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Absolute Convergence of the Twisted Arthur-Selberg Trace Formula

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To Pooja.

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ABSTRACT

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We show that the distributions occurring in the geometric and spectral side of the twisted Arthur-Selberg trace formula extend to non-compactly supported test functions. The geometric assertion is modulo a hypothesis on root systems proven among other cases, when the group is split. The result extends the work of Finis-Lapid (and Müller, spectral side) to the twisted setting. We also give an application towards finiteness of residues of certain Rankin-Selberg L-functions.

1. INTRODUCTION

Understanding the automorphic representations of a connected reductive group over a global field has been a central problem in the theory of automorphic forms and the Arthur-Selberg trace formula has been an indispensable tool in doing so. The trace formula was conceived by Selberg [Sel56] to show the existence of Maass forms via a Weyl law, and vastly generalized by Arthur to any connected reductive group G over a number field F. It is an identity of two distributions on $G(\mathbb{A})$ where $\mathbb{A} = \mathbb{A}_F$ is the ring of adeles of F viz.,

$$J^G_{\text{spec}}(f) = J^G_{\text{geom}}(f),$$

f being a smooth function on $G(\mathbb{A})$ of compact support. The spectral side is a sum-integral over automorphic representations of G and the geometric side contains weighted orbital integrals. It was realized early on that an appropriate 'twisted' trace formula developed for a connected component $\widetilde{G}$ of a reductive group G would be useful in proving the 'endoscopic' cases of Langlands' Functoriality conjectures. If θ is an F-automorphism of G of finite order, we can form the reductive group $G \rtimes \langle\theta\rangle$ of which $\widetilde{G} = G \rtimes \theta$ is a connected component. The trace formula for $\widetilde{G}$ or the twisted trace formula was developed in the lectures given at the Friday Morning seminar at IAS organized by Clozel, Labesse and Langlands [CLL84] and has been exposed and improved upon in the book [LW13]. The twisted trace formula has also been instrumental in proving the cyclic (solvable) base change case in the book of Arthur and Clozel [AC89].

If $\phi : {}^LH \to {}^LG$ is an L-homomorphism between the L-groups of quasisplit connected reductive groups H and G, Langlands' Functoriality predicts a transfer of automorphic representations of H to G. Among these are the L-homomorphisms arising out of endoscopic groups of which the classical groups (orthogonal and symplectic) are prototypical examples when $G = GL(n)$ and $\theta(g) = {}^t g^{-1}$. Arthur [Art13] proved functoriality in this case and Mok [Mok15] extended it to unitary groups. This had been conditional on the Fundamental Lemma which was resolved by Ngô [Ngô10] and the stabilization of the twisted trace formula by Mœglin-Waldspurger [MW16a, MW16b]. Arthur proved the geometric side of the trace formula converges for $f \in \mathcal{C}_c^\infty(G(\mathbb{A}))$ but he deftly didn't make use of the convergence of the spectral side. The convergence was proven in [MS04] for $G = GL(n)$ and later by Finis, Lapid and Müller [FL11a, FLM11] for general G. Our work extends their result to the twisted trace formula.

Finis-Lapid have proven the absolute convergence of the spectral and geometric sides for more general test functions than those of compact support whose extension to the twisted setting is the main theme of this paper. Let $\mathbf{K}$ be a "good" maximal compact subgroup of $G(\mathbb{A})$ and K be an open compact subgroup of the finite adeles $G(\mathbb{A}_f)$ in $\mathbf{K}$. They consider a class $\mathcal{C}(G(\mathbb{A}), K)$ of test functions f on $G(\mathbb{A})$ which are right K-invariant at non-Archimedean places and at the Archimedean places satisfy, $\|f * X\|_{L^1(G(\mathbb{A}))} < \infty$ for every $X \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$, the universal enveloping algebra of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ acting as differential operators. For such test functions they prove the convergence on the spectral side [FL11b, FLM11] and also the geometric side [FL11a, FL16], thus constructing an invariant trace formula for a broader class of test functions. In the case when $\widetilde{G}$ is a component of $G \rtimes \langle\theta\rangle$, the space $\mathcal{C}(\widetilde{G}(\mathbb{A}), K)$ can be defined similarly by considering the action of X on a smooth function f defined on $\widetilde{G}(\mathbb{A})$. The main result of this exposition is to derive the convergence and hence the continuity of the distributions occurring in the twisted Arthur-Selberg trace formula with Theorem 4.1.1 and Theorem 7.2.2 being the statements for the geometric and spectral sides respectively.

The proof of convergence in the geometric side involves estimating the sums over certain twisted Bruhat cells and using the slow decay of intertwining operators and is carried out in Sections 4 and 5. The main steps follow [FL11a, FL16] except that the twisted equivalent of the crucial lemma 2.2 of [FL11a] does not hold. The corresponding modification is discussed in Chapter 6 as the Root Cone Lemma. The convergence on the geometric side is modulo this geometric lemma but we prove it completely when G is split and also for cyclic base change. It is a lemma about root systems involving automorphisms of the Dynkin diagram of G so depends only on the semisimple part of G. We reduce it to split simple groups and then prove all cases in the Cartan-Killing classification. A crucial step in the proof when G is of type A_n was shown to us by P. Majer as an answer to a question [PM] on MathOverflow. The lemma for E_6 type is proven using the Mathematical software `SageMath` [TSD17].

The convergence results usually involve a parameter T in a finite dimensional vector space $\mathfrak{a}_0$ (see Chapter 2 for definition) chosen sufficiently away from the origin. In our setting however, T is allowed to vary only along a line but it is enough to ensure the convergence. Since the distributions involved are polynomials in T, they can be extended to any $T \in \mathfrak{a}_0$ and in particular to the special point T_0 that makes the distributions independent of the chosen minimal parabolic subgroup, see [Art81, Lemma 1.1]. For applications to limit multiplicities it is essential to keep track on the dependence the compact open subgroup K of $G(\mathbb{A}_f)$ in proving the bound on the seminorm μ but the non-twisted bounds work verbatim.

Chapter 7 is devoted to proving the spectral side. The convergence of the spectral side in the non-twisted setting involved estimating the derivatives of certain intertwining operators appearing in the spectral expansion and has been the main result of [FL11b]. The crucial difference on the spectral side of the twisted trace formula is that the trace here is a composition of operators on different spaces. We introduce a unitary shift operator which converts the twisted trace formula to the usual one and invoke the estimates in [ibid].

In Chapter 8 we give an application of the continuity of the spectral side towards proving the finiteness of residues of certain Rankin-Selberg L-functions that was suggested to us by J. Getz. One possible application is the Weyl Law for self-dual automorphic representations in the style of [LM09] which would explain the endoscopic classification of classical groups in a more quantitative way. This is currently a work in progress.

2. NOTATION

We will follow the notations of Labesse and Waldspurger [LW13] rather than of Arthur. Throughout, G will denote a connected reductive group over $\mathbb{Q}$ and $\widetilde{G}$, a twisted G-space [ibid., Chