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Sublaplacians on Real Flag Manifolds

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This dissertation is dedicated to my mother and father, Nina and Vincent Ursitti.

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ABSTRACT

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Hypoelliptic differential operators and associated geometries with origins in Lie theory are studied. We prove upper bounds on the dimension of Killing fields of analytic pseudosubriemannian manifolds under certain technical hypotheses. Existence and uniqueness results for adapted complex structures in open subsets of cotangent bundles of analytic subriemannian manifolds are proved. A generalized Lichnerowicz theorem expressing the difference between a connection laplacian and a Dirac laplacian for arbitrary linear connections and quadratic forms is proved, along with a preliminary result in local index theory for subriemannian metrics. We prove general results on the ubiquity of hypoelliptic sublaplacians arising in reductive Lie theory from the natural filtered structure of the tangent bundle of flag manifolds. A framework for studying the heat kernels of such operators from the standpoint of abstract harmonic analysis is developed involving branching the regular representation with respect to the inclusion of a closed subgroup which is transverse to the horizontal distribution of a given sublaplacian. In the compact case explicit formulæ are given.

1. INTRODUCTION

The goal of this dissertation is to describe and study differential operators on real flag manifolds, i.e. the coset spaces G/P where G is a reductive Lie group and $P \subset G$ is a parabolic subgroup. The author's point of view is that flag manifolds are natural places to discover and study subelliptic operators in particular, if only because many examples with completely explicit algebraic structure can be found and because a systematic method for comparing general parabolic geometries to the flat model spaces (i.e. G/P) via rather sophisticated types of curvature has been developed [1].

Although some work appeared earlier, the study of hypoelliptic differential operators began in earnest in the 1960s, with the work of J.J. Kohn and his coauthors on the $\overline{\partial}_b$-laplacian on the boundary of a strongly pseudoconvex domain in $\mathbf{C}^n$. This boundary laplacian, denoted $\Box_b$, is similar to the standard laplacian Δ on $\mathbf{R}^n$ in the sense that it can be locally expressed as a "sum of squares" of real vector fields, but it is also dissimilar because in the case of $\Box_b$ these vector fields only span a real hyperplane in the tangent space at each point.

Yet, $\Box_b$ manages to retain the essential qualitative property of Δ: it is hypoelliptic, meaning that for any distribution u, the smoothness of $\Box_b u$ implies that of u itself. This is significant, because the hypoellipticity of Δ on $\mathbf{R}^n$ depends crucially on the fact that the operator differentiates in each coordinate direction, for even the slightly modified operator $\partial^2_{x_1} + \cdots + \partial^2_{x_{n-1}}$ is *not* hypoelliptic on $\mathbf{R}^n$ as elementary examples demonstrate. It was Hörmander who in 1967 explained the necessary and sufficient condition for hypoellipticity which is satisfied by $\Box_b$ and not by $\partial^2_{x_1} + \cdots + \partial^2_{x_{n-1}}$,

Theorem 1.0.1 (Hörmander, [2]**)** *If $X_0, \ldots, X_r$ are real vector fields and c is a smooth function in an open subset $\Omega \subset \mathbf{R}^n$, then the operator $P = \sum_{i=1}^r X_i^2 + X_0 + c$*

is hypoelliptic if and only if the Lie algebra generated over $\mathbf{R}$ *by* $X_0, \ldots, X_r$ *spans the tangent space to* Ω *in every point.*

Hörmander's theorem immediately gives us a way to construct examples of sub-laplacians. A real Lie algebra $\mathfrak{n}$ will be called *stratified* if

1. $\mathfrak{n}$ admits a grading of the form $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k$, and is thus nilpotent since all lie monomials of homogeneous degree greater than k are zero,

2. $\mathfrak{n}_1$ generates $\mathfrak{n}$ as a Lie algebra.[1]

For a stratified algebra $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k$ we can choose a basis $X_1, \ldots, X_r$ of $\mathfrak{n}_1$, so that the operator $\sum_{i=1}^{r} X_i^2$ on the associated nilpotent Lie group (i.e. the vector space $\mathfrak{n}$ with the polynomial group law given by the baker-campbell-hausdorff formula) is hypoelliptic by Hörmander's theorem. Among the standard examples are the free nilpotent algebras and the Heisenberg algebras. The latter will be especially important, so we will explain their structure. Heisenberg algebras are the simplest nonabelian Lie algebras, for their construction only three ingredients are needed: two vector spaces V, W over $\mathbf{R}$, and a surjective skew-symmetric $\mathbf{R}$ bilinear form $\langle \cdot, \cdot \rangle : V \times V \to W$. The associated Heisenberg algebra is then $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ with $\mathfrak{n}_1 = V$, $\mathfrak{n}_2 = W$ and $[v_1 \oplus w_1, v_2 \oplus w_2] = 0 \oplus \langle v_1, v_2 \rangle$. Note that $\mathfrak{n}_2$ is central so all iterated brackets of three or more arguments are zero and the Jacobi identity is trivially satisfied. The associated Heisenberg group is the vector space $\mathfrak{n}$ with group law given by the baker-campbell-hausdorff formula: $(v_1 \oplus w_1)(v_2 \oplus w_2) = (v_1 + v_2) \oplus (w_1 + w_2 + \frac{1}{2} \langle v_1, v_2 \rangle)$.

In particular for any unital $\mathbf{R}$-algebra A, associative or not, with an anti-automorphic involution $x \mapsto x^*$, the involution extends in the usual way to the direct sum $\bigoplus_{m,n \geq 1} A^{m \times n}$ of all finite dimensional matrices with coefficients in A, with any matrix $T \in A^{m \times n}$ mapping to its conjugate transpose $T^* \in A^{n \times m}$. As usual we have the $\mathbf{R}$-bilinear product $A^{m \times n} \times A^{m \times n} \to A^{m \times m}$ given by $(T, U) \mapsto TU^*$. More generally if $B \subset A$ is a $*$-closed unital subalgebra which is associative and such that

[1] The term stratified is also sometimes used to indicate such algebras in which the equality $[\mathfrak{n}_1, \mathfrak{n}_l] = \mathfrak{n}_{l+1}$ holds for each $l = 1, \ldots, k-1$, here we require only that $\mathfrak{n}_1$ generates $\mathfrak{n}$.

the product in A is unique (i.e. associative) on $B \otimes A \otimes B$ then for any $\alpha \in B^{m\times m}$ and $\beta \in B^{n\times n}$, both units with β hermitian and $\alpha = \beta$ if $m = n$, the involution $U \mapsto U^*$ on $A^{m\times n} \oplus A^{n\times m}$ (if $m \neq n$) or $A^{m\times m}$ (if $m = n$) can be twisted by (α, β), i.e. $U \mapsto U^{*_{\alpha,\beta}} = (\alpha^* U \beta^{-1})^* = \beta^{-1} U^* \alpha$, thus producing a new $\mathbf{R}$-bilinear product $(T, U) \mapsto TU^{*_{\alpha,\beta}} = T\beta^{-1}U^*\alpha$.

The square matrix algebra $A^{m\times m}$ admits a decomposition into real and imaginary parts by way of the ± 1 eigenspace decomposition under the involution $*_{\alpha,\alpha}$ (this is independent of α). The resulting Heisenberg algebra is $\mathfrak{n} = A^{m\times n} \oplus \operatorname{Im} A^{m\times m}$ with the Lie bracket arising from the skew symmetric form

$$(T, U) \mapsto \operatorname{Im}(TU^*) = \frac{1}{2}(TU^{*_{\alpha,\beta}} - (TU^{*_{\alpha,\beta}})^{*_{\alpha,\alpha}}) = \frac{1}{2}(TU^{*_{\alpha,\beta}} - UT^{*_{\alpha,\beta}}),$$

provided that it is surjective (or more generally if one reduces consideration to the subalgebra therein which is generated by $A^{m\times n}$). In particular with $m = 1$, $n = p+q$, $\alpha = 1$, and $\beta \in \mathbf{R}^{n\times n}$ equal to a strategically chosen diagonal ± 1 matrix we obtain the split-signature affine Heisenberg groups $A^{p+q} \oplus \operatorname{Im} A$. By explicitly describing the nilradicals of maximal parabolic subgroups in various classical groups, Wolf has identified a large family of real flag varieties locally equivalent to these generalized Heisenberg groups in many cases [3,4]. In particular we can consider this construction using the $\mathbf{R}$-algebras listed in Table 1.1 related to the Freudenthal magic square [5].[2]

As above, for each of these Heisenberg algebras we could choose a basis of $\mathfrak{n}_1$ and study the associated sublaplacian on the group. However, we are more interested in geometries which are locally equivalent to these Heisenberg groups, but not globally so. For $\mathbf{R}, \mathbf{C}$ and $\mathbf{H}$, these generalized Heisenberg algebras are associated to Hopf fibrations, the total space of which will be locally equivalent to one of the previously described Heisenberg groups. For $\mathbf{C}$ and $\mathbf{H}$ these Hopf fibrations come in an infinite series

$$S^1 \hookrightarrow S^{2n+1} \to \mathbf{P}^n(\mathbf{C}) \quad \text{and} \quad S^3 \hookrightarrow S^{4n+3} \to \mathbf{P}^n(\mathbf{H}),$$

respectively. For the octonions there is a unique fibration $S^7 \hookrightarrow S^{15} \to S^8$ of S^{15}.

[2] The rather unfortunate names appearing in the table are taken from [5].

Table 1.1.
Normed algebras over **R**.

symbol	**R**-algebra
$\mathbf{C}$	the complex numbers
$\mathbf{H}$	the quaternions
$\mathbf{O}$	the octonions
$\mathbf{C} \otimes \mathbf{O}$	the bi-octonions
$\mathbf{H} \otimes \mathbf{O}$	the quater-octonions
$\mathbf{O} \otimes \mathbf{O}$	the octo-octonions

Each of these Hopf fibrations has a rather satisfactory explanation in terms of Lie theory. In each case the total space of the fibration arises as the boundary of the associated $n+1$-dimensional hyperbolic space (with $n > 1$ for $\mathbf{C}$ and $\mathbf{H}$ only), and it is the realization of this hyperbolic space as an open domain in projective space[3] which explains the fibration. Indeed, in each case the split signature quadratic form $-|z_0|^2 + |z_1|^2 + \cdots + |z_{n+1}|^2$ is defined on $n+1$-dimensional projective space as a section of a real line bundle and the relevant hyperbolic space is identical to the open domain which corresponds to the lines in $n+2$-dimensional affine space which are positive for this form. The boundary is thus the projective locus of the null cone for the given quadratic form, and in each case the isometry group for hyperbolic space acts on the boundary as well.

Some examples of this type of construction are listed in Table 1.2. In the $\mathbf{C}$ and $\mathbf{H}$ cases, the subgroups $P^{\mathbf{C}}_{n+2}, P^{\mathbf{H}}_{n+2}$, etc. are the isotropy subgroups of a null line for the associated quadratic form in $n+2$-dimensional affine space, in the other cases the groups $P^{\mathbf{O}}, P^{\mathbf{C}\otimes\mathbf{O}}, P^{\mathbf{H}\otimes\mathbf{O}}, P^{\mathbf{O}\otimes\mathbf{O}}$ are strategically chosen parabolic subgroups in the respective noncompact semisimple groups. Now the crucial observation is that, for

[3]The projective space for the exceptional algebras is defined directly as an appropriate quotient of Lie groups.

Table 1.2.
Symmetric spaces associated to composition algebras over $\mathbf{R}$.

$\mathbf{R}$-algebra	hyperbolic space	boundary
$\mathbf{C}$	$\mathrm{SU}(1,n+1)/\,\mathrm{U}(n+1)$	$\mathrm{SU}(1,n+1)/P^{\mathbf{C}}_{n+2}$
$\mathbf{H}$	$\mathrm{Sp}_1(1,n+1)/\,\mathrm{Sp}(n+1)$	$\mathrm{Sp}_1(1,n+1)/P^{\mathbf{H}}_{n+2}$
$\mathbf{O}$	$\mathrm{F}_4\,/\,\mathrm{Spin}(9)$	$\mathrm{F}_4\,/P^{\mathbf{O}}$
$\mathbf{C}\otimes\mathbf{O}$	$\mathrm{E}_6\,/K_6$	$\mathrm{E}_6\,/P^{\mathbf{C}\otimes\mathbf{O}}$
$\mathbf{H}\otimes\mathbf{O}$	$\mathrm{E}_7\,/K_7$	$\mathrm{E}_7\,/P^{\mathbf{H}\otimes\mathbf{O}}$
$\mathbf{O}\otimes\mathbf{O}$	$\mathrm{E}_8\,/K_8$	$\mathrm{E}_8\,/P^{\mathbf{O}\otimes\mathbf{O}}$

instance in the complex case, a null line cannot be contained entirely in any positive defininte or negative definite subspace of $\mathbf{C}^{n+2}$, so it must have a full rank projection onto either summand of any positive/negative definite splitting of $\mathbf{C}^{n+2}$, and viewing this projected line as a point in the projective space $\mathbf{P}^n(\mathbf{C})$ one obtains the map $\mathrm{SU}(1,n+1)/P^{\mathbf{C}}_{n+2}\to\mathbf{P}^n(\mathbf{C})$, this is the Hopf fibration. Similar observations apply to the other cases, the boundary of the associated hyperbolic space is a projectivized null cone, but a line in the null cone projects to a line in the positive definite factor for the split quadratic form and this line is evidently a point in the associated projective space. This map is the desired fibration.

The main qualitative feature of these generalized fibrations is that they all have a total space equal to a generalized flag manifold, i.e. the total space is a quotient of a reductive group by a parabolic subgroup. If G is a symplectic group or a split signature unitary group with defining action on a vector space V then the parabolic subgroups $P\subset G$ are the isotropy groups of totally isotropic flags $V_1\subset\ldots\subset V_k$ (i.e. the bilinear form under consideration must vanish when restricted to each V_i). Given any orthogonal decomposition $V=Q\oplus R$ with Q and R nondegerate for the bilinear form under consideration, a totally isotropic flag $V_1\subset\ldots\subset V_k$ can be projected into either summand, say Q, and the resulting map $(V_1\subset\ldots\subset V_k)\mapsto(P_QV_1\subset\ldots\subset P_QV_k)$

is generally not injective. The subgroup of G which acts invariantly on *either* Q *or* R must also act invariantly on *both* Q *and* R, and as such it is the direct product $G_Q \times G_R$ of its projected actions. If P is the isotropy group of the totally isotropic flag $V_1 \subset \ldots \subset V_k$, the action of this direct product partitions the flag variety G/P in a natural way. If, for instance, $V = \mathbf{C}^{n+2}$ with metric $-|z_0|^2 + |z_1|^2 + \ldots + |z_{n+1}|^2$, $G = \mathrm{U}(1, n+1)$, $Q \oplus R = \mathbf{C} \oplus \mathbf{C}^{n+1}$ with $\mathbf{C}$ negative and $\mathbf{C}^{n+1}$ positive, then $G_Q \times G_R = \mathrm{U}(1) \times \mathrm{U}(n+1)$ and this direct product acts transitively on the flag variety of all null lines and projecting a null line on R is the surjection which defines the Hopf fibration with base $\mathbf{P}^n(\mathbf{C})$ and fiber $\mathrm{U}(1)$. In group theoretic terms, the isotropy group of a projected line in $\mathbf{C}^{n+1}$ is $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(n)$ and the isotropy group of a null line above it is $\Delta\,\mathrm{U}(1) \times \mathrm{U}(n) \subset \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(n)$ where $\Delta\,\mathrm{U}(1)$ indicates the diagonal injection of $\mathrm{U}(1)$ into $\mathrm{U}(1)^2$. So, the Hopf fibration is the usual three term fibration associated to the three term inclusion

$$\Delta\,\mathrm{U}(1) \times \mathrm{U}(n) \subset \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(n) \subset \mathrm{U}(1) \times \mathrm{U}(n+1).$$

On the other hand if the metric is changed to $+|z_0|^2 - |z_1|^2 + |z_2|^2 + \ldots + |z_{n+1}|^2$ and $Q \oplus R = \mathbf{C} \oplus \mathbf{C}^{n+1}$ with Q positive and R of split signature, then $G_Q \times G_R = \mathrm{U}(1) \times \mathrm{U}(1, n)$ acting with two orbits on the variety of null lines:

1. the null lines contained in R,
2. the null lines with positive projection in Q and negative projection in R.

Isolating the second orbit, for instance, the projection of a null line into R is a fibration of an open domain in the associated flag variety onto an open domain in the projective space $\mathbf{P}^n(\mathbf{C})$, again with fiber $\mathrm{U}(1)$. As above the isotropy group of a negative line in R is $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(n)$, and the isotropy group of a null line above it is $\Delta\,\mathrm{U}(1) \times \mathrm{U}(n) \subset \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(n)$ so as before this open domain is the total space of the usual three term fibration associated to the three term inclusion

$$\Delta\,\mathrm{U}(1) \times \mathrm{U}(n) \subset \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(n) \subset \mathrm{U}(1) \times \mathrm{U}(1, n).$$

This discussion raises a basic question regarding the geometries described above: how does the algebraic structure of the parabolic homogeneous space G/P interact with and elucidate the nature of the various hypoelliptic sublaplacians defined via Lie theory? In this dissertation, we will begin to answer this question. On the one hand L. Bérard-Bergery has proved in [6] (see also [7]) that for any three term inclusion $K \subset H \subset G$ with K and H compact the standard fibration $H/K \hookrightarrow G/K \to G/H$ is a riemannian submersion with totally geodesic fibers for any metric on G/K defined by splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$ as an $\mathfrak{h}$-module and further splitting $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{k}'$ as a $\mathfrak{k}$-module and choosing metrics on $\mathfrak{h}'$ and $\mathfrak{k}'$ which are respectively $\mathfrak{h}$ and $\mathfrak{k}$ invariant. As such, the Laplace operator on the total space commutes with the vertical Laplace operator and a suitable linear combination of the two is a degenerate horizontal laplacian for the tangent distribution metrically orthogonal to the fibers, i.e. it is equal to $f \mapsto \operatorname{div}(Ldf)$ where L is the linear map $L : T^*(G/K) \to T(G/K)$ defined by projecting the cotangent fiber at every point into the orthogonal to the vertical tangent space. However, without viewing the total space of the fibration as an open domain in a flag variety there is no reason to suspect that the horizontal distribution is bracket-generating so that the above described operator is hypoelliptic.

On the other hand if, as in the examples described above, the total space is identifiable with an open domain in a flag variety of the form G/P with G reductive and $P \subset G$ parabolic, then there is a strong reason to suspect that the horizontal distribution is bracket-generating. The relevant initial observation concerns the structure of the Lie algebra of G in relation to that of P. Indeed, the Lie algebra $\mathfrak{p}$ of the parabolic subgroup P is a semidirect product $\mathfrak{g}_0 \ltimes (\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k)$ where $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is a stratified nilpotent algebra (as defined above) and $\mathfrak{g}_0$ is a reductive Lie algebra of derivations of the stratified factor. In practice, in the semisimple case one obtains this type of structure as follows:

1. start with a semisimple real Lie algebra $\mathfrak{g}$,

2. identify a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$,

3. identify a θ-stable Cartan subalgebra $\mathfrak{t} \oplus \mathfrak{a}$,

4. identify the $\mathfrak{a}$-restricted roots $\lambda \in \mathfrak{a}^*$,

5. choose a system Δ_0^+ of simple positive restricted roots associated to $\mathfrak{a}$,

6. choose a subset $\Sigma \subset \Delta_0^+$ of simple positive restricted roots (equivalently, a subset of noncompact simple positive roots invariant under the involution induced by the Satake involution) to define the parabolic,

7. for $i \neq 0$ define $\mathfrak{g}_i$ to be the direct sum of all restricted root spaces of height i with respect to Σ,[4]

8. define $\mathfrak{g}_0$ to be the common normalizer in $\mathfrak{g}$ of each of the spaces $\mathfrak{g}_i$, $i \geq 1$.

In the reductive case the same procedure is applied to the derived algebra $[\mathfrak{g}, \mathfrak{g}]$. The subalgebras

$$\mathfrak{p}_+ = \mathfrak{g}^{\geq 0} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \quad \text{and} \quad \mathfrak{p}_- = \mathfrak{g}^{\leq 0} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$$

are both parabolic with nilradicals (here we assume that k is the largest integer such that $\mathfrak{g}_k$ is nontrivial)

$$\mathfrak{n}_+ = \mathfrak{g}^{\geq 1} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \quad \text{and} \quad \mathfrak{n}_- = \mathfrak{g}^{\leq -1} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1},$$

respectively, each with $\mathfrak{g}_0$ as a reductive Levi factor. If G is a group with algebra $\mathfrak{g}$ then a subgroup $P \subset G$ is said to be a *parabolic subgroup* if it is an open subgroup of the normalizer $N_G(\mathfrak{p}_+)$.

Another crucial observation is that the direct sum decomposition of $\mathfrak{g}$ given by $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is compatible with the Lie bracket in $\mathfrak{g}$, so in fact this decomposition makes $\mathfrak{g}$ into a *graded* Lie algebra. This is essentially the only way to give $\mathfrak{g}$ the structure of a graded Lie algebra - for any such structure on a reductive algebra will come from the procedure outlined above (for a proof, see [1]).

[4] A rootspace $\mathfrak{g}_\lambda$ is said to have height i with respect to Σ if λ is a sum of simple positive roots with either all positive or all negative coefficients, with sum i, of roots in the complement $\Delta_0^+ \setminus \Sigma$.

For a reductive group G, a gradation on the Lie algebra $\mathfrak{g}$, and a parabolic subgroup $P \subset G$ with respect to the given gradation, we can construct natural fiber bundles on G/P using the usual associated bundle construction: G is the total space of a principle bundle over G/P with fiber P, and there is an associated bundle for any P module. In particular, for every linear representation of P there is a canonical vector bundle on G/P and the most obvious representation is $\mathfrak{g}/\mathfrak{p}_+$, for which the associated bundle is the tangent bundle to G/P. However, P also normalizes $\mathfrak{g}^{\geq i}$ for each $i \leq 0$, so if $i < 0$ then $\mathfrak{g}^{\geq i}/\mathfrak{p}_+$ is a subrepresentation of $\mathfrak{g}/\mathfrak{p}_+$ and the associated vector bundle on G/P is therefore a subbundle of the tangent bundle. In this way, the tangent bundle $T(G/P)$ admits a natural increasing filtration

$$T^{-1}(G/P) \subset T^{-2}(G/P) \subset \cdots \subset T^{-k}(G/P) = T(G/P)$$

which is associated to the increasing sequence of P-subrepresentations

$$\mathfrak{g}^{\geq -1}/\mathfrak{p}_+ \subset \mathfrak{g}^{\geq -2}/\mathfrak{p}_+ \subset \cdots \subset \mathfrak{g}^{\geq -k}/\mathfrak{p}_+ = \mathfrak{g}/\mathfrak{p}_+$$

arising from the root height gradation of the algebra $\mathfrak{g}$ described above. Moreover, this filtration of $T(G/P)$ is compatible with the Lie bracket of vector fields, and so the fact that $\mathfrak{g}_{-1}$ generates the nilradical of $\mathfrak{p}_-$ means that we can refer back to Hörmander's theorem to construct a natural sublaplacian on G/P. To do this, we observe that a maximal compact subgroup $K \subset G$ acts transitively on G/P, so in fact $G/P = K/K_P$ where $K_P = K \cap P$. In particular G/P is a compact manifold (in fact, it is a smooth projective variety). Thus, there exists a K-invariant metric on $T(G/P)$ and one can construct the sublaplacian Δ_{-1} which is in every point a sum of squares of an orthonormal frame of the tangent fiber of the subbundle $T^{-1}(G/P)$, within a perturbation of differential order one. In fact, there is a sublaplacian for every isotropy orbit in $\mathfrak{g}^{\geq -1}/\mathfrak{p}_+$ containing $\mathfrak{g}^{-1}/\mathfrak{p}_+$.

Before addressing this topic we will prove some new results pertaining to general pseudosubriemannian geometry. In chapter 2 we begin by proving in Theorem 2.1.3 that the bound $n+n^2$ on the dimension of complete Killing fields for a subriemannian metric on a connected manifold also holds for analytic pseudosubriemannian metrics provided that the cotangent bundle contains at least one vertically regular point for the hamitonian flow post-composed with the base projection. We are able to prove the existence of such points given additional hypotheses. Specifically, in Theorem 2.2.4 and the preparatory results leading to it, it is proved that vertically regular points exist in the cotangent fiber above any point which admits a so-called *preferred frame*, which is a tangent frame satisfying certain bracket conditions which ensure that there exists a partially transverse *subriemannian* manifold in a bruhat-whitney complexification on which the standard metric argument due to Agrachev for the existence of vertically regular points can be used.

Moving on to the second main topic of chapter 2, we prove the existence and uniqueness of adapted complex structures in conic open subsets of the cotangent bundle for analytic subriemannian metrics. A complex structure in such an open set is *adapted* if it stabilizes the two-dimensional subspace of vector fields generated by the radial dilation field and the metric hamiltonian vector field. This notion is due to Lempert and Szőke [8–10] and Guillemin and Stenzel [11, 12]. After several preparatory results we prove uniqueness of such structures in Theorem 2.3.4 and existence in Theorem 2.3.8.

The final topic of chapter 2 concerns connections on subriemannian manifolds and possible adaptations of existing techniques of local index theory to the subriemannian case. First it is proved in Lemma 2.4.1 that the horizontal distribution in TT^*M of any partial connection which is lagrangian and annihilates the metric must contain the hamiltonian vector field. In Proposition 2.4.2 it is proved that if such a connection is also linear, then geodesics are determined by their initial tangent vector so no such connection can be linear in the interior of the set in which the metric is degenerate for it is well known that geodesics are *not* determined by their tangent vectors in this set. Because of this, it is not possible to develop a theory of Dirac operators built from canonically chosen connections as in the riemannian case (see, e.g. [13]). Thus, we begin the study of Dirac operators built from completely general connections and to this end we prove in Theorem 2.4.3 a completely generalized Lichnerowicz formula in which nontrivial terms involving the torsion and covariant derivative of the metric appear. By the preceding remarks these terms cannot be completely gotten rid of through a judicious choice of connection as in the nondegenerate riemannian case.

The final result of chapter 2 begins the process of adapting E. Getzler's rescaling method to calculate the supertrace of the diagonal heat kernel of the square of a Dirac operator on a graded vector bundle. In the nondegenerate riemannian case, Getzler was able to calculate the supertrace by decomposing the endomorphism bundle $\mathrm{End}(E)$ of a graded Clifford module into the tensor product $\mathrm{End}(E) = \mathrm{Cl}(T^*M) \otimes W$ by expressing E as the tensor product of the spinor bundle with a twisting bundle and likewise decomposing a Clifford compatible superconnection into the tensor product of the riemannian connection with an arbitrary superconnection on the twisting bundle. The aforementioned rescaling of the heat kernel on E results from parallelizing the kernel along geodesic radii, contracting the spatial variable by $\sqrt{u}$ and the temporal variable by u, and simultaneously dilating the $\mathrm{Cl}(T^*M)$ factor by the functorial action of $1/\sqrt{u}$ on $\mathrm{Cl}(T^*M)$ after identifying it with $\bigwedge T^*M$ by the natural symbol map. Once this is done, the entire kernel is multiplied by $u^{n/2}$. For $k < n$ this kills off the contribution from $\bigwedge^k T^*M$ in the $u \to 0$ limit and the top degree contribution from $\bigwedge^n T^*M$ is constant in u - but this does not affect the supertrace because any Clifford element in $\bigwedge^k T^*M$ for $k < n$ is a sum of supercommutators and as such it must be in the kernel of any supertrace. These rescaled heat kernels are themselves heat kernels to corresponding rescaled operators and by showing that the rescaled operators have a $u \to 0$ limit, Getzler was able to identify the supertrace of the heat kernel at any given point with the heat kernel of a polynomial coefficient operator on a euclidean vector space which is explicitly calclulable.

In the degenerate case the Clifford algebra decomposes as $\mathrm{Cl}(T^*M) = \mathrm{Cl}(P) \otimes \bigwedge N$ where N is the kernel of the degenerate form and P is any complementary nondegenerate subspace. Elements of $\bigwedge N$ with no scalar component must be nilpotent in any representation and as such they must have zero supertrace. Likewise, elements of $\mathrm{Cl}(P)$ of exterior degree less than $\dim P$ are supercommutators and therefore must have zero supertrace. Only elements of the subspace $\bigwedge^{\dim P} P \subset \mathrm{Cl}(T^*M)$ can have nonzero supertrace. Thus, in order to successfully generalize Getzler's rescaling method we must modify the dilations so that $\bigwedge^{\dim P} P$ is unaffected *and* such that the corresponding rescaled operators have a limit as $u \to 0$. The first part of this strategy is achieved in Theorem 2.4.4. However, there is alot of apparent freedom in how the dilations are chosen to affect the $\bigwedge N$ factor, and it remains to be seen if such an intricate apparatus (i.e. connections on E and T^*M, local spatial dilations, and Clifford algebra dilations) can be chosen so that the resulting rescaled operators have a $u \to 0$ limit. In any case where this limit exists it will be realized as a polynomial coefficient operator on a nilpotent Lie group with dilations, so in principle the supertrace can be computed by computing the heat kernel of this operator.

