac{\partial^2}{\partial x^2} + (2\mu+2)\frac{\partial}{\partial y} - (2\mu+b+3)x\frac{\partial}{\partial x} \\ &\quad -(2\mu+b+3)y\frac{\partial}{\partial y} + (n-2)(n+2\mu+b)\mathcal{I}\Bigg] \left[\frac{\partial^2}{\partial x^2} + 2\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}\right] P = 0, \end{aligned}$$

where $\mathcal{I}$ is the identity operator.

The following corollary is a consequence of Theorem 3.1 and Proposition 3.9 for $\kappa = d = 2$.

Corollary 4.7. *Let $\{S_j^n : 0 \le j \le n\}$ denote a monic orthogonal basis of $\mathscr{V}_n^2(\nabla^2, W_{a,b,\mu}^J)$. Then, a monic orthogonal basis $\{\mathcal{S}_j^n : 0 \le j \le n\}$ of $\mathscr{V}_n^2(S, W_{a,b,\mu}^J)$ is given by:*

$$\mathcal{S}_0^0(x,y) = 1,$$
$$\mathcal{S}_0^1(x,y) = x - p_1, \quad \mathcal{S}_1^1(x,y) = y - p_2,$$
$$\mathcal{S}_j^n(x,y) = S_j^n(x,y) - \mathcal{T}^1(S_j^n, \mathbf{p}; x, y), \quad n \ge 2,$$

where $\mathcal{T}^1(S_j^n, \mathbf{p})$ is the Taylor polynomial of first degree of S_j^n at $\mathbf{p} = (p_1, p_2)$, and where

$$\langle \mathcal{S}_0^0, \mathcal{S}_0^0 \rangle_S = \lambda_0,$$
$$\langle \mathcal{S}_0^1, \mathcal{S}_0^1 \rangle_S = \langle \mathcal{S}_1^1, \mathcal{S}_1^1 \rangle_S = \lambda_1,$$
$$\langle \mathcal{S}_j^n, \mathcal{S}_j^n \rangle_S = \langle S_j^n, S_j^n \rangle_{\nabla^2}, \quad 0 \le j \le n, \quad n \ge 2.$$

.

Let us arrange the elements of the basis $\{S_j^n : 0 \le j \le n\}$ of $\mathscr{V}_n^2(\nabla^2, W_{a,b,\mu}^J)$ in a vector form. We denote by $\mathbb{S}_n$ the column vector of size $n+1$:

$$\mathbb{S}_n = \Big(S_0^n(x,y), \quad S_1^n(x,y), \quad \ldots, \quad S_n^n(x,y)\Big)^T,$$

and by $\mathbb{Q}_n$ the column vector of size $n+1$:

$$\mathbb{Q}_n = \Big(x^n, \quad x^{n-1}y, \quad x^{n-2}y^2, \quad \ldots, \quad y^n\Big)^T. \tag{4.61}$$

As we mentioned in Section 3.4.4.1, in order to simplify the computation of the matrix $\langle \mathbb{Q}_n, \mathbb{Q}_m^T \rangle_{\nabla^2}$ we only need that each entry of $\mathbb{Q}_n$ to be defined as a monomial. The following corollary is a consequence of Proposition 3.17 and (3.81) with $\kappa = d = 2$.

Corollary 4.8. *Let $n, m \ge 2$, $a, b, \mu > -1$, and let $\mathbb{Q}_n$ be defined in (4.61). Then, each entry of the matrix $\langle \mathbb{Q}_n, \mathbb{Q}_m^T \rangle_{\nabla^2} = \Big(\langle x^{n-i}y^i, x^{m-j}y^j \rangle_{\nabla^2}\Big)_{0 \le i \le n, 0 \le j \le m}$ of size $(n+1) \times (m+1)$ can be computed for $0 \le i \le n$ and $0 \le j \le m$ by:*

$$\langle x^{n-i}y^i, x^{m-j}y^j \rangle_{\nabla^2} = \begin{cases} \dfrac{(a+2\mu+2)_{n+m-4}(A_{i,j}^{n,m,\mu} + 2B_{i,j}^{n,m,\mu} + C_{i,j}^{n,m,\mu})}{(a+b+2\mu+3)_{n+m-4}}, & n+m-i-j \text{ is even}, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$A_{i,j}^{n,m,\mu} = \begin{cases} \dfrac{(-i)_2(-j)_2}{(\mu+3/2)_{\frac{n+m-i-j}{2}}} \left(\dfrac{1}{2}\right)_{\frac{n+m-i-j}{2}}, & 2 \le i \le n, \quad 2 \le j \le m, \\ 0, & \text{otherwise}, \end{cases}$$

$$B_{i,j}^{n,m,\mu} = \begin{cases} \dfrac{ij(n-i)(m-j)}{(\mu+3/2)_{\frac{n+m-i-j}{2}-1}} \left(\dfrac{1}{2}\right)_{\frac{n+m-i-j}{2}-1}, & 1 \le i \le n-1, \quad 1 \le j \le m-1, \\ 0, & \text{otherwise}, \end{cases}$$

$$C_{i,j}^{n,m,\mu} = \begin{cases} \dfrac{(i-n)_2(j-m)_2}{(\mu+3/2)_{\frac{n+m-i-j}{2}-2}} \left(\dfrac{1}{2}\right)_{\frac{n+m-i-j}{2}-2}, & 0 \le i \le n-2, \quad 0 \le j \le m-2, \\ 0, & \text{otherwise}. \end{cases}$$

On the bounded cone $\mathbb{V}_1^2$ the polynomials in $\mathbb{S}_n$ can be computed recursively by means of the relations in Proposition 3.10, Proposition 3.11 and Corollary 4.8. Next we show a numerical example.

Example 4.5 (Numerical, see Note 4.1). Let $a = b = \mu = -1/2$ the parameters for the weight function $W^J_{a,b,\mu}$ on the bounded cone, which is defined in (4.56). From (4.61) we have for $0 \le n \le 5$:

$$\mathbb{Q}_0 = 1, \quad \mathbb{Q}_1 = \left(x, \;\; y\right)^T, \quad \mathbb{Q}_2 = \left(x^2, \;\; xy, \;\; y^2\right)^T, \quad \mathbb{Q}_3 = \left(x^3, \;\; x^2y, \;\; xy^2, \;\; y^3\right)^T,$$
$$\mathbb{Q}_4 = \left(x^4, \;\; x^3y, \;\; x^2y^2, \;\; xy^3, \;\; y^4\right)^T, \quad \mathbb{Q}_5 = \left(x^5, \;\; x^4y, \;\; x^3y^2, \;\; x^2y^3, \;\; xy^4, \;\; y^5\right)^T.$$

From Proposition 3.11 and Corollary 4.8, with $\kappa = 2$, we have the following iterations:

1. First iteration:

$$\mathbf{B}_{3,2} = \left\langle \mathbb{Q}_3, \mathbb{Q}_2^T \right\rangle_{\nabla^2} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \quad \mathbf{B}_{4,2} = \left\langle \mathbb{Q}_4, \mathbb{Q}_2^T \right\rangle_{\nabla^2} = \begin{pmatrix} \frac{9}{2} & 0 & 0 \\ 0 & \frac{9}{8} & 0 \\ \frac{3}{2} & 0 & \frac{3}{4} \\ 0 & \frac{9}{4} & 0 \\ 0 & 0 & 9 \end{pmatrix},$$

$$\mathbf{B}_{5,2} = \left\langle \mathbb{Q}_5, \mathbb{Q}_2^T \right\rangle_{\nabla^2} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{15}{4} & 0 & 0 \\ 0 & \frac{15}{8} & 0 \\ \frac{5}{4} & 0 & \frac{15}{8} \\ 0 & \frac{5}{2} & 0 \\ 0 & 0 & \frac{25}{2} \end{pmatrix}, \quad \mathbf{H}_2^{\nabla^2} = \left\langle \mathbb{Q}_2, \mathbb{Q}_2^T \right\rangle_{\nabla^2} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

2. Second iteration:

$$\mathbf{B}_{4,3} = \left\langle \mathbb{Q}_4, \mathbb{Q}_3^T \right\rangle_{\nabla^2} - \mathbf{B}_{4,2} \left(\mathbf{H}_2^{\nabla^2}\right)^{-1} \mathbf{B}_{3,2}^T = \begin{pmatrix} 0 & \frac{3}{2} & 0 & 0 \\ \frac{45}{8} & 0 & \frac{3}{4} & 0 \\ 0 & 3 & 0 & \frac{3}{4} \\ 0 & 0 & \frac{27}{8} & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix},$$

$$\mathbf{B}_{5,3} = \left\langle \mathbb{Q}_5, \mathbb{Q}_3^T \right\rangle_{\nabla^2} - \mathbf{B}_{5,2} \left(\mathbf{H}_2^{\nabla^2}\right)^{-1} \mathbf{B}_{3,2}^T = \begin{pmatrix} \frac{1575}{128} & 0 & 0 & 0 \\ 0 & \frac{195}{64} & 0 & 0 \\ \frac{315}{64} & 0 & \frac{465}{256} & 0 \\ 0 & \frac{15}{4} & 0 & \frac{135}{64} \\ 0 & 0 & \frac{165}{32} & 0 \\ 0 & 0 & 0 & \frac{225}{16} \end{pmatrix},$$

$$\mathbf{H}_3^{\nabla^2} = \left\langle \mathbb{Q}_3, \mathbb{Q}_3^T \right\rangle_{\nabla^2} - \mathbf{B}_{3,2} \left(\mathbf{H}_2^{\nabla^2}\right)^{-1} \mathbf{B}_{3,2}^T = \begin{pmatrix} \frac{27}{4} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \frac{7}{4} & 0 \\ 0 & 0 & 0 & \frac{9}{2} \end{pmatrix}.$$

3. Third iteration:

$$\mathbf{B}_{5,4} = \left\langle \mathbb{Q}_5, \mathbb{Q}_4^T \right\rangle_{\nabla^2} - \sum_{j=2}^{3} \mathbf{B}_{5,j} \left(\mathbf{H}_j^{\nabla^2}\right)^{-1} \mathbf{B}_{4,j}^T = \begin{pmatrix} 0 & \frac{105}{128} & 0 & 0 & 0 \\ \frac{1737}{256} & 0 & -\frac{9}{128} & 0 & 0 \\ 0 & \frac{6513}{3584} & 0 & -\frac{135}{1792} & 0 \\ -\frac{81}{64} & 0 & \frac{615}{512} & 0 & \frac{27}{64} \\ 0 & -\frac{297}{448} & 0 & \frac{225}{112} & 0 \\ 0 & 0 & \frac{15}{64} & 0 & \frac{45}{16} \end{pmatrix},$$

$$\mathbf{H}_4^{\nabla^2} = \left\langle \mathbb{Q}_4, \mathbb{Q}_4^T \right\rangle_{\nabla^2} - \sum_{j=2}^{3} \mathbf{B}_{4,j} \left(\mathbf{H}_j^{\nabla^2}\right)^{-1} \mathbf{B}_{4,j}^T = \begin{pmatrix} \frac{549}{64} & 0 & -\frac{21}{32} & 0 & 0 \\ 0 & \frac{4035}{3584} & 0 & -\frac{225}{896} & 0 \\ -\frac{21}{32} & 0 & \frac{141}{256} & 0 & \frac{3}{32} \\ 0 & -\frac{225}{896} & 0 & \frac{45}{56} & 0 \\ 0 & 0 & \frac{3}{32} & 0 & \frac{9}{8} \end{pmatrix}.$$

Therefore, from Proposition 3.11, we have:

$$\mathbf{A}_{3,2} = \mathbf{B}_{3,2} \left(\mathbf{H}_2^{\nabla^2}\right)^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}, \quad \mathbf{A}_{4,2} = \mathbf{B}_{4,2} \left(\mathbf{H}_2^{\nabla^2}\right)^{-1} = \begin{pmatrix} \frac{9}{8} & 0 & 0 \\ 0 & \frac{9}{16} & 0 \\ \frac{3}{8} & 0 & \frac{3}{16} \\ 0 & \frac{9}{8} & 0 \\ 0 & 0 & \frac{9}{4} \end{pmatrix},$$

$$\mathbf{A}_{5,2} = \mathbf{B}_{5,2} \left(\mathbf{H}_2^{\nabla^2}\right)^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{15}{16} & 0 & 0 \\ 0 & \frac{15}{16} & 0 \\ \frac{5}{16} & 0 & \frac{15}{32} \\ 0 & \frac{5}{4} & 0 \\ 0 & 0 & \frac{25}{8} \end{pmatrix}, \quad \mathbf{A}_{4,3} = \mathbf{B}_{4,3} \left(\mathbf{H}_3^{\nabla^2}\right)^{-1} = \begin{pmatrix} 0 & \frac{3}{4} & 0 & 0 \\ \frac{5}{6} & 0 & \frac{3}{7} & 0 \\ 0 & \frac{3}{2} & 0 & \frac{1}{6} \\ 0 & 0 & \frac{27}{14} & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$\mathbf{A}_{5,3} = \mathbf{B}_{5,3}\left(\mathbf{H}_3^{\nabla^2}\right)^{-1} = \begin{pmatrix} \frac{175}{96} & 0 & 0 & 0 \\ 0 & \frac{195}{128} & 0 & 0 \\ \frac{35}{48} & 0 & \frac{465}{448} & 0 \\ 0 & \frac{15}{8} & 0 & \frac{15}{32} \\ 0 & 0 & \frac{165}{56} & 0 \\ 0 & 0 & 0 & \frac{25}{8} \end{pmatrix},$$

$$\mathbf{A}_{5,4} = \mathbf{B}_{5,4}\left(\mathbf{H}_4^{\nabla^2}\right)^{-1} = \begin{pmatrix} 0 & \frac{112}{143} & 0 & \frac{35}{143} & 0 \\ \frac{26491}{30780} & 0 & \frac{2336}{2565} & 0 & -\frac{584}{7695} \\ 0 & \frac{2449}{1430} & 0 & \frac{505}{1144} & 0 \\ \frac{142}{7695} & 0 & \frac{11131}{5130} & 0 & \frac{5977}{30780} \\ 0 & -\frac{24}{715} & 0 & \frac{356}{143} & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{2} \end{pmatrix}.$$

If $\overset{1}{=}$ denotes the congruence relation (4.2) then, from Proposition 3.10, we have for $0 \leq n \leq 5$ that:

$$\mathbb{S}_0 \overset{1}{=} \mathbb{Q}_0 = 1,$$
$$\mathbb{S}_1 \overset{1}{=} \mathbb{Q}_1 = \begin{pmatrix} x, & y \end{pmatrix}^T,$$
$$\mathbb{S}_2 \overset{1}{=} \mathbb{Q}_2 = \begin{pmatrix} x^2, & xy, & y^2 \end{pmatrix}^T,$$
$$\mathbb{S}_3 \overset{1}{=} \mathbb{Q}_3 - \mathbf{A}_{3,2}\mathbb{S}_2 \overset{1}{=} \begin{pmatrix} x^3, & x^2y - \frac{1}{2}x^2, & xy^2 - xy, & y^3 - \frac{3}{2}y^2 \end{pmatrix}^T,$$
$$\mathbb{S}_4 \overset{1}{=} \mathbb{Q}_4 - \mathbf{A}_{4,3}\mathbb{S}_3 - \mathbf{A}_{4,2}\mathbb{S}_2 \overset{1}{=} \begin{pmatrix} x^4 - \frac{3}{4}x^2y - \frac{3}{4}x^2 \\ x^3y - \frac{5}{6}x^3 - \frac{3}{7}xy^2 - \frac{15}{112}xy \\ x^2y^2 - \frac{3}{2}x^2y - \frac{1}{6}y^3 + \frac{3}{8}x^2 + \frac{1}{16}y^2 \\ xy^3 - \frac{27}{14}xy^2 + \frac{45}{56}xy \\ y^4 - 2y^3 + \frac{3}{4}y^2 \end{pmatrix},$$
$$\mathbb{S}_5 \overset{1}{=} \mathbb{Q}_5 - \mathbf{A}_{5,4}\mathbb{S}_4 - \mathbf{A}_{5,3}\mathbb{S}_3 - \mathbf{A}_{5,2}\mathbb{S}_2$$
$$\overset{1}{=} \begin{pmatrix} x^5 - \frac{112}{143}x^3y - \frac{35}{143}xy^3 - \frac{5355}{4576}x^3 + \frac{21}{26}xy^2 - \frac{105}{1144}xy \\ x^4y - \frac{26491}{30780}x^4 - \frac{2336}{2565}x^2y^2 + \frac{584}{7695}y^4 + \frac{1687}{3456}x^2y + \frac{16835}{131328}x^2 \\ x^3y^2 - \frac{2449}{1430}x^3y - \frac{505}{1144}xy^3 + \frac{1597}{2288}x^3 + \frac{2277}{4160}xy^2 - \frac{57}{2288}xy \\ x^2y^3 - \frac{142}{7695}x^4 - \frac{11131}{5130}x^2y^2 - \frac{5977}{30780}y^4 + \frac{301}{216}x^2y + \frac{9}{32}y^3 - \frac{1435}{8208}x^2 - \frac{3}{64}y^2 \\ xy^4 + \frac{24}{715}x^3y - \frac{356}{143}xy^3 - \frac{4}{143}x^3 + \frac{957}{520}xy^2 - \frac{353}{1144}xy \\ y^5 - \frac{5}{2}y^4 + \frac{15}{8}y^3 - \frac{5}{16}y^2 \end{pmatrix}.$$

Finally, let $\mathbf{p} = (p_1, p_2) = (-1, 1)$. The Taylor polynomials of first degree at $\mathbf{p}$ of

$\mathbb{S}_2, \mathbb{S}_3, \mathbb{S}_4$ and $\mathbb{S}_5$ are given by:

$$\mathcal{T}^1(\mathbb{S}_2, \mathbf{p}; x, y) = \begin{pmatrix}-2x-1, & x-y+1, & 2y-1\end{pmatrix}^T,$$

$$\mathcal{T}^1(\mathbb{S}_3, \mathbf{p}; x, y) = \begin{pmatrix}3x+2, & -x+y-\frac{3}{2}, & -y+1, & -\frac{1}{2}\end{pmatrix}^T,$$

$$\mathcal{T}^1(\mathbb{S}_4, \mathbf{p}; x, y) = \begin{pmatrix} -x-\frac{3}{4}y-\frac{3}{4} \\ -\frac{1}{16}x-\frac{1}{112}y+\frac{115}{336} \\ \frac{1}{4}x+\frac{1}{8}y-\frac{5}{48} \\ -\frac{1}{8}x+\frac{3}{56}y-\frac{3}{56} \\ -\frac{1}{2}y+\frac{1}{4} \end{pmatrix},$$

$$\mathcal{T}^1(\mathbb{S}_5, \mathbf{p}; x, y) = \begin{pmatrix} -\frac{137}{352}x-\frac{7}{1144}y+\frac{227}{2288} \\ \frac{30941}{984960}x-\frac{5857}{196992}y-\frac{11827}{656640} \\ \frac{131}{3520}x-\frac{753}{22880}y+\frac{17}{4576} \\ -\frac{1477}{61560}x+\frac{335}{12312}y+\frac{3191}{164160} \\ \frac{13}{220}x+\frac{359}{5720}y-\frac{59}{1144} \\ \frac{1}{16} \end{pmatrix}.$$

Then, from Corollary 4.7, the following polynomials are a monic orthogonal basis $\left\{\mathcal{S}_j^n : 0 \le j \le n\right\}$ for the space $\mathscr{V}_n^2(S, W_{a,b,\mu}^J)$ on the bounded cone for $0 \le n \le 5$:

- For the space $\mathscr{V}_0^2(S, W_{a,b,\mu}^J)$:

$$\mathcal{S}_0^0(x, y) = 1.$$

- For the space $\mathscr{V}_1^2(S, W_{a,b,\mu}^J)$:

$$\mathcal{S}_0^1(x, y) = x+1,$$
$$\mathcal{S}_1^1(x, y) = y-1.$$

- For the space $\mathscr{V}_2^2(S, W_{a,b,\mu}^J)$:

$$\mathcal{S}_0^2(x, y) = x^2+2x+1,$$
$$\mathcal{S}_1^2(x, y) = xy-x+y-1,$$
$$\mathcal{S}_2^2(x, y) = y^2-2y+1.$$

- For the space $\mathscr{V}_3^2(S, W_{a,b,\mu}^J)$:

$$\mathcal{S}_0^3(x, y) = x^3-3x-2,$$
$$\mathcal{S}_1^3(x, y) = x^2y-\frac{1}{2}x^2+x-y+\frac{3}{2},$$
$$\mathcal{S}_2^3(x, y) = xy^2-xy+y-1,$$
$$\mathcal{S}_3^3(x, y) = y^3-\frac{3}{2}y^2+\frac{1}{2}.$$

- For the space $\mathscr{V}_4^2(S, W_{a,b,\mu}^J)$:

$$\begin{aligned}
\mathcal{S}_0^4(x,y) &= x^4 - \frac{3}{4}x^2y - \frac{3}{4}x^2 + x + \frac{3}{4}y + \frac{3}{4},\\
\mathcal{S}_1^4(x,y) &= x^3y - \frac{5}{6}x^3 - \frac{3}{7}xy^2 - \frac{15}{112}xy + \frac{1}{16}x + \frac{1}{112}y - \frac{115}{336},\\
\mathcal{S}_2^4(x,y) &= x^2y^2 - \frac{3}{2}x^2y - \frac{1}{6}y^3 + \frac{3}{8}x^2 + \frac{1}{16}y^2 - \frac{1}{4}x - \frac{1}{8}y + \frac{5}{48},\\
\mathcal{S}_3^4(x,y) &= xy^3 - \frac{27}{14}xy^2 + \frac{45}{56}xy + \frac{1}{8}x - \frac{3}{56}y + \frac{3}{56},\\
\mathcal{S}_4^4(x,y) &= y^4 - 2y^3 + \frac{3}{4}y^2 + \frac{1}{2}y - \frac{1}{4}.
\end{aligned}$$

- For the space $\mathscr{V}_5^2(S, W_{a,b,\mu}^J)$:

$$\begin{aligned}
\mathcal{S}_0^5(x,y) &= x^5 - \frac{112}{143}x^3y - \frac{35}{143}xy^3 - \frac{5355}{4576}x^3 + \frac{21}{26}xy^2 - \frac{105}{1144}xy + \frac{137}{352}x\\
&\quad + \frac{7}{1144}y - \frac{227}{2288},\\
\mathcal{S}_1^5(x,y) &= x^4y - \frac{26491}{30780}x^4 - \frac{2336}{2565}x^2y^2 + \frac{584}{7695}y^4 + \frac{1687}{3456}x^2y + \frac{16835}{131328}x^2\\
&\quad - \frac{30941}{984960}x + \frac{5857}{196992}y + \frac{11827}{656640},\\
\mathcal{S}_2^5(x,y) &= x^3y^2 - \frac{2449}{1430}x^3y - \frac{505}{1144}xy^3 + \frac{1597}{2288}x^3 + \frac{2277}{4160}xy^2 - \frac{57}{2288}xy\\
&\quad - \frac{131}{3520}x + \frac{753}{22880}y - \frac{17}{4576},\\
\mathcal{S}_3^5(x,y) &= x^2y^3 - \frac{142}{7695}x^4 - \frac{11131}{5130}x^2y^2 - \frac{5977}{30780}y^4 + \frac{301}{216}x^2y + \frac{9}{32}y^3\\
&\quad - \frac{1435}{8208}x^2 - \frac{3}{64}y^2 + \frac{1477}{61560}x - \frac{335}{12312}y - \frac{3191}{164160},\\
\mathcal{S}_4^5(x,y) &= xy^4 + \frac{24}{715}x^3y - \frac{356}{143}xy^3 - \frac{4}{143}x^3 + \frac{957}{520}xy^2 - \frac{353}{1144}xy - \frac{13}{220}x\\
&\quad - \frac{359}{5720}y + \frac{59}{1144},\\
\mathcal{S}_5^5(x,y) &= y^5 - \frac{5}{2}y^4 + \frac{15}{8}y^3 - \frac{5}{16}y^2 - \frac{1}{16}.
\end{aligned}$$

This completes our numerical example on the bounded cone.

4.5.2 Laguerre weight function

The unbounded cone of $\mathbb{R}^2$ is the set:

$$\mathbb{V}_\infty^2 := \left\{(x,y) \in \mathbb{R}^2 : |x| \le y, 0 \le y < \infty\right\}.$$

In this section we construct the Sobolev orthogonal polynomials in two variables with respect to the inner product:

$$\langle f, g\rangle_S = c_{a,\mu} \int_{\mathbb{V}^2_\infty} \nabla^3 f(x,y) \cdot \nabla^3 g(x,y) W^L_{a,\mu}(x,y) dxdy + \lambda_2 \nabla^2 f(p_1,p_2) \cdot \nabla^2 g(p_1,p_2) + \lambda_1 \nabla f(p_1,p_2) \cdot \nabla g(p_1,p_2) + \lambda_0 f(p_1,p_2) g(p_1,p_2), \quad (4.62)$$

where (p_1, p_2) is a given point in $\mathbb{R}^2$, $\lambda_0, \lambda_1, \lambda_2 > 0$, ∇f, $\nabla^2 f$ and $\nabla^3 f$ are given in (4.1), $W^L_{a,\mu}$ is the weight function on the unbounded cone:

$$W^L_{a,\mu}(x,y) = y^a e^{-y}(y^2 - x^2)^\mu, \quad a, \mu > -1, \quad (x,y) \in \mathbb{V}^2_\infty, \quad (4.63)$$

$c_{a,\mu}$ is the normalization constant (see (1.59)):

$$c_{a,\mu} := \left(\int_{\mathbb{V}^2_\infty} W^L_{a,\mu}(x,y) dxdy \right)^{-1} = \frac{\Gamma(\mu + 3/2)}{\sqrt{\pi}\Gamma(\mu+1)\Gamma(a+2\mu+2)},$$

and the main part of (4.62) is denoted by:

$$\langle f, g\rangle_{\nabla^3} = c_{a,\mu} \int_{\mathbb{V}^2_\infty} \nabla^3 f(x,y) \cdot \nabla^3 g(x,y) W^L_{a,\mu}(x,y) dxdy. \quad (4.64)$$

We denote by $\mathscr{V}^2_n(S, W^L_{a,\mu})$ and $\mathscr{V}^2_n(\nabla^3, W^L_{a,\mu})$ the spaces of orthogonal polynomials of degree n with respect to (4.62) and (4.64), respectively, where $\dim \mathscr{V}^2_n(S, W^L_{a,\mu}) = \dim \mathscr{V}^2_n(\nabla^3, W^L_{a,\mu}) = n + 1$. Similarly, we denote by $\mathscr{V}^2_n(W^L_{a,\mu})$ the orthogonal polynomials with respect to:

$$\langle f, g\rangle_{W^L_{a,\mu}} = c_{a,\mu} \int_{\mathbb{V}^2_\infty} f(x,y) g(x,y) W^L_{a,\mu}(x,y) dxdy. \quad (4.65)$$

From (1.60)–(1.61), the polynomials in the space $\mathscr{V}^2_n(W^L_{0,\mu})$, when the parameter $a = 0$, satisfy the partial differential equation:

$$y\frac{\partial^2 P}{\partial x^2} + y\frac{\partial^2 P}{\partial y^2} + 2x\frac{\partial^2 P}{\partial x \partial y} - x\frac{\partial P}{\partial x} + (2\mu + 2 - y)\frac{\partial P}{\partial y} = -nP, \quad P \in \mathscr{V}^2_n(W^L_{0,\mu}), \quad \mu > -1. \quad (4.66)$$

From Proposition 3.7, we know that if $P \in \mathscr{V}^2_n(S, W^L_{a,\mu})$ or $P \in \mathscr{V}^2_n(\nabla^3, W^L_{a,\mu})$ then:

$$(\partial_1 + \partial_2)^3 P := \frac{\partial^3 P}{\partial x^3} + 3\frac{\partial^3 P}{\partial x^2 \partial y} + 3\frac{\partial^3 P}{\partial x \partial y^2} + \frac{\partial^3 P}{\partial y^3} \in \mathscr{V}^2_{n-3}(W^L_{a,\mu}). \quad (4.67)$$

Putting (4.66) and (4.67) together, then they prove the following result.