In chapter 3 we prove the main structural theorems for sublaplacians on flag manifolds. After various preparatory results we prove Proposition 3.2.1 and Proposition 3.2.2 regarding bracket generating subbundles of $T(G/P)$ corresponding to direct sums of root spaces. Finally in chapter 4 we address the main strategy for developing explicit expressions for heat kernels of sublaplacians on homogeneous spaces. This involves recasting the theory of heat flow in the language of abstract harmonic analysis, in which heat kernels form a semigroup of positive operators in a separable Hilbert space and their pointwise values are given by integration over the unitary dual with respect to the Plancherel measure.

Many sublaplacians of interest on groups or homogeneous spaces can be expressed as linear combinations of Casimirs from nested subgroups. As a basic example of this we can return to the parabolic homogeneous space G/P discussed earlier. Taking the Cartan involution invariants in the decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ results in the decomposition $\mathfrak{k} = \mathfrak{k}_k \oplus \cdots \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_0$ with $\mathfrak{k}_j = (1+\theta)(\mathfrak{g}_{-j} \oplus \mathfrak{g}_j)$. Apparently $\mathfrak{k}^+ = \mathfrak{k}_0 \oplus \mathfrak{k}_2 \oplus \cdots \oplus \mathfrak{k}_{2j} \oplus \cdots$ is a Lie subalgebra and the difference $\Delta_{\mathfrak{k}} - \Delta_{\mathfrak{k}^+}$ is the sublaplacian arising as a sum of squares of an orthonormal basis of the odd degree subspaces of $\mathfrak{k}$ in a bi-invariant riemannian metric. The heat kernel for such a sublaplacian can be expressed by finding its spectral projections and eigenvectors. Thus, each constituent reprepresention of K must be branched into representations of K^+ and the eigenfunctions of the sublaplacian $\Delta_{\mathfrak{k}} - \Delta_{\mathfrak{k}^+}$ on $G/P = K/K_0$ are matrix coefficients arising from K_0 invariants embedded in irreducible representations of K^+ which are in turn embedded into irreducible representations of K. In this manner, one can obtain explicit expressions for heat kernels as infinite series of matrix coefficients which can be written out as classical special functions associated to root systems. The same method can be used for noncompact groups for which the Plancherel measure is explicitly known. With this in mind we develop this method from the standpoint of abstract harmonic analysis as desrcibed above. In particular we prove in Theorem 4.2.2 a criterion for essential self-adjointness of spectrally defined operators such as the aforementioned sublaplacians and in Theorems 4.4.1 and 4.4.2 we formulate the main results for compact groups which follow from the development of ideas described above.

2. GEODESICS AND ADAPTED COMPLEX STRUCTURES

By a *pseudosubriemannian manifold* we mean a smooth manifold M with a quadratic form $H : T^*M \to \mathbf{R}$ (i.e. a smooth function, homogeneous of degree two in the fibers, which in every fiber defines a symmetric bilinear form via polarization) such that the tangent distribution equal to the annihilator D of $\ker H$ is bracket-generating. Here and below, $\ker H$ will denote the *form kernel* of the hamiltonian, which can be properly smaller than the level set of zero if H takes both positive and negative values. By definition *bracket-generating* means that for any list $\xi_1, \ldots, \xi_k$ of one forms which spans the cotangent fiber in every point, the vector fields $X_1, \ldots, X_k$ obtained by respectively tracing the ξ_i through the bilinear form defined by H generate a Lie algebra over $\mathbf{R}$ which spans the tangent fiber in every point. If H takes only nonpositive or nonnegative values then the given pseudoriemannian manifold is said to be *subriemannian*, in accordance with the standard specialization from pseudoriemannian to riemannian geometry which is the special case of the scenario discussed here in which $\ker H$ is trivial.

2.1 Geodesics

In the subriemannian case we can without loss of generality assume that the hamiltonian H takes only nonnegative values, and in this case much of the standard metric theory from riemannian geometry can be generalized. We define $H^1([0,1], D)$ to be the set of measurable maps from the interval $[0,1]$ into the manifold M such that every element $\gamma \in H^1([0,1], D)$ is absolutely continuous and its derivative is a.e. in the distribution D and has finite energy: $E(\gamma) = \frac{1}{2}\int |\dot{\gamma}|^2 < \infty$, and therefore also finite length $l(\gamma) = \int |\dot{\gamma}| < \infty$. The set $H^1([0,1], D)$ of finite energy paths can be given the natural structure of a Hilbert manifold (c.f. [14]) and with this structure the point evaluations $\gamma \mapsto \gamma(s) \in M$ and the energy $\gamma \mapsto E(\gamma) \in [0, \infty)$ are differentiable maps. The cauchy-schwarz inequality shows that $l(\gamma) \leq \sqrt{2E(\gamma)}$ with equality if and only if $|\dot{\gamma}|$ is constant. Thus, since $l(\gamma)$ is parameter independent evidently a path minimizes energy among all paths joining two points if and only if it minimizes length and has constant speed.

By Chow's theorem (c.f. [14]), every pair of points in M (as usual, we shall assume by default that M is connected) can be joined by an element of $H^1([0,1], D)$, and with this in mind the intrinsic metric distance on M is defined in the usual manner:

$$d(x, y) := \inf\{l(\gamma) : \gamma \in H^1([0,1], D), \gamma(0) = x, \gamma(1) = y\}.$$

Symmetry, off-diagonal positivity, and the triangle inequality are immediately verified, so d is indeed a metric. The metric topology coincides with the manifold topology, but the Hausdorff dimension of the resulting metric space (M, d) is typically larger than the manifold dimension. In particular, $d : M \times M \to [0, \infty)$ is continuous. As in riemannian geometry, a path $\gamma \in H^1([0,1], D)$ is said to be a *geodesic* if it has constant speed and if every open segment therein contains a closed segment which realizes the distance between its endpoints.

Standard computations (c.f. [14]) demonstrate that projections into M of hamiltonian integral curves are geodesics, however in subriemannian geometry the converse is not true. This deficiency arises because, unlike in riemannian geometry, the horizontal endpoint map may have critical points. In other words, it is not *always* possible to perturb the endpoints of an element $\gamma \in H^1([0,1], D)$ in all directions through infinitesimal variations. In any case, this type of pathology can only occur on a closed and nowhere dense set. In fact there is a result due to A. Agrachev which shows that the subriemannian distance is generically smooth.

Definition 2.1.1 *For a smooth manifold M and hamiltonian function $H : T^*M \to$ $\mathbf{R}$ with hamiltonian vector field X_H, a point $\xi \in T^*M$ will be called* vertically regular *if it is in the domain of* $\exp(X_H)$ *and if it is a regular point of the map* $\pi \circ \exp(X_H)$ *restricted to* $T^*_{\pi(\xi)}M$*.*

Definition 2.1.2 *For a given subriemannian manifold* M*, a pair* $(x, y) \in M \times M$ *is said to be* smooth *if* x *and* y *are joined by a unique length minimizing path which is the projection of a hamiltonian integral curve beginning in a vertically regular point of* $T^*_x M$*.*

Evidently the set of smooth pairs in $M \times M$ is a symmetric subset, since if $\xi_x \in T^*_x M$ and $\xi_y \in T^*_y M$ are connected by the hamiltonian flow then the forward image through the hamiltonian flow of the vertical tangent space at ξ_x is transverse to the vertical tangent space at ξ_y if and only if the forward image of the vertical tangent space at $-\xi_y$ is transverse to the vertical tangent space at $-\xi_x$. Note that a diagonal pair (x, x) is smooth if and only if the metric is nondegenerate (i.e. riemannian) in $T^*_x M$, for otherwise x is not a regular value.

With this definition in place, Agrachev has proved the following theorem.

Theorem 2.1.1 (Agrachev) *For a given metrically complete subriemannian manifold* M*, the set of smooth pairs in* $M \times M$ *is symmetric, open, and dense. Every cross section of this set is open and dense in* M*. The distance function is smooth on the set of smooth pairs and it is analytic if the subriemannian metric is analytic.*

For a proof see [15] and also the Arxiv preprint of the same title which contains some additional material [16]. If M is not necessarily complete, then it is still true that there is a dense open set of regular exponential values locally speaking as can be seen by patching a small neighborhood of a given point into a complete (compact, e.g.) subriemannian manifold and observing that this modification does not affect the hamiltonian geodesics which are restricted to the patch.

In particular this proves that, as in riemannian geometry, an isometry of any connected subriemannian manifold is determined by its differential action restricted to any single cotangent (or equivalently tangent) fiber. Indeed, if $F_1, F_2 : M \to M$ are isometries mapping x to y with equal differential pullbacks from T^*_yM to T^*_xM then $F_1^{-1}F_2$ maps x to x and fixes every element of T^*_xM, so it must fix every hamiltonian geodesic in M emanating from x. By Agrachev's theorem such endpoints are locally dense so $F_1^{-1}F_2$ must leave an entire open neighborhood of x elementwise fixed - and therefore every cotangent vector lying above this neighborhood is also fixed. From this one sees that the set of cotangent vectors which are fixed by $F_1^{-1}F_2$ is, in addition to being nonempty and closed for elementary topological reasons, an open set. It is therefore an entire connected component of T^*M. Therefore, if M is connected and if X is a complete vector field on M which exponentiates to an isometry then it is globally determined by its value in any single point along with the action of its Lie derivative on the cotangent fiber above that point - for if both of these are trivial then X must exponentiate to the identity by the preceding argument.

A similar result for pseudoriemannian manifolds admits an equally simple proof, for the differential of the exponential map restricted to vertical tangent vectors on T^*M is given by metric duality - since the metric in this case is required to be nondegenerate the inverse function theorem shows that hamiltonian geodesics cover a full open neighborhood of the point from which they emanate and the preceding argument goes through in exactly the same way as before.

There appears to be no similarly easy proof in the pseudosubriemannian case even though the result itself should be expected to be true. Intuitively it is fairly easy to see how the hamiltonian exponential map works locally: if $\ker H \subset T^*_xM$ denotes the annihilator of the horizontal distribution D then an open neighborhood U of $0 \in T^*_xM/(\ker H \cap T^*_xM)$ can be pulled back to an open neighborhood of $\ker H$ in T^*_xM, which will be denoted $\widetilde{U}$. If U is sufficiently small then $\widetilde{U}$ is in the domain of the exponential map wether M is complete or not, and this map collapses the entire kernel $\ker H \cap T^*_xM$ into the point x. Thus, in the degenerate case no point in $\ker H \cap T^*_xM$ is a point of maximal differential rank for the exponential map.

However, each such point *is* always of maximal *transverse* differential rank, the transverse differential being given by projection into $T^*_xM/(\ker H \cap T^*_xM)$ followed by metric duality. In particular if one fixes a foliation of the neighborhood $\widetilde{U}$ which is transverse to $\ker H \cap T_x^*M$ (such as the foliation defined by any splitting of the quotient $T_x^*M \to T_x^*M/(\ker H \cap T_x^*M)$), then the exponential map has maximal rank when restricted to any given leaf. In this manner one sees the general behavior of the subriemannian exponential map: if U is sufficiently small then each leaf of a given transverse foliation is mapped diffeomorphically onto an embedded submanifold which is tangent to the horizontal distribution at x, moreover the centerpoint of each leaf is mapped on x. So the forward image of any sufficiently small transverse foliation in T_x^*M looks like that same foliation with the centerpoints from each leaf crushed into the point x.

In the pseudosubriemannian case the behavior should be more or less the same - because the transverse differential of the exponential map will again be given by metric duality which is assumed to be nondegenerate. Thus, one is led to suspect that the local density of the image of the exponential map in the positive definite case is a result of the bracket-generating property of the horizontal distribution and it should therefore persist for any nondegenerate metric of arbitrary signature.

If we restrict attention to analytic pseudosubriemannian structures then it is simple to prove that any analytic isometry which fixes an open subset of vertically regular points in a single cotangent fiber must be trivial.

Lemma 2.1.2 *If M is a connected analytic pseudosubriemannian manifold then the only analytic isometry of M which leaves elementwise fixed any open set of vertically regular points in any cotangent fiber is the identity map.*

Proof Any such isometry must be the identity when restricted to the forward image in M of the described open set. This being an open set we conclude that the specified isometry can only be the identity, for by the usual argument involving convergence of Taylor series the interior of the set on which two analytic maps are equal must be a connected component, but we've assumed that M is connected. ■

Theorem 2.1.3 *If M is a connected analytic pseudosubriemannian manifold of dimension n such that T^*M contains at least one vertically regular point then the Lie algebra of complete analytic vector fields which annihilate the metric has dimension at most $n + n^2$.*

Proof This result follows from the lemma in the usual fashion. Let $\xi \in T_x^*M$ be a vertically regular point. By the lemma, the map which takes an analytic Killing field to its value in $T_xM \oplus \mathfrak{gl}(T_x^*M)$ (by way of the Lie derivative in the second summand) is injective. This is easily seen because the difference of two Killing fields with equal values in $T_xM \oplus \mathfrak{gl}(T_x^*M)$ must exponentiate to an isometry which fixes the entire cotangent fiber T_x^*M and in particular fixes an open neighborhood of ξ. By the lemma we conclude that the given isometry is the identity so the described vector field must be trivial. ■

The existence of vertically regular points is not automatic. A simple appeal to Sard's theorem does not work because we require differential regularity *along the fiber* and not globally. Globally speaking of course the exponential map is surjective since it fixes the zero section elementwise and as a result there must be many regular points for otherwise the image would have measure zero by Sard's theorem and this is demonstrably false. However, one cannot conclude from Sard's theorem that any such point is *vertically* regular.

2.2 Vertically Regular Points for Analytic Metrics

With further hypotheses it is possible to prove that such vertically regular points exist generically. A result of Bruhat and Whitney [17] states that a paracompact analytic manifold M of dimension n can always be analytically embedded into a complex manifold X of dimension n as a totally real submanifold which is the fixed point set of an antiholomorphic involution which negates $JT_xM \subset T_xX$ for every $x \in M$. In addition to this, the resulting complexification is essentially unique, for given two such complexifications the identity map between the two analytically emebedded copies of the original compact manifold extends to a biholomorphism of open neighborhoods. The existence of such complexifications will be used below to demonstrate the existence of vertically regular points provided certain additional hypotheses are met.

It will be necessary to refer to the stefan-sussmann theory of generalized distributions and foliations, the following survey of that theory is taken primarily from [18–22]. A *generalized distribution* on a manifold M is quite simply a subset $D \subset TM$ which is a linear subspace in every fiber. The dual notion is that of a *generalized pfaffian system* which is, analogously, a subset $E \subset T^*M$ which is a linear subspace in every fiber. Evidently one can always pass from a given generalized distribution to its fiberwise annihlator, which is a generalized pfaffian system, and vice versa. Concerning regularity, a generalized distribution or pfaffian system is said to be *smooth* or *differentiable* if each of its points is the value in its fiber of a smooth section (of the given distribution or pfaffian system). In the differentiable case the rank of a generalized distribution or generalized pfaffian system is lower semicontinuous and therefore the rank of the annihilator (whether it be a generalized distribution or pfaffian system) is upper semicontinuous and as such cannot be differentiable unless it is constant on connected components.

An *integral* of a generalized distribution D is an immersion $\iota : N \to M$ such that the direct image of every vector in TN lies in D. An *integral manifold* of D is the image of an injective integral, i.e. an immersed submanifold every tangent vector of which lies in D. A smooth generalized distribution D is said to be *completely integrable* if every point of M is contained in an integral manifold of D which is "tangentially maximal" in the sense that each of its points has the entire fiber of D as its tangent space. Note that if D has varying rank along any given integral manifold then it cannot be tangentially maximal for obvious dimensional reasons. Thus, whereas it is possible for smooth generalized distributions of varying rank to be completely integrable, the rank must be constant on any of the above described tangentially maximal manifolds.

The main theorem on the existence of an associated foliation states that for a given generalized distribution which is smooth and completely integrable, every point is contained in a unique tangentially maximal integral manifold which is not properly contained in any other tangentially maximal integral manifold. Thus, these integral manifolds which are both spatially and tangentially maximal partition the ambient manifold into equivalence classes - i.e. the leaves of the generalized foliation associated with the given generalized distribution. A proof can be found in [21].

Now let $\{X_i\}_{i\in I}$ be an indexed collection of topological spaces and for each i let $\{T_{ij}\}_{j\in J_i}$ be an indexed collection of maps $T_{ij} : X_i \to \mathrm{Diff}_{\mathrm{loc}}(M)$ where $\mathrm{Diff}_{\mathrm{loc}}(M)$ is the set of diffeomorphisms between open subsets of a given manifold M. For each $i \in I$, $j \in J_i$ and $m \in M$ there is a map taking $x \in X_i$ to $(T_{ij}x)m \in M$ defined on the subset of X_i such that m lies in the domain of T_{ij}. For clarity we shall assume that this is always an open subset of X_i although this is probably not absolutely necessary. The topology on M defined by the data $\{X_i, T_{ij}\}$ is the final topology defined by all maps of this type, i.e. the finest topology with the property that each such map is continuous.

This topology is always finer than the manifold topology, but can be strictly finer. For instance it contains every subset of M which does not intersect the range of any elements of the collection $\{T_{ij}x\}_{i\in I, j\in J_i, x\in X_i}$ of local diffeomorphisms. In the case that this collection contains the identity and is invariant under inversion and pseudo-composition (i.e. composition combined with domain reduction so that the relevant expressions make sense), the relation $m \sim p$ if $p = (T_{ij}x)m$ for some $i \in I$, $j \in J_i$ and $x \in X_i$ is an equivalence relation on M and the equivalence classes are called the *orbits* of the collection $\{T_{ij}x\}_{i\in I, j\in J_i, x\in X_i}$ of local diffeomorphisms. These orbits together with every point in the common complement of the images of the $T_{ij}x$ form the collection of connected components of the above described topology.

At this level of generality this construction can be quite pathological, as there is no initial requirement on the continuity of the maps T_{ij}. If all of the local diffeomorphisms in the range of T_{ij} have the same domain then it's possible to impose such a requirement, but in general one would have to account for the "movement" of the domain. Here, however, we will only be concerned with local diffeomorphisms of the form

$$\exp(t_k Y_k)\exp(t_{k-1}Y_{k-1})\cdots\exp(t_1 Y_1)$$

where each Y_s is a vector field on an open subset of M. This is a special case of the framework described above by taking $\{Y_s\}_{s\in S}$ to be any indexed collection of smooth vector fields each defined in an (s-dependent) open subset of M. With $\{Y_s\}_{s\in S}$ determined, set

1. $I = \mathbf{N}_1 = \{1, 2, \ldots\}$,

2. $X_i = \mathbf{R}^i$,

3. $J_i = S^i$, and

4. $T_{ij}(t_1, \ldots, t_i) = \exp(t_i Y_{j_i})\exp(t_{i-1}Y_{j_{i-1}})\cdots\exp(t_1 Y_{j_1})$.

In the manner described above the expressions

$$(t_1, \ldots, t_i) \mapsto T_{ij}(t_1, \ldots, t_i)m$$

for m in the domain of $T_{ij}(t_1, \ldots, t_i)$ define an indexed collection of maps from open subsets of the vector spaces $\{\mathbf{R}^i\}_{i\geq 1}$ into M. The final topology associated to this collection of maps into M has the "orbits" of the finite concatenation of flows of the fields $\{Y_s\}_{s\in S}$ as its connected components. For clarity we shall assume that each orbit has dimension at least one, or equivalently there are no discretely embedded points in the described topology, or equivalently that every point in M is in the support of at least one element of $\{Y_s\}_{s\in S}$ (note, the fields Y_s are not required to be smoothly extendible to M). The set of all $T_{ij}(t_1, \ldots, t_i)$ associated to the collection $\{Y_s\}_{s\in S}$ of local vector fields is generally referred to as a *pseudogroup*, i.e. a collection of bijections between sets which is stable under inversion and which is closed under composition *provided* the domain and range are appropriately reduced. If all domains are subsets of the same set (as in the collection of open subsets of a single manifold

M which is our case of interest), then the notion of orbit is essentially the same as for a usual group action.

So far we've introduced two ideas:

1. smooth generalized distributions of (potentially) varying rank,
2. indexed collections $\{Y_s\}_{s\in S}$ of local vector fields and the associated pseudogroups of local diffeomorphisms which they generate.

There is an obvious way to generate an indexed collection of local vector fields from a generalized distribution and *two* apparent ways to generate a generalized distribution from a collection of local vector fields:

1. starting with a smooth generalized distribution, one takes for the collection $\{Y_s\}_{s\in S}$ the set of all smooth local sections of the given distribution,
2. starting with an indexed collection $\{Y_s\}_{s\in S}$ of local vector fields which for clarity we shall assume contains the zero vector field, one takes the generalized distribution arising as the linear span in every tangent fiber of the values of the given vector fields, or
3. starting with an indexed collection $\{Y_s\}_{s\in S}$ of local vector fields which for clarity we shall assume contains the zero vector field, one takes the generalized distribution arising as the linear span in every tangent fiber of the direct images of the given vector fields through the associated pseudogroup of local diffeomorphisms.

For a given collection $\{Y_s\}_{s\in S}$, the distribution defined by the former method will be called the *naïve distribution generated by* $\{Y_s\}_{s\in S}$ and the latter distribution will be called the *invariant distribution generated by* $\{Y_s\}_{s\in S}$. The main result of Stefan [18, 19] and Sussmann [20] is the following.

Theorem 2.2.1 (Stefan, Sussmann) *If* $\{Y_s\}_{s\in S}$ *is an everywhere defined collection of local vector fields then the invariant distribution it defines is completely integrable. The leaves of the associated foliation are the orbits of the associated pseudogroup of local diffeomorphisms and the restriction of the associated final topology* *to each leaf is identical to the topology it inherits from its source as the image of an injective immersion.*

So, for instance if $\{Y_s\}_{s\in S}$ consists of a single vector field $Y = a\partial_x + b\partial_y$ on the torus $\mathbf{R}^2/\mathbf{Z}^2$ and if, furthermore, a/b is irrational then the associated foliation will consist of uncountably many leaves of dimension one each of which is dense in the standard topology. However, the final topology separates these leaves into connected components each homeomorphic to $\mathbf{R}$.

In addition to this, Sussmann [20] proved the following.

Theorem 2.2.2 (Sussmann) *For any everywhere defined collection $\{Y_s\}_{s\in S}$ of local vector fields, the following are equivalent:*

1. *the naïve distribution generated by $\{Y_s\}_{s\in S}$ is invariant under the pseudogroup of local diffeomorphisms generated by $\{Y_s\}_{s\in S}$,*

2. *the naïve distribution generated by $\{Y_s\}_{s\in S}$ is equal to the invariant distribution generated by $\{Y_s\}_{s\in S}$,*

3. *the naïve distribution generated by $\{Y_s\}_{s\in S}$ is completely integrable,*

4. *for every point $x \in M$ there exists a finite set $Y_1, \ldots, Y_k \in \{Y_s\}_{s\in S}$ which spans the fiber of the naïve distribution at x, such that for any other local section Z of the naïve distribution and any linear combination $W = c_1Y_1 + \ldots + c_kY_k$, $[Z, W]$ is a linear combination of $Y_1, \ldots, Y_k$ along the flow line of Z emanating from x.*

In addition these four equivalent conditions imply

5. *smooth sections of the naïve distribution generated by $\{Y_s\}_{s\in S}$ are closed under the commutator bracket,*

and conversely provided the elements of $\{Y_s\}_{s\in S}$ are analytic.

Proceeding now to the main results of this section, for a given connected and analytic pseudosubriemannian manifold M embedded in a bruhat-whitney complexification X, a local horizontal (i.e. contained in the horizontal distribution D) analytic orthogonal frame $Y_1, \ldots, Y_l, Z_1, \ldots, Z_m$ on M such that $\|Y_i\|^2 = 1$ for all $i = 1, \ldots, l$ and $\|Z_i\|^2 = -1$ for all $i = 1, \ldots, m$ extends uniquely to a holomorphic complex frame in an open neighborhood in X of the frame domain by viewing TX as a holomorphic vector bundle by way of its natural identification with $T^{(1,0)}_{\mathbf{C}} X$ via the projection $(i + J)/2i$ (such a local frame will be called a *local horizontal analytic orthonormal frame*). Such a frame defines, for any x in the (extended) frame domain, a linear map from the complex free Lie algebra on $l + m$ generators into $T_x X$ by evaluation of a Lie polynomial at x. The kernel of this map is a subalgebra (but not necessarily an ideal) of the free Lie algebra which we will call the *subalgebra of relations at x*.

We now consider the possibility that either one of the modified frames $JY_1, \ldots, JY_l, Z_1, \ldots, Z_m$ or $Y_1, \ldots, Y_l, JZ_1, \ldots, JZ_m$ generates a totally real (and therefore maximally real) subspace of $T_x X$. First, since the listed vector fields are *holomorphically* extended to a neighborhood of their original domain, the Lie bracket is bilinear over $\mathbf{C} = \mathbf{R} \oplus J\mathbf{R}$. For this reason, with any choice of $V_1, \ldots, V_r \in \{Y_1, \ldots, Y_l, Z_1, \ldots, Z_m\}$ and $e_1, \ldots, e_r \in \{0, 1, 2, 3\}$,

$$[J^{e_r} V_r, \ldots, J^{e_1} V_1] = J^{e_r + \ldots + e_1} [V_r, \ldots, V_1]$$

where $V_r \otimes \ldots \otimes V_1 \mapsto [V_r, \ldots, V_1]$ denotes the iterated Lie bracket derived from any choice of recursive binary interval partitioning of $\{1, \ldots, r\}$ (i.e. partition $\{1, \ldots, r\}$ into two intervals, then for each of these intervals consisting of two or more elements, choose a partition of that interval into two intervals and continue recursively). It is immediately clear that multiplying a Lie monomial by ± 1 according to the parity of the number of Y_i factors or (respectively) Z_i factors it contains is an automorphism of the free Lie algebra which we will label as N_Y or N_Z (respectively). By splitting a Lie polynomial into even and odd monomials, a generic element of the free Lie algebra generated by $JY_1, \ldots, JY_l, Z_1, \ldots, Z_m$ in $T_x X$ can be written as $W_+ + JW_-$ where W_+ and W_- are, respectively, N_Y even and N_Y odd Lie polynomials in the variables

$Y_1, \ldots, Y_l, Z_1, \ldots, Z_m$ (i.e. every constituent monomial in W_+ has an even number of Y_i factors and every constituent monomial in W_- has an odd number of Y_i factors), and likewise an analogous statement holds for $Y_1, \ldots, Y_l, JZ_1, \ldots, JZ_m$ by a similar argument.

Definition 2.2.1 *If M is an analytic pseudosubriemannian manifold, a local horizontal analytic orthonormal frame $Y_1, \ldots, Y_l, Z_1, \ldots, Z_m$ is said to be a Y*-preferred frame *or respectively a Z*-preferred frame *at $x \in M$ if the subalgebra of relations at x is invariant under the automorphism N_Y or respectively N_Z.*

It is worth mentioning that if $m = 0$ (i.e. there are no Z_i terms), then the frame $Y_1, \ldots, Y_l$ can only be Y-preferred at x if all of the commutators vanish at x, for apparently $[Y_i, Y_j]$ is in the linear span of $Y_1, \ldots, Y_l$ in a neighborhood of x and the algebraic expression of this is a relation of degree two in the free Lie algebra. If the frame is preferred at x then apparently $[Y_i, Y_j]$ is equal to its negative and is therefore zero. Such a frame is of course Z-preferred in any case, vacuously. For any finite list $V_1, \ldots, V_r$ of vector fields on a smooth manifold, the *Lie hull* of the V_i in any tangent fiber is the linear span in that fiber of all Lie polynomials in the V_i.

Lemma 2.2.3 *If M is an analytic pseudosubriemannian manifold of dimension n, X is a bruhat-whitney complexification of M, and $Y_1, \ldots, Y_l, Z_1, \ldots, Z_m$ is a local horizontal analytic orthonormal frame then for any x in the frame domain, the Lie hull of $JY_1, \ldots, JY_l, Z_1, \ldots, Z_m$ in T_xX is maximally real in T_xX provided that the frame is Y-preferred at x and likewise the Lie hull of $Y_1, \ldots, Y_l, JZ_1, \ldots, JZ_m$ is maximally real in T_xX provided the frame is Z-preferred at x.*

Proof As described above, by splitting a Lie polynomial into N_Y-even and N_Y-odd monomials a generic element of the Lie hull of $JY_1, \ldots, JY_l, Z_1, \ldots, Z_m$ in T_xX can be written as $W_+ + JW_-$ where W_+ and W_- are, respectively, N_Y even and N_Y odd Lie polynomials in the vector fields $Y_1, \ldots, Y_l, Z_1, \ldots, Z_m$. We can write two generic elements as $W_+^1 + JW_-^1$ and $W_+^2 + JW_-^2$. If the Lie hull of $JY_1, \ldots, JY_l, Z_1, \ldots, Z_m$ in

T_xX contains a complex line then $W^1_\pm$ and $W^2_\pm$ can be chosen such that $W^1_+ + JW^1_- = J(W^2_+ + JW^2_-) = JW^2_+ - W^2_-$ with $W^1_+ + W^2_-$ and $W^1_- - W^2_+$ tangent to M at x so $W^1_+ + W^2_- = W^1_- - W^2_+ = 0$ in T_xX. If, further, $Y_1, \dots, Y_l, Z_1, \dots, Z_m$ is Y-preferred at x then the relation subalgebra in the free Lie algebra is invariant under the grading automorphism N_Y which negates W^1_- and W^2_- and leaves W^1_+ and W^2_+ fixed, so in fact $W^1_+ \pm W^2_- = \pm W^1_- - W^2_+ = 0$ and therefore $W^1_+ = W^2_- = W^1_- = W^2_+ = 0$. In particular $W^1_+ + JW^1_- = 0 \in T_xX$. We conclude that the Lie hull of $JY_1, \dots, JY_l, Z_1, \dots, Z_m$ in T_xX cannot contain a complex line and is therefore totally real. Since $Y_1, \dots, Y_l, Z_1, \dots, Z_m$ is assumed to be Y-preferred at x the grading automorphism N_Y is defined on T_xM so the Lie hull of $JY_1, \dots, JY_l, Z_1, \dots, Z_m$, being obtained from T_xM by multiplying the even and odd summands respectively by 1 and J, must have dimension n. The analogous statements in the Z-preferred case are proved in the same way. ■

For any local horizontal analytic orthonormal frame $Y_1, \dots, Y_l, Z_1, \dots, Z_m$, the real span in every tangent fiber of the holomorphic vector fields $JY_1, \dots, JY_l, Z_1, \dots, Z_m$ or respectively $Y_1, \dots, Y_l, JZ_1, \dots, JZ_m$ defines a distribution in a full open neighborhood of $x \in X$ which will be denoted respectively by D_Y and D_Z. It should be noted that D_Y and D_Z evidently depend on the elements of the chosen frame and are not invariants of the distribution $D \subset TM$ or even invariants of the individual real spans of $Y_1, \dots, Y_l$ and $Z_1, \dots, Z_m$. Indeed, even two distinct analytic vector fields on M which are real analytic multiples of each other have different holomorphic extensions to X which do not differ by a real factor even though they define the same one dimensional distribution in TM.