Proposition 4.14. *Let $P \in \mathscr{V}^2_n(S, W^L_{0,\mu})$ or $P \in \mathscr{V}^2_n(\nabla^3, W^L_{0,\mu})$, with $\mu > -1$. Then P satisfies the fifth-order partial differential equation:*

$$\left[y\frac{\partial^2}{\partial x^2} + y\frac{\partial^2}{\partial y^2} + 2x\frac{\partial^2}{\partial x \partial y} - x\frac{\partial}{\partial x} + (2\mu + 2 - y)\frac{\partial}{\partial y} + (n-3)\mathcal{I} \right] \left[\frac{\partial^3}{\partial x^3} + 3\frac{\partial^3}{\partial x^2 \partial y} + 3\frac{\partial^3}{\partial x \partial y^2} + \frac{\partial^3}{\partial y^3} \right] P = 0,$$

where $\mathcal{I}$ is the identity operator.

The following corollary is a consequence of Theorem 3.1 for $\kappa = 3$ and $d = 2$.

Corollary 4.9. *Let* $\left\{S_j^n : 0 \le j \le n\right\}$ *denote a monic orthogonal basis of* $\mathscr{V}_n^2(\nabla^3, W_{a,\mu}^L)$. *Then, a monic orthogonal basis* $\left\{\mathcal{S}_j^n : 0 \le j \le n\right\}$ *of* $\mathscr{V}_n^2(S, W_{a,\mu}^L)$ *is given by:*

$$\begin{aligned}
&\mathcal{S}_0^0(x,y) = 1,\\
&\mathcal{S}_0^1(x,y) = x - p_1, \quad \mathcal{S}_1^1(x,y) = y - p_2,\\
&\mathcal{S}_0^2(x,y) = (x-p_1)^2, \quad \mathcal{S}_1^2(x,y) = (x-p_1)(y-p_2), \quad \mathcal{S}_2^2(x,y) = (y-p_2)^2,\\
&\mathcal{S}_j^n(x,y) = S_j^n(x,y) - \mathcal{T}^2(S_j^n, \mathbf{p}; x, y), \quad n \ge 3.
\end{aligned}$$

where $\mathcal{T}^2(S_j^n, \mathbf{p})$ *is the Taylor polynomial of second degree of* S_j^n *at* $\mathbf{p} = (p_1, p_2)$.

Let us arrange the elements of the basis $\left\{S_j^n : 0 \le j \le n\right\}$ of $\mathscr{V}_n^2(\nabla^3, W_{a,\mu}^L)$ in a vector form. We denote by $\mathbb{S}_n$ the column vector of size $n+1$:

$$\mathbb{S}_n = \Big(S_0^n(x,y), \quad S_1^n(x,y), \quad \ldots, \quad S_n^n(x,y)\Big)^T,$$

and by $\mathbb{Q}_n$ the column vector of size $n+1$:

$$\mathbb{Q}_n = \Big(x^n, \quad x^{n-1}y, \quad x^{n-2}y^2, \quad \ldots, \quad y^n\Big)^T. \tag{4.68}$$

The following corollary is a consequence of Proposition 3.17 and (3.82) with $\kappa = 3$ and $d = 2$.

Corollary 4.10. *Let* $n, m \ge 3$, $a, \mu > -1$, *and let* $\mathbb{Q}_n$ *be defined in* (4.68). *Then, each entry of the matrix* $\left\langle \mathbb{Q}_n, \mathbb{Q}_m^T \right\rangle_{\nabla^3} = \Big(\langle x^{n-i}y^i, x^{m-j}y^j\rangle_{\nabla^3}\Big)_{0\le i\le n, 0\le j\le m}$ *of size* $(n+1)\times(m+1)$ *can be computed for* $0 \le i \le n$ *and* $0 \le j \le m$ *by*

$$\left\langle x^{n-i}y^i, x^{m-j}y^j\right\rangle_{\nabla^3} = \begin{cases} (a+2\mu+2)_{n+m-6}\left(A_{i,j}^{n,m,\mu} + 3B_{i,j}^{n,m,\mu} + 3C_{i,j}^{n,m,\mu} + D_{i,j}^{n,m,\mu}\right), & n+m-i-j \text{ is even},\\ 0, & \text{otherwise}, \end{cases}$$

where

$$A_{i,j}^{n,m,\mu} = \begin{cases} \dfrac{(-i)_3(-j)_3}{(\mu+3/2)_{\frac{n+m-i-j}{2}}}\left(\dfrac{1}{2}\right)_{\frac{n+m-i-j}{2}}, & 3 \le i \le n, \quad 3 \le j \le m,\\ 0, & \text{otherwise}, \end{cases}$$

$$B_{i,j}^{n,m,\mu} = \begin{cases} \dfrac{(n-i)(m-j)(-i)_2(-j)_2}{(\mu+3/2)_{\frac{n+m-i-j}{2}-1}}\left(\dfrac{1}{2}\right)_{\frac{n+m-i-j}{2}-1}, & 2 \le i \le n-1, \quad 2 \le j \le m-1,\\ 0, & \text{otherwise}, \end{cases}$$

$$C_{i,j}^{n,m,\mu} = \begin{cases} \dfrac{ij(i-n)_2(j-m)_2}{(\mu+3/2)_{\frac{n+m-i-j}{2}-2}}\left(\dfrac{1}{2}\right)_{\frac{n+m-i-j}{2}-2}, & 1 \le i \le n-2, \quad 1 \le j \le m-2,\\ 0, & \text{otherwise}, \end{cases}$$

$$D_{i,j}^{n,m,\mu} = \begin{cases} \dfrac{(i-n)_3(j-m)_3}{(\mu+3/2)_{\frac{n+m-i-j}{2}-3}}\left(\dfrac{1}{2}\right)_{\frac{n+m-i-j}{2}-3}, & 0 \le i \le n-3, \quad 0 \le j \le m-3,\\ 0, & \text{otherwise}. \end{cases}$$

On the unbounded cone $\mathbb{V}^2_\infty$ the polynomials in $\mathbb{S}_n$ can be computed recursively by means of the relations in Proposition 3.10, Proposition 3.11 and Corollary 4.10. Next we show a numerical example.

Example 4.6 (Numerical, see Note 4.1). Let $a = \mu = -1/2$ the parameters for the weight function $W^L_{a,\mu}$ on the unbounded cone, which is defined in (4.63). From (4.68) we have for $0 \le n \le 5$:

$$\mathbb{Q}_0 = 1, \quad \mathbb{Q}_1 = \begin{pmatrix} x, & y \end{pmatrix}^T, \quad \mathbb{Q}_2 = \begin{pmatrix} x^2, & xy, & y^2 \end{pmatrix}^T, \quad \mathbb{Q}_3 = \begin{pmatrix} x^3, & x^2y, & xy^2, & y^3 \end{pmatrix}^T,$$
$$\mathbb{Q}_4 = \begin{pmatrix} x^4, & x^3y, & x^2y^2, & xy^3, & y^4 \end{pmatrix}^T, \quad \mathbb{Q}_5 = \begin{pmatrix} x^5, & x^4y, & x^3y^2, & x^2y^3, & xy^4, & y^5 \end{pmatrix}^T.$$

From Proposition 3.11 and Corollary 4.10, with $\kappa = 3$, we have the following iterations:

1. First iteration:

$$\mathbf{B}_{4,3} = \left\langle \mathbb{Q}_4, \mathbb{Q}_3^T \right\rangle_{\nabla^3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 18 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 18 & 0 \\ 0 & 0 & 0 & 72 \end{pmatrix}, \ \mathbf{H}_3^{\nabla^3} = \left\langle \mathbb{Q}_3, \mathbb{Q}_3^T \right\rangle_{\nabla^3} = \begin{pmatrix} 36 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix},$$
$$\mathbf{B}_{5,3} = \left\langle \mathbb{Q}_5, \mathbb{Q}_3^T \right\rangle_{\nabla^3} = \begin{pmatrix} 135 & 0 & 0 & 0 \\ 0 & 27 & 0 & 0 \\ 27 & 0 & \frac{27}{2} & 0 \\ 0 & 27 & 0 & \frac{27}{2} \\ 0 & 0 & 54 & 0 \\ 0 & 0 & 0 & 270 \end{pmatrix}.$$

2. Second iteration:

$$\mathbf{B}_{5,4} = \left\langle \mathbb{Q}_5, \mathbb{Q}_4^T \right\rangle_{\nabla^3} - \mathbf{B}_{5,3} \left(\mathbf{H}_3^{\nabla^3}\right)^{-1} \mathbf{B}_{4,3}^T = \begin{pmatrix} 0 & 270 & 0 & 0 & 0 \\ 540 & 0 & 108 & 0 & 0 \\ 0 & \frac{513}{2} & 0 & 81 & 0 \\ 0 & 0 & 243 & 0 & 108 \\ 0 & 0 & 0 & 459 & 0 \\ 0 & 0 & 0 & 0 & 2160 \end{pmatrix},$$
$$\mathbf{H}_4^{\nabla^3} = \left\langle \mathbb{Q}_4, \mathbb{Q}_4^T \right\rangle_{\nabla^3} - \mathbf{B}_{4,3} \left(\mathbf{H}_3^{\nabla^3}\right)^{-1} \mathbf{B}_{4,3}^T = \begin{pmatrix} 216 & 0 & 0 & 0 & 0 \\ 0 & \frac{117}{2} & 0 & 0 & 0 \\ 0 & 0 & 42 & 0 & 0 \\ 0 & 0 & 0 & \frac{135}{2} & 0 \\ 0 & 0 & 0 & 0 & 288 \end{pmatrix}.$$