Nevertheless the $l+m$ dimensional distributions D_Y and D_Z are well defined in a neighborhood of $x \in X$. To make things more concrete, let $U_x \subset X$ be a specific open neighborhood of x in which the holomorphic continuations of the listed vector fields remain independent. Let $V^x_Y, V^x_Z \subset U_x$ denote the accessible sets from x determined respectively by the distributions D_Y and D_Z, i.e. the set of all endpoints of absolutely continuous curves in U_x emanating from x and with derivatives almost everywhere in

D_Y or respectively D_Z, or alternatively the orbits of the pseudogroup associated to the collection $\{JY_1, \ldots, JY_l, Z_1, \ldots, Z_m\}$ or respectively $\{Y_1, \ldots, Y_l, JZ_1, \ldots, JZ_m\}$ defined in the preceding overview of the stefan-sussmann theory.

Sussmann's theorem (Theorem 2.2.2) shows that the tangent space to V_Y^x or V_Z^x at any point is generated as a real vector space by the direct images of D_Y or D_Z through all concatenated flows through real multiples of the given vector fields, and that this tangent space contains the Lie hull of D_Y or D_Z at every point (i.e. the real linear span of all iterated Lie brackets of local sections of D_Y or D_Z). Furthermore, since the given vector fields are analytic the tangent space is precisely equal to the Lie hull, this is not true generally speaking in the smooth case (see [14, 21, 22]).

Theorem 2.2.4 *If M is an analytic pseudosubriemannian manifold and $Y_1, \ldots, Y_l, Z_1, \ldots, Z_m$ is a local horizontal analytic orthonormal frame which is Y-preferred at x then V_Y^x is a totally real subriemannian manifold of dimension n which is totally geodesic for the holomorphically extended hamiltonian flow and likewise if the given frame is Z-preferred at x then V_Z^x is a totally real subriemannian manifold of dimension n which is totally geodesic for the holomorphically extended hamiltonian flow.*

By the holomorphically extended hamiltonian flow we mean the flow in the cotangent bundle of the real part of the holomorphically continued hamiltonian, given by

$$2H = \mathrm{Re}\left(\sum_{i=1}^{l}(P_{Y_i} + iP_{JY_i})^2 - \sum_{i=1}^{m}(P_{Z_i} + iP_{JZ_i})^2\right) = \sum_{i=1}^{l} P_{Y_i}^2 - P_{JY_i}^2 + \sum_{i=1}^{m} P_{JZ_i}^2 - P_{Z_i}^2$$

where for any vector field W, P_W is the fiberwise linear momentum on the cotangent bundle which is defined by W. This will be discussed more extensively in the next section, but for now it is sufficient to observe that the holomorphic extension of the flow of an analytic vector field on a maximally real submanifold such as $T^*M \subset T^*X$ is the flow of the real part of the holomorphic extension of the holomorphic part of the original vector field. Equivalently, it is the flow of the hamiltonian vector field on T^*X associated to the real part of the holomorphically continued hamiltonian function.

Proof The assertions in the theorem follow directly from the preceding results along with Sussmann's theorem, because Sussmann's theorem implies that in the Y-preferred case the tangent space to V_Y^x at any point is the direct image of $T_x V_Y^x$ through a finite concatenation of holomorphic flows and as such it must be totally real in X with dimension equal to that of $T_x V_Y^x$ and likewise for the Z-preferred case. For any vector field V, the vertical component of the holomorphically extended hamiltonian flow defined by Hamilton's equations is

$$\dot{P_V} = \sum_{i=1}^{l} P_{[V,JY_i]} P_{JY_i} - P_{[V,Y_i]} P_{Y_i} + \sum_{i=1}^{m} \Big(P_{[V,Z_i]} P_{Z_i} - P_{[V,JZ_i]} P_{JZ_i},$$

and from this expression it is clear that for any Lie polynomial W in $JY_1, \ldots, JY_l$, $Z_1, \ldots, Z_m$,

$$\dot{P_{JW}} = \sum_{i=1}^{l} -P_{[W,Y_i]} P_{JY_i} - P_{J[W,Y_i]} P_{Y_i} + \sum_{i=1}^{m} \Big(P_{J[W,Z_i]} P_{Z_i} + P_{[W,Z_i]} P_{JZ_i}.$$

If we identify $T^* V_Y^x$ with the annihilator of JTV_Y^x then this expression vanishes on $T^* V_Y^x \cup JT^* V_Y^x$. In other words the extended hamiltonian flow begun at any point in $T^* V_Y^x \cup JT^* V_Y^x$ is contained in a level set of the function P_{JW}. However P_{JW} vanishes on $T^* V_Y^x$ so apparently the holomorphically continued hamiltonian flow is tangent to $T^* V_Y^x$ so V_Y^x is a totally real and totally geodesic subriemannian manifold and likewise for V_Z^x in the Z-preferred case with the same proof. ■

The main conclusion here is that any $x \in M$ which admits a preferred local horizontal analytic orthonormal frame in a neighborhood is contained in a totally real submanifold of maximal dimension for which the holomorphic continuation of the pseudosubriemannian hamiltonian on M is (tangentially) strictly positive or negative and so the usual metric theory for subriemannian manifolds can be applied locally to this submanifold.

Theorem 2.2.5 *If M is a connected analytic pseudosubriemannian manifold and $Y_1, \ldots, Y_l, Z_1, \ldots, Z_m$ is a local horizontal analytic orthonormal frame which is Y-preferred or Z-preferred at any point then the Lie algebra of complete analytic vector fields which annihilate the metric has dimension at most $n + n^2$.*

Proof As a consequence of Theorem 2.1.3, the proof is reduced to the existence of at least one vertically regular point. However, the existence of such a point follows from Theorem 2.2.4 which together with the given hypotheses ensures that there is a maximally real and totally geodesic subriemannian submanifold at x in the complexification X, so Agrachev's theorem applies to this submanifold, i.e. there must exist vertically regular points in every neighborhood of the kernel of the hamiltonian in $T_x^*V_Y^x$ or $T_x^*V_Z^x$ as the case may be. Without loss of generality assume that the given frame is Y-preferred at $x \in M$ so $T_x^*V_Y^x \subset T_x^*X$ is maximally real. As a result the vertically regular points in $T_x^*V_Y^x$ for the submanifold V_Y^x are also vertically regular for the holomorphic hamiltonian exponential map. In particular the set of critical points is a *nontrivial* divisor in the complex fiber T_x^*X, but this means that the critical divisor has a nowhere dense intersection in *every* maximally real subspace of T_x^*X and T_x^*M is one of these. ■

2.3 Holomorphic Hamiltonian Flow and Adapted Complex Structures

The relationship between a hamiltonian function on a cotangent bundle and its hamiltonian vector field can be somewhat easier to grasp when explained by way of the associated Poisson manifold structure. The *Poisson product* is the bilinear product on $\mathscr{C}^\infty(T^*M)$ defined by $\{f, g\} = df(X_g) = X_g f$ where X_g is the hamiltonian vector field for g, equivalently $\{f, g\} = \langle df, dg\rangle$ where $\langle\cdot,\cdot\rangle$ is (one of the two) symplectic forms on T^*T^*M which is dual to the natural symplectic form on TT^*M. Moreover it is a Lie bracket on functions: it is antisymmetric, bilinear under scalar multiplication, and satisfies the Jacobi identity. Furthermore in adjoint form it is a derivation of the usual pointwise product commutative algebra structure on $\mathscr{C}^\infty(T^*M)$.

The action of $\mathbf{R}_+^\times$ on T^*M normalizes the Poisson bracket: $m_r\{m_{r^{-1}}f, m_{r^{-1}}g\} = r\{f, g\}$, where $r \in \mathbf{R}_+^\times$ and m_r denotes the pullback on $\mathscr{C}^\infty(T^*M)$ by the fiberwise multiplicative action of r on T^*M. Thus, if Q is a homogeneous hamiltonian of degree $\alpha \in \mathbf{R}^\times$ in an open subset of T^*M then

$$\{Q, g\} = r^{-1} m_r \{m_{r^{-1}} Q, m_{r^{-1}} g\} = r^{\alpha - 1} m_r \{Q, m_{r^{-1}} g\}$$

In other words $m_r \operatorname{ad}_Q m_{r^{-1}} = \operatorname{ad}_{r^{1-\alpha} Q} = r^{1-\alpha} \operatorname{ad}_Q$, so the Poisson adjoint operator ad_Q is projectively normalized by the homothetic action of $\mathbf{R}_+^\times$. Dualizing this, one obtains the normalization $[\mathcal{R}, X_Q] = (\alpha - 1) X_Q$ where X_Q is the hamiltonian vector field symplectically dual to dQ and $\mathcal{R}$ is the Euler vector field (i.e. the direct image through the homothetic action of the unit tangent vector $\partial/\partial x \in T_1(\mathbf{R}_+^\times)$). Therefore

$$\begin{aligned}
[f(Q)\mathcal{R}, g(Q)X_Q] &= f(Q)(\mathcal{R}g(Q))X_Q - g(Q)(X_Q f(Q))\mathcal{R} + f(Q)g(Q)[\mathcal{R}, X_Q] \\
&= f(Q)g'(Q)\alpha Q X_Q + f(Q)g(Q)(\alpha - 1)X_Q \\
&= f(Q)(\alpha g'(Q)Q + (\alpha - 1)g(Q))X_Q \\
&= f(Q)\left(\alpha \frac{g'(Q)}{g(Q)} Q + (\alpha - 1)\right) g(Q) X_Q
\end{aligned} \tag{2.1}$$

for smooth functions f and g on the range of Q. Consequently, for any smooth functions f, g on the range of Q the flow of $f(Q)\mathcal{R}$ normalizes that of $g(Q)X_Q$ as follows,

$$\begin{aligned}
&\exp(f(Q)\mathcal{R}) \exp(g(Q)X_Q) \exp(-f(Q)\mathcal{R}) \\
&\qquad = \exp\left(e^{-f(Q)(\alpha g'(Q)Q + (\alpha-1)g(Q))/g(Q)} g(Q) X_Q\right)
\end{aligned} \tag{2.2}$$

for all points at which both sides of the expression make sense. The change in sign in the exponent on the right side, which seems unnatural, is due to the fact that the natural geometric action of the diffeomorphism group on the manifold T^*M corresponds to a right action of the diffeomorphism group on functions. The most obvious preliminary conclusion from this fact is that $\mathcal{R}$ and X_Q span an involutive distribution in their common domain of definition, having two-dimensional leaves in the open subset where they are independent over $\mathbf{R}$.

Here and below, we use the term *truncated conic open subset* to indicate an open subset of T^*M which is invariant under contractions (i.e. all dilations in $(0, 1]$).

Definition 2.3.1 *For any smooth Q, nonvanishing and homogeneous of degree one on a truncated conic open subset $U \subset T^*M \setminus M$ such that X_Q is independent of $\mathcal{R}$ throughout U, and any 2×2 real matrix J such that $J^2 + 1 = 0$, a complex structure on U will be called* J-adapted *if it acts by J on the two dimensional subspace of vector fields $\mathbf{R}X_Q \oplus \mathbf{R}Q^{-1}\mathcal{R}$ after identifying it with $\mathbf{R}^2$ by way of the basis $X_Q, Q^{-1}\mathcal{R}$.*

The normalization (2.1) shows that for any Q which is homogeneous of degree one, $[f(Q)\mathcal{R}, X_Q] = 0$ for any smooth f, so $f(Q)\mathcal{R}$ and X_Q generate a two dimensional *abelian* Lie algebra of vector fields in U, tangent to the $(\mathcal{R}, X_Q)$ foliation. In this case these fields exponentiate to an abelian *pseudogroup* of diffeomorphisms, i.e. a set of diffeomorphisms having all the natural properties of a group except that the domains will in general be proper subsets of the entire set U. Here and below, Q_β will denote any smooth function $\psi(Q)$ of Q with ψ homogeneous of degree $\beta \in \mathbf{R}$ on the range of Q. Thus, any such Q_β is homogeneous of degree β. Evidently $[Q_\beta\mathcal{R}, X_Q] = 0$ for any such choice of Q_β, but $\beta = -1$ (i.e. such that QQ_β is constant) is the unique choice for which $Q_\beta\mathcal{R}$ has nonzero radial limits on the zero section. This fact will be used later on.

Lemma 2.3.1 *For any analytic nonvanishing Q, defined and positively homogeneous of degree one in a conic open subset $U \subset T^*M$,*

1. *for any J-adapted complex structure in U and any point $\xi \in U$ the map*

$$\tau + i\sigma \mapsto \exp(\tau X_Q + \sigma J X_Q)\xi = \exp((\tau + \sigma J_{11})X_Q + \sigma J_{21}Q^{-1}\mathcal{R})\xi$$

   is a holomorpic immersion from a neighborhood of zero in $\mathbf{C}$ to a neighborhood of ξ in the leaf containing it,

2. *the $(Q^{-1}\mathcal{R}, X_Q)$-pseudogroup orbit of any tangent vector at ξ is a holomorphic section of the pullback of TU through this immersion.*

Proof The fact that $\tau + i\sigma \mapsto \exp(\tau X_Q + \sigma J X_Q)\xi$ is a holomorphic immersion from a neighborhood of zero in $\mathbf{C}$ to a neighborhood of ξ in its leaf for any J-adapted complex structure in U is clear by inspection. To prove assertion 2, observe that if $\gamma : (-\epsilon, +\epsilon) \to U$ is a smooth segment with $\gamma(0) = \xi$ then the pseudogroup orbit of the kinematic tangent vector $\dot{\gamma}(0)$ is given by $\dot{\gamma}_{\tau,\sigma}(0)$ at $\exp(\tau X_Q + \sigma J X_Q)\xi$ where

$$\gamma_{\tau,\sigma} = \exp(\tau X_Q + \sigma J X_Q)\gamma : (-\epsilon, +\epsilon) \to U.$$

Now, if F is a holomorphic function on a neighborhood of the range of γ then it is apparently defined and holomorphic on the range of $\gamma_{\tau,\sigma}$ for $\tau + i\sigma$ in some open neighborhood of zero $\Omega \subset \mathbf{C}$, the composite function $\tau + i\sigma \mapsto F \circ \gamma_{\tau,\sigma}(t)$ is apparently holomorphic in Ω with t fixed for every t in some interval containing zero and as such its derivative in t is also holomorphic in Ω, but this is nothing other than $dF(\dot{\gamma}_{\tau,\sigma}(t))$. In other words for holomorphic F with domain contained in the leaves intersecting γ, $dF(\dot{\gamma}_{\tau,\sigma}(t))$ is holomorphic when restricted to leaves so for any t in the domain of γ the orbit $\dot{\gamma}_{\tau,\sigma}(t)$ apparently defines a holomorphic section of the pullback to Ω of TU. This argument, adapted from Lempert and Szőke [8], completes the proof of assertion 2. ■

The concept of *geodesic limits* will be used in the following lemma and below. A vector field X on a truncated conic open subset of T^*M will be said to have geodesic limits along the zero section if it has limits along the zero section when restricted to any open subset of a geodesic leaf invariant under contractive dilations (i.e. those arising from scalars in $(0, 1]$) therein which does not intersect the corresponding reflected (i.e. negated) leaf.

Lemma 2.3.2 *For any analytic nonvanishing Q, defined and positively homogeneous of degree one in a conic open subset $U \subset T^*M\backslash M$ and any ψ defined and homogeneous of degree $\beta \in \mathbf{R}^\times$ on the range of Q,*

1. *with $Q_\beta = \psi(Q)$, the $(Q_\beta\mathcal{R}, X_Q)$-pseudogroup orbit of any tangent vector which annihilates Q is equal to the $(Q^{-1}\mathcal{R}, X_Q)$-pseudogroup orbit of that same tangent vector,*

2. *each pseudogroup orbit of a tangent vector which annihilates Q has geodesic limits along the zero section which are tangent to the zero section and equal to the fiber projection into TM of any element of the orbit.*

Proof The equality

$$\exp(\lambda X + tY) = \exp(\lambda X)\exp\left(\frac{1 - e^{-\lambda\,\mathrm{ad}_X}}{\lambda\,\mathrm{ad}_X} tY\right)$$

for exponentiated (i.e. the exponentiated action *on functions*) vector fields which generate a finite dimensional Lie algebra is a special case of the baker-campbell-hausdorff formula and can be proved by manipulation of power series. In the case $X = \mathcal{R}$ and $Y = Q_\beta \mathcal{R}$, apparently $\mathrm{ad}_X(Y) = [\mathcal{R}, Q_\beta \mathcal{R}] = \beta Q_\beta \mathcal{R}$ and therefore

$$\exp(\lambda\mathcal{R} + tQ_\beta\mathcal{R}) = \exp(\lambda\mathcal{R})\exp\left(\frac{1 - e^{-\beta\lambda}}{\beta\lambda} tQ_\beta\mathcal{R}\right)$$

on functions. As in the normalization (2.2), the geometric action of the diffeomorphism group has opposite variance to the corresponding action on functions, so the equality of *diffeomorphisms*

$$\exp(\lambda\mathcal{R} + tQ_\beta\mathcal{R}) = \exp\left(\frac{1 - e^{-\beta\lambda}}{\beta\lambda} tQ_\beta\mathcal{R}\right)\exp(\lambda\mathcal{R})$$

holds in the intersection of domains of either side. Converting the above equality of maps into an equality of their differential actions, one finds that

$$\exp(\lambda\mathcal{R} + tQ_\beta\mathcal{R})_* X = \exp\left(\frac{1 - e^{-\beta\lambda}}{\beta\lambda} tQ_\beta\mathcal{R}\right)_* \exp(\lambda\mathcal{R})_* X$$

for any tangent vector X. If X happens to be tangent to the level set $\lambda + tQ_\beta = 0$ then apparently the left side, and therefore the right side, leaves X fixed. We conclude that

$$\exp\left(\frac{e^{-\beta\lambda} - 1}{\beta\lambda} tQ_\beta\mathcal{R}\right)_* X = \exp(\lambda\mathcal{R})_* X$$

for all tangent vectors which are tangent to the level set $\lambda + tQ_\beta = 0$. Since the $(Q_\beta\mathcal{R}, X_Q)$ and $(Q^{-1}\mathcal{R}, X_Q)$-pseudogroup actions can be distinguished by their actions along radial lines, the first assertion is proved.

For the second assertion, we merely observe that as a consequence of the first assertion all such orbits are equal to the corresponding $(\mathcal{R}, X_Q)$ orbits - but the action of $\mathcal{R}$ on tangent vectors to T^*M is simple to describe. As always any tangent vector can be described kinematically by an arc γ in T^*M with $\dot{\gamma}(0)$ chosen appropriately. The direct image of $\dot{\gamma}(0)$ through $\exp(\lambda\mathcal{R})$ is the derivative of the arc $\exp(\lambda\mathcal{R})\gamma$ at zero, but $\lim_{\lambda\downarrow-\infty}\exp(\lambda\mathcal{R})\gamma$ is precisely the arc in the zero section arising as the fiber projection of the original arc γ. This proves the second assertion. ∎

Lemma 2.3.3 *If W is a complex vector space of dimension n and $V \subset W$ is any real subspace of real dimension n then there exists a maximally real subspace in W having trivial intersection with V.*

Proof Let $z_1, \ldots, z_n \in W$ be a complex basis and let $x_1, \ldots, x_n$ be such that their real span has dimension n and has trivial intersection with V. Let m_{ij} be the unique $n \times n$ complex matrix defined by $x_i = \sum_{j=1}^{n} m_{ij} z_j$. This matrix may or may not be invertible, depending on wether or not $x_1, \ldots, x_n$ is or is not independent over $\mathbf{C}$. However, the characteristic polynomial $\lambda \mapsto \det(\lambda + m)$ is necessarily nonzero in a punctured neighborhood of $0 \in \mathbf{C}$, and for all values of λ in this punctured neighborhood apparently $x_1^\lambda, \ldots, x_n^\lambda$ defined by $x_i^\lambda = (\lambda + m)z_i = \lambda z_i + m z_i$ is a complex basis and therefore generates a maximally real subspace over $\mathbf{R}$. Furthermore for $|\lambda|$ sufficiently small the real subspace generated by $x_1^\lambda, \ldots, x_n^\lambda$ must be transverse to V, since this is true by hypothesis for $x_1^0, \ldots, x_n^0$ and it is an open condition. ∎

Theorem 2.3.4 *For any analytic nonvanishing Q, defined and homogeneous of degree one in a truncated conic open subset $U \subset T^*M \setminus M$, and any 2×2 real matrix J such that $J^2 + 1 = 0$, there is at most one J-adapted complex structure in U.*

Proof This proof is adapted from Lempert and Szőke [8]. Assume that there is a given J-adapted complex structure in U, written $\mathscr{J}$. For any $\xi \in U$ let $L \subset T_\xi U$ be a maximally real subspace which is transverse to the vertical subspace $VT_\xi U$. The preceding lemma ensures that such a subspace exists. We claim that it is actually possible to choose a (potentially distict) maximally real subspace $\widetilde{L}$, also transverse to $VT_\xi U$, such that $X_Q \in \widetilde{L} \subset \ker dQ$. First, observe that the complex line generated by X_Q, i.e. $\mathbf{R}Q^{-1}\mathcal{R} \oplus \mathbf{R}X_Q$ must intersect any maximally real subspace in a real line. In particular this is true for L so we can choose X_n to be the unique element of L congruent to X_Q in $T_\xi U/VT_\xi U$. Let $X_1, \ldots, X_n$ be an extension of X_n to a real basis of L and therefore a complex basis of $T_\xi U$. Define $r_i \in \mathbf{R}$ for $1 \leq i \leq n-1$ to be the unique coefficients such that $X_i + r_i Q^{-1}\mathcal{R} \in \ker dQ$ for $1 \leq i \leq n-1$. We claim that the list $X_1 + r_1 Q^{-1}\mathcal{R}, \ldots, X_{n-1} r_{n-1} Q^{-1}\mathcal{R}, X_Q$ is a complex basis generating a maximally real subspace transverse to $VT_\xi U$. First, by reducing each element into the quotient $T_\xi U/VT_\xi U$ it is immediately apparent that the real subspace generated by this new list must be transverse to $VT_\xi U$ - since the new list is congruent to a basis of the quotient. To show that the new list is a complex basis, we consider the complex exterior product $(X_1 + r_1 Q^{-1}\mathcal{R}) \wedge \ldots \wedge (X_{n-1} + r_{n-1} Q^{-1}\mathcal{R}) \wedge X_Q$. By hypothesis, $Q^{-1}\mathcal{R}, X_Q$, and X_n lie on the same complex line so that in fact there are complex numbers $\zeta_1, \ldots, \zeta_n$ such that

$$\begin{aligned}
(X_1 + r_1 Q^{-1}\mathcal{R}) \wedge \ldots \wedge (X_{n-1} + r_{n-1} Q^{-1}\mathcal{R}) \wedge X_Q& \\
&= (X_1 + \zeta_1 X_n) \wedge \ldots \wedge (X_{n-1} + \zeta_{n-1} X_n) \wedge \zeta_n X_n \\
&= \zeta_n (X_1 \wedge \ldots \wedge X_n)
\end{aligned}$$

with $\zeta_n \neq 0$. Thus, the real span of $X_1 + r_1 Q^{-1}\mathcal{R}, \ldots, X_{n-1} + r_{n-1} Q^{-1}\mathcal{R}, X_Q$ is a maximally real subspace of $T_\xi U$ which is transverse to $VT_\xi U$, contains X_Q, and is contained in $\ker dQ$.

To simplify things, we can relabel everything so that the list $X_1, \ldots, X_n \in T_\xi U$ is such that

1. the real subspace generated by $X_1, \ldots, X_n$ is maximally real, is contained in $\ker dQ$, and is transverse to $VT_\xi U$,

2. $X_n = X_Q$.

In addition we can choose $Y_1, \ldots, Y_n \in T_\xi U$ such that

1. $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ is a real basis of $T_\xi U$,

2. $Y_1, \ldots, Y_{n-1} \in \ker dQ$,

3. $Y_n = Q^{-1}\mathcal{R}$,

4. $Y_1, \ldots, Y_{n-1}$ are $\mathbf{R}$-linearly independent in the quotient $T_\xi U / VT_\xi U$.

The Y_i can be chosen by first choosing $\widetilde{Y}_i$ for $1 \leq i \leq n-1$ to be a basis of the real subspace $\ker dQ \cap VT_\xi U$, and then defining $Y_i = \widetilde{Y}_i + X_i$.

The same symbols $X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1}$ will be used to denote the corresponding $(Q^{-1}\mathcal{R}, X_Q)$-pseudogroup orbits of the chosen elements of $T_\xi U$. By the foregoing result Lemma 2.3.1, the section $X_1 \wedge \ldots \wedge X_n$ of the complex line $\bigwedge^n_{\mathbb{C}} TU$ has a holomorphic restriction to the geodesic leaf containing ξ and as such it must vanish on a divisor, and this divisor must be nontrivial since the value of $X_1 \wedge \ldots \wedge X_n$ in $\bigwedge^n_{\mathbb{C}} T_\xi U$ is nonzero. Let $\psi_{rs} + i\varphi_{rs}$ be the $n \times n$ holomorphic matrix defined on the complement of the aforementioned divisor by $Y_r = \sum_{s=1}^n (\psi_{rs} + \mathscr{J}\varphi_{rs}) X_s$. By the foregoing result Lemma 2.3.2, all of the vectors $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ have (analytic) geodesic limits on the zero section, all other than $Y_n = Q^{-1}\mathcal{R}$ are tangent to the zero section, and $X_1, \ldots, X_n$ is a basis of $T_{\pi(\xi)} M$. As a result, the matrix $\psi_{rs} + i\varphi_{rs}$ has an analytic limit on an open subset of the geodesic boundary of the leaf in the zero section and, crucially, this analytic limit depends only on $Q, X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1}$ and J, but *not* $\mathscr{J}$. Therefore, $\psi_{rs} + i\varphi_{rs}$ is determined at all points in its domain by data which are independent of $\mathscr{J}$.

Finally, since $Y_r = \sum_{s=1}^n (\psi_{rs} + \mathscr{J}\varphi_{rs}) X_s$ and $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ is a real basis of $T_\xi U$, φ must have full rank at ξ, so

$$\mathscr{J} X_p = \sum_{r=1}^n \varphi^{-1}_{pr} Y_r - \sum_{r,s=1}^n \varphi^{-1}_{pr} \psi_{rs} X_s.$$

In this manner, we find that the action of the J-adapted complex structure $\mathscr{J}$ is determined on the complex basis $X_1, \ldots, X_n$ entirely by J and by the projections of $X_1, \ldots, X_n, Y_1, \ldots Y_{n-1}$ into TM in any open interval of the boundary geodesic containing $\pi(\xi)$. This proves the theorem. ∎

Corollary 2.3.5 *For any analytic and strictly positive H, defined and homogeneous of degree two in a truncated conic open subset $U \subset T^*M \setminus M$, and any 2×2 real matrix J such that $J^2 + 1 = 0$, there is at most one complex structure in U which acts by J on the two dimensional subspace $\mathbf{R}X_H \oplus \mathbf{R}\mathcal{R}$ with basis $X_H, \mathcal{R}$.*

Proof If m denotes the diagonal 2×2 matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, any complex structure acting by J on $\mathbf{R}X_H \oplus \mathbf{R}\mathcal{R}$ must act by mJm^{-1} on the basis $(\frac{1}{2\sqrt{H}}X_H, \frac{1}{\sqrt{H}}\mathcal{R})$ obtained by multiplying the basis $(mX_H, m\mathcal{R}) = (X_H, 2\mathcal{R})$ by $\frac{1}{2\sqrt{H}}$. However, $\frac{1}{2\sqrt{H}}X_H$ is equal to $X_{\sqrt{H}}$, i.e. the vector field for the degree one homogeneous hamiltonian $\sqrt{H}$. Thus, any complex structure in U acting by J on $\mathbf{R}X_H \oplus \mathbf{R}\mathcal{R}$ apparently acts by mJm^{-1} on $\mathbf{R}X_{\sqrt{H}} \oplus \mathbf{R}\frac{1}{\sqrt{H}}\mathcal{R}$, by Theorem 2.3.4 there can be at most one such structure. ∎

Adapted complex structures were originally introduced by Lempert and Szőke [8–10] and separately from a different perspective by Guillemin and Stenzel [11, 12], see also [23]. The original definition was the one described in the preceding corollary, i.e. it used X_H and $\mathcal{R}$ as a basis rather than $X_{\sqrt{H}}$ and $\frac{1}{\sqrt{H}}\mathcal{R}$. Here we've chosen to change the perspective so as to use hamiltonian functions which are homogeneous of degree one, since this makes the associated pseudogroup abelian and clarifies the existence of limits of the various orbits on the geodesic boundary.

The positivity hypothesis could be removed at the cost of more extensive notation and details. Having given a general uniqueness proof, we now proceed to various existence proofs for adapted complex structures for analytic subriemannian manifolds, first from the extrinsic perspective by way of bruhat-whitney complexifications and then from the intrinsic perspective through holomorphically continued lagrangian polarizations. This will involve a more detailed examination of the complexified hamiltonian flow. Hamiltonian functions will now be denoted by H as in the preceding corollary (generally speaking Q denotes a hamiltonian homogeneous of degree one whereas H denotes a hamiltonian homogeneous of unspecified degree). Let M denote an analytic pseudosubriemannian manifold and let X denote a bruhat-whitney complexification of M. There are four equivalent descriptions of the complexified hamiltonian flow on T^*X.

1. The cotangent bundle T^*M is a totally real submanifold of the real cotangent bundle T^*X of the complexification, itself a complex manifold via identification with $T^{(1,0)*}X$, and in the usual fashion the real analytic diffeomorphism of T^*M resulting from exponentiating the hamiltonian vector field corresponding to the analytic hamiltonian defining the subriemannian structure of M extends to a biholomorphism of an open neighborhood of $T^*M \subset T^*X$, this biholomorphism is the complexified hamiltonian flow.

2. The hamiltonian vector field X_H on T^*M is analytic and therefore extends to a holomorphic (real) vector field on a neighborhood of T^*M in T^*X, exponentiating this vector field defines the complexified hamiltonian flow.

3. The hamiltonian function H on T^*M extends to a holomorphic hamiltonian on T^*X. The corresponding hamiltonian vector field symplectically dual to $\operatorname{Re} dH$ is equal to the (real) holomorphic continuation of the original hamiltonian vector field. As before exponentiating this vector field defines the complexified hamiltonian flow.

4. Any local analytic frame $X_1, \ldots, X_l$ on an open subset $U \subset M$ with dual coframe $\xi_1, \ldots, \xi_l$ can be holomorphically continued to an open neighborhood of U in X. If H is a quadratic form in the fibers then the same is true of the metric coefficients $g(\xi_i, \xi_j)$, with U reduced if necessary. The holomorphically continued hamiltonian is given by

$$2H = \sum_{ij} g(\xi_i, \xi_j)(P_{X_i} + iP_{JX_i})(P_{X_j} + iP_{JX_j}).$$

   As before the hamiltonian vector field is symplectically dual to the differential of the real part of this function.

The last definition shows that, in the case of a quadratic form hamiltonian H, the hamiltonian vector field X_H can always be holomorphically continued to an open neighborhood of T^*M in T^*X which contains the entirety of every cotangent fiber it intersects and as such, we can write a chosen maximal domain of holomorphic continuation for X_H as $T^*U_{X,H}$ for some open neighborhood $U_{X,H}$ of $M \subset X$. If $\alpha, \beta \in \mathbf{R}$ the exponential map $\xi \mapsto \pi \circ \exp((\alpha + \beta\mathcal{J})X_H)\xi$ (where $\mathcal{J}$ is the complex structure on TT^*X) is holomorphic, its restriction to any given fiber T_x^*X, where defined, is a holomorphic map between complex manifolds of equal dimension so the critical set of any such restriction is a divisor.

To condense notation, write $F_{\alpha,\beta}(\xi) = \pi \circ \exp((\alpha + \beta\mathcal{J})X_H)\xi$ for ξ in the flow domain for $(\alpha+\beta\mathcal{J})X_H$ inside of a maximal domain $T^*U_{X,H}$ of holomorphic continuation of X_H. Naturally the domain $\Omega_{\alpha,\beta}$ of this map is a proper open submanifold of $T^*U_{X,H}$ and depends on α and β. The critical set $\mathrm{Crit}(F_{\alpha,\beta})$ in $\Omega_{\alpha,\beta}$ is a closed and nowhere dense subset which is locally a finite intersection of of divisors (specifically the central binomial coefficient $(2n)!/(n!)^2$ arising as the number of $n \times n$ minor determinants in a $n \times 2n$ matrix). Again to condense notation, we will write $\widetilde{\Omega}_{\alpha,\beta} = \Omega_{\alpha,\beta} \setminus \mathrm{Crit}(F_{\alpha,\beta})$. Furthermore, we denote by $\mathrm{Crit}_V(F_{\alpha,\beta})$ the set of vertically critical points for $F_{\alpha,\beta}$, i.e. those points at which $\ker DF_{\alpha,\beta}$ intersects the vertical tangent space. As noted above the intersection $\mathrm{Crit}_V(F_{\alpha,\beta}) \cap T_x^*X$ is a divisor in the n-dimensional complex manifold $\Omega_{\alpha,\beta} \cap T_x^*X$.

Any immersed submanifold $Y \subset \widetilde{\Omega}_{\alpha,\beta}$ of real dimension $2n = \dim_{\mathbf{R}} X$ which is tangentially transverse to $\ker DF_{\alpha,\beta}$ inherits an implied complex structure by using $F_{\alpha,\beta}$ to identify it locally with its image in the complex manifold X. These complex structures satisfy certain further properties which in more specific cases characterize them uniquely.

Lemma 2.3.6 *If H is analytic and homogeneous of degree $k \in \mathbf{Z}$ then the direct image of $\mathcal{R}$ through the biholomorphism* $\exp((\alpha + \beta\mathcal{J})X_H)$ *(wherever defined) is* $\mathcal{R} + (k-1)(\alpha + \beta\mathcal{J})X_H$.

Proof This follows from the holomorphically continued normalization 2.2:

$$\begin{aligned}
&\exp((\alpha+\beta\mathcal{J})X_H)\exp(t\mathcal{R})\\
&\qquad= \exp(t\mathcal{R})\exp(e^{(k-1)t}(\alpha+\beta\mathcal{J})X_H)\\
&\qquad= \exp(t\mathcal{R})\exp((e^{(k-1)t}-1)(\alpha+\beta\mathcal{J})X_H)\exp((\alpha+\beta\mathcal{J})X_H)\\
&\qquad= \exp(t(\mathcal{R}+(k-1)(\alpha+\beta\mathcal{J})X_H)+o(t))\exp((\alpha+\beta\mathcal{J})X_H).
\end{aligned}$$

■

Lemma 2.3.7 *If H is analytic and homogeneous of degree $k \in \mathbf{Z}$, then $\mathcal{R}-(k-1)(\alpha+\beta\mathcal{J})X_H \in \ker DF_{\alpha,\beta}$ in all fibers of $T\widetilde{\Omega}_{\alpha,\beta}$. In particular, in the implied complex structure on the quotient bundle $T\widetilde{\Omega}_{\alpha,\beta}/\ker DF_{\alpha,\beta}$, the complex number $\alpha+\beta i$ maps $(k-1)X_H + \ker DF_{\alpha,\beta}$ into $\mathcal{R}+\ker DF_{\alpha,\beta}$.*

Proof By Lemma 2.3.6, the direct images through $\exp((\alpha+\beta\mathcal{J})X_H)$ of $\mathcal{R}$ and $(k-1)(\alpha+\beta\mathcal{J})X_H$ differ by a multiple of $\mathcal{R}$, which is annihilated by the fiber projection π. ■

Thus, at any point in $T^*M \cap \widetilde{\Omega}_{\alpha,\beta}$ at which TT^*M is transverse to $\ker DF_{\alpha,\beta}$, apparently in the implied complex structure on TT^*M obtained by identifying T^*M locally with an open subset of X through $F_{\alpha,\beta}$, the complex number $(k-1)(\alpha+\beta i)$ maps X_H into $\mathcal{R}$. In other words the implied complex structure itself (i.e. the number i) maps $(k-1)\beta X_H$ to $\mathcal{R}-(k-1)\alpha X_H$. If, furthermore, the homogeneity degree k is equal to two then βX_H is mapped to $\mathcal{R}-\alpha X_H$, this is the case of interest.

According to Corollary 2.3.5, complex structures acting by a matrix J on $\mathbf{R}X_H \oplus \mathbf{R}\mathcal{R}$ are uniquely determined by J and H, if they exist. Furthermore, Lemma 2.3.7 hints at a method to prove that they do indeed exist, i.e. they can be obtained by embedding M into a bruhat-whitney complexification X, exponentiating the vector field $(\alpha+\beta\mathcal{J})X_H$ with $\beta \neq 0$ in a an open neighborhood of $\ker H \subset T^*X|_M$, projecting to X and then identifying the open set of regular points with X locally so that the complex structure on X defines a complex structure on this set. For H homogeneous of degree two, according to the normalization 2.2,

$$\exp(-z_{\alpha,\beta}\mathcal{R})\exp(X_H)\exp(z_{\alpha,\beta}\mathcal{R}) = \exp((\alpha+\beta\mathcal{J})X_H)$$

for any logarithm $z_{\alpha,\beta}$ of $\alpha+\beta\mathcal{J} \in \mathbf{C} = \mathbf{R}\oplus\mathbf{R}\mathcal{J}$. However, $\mathcal{J}$ acts on $\mathcal{R}$ by rotating $\mathcal{R}$ into the infinitesimal rotation field tangent to the action of $\mathrm{U}(1) \subset \mathbf{C}^\times$ coming from the complex vector space structure of each fiber T_xX, and so $\exp(z_{\alpha,\beta}\mathcal{R}) = (\alpha+\beta\mathscr{J})$ where $\mathscr{J}$ denotes the complex structure on T^*X acting linearly in every fiber. Therefore, $\pi\circ\exp((\alpha+\beta\mathcal{J})X_H) = \pi\circ\exp(X_H)\circ(\alpha+\beta\mathscr{J})$. With this preparation, we are able to prove the following theorem.

Theorem 2.3.8 *If M is an analytic subriemannian manifold and $\alpha+\beta i \in \mathbf{C}$ with $\beta \neq 0$, then there exists a complex structure mapping βX_H to $\mathcal{R}-\alpha X_H$ in the intersection of the open submanifold $(\alpha+\beta\mathscr{J})^{-1}(\widetilde{\Omega}_{1,0}\setminus \mathrm{Crit}_V(F_{1,0}))\cap T^*M \subset T^*M$ with a sufficiently small open neighborhood of* $\ker H$*.*

Proof Most aspects of this result have been developed in the foregoing exposition. We've shown that such a complex structure can be defined by pulling back the complex structure of a bruhat-whitney complexification X through the map $F_{\alpha,\beta}$ in the open subset of $T^*M\cap\widetilde{\Omega}_{\alpha,\beta}$ where TT^*M is transverse to $\ker DF_{\alpha,\beta}$. Alternatively, the equality $F_{\alpha,\beta} = \pi\circ\exp(X_H)\circ(\alpha+\beta\mathscr{J})$ shows that this is the same as the preimage through the fiberwise linear map $(\alpha+\beta\mathscr{J})$ of the set of points in $(\alpha+\beta\mathscr{J})T^*M$ at which this "rotated" copy of the submanifold $T^*M\subset T^*X$ is transverse to $\ker DF_{1,0}$. However, since points in $(\alpha+\beta\mathscr{J})\ker H$ are stationary for the flow $\exp(X_H)$, there must exist an open neighborhood of $(\alpha+\beta\mathscr{J})\ker H \subset (\alpha+\beta\mathscr{J})T^*M$ such that for any ξ in said neighborhood, $\ker DF_{1,0}\subset T_\xi((\alpha+\beta\mathscr{J})T^*M)$ consists only of vertical tangent vectors. This is a consequence of the fact that β is assumed to be nonzero so the direct image of any vertical tangent vector in $T_\xi((\alpha+\beta\mathscr{J})T^*M)$, if nonzero, must be "rotated" away from the direct image of any fixed subspace of horizontal tangent vectors. Because of this, for any splitting $T_\xi((\alpha+\beta\mathscr{J})T^*M) = V_\xi \oplus L_\xi$ with V_ξ equal to the vertical subspace,

$$\ker DF_{1,0}|_\xi = (\ker DF_{1,0}|_\xi \cap V_\xi) \oplus (\ker DF_{1,0}|_\xi \cap H_\xi)$$

for points ξ sufficiently close to $(\alpha+\beta\mathscr{J})\ker H$. However, since points in $(\alpha+\beta\mathscr{J})\ker H$ are stationary for the hamiltonian flow, $DF_{1,0} = D\pi$ at these points, where π is the fiber projection. As a result, for ξ in some (potentially smaller) open neighborhood of $(\alpha+\beta\mathscr{J})\ker H$, $DF_{1,0}$ cannot annihilate nonvertical tangent vectors, so apparently $\ker DF_{1,0}|_\xi = (\ker DF_{1,0}|_\xi \cap V_\xi)$, i.e. a point in such a sufficiently small neighborhood of $(\alpha+\beta\mathscr{J})\ker H$ can be critical for $F_{1,0}$ only if it is vertically critical. However, the vertically critical points for $F_{1,0}$ in any fiber T_xX with $x \in M$ form a divisor, which must be nontrivial by Agrachev's theorem. As a result, the vertically critical divisor cannot intersect any maximally real subspace of T_xX in an open set, and $(\alpha+\beta\mathscr{J})T^*_xM \subset T_xX$ is maximally real. We conclude that if $\xi \in (\alpha+\beta\mathscr{J})T^*M \subset T^*X$ is in the complement of the $F_{1,0}$ vertically critical divisor in its fiber and if, in addition, ξ is sufficiently close to $(\alpha+\beta\mathscr{J})\ker H$, then $(\alpha+\beta\mathscr{J})T^*M \subset T^*X$ is tangentially transverse to $\ker DF_{1,0}$ at ξ so there is an implied complex structure on $(\alpha+\beta\mathscr{J})T^*M \subset T^*X$ gotten by pulling back the structure on X through $F_{1,0}$. Furthermore, multiplication by $(\alpha+\beta\mathscr{J})^{-1}$ in the fibers of T^*X transfers this complex structure on $(\alpha+\beta\mathscr{J})T^*M \subset T^*X$ to a complex structure mapping βX_H to $\mathcal{R}-\alpha X_H$ in the intersection of the open submanifold $(\alpha+\beta\mathscr{J})^{-1}(\widetilde{\Omega}_{1,0} \setminus \mathrm{Crit}_V(F_{1,0})) \cap T^*M \subset T^*M$ with a sufficiently small open neighborhood of $\ker H$. ■

We now proceed to give an existence proof from an intrinsic perspective, i.e. without embedding M into a bruhat-whitney complexification. Here we mainly follow Lempert and Szőke [9] and Hall and Kirwin [23]. Theorem 2.3.8 proves existence of a complex structure in an open submanifold of T^*M mapping βX_H to $\mathcal{R}-\alpha X_H$ by selecting a bruhat-whitney complexification X and proving that the open submanifold of regular points in T^*M for the map $\pi \circ \exp((\alpha+\beta\mathcal{J})X_H)$ is nonempty. Since this map must be a local diffeomorphism at such points the complex structure in X can be pulled back to the regular set in T^*M and Lemmas 2.3.6 and 2.3.7 show that this complex structure indeed maps βX_H to $\mathcal{R} - \alpha X_H$. However, Theorem 2.3.8 provides no information on the nature of the described open submanifold of regular points, which is still very obscure. The intrisic perspective provides much greater clarity in this respect.

Asserting that a point in $\xi \in T^*M$ is regular for $\pi \circ \exp((\alpha+\beta\mathcal{J})X_H)$ is equivalent to asserting that the direct image of $T_\xi T^*M$ at any such point is transverse to the vertical tangent space at the image $\xi_{\alpha,\beta} = \exp((\alpha+\beta\mathcal{J})X_H)\xi$, i.e.

$$\exp((\alpha+\beta\mathcal{J})X_H)_* T_\xi T^*M \bigcap VT_{\xi_{\alpha,\beta}} T^*X = \{0\}$$

Furthermore, the vertical tangent space $VT_{\xi_{\alpha,\beta}}T^*X = T^*_{\pi(\xi_{\alpha,\beta})}X$ is complex and pulling back the complex structure from X at $\pi(\xi_{\alpha,\beta})$ in the manner described above is equivalent to identifying the transverse subspace $\exp((\alpha+\beta\mathcal{J})X_H)_* T_\xi T^*M \subset T_{\xi_{\alpha,\beta}}T^*X$ with the complex quotient $T_{\xi_{\alpha,\beta}}T^*X / VT_{\xi_{\alpha,\beta}}T^*X$. In other words, if we identify $\mathbf{C} \otimes \exp((\alpha+\beta\mathcal{J})X_H)_* T_\xi T^*M$ with $T_{\xi_{\alpha,\beta}}T^*X$ by equating i and $\mathcal{J}$, then $VT_{\xi_{\alpha,\beta}}T^*X$ is the antiholomorphic tangent space in the described complex structure.

In other words, we could equivalently consider the direct image $\exp(-(\alpha+\beta\mathcal{J})X_H)_* VTT^*X$ of the vertical tangent subbundle, restrict it to the open submanifold of T^*M where it is transverse to TT^*M, and define a complex structure on T^*M by declaring $\exp(-(\alpha+\beta\mathcal{J})X_H)_* VTT^*X$ to be the antiholomorphic subspace. However, after reinterpreting the situation in this manner it is immediately apparent that we don't need the bruhat-whitney complexification at all, because $\exp(-(\alpha+\beta\mathcal{J})X_H)_* VTT^*X$ can be identified with the value at $\alpha+\beta i$ of the holomorphic continuation of the complex subbundle of $T^{\mathbf{C}}T^*M$ defined for sufficiently small $t \in \mathbf{R}$ by $t \mapsto \exp(-tX_H)_*(VT^{\mathbf{C}}T^*M)$.

With this in mind, for any analytic subriemannian manifold M, the map $P : \Omega \to \mathrm{Lag}_{\mathbf{C}}(T^{\mathbf{C}}T^*M)$ defined on the open flow domain $\Omega \subset T^*M \times \mathbf{R}$ for the hamiltonian vector field X_H and taking values in the complex lagrangian grassmannian bundle $\mathrm{Lag}_{\mathbf{C}}(T^{\mathbf{C}}T^*M)$, defined by

$$P_\xi(t) = \exp(-tX_H)_*(VT^{\mathbf{C}}_{\exp(tX_H)\xi}T^*M)$$

is analytic on account of the fact that the hamiltonian H is analytic and as such must extend to an open neighborhood $\Omega_{\mathbf{C}} \subset T^*M \times \mathbf{C}$ of Ω, holomorphically in the variable t for fixed $\xi \in T^*M$. We shall assume that $\Omega_{\mathbf{C}}$ is maximal, in that P does not extend to any properly larger open set.

Lemma 2.3.9 *For all $(\xi, z) \in \Omega_{\mathbf{C}}$,*

1. *$(\xi, \overline{z}) \in \Omega_{\mathbf{C}}$ and $P_\xi(\overline{z}) = \overline{P_\xi(z)}$,*
2. *for fixed z, the subbundle $P_\xi(z)$ is involutive and contains $\mathcal{R} - zX_H$,*
3. *for $t \in \mathbf{R}$ such that ξ is in the domain of $\exp(tX_H)$, $(\exp(tX_H)\xi, z-t) \in \Omega_{\mathbf{C}}$ and $\exp(tX_H)_*P_\xi(z) = P_{\exp(tX_H)\xi}(z-t)$,*
4. *for $\tau \in \mathbf{R}$, $(e^\tau\xi, e^{-\tau}z) \in \Omega_{\mathbf{C}}$, $\exp(\tau\mathcal{R})_*P_\xi(z) = P_{e^\tau\xi}(e^{-\tau}z)$.*

Proof Assertion 1 is trivial, since $z \mapsto P_\xi(z)$ and $z \mapsto \overline{P_\xi(\overline{z})}$ are both holomorphic curves in $\mathrm{Lag}_{\mathbf{C}}(T^{\mathbf{C}}_\xi T^*M)$ which are equal on an interval in $\mathbf{R}$. For assertion 2, let $\eta_1, \ldots, \eta_n$ be a local analytic frame for T^*M defined in an open set $U \subset M$. This frame naturally defines an analytic frame for the vertical subbundle VTT^*M in all fibers above U, we will use the same symbols $\eta_1, \ldots, \eta_n$ for this vertical frame. After reducing the domain of the η_j to any connected open subset W of their original domain, define $\eta^z_j = \exp(zX_H)_*\eta_i$ for z in a connected W-dependent domain in $\mathbf{C}$ such that the expression makes sense and intersects a connected interval in $\mathbf{R}$. For $z \in \mathbf{R}$, the η^z_j clearly span an involutive subbundle (the tangent bundle to the direct image through $\exp(zX_H)$ of the vertical foliation). As a result, for such real z there exist analytic functions $c^{ij}_{k,z}$ on $\exp(zX_H)W$ such that $[\eta^z_i, \eta^z_j] = \sum_{k=1}^n c^{ij}_{k,z}\eta^z_k$. This expression continues holomorphically to admissible $z \notin \mathbf{R}$ and as such it expresses the commutators of a local frame for $P_\xi(z)$ for fixed z as elements of $P_\xi(z)$ for $\xi \in \exp((\operatorname{Re} z)X_H)W$. Since W can be chosen arbitrarily, $P_\xi(z)$ is apparently involutive where it is defined, for all fixed z.

The second statement in 2 is a direct consequence of Lemma 2.3.6, which shows that $\exp(zX_H)_*\mathcal{R} = \mathcal{R} + zX_H$ for $z \in \mathbf{R}$. Since $\mathcal{R}$ is vertical in every fiber, $z \mapsto \mathcal{R} - zX_H \subset T^{\mathbf{C}}_\xi T^*M$ is holomorphic and is included in $P_\xi(z)$ for z in an open interval of $\mathbf{R}$, therefore $\mathcal{R} - zX_H \in P_\xi(z)$ for all $(\xi, z) \in \Omega_{\mathbf{C}}$.

Assertion 3 holds by definition for real z, hence for all z at which both sides of the given expression make sense by analytic continuation. Assertion 4 follows from holomorphic continuation of the following chain of equalities for $\tau, z \in \mathbf{R}$,

$$\begin{aligned}
\exp(\tau\mathcal{R})_* P_\xi(z) &= \exp(\tau\mathcal{R})_* \exp(-zX_H)_* P_{\exp(zX_H)\xi}(0) \\
&= \exp(\tau\mathcal{R})_* \exp(-zX_H)_* \exp(-\tau\mathcal{R})_* \exp(\tau\mathcal{R})_* P_{\exp(zX_H)\xi}(0) \\
&= \exp(\tau\mathcal{R})_* \exp(-zX_H)_* \exp(-\tau\mathcal{R})_* P_{\exp(\tau\mathcal{R})\exp(zX_H)\xi}(0) \\
&= \exp(-e^{-\tau} zX_H)_* P_{\exp(\tau\mathcal{R})\exp(zX_H)\xi}(0) \\
&= P_{\exp(\tau\mathcal{R})\xi}(e^{-\tau} z) \\
&= P_{e^\tau \xi}(e^{-\tau} z).
\end{aligned}$$

∎

Thus, evidently the affine group $\mathbf{R}_+ \ltimes \mathbf{R}$ acts invariantly on $\Omega_{\mathbf{C}}$, at least if one gives proper attention to domain considerations. If M is a complete manifold, so that X_H is a complete vector field, then this is a true group action. Define $\widetilde{\Omega}_{\mathbf{C}} \subset \Omega_{\mathbf{C}}$ to be the open subset containing points (ξ, z) such that $P_\xi(z) \cap \overline{P_\xi(z)} = \{0\}$.

Corollary 2.3.10 *For all* $(\xi, z) \in \widetilde{\Omega}_{\mathbf{C}}$,

1. *for* $t \in \mathbf{R}$ *such that* ξ *is in the domain of* $\exp(tX_H)$, $(\exp(tX_H)\xi, z - t) \in \widetilde{\Omega}_{\mathbf{C}}$,
2. *for* $\tau \in \mathbf{R}$, $(e^\tau \xi, e^{-\tau} z) \in \widetilde{\Omega}_{\mathbf{C}}$,
3. $(r\xi, z) \in \widetilde{\Omega}_{\mathbf{C}}$ *for all* $r > 0$ *such that* $(\xi, rz) \in \widetilde{\Omega}_{\mathbf{C}}$, *in particular for any* $z \in \mathbf{C}$ *the* z *cross-section of* $\widetilde{\Omega}_{\mathbf{C}}$ *is a truncated conic open subset of* T^*M,

4. $P_\xi(z)$ is the antiholomorphic tangent space for a complex structure $\mathcal{J}$ on the z cross-section of $\widetilde{\Omega}_{\mathbf{C}}$ which maps $(\operatorname{Im} z)X_H$ to $\mathcal{R} - (\operatorname{Re} z)X_H$.

Proof Assertions 1 and 2 follow from the analogous assertions in Lemma 2.3.9 and the fact that $\mathcal{R}$ and X_H are real vector fields and as such they transform real tangent vectors to real tangent vectors. So in other words, not only does $\exp(tX_H)_*$ map $P_\xi(z)$ to $P_{\exp(tX_H)\xi}(z-t)$ but also it must map real tangent vectors in the former space to real tangent vectors in the latter. Thus, it is not possible for precisely one of $P_\xi(z), P_{\exp(tX_H)\xi}(z-t)$ to contain a nonzero real tangent vector, and likewise for $P_\xi(z), P_{e^\tau z}(e^{-\tau}z)$ with a similar proof. Assertion 3 follows immediately from 2.

The fact that $P_\xi(z)$ is the antiholomorphic tangent space for a complex structure $\mathcal{J}$ on the z cross-section of $\widetilde{\Omega}_{\mathbf{C}}$ follows directly from the fact that this subbundle is involutive and has trivial intersection with its conjugate or, equivalently, trivial intersection with the real tangent space. The former fact has been proved in Lemma 2.3.9 and the domain $\widetilde{\Omega}_{\mathbf{C}}$ has been defined so that the latter fact is true. It remains only to prove the stated action of $\mathcal{J}$ on $\mathbf{R}X_H \oplus \mathbf{R}\mathcal{R}$. As proved in Lemma 2.3.9, $\mathcal{R} - zX_H \in P_\xi(z)$ so if $\mathcal{J}$ is the complex structure having $P_\xi(z)$ as the antiholomorphic tangent space, $i(\mathcal{R} - zX_H) = -\mathcal{J}(\mathcal{R} - zX_H)$. Equating imaginary parts shows $\mathcal{R} - (\operatorname{Re} z)X_H = \mathcal{J}(\operatorname{Im} z)X_H$, as desired. ∎

Theorem 2.3.11 *For any analytic subriemannian manifold M, if $(x,y) \in M \times M$ is a smooth pair in the sense of Agrachev and $2\xi \in T^*M$ is the midpoint of the hamiltonian lift of the unique geodesic segment connecting x and y, then $(r\xi, i) \in \widetilde{\Omega}_{\mathbf{C}}$ for sufficiently small $r > 0$.*

Proof Let $\eta_1, \ldots, \eta_n$ be an analytic cotangent frame in an open neighborhood of x and let $\zeta_1, \ldots, \zeta_n$ be an analytic cotangent frame in an open neighborhood of y. Using the same symbols η_j and ζ_j we view $\eta_1, \ldots, \eta_n$ and $\zeta_1, \ldots, \zeta_n$ as analytic frames for the vertical tangent bundle VTT^*M in the respective preimages through the fiber projection of their original domains. For $1 \le j \le n$ and some sufficiently small $\epsilon > 0$, define $\eta_j^z = \exp(-zX_H)_*\eta_j \in T_\xi T^*M$ for $z \in (-1-\epsilon, -1+\epsilon)$ and

$\zeta_j^w = \exp(-wX_H)_*\zeta_j \in T_\xi T^*M$ for $w \in (1-\epsilon, 1+\epsilon)$. Strictly speaking $\exp(-zX_H)_*\eta_j$ and $\exp(-wX_H)_*\zeta_j$ are vector fields defined in a full open neighborhood of ξ but by η_j^z and ζ_j^w we indicate the isolated values of these fields in the fiber $T_\xi T^*M$. Apparently $\eta_1^z, \ldots, \eta_n^z$ and $\zeta_1^w, \ldots, \zeta_n^w$ are bases for $P_\xi(z)$ and $P_\xi(w)$ respectively, for appropriate values of z and w in each case.