Therefore, from Proposition 3.11, we have:

$$\mathbf{A}_{4,3} = \mathbf{B}_{4,3}\left(\mathbf{H}_3^{\nabla^3}\right)^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{A}_{5,3} = \mathbf{B}_{5,3}\left(\mathbf{H}_3^{\nabla^3}\right)^{-1} = \begin{pmatrix} \frac{15}{4} & 0 & 0 & 0 \\ 0 & \frac{9}{4} & 0 & 0 \\ \frac{3}{4} & 0 & \frac{9}{8} & 0 \\ 0 & \frac{9}{4} & 0 & \frac{3}{8} \\ 0 & 0 & \frac{9}{2} & 0 \\ 0 & 0 & 0 & \frac{15}{2} \end{pmatrix},$$

$$\mathbf{A}_{5,4} = \mathbf{B}_{5,4}\left(\mathbf{H}_4^{\nabla^3}\right)^{-1} = \begin{pmatrix} 0 & \frac{60}{13} & 0 & 0 & 0 \\ \frac{5}{2} & 0 & \frac{18}{7} & 0 & 0 \\ 0 & \frac{57}{13} & 0 & \frac{6}{5} & 0 \\ 0 & 0 & \frac{81}{14} & 0 & \frac{3}{8} \\ 0 & 0 & 0 & \frac{34}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{15}{2} \end{pmatrix}.$$

If $\stackrel{2}{=}$ denotes the congruence relation (4.3) then, from Proposition 3.10, we have for $0 \leq n \leq 5$ that:

$$\mathbb{S}_0 \stackrel{2}{=} \mathbb{Q}_0 = 1,$$
$$\mathbb{S}_1 \stackrel{2}{=} \mathbb{Q}_1 = \begin{pmatrix} x, & y \end{pmatrix}^T,$$
$$\mathbb{S}_2 \stackrel{2}{=} \mathbb{Q}_2 = \left(x^2, \quad xy, \quad y^2\right)^T,$$
$$\mathbb{S}_3 \stackrel{2}{=} \mathbb{Q}_3 = \left(x^3, \quad x^2y, \quad xy^2, \quad y^3\right)^T,$$
$$\mathbb{S}_4 \stackrel{2}{=} \mathbb{Q}_4 - \mathbf{A}_{4,3}\mathbb{S}_3 \stackrel{2}{=} \left(x^4, \quad x^3y - \tfrac{1}{2}x^3, \quad x^2y^2 - x^2y, \quad xy^3 - \tfrac{3}{2}xy^2, \quad y^4 - 2y^3\right)^T,$$
$$\mathbb{S}_5 \stackrel{2}{=} \mathbb{Q}_5 - \mathbf{A}_{5,4}\mathbb{S}_4 - \mathbf{A}_{5,3}\mathbb{S}_3 \stackrel{2}{=} \begin{pmatrix} x^5 - \frac{60}{13}x^3y - \frac{75}{52}x^3 \\ x^4y - \frac{5}{2}x^4 - \frac{18}{7}x^2y^2 + \frac{9}{28}x^2y \\ x^3y^2 - \frac{57}{13}x^3y - \frac{6}{5}xy^3 + \frac{75}{52}x^3 + \frac{27}{40}xy^2 \\ x^2y^3 - \frac{81}{14}x^2y^2 - \frac{3}{8}y^4 + \frac{99}{28}x^2y + \frac{3}{8}y^3 \\ xy^4 - \frac{34}{5}xy^3 + \frac{57}{10}xy^2 \\ y^5 - \frac{15}{2}y^4 + \frac{15}{2}y^3 \end{pmatrix}.$$

Finally, let $\mathbf{p} = (p_1, p_2) = (0, 1)$. The Taylor polynomials of second degree at $\mathbf{p}$ of $\mathbb{S}_3, \mathbb{S}_4$ and $\mathbb{S}_5$ are given by:

$$\mathcal{T}^2(\mathbb{S}_3, \mathbf{p}; x, y) = \left(0, \quad x^2, \quad 2xy - x, \quad 3y^2 - 3y + 1\right)^T,$$
$$\mathcal{T}^2(\mathbb{S}_4, \mathbf{p}; x, y) = \left(0, \quad 0, \quad 0, \quad -\tfrac{1}{2}x, \quad -2y + 1\right)^T,$$

$$\mathcal{T}^2(\mathbb{S}_5, \mathbf{p}; x, y) = \begin{pmatrix} 0 \\ -\frac{9}{4}x^2 \\ -\frac{9}{4}xy + \frac{69}{40}x \\ -\frac{5}{4}x^2 - \frac{9}{8}y^2 + \frac{15}{8}y - \frac{3}{4} \\ -5xy + \frac{49}{10}x \\ -\frac{25}{2}y^2 + \frac{45}{2}y - 9 \end{pmatrix}.$$

Then, from Corollary 4.9, the following polynomials are a monic orthogonal basis $\left\{\mathcal{S}_j^n : 0 \leq j \leq n\right\}$ for the space $\mathscr{V}_n^2(S, W_{a,\mu}^L)$ on the unbounded cone for $0 \leq n \leq 5$:

- For the space $\mathscr{V}_0^2(S, W_{a,\mu}^L)$:

  $\mathcal{S}_0^0(x, y) = 1.$

- For the space $\mathscr{V}_1^2(S, W_{a,\mu}^L)$:

  $\mathcal{S}_0^1(x, y) = x,$
  $\mathcal{S}_1^1(x, y) = y - 1.$

- For the space $\mathscr{V}_2^2(S, W_{a,\mu}^L)$:

  $\mathcal{S}_0^2(x, y) = x^2,$
  $\mathcal{S}_1^2(x, y) = xy - x,$
  $\mathcal{S}_2^2(x, y) = y^2 - 2y + 1.$

- For the space $\mathscr{V}_3^2(S, W_{a,\mu}^L)$:

  $\mathcal{S}_0^3(x, y) = x^3,$
  $\mathcal{S}_1^3(x, y) = x^2y - x^2,$
  $\mathcal{S}_2^3(x, y) = xy^2 - 2xy + x,$
  $\mathcal{S}_3^3(x, y) = y^3 - 3y^2 + 3y - 1.$

- For the space $\mathscr{V}_4^2(S, W_{a,\mu}^L)$:

  $\mathcal{S}_0^4(x, y) = x^4,$
  $\mathcal{S}_1^4(x, y) = x^3y - \dfrac{1}{2}x^3,$
  $\mathcal{S}_2^4(x, y) = x^2y^2 - x^2y,$
  $\mathcal{S}_3^4(x, y) = xy^3 - \dfrac{3}{2}xy^2 + \dfrac{1}{2}x,$
  $\mathcal{S}_4^4(x, y) = y^4 - 2y^3 + 2y - 1.$

- For the space $\mathscr{V}_5^2(S, W_{a,\mu}^L)$:

$$
\begin{aligned}
\mathcal{S}_0^5(x,y) &= x^5 - \frac{60}{13}x^3y - \frac{75}{52}x^3,\\
\mathcal{S}_1^5(x,y) &= x^4y - \frac{5}{2}x^4 - \frac{18}{7}x^2y^2 + \frac{9}{28}x^2y + \frac{9}{4}x^2,\\
\mathcal{S}_2^5(x,y) &= x^3y^2 - \frac{57}{13}x^3y - \frac{6}{5}xy^3 + \frac{75}{52}x^3 + \frac{27}{40}xy^2 + \frac{9}{4}xy - \frac{69}{40}x,\\
\mathcal{S}_3^5(x,y) &= x^2y^3 - \frac{81}{14}x^2y^2 - \frac{3}{8}y^4 + \frac{99}{28}x^2y + \frac{3}{8}y^3 + \frac{5}{4}x^2 + \frac{9}{8}y^2 - \frac{15}{8}y + \frac{3}{4},\\
\mathcal{S}_4^5(x,y) &= xy^4 - \frac{34}{5}xy^3 + \frac{57}{10}xy^2 + 5xy - \frac{49}{10}x,\\
\mathcal{S}_5^5(x,y) &= y^5 - \frac{15}{2}y^4 + \frac{15}{2}y^3 + \frac{25}{2}y^2 - \frac{45}{2}y + 9.
\end{aligned}
$$

This completes our numerical example on the unbounded cone.

Chapter 5

Open problems

In this chapter we mention some open problems related to the Sobolev orthogonal polynomials with respect to the inner product (3.12). On the following problems, we suppose that the reader is familiar with Chapter 3.