Now, on account of the fact that we've assumed (x, y) to be a smooth pair in the sense of Agrachev, the real span of $\eta_1^{-1}, \ldots, \eta_n^{-1}$ is transverse to the real span of $\zeta_1^1, \ldots, \zeta_n^1$, for if these two real subspaces were to contain a common nonzero vector u then $v = \exp(-X_H)_*u$ would be a vertical tangent vector in the fiber $T_{\exp(-X_H)\xi}T^*M$ such that $\exp(2X_H)w \in T_{\exp(2X_H)}T^*M$ is also vertical - but there can be no such vector on account of the fact that (x, y) is assumed to be a smooth pair. Having shown that $\eta_1^{-1}, \ldots, \eta_n^{-1}, \zeta_1^1, \ldots, \zeta_n^1$ is a real basis of $T_\xi T^*M$ we observe that it is also a complex basis of $T_\xi^{\mathbf{C}} T^*M$ since $T_\xi T^*M \subset T_\xi^{\mathbf{C}} T^*M$ is maximally real. Thus, for $(z, w) \in (-1-\epsilon, -1+\epsilon) \times (1-\epsilon, 1+\epsilon) \subset \mathbf{C}^2$,

$$(z, w) \mapsto \eta_1^z \wedge \ldots \wedge \eta_n^z \wedge \zeta_1^w \wedge \ldots \wedge \zeta_n^w$$

is an analytic map taking values in the complex line $\bigwedge^{2n} T_\xi^{\mathbf{C}} T^*M$ which does not vanish at $(z, w) = (-1, 1)$. The holomorphic continuation of this map vanishes at $(0, 0)$ (since both the η_j^0 and ζ_j^0 form a basis of the vertical tangent space at ξ). However, since it does not vanish at $(-1, 1)$ we conclude that its divisor must have a discrete intersection in the complex line defined by $z + w = 0$. In particular, there must exist $\rho > 0$ such that it is nonzero at all points of the form $(ri, -ri)$ for $0 < r < \rho$. Let ρ_ξ denote the supremum of all ρ with this property. Since $-ri = \overline{ri}$, we find that $P_\xi(ri) \cap \overline{P_\xi(ri)} = \{0\}$ for $0 < r < \rho_\xi$. In other words, $(\xi, ri) \in \widetilde{\Omega}_{\mathbf{C}}$ for $0 < r < \rho_\xi$. Thus, by assertion 3 from Corollary 2.3.10, $(r\xi, i) \in \widetilde{\Omega}_{\mathbf{C}}$ for $0 < r < \rho_\xi$.

■

Theorem 2.3.11 proves that, for any fixed $\rho > 0$, a complex structure mapping ρX_H to $\mathcal{R}$ exists on some sufficiently small truncated conic open subset $\Omega^\rho \subset T^*M$ for any analytic subriemannian manifold M. Furthermore, by Corollary 2.3.10 a "*t*-

sheared" complex structure mapping ρX_H to $\mathcal{R} - tX_H$ exists on the forward image through $\exp(-tX_H)$ of the intersection of Ω^ρ with the domain of $\exp(-tX_H)$. By Corollary 2.3.5, these complex structures are unique. We anticipate that most of the results from Lempert and Szőke [8–10] and Guillemin and Stenzel [11, 12] can be adapted in one way or another to the present situation of subriemannian manifolds.

2.4 Connections

The well known "fundamental theorem of riemannian geometry" states that a riemannian manifold admits a unique torsion free affine connection which annihilates the metric. The common elementary proof of this fact involves writing the expression $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ with the vector fields cyclically permuted. From there using various strategic additions and subtractions of these equalities one can isolate expressions of the form $\nabla_X Y - \nabla_Y X$ which can be replaced with the Lie bracket $[X, Y]$ on account of the fact that the connection is assumed to be torsion free. Further manipulating the resulting expressions, one obtains the so-called *Koszul formula*:

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle .$$

Since the right side depends only on the metric, it is sufficient to uniquely define the described connection. The same argument of course works in the pseudoriemannian case. In the pseudosubriemannian case the same argument cannot work. First, the righthand expression written above which defines $2\langle \nabla_X Y, Z\rangle$ in the pseudoriemannian case does not even make sense in the pseudosubriemannian case since the inner product is not necessarily defined on the brackets, and secondly an expression of the form $2\langle \nabla_X Y, Z\rangle$ is sufficient to define $\nabla_X Y$ in the pseudoriemannian case since the metric is everywhere defined and nondegenerate - but no such implicit expression will work in the pseudosubriemannian case.

Indeed, there seems to be no natural choice of connection to use in pseudosubriemannian geometry. However, there is evidence that certain connections should be preferred over others - but that such connections are not linear except in the nondegenerate pseudoriemannian case. To begin with, by a *partial linear connection* (on the cotangent bundle) we mean a linear differential operator of order one $\nabla : \mathscr{C}^\infty(M, T^*M) \longrightarrow \mathscr{C}^\infty(M, D^* \otimes T^*M)$ where D^* is the dual of a smooth distribution in the tangent bundle, which satisfies the Leibniz rule: $\nabla(f\eta) = df|_D \otimes \eta|_D + f\nabla\eta$ for $f \in \mathscr{C}^\infty(M)$ and $\eta \in \mathscr{C}^\infty(M, T^*M)$. We observe that since D is assumed to be a smooth distribution, every point in D is the value of a smooth local section and so a section of D^* is smooth if its evaluation against any smooth section of D is a smooth function. In other words, the space of smooth sections $\mathscr{C}^\infty(M, D^* \otimes T^*M)$ is perfectly well-defined. Alternatively, one can view a partial linear connection in the geometric sense, i.e. as a splitting of the sequence $VTT^*M \longrightarrow \pi^*(D) \subset TT^*M \longrightarrow D$ defined by a transverse distribution $\pi^*(D) \subset TT^*M$ which is dilation invariant and additive (thus linear) and which is likewise smooth and has potentially varying rank.

The torsion of such a partial connection is defined by the usual expression $T(X, Y) = \nabla^*_X Y - \nabla^*_Y X - [X, Y]$ for sections X, Y of D, where ∇^* is the dual partial connection. Alternatively, the torsion can be defined as the section of $TM \otimes (D^* \bigwedge D^*)$ which when traced against the generic one-form η gives the skew-symmetric form $\wedge \circ \nabla\eta - d\eta$ on D obtained by subtracting $d\eta$ from the skew symmetric part of $\nabla\eta \in D^* \otimes D^*$. This is the natural symplectic inner product in TT^*M restricted to the ∇-horizontal lift of D at η. Thus, for a torsion free connection the horizontal space of ∇ in TT^*M is isotropic in that it is contained in its symplectic orthogonal.

It seems natural to seek out partial connections for which the horizontal subbundle $H \subset TT^*M$ is contained in a horizontal lagrangian subspace which annihilates the metric, since these conditions mimic the familiar characteristic properties of the canonical connection in the pseudoriemannian case. With this in mind we have the following result.

Lemma 2.4.1 *If M is a pseudosubriemannian manifold then any lagrangian subbundle of TT^*M which is transverse to the vertical and annihilates the hamiltonian function contains the hamiltonian vector field.*

Proof For any hamiltonian $H \in \mathscr{C}^\infty(T^*M)$, the value of the hamiltonian vector field in $T_\xi T^*M$ is the unique tangent vector such that when projected into any lagrangian splitting of $T_\xi T^*M$, it reproduces the restrictions of dH to each direct summand. In other words if $T_\xi T^*M = L_1 \oplus L_2$ is a lagrangian splitting then one can split the one form $dH = dH_1 + dH_2$ uniquely with dH_1 annihilating L_2 and dH_2 annihilating L_1, but the symplectic structure identifies L_1 with the dual of L_2 and L_2 with the dual of L_1, so dH_1 is equal to a unique element of L_2 and dH_2 is a unique element of L_1. Adding these components reproduces the hamiltonian field for H. For any global lagrangian splitting of TT^*M by horizontal and vertical subspaces, at any point the horizontal component of the hamiltonian field at $\xi \in T^*_xM$ is defined by requiring its projection in T_xM to reproduce the restriction of dH at ξ to directions tangent to the fiber T^*_xM. However, for a pseudosubriemannian manifold the hamiltonian is a quadratic form, and a simple computation verifies that the differential of such forms are at every point defined by the natural mapping into the dual given by the form itself (actually twice that, but this factor is accounted for by halving the diagonal restriction of the quadratic form to define the hamiltonian). Therefore, if a given horizontal subbundle $H \subset TT^*M$ is as hypothesized in the lemma, the hamiltonian is annihilated by H, so the hamiltonian vector field has no vertical component with respect to the $H \oplus VTT^*M$ splitting. The hamiltonian field is therefore contained in H. ■

A canonical splitting $VTT^*M \longrightarrow \pi^*(D) \subset TT^*M \longrightarrow D$ defined by a metric-annihilating lagrangian distribution as described in the lemma has been found by Zelenko and Li [24], see also Barilari and Rizzi [25]. However, this splitting is only defined in a dense open subset of T^*M and is generally not the restriction of a linear connection there. In fact, as the next result shows a partial linear connection which permits differentiation in all metrically horizontal directions and has a horizontal distribution contained in a metric-annihilating lagrangian distribution cannot exist in the interior of the set where the metric is degenerate.

Proposition 2.4.2 *If M is a pseudosubriemannian manifold then in the interior of the set where the metric is degenerate there does not exist any partial connection which permits differentiation in all metrically horizontal directions, and which has a horizontal distribution in TT^*M contained in a horizontal lagrangian distribution which annihilates the hamiltonian.*

Proof Denote by L the metric trace, i.e. $L \in \mathrm{Hom}(T^*M, TM)$. Furthermore, let ∇ be a partial connection as described in the proposition, i.e. ∇ permits differentiation in all metrically horizontal directions and has a horizontal distribution in TT^*M contained in a horizontal lagrangian distribution which annihilates the hamiltonian H. Since the horizontal distribution for ∇ annihilates H, it also parallelizes L. Therefore for X such that ∇_X makes sense, $\nabla^*_X(L\eta) = L(\nabla_X \eta)$. If $\gamma : (-\epsilon, \epsilon) \to T^*M$ is a hamiltonian integral curve then $\nabla_{\dot{\pi\gamma}}\gamma = 0$ on account of the fact that the horizontal distribution of ∇ contains the hamiltonian vector field. Furthermore, $L\gamma = \dot{\pi\gamma}$ according to the definition of the hamiltonian vector field. Therefore, $0 = L(\nabla_{\dot{\pi\gamma}}\gamma) = \nabla^*_{\dot{\pi\gamma}}(L\gamma) = \nabla^*_{\dot{\pi\gamma}}(\dot{\pi\gamma})$. The conclusion is that for any such γ, $\pi\gamma$ is a geodesic in the usual sense for the dual partial connection ∇^*, i.e. it is an integral curve of the vector field in TM equal at every point to the ∇^*-horizontal lift of that point. In particular $\pi\gamma$ is completely determined by any single one of its tangent vectors, but this property is manifestly false for hamiltonian geodesics in the interior of the degenerate set for the pseudosubriemannian metric as can be seen by translating any point in any given hamiltonian integral curve by an element of the kernel of the hamiltonian - the integral curve passing through this new point will have the same derivative at the point in question but will in general have a projection into M which is distinct from the projection of the original curve. ∎

Nevertheless, alot can be accomplished even with an arbitrary linear connection which is *a priori* completely unrelated to the pseudosubriemannian structure. Here we adapt some results from [13] to the subriemannian case. The standing assumptions will be that

- $g \in TM \otimes TM$ denotes the symmetric section obtained by polarizing the subriemannian hamiltonian into a symmetric bilinear form,
- E denotes any vector bundle over the subriemannian manifold M,
- ∇^E is a linear connection on E,
- ∇^{T^*M} is any linear connection on T^*M (i.e. having no evident relation to or dependence on the subriemannian structure).

With connections ∇^E and ∇^{T^*M} fixed on E and T^*M, the symbols ∇^{E^*} and ∇^{TM} will always denote the respective dual connections. The unmodified symbol ∇ will be used to represent any one of $\nabla^E, \nabla^{T^*M}, \nabla^{E^*}$, or ∇^{TM} when it is clear from context which is meant and likewise Tr will be used to denote the natural pairing of a bundle with its dual *or* the pairing of a bundle with itself when a metric is present. With these connections fixed the associated Laplace operator is defined for any section s of E by the negated trace of the $(\nabla^E, \nabla^{T^*M})$-covariant hessian, i.e.

$$\Delta s = \Delta^{\nabla^E, \nabla^{T^*M}} s = -\operatorname{Tr}(\nabla^{E \otimes T^*M} \nabla^E s).$$

This is apparently defined for any quadratic form hamiltonian whatsoever, there is no restriction regarding nondegeneracy or positivity. However, hypoellipticity is another matter and in the subriemannian case it is ensured by the fact that in any local frame domain for TM and E, Δ is given by

$$\Delta = -\sum_{ij} g(X_i^*, X_j^*) \left(\nabla^E_{X_i} \nabla^E_{X_j} - \nabla^E_{\nabla^{TM}_{X_i} X_j} \right)$$

which is, within a perturbation of differential order one, a Hörmander sum of squares type operator. It is easily proved that for two bundles E_1 and E_2 with connections ∇^{E_1} and ∇^{E_2},

$$\Delta(s_1 \otimes s_2) = (\Delta s_1) \otimes s_2 - 2 \operatorname{Tr}(\nabla s_1 \otimes \nabla s_2) + s_1 \otimes (\Delta s_2).$$

A *g-Dirac operator* on E is a differential operator $\mathbf{D}$ of order one on E such that $\mathbf{D}^2$ is a g-sublaplacian, i.e. the principal symbol $[[\mathbf{D}^2, f], f] = -2g(df, df)$ must be given by the metric. Naturally the principal symbol $[\mathbf{D}, f] \in \mathrm{End}(E)$ of $\mathbf{D}$ itself defines an action of the complete tensor algebra $\mathrm{T}(T^*M)$ on E, and as with any operator of differential order one, for any linear connection ∇^E on E it differs from the contraction $\mathrm{Tr}([\mathbf{D}, \cdot]\nabla^E)$ by an operator of order zero (note that the symbol $[\mathbf{D}, \cdot]$ defines an action of T^*M on E and as such it is a section of $\mathrm{End}(E) \otimes TM$ and can be traced with $\nabla s \in T^*M \otimes E$ for s any section of E). Because we've assumed that $\mathbf{D}^2$ is a g-sublaplacian, this is necessarily a *Clifford action*, i.e. it factors through the Clifford algebra bundle $\mathrm{Cl}(T^*M)$ of T^*M defined by the metric g. This bundle is defined by quotienting the complete tensor algebra of T^*M by the ideal generated by $\eta^{\otimes 2} + g(\eta, \eta)$ for every one form η.

For any linear connection we will denote by $\mathbf{D}_{\nabla^E} = \mathrm{Tr}(\mathbf{c}\nabla_E)$ the ∇^E Dirac operator described above, i.e. $\mathbf{c}$ is used to denote the Clifford symbol $[\mathbf{D}, \cdot] \in \mathrm{End}(E) \otimes TM$ and $\mathbf{D}_{\nabla^E} = \mathbf{c}\nabla_E$, where we've omitted the $\mathrm{Tr}(\cdot)$ to condense notation. Expressions for the difference $\mathbf{D}^2_{\nabla^E} + \mathrm{Tr}(\nabla^{E\otimes T^*M}\nabla^{T^*M})$ are called *Lichnerowicz* formulas. Typically one assumes that more structure is present such as a metric preserving and/or torsion free connection on T^*M. However, as we've already proved that these do not exist for degenerate subriemannian metrics, it seems optimal to give a completely general Lichnerowicz formula depending only on g, ∇^E and ∇^{T^*M}.

Theorem 2.4.3 *For any symmetric bilinear form g on T^*M, and any two connections ∇^E and ∇^{T^*M} on E and T^*M respectively,*

$$(\mathbf{c}\nabla^E)^2 = -\,\mathrm{Tr}(\nabla^{E\otimes T^*M}\nabla^{T^*M}) + \left(\mathbf{c}(\nabla\mathbf{c}) - \frac{1}{2}\mathbf{c}T^{\nabla^{TM}}\right)\left(\nabla^E + \frac{1}{2}\mathbf{c}F^{\nabla^E}.\right.$$

Rather than embellish the right side of the expression in Theorem 2.4.3 with clarifying yet excessive notation, we will discuss here how it should be interpreted. The Clifford symbol $\mathbf{c}$ is a section of $\mathrm{End}(E)\otimes TM$ and the torsion $T^{\nabla^{T^*M}}$ is a section of $\bigwedge^2 T^*M \otimes TM$. Thus, $\mathbf{c}(\nabla\mathbf{c})$ denotes the section of $\mathrm{End}(E) \otimes TM$ resulting from tracing the TM factor of $\mathbf{c}$ with the T^* factor of $\nabla\mathbf{c}$ and likewise tracing the inner pair $E^* \otimes E$ (i.e. composing endomorphisms) to give the section $\mathbf{c}(\nabla\mathbf{c})$ of $\mathrm{End}(E) \otimes TM$. From this, the section $\mathbf{c}T^{\nabla^{T^*M}}$ obtained by quantizing the two-form factor into a Clifford endomorphism of E is subtracted and we're left with a single section of $\mathrm{End}(E)\otimes TM$ which can then be traced with the one-form factor prepended to any section of E by way of the action of ∇. Thus, $(\mathbf{c}(\nabla\mathbf{c}) - \mathbf{c}T^{\nabla^{T^*M}}/2)\nabla^E$ is a well-defined differential operator of order one. Likewise, the endomorphism $\mathbf{c}F^{\nabla^E}$ is obtained by simply quantizing the two-form factor of the curvature operator F^{∇^E} of ∇^E into a Clifford endomorphism of E.

Proof The desired equality follows by direct computation in a local frame:

$$
\begin{aligned}
(\mathbf{c}\nabla^E)^2 &= \mathbf{c}(\nabla\mathbf{c})\nabla^E + \mathbf{c}\nabla\nabla^E \\
&= \mathbf{c}(\nabla\mathbf{c})\nabla^E + c(X_i^*)c(X_j^*)\left(\nabla^E_{X_i}\nabla^E_{X_j} - \nabla^E_{\nabla^{TM}_{X_i}X_j}\right) \\
&= \mathbf{c}(\nabla\mathbf{c})\nabla^E \\
&\qquad + \frac{1}{2}c(X_i^*)c(X_j^*)\left(\nabla^E_{X_i}\nabla^E_{X_j} + \nabla^E_{X_j}\nabla^E_{X_i} - \nabla^E_{\nabla^{TM}_{X_i}X_j} - \nabla^E_{\nabla^{TM}_{X_j}X_i}\right) \\
&\qquad + \frac{1}{2}c(X_i^*)c(X_j^*)\left(\nabla^E_{X_i}\nabla^E_{X_j} - \nabla^E_{X_j}\nabla^E_{X_i} - \nabla^E_{\nabla^{TM}_{X_i}X_j} + \nabla^E_{\nabla^{TM}_{X_j}X_i}\right) \\
&= \mathbf{c}(\nabla\mathbf{c})\nabla^E \\
&\qquad - \frac{g(X_i^*, X_j^*)}{2}\left(\nabla^E_{X_i}\nabla^E_{X_j} + \nabla^E_{X_j}\nabla^E_{X_i} - \nabla^E_{\nabla^{TM}_{X_i}X_j} - \nabla^E_{\nabla^{TM}_{X_j}X_i}\right) \\
&\qquad + \frac{1}{2}c(X_i^*)c(X_j^*)\left(F^{\nabla^E}(X_i, X_j) + \nabla^E_{[X_i,X_j]} - \nabla^E_{T^{\nabla^{TM}}(X_i,X_j)+[X_i,X_j]}\right) \\
&= \mathbf{c}(\nabla\mathbf{c})\nabla^E - \mathrm{Tr}(\nabla^{E\otimes T^*M}\nabla^{T^*M}) \\
&\qquad + \frac{1}{2}c(X_i^*)c(X_j^*)\left(F^{\nabla^E}(X_i, X_j) - \nabla^E_{T^{\nabla^{TM}}(X_i,X_j)}\right)
\end{aligned}
$$

$\blacksquare$

The subalgebra generated by the kernel $\ker H$ of the hamiltonian is apparently the usual exterior algebra $\bigwedge \ker H$ in every fiber, and it is central in the entire Clifford algebra bundle in the graded sense. Thus, if one has the local direct sum decomposition $T^*U = R \bigoplus \ker H$ over some open subset $U \subset M$ then apparently $\mathrm{Cl}(T^*U) = \mathrm{Cl}(R) \bigotimes \bigwedge \ker H$ *as a graded algebra*, meaning that the factors commute with one another provided the expression is multiplied by the proper power of -1

to account for the grading. For any metric, degenerate or not, it is always possible to find a basis of any fiber T_x^*M which is mutually orthogonal, i.e. scalar products of any two elements are zero. Any pair of elements in such a basis are necessarily anti-commuting, so as for the exterior algebra (which is a special case), products of basis elements which are monotone in any chosen ordering form a basis of $\mathrm{Cl}(T_x^*M)$. Thus, there are mutually inverse natural maps (i.e. η in T^*M corresponds to η in $\mathrm{Cl}(T^*M)$)

$$\sigma : \mathrm{Cl}(T^*M) \to \bigwedge T^*M \quad \text{and} \quad \mathbf{c} : \bigwedge T^*M \to \mathrm{Cl}(T^*M)$$

extending the identity on the common linear subspace T^*M called respectively the *symbol* and *quantization* maps, which are linear isomorphisms but *not* algebra isomorphisms.

Assume now that x_o is a regular point for the subriemannian structure (i.e. it is a point of continuity for the vector of dimensions of the flag generated by lie brackets of horizontal vector fields of respectively ascending degrees) and that a set $x_1, \ldots, x_n$ of privleged coordinates has been chosen [26]. Any set of privleged coordinates identifies an open neighborhood of $x_o \in M$ with an open neighborhood of the identity in a nilpotent Lie group with dilations δ_u, and as such, these dilations can be viewed as acting on a neighborhood of $x_o \in M$. Generally speaking only the contractive dilations for $u \in (0,1)$ will be defined on the entire domain, in any case these are all that's necessary. If $\widehat{d}$ denotes the subriemannian metric distance in the described nilpotent Lie group and d denotes the metric for the subriemannian manifold M then there exist $C, r > 0$ such that

$$-C\widehat{d}(p,q)d(q,q')^{1/r} \leq d(q,q') - \widehat{d}(q,q') \leq C\widehat{d}(p,q)\widehat{d}(q,q')^{1/r}$$

for p, q, q' sufficiently close to x_o, this is proved in [26]. With $p = q = x_o$, apparently $d(x_o, q') - \widehat{d}(x_o, q')$ and consequently $d(x_o, \delta_u q') = ud(x_o, q')$ for $u \in (0, \infty)$ and $q \in M$ such that both expressions make sense. Note that even though this identification preserves the radial distance to the basepoint x_o, is *not* a metric isometry everywhere - for that there will be curvature obstructions.

A heat kernel $k_{x_o}(t,x)$ for $\mathbf{D}^2_{\nabla^E}$ with pole at x_o, if it exists, is a time dependent section of $\mathrm{Hom}(E_{x_o}, E_x)$ annihilated by the operator $\partial_t + \mathbf{D}^2_{\nabla^E}$ and having a limit equal to $1 \in \mathrm{End}(E_{x_o})$ times a point mass at x_o as $t \to 0$. Any heat kernel can be converted to a function taking values in $\mathrm{End}(E_{x_o})$ by post-composing with the ∇^E parallel translation from E_x to E_{x_o} *along orbits for the dilations* δ_u, which we denote with $\mathrm{Pt}^{x_o}_\delta$. Thus, $\mathrm{Pt}^{x_o}_\delta\, k_{x_o}(t,x)$ is a time dependent function in the domain of the dilations taking values in $\mathrm{End}(E_{x_o})$, which is annihilated by the operator $\partial_t + \mathbf{D}^2_{\nabla^E}$ by viewing $\mathbf{D}_{\nabla^E}$ as an operator on E_{x_o}-valued functions by way of parallel translation.

For the remainder of this section we make the following simplifying assumptions:

1. $\mathrm{Cl}(T^*M)$ admits a (naturally graded) spinor module S, i.e. a module such that the action map $\mathrm{Cl}(T^*M) \to \mathrm{End}(S)$ is bijective,

2. locally in an open neighborhood of any point of M, $E = S \otimes W$ is the graded tensor product of the spinor module S and a graded vector space W,

3. $\nabla^E = \nabla^S \otimes \nabla^W$ for some choice of ∇^S and ∇^W.

For nondegenerate metrics the existence of the spinor module is well-established and it is associated to the orthonormal frame bundle so it inherits a natural connection from the riemannian connection on T^*M. This simplifies matters in the nondegenerate case, but in the degenerate case it must be taken as a hypothesis. With these assumptions in place, $\mathrm{Pt}^{x_o}_\delta\, k_{x_o}(t,x)$ is a time dependent function in the domain of the dilations taking values in the subalgebra $\mathrm{Cl}(T^*_{x_o}M) \otimes \mathrm{End}(W_{x_o}) \subset \mathrm{End}(E_{x_o})$, which can be identified with $\bigwedge T^*_{x_o}M \otimes \mathrm{End}(W_{x_o})$ by way of the symbol map on the first factor. With this assumption in place we can view $\mathrm{Pt}^{x_o}_\delta\, k_{x_o}(t,x)$ as a time dependent function in the domain of the dilations, valued in $\bigwedge T^*_{x_o}M \otimes \mathrm{End}(W_{x_o})$. Now we extend the spatial dilations to the heat kernel k by identifying *any* action β of $(0,\infty)$ on $T^*_{x_o}M$ which acts invariantly on $\ker H \subset T^*_{x_o}M$ and dilates the quotient in the usual way. Finally, define the dilations acting on $\mathrm{Pt}^{x_o}_\delta\, k$ by

$$\alpha_u\, \mathrm{Pt}^{x_o}_\delta\, k_{x_o}(t,x) = (\beta^{-Q/d}_{\sqrt{u}} \otimes 1)\, \mathrm{Pt}^{x_o}_\delta\, k_{x_o}(ut, \delta_{\sqrt{u}} x).$$

Here Q denotes the homogeneous dimension at x_o and d is the dimension of the horizontal space at x_o. The meaning of $\beta_{\sqrt{u}} \otimes 1$ should be clear: since $\mathrm{Pt}_\delta^{x_o} k_{x_o}$ takes values in

$$\mathrm{Cl}(T^*_{x_o}M) \otimes \mathrm{End}(W_{x_o}) = \bigwedge(T^*_{x_o}M \otimes \mathrm{End}(W_{x_o})$$

via the symbol map, the functorial extension of the action β to $\mathrm{Cl}(T^*_{x_o}M)$ by way of its identification with the exterior algebra is perfectly well-defined. The $\mathrm{End}(W_{x_o})$ factor is unaffected by $\beta_{\sqrt{u}}^{-Q/d} \otimes 1$ and the factor $\mathrm{Cl}(T^*_{x_o}M)$ is acted upon by this functorial extension.

Theorem 2.4.4 *If $E = S \otimes W$ is a graded g-Clifford module as described above, then* $\mathrm{Str}\, k_{x_o}(1, x_o) = \mathrm{Str}\, u^{Q/2}\alpha_u \,\mathrm{Pt}_\delta^{x_o} k_{x_o}(1, x_o)$ *for all u. Furthermore, $u^{Q/2}\alpha_u \,\mathrm{Pt}_\delta^{x_o} k_{x_o}(t, x)$ is a heat kernel for $u\alpha_u \mathbf{D}^2_{\nabla^E}\alpha_u^{-1}$ acting on functions taking values in* $\mathrm{End}(E_{x_o})$ *and defined in the domain of the chosen privleged coordinates. If, furthermore, $\lim_{u\to 0} u\alpha_u \mathbf{D}^2_{\nabla^E}\alpha_u^{-1}$ exists and has heat kernel k^0 then* $\mathrm{Str}\, k_{x_o}(1, x_o) = \mathrm{Str}\, k^0_{x_o}(1, x_o)$.