Problem 5.1. *To find an explicit basis for the space* $\mathscr{V}_n^d(\nabla^\kappa, W)$*, and therefore for* $\mathscr{V}_n^d(S, W)$*, of Sobolev orthogonal polynomials of degree* n *in* d *variables with respect to the bilinear form* (3.13)*, without appealing to iterative methods.*

Problem 5.2. *To study the Sobolev orthogonal polynomials in* d *variables with respect to an inner product of the form:*

$$\langle f, g\rangle_S = \langle f, g\rangle_{\nabla^\kappa} + \langle f, g\rangle ,$$

where $\langle \cdot, \cdot\rangle_{\nabla^\kappa}$*,* $\kappa \in \mathbb{N}$*, is the bilinear form* (3.13)*, and* $\langle \cdot, \cdot\rangle$ *is an additional term that makes the inner product* $\langle \cdot, \cdot\rangle_S$ *well-defined on* Π^d*. In particular, we could begin this study for a continuous-discrete inner product of the form:*

$$\langle f, g\rangle_S = \langle f, g\rangle_{\nabla^\kappa} + \sum_{i=0}^{\kappa-1} (\nabla^i f(\mathbf{p}))^T \mathbf{M}_i (\nabla^i g(\mathbf{p})), \tag{5.1}$$

where $\mathbf{M}_i$ *is a positive definite matrix of size* $d^i \times d^i$*,* $0 \le i \le \kappa - 1$*. Notice that the inner product* (3.12) *is a particular case of* (5.1) *by choosing* $\mathbf{M}_i = \lambda_i \mathbf{I}$*,* $\lambda_i > 0$*, where* $\mathbf{I}$ *is the identity matrix. Many of the results in Chapter 3 can be applied to this case.*

Problem 5.3. *Even though the equations* (3.61)*,* (3.63)*,* (3.65)*,* (3.69)*,* (3.70)*,* (3.84) *and* (3.86)*, show particular examples of partial differential equations, it is still an open problem to know if there is a differential operator for which the polynomials in the space* $\mathscr{V}_n^d(S, W)$ *are eigenfunctions.*

Problem 5.4. *It is still an open problem to study the zeros for the polynomials in the space* $\mathscr{V}_n^d(S, W)$*.*

Problem 5.5. *To study Fourier orthogonal series by using the polynomials in the space* $\mathscr{V}_n^d(S, W)$ *and related problems (for example, approximation theory and reproducing kernels).*

Bibliography

[1] Abramowitz, Milton and Stegun, Irene Ann, eds. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables.* Applied mathematics series - 55. Washington, D.C.: National Bureau of Standards, June 1964.

[2] Aktaş, Rabia and Xu, Yuan. "Sobolev orthogonal polynomials on a simplex". In: *Int. Math. Res. Not. IMRN* 2013.13 (2012), pp. 3087–3131.

[3] Alfaro, Manuel, Álvarez de Morales, María, and Rezola, M. Luisa. "Orthogonality of the Jacobi polynomials with negative integer parameters". In: *J. Comput. Appl. Math.* 145.2 (2002), pp. 379–386.

[4] Alfaro, Manuel, López, Guillermo, and Rezola, M. Luisa. "Some properties of zeros of Sobolev-type orthogonal polynomials". In: *J. Comput. Appl. Math.* 69.1 (1996), pp. 171–179.

[5] Alfaro, Manuel, Marcellán, Francisco, Peña, Ana, and Rezola, M. Luisa. "On linearly related orthogonal polynomials and their functionals". In: *J. Math. Anal. Appl.* 287.1 (2003), pp. 307–319.

[6] Alfaro, Manuel, Pérez, Teresa E., Piñar, Miguel A., and Rezola, M. Luisa. "Sobolev orthogonal polynomials: the discrete-continuous case". In: *Methods Appl. Anal.* 6.4 (1999), pp. 593–616.

[7] Althammer, Peter. "Eine Erweiterung des Orthogonalitätsbegriffes bei Polynomen und deren Anwendung auf die beste Approximation". In: *J. Reine Angew. Math.* 211 (Apr. 1962), pp. 192–204.

[8] Álvarez de Morales, María, Pérez, Teresa E., and Piñar, Miguel A. "Sobolev orthogonality for the Gegenbauer polynomials $\{C_n^{(-N+1/2)}\}_{n\geq 0}$". In: *J. Comput. Appl. Math.* 100.1 (1998), pp. 111–120.

[9] Álvarez-Nodarse, Renato. *Polinomios hipergeométricos clásicos y q-polinomios.* Monografías del seminario matemático García de Galdeano. Prensas Universitarias de Zaragoza. Departamento de Matemáticas, Universidad de Zaragoza, 2003.

[10] Álvarez-Nodarse, Renato and Marcellán, Francisco. "A generalization of the class Laguerre polynomials: asymptotic properties and zeros". In: *Appl. Anal.* 62.3–4 (1996), pp. 349–366.

[11] Álvarez-Nodarse, Renato and Moreno-Balcázar, Juan José. “Asymptotic properties of generalized Laguerre orthogonal polynomials”. In: *Indag. Math. (N.S.)* 15.2 (2004), pp. 151–165.

[12] Arvesú, Jorge, Álvarez-Nodarse, Renato, Marcellán, Francisco, and Pan, Ke-Lin. “Jacobi-Sobolev-type orthogonal polynomials: Second-order differential equation and zeros”. In: *J. Comput. Appl. Math.* 90.2 (1998), pp. 135–156.

[13] Atkinson, Kendall and Hansen, Olaf. “Solving the nonlinear Poisson equation on the unit disk”. In: *J. Integral Equations Appl.* 17.3 (2005), pp. 223–241.

[14] Bavinck, Herman and Meijer, Hendrik Gerrit. “Orthogonal polynomials with respect to a symmetric inner product involving derivatives”. In: *Appl. Anal.* 33.1–2 (1989), pp. 103–117.

[15] Bavinck, Herman and Meijer, Hendrik Gerrit. “On orthogonal polynomials with respect to an inner product involving derivatives: zeros and recurrence relations”. In: *Indag. Math. (N.S.)* 1.1 (1990), pp. 7–14.

[16] Berti, Andrea C., Bracciali, Cleonice F., and Sri Ranga, Alagacone. “Orthogonal polynomials associated with related measures and Sobolev orthogonal polynomials”. In: *Numer. Algorithms* 34 (2003), pp. 203–216.

[17] Berti, Andrea C. and Sri Ranga, Alagacone. “Companion orthogonal polynomials: some applications”. In: *Appl. Numer. Math.* 39.2 (2001), pp. 127–149.

[18] Bracciali, Cleonice F., Delgado, Antonia M., Fernández, Lidia, Pérez, Teresa E., and Piñar, Miguel A. “New steps on Sobolev orthogonality in two variables”. In: *J. Comput. Appl. Math.* 235.4 (2010), pp. 916–926.

[19] Brenner, J. “Über eine Erweiterung des Orthogonaltätsbegriffes bei Polynomen”. In: *Proceedings of the Conference on Constructive Theory of Functions.* Ed. by Alexits, G. and Stechkin, S. B. Budapest: Akadémiai Kiadó, Aug. 1972, pp. 77–83.

[20] Burris, Stanley and Sankappanavar, Hanamantagouda P. *A Course in Universal Algebra.* Vol. 78. Graduate Texts in Mathematics. New York: Springer-Verlag, 1981.

[21] Chihara, Theodore S. *An introduction to orthogonal polynomials.* Gordon and Breach, Science Publishers, Inc., 1978.

[22] Cohen, Edgar A. “Zero distribution and behavior of orthogonal polynomials in the Sobolev space $W^{1,2}[-1,1]$”. In: *SIAM J. Math. Anal.* 6.1 (1975), pp. 105–116.

[23] Dai, Feng and Xu, Yuan. “Polynomial approximation in Sobolev spaces on the unit sphere and the unit ball”. In: *J. Approx. Theory* 163.10 (2011), pp. 1400–1418.

[24] Dai, Feng and Xu, Yuan. *Approximation theory and harmonic analysis on spheres and balls.* Springer monographs in mathematics. New York: Springer, 2013.

[25] De Bruin, Marcel G. "A tool for locating zeros of orthogonal polynomials in Sobolev inner product spaces". In: *J. Comput. Appl. Math.* 49 (1993), pp. 27–35.

[26] De Bruin, Marcel G. and Meijer, Hendrik Gerrit. "Zeros of orthogonal polynomials in a non-discrete Sobolev space". In: *Special functions (Torino, 1993).* Vol. 2. Ann. Numer. Math. 1–4. 1995, pp. 233–246.

[27] de Jesus, Márcio Dinis do Nascimento, Marcellán, Francisco, Petronilho, Jose Carlos, and Pinzón-Cortés, Natalia C. "(M, N)-coherent pairs of order (m, k) and Sobolev orthogonal polynomials". In: *J. Comput. Appl. Math.* 256 (2014), pp. 16–35.

[28] de Jesus, Márcio Dinis do Nascimento and Petronilho, Jose Carlos. "On linearly related sequences of derivatives of orthogonal polynomials". In: *J. Math. Anal. Appl.* 347.2 (2008), pp. 482–492.

[29] de Jesus, Márcio Dinis do Nascimento and Petronilho, Jose Carlos. "Sobolev orthogonal polynomials and (M, N)-coherent pairs of measures". In: *J. Comput. Appl. Math.* 237.1 (2013), pp. 83–101.

[30] Delgado, Antonia M., Fernández, Lidia, Lubinsky, Doron S., Pérez, Teresa E., and Piñar, Miguel A. "Sobolev orthogonal polynomials on the unit ball via outward normal derivatives". In: *J. Math. Anal. Appl.* 440.2 (2016), pp. 716–740.

[31] Delgado, Antonia M. and Marcellán, Francisco. "Companion linear functionals and Sobolev inner products: A case study". In: *Methods Appl. Anal.* 11.2 (2004), pp. 237–266.

[32] Delgado, Antonia M., Pérez, Teresa E., and Piñar, Miguel A. "Sobolev-type orthogonal polynomials on the unit ball". In: *J. Approx. Theory* 170 (2013), pp. 94–106.

[33] Dimitrov, Dimitar K., Marcellán, Francisco, and Rafaeli, Fernando R. "Monotonicity of zeros of Laguerre-Sobolev-type orthogonal polynomials". In: *J. Math. Anal. Appl.* 368.1 (2010), pp. 80–89.