Proof If $\eta_1, \ldots, \eta_d$ is an orthonormal set in $T^*_{x_o}M$ which is a basis of $T^*_{x_o}M/\ker H$ then as with the exterior algebra the products of distinct elements of $\eta_1, \ldots, \eta_d$ with strictly increasing indices forms a basis of $\mathrm{Cl}(T^*_{x_o}M/\ker H)$. Furthermore for any such product $\eta_{i_1} \cdots \eta_{i_k}$,

$$-\eta_{i_2} \cdots \eta_{i_k} = \eta_{i_1}(\eta_{i_1} \cdots \eta_{i_k}) = (-1)^{k-1}(\eta_{i_1} \cdots \eta_{i_k})\eta_{i_1}$$

so

$$[\eta_{i_1}, \eta_{i_1} \cdots \eta_{i_k}] = \eta_{i_1}(\eta_{i_1} \cdots \eta_{i_k}) - (-1)^k(\eta_{i_1} \cdots \eta_{i_k})\eta_{i_1} = -2\eta_{i_2} \cdots \eta_{i_k}.$$

In other words: every basis element of degree strictly less than d is a supercommutator. It follows that $\mathrm{Cl}(T^*_{x_o}M/\ker H)$ admits a projectively unique supertrace, since any supertrace must vanish on all basis elements except $\eta_1 \ldots \eta_d$. The same statement is *not* true after incorporating the factor $\bigwedge(\ker H$ in the decomposition

$$\mathrm{Cl}(T^*_{x_o}M) = \mathrm{Cl}(T^*_{x_o}M/\ker H) \otimes \bigwedge(\ker H,$$

since the only supercommutator in $\bigwedge \ker H$ is zero. However, in any representation the elements of $\bigwedge \ker H$ of positive degree are nilpotent and therefore cannot have a nonzero supertrace. Likewise any product ab with $a \notin \bigwedge \ker H$ and $b \in \bigwedge \ker H$ must be nilpotent if b is has positive degree since any power $(ab)^n$ is equal to cb^n for some c. Such elements cannot have a nonzero supertrace and we therefore conclude that the only elements $\mathrm{Cl}(T^*_{x_o}M)$ having nonzero supertrace are those having maximal degree d in the quotient $\mathrm{Cl}(T^*_{x_o}M/\ker H)$. Also, for any tensor $\eta \otimes \nu \in \mathrm{Cl}(T^*_{x_o}M) \otimes \mathrm{End}(W_{x_o})$, $\mathrm{Str}(\eta \otimes \nu) = \mathrm{Str}(\eta)\,\mathrm{Str}(\nu)$. Thus, for $\mathrm{Str}\, k_{x_o}(1, x_o) = \mathrm{Str}\, u^{Q/2} \alpha_u \, \mathrm{Pt}_\delta^{x_o} \, k_{x_o}(1, x_o)$ to be true the only necessary condition is that the functorial extension of the dilations $\beta_{\sqrt{u}}^{-Q/d}$ on $\bigwedge^d T^*_{x_o}M/\ker H$ cancel the constant $u^{Q/2}$ necessary to maintain the approximate identity property, but the exponent Q/d has been chosen precisely so that this is so. The second and third assertions follow directly from the first. ■

3. GEOMETRY OF REAL FLAG MANIFOLDS

In this chapter it will be a standing assumption that the various data used to specify flag manifolds described in the introduction have been chosen. Namely, it will be said that a list $(G, \theta, \mathfrak{h}, \Delta^+(\mathfrak{g}, \mathfrak{h}), P_\Sigma)$ of data is *admissible* if

1. G is a connected Lie group with reductive Lie algebra $\mathfrak{g}_{\mathbf{R}}$, a real form of $\mathfrak{g} = \mathfrak{g}_{\mathbf{R}} \otimes \mathbf{C}$, with $\mathrm{Ad}(G) \subset \mathrm{Aut}(\mathfrak{g})$ inner and such that the identity component of the derived group $[G, G]$ has finite center,

2. θ is a Cartan involution on the derived ideal $[\mathfrak{g}_{\mathbf{R}}, \mathfrak{g}_{\mathbf{R}}]$,

3. $\mathfrak{h}_{\mathbf{R}} = \mathfrak{h} \cap \mathfrak{g}_{\mathbf{R}}$ is a maximally noncompact θ-stable Cartan subalgebra of $\mathfrak{g}_{\mathbf{R}}$ (much of the theory remains true even if $\mathfrak{h}_{\mathbf{R}}$ is not necessarily maximally noncompact, but it is a customary hypothesis when discussing parabolic subalgebras of real reductive Lie algebras),

4. $\Delta^+(\mathfrak{g}, \mathfrak{h})$ is a simple system of roots of $\mathfrak{h}$ which is admissible for the real form $\mathfrak{g}_{\mathbf{R}}$ (i.e. the associated set of positive noncompact roots is invariant under the Satake involution σ^* arising from the complex conjugation for the real form $\mathfrak{g}_{\mathbf{R}}$) and,

5. $P_\Sigma \subset G$ is a parabolic subgroup with Lie algebra $\mathfrak{p}^\Sigma_{\mathbf{R}}$ constructed from the subset Σ of the noncompact roots in $\Delta^+(\mathfrak{g}, \mathfrak{h})$ which is stable under the action of the Satake involution.

In this chapter we will define and study differential operators on G/P_Σ which arise naturally after a list $(G, \theta, \mathfrak{h}, \Delta^+(\mathfrak{g}, \mathfrak{h}), P_\Sigma)$ of data satisfying the standard hypotheses has been chosen. Such a list will be called an *admissible datum*.

The Cartan decomposition $[\mathfrak{g}_{\mathbf{R}}, \mathfrak{g}_{\mathbf{R}}] = \mathfrak{k}_{\mathbf{R}} \oplus \mathfrak{s}_{\mathbf{R}}$ arises in the standard way as the ± 1 eigenspace decomposition of the involution θ. Since the Cartan subalgebra $\mathfrak{h}_{\mathbf{R}}$ is

θ-stable, evidently $\mathfrak{h}_{\mathbf{R}} = (\mathfrak{h}_{\mathbf{R}} \cap \mathfrak{k}_{\mathbf{R}}) \oplus (\mathfrak{h}_{\mathbf{R}} \cap \mathfrak{s}_{\mathbf{R}}) = \mathfrak{t}_{\mathbf{R}} \oplus \mathfrak{a}_{\mathbf{R}}$ with $\mathfrak{a}_{\mathbf{R}}$ maximal abelian in $\mathfrak{s}_{\mathbf{R}}$ since $\mathfrak{h}_{\mathbf{R}}$ is maximally noncompact by hypothesis. If $\mathfrak{n}^+_{\mathbf{R}}$ is the real part of the nilradical of the minimal parabolic associated to the simple root system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ then the projection $(1-\theta)/2$ maps $\mathfrak{n}^+_{\mathbf{R}}$ bijectively onto the orthogonal complement of $\mathfrak{a}_{\mathbf{R}}$ in $\mathfrak{s}_{\mathbf{R}}$, so that $\mathfrak{g}_{\mathbf{R}}$ admits the Iwasawa decomposition $\mathfrak{g}_{\mathbf{R}} = \mathfrak{k}_{\mathbf{R}} \oplus \mathfrak{a}_{\mathbf{R}} \oplus \mathfrak{n}^+_{\mathbf{R}} \oplus Z_{\mathfrak{g}_{\mathbf{R}}}$ with $\mathfrak{a}_{\mathbf{R}} \oplus \mathfrak{n}^+_{\mathbf{R}} \oplus Z_{\mathfrak{g}_{\mathbf{R}}} \subset \mathfrak{p}^{\Sigma}_{\mathbf{R}}$ so if $K \subset G$ is the (not necessarily compact) analytic subgroup with Lie algebra $\mathfrak{k}_{\mathbf{R}}$, then the orbit map from K to its orbit $K/M_{\Sigma} = K1P_{\Sigma} \subset G/P_{\Sigma}$ ($M_{\Sigma} = K \cap P_{\Sigma}$) is a submersion onto its image. In particular the orbit must be open in G/P_{Σ}. On the other hand if K is compact then the orbit must also be compact, hence closed, and it must therefore be a connected component of G/P_{Σ}. However, we've assumed that G is connected so G/P_{Σ} is connected as well so if K is compact then $K/M_{\Sigma} = G/P_{\Sigma}$. A standard result in Lie theory states that K is compact if and only if the analytic subgroup associated to the subalgebra $[\mathfrak{g}_{\mathbf{R}}, \mathfrak{g}_{\mathbf{R}}]$ has finite center, but we've taken this criterion as a hypothesis by requiring the identity component of the derived group $[G, G]$ to have finite center.

3.1 Structure of Homogeneous Spaces

Let $L \subset G$ be Lie groups with G connected and L closed in G, and with Lie algebras $\mathfrak{l}_{\mathbf{R}} \subset \mathfrak{g}_{\mathbf{R}}$, each respectively a real form of the complexifications $\mathfrak{l} = \mathfrak{l}_{\mathbf{R}} \otimes \mathbf{C}$ and $\mathfrak{g} = \mathfrak{g}_{\mathbf{R}} \otimes \mathbf{C}$. In this subsection we make no further assumptions on G and L (e.g. $\mathfrak{g}_{\mathbf{R}}$ is not necessarily reductive, although the results proved here will be applied to the reductive case). We are interested in identifying involutive or bracket-generating subbundles of the real tangent bundle $T(G/L)$ which are invariant under the natural action of G on G/L. It is an easily provable fact that general left G-invariant subbundles of the real tangent bundle are in bijective correspondence with subspaces of $\mathfrak{g}_{\mathbf{R}}/\mathfrak{l}_{\mathbf{R}}$ which are Ad_L-invariant, or equivalently subspaces of $\mathfrak{g}_{\mathbf{R}}$ which are Ad_L-invariant and contain $\mathfrak{l}_{\mathbf{R}}$, but in order to discuss Lie brackets of sections more details are needed.

Vector fields (and differential operators generally) on G which commute with right multiplication by L act invariantly on right L-invariant functions and therefore descend naturally to differential operators on G/L. However, such operators need not be invariant under the left action of G on G/L. Thus, the direct image of a subspace of right L-invariant vector fields will not do, but the consideration thereof indicates the correct idea: one should at least attempt to consider the direct image of *left* invariant vector fields on G since we want them to commute with the action of G on the quotient, within an error which lies in the subbundle in question. However, left invariant vector fields do not descend naturally to G/L unless they are also right L-invariant, in general they only do so modulo the action of the adjoint representation of the isotropy group L. In other words, if $X \in T_1(G)$ then $\lim_{t\to 0} t^{-1}[f(ue^{tX}L) - f(uL)]$ defines an element of $T_{uL}(G/L)$ for any $u \in G$ but this definition is *not* independent of the specific element u used to represent the coset uL unless X is invariant under the action of Ad_L. Generally speaking, for any $y \in L$, uyL evidently defines the same coset and the resulting tangent vector is

$$\begin{aligned}\lim_{t\to 0} t^{-1}[f(uye^{tX}L) - f(uL)] &= \lim_{t\to 0} t^{-1}[f(uye^{tX}y^{-1}L) - f(uL)] \\ &= \lim_{t\to 0} t^{-1}[f(ue^{t\,\mathrm{Ad}_y X}L) - f(uL)].\end{aligned}$$

Thus, while the tangent vector $\lim_{t\to 0} t^{-1}[f(ue^{tX}L) - f(uL)]$ is not independent of $u \in G$, its orbit in $T_{uL}(G/L)$ under Ad_L is perfectly well defined, and this argument essentially constitutes a proof of the fact stated earlier, that subbundles of the real tangent bundle $T(G/L)$ invariant under G are naturally in bijection with Ad_L-invariant subspaces of $\mathfrak{g}_{\mathbf{R}}/\mathfrak{l}_{\mathbf{R}}$, or equivalently Ad_L-invariant subspaces of $\mathfrak{g}_{\mathbf{R}}$ which contain $\mathfrak{l}_{\mathbf{R}}$.

In order to define a local section of such a subbundle, or for that matter to define vector fields locally on G/L, one must first choose a gauge, i.e. an embedded submanifold $U \subset G$ which intersects every left coset of L at most once and is transverse to each coset which it intersects. The intersection hypothesis means that U can be smoothly identified with an open subset of G/L (i.e. it defines a section of the quotient $G \to G/L$), and that $(u, y) \mapsto uy$ is a diffeomorphism from $U \times L$ onto the preimage in G of this open subset of G/L. After a gauge has been chosen, we can compute the direct image of a left invariant field by way of the infinitesimal action of $\mathfrak{g}_{\mathbf{R}}$ on the right, which amounts to projecting a tangent vector to G at a point $u \in U$ onto the tangent space of U via the transverse subspace tangent to the left action of L, and then computing the direct image. Equivalently, to any given $X \in \mathfrak{g}_{\mathbf{R}}$ we associate the tangent vector $\lim_{t\to 0} t^{-1}[f(ue^{tX}L) - f(uL)] \in T_{uL}(G/L)$ for any $u \in U$. More generally if $u \mapsto X_u$ is a smooth map from U into $\mathfrak{g}_{\mathbf{R}}$, then $\lim_{t\to 0} t^{-1}[f(ue^{tX_u}L) - f(uL)]$ defines a section of the tangent bundle over the image of U in G/L which vanishes in the fiber $T_{uL}(G/L)$ if and only if $X_u \in \mathfrak{l}_{\mathbf{R}} \subset \mathfrak{g}_{\mathbf{R}}$.

There is an established formalism to handle Lie bracket computations involving such sections using what is generally known as the *maurer-cartan form*, denoted ω_G. This is the $\mathfrak{g}$-valued one form which maps an element $X \in T_x(G) \otimes \mathbf{C}$ into the element $l_{x^{-1}*}X \in T_1(G) \otimes \mathbf{C} = \mathfrak{g}$, i.e. the direct image of X through left multiplication by x^{-1} or equivalently the value in $T_1(G) \otimes \mathbf{C}$ of the unique left invariant vector field on G which extrapolates X. The maurer-cartan form ω_U on the image of the gauge U in G/L is the pullback of ω_G through the gauge, which maps the real tangent vector $\lim_{t\to 0} t^{-1}[f(ue^{tX_u}L) - f(uL)] \in T_{uL}(G/L)$ to the projection of X_u in $l_{u^{-1}*}T_u(U)$ via the direct sum decomposition $\mathfrak{g}_{\mathbf{R}} = l_{u^{-1}*}T_u(U) \oplus \mathfrak{l}_{\mathbf{R}}$.

Lemma 3.1.1 *If U is any gauge, then for any smooth map $u \mapsto X_u \in \mathfrak{g}_{\mathbf{R}}$, there exists a unique vector field $Z \in \mathscr{C}^\infty(T(G/L))$ over U such that $\omega_U(Z) - X_u \in \mathfrak{l}_{\mathbf{R}}$ at every point in U.*

The proof should be obvious: since $l_{u^{-1}*}T_u(U) \subset \mathfrak{g}_{\mathbf{R}}$ is a transverse complement to $\mathfrak{l}_{\mathbf{R}}$ for all $u \in U$, the construction of Z amounts to the determination of the components of X_u in the direct sum decomposition $\mathfrak{g}_{\mathbf{R}} = l_{u^{-1}*}T_u(U) \oplus \mathfrak{l}_{\mathbf{R}}$ for all $u \in U$. All of this evidently depends on the choice of gauge U. Elementary arguments demonstrate that if $\lambda : U \to L$ is a smooth map from U into L then the pointwise product $U\lambda$ is another gauge and all gauges arise in this fashion from a unique gauge transformation λ. Furthermore, the two forms ω_U and $\omega_{U\lambda}$ are related by the equation $\omega_{U\lambda} = \mathrm{Ad}_{\lambda^{-1}}\,\omega_U + \lambda^*\omega_L$, where the latter summand is the $\mathfrak{l}$-valued pullback through λ of the maurer-cartan form on L. Thus, we have the following result.

Lemma 3.1.2 *If $T_V(G/L) \subset T(G/L)$ is the G-invariant subbundle associated to any Ad_L-invariant subspace $V \subset \mathfrak{g}_{\mathbf{R}}$ containing $\mathfrak{l}_{\mathbf{R}}$, then a tangent vector $X \in T_{xL}(G/L)$ is an element of $T_V(G/L)$ if and only if $\omega_U(X) \in V$ for any gauge $U \subset G$ which intersects the coset xL.*

In particular, the gauge transformation equation $\omega_{U\lambda} = \mathrm{Ad}_{\lambda^{-1}}\,\omega_U + \lambda^*\omega_L$ shows that this criterion is independent of the particular gauge used to test the inclusion. Standard computations demonstrate

$$d\omega_G(X,Y) = X\omega_G(Y) - Y\omega_G(X) - \omega_G([X,Y]).$$

However, $X\omega_G(Y) = Y\omega_G(X) = 0$ whenever X and Y are left invariant, so in that case $d\omega_G(X,Y) = -\omega_G([X,Y]) = -[\omega_G(X), \omega_G(Y)]$. On the other hand both sides of this latter equation are tensors, so evidently $d\omega_G(X,Y) = -[\omega_G(X), \omega_G(Y)]$ and

$$\omega_G([X,Y]) = X\omega_G(Y) - Y\omega_G(X) + [\omega_G(X), \omega_G(Y)]$$

for all vector fields X, Y.

In particular we can use this formula to deduce information about Lie brackets of vector fields on G/L. Using a gauge U, vector fields on the image of U in G/L can be extended to the open subset $UL = \{ux : u \in U, x \in L\} \subset G$ by requiring them to be tangent to every constant right L translate of U (i.e. so that they are right invariant under L). On such vector fields $\omega_G = \omega_U$ so that

$$\omega_U([X,Y]) = X\omega_U(Y) - Y\omega_U(X) + [\omega_U(X), \omega_U(Y)]. \tag{3.1}$$

Lemma 3.1.3 *If $V \subset \mathfrak{g}_{\mathbf{R}}$ contains $\mathfrak{l}_{\mathbf{R}}$ and is Ad_L-invariant and U is any gauge, then for any two vector fields $X, Y \in \mathscr{C}^\infty(T_V(G/L))$ over U there exists a unique vector field $Z \in \mathscr{C}^\infty(T_V(G/L))$ over U such that $\omega_U([X,Y]) - [\omega_U(X), \omega_U(Y)] = \omega_U(Z)$.*

Proof The evaluations $\omega_U(X)$ and $\omega_U(Y)$ of X and Y in the form ω_U define smooth maps from U into $V \subset \mathfrak{g}_{\mathbf{R}}$ by Lemma 3.1.2. Since the derivatives of such maps must also take values in V, evidently $X\omega_U(Y) - Y\omega_U(X)$ also defines a smooth map from U into V. So, by Lemma 3.1.1 there exists a unique vector field $Z \in \mathscr{C}^\infty(T(G/L))$ over U such that $\omega_U(Z) - X\omega_U(Y) + Y\omega_U(X) \in \mathfrak{l}_{\mathbf{R}}$ at every point in U. Thus, $Z \in \mathscr{C}^\infty(T_V(G/L))$ over U and substituting this equality into (3.1) completes the proof. ■

Corollary 3.1.4 *If $V \subset \mathfrak{g}_{\mathbf{R}}$ is an Ad_L-invariant subalgebra which contains $\mathfrak{l}_{\mathbf{R}}$, then $T_V(G/L)$ is an involutive subbundle.*

Proof For two vector fields $X, Y \in \mathscr{C}^\infty(T_V(G/L))$ over a gauge U, $\omega_U(X)$ and $\omega_U(Y)$ take values in V by Lemma 3.1.2, so by hypothesis $[\omega_U(X), \omega_U(Y)]$ also takes values in V, so by Lemma 3.1.3 $\omega_U([X,Y])$ must also take values in V. ■

Of course, the subbundle $T_V(G/L)$ must correspond to the analytic subgroups G_V with algebra V, so the leaves of the foliation associated to $T_V(G/L)$ by the theorem of Frobenius are orbits of G_V and its conjugates. If $D \subset T(G/L)$ is an arbitrary subbundle of constant rank, a tangent vector $X \in T_{xL}(G/L)$ is said to be an element of the *Lie hull* of D if there exists a finite number of local sections $X_1, \ldots, X_n$ of D in a neighborhood of X such that X is the value in $T_{xL}(G/L)$ of an element of the Lie algebra generated over $\mathbf{R}$ by $X_1, \ldots, X_n$.

Proposition 3.1.1 *If $V \subset \mathfrak{g}_{\mathbf{R}}$ contains $\mathfrak{l}_{\mathbf{R}}$ and is Ad_L-invariant, then in every fiber of $T(G/L)$ the Lie hull of $T_V(G/L)$ is $T_{\overline{V}}(G/L)$ where $\overline{V} \subset \mathfrak{g}_{\mathbf{R}}$ is the subalgebra generated by V.*

Proof Since $T_{\overline{V}}(G/L)$ is involutive by Corollary 3.1.4, the hull of $T_V(G/L)$ in any tangent fiber can be no larger than the fiber of $T_{\overline{V}}(G/L)$. On the other hand by Lemma 3.1.3, for any gauge U and any $u \in U$ the subalgebra $\mathfrak{l}_{\mathbf{R}}$ together with the values $\omega_U(X) \in \mathfrak{g}_{\mathbf{R}}$ for X in the hull of $T_V(G/L)$ at u must form a Lie subalgebra of $\mathfrak{g}_{\mathbf{R}}$. Thus, the fiber of $T_{\overline{V}}(G/L)$ at u must be contained in the hull of $T_V(G/L)$ at u. ■

Corollary 3.1.5 *If $V \subset \mathfrak{g}_{\mathbf{R}}$ is any Ad_L-invariant subspace which contains $\mathfrak{l}_{\mathbf{R}}$, then the subbundle $T_V(G/L)$ is bracket-generating if and only if V generates the Lie algebra $\mathfrak{g}_{\mathbf{R}}$.*

3.2 Applications to Flag Manifolds

Let $(G, \theta, \mathfrak{h}, \Delta^+(\mathfrak{g}, \mathfrak{h}), P_\Sigma)$ be an admissible datum as defined at the beginning of this chapter. The Σ-height defines a symmetric grading of the Lie algebra, $\mathfrak{g} = \mathfrak{g}^\Sigma_{-k} \oplus \cdots \oplus \mathfrak{g}^\Sigma_0 \oplus \cdots \oplus \mathfrak{g}^\Sigma_k$, with $\mathfrak{p}^\Sigma = \mathfrak{g}^\Sigma_0 \oplus \cdots \oplus \mathfrak{g}^\Sigma_k$. Since Σ is invariant under the root involution σ^* associated to the given real form, the Σ-height of root spaces is also invariant, so the grading is compatible with the real structure and thus defines a grading on the real part $\mathfrak{g}_{\mathbf{R}} = \mathfrak{g}^\Sigma_{-k\mathbf{R}} \oplus \cdots \oplus \mathfrak{g}^\Sigma_{0\mathbf{R}} \oplus \cdots \oplus \mathfrak{g}^\Sigma_{k\mathbf{R}}$ with $\mathfrak{g}^\Sigma_{j\mathbf{R}} = \mathfrak{g}_{\mathbf{R}} \cap \mathfrak{g}^\Sigma_j$.

Proposition 3.2.1 *The nilradical $\mathfrak{n}^- = \mathfrak{g}^\Sigma_{-k} \oplus \cdots \oplus \mathfrak{g}^\Sigma_{-1}$ and its real form $\mathfrak{n}^-_{\mathbf{R}} = \mathfrak{g}^\Sigma_{-k\mathbf{R}} \oplus \cdots \oplus \mathfrak{g}^\Sigma_{-1\mathbf{R}}$ are lie-generated by $\mathfrak{g}^\Sigma_{-1}$ and $\mathfrak{g}^\Sigma_{-1\mathbf{R}}$ respectively.*

By the subalgebra of $\mathfrak{n}^-$ (respectively $\mathfrak{n}^-_{\mathbf{R}}$) which is *lie-generated* by $\mathfrak{g}^\Sigma_{-1}$ (respectively $\mathfrak{g}^\Sigma_{-1\mathbf{R}}$) we mean the set of all finite sums of Lie monomials with entries in $\mathfrak{g}^\Sigma_{-1}$ (respectively $\mathfrak{g}^\Sigma_{-1\mathbf{R}}$). This is evidently a Lie algebra over $\mathbf{C}$ (respectively $\mathbf{R}$) because $\mathfrak{g}^\Sigma_{-1}$ (respectively $\mathfrak{g}^\Sigma_{-1\mathbf{R}}$) is a vector space over $\mathbf{C}$ (respectively $\mathbf{R}$). In fact, since $\mathfrak{g}^\Sigma_{-1}$ (respectively $\mathfrak{g}^\Sigma_{-1\mathbf{R}}$) is a module for the adjoint action of $\mathfrak{g}^\Sigma_0$ (respectively $\mathfrak{g}^\Sigma_{0\mathbf{R}}$), so is the subalgebra which is lie-generated by it.

Proof First, the fact that $\mathfrak{n}^-_{\mathbf{R}}$ is lie-generated by $\mathfrak{g}^\Sigma_{-1\mathbf{R}}$ follows from the fact that $\mathfrak{n}^-$ is lie-generated by $\mathfrak{g}^\Sigma_{-1}$ because each entry in any given Lie monomial can be split into its real and imaginary parts. Thus, it is sufficient to prove that every negative root space is included in the subalgebra of $\mathfrak{n}^-$ lie-generated by $\mathfrak{g}^\Sigma_{-1}$. Suppose to the contrary that α is a negative root such that $\mathfrak{g}^\alpha$ is not in this subalgebra. Let β be a sum of roots in $-\Sigma$ (possibly with repetitions) and let γ be a root of Σ-height zero such that $\alpha = \beta + \gamma$. The subalgebra in question is closed under the adjoint action of $\mathfrak{g}^\Sigma_0$, so the assumption that it does not include $\mathfrak{g}^\alpha$ implies that it does not include $\mathfrak{g}^\beta$.

In turn, this implies that it does not include $\mathfrak{g}^{\delta}$ for any root δ such that δ and $\gamma - \delta$ are sums of elements of $-\Sigma$, but this would imply that even the $-\Sigma$ root spaces in $\mathfrak{g}^{\Sigma}_{-1}$ are not in the subalgebra in question, an obvious contradiction. ■

The following result is proved by directly applying the preceding result of this subsection.

Proposition 3.2.2 *The tangent bundle to the flag variety G/P_{Σ} associated to an admissible datum $(G, \theta, \mathfrak{h}, \Delta^{+}(\mathfrak{g}, \mathfrak{h}), P_{\Sigma})$ admits a natural filtration*

$$T_{-1}(G/P_{\Sigma}) \subset \ldots \subset T_{-k}(G/P_{\Sigma}) = T(G/P_{\Sigma})$$

by bracket-generating subbundles. A tangent vector $\xi \in T_{xP_{\Sigma}}(G/P_{\Sigma})$ is an element of $T_{-j}(G/P_{\Sigma})$ if and only if its value in the maurer-cartan form associated to any gauge intersecting the coset xP_{Σ} is an element of $\mathfrak{g}^{\Sigma}_{-j\mathbf{R}} \oplus \ldots \oplus \mathfrak{g}^{\Sigma}_{k\mathbf{R}}$.

4. DIAGONALIZATION OF BRANCHED INFINITESIMAL CHARACTERS

If $\mathfrak{g}$ is any complex finite dimensional Lie algebra then a well known lemma due to Dixmier shows that the center $Z_{\mathfrak{g}}$ of $\mathcal{U}(\mathfrak{g})$ must act by scalars in any irreducible representation.

Lemma 4.0.1 (Dixmier) *If V is a vector space over a field k and $\mathrm{End}_k(V)$ contains an element T such that every monic irreducible polynomial in T is invertible, then $k(x)$ injects into V as a vector space over k.*

Proof (c.f. [27, 28]) The hypotheses clearly imply that every nonzero polynomial in T is invertible and therefore every rational function in T is well defined and all other than zero are invertible. Thus, V is a vector space for the rational function field in one variable over k realized as $k(T)$. Since any nonzero element of $k(T)$ is invertible and therefore has trivial kernel, $k(T)^{\times}v$ is a faithful orbit for every nonzero $v \in V$. ■

Corollary 4.0.2 *If V is a vector space over an uncountable field k and $\mathrm{End}_k(V)$ contains an element T such that every monic irreducible polynomial in T is invertible, then V has uncountable dimension over k.*

Proof If k is uncountable the rational function field $k(x)$, being the function field of the projective line, must have uncountable dimension over k. So, if V has countable dimension over k then $k(x)$ cannot inject into V and therefore an element of $\mathrm{End}_k(V)$ which satisfies the above hypotheses would violate the lemma. ■

Theorem 4.0.3 (Dixmier) *If $\mathfrak{g}$ is a finite dimensional Lie algebra over an uncountable algebraically closed field k of characteristic zero and if V is an irreducible $\mathfrak{g}$ module, then the commutant of $\mathcal{U}(\mathfrak{g})$ in $\mathrm{End}_k(V)$ is k.*

Proof (c.f. [27,28]) If $v \in V$ is any nonzero element then it must cyclically generate the entire module V under the action of $\mathcal{U}(\mathfrak{g})$, for otherwise its orbit would constitute an invariant subspace. Thus, V must have countable dimension over k so by the lemma for any k endomorphism T there exists $c \in k$ such that $T - c$ is not invertible, but if T also commutes with $\mathcal{U}(\mathfrak{g})$ then the kernel and image of $T - c$ are invariant subspaces for $\mathcal{U}(\mathfrak{g})$ so $T - c = 0$ is the only possibility. ∎

In particular if $\mathfrak{g}$ is a complex finite dimensional Lie algebra we conclude that the center $Z_\mathfrak{g}$ of $\mathcal{U}(\mathfrak{g})$ in any irreducible $\mathfrak{g}$ module must act by scalar multiples of the identity, each realized as a character $Z_\mathfrak{g} \to \mathbf{C}$, called the *infinitesimal character* of the module. Now if $\mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n \subset \mathfrak{g}$ is a nested list of subalgebras of $\mathfrak{g}$, then the subalgebra $Z = Z_{\mathfrak{g}_1,\ldots,\mathfrak{g}_n}$ generated by the centers $Z_{\mathfrak{g}_1}, \ldots, Z_{\mathfrak{g}_n}$ is evidently commutative. Indeed, for any inclusion $\mathfrak{g}_i \subset \mathfrak{g}_j$, $Z_{\mathfrak{g}_j}$ commutes with every element of $\mathfrak{g}_i$ so in particular it must commute with $Z_{\mathfrak{g}_i}$ which is itself commutative so the pair $Z_{\mathfrak{g}_i}, Z_{\mathfrak{g}_j}$ must generate a commutative subalgebra of $\mathcal{U}(\mathfrak{g})$. A straightforward generalization of this argument shows that Z as defined above must be commutative.

Having defined the commutative algebra $Z \subset \mathcal{U}(\mathfrak{g})$, we consider the task of diagonalizing a given $\mathfrak{g}$ module into character spaces for Z. The appropriate strategy should be obvious: isolate an irreducible $\mathfrak{g}_n$ submodule or more generally a maximal direct sum of isomorphic $\mathfrak{g}_n$ submodules (i.e. a maximal isotype) so that $Z_{\mathfrak{g}_n}$ has an infinitesimal character on that submodule according to Dixmier's lemma. Such a submodule will in general not be irreducible for $\mathfrak{g}_{n-1}$, so we isolate a maximal isotype for $\mathfrak{g}_{n-1}$ within the specified $\mathfrak{g}_n$ isotype and on this subspace $Z_{\mathfrak{g}_n}$ and $Z_{\mathfrak{g}_{n-1}}$ and therefore the entire subalgebra of $\mathcal{U}(\mathfrak{g})$ generated thereby will act by scalars. Continuing in this manner by passing to progressively smaller subtypes, we obtain an isotype for $\mathfrak{g}_1$ on which each center $Z_{\mathfrak{g}_1}, \ldots, Z_{\mathfrak{g}_n}$, and therefore the entire subalgebra $Z \subset \mathcal{U}(\mathfrak{g})$ generated thereby, must act by scalars.

In other words, the representation of $\mathfrak{g}$ must be branched into isotypic subspaces for the specified subalgebras. Since we are only interested in Lie algebra representations arising infinitesimally from representations of groups, we begin by considering branching properties of nested lists of closed subgroups in the general setting of abstract C^* algebras and locally compact Hausdorff groups. This is more generality than is needed but it does not amount to much added difficulty and in any case it is the customary setting for the functional analysis and spectral theory which will be utilized.

4.1 General Representation Theory for C^*-Algebras

A Banach $*$-algebra is a complex Banach algebra with a $*$-involution, i.e. a complex anti-linear anti-homomorphism of the algebra. A C^* algebra is a Banach $*$-algebra which satisfies the C^* identity: $\|x^*x\| = \|x\|^2$. The ramifications of the C^* identity are quite deep, especially when one also considers its brevity and simplicity. The motivating idea for C^* theory is the Gelfand transform for commutative Banach algebras. If A is such an algebra then the algebra homomorphisms from A into $\mathbf{C}$ are automatically continuous and form a locally compact Hausdorff space, denoted here by $\widehat{A}$, when equipped with the topology induced from the weak dual of A. Any element $x \in A$ defines a continuous function on $\widehat{A}$ in the obvious way: $\varphi \mapsto \varphi(x)$. This is the Gelfand transform for commutative Banach algebras. Moreover, $\widehat{A}$ is compact if A has a unit (and conversely provided that the Gelfand transform is injective), and if no unit exists then the function defined by x vanishes at infinity in the one point compactification.

Thus, the Gelfand transform $x \mapsto [\varphi \mapsto \varphi(x)]$ is a contractive homomorphism from any commutative Banach algebra A into the commutative Banach algebra $\mathscr{C}_0(\widehat{A})$. However, in this general setting the transform has a few less than optimal qualities. First, it is not necessarily injective and even though it is a metric contraction, it is not necessarily an isometry. Furthermore, the target $\mathscr{C}_0(\widehat{A})$ is not simply a commutative Banach algebra, it is also a $*$-algebra (and even a C^* algebra), and even when A is a commutative Banach $*$-algebra the Gelfand transform is not necessarily a $*$-homomorphism. All of these deficiencies disappear when A is a C^* algebra, as the

C^* identity can be shown to be a sufficient criterion for the Gelfand transform to be an isometric isomorphism of Banach $*$-algebras (on the other hand, if for a particular Banach $*$-algebra the Gelfand transform is known to be an injective $*$-homomorphism then the C^* identity is clearly necessary for it to be an isometry).

Consequently, one can unambiguously define the *positive cone* of a commutative C^* algebra to be the set of elements which have an everywhere nonnegative Gelfand transform. The algebra then inherits a partial ordering in the usual way: $x \leq y$ if $y - x$ is in the positive cone. One of the principal implications of the C^* identity in the general (i.e. not necessarily commutative) case is the fact that there is still a well-defined notion of positivity, along with the resulting partial order relation. For such algebras, positivity of a generic element x is defined by requiring x to be hermitian with nonnegative spectrum. Such elements must have hermitian roots of all positive orders, so every positive element x is of the form $x = y^*y$ for some y. Conversely, it can be shown (see, e.g. [29]) that every element of the form y^*y is positive, so the positive cone A_+ of any given C^* algebra A is precisely the set $A_+ = \{y^*y : y \in A\}$.

The notion of positivity greatly simplifies the representation theory of C^* algebras, i.e. the study of $*$-homomorphisms from a generic C^* algebra into the C^* algebras $\mathscr{L}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space. In the commutative case any element of $\widehat{A}$ is such a representation and these are the only irreducible representations. So, the Gelfand transform realizes an isometric $*$-isomorphism from any commutative C^* algebra into the algebra of continuous functions on the set of its irreducible $*$-representations, with the appropriate topology, the value of the function $x \in A$ at any point being the operator associated to x in the given representation which in the commutative case is simply a complex number.

Accordingly, one would hope to have a similarly appealing result in the noncommutative case. In other words, there should be a more or less natural topology on the set of inequivalent irreducible Hilbert space representations of any not necessarily commutative C^* algebra A, and mapping each such representation to the value of an element x should define a continuous "function". The main issue is clearly that in the noncommutative case the various target spaces for the representations cannot be identified with one another, so the desired "function" cannot take values in a single codomain, instead it must be viewed as a section of a bundle of algebras. Nevertheless, a very satisfactory and intricate theory has grown out of these ideas.

4.1.1 Positive Functionals

The representation theory of C^* algebras begins with the observation that if v is any nonzero vector in a representation $\pi : A \to \mathscr{L}(\mathcal{H})$ of a C^* algebra A on a Hilbert space $\mathcal{H}$ then $x \mapsto \langle \pi(x)v, v\rangle$ is a positive linear functional (i.e. it maps the positive cone A_+ into the nonnegative real numbers) and the kernel I_v of the resulting seminorm $\|x\|_v^2 = \|\pi(x)v\|^2$ is the left ideal of elements of A which annihilate v. This gives A/I_v the structure of a pre-hilbert space equipped with a natural representation of A via multiplication on the left. The resulting metrically complete Hilbert space is isometric as an A module to the closed cyclic subspace of $\mathcal{H}$ generated by v under the action of A.

In this manner, one realizes the closed cyclic subspace generated by any $v \in \mathcal{H}$ as the Hilbert space completion of a quotient of A itself in a pre-hilbert norm obtained from a positive linear functional. However, a representation can be manufactured in precisely the same way from *any* positive functional, and the representation constructed in this manner from a given functional ρ is called the *Gelfand-Naimark-Segal* or GNS representation associated to ρ.

Naturally, an irreducible representation is cyclic and is generated by any nonzero vector. Thus, the irreducible representations are among those obtained from positive functionals and there is an explicit criterion for determining when this occurs. The set of rays ($\mathbf{R}_+$ orbits) in the cone of positive functionals on A is partially ordered, with $\mathbf{R}_+\rho \leq \mathbf{R}_+\omega$ if $\omega - \lambda\rho$ is positive for at least one $\lambda \in \mathbf{R}_+$, in which case $\mathbf{R}_+\rho$ is said to be *subordinate* to $\mathbf{R}_+\omega$. Both of the rays $\mathbf{R}_+\rho$ and $\mathbf{R}_+\omega$ are subordinate to $\mathbf{R}_+(\rho+\omega)$, for example. A ray is said to be *pure* if no ray other than itself is subordinate to it, and the pure rays are precisely those which produce irreducible representations via the GNS construction. Proofs of this fact can be found throughout the literature, but from an intuitive perspective it should be obvious: if $\mathbf{R}_+\rho$ is subordinate to $\mathbf{R}_+\omega$ then the left ideal I_ρ contains the left ideal I_ω, so A/I_ρ is a submodule of A/I_ω. So, algebraically speaking the submodules of the GNS representation arising from a given positive functional correspond with those functionals which are subordinate to it, and standard arguments show that this still true when one passes to the Hilbert space completion.

In this manner, one can construct a natural topology on the set of irreducible representations in the following way. Since GNS representations arising from two elements of the same ray $\mathbf{R}_+\rho$ of positive functionals are equivalent, it amounts to no loss of generality to consider only those functionals of norm not greater than one. Such functionals form a convex subset of A^* which is compact in the weak topology, and the set of extreme points of this set is precisely $E(A) \cup \{0\}$ where $E(A)$ denotes the pure states (i.e. pure positive functionals of norm one).

One of the foundational results of the entire theory is the equivalence between abstract C^* algebras and C^* algebras which are presented as uniformly closed $*$-subalgebras of $\mathscr{L}(\mathcal{H})$. In other words: every abstract C^* algebra admits a faithful representation. To prove this one first proves that a positive linear functional on a closed subalgebra $B \subset A$ can be extended to a positive functional on A. Having proved this, for any nonzero $x \in A$ one defines the subalgebra B_x to be the C^* subalgebra of A generated by x^*x. This is commutative, so x^*x defines a nonzero function on its spectrum $\widehat{B_x}$, and any point mass in the support of x^*x defines a positive linear functional in which x^*x does not vanish. Now, such a functional can then be extended to a positive functional on the entire algebra A, and evidently x^*x cannot be in the kernel of the resulting GNS representation, but this means that x itself cannot be in the kernel.

Having proved that for any $x \in A$ there is at least one GNS representation in which x does not vanish. One can form the direct sum of all GNS representations (the so-called *universal* representation), which evidently must be faithful. This fundamental theorem is generally attributed to Gelfand and Naimark.

4.1.2 The Structure Space $\widehat{A}$

Let $\mathscr{R}$ denote a set of representations of the C^* algebra A. In general no other conditions are necessary for the following construction to work, in particular $\mathscr{R}$ may contain two or more distinct representations which are unitarily equivalent. If one is unconcerned with set-theoretic issues then $\mathscr{R}$ could be simply the collection of all possible representations of A, as in all possible $*$-homomorphisms $A \to \mathscr{L}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space. However, in light of the preceding comments on GNS representations it is worthwhile to point out that it is possible to include a complete set of inequivalent irreducible representations by setting $\mathscr{R}$ equal to the set of GNS representations corresponding to positive functionals of norm one, or even just the extreme points $E(A)$ of this set. Either of these are well defined subsets of a Banach space and as such they are fairly concrete.

Given a set $\mathscr{R}$ of representations of A, let (π, p, ϵ, S) be a datum consisting of

1. a representation $[\pi : A \to \mathscr{L}(\mathcal{H}_\pi)] \in \mathscr{R}$,
2. an orthogonal projection $p \in \mathscr{L}(\mathcal{H}_\pi)$,
3. a positive number ϵ,
4. a nonempty subset $S \subset A$.

For such a datum define the set $\mathscr{U}(\pi, p, \epsilon, S) \subset \mathscr{R}$ to be the set of $\pi' \in \mathscr{R}$ such that there exists a continuous map $T : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ satisfying

$$\|p(1 - T^*T)p\|_{\mathscr{L}(\mathcal{H}_\pi)} < \epsilon \quad \text{and} \quad \|p(\pi(x) - T^*\pi'(x)T)p\|_{\mathscr{L}(\mathcal{H}_\pi)} < \epsilon$$

for every $x \in S$. Descriptively speaking this means that the action of A on the range of p, i.e. the action of the localized representation $p\pi(A)p$ must be *nearly* unitarily equivalent to a localized representation in π', in particular the requirement that $\|p(1 - T^*T)p\|_{\mathscr{L}(\mathcal{H}_\pi)} < \epsilon$ indicates that the restriction of T to the range of p differs from an isometry within an error strictly less than ϵ.

The *regional* or *Fell* topology on $\mathscr{R}$ is defined by using the sets $\mathscr{U}(\pi, p, \epsilon, S)$, where

1. p is a finite rank projection,

2. $\epsilon > 0$ is arbitrary,

3. $S \subset A$ is finite,

as a basis of neighborhoods of π. Clearly, this topology does not distinguish unitarily equivalent representations, so the *structure space* or *dual* of A, denoted $\widehat{A}$, is unambiguously defined as the set of unitary equivalence classes of irreducible representations of A equipped with the regional topology.

4.1.3 The Primitive Ideal Space $\mathrm{Pr}(A)$

A natural point of view to take when seeking a topology on the set of unitary equivalence classes of irreducible representations of a C^* algebra A is to examine their factorization properties. In other words, quotient algebras of A arise from closed ideals $J \subset A$ and an irreducible unitary representation of A/J evidently defines an irreducible unitary representation of A by factoring through the quotient. For a given ideal J, such representations of A are precisely those which vanish on J. The natural topology on the set of representation classes should interpret those representations which factor through a fixed quotient A/J as a closed set.

With this in mind, one isolates the closed ideals $\ker \pi$ for irreducible π, which are said to be *primitive*. The set of all primitive ideals will be denoted $\mathrm{Pr}(A)$ and the dual $\widehat{A}$ evidently surjects onto $\mathrm{Pr}(A)$ by mapping a given equivalence class of representations to its kernel. In general, this map is *not* injective as inequivalent representations can have the same kernel. Defining a topology on $\mathrm{Pr}(A)$ in the manner described above amounts to first isolating the collection of subsets of $\mathrm{Pr}(A)$ indexed by the closed ideals of A with each ideal J mapping to the subset of primitive ideals which contain it or equivalently the set of irreducible representations which annihilate it, and then considering the coarsest topology on $\mathrm{Pr}(A)$ in which each of these sets is closed.

In fact, the above described collection of sets is already the collection of closed sets for a topology - no extra closed sets are needed. To prove this, recall that by the Gelfand-Naimark theorem there is always a faithful representation, so one sees that there is no nonzero element which is annihilated in every irreducible representation and as a result the intersection of all primitive ideals is trivial. Furthermore, by considering the quotient by a given closed ideal J, J is evidently the intersection of the primitive ideals which contain it. Therefore, the abstract closure operation defined by

$$\overline{X} = \left\{ I : I \text{ is primitive and } \bigcap_{J \in X} \left(J \subset I \right\} \right.$$

realizes the above described closed sets (i.e. sets of primitive ideals which contain a given closed ideal) as the formally closed sets under this operation (i.e. those for which $\overline{X} = X$). Furthermore, this operation satisfies Kuratowski's closure axioms, meaning that the sets X such that $\overline{X} = X$ form the collection of closed sets for a topology. The resulting topology on $\mathrm{Pr}(A)$ is called the $hull - kernel$ or $Jacobson$ topology. The hull-kernel topology is equal to the regional topology when pulled back to $\widehat{A}$ (however, it can also be defined for more general Banach $*$-algebras and generally speaking the regional topology may be strictly finer).

4.2 Unbounded Operators

The theory of unbounded operators (or more precisely, not necessarily bounded operators) is based on the fact that for any operator $T : D(T) \subset \mathcal{H} \to \mathcal{R}$ from a complex linear subspace (not necessarily closed, not necessarily dense) of a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{R}$, the adjoint domain $D^*(T)$ is unambiguously defined as the set of $v \in \mathcal{R}$ such that $\langle T(\cdot), v \rangle_{\mathcal{R}}$ is a continuous linear functional on $D(T)$, i.e. those vectors whose corresponding rank one projections make T continuous after post-composition. The notation $D^*(T)$ as opposed to $D(T^*)$ is chosen specifically because the actual adjoint operator for T is not unique (unless $D(T)$ is dense), but the domain of the adjoint is well-defined in spite of this ambiguity (in fact, even if $\mathcal{H}$ and $\mathcal{R}$ are general topological vector spaces, the adjoint of a closed subspace of $\mathcal{H} \oplus \mathcal{R}$ can always be defined as a closed subspace of $\mathcal{R}^* \oplus \mathcal{H}^*$, by taking the annihilator of the given subspace in $\mathcal{H}^* \oplus \mathcal{R}^*$, negating the first summand and switching the order, the above described ambiguity is then manifest by the possibility that the adjoint is not single-valued).

Now T can be decomposed as $T = P_{\overline{D^*(T)}}T + P_{D^*(T)^\perp}T$ and most of the complexity inherent in the theory of unbounded operators is more or less summarized by this decomposition:

1. $P_{\overline{D^*(T)}}T$, while still unbounded, is well-behaved in many respects: its graph closure is single valued (i.e. it is a closable operator) and no matter how T^* is defined (i.e. within the above described ambiguity) it satisfies the expected equality $\overline{D^*(T)} \quad R(P_{\overline{D^*(T)}}T) = \ker(T^*)$, and moreover $D^*(T)$ has dense intersection in $\overline{R(P_{\overline{D^*(T)}}T)}$ and any admissible version of T^* will be injective there.

2. $P_{D^*(T)^\perp}T$ is pathological, it is not closable and it has trivial adjoint domain.

The existence of the second type of map is somewhat confusing and has no analog in finite dimensional linear algebra, wherein every nontrivial map has a nontrivial adjoint. Generally speaking they are those maps which are discontinuous and remain discontinuous after composition with any rank one orthogonal projection in the codomain. This way of imposing continuity by composing with projections is the heart of the theory of unbounded operators. For instance, one can take the operator $i\frac{d}{dx}$ on $L^2(\mathbf{R})$ - it is discontinuous in general but *not* if it is post-composed with a projection onto a subspace of elements having first derivative in $L^2(\mathbf{R})$ and on which the operator $f \mapsto i\frac{d}{dx}f$ is bounded in the topology inherited from $L^2(\mathbf{R})$. In particular any finite rank projection with range consisting only of $f \in L^2(\mathbf{R})$ such that also $i\frac{d}{dx}f \in L^2(\mathbf{R})$ will work.

This description of the situation, while satisfactory for most purposes, is somewhat asymmetrical insofar as it confers a certain logical precedence upon T which it should not really have. In particular T may not be closed even when projected into $D^*(T)$, but any admissible version of the adjoint T^* is closed on $D^*(T)$ after projection into $D(T)$. Thus, one might envision the ideal situation as that in which *two* operators are given: $T : D(T) \subset \mathcal{H} \to \mathcal{R}$ and $S : D(S) \subset \mathcal{R} \to \mathcal{H}$ such that

1. $D(T) \subset D^*(S)$,
2. $D(S) \subset D^*(T)$, and
3. the sesquilinear forms $\langle T(\cdot), \cdot\rangle_{\mathcal{R}}$ and $\langle \cdot, S(\cdot)\rangle_{\mathcal{H}}$ are equal on $D(T) \times D(S)$.

With these data given,

1. $P_{D^*(S)^\perp}S$ and $P_{D^*(T)^\perp}T$ are pathological in the sense described above (i.e. the have trivial adjoint domain or equivalently they remain discontinuous after composition with every rank one orthogonal projection in the codomain).
2. the closures of $P_{\overline{D^*(T)}}T$ and $P_{\overline{D^*(ST)}}S$ define an adjoint pair of closed operators on $D(T) \times D(S)$.

Naturally one is interested in extending T and S to closed operators on $D^*(S)$ and $D^*(T)$ respectively and in such a way that the extensions remain an adjoint pair. However, most examples which arise in practice have $D(T)$ dense in the closure of $D^*(S)$ and $D(S)$ dense in the closure of $D^*(T)$. By throwing away the pathological parts of the operators, we can assume that T is densely defined with a densely defined adjoint and S is densely defined with a densely defined adjoint, or in other words both T and S are densely defined and closable so T^* is the closure of S and S^* is the closure of T.

Thus, one can unambiguously say that a pair $T : D(T) \subset \mathcal{H} \to \mathcal{R}$ and $S : D(S) \subset \mathcal{R} \to \mathcal{H}$ of densely defined *closed* operators is an *adjoint pair* if $T^* = S$ and $S^* = T$.

If $T : D(T) \subset \mathcal{H} \to \mathcal{R}$, $P : D(P) \subset \mathcal{H} \to \mathcal{R}$ are any operators whatsoever, the notation $T \subset P$ indicates that $D(T) \subset D(P)$ and that $T = P$ on $D(T)$, if $D(T)$ is dense then the inclusion $P^* \subset T^*$ is easily proved. The basic extension problem for a pair T, P of densely defined closed operators such that $T \subset P$ is to describe all intermediate closed operators, i.e. one wants a description of the set of all closed operators A such that $T \subset A \subset P$. This problem has a very satisfactory answer if there exists at least one intermediate closed operator A_1 which is injective with dense range and bounded inverse.

Theorem 4.2.1 *If T, A_1 and P are closed densely defined operators on $\mathcal{H}$ taking values in $\mathcal{R}$ such that $T \subset A_1 \subset P$ and if A_1 is injective with dense range and a bounded inverse then there exists a natural adjoint-compatible bijective correspondence from the set of all closed operators A such that $T \subset A \subset P$ to the set of closed operators from* $\ker(P)$ *into* $\ker(T^*)$.

For a proof, see [30]. Note that closed operators from $\ker(P)$ to $\ker(T^*)$ in this parameterization are not necessarily densely defined, the only requirement is that they are closed. Also, the operator A_1 must be surjective, since it has a bounded inverse the inverse A_1^{-1} must be everywhere defined with dense range since A_1 is densely defined (but by the closed graph theorem the range of A_1^{-1} will be surjective if and only if A_1 is bounded, and in that case $T = A_1 = P$).

A densely defined closed operator $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ is *symmetric* if $T \subset T^*$ and *self-adjoint* if $T = T^*$. Evidently self-adjoint operators are symmetric but the converse is false: there exist closed operators which are nontrivially extended by their adjoints. The main example of this is a differential operator T on a smooth manifold X equipped with a smooth measure μ (a measure which is a smooth positive deformation of lebesgue measure in every coordinate system or equivalently a smooth positive section of the line bundle of one-densities). In this situation one can set $D(T) = \mathscr{C}_0^\infty(X) \subset L^2(X)$, which is dense, and then $D^*(T) \supset \mathscr{C}_0^\infty(X)$ so $D^*(T)$ is dense thus T is closable and T^* is densely defined. Now the inclusion $D(T) \subset D^*(T)$

is strict, since $D^*(T)$ will contain any function which is smooth enough and decays rapidly enough at ∞ so that its image in the operator T is still in $L^2(X)$. If the differential adjoint of T as computed in any coordinate chart is equal to T then $T = T^*$ as differential operators, but in general *not* as operators in $L^2(X)$, for even after one passes to the domain of the operator closure of T, the extension $T \subset T^*$ may still be strict. An example is the positive Laplace operator on a bounded domain in $\mathbf{R}^n$ or more generally a riemannian manifold which is not metrically complete, for then there will be harmonic functions in $L^2(X)$ which are not in the domain of the graph closure of the restriction to smooth functions of compact support.

In general, if $T \subset T^*$ is a strict inclusion then the basic extension theorem discussed above can be considered with $P = T^*$, so if there is a closed intermediate operator $T \subset A_1 \subset T^*$ which has zero as a resolvent value then the set of all intermediate operators is in natural adjoint-compatible bijective correspondence with the set of closed operators on $\ker(T^*)$. However, this correspondence is too general to be of use in most specific situations and in any case we've not given a proof or a description of it. For specific classes of symmetric operators more precise tools are available, which we now proceed to describe.

The set of self-adjoint extensions of a closed symmetric operator T is of great interest, any such extension must lie between T and T^*, since adjunction is inclusion reversing. Thus, by the main extension theorem if T admits a single intermediate extension $T \subset A_1 \subset T^*$ having zero as a resolvent value then A_1 must be self-adjoint (since its inverse is bounded and symmetric therefore self-adjoint, so A_1 is self-adjoint) and in this case the set of all self-adjoint extensions of T is in bijection with self-adjoint operators, bounded or not, on closed subspaces of $\ker(T^*)$, so if T has dense range then there is at most one self-adjoint extension with a bounded inverse.

This is indicative of the general situation which is most frequently of interest in analysis and geometry, there one typically has a symmetric hypoelliptic differential operator T which is bounded below by $-\alpha \in \mathbf{R}$ in its natural quadratic form, $T = \Delta + L$ where Δ is the *nonnegative* Laplace operator on any riemannian manifold and

L is a first order operator which is formally symmetric and bounded below in its quadratic form by $-\alpha$, is an example. With this hypothesis in place one can be sure that the closure of $T+\alpha+1$ is bounded below by 1 and therefore it is injective and, crucially, its inverse (which is *a priori* defined only on the range) is bounded. This means that $T+\alpha+1$ has closed range and $(T+\alpha+1)^{-1} : R(T+\alpha+1) \to \mathcal{H}$ is a contraction. However, it is fairly simple to prove (via a lax-milgram type argument) that the completion of $D(T)$ in the quadratic form $\langle (T+\alpha+1)\cdot, \cdot\rangle$ (i.e. the range of the inverse square root $(T+\alpha+1)^{-1/2}$) is the domain of a self-adjoint extension $T = (T+\alpha+1)-\alpha-1$ with the same lower bound as T, this is the so-called *Friedrichs extension* of T.

All self-adjoint extensions of T can be obtained in more or less the same fashion. The kernel of any such extension must be a closed subspace K such that $\ker(T) \subset K \subset \ker(T^*)$, so the domain of such an extension in $K^\perp$ contains the projection of $D(T)$, and we shall call the restriction of T to this projected domain the *compression* of T (into $K^\perp$). The self-adjoint extensions of T with kernel containing K are evidently in bijective correspondence with the self-adjoint extensions of the compression into $K^\perp$ and in particular, K is the kernel of a self-adjoint extension of T if and only if the compression of T into $K^\perp$ admits a self-adjoint extension with dense range.

A symmetric operator which is not necessarily closed is said to be *essentially self-adjoint* if its closure is self-adjoint, in which case the closure is the unique self-adjoint extension and is equal to the adjoint of the originally given (not necessarily closed) operator. The rest of this section will be devoted to the proof of the following theorem.

Theorem 4.2.2 *If B is a C^* algebra and $A \subset B^{**}$ is a commutative C^* subalgebra of the enveloping algebra, and if furthermore ω is a normalized positive functional on B with separable GNS representation $\mathcal{H}_\omega$ then for any strictly positive Borel measurable function L on $\widehat{A}$, the positive powers $L^{s/2}$ each define an essentially self-adjoint positive operator on $\mathcal{H}_\omega$. Furthermore, for any $x \in B^{**}$ the vector $\xi \in \mathcal{H}_\omega$ is in the* *domain of* $xL^{s/2}$ *if and only if the holomorphic function of* z *defined by* $xe^{-zL}L^{s/2}\xi$ *in the right half-plane has a continuous limit on* $\mathrm{Re}\, z = 0$.

Proof To prove the first statement, we will use the well known criterion of E. Nelson on analytic vectors. A vector in the domain of all powers $L^{ns/2}$ is said to be an *analytic vector* if $\sum \frac{\|L^{ns/2}v\|}{n!}t^n$ is finite for at least one $t > 0$. Evidently this must be true since if $\mathcal{H}_\omega = \bigoplus_{k\geq 1} A\xi_k$ is a direct sum decomposition into cyclic representations then for any $w \in \mathcal{H}_\omega$ there are Borel measurable functions $w_k : \widehat{A} \to \mathbf{C}$ which are respectively in L^2 with respect to the vector state ξ_k, such that $w = \oplus w_k$. In this case define an operator T_z on $\mathcal{H}$ for $\mathrm{Re}\, z > 0$ by $T_z w = \oplus \frac{1}{k} e^{-zL^{s/2}} w_k$ and clearly each such vector is in the domain of $L^{ns/2}$ with $L^{ns/2}T_z w = \oplus \frac{1}{k} e^{-zL^{s/2}} L^{ns/2} w_k$. Thus,

$$\begin{aligned} \sum_n \frac{\|L^{ns/2}T_z w\|_{\mathcal{H}_\omega}}{n!} t^n &\leq \sum_n \sum_k \frac{1}{k} \frac{\|L^{ns/2}e^{-zL^{s/2}}w_k\|_{\mathcal{H}_\omega}}{n!} t^n \\ &\leq \sum_k \frac{1}{k} \|e^{(t-z)L^{s/2}} w_i\|_{\mathcal{H}_\omega} \\ &\leq \left(\sum_k \frac{1}{k^2} \right)^{1/2} \left(\sum_k \|e^{(t-z)L^{s/2}} w_i\|^2_{\mathcal{H}_\omega} \right)^{1/2} \\ &\leq \frac{\pi}{\sqrt{6}} \|e^{(t-z)L^{s/2}} w\|_{\mathcal{H}_\omega}. \end{aligned}$$

Provided $\mathrm{Re}(t - z) < 0$ the final figure is finite, so we conclude that the range of T_z consists of analytic vectors provided $\mathrm{Re}\, z < 0$. However, $T_z^* = T_{\overline{z}}$ which is injective, so T_z has dense range and $\mathcal{H}_\omega$ thus contains a dense set of analytic vectors. By Nelson's criterion [31], $L^{s/2}$ is essentially self-adjoint.

The second statement follows from a typical trick: the domain of $xL^{s/2}$ is the adjoint domain to $L^{s/2}x^*$, and ξ is an element if and only if $\xi, L^{s/2}x^*(\cdot)$ is continuous. However, if this is so then $e^{-zL}\xi, L^{s/2}x^*(\cdot)$ has a limit on $\mathrm{Re}\, z = 0$ but for $\mathrm{Re}\, z < 0$ it is equal to $xe^{-zL}L^{s/2}\xi, \cdot$. Note that a limit at any point on $\mathrm{Re}\, z = 0$ implies a limit at all such points, since the semigroup is unitary on this line. This proves the theorem. ∎

The setting we have in mind is that which was discussed at the beginning of the chapter, i.e. that in which a nested sequence $G \supset G_1 \supset \ldots \supset G_r$ of connected reductive groups is given, each closed in its predecessor. The algebra $\mathbf{C}[\Delta, \Delta_1, \ldots, \Delta_r]$ generated by the respective Casimirs is commutative, but it consists of unbounded operators. However, since these operators are formally self-adoint any closed extensions for them must have real spectrum, so they will generate unitary groups $e^{it\Delta}, e^{it\Delta_1}, \ldots, e^{it\Delta_r}$ and for $B = \mathscr{C}^*(G)$ the C^*-subalgebra in B^{**} generated by these groups is the guiding example of the subalgebra A in the theorem.