[34] Dimitrov, Dimitar K., Mello, Mirela V., and Rafaeli, Fernando R. "Monotonicity of zeros of Jacobi-Sobolev type orthogonal polynomials". In: *Appl. Numer. Math.* 60.3 (2010), pp. 263–276.

[35] Dueñas, Herbert A. and Garza, Luis E. "Jacobi-Sobolev-type orthogonal polynomials: holonomic equation and electrostatic interpretation - a non-diagonal case". In: *Integral Transforms Spec. Funct.* 24.1 (2013), pp. 70–83.

[36] Dueñas, Herbert A., Garza, Luis E., and Piñar, Miguel A. "A higher order Sobolev-type inner product for orthogonal polynomials in several variables". In: *Numer. Algorithms* 68.1 (2015), pp. 35–46.

[37] Dueñas, Herbert A., Huertas, Edmundo J., and Marcellán, Francisco. "Asymptotic properties of Laguerre-Sobolev type orthogonal polynomials". In: *Numer. Algorithms* 60.1 (2012), pp. 51–73.

[38] Dueñas, Herbert A. and Marcellán, Francisco. "The Laguerre-Sobolev-type orthogonal polynomials". In: *J. Approx. Theory* 162.2 (2010), pp. 421–440.

[39] Dueñas, Herbert A. and Marcellán, Francisco. "The holonomic equation of the Laguerre-Sobolev-type orthogonal polynomials: a non-diagonal case". In: *J. Difference Equ. Appl.* 17.6 (2011), pp. 877–887.

[40] Dueñas, Herbert A. and Marcellán, Francisco. "The Laguerre-Sobolev-type orthogonal polynomials. Holonomic equation and electrostatic interpretation". In: *Rocky Mountain J. Math.* 41.1 (2011), pp. 95–131.

[41] Dueñas, Herbert A., Pinzón-Cortés, Natalia, and Salazar-Morales, Omar. "Sobolev orthogonal polynomials of high order in two variables defined on product domains". In: *Integral Transforms Spec. Funct.* 28.12 (Oct. 2017), pp. 988–1008.

[42] Dueñas, Herbert A., Salazar-Morales, Omar, and Piñar, Miguel A. "Sobolev orthogonal polynomials of several variables on product domains". In: *Mediterr. J. Math.* 18.5 (2021). Article 227, pp. 1–21.

[43] Duistermaat, Johannes Jisse and Kolk, Johan A. C. *Distributions: theory and applications*. Cornerstones. Birkhäuser Basel, 2010.

[44] Dunkl, Charles F. and Xu, Yuan. *Orthogonal polynomials of several variables*. 2nd ed. Vol. 155. Encyclopedia of mathematics and its applications. Cambridge, United Kingdom: Cambridge University Press, 2014.

[45] Durán, Antonio J. "A generalization of Favard's theorem for polynomials satisfying a recurrence relation". In: *J. Approx. Theory* 74.1 (1993), pp. 83–109.

[46] Evans, W. D., Littlejohn, Lance L., Marcellán, Francisco, Markett, Clemens, and Ronveaux, André. "On recurrence relations for Sobolev orthogonal polynomials". In: *SIAM J. Math. Anal.* 26.2 (1995), pp. 446–467.

[47] Everitt, William Norrie, Kwon, Kil Hyun, Littlejohn, Lance L., and Wellman, Richard. "On the spectral analysis of the Laguerre polynomials $\{L_n^{-k}(x)\}$ for positive integers k". In: *Spectral theory and computational methods of Sturm-Liouville problems (Knoxville, TN, 1996)*. Ed. by Hinton, Don and Schaefer, Philip W. Vol. 191. Lect. Notes Pure Appl. Math. New York: Marcel Dekker, Inc., 1997, pp. 251–283.

[48] Everitt, William Norrie, Littlejohn, Lance L., and Wellman, Richard. "The Sobolev orthogonality and spectral analysis of the Laguerre polynomials $\{L_n^{-k}\}$ for positive integers k". In: *J. Comput. Appl. Math.* 171.1–2 (Oct. 2004), pp. 199–234.

[49] Fernández, Lidia, Marcellán, Francisco, Pérez, Teresa E., Piñar, Miguel A., and Xu, Yuan. "Sobolev orthogonal polynomials on product domains". In: *J. Comput. Appl. Math.* 284 (2015), pp. 202–215.

[50] Fernández, Lidia, Pérez, Teresa E., and Piñar, Miguel A. "Orthogonal polynomials in two variables as solutions of higher order partial differential equations". In: *J. Approx. Theory* 163 (2011), pp. 84–97.

[51] Gröbner, Wolfgang. "Orthogonale Polynomsysteme, die gleichzeitig mit $f(x)$ auch deren Ableitung $f'(x)$ approximieren". In: *Funktionalanalysis, Approximationstheorie, Numerische Mathematik*. Vol. 7. International series of numerical mathematics. Springer Basel AG, 1967, pp. 24–32.

[52] Guo, Ben-Yu, Shen, Jie, and Wang, Li-Lian. "Generalized Jacobi polynomials/functions and their applications". In: *Appl. Numer. Math.* 59.5 (2009), pp. 1011–1028.

[53] Horn, Roger A. and Johnson, Charles R. *Matrix analysis*. Cambridge University Press, 1985.

[54] Horn, Roger A. and Johnson, Charles R. *Topics in matrix analysis*. Cambridge University Press, 1991.

[55] Huertas, Edmundo J., Marcellán, Francisco, and Rafaeli, Fernando R. "Zeros of orthogonal polynomials generated by canonical perturbations of measures". In: *Appl. Math. Comput.* 218.13 (2012), pp. 7109–7127.

[56] Iserles, Arieh, Koch, Per Erik, Norsett, Syvert Paul, and Sanz-Serna, Jesús María. "On polynomials orthogonal with respect to certain Sobolev inner products". In: *J. Approx. Theory* 65.2 (1991), pp. 151–175.

[57] Jung, Il Hyo, Kwon, Kil Hyun, and Lee, Jeong Keun. "Sobolev orthogonal polynomials relative to $\lambda p(c)q(c)+\langle\tau, p'(x)q'(x)\rangle$". In: *Commun. Korean Math. Soc.* 12.3 (1997), pp. 603–617.

[58] Kim, D. H., Kwon, Kil Hyun, Marcellán, Francisco, and Yoon, Gang Joon. "Sobolev orthogonality and coherent pairs of moment functionals: An inverse problem". In: *Int. Math. J.* 2.9 (2002), pp. 877–888.

[59] Kim, D. H., Kwon, Kil Hyun, Marcellán, Francisco, and Yoon, Gang Joon. "Zeros of Jacobi-Sobolev orthogonal polynomials". In: *Int. Math. J.* 4.5 (2003), pp. 413–422.

[60] Koekoek, J. and Koekoek, Roelof. "Differential equations for generalized Jacobi polynomials". In: *J. Comput. Appl. Math.* 126.1–2 (2000), pp. 1–31.

[61] Koekoek, J., Koekoek, Roelof, and Bavinck, Herman. "On differential equations for Sobolev-type Laguerre polynomials". In: *Trans. Amer. Math. Soc.* 350.1 (1998), pp. 347–393.

[62] Koekoek, Roelof. "Generalizations of Laguerre polynomials". In: *J. Math. Anal. Appl.* 153.2 (1990), pp. 576–590.

[63] Koekoek, Roelof. "The search for differential equations for certain sets of orthogonal polynomials". In: *J. Comput. Appl. Math.* 49 (1993), pp. 111–119.

[64] Koekoek, Roelof and Meijer, Hendrik Gerrit. "A generalization of Laguerre polynomials". In: *SIAM J. Math. Anal.* 24.3 (1993), pp. 768–782.

[65] Koornwinder, Tom H. "Orthogonal polynomials with weight function $(1 - x)^\alpha(1+x)^\beta + M\delta(x+1) + N\delta(x-1)$". In: *Canad. Math. Bull.* 27.2 (1984), pp. 205–214.

[66] Krall, Allan M. "Orthogonal polynomials satisfying fourth order differential equations". In: *Proc. Roy. Soc. Edinburgh Sect. A* 87.3–4 (1981), pp. 271–288.

[67] Krall, Harry LeVern. "On orthogonal polynomials satisfying a certain fourth order differential equation". In: *Pennsylvania State College Studies* 1940.6 (1940), pp. 1–24.

[68] Krall, Harry LeVern and Sheffer, Isador Mitchell. "Orthogonal polynomials in two variables". In: *Ann. Mat. Pura Appl.* 76.4 (1967), pp. 325–376.

[69] Kwon, Kil Hyun, Lee, J. H., and Marcellán, Francisco. "Generalized coherent pairs". In: *J. Math. Anal. Appl.* 253.2 (2001), pp. 482–514.

[70] Kwon, Kil Hyun and Littlejohn, Lance L. "The orthogonality of the Laguerre polynomials $\{L_n^{-k}(x)\}$ for positive integers k". In: *Special functions (Torino, 1993).* Vol. 2. Ann. Numer. Math. 1–4. 1995, pp. 289–303.

[71] Kwon, Kil Hyun and Littlejohn, Lance L. "Sobolev orthogonal polynomials and second-order differential equations". In: *Rocky Mountain J. Math.* 28.2 (1998), pp. 547–594.

[72] Lee, Jeong Keun and Littlejohn, Lance L. "Sobolev orthogonal polynomials in two variables and second order partial differential equations". In: *J. Math. Anal. Appl.* 322.2 (2006), pp. 1001–1017.