Here we require L to be strictly positive everywhere so as to be sure that $L^{s/2}e^{-zL}$ is a bounded operator on $\mathcal{H}_\omega$. It would be enough to identify a countable list $\xi_0, \xi_1, \ldots$ of unit vectors which are cyclic generators of a direct sum decomposition into A modules.

4.3 Locally Compact Groups

Let G be a locally compact group Hausdorff group. In this dissertation we are interested only in Lie groups, furthermore locally compact groups which are not Hausdorff are too pathological to be of general interest. In any case a topological group which satisfies the T_1 separation axiom (for every point pair there is a neighborhood of one point not containing the other) is automatically Hausdorff, and quotients of topological groups by closed subgroups are Hausdorff so in non-hausdorff groups $\overline{\{1\}}$ is a closed normal subgroup and one can pass to the Hausdorff quotient $G/\overline{\{1\}}$ [32]. The standard approach to the representation theory of G is to instead consider the group C^* algebra of G, which we will denote $\mathscr{C}^*(G)$, and more generally the C^* completion of the bounded measure algebra $\mathscr{C}_0(G)'$. These algebras are defined as follows.

The abelian C^* algebra $\mathscr{C}_0(G)$ admits a coassociative coproduct $f \mapsto \Delta f \in \mathscr{C}_b(G \times G)$ with $\Delta f(x, y) = f(xy)$, so the Banach dual $\mathscr{C}_0(G)'$ inherits a dual associative product given by evaluation of the direct product of functionals on the coproduct of a function, i.e. convolution. In addition to the involution which defines the $*$-algebra structure on $\mathscr{C}_0(G)$ (i.e. pointwise conjugation), the group structure of G induces another involution via an additional precomposition with the inversion map ι, i.e. $\tilde{f}(g) = f^* \circ \iota(g) = \overline{f(g^{-1})}$, and the dual of this involution gives $\mathscr{C}_0(G)'$ the structure of a Banach $*$-algebra: $\langle \mu^*, f \rangle = \overline{\left\langle \mu, \tilde{f} \right\rangle}$. In concrete terms, elements of the Banach algebra $\mathscr{C}_0(G)'$ are bounded complex Borel measures on G and for a given Borel set E, $\mu^*(E) = \overline{\mu(E^{-1})}$.

If λ is a left-invariant Haar measure then the convolution algebra $L^1(\lambda)$ embeds into $\mathscr{C}_0(G)'$ as the closed $*$-ideal of elements which are absolutely continuous with respect to λ [29]. All convolution products with both factors in $L^2(\lambda)$ are continuous and tend to zero at infinity, i.e. they are elements of $\mathscr{C}_0(G)$, in fact they form a dense subalgebra of $\mathscr{C}_0(G)$ called the *Fourier algebra* $A(G) = L^2(\lambda) * L^2(\lambda)$, introduced originally by Eymard [33] (see also [34]). Furthermore, the measure algebra $\mathscr{C}_0(G)'$ acts on any unitary representation of G by integration and in particular it acts on $L^2(\lambda)$ wherein it satisfies the equality

$$\langle \mu \cdot \psi, \varphi \rangle_{L^2(\lambda)} = \ \mu, \overline{\psi} * (\varphi \circ \iota) \ _{\mathscr{C}_0(G)' \times \mathscr{C}_0(G)} .$$

Since $A(G) \subset \mathscr{C}_0(G)$ is dense, for any given nontrivial element $\mu \in \mathscr{C}_0(G)'$ there exist $\psi, \varphi \in L^2(\lambda)$ such that the right side of this equality is nonzero, hence the integrated form of μ in $\mathscr{L}(L^2(\lambda))$ cannot be zero. This being true for every nonzero element, we conclude that the $*$-algebra $\mathscr{C}_0(G)'$ and its subalgebra $L^1(\lambda)$ (or any other subalgebra, for that matter) are *reduced*, i.e. zero is the only element which vanishes in *every* $*$-representation. As with any reduced Banach $*$-algebra we can form the C^* completions of $\mathscr{C}_0(G)'$ and $L^1(\lambda)$, denoted respectively by $\mathscr{M}^*(G)$ (the measure algebra) and $\mathscr{C}^*(G)$, which is in each case the metric completion in the unique norm satisfying the C^* identity, i.e. is the norm equal to the supremum of the norms over all $*$-representations [29] (or equivalently, via the GNS construction, all Hilbert space $*$-representations). With these respective C^* algebra structures, $\mathscr{C}^*(G)$ becomes an isometrically embedded $*$-ideal in $\mathscr{M}^*(G)$.

There is another important interpretation of the realization of $\mathscr{C}^*(G)$ as a $*$-ideal in $\mathscr{M}^*(G)$. If $\mathcal{A}$ is any associative algebra over a field k, then a pair (L, R) of k endomorphisms of the vector space $\mathcal{A}$ is called a *multiplier* of $\mathcal{A}$ if L and R behave, respectively, like left and right multiplication by an element of an algebra which contains $\mathcal{A}$ as an ideal: for $x, y \in \mathcal{A}$

$$L(xy) = (Lx)y \quad x(Ly) = (Rx)y \quad R(xy) = x(Ry).$$

By construction, such pairs can be composed in the obvious way so as to form an algebra which contains $\mathcal{A}$ as an ideal. For C^* algebras we require R and L to be bounded endomorphisms and the maximum of their operator norms together with the adjunction $L^*(x) = (R(x^*))^*$ (and likewise for R^*) gives a C^* algebra structure to the bounded multiplier algebra, denoted $\mathcal{M}(\mathcal{A})$ [35]. Every element of the enveloping von Neumann algebra $\widetilde{\mathcal{A}}$ which maps $\mathcal{A}$ into itself under both left and right multiplication is evidently a multiplier, in fact every multiplier arises in this fashion. To see this, form a faithful Hilbert space representation $\mathcal{H}$ of $\mathcal{M}(\mathcal{A})$. If $y \in \mathcal{M}(\mathcal{A}) \subset \mathscr{L}(\mathcal{H})$ and $z \in \mathcal{A}' \subset \mathscr{L}(\mathcal{H})$ then a short computation shows that the commutator $yz - zy$ annihilates $\mathcal{A}$ on both sides, but this implies that $zy - zy = 0$ since the strong closure of the unit ball of $\mathcal{A}$ contains the identity by the Kaplansky density theorem. Thus, $\mathcal{M}(\mathcal{A}) \subset \mathcal{A}'' \subset \mathscr{L}(\mathcal{H})$. In fact, $\mathcal{M}(\mathcal{A})$ is the largest C^* algebra into which $\mathcal{A}$ embeds as an *essential* (or sometimes called *thick* [36]) $*$-ideal - i.e. one which intersects all $*$-ideals nontrivially. Regarding the isometric inclusion $\mathscr{C}^*(G) \subset \mathscr{M}^*(G)$, Wendel has proved that $\mathscr{M}^*(G) = \mathcal{M}(\mathscr{C}^*(G))$ [37].

Now we come to the main point, which is that unitary representations of G, $*$-representations of $\mathscr{M}^*(G)$ and $*$-representations of $\mathscr{C}^*(G)$ are essentially equivalent objects. Indeed, as described above a unitary representation of G gives rise to a representation of $\mathscr{C}_0(G)'$ (respectively $L^1(\lambda)$) by integration, this is evidently continuous in the C^* norm on $\mathscr{C}_0(G)'$ (respectively $L^1(\lambda)$) so it is uniquely defined on the completion $\mathscr{M}^*(G)$ (respectively $\mathscr{C}^*(G)$). Conversely, the elements of G realized as point masses in $\mathscr{M}^*(G)$ form a unitary representation of G, so the restriction of any $*$-representation of the former is a unitary representation of the latter. Furthermore, since $\mathscr{C}^*(G)$ embeds isometrically into $\mathscr{M}^*(G)$, a $*$-representation of the latter automatically gives a $*$-representation of the former by precomposition with the embedding. The last assertion to be justified is the extension of a $*$-representation of

$\mathscr{C}^*(G)$ to $\mathscr{M}^*(G)$. This follows from the aforementioned fact that the embedding $\mathscr{C}^*(G) \hookrightarrow \mathscr{M}^*(G)$ is a closed $*$-ideal, and nondegenerate (defined below) representations of $*$-ideals always extend uniquely to the entire algebra in which they are contained [29]. To summarize:

1. From a unitary representation of G one obtains $*$-representations of $\mathscr{C}_0(G)'$ and $L^1(\lambda)$ via integration. Both are continuous in their respective C^* norms so they are defined on the completions $\mathscr{M}^*(G)$ and $\mathscr{C}^*(G)$.

2. From a $*$-representation of $\mathscr{M}^*(G)$ one obtains a unitary representation of G by restriction to point masses and a $*$-representation of $\mathscr{C}^*(G)$ by precomposition with its continuous embedding into $\mathscr{M}^*(G)$.

3. From a *nondegenerate* $*$-representation of $\mathscr{C}^*(G)$, one obtains a unique extension to $\mathscr{M}^*(G)$ since $\mathscr{C}^*(G) \subset \mathscr{M}^*(G)$ is a $*$-ideal and thereby also a unitary representation of G via restriction.

If $\mathcal{H}$ is a Hilbert space $*$-representation of a $*$-algebra $\mathcal{A}$, then the subspaces

- $N(\mathcal{A}) = \{\xi : T\xi = 0 \text{ for all } T \in \mathcal{A}\}$,
- $R(\mathcal{A}) = \overline{\{T\xi : \xi \in \mathcal{H}, T \in \mathcal{A}\}}$ (linear closure)

are orthogonal and the representation is said to be *nondegenerate* if $N(\mathcal{A})$ is trivial. In particular any nontrivial irreducible Hilbert space representation must be nondegenerate, since $N(\mathcal{A})$ and $R(\mathcal{A})$ are closed invariant subspaces for $\mathcal{A}$, and in any case a degenerate $*$-representation of a $*$-ideal $\mathcal{I} \subset \mathcal{A}$ can be uniquely extended to $\mathcal{A}$ on the closed subspace $R(\mathcal{I})$.

Thus, the representation theory of G can be viewed as being essentially equivalent to that of $\mathscr{C}^*(G)$ and to each irreducible representation of $\mathscr{C}^*(G)$ we can associate the kernel, which is a $*$-ideal. Such ideals (i.e. the kernels of irreducible representations) are said to be primitive. The correspondence from irreducible representations to primitive ideals is not injective, but in favorable circumstances it is injective on unitary equivalence classes - i.e. when $\mathscr{C}^*(G)$ is of type I, a condition which we now describe. For any C^* algebra $\mathcal{A}$,

- denote by $\check{\mathcal{A}}$ the set of unitary equivalence classes of irreducible Hilbert space representations of $\mathcal{A}$,

- denote by $\hat{\mathcal{A}}$ the set of primitive ideals of $\mathcal{A}$, i.e. kernels of irreducible Hilbert space representations of $\mathcal{A}$.

The natural topologies on both spaces have been discussed previously, the natural topology on $\check{\mathcal{A}}$ is called the *regional* or *Fell* topology and the natural topology on $\hat{\mathcal{A}}$ is called the *hull-kernel* or *Jacobson* topology. As mentioned above there is an obvious surjection $\check{\mathcal{A}} \to \hat{\mathcal{A}}$ which maps an equivalence class of representations to its common kernel, so there is also a hull-kernel topology naturally defined on $\check{\mathcal{A}}$ via the quotient map. Since we've assumed that $\mathcal{A}$ is a C^* algebra, these two topologies on $\check{\mathcal{A}}$ coincide [29], but for more general Banach algebras the regional topology can be strictly finer than the hull-kernel topology.

Ultimately, we would like to associate a given primitive ideal with a unique equivalence class of representations, i.e. we would like the surjection $\check{\mathcal{A}} \to \hat{\mathcal{A}}$ to be a bijection, and this is where the type I condition comes in. The type I condition is really an amalgam of various more or less equivalent conditions which are described differently in different sources, the equivalence thereof being due to Glimm [38] in the separable case and later to Sakai [39–41]. It seems that Blackadar [42] has compiled the most detailed summary, so we will quote the result recorded there.

Theorem 4.3.1 (Glimm-Sakai) *For any C^* algebra $\mathcal{A}$, the following are equivalent:*

1. *(internal type I) every quotient $\mathcal{B}$ of $\mathcal{A}$ contains an element x such that the hereditary subalgebra $\overline{x^*\mathcal{B}x}$ is commutative,*

2. *(bidual type I) the second dual $\mathcal{A}^{**}$ (note $\mathcal{A}$ is arens-regular) is a von Neumann algebra of type I, equivalently the bicommutant $\mathcal{A}''$ in any representation is a von Neumann algebra of type I,*

3. *(postliminal) $\mathcal{A}$ admits a composition series $\{\mathcal{A}_i\}$ such that every irreducible representation of every simple subquotient takes values in the compact operators (i.e. the simple subquotients are CCR),*

4. *(GCR) every irreducible representation of $\mathcal{A}$ has nontrivial intersection with the compact operators, or equivalently contains the compact operators.*

If these equivalent conditions are met then the map $\check{\mathcal{A}} \to \hat{\mathcal{A}}$ is a bijection. Conversely if $\check{\mathcal{A}} \to \hat{\mathcal{A}}$ is a bijection and in addition $\mathcal{A}$ is separable, then the listed conditions hold.

Regarding the second condition, recall that a von Neumann algebra is said to be of type I if every nonzero central projection majorizes a nonzero abelian projection (i.e. such that the associated hereditary subalgebra is commutative).

A locally compact Hausdorff group is said to be of type I or have type I representation theory if its group C^* algebra has this property. There are many familiar classes of groups which are of type I - and also some which are not. In particular, connected reductive Lie groups are of type I, in fact they are CCR groups, which is a stronger condition. A group G is *CCR* (i.e. "completely continuous representation theory") or *liminal* if $\mathscr{C}^*(G)$ is a CCR C^* algebra, which means that all of its irreducible $*$-representations are contained in the compact operators.

The proof of this fact is outlined in [28, 43] and is more or less a consequence of the standard admissibility theorem of Harish-Chandra. If G is a connected reductive group with maximal compact subgroup K, then for any pair π_1, π_2 of unitary representations of K define

$$E_{\pi_1,\pi_2} f(x) = \iint_{K\times K} \chi_{\pi_1}(k_1) f(k_1^{-1} x k_2) \overline{\chi_{\pi_2}(k_2)} dk_1 dk_2$$

for suitably nice functions f on G. Clearly, $E_{\pi_1,\pi_2} f$ is a left and right K-finite vector in any vector space of functions containing f, since all irreducible representations of compact groups are finite dimensional. Now if for instance f is in the Schwartz space of Harish-Chandra (c.f. [28]) then a suitable linear combination of the $E_{\pi_1,\pi_2} f$ will converge back to f, in other words the Schwartz space, and therefore the L^1

convolution algebra of G, contains a dense subspace of left and right K-finite vectors. Such K-finite vectors must take values in the finite rank operators in any admissible representation of G (i.e. one for which the space of equivariant injections from any finite dimensional K representation is itself finite dimensional). Since Harish-Chandra proved that all irreducible unitary representations of G are admissible, a dense subspace of $\mathscr{C}^*(G)$ must take values in the finite rank operators in any such representation. This means that the image of $\mathscr{C}^*(G)$ must be contained in the operator norm closure of the finite rank operators, which is the ideal of compact operators.

Thus, connected reductive Lie groups are CCR. In addition to this we mention without proof that connected nilpotent groups are CCR, connected real algebraic groups are type I, exponential solvable Lie groups are type I and more generally there is a detailed criterion which describes necessary and sufficient conditions concerning the topology of the coadjoint orbit space for a simply connected solvable Lie group to be of type I [32].

4.4 Main Results For Compact Groups

The representation theory of compact groups is especially simple. The main result is the famous peter-weyl theorem.

Theorem 4.4.1 (Peter,Weyl) *The unitary dual of a compact group is discrete, every irreducible unitary representation is finite dimensional, and the Plancherel measure is given by the multiplicity function* $\dim \pi$.

In other words, for any irreducible unitary representation π of a compact group K, $\dim \operatorname{Hom}_K(V_\pi, L^2(K)) = \dim V_\pi$. Thus, $L^2(K) = \bigoplus_\pi V_\pi^{\oplus \dim V_\pi}$ as K modules. However, there is more structure. The Fourier transform for $f \in L^2(K)$ is given by

$$\hat{f}(\pi) = \int_K f(k)\pi(k)dk \in \operatorname{End}(V_\pi)$$

(normalized Haar measure) for any irreducible unitary representation V and therefore $L^2(K) = \bigoplus_\pi \operatorname{End}(V_\pi)$, the Fourier transform being an isometry with respect to normalized Haar measure on the left and Plancherel measure (i.e. $\dim \pi$ on every summand) on the right, which is much more natural and incorporates the action of K on the left and right.

Theorem 4.4.2 *Let K be a connected compact Lie group and let $K \supset K_1 \supset \ldots \supset K_r$ be a descending sequence of connected subgroups, each closed in its predecessor, with $\Delta, \Delta_1, \ldots, \Delta_r$ the respective positive casimirs (relative to the normalized Haar measure from K).*

1. *In any finite dimensional irreducible representation of K, $\Delta_{t_0,\ldots,t_r} = t_0\Delta + t_1\Delta_1 + \ldots + t_r\Delta_r$ is given by $\sum_{\pi,\pi_1,\ldots,\pi_r} t_{\pi,\ldots,\pi_r} P_{\pi,\pi_1,\ldots,\pi_r}$ where $P_{\pi,\pi_1,\ldots,\pi_r}$ is the projection into the π_r subtype of K_r contained in the π_{r-1} subtype of K_{r-1}, etcetera, the constants c_i are the respective Casimir eigenvalues and $t_{\pi,\ldots,\pi_r} = t_0c_0 + t_1c_1 + \ldots + t_rc_r$,*

2. *The Fourier transform of the point mass δ_1 is given by $\widehat{\delta_1} = \bigoplus_\pi \mathrm{id}_\pi$, thus if $t_0\Delta + t_1\Delta_1 + \ldots + t_r\Delta_r$ is hypoelliptic then its integral kernel with initial point $x \in K$ evaluated at $y \in K$ is given by*

$$e^{-\Delta_{t_0,\ldots,t_r}} x\delta_1, y\delta_1 = \sum_{\pi,\pi_1,\ldots,\pi_r} e^{-t_{\pi,\ldots,\pi_r}} (\dim \pi) \operatorname{Tr}_\pi(y^{-1}xP_{\pi,\pi_1,\ldots,\pi_r}).$$

Proof The equality $\widehat{\delta_1} = \bigoplus_\pi \mathrm{id}_\pi$ is a restatement of the Plancherel formula for compact groups, i.e. $f(1) = \sum_\pi (\dim \pi) \int_K f\overline{\chi_\pi}$ where $\chi_\pi(k) = \operatorname{Tr}_\pi(k)$ is the character, all other statements follow from the peter-weyl isometry $L^2(K) = \bigoplus_\pi \mathrm{End}(V_\pi)$. ■

With this and the preparatory results from chapter 3 in mind, we have the following result.

Theorem 4.4.3 *If $(G, \theta, \mathfrak{h}, \Delta^+(\mathfrak{g}, \mathfrak{h}), P_\Sigma)$ is an admissible datum, then for any two connected compact Θ-stable subgroups $L, K \subset G$ such that*

1. *K acts transitively on G/P_Σ,*

2. *$K \cap P_\Sigma \subset L \subset K$,*

3. $\mathfrak{l}\cap[\mathfrak{g},\mathfrak{g}]$ *is orthogonal to* $\mathfrak{g}^{\Sigma}_{-1}$ *in the Killing form,*

the horizontal distribution for the associated fibration $L/(K\cap P_\Sigma)\hookrightarrow K/(K\cap P_\Sigma)\to K/L$ *is bracket-generating and the heat kernel for the operator* $\Delta_K-\Delta_L$ *on left* $K\cap P_\Sigma$ *invariant functions in* $L^2(K)$ *is given by*

$$e^{-t(\Delta_K-\Delta_L)}x\delta_{K\cap P_\Sigma}, y\delta_{K\cap P_\Sigma} = \sum_{\pi_K,\pi_L}\Big(e^{-t(c_{\pi_K}-c_{\pi_L})}(\dim\pi_K)\,\mathrm{Tr}_{\pi_K}(y^{-1}xP_{\pi_L,K\cap P_\Sigma})$$

where the sum runs over all pairs π_K,π_L *of irreducible representations of* K *and* L *respectively, and*

1. c_{π_K}, c_{π_L} *are the respective Casimir eigenvalues,*
2. $P_{\pi_L,K\cap P_\Sigma}$ *is the projection into the* $K\cap P_\Sigma$ *invariants embedded in the* π_L *subtype of* π_K*.*

Proof Since $\mathfrak{g}^{\Sigma}_{-1}\oplus\mathfrak{p}^{\Sigma}$ is bracket-generating in $[\mathfrak{g},\mathfrak{g}]$, the orthogonality hypothesis $\mathfrak{l}\cap[\mathfrak{g},\mathfrak{g}],\mathfrak{g}^{\Sigma}_{-1} = 0$ ensures that $\Delta_K-\Delta_L$ is hypoelliptic on the total space $K/(K\cap P_\Sigma)=G/P$, or equivalently on left $K\cap P_\Sigma$ invariant functions on K, since $\mathfrak{g}^{\Sigma}_{-1}$ defines a bracket-generating subbundle of $T(G/P)$. Thus, the given expression for the heat kernel follows from Theorem 4.4.2. ■

There are many results available to study branching multiplicities for compact groups so it is in principal possible to give an entirely explicit expression for the heat kernel using the formula in Theorem 4.4.3. It will involve classical special functions. Continuing the discussion from the introduction, we find that for $G=\mathrm{U}(1,n+1)$ and $P\subset G$ equal to the parabolic isotropy group of the null line in $\mathbf{C}^{n+2}$ defined by $z_0-z_1=z_2=\ldots=z_{n+1}=0$, the compact isotropy group $\mathrm{U}(1)\times\mathrm{U}(n+1)$ of the orthogonal positive/negative splitting $\mathbf{C}\oplus\mathbf{C}^{n+1}$ acts transitively on G/P. Thus, Theorem 4.4.3 applies to the standard Hopf fibration

$$\begin{aligned}(\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(n))/(\Delta\,\mathrm{U}(1)\times\mathrm{U}(n))&\\ \hookrightarrow(\mathrm{U}(1)\times\mathrm{U}(n+1))/(\Delta\,\mathrm{U}(1)\times\mathrm{U}(n))&\\ \to(\mathrm{U}(1)\times\mathrm{U}(n+1))/(\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(n))&\end{aligned}$$

arising from the sequence $\Delta\,\mathrm{U}(1)\times\mathrm{U}(n)\subset\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(n)\subset\mathrm{U}(1)\times\mathrm{U}(n+1)$. This is because the isotropy group is $\Delta\,\mathrm{U}(1)\times\mathrm{U}(n)$ and $\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(n)$ acts transversely to the bracket-generating subbundle of $T(G/P)$ (its orbits are the sets of null lines projecting to the same line in $\mathbf{C}^{n+1}=\{z_0=0\}$).

Likewise, if we replace $\mathbf{C}$ with the quaternion algebra $\mathbf{H}$, then an exactly analogous argument goes through for the quaternion Hopf fibration

$$\begin{aligned}(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(n))/(\Delta\,\mathrm{Sp}(1)\times\mathrm{Sp}(n))&\\ &\hookrightarrow(\mathrm{Sp}(1)\times\mathrm{Sp}(n+1))/(\Delta\,\mathrm{Sp}(1)\times\mathrm{Sp}(n))\\ &\to(\mathrm{Sp}(1)\times\mathrm{Sp}(n+1))/(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(n))\end{aligned}$$

arising from the sequence $\Delta\,\mathrm{Sp}(1)\times\mathrm{Sp}(n)\subset\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(n)\subset\mathrm{Sp}(1)\times\mathrm{Sp}(n+1)$. So, Theorem 4.4.3 applies to this fibration as well.

For the normed algebras $\mathbf{O}$, $\widetilde{\mathbf{O}}$ (the *split octonions*) and $\mathbf{C}\otimes\mathbf{O}$, the Hopf fibrations are not as straightforward. As sketched out in [44], one defines

$$\mathrm{Herm}_3(\mathbf{C}\otimes\mathbf{O})=\left\{\left(\begin{smallmatrix}r_1&\overline{x}&\overline{y}\\ x&r_2&z\\ y&\overline{z}&r_3\end{smallmatrix}\right):r_1,r_2,r_3\in\mathbf{R},x,y,z\in\mathbf{C}\otimes\mathbf{O}\right\}$$

along with its real forms

$$\begin{aligned}\mathrm{Herm}_3(\mathbf{O})&=\left\{\left(\begin{smallmatrix}r_1&\overline{x}&\overline{y}\\ x&r_2&z\\ y&\overline{z}&r_3\end{smallmatrix}\right):r_1,r_2,r_3\in\mathbf{R},x,y,z\in\mathbf{O}\right\}\\ \mathrm{Herm}_3(\widetilde{\mathbf{O}})&=\left\{\left(\begin{smallmatrix}r_1&\overline{x}&\overline{y}\\ x&r_2&z\\ y&\overline{z}&r_3\end{smallmatrix}\right):r_1,r_2,r_3\in\mathbf{R},x,y,z\in\widetilde{\mathbf{O}}\right\}\\ \mathrm{Herm}_3'(\mathbf{O})&=\left\{\left(\begin{smallmatrix}r_1&-i\overline{x}&-i\overline{y}\\ ix&r_2&z\\ iy&\overline{z}&r_3\end{smallmatrix}\right):r_1,r_2,r_3\in\mathbf{R},x,y,z\in\mathbf{O}\right\}\end{aligned}$$

In each of these respective cases, the automorphisms of the respective Jordan algebra structures are the simply connected groups $\mathrm{F}_4^{\mathbf{C}}$ (the complex form), F_4^{c} (the compact form), $\mathrm{F}_4^{(4)}$ (the split form), $\mathrm{F}_4^{(-20)}$ (the unique noncompact and nonsplit real form). Each of these groups acts transitively on the idempotents of trace one, with isotropy conjugate to an injected copy of $\mathrm{Spin}(9)$ in the real cases. The total space of the octonion Hopf fibration $S^7\hookrightarrow S^{15}\to S^8$ arises as the boundary of the exceptional symmetric space $\mathrm{F}_4^{(-20)}/\,\mathrm{Spin}(9)$. The isotropy group of a boundary point is a parabolic $P\subset\mathrm{F}_4^{(-20)}$ and it can be shown that $P\cap\mathrm{Spin}(9)$ is isomorphic to

Spin(7), embedded by way of the sequence $\mathrm{Spin}(7) \to \mathrm{Spin}(8) \to \mathrm{Spin}(9)$ where the second embedding is the usual one but the first embedding is the usual one followed by a triality automorphism of Spin(8) [5, 44]. Thus, Theorem 4.4.3 applies to this fibration as well.

Table 4.1.
Compact fibrations $L/M \hookrightarrow K/M \to K/L$ with hidden symmetry.

M	L	K
$\mathrm{U}(n)$	$\mathrm{U}(n) \times \mathrm{U}(1)$	$\mathrm{U}(n+1)$
$\mathrm{U}(n)$	$\mathrm{SO}(2n)$	$\mathrm{SO}(2n+1)$
$\mathrm{Sp}(n) \times \mathrm{U}(1)$	$\mathrm{Sp}(n) \times \mathrm{Sp}(1)$	$\mathrm{Sp}(n+1)$
$\mathrm{Sp}(n) \times \mathrm{U}(1)$	$\mathrm{U}(2n) \times \mathrm{U}(1)$	$\mathrm{U}(2n+1)$
$\mathrm{Sp}(n) \times \Delta\,\mathrm{Sp}(1)$	$\mathrm{Sp}(n) \times \mathrm{Sp}(1)^2$	$\mathrm{Sp}(n+1) \times \mathrm{Sp}(1)$
$\mathrm{Spin}(7)$	$\mathrm{Spin}(8)$	$\mathrm{Spin}(9)$
$\mathrm{SU}(2) \times \Delta\,\mathrm{SU}(2)$	$\mathrm{SO}(4) \times \mathrm{SO}(3)$	$\mathrm{SO}(5) \times \mathrm{SO}(3)$
$\mathrm{SU}(2) \times \Delta\,\mathrm{SO}(2)$	$\mathrm{SO}(4) \times \mathrm{SO}(2)$	$\mathrm{SO}(5) \times \mathrm{SO}(2)$
$\mathrm{SU}(3) \times \Delta\,\mathrm{SO}(2)$	$\mathrm{U}(3) \times \mathrm{SO}(2)$	$\mathrm{SO}(6) \times \mathrm{SO}(2)$
$\mathrm{SU}(3) \times \Delta\,\mathrm{SO}(2)$	$\mathrm{Spin}(6)$	$\mathrm{SO}(6) \times \mathrm{Spin}(7)$

We anticipate that many of the fibrations identified by T. Kobayashi [45] as having "hidden symmetry" will have total spaces identifiable with real flag varieties and as such, will have hypoelliptic horizontal sublaplacians with heat kernels described as in Theorem 4.4.3. These are listed in Table 4.1, which contains the above described Hopf fibrations.

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