[73] Lewis, Daniel C. "Polynomial least square approximations". In: *Amer. J. Math.* 69.2 (Apr. 1947), pp. 273–278.

[74] Li, Huiyuan and Shen, Jie. "Optimal error estimates in Jacobi-weighted Sobolev spaces for polynomial approximations on the triangle". In: *Math. Comp.* 79.271 (2010), pp. 1621–1646.

[75] Li, Huiyuan and Xu, Yuan. "Spectral approximation on the unit ball". In: *SIAM J. Numer. Anal.* 52.6 (2014), pp. 2647–2675.

[76] Marcellán, Francisco, Branquinho, Amilcar, and Petronilho, Jose Carlos. "Classical orthogonal polynomials: a functional approach". In: *Acta Appl. Math.* 34.3 (1994), pp. 283–303.

[77] Marcellán, Francisco, Martínez-Finkelshtein, Andrei, and Moreno-Balcázar, Juan José. "k-coherence of measures with non-classical weights". In: *Margarita mathematica en memoria de José Javier (Chicho) Guadalupe Hernández.* Ed. by Español, Luis and Varona, Juan L. Universidad de La Rioja. Logroño, Spain: Servicio de Publicaciones, 2001, pp. 77–83.

[78] Marcellán, Francisco, Pérez, Teresa E., and Piñar, Miguel A. "Gegenbauer-Sobolev Orthogonal Polynomials". In: *Nonlinear Numerical Methods and Rational Approximation II.* Ed. by Cuyt, Annie. Vol. 296. Mathematics and Its Applications. Dordrecht: Springer, 1994, pp. 71–82.

[79] Marcellán, Francisco, Pérez, Teresa E., and Piñar, Miguel A. "Orthogonal polynomials on weighted Sobolev spaces: the semiclassical case". In: *Special functions (Torino, 1993)*. Vol. 2. Ann. Numer. Math. 1–4. 1995, pp. 93–122.

[80] Marcellán, Francisco, Pérez, Teresa E., and Piñar, Miguel A. "Laguerre-Sobolev orthogonal polynomials". In: *J. Comput. Appl. Math.* 71.2 (1996), pp. 245–265.

[81] Marcellán, Francisco and Petronilho, Jose Carlos. "Orthogonal polynomials and coherent pairs: the classical case". In: *Indag. Math. (N.S.)* 6.3 (1995), pp. 287–307.

[82] Marcellán, Francisco and Pinzón-Cortés, Natalia C. "Higher order coherent pairs". In: *Acta Appl. Math.* 121 (2012), pp. 105–135.

[83] Marcellán, Francisco and Pinzón-Cortés, Natalia C. "Generalized coherent pairs on the unit circle and Sobolev orthogonal polynomials". In: *Publ. Inst. Math. (Beograd) (N.S.)* 96.110 (2014), pp. 193–210.

[84] Marcellán, Francisco and Rafaeli, Fernando R. "Monotonicity and asymptotics of zeros of Laguerre-Sobolev-type orthogonal polynomials of higher order derivatives". In: *Proc. Amer. Math. Soc.* Vol. 139. 11. Nov. 2011, pp. 3929–3936.

[85] Marcellán, Francisco and Ronveaux, André. "On a class of polynomials orthogonal with respect to a discrete Sobolev inner product". In: *Indag. Math. (N.S.)* 1.4 (1990), pp. 451–464.

[86] Marcellán, Francisco and Xu, Yuan. "On Sobolev orthogonal polynomials". In: *Expo. Math.* 33 (2015), pp. 308–352.

[87] Martínez, Clotilde and Piñar, Miguel A. "Orthogonal Polynomials on the Unit Ball and Fourth-Order Partial Differential Equations". In: *SIGMA Symmetry Integrability Geom. Methods Appl.* 12 (2016). Paper No. 020, pp. 1–11.

[88] Martínez-Finkelshtein, Andrei. "Asymptotic properties of Sobolev orthogonal polynomials". In: *J. Comput. Appl. Math.* 99 (1998), pp. 491–510.

[89] Martínez-Finkelshtein, Andrei. "Analytic aspects of Sobolev orthogonal polynomials revisited". In: *J. Comput. Appl. Math.* 127 (2001), pp. 255–266.

[90] Meijer, Hendrik Gerrit. "Coherent pairs and zeros of Sobolev-type orthogonal polynomials". In: *Indag. Math. (N.S.)* 4.2 (1993), pp. 163–176.

[91] Meijer, Hendrik Gerrit. "Sobolev orthogonal polynomials with a small number of real zeros". In: *J. Approx. Theory* 77.3 (1994), pp. 305–313.

[92] Meijer, Hendrik Gerrit. "A short history of orthogonal polynomials in a Sobolev space. I. The non-discrete case". In: *31st Dutch Mathematical Conference.* Vol. 14. 1. Groningen, 1995: Nieuw Arch. Wiskd. (4), 1996, pp. 93–112.

[93] Meijer, Hendrik Gerrit. "Determination of all coherent pairs". In: *J. Approx. Theory* 89.3 (1997), pp. 321–343.

[94] Mello, Mirela V., Paschoa, Vanessa G., Pérez, Teresa E., and Piñar, Miguel A. "Multivariate Sobolev-type orthogonal polynomials". In: *Jaen Journal on Approximation* 3.2 (2011), pp. 241–259.

[95] Molano-Molano, Luis Alejandro. "On asymptotic properties of Laguerre-Sobolev type orthogonal polynomials". In: *Arab J. Math. Sci.* 19.2 (2013), pp. 173–186.

[96] Molano-Molano, Luis Alejandro. "On Laguerre-Sobolev type orthogonal polynomials: zeros and electrostatic interpretation". In: *ANZIAM J.* 55.1 (2013), pp. 39–54.

[97] Pérez, Teresa E. and Piñar, Miguel A. "On Sobolev orthogonality for the generalized Laguerre polynomials". In: *J. Approx. Theory* 86.3 (Sept. 1996), pp. 278–285.

[98] Pérez, Teresa E., Piñar, Miguel A., and Xu, Yuan. "Weighted Sobolev orthogonal polynomials on the unit ball". In: *J. Approx. Theory* 171 (2013), pp. 84–104.

[99] Piñar, Miguel A. and Xu, Yuan. "Orthogonal polynomials and partial differential equations on the unit ball". In: *Proc. Amer. Math. Soc.* 137.9 (Sept. 2009), pp. 2979–2987.

[100] Saint Raymond, Xavier. *Elementary introduction to the theory of pseudodifferential operators.* Studies in advanced mathematics. CRC Press, Inc., 1991.

[101] Salazar-Morales, Omar and Dueñas, Herbert A. "Laguerre-Gegenbauer-Sobolev orthogonal polynomials in two variables on product domains". In: *Rev. Colombiana Mat.* (2021). Accepted. To appear.

[102] Salazar-Morales, Omar and Dueñas, Herbert A. "Partial differential equations for some families of Sobolev orthogonal polynomials". Submitted to journal. 2022.

[103] Schäfke, Friedrich Wilhelm. "Zu den Orthogonalpolynomen von Althammer". In: *J. Reine Angew. Math.* 252 (1972), pp. 195–199.

[104] Schäfke, Friedrich Wilhelm and Wolf, Gerhard. "Einfache verallgemeinerte klassische Orthogonalpolynome". In: *J. Reine Angew. Math.* 262/263 (1973), pp. 339–355.

[105] Sharapudinov, Idris Idrisovich. "Sobolev orthogonal polynomials associated with Chebyshev polynomials of the first kind and the Cauchy problem for ordinary differential equations". In: *Differ. Equ.* 54.12 (2018), pp. 1602–1619.

[106] Sharapudinov, Idris Idrisovich. "Sobolev orthogonal polynomials generated by Jacobi and Legendre polynomials, and special series with the sticking property for their partial sums". In: *Sb. Math.* 209.9 (2018), pp. 1390–1417.

[107] Sharapudinov, Idris Idrisovich. "Sobolev-orthogonal systems of functions associated with an orthogonal system". In: *Izv. Math.* 82.1 (2018), pp. 212–244.

[108] Sharapudinov, Idris Idrisovich. "Sobolev-orthogonal systems of functions and the Cauchy problem for ODEs". In: *Izv. Math.* 83.2 (2019), pp. 391–412.

[109] Shen, Jie, Wang, Li-Lian, and Li, Huiyuan. "A triangular spectral element method using fully tensorial rational basis functions". In: *SIAM J. Numer. Anal.* 47.3 (2009), pp. 1619–1650.

[110] Suetin, P. K. *Orthogonal polynomials in two variables.* Vol. 3. Analytical Methods and Special Functions. Gordon and Breach Science Publishers, 1988.

[111] Szegö, Gabor. *Orthogonal polynomials.* 4th ed. Vol. 23. Amer. Math. Soc. Colloq. Publ. Providence, Rhode Island: American Mathematical Society, 1975.

[112] Xu, Yuan. "A family of Sobolev orthogonal polynomials on the unit ball". In: *J. Approx. Theory* 138 (2006), pp. 232–241.

[113] Xu, Yuan. "Sobolev orthogonal polynomials defined via gradient on the unit ball". In: *J. Approx. Theory* 152 (2008), pp. 52–65.

[114] Xu, Yuan. "Approximation and orthogonality in Sobolev spaces on a triangle". In: *Constr. Approx.* 46.2 (2017), pp. 349–434.

[115] Xu, Yuan. "Orthogonal polynomials and Fourier orthogonal series on a cone". In: *J. Fourier Anal. Appl.* 26.3 (2020). Paper 36, pp. 1–42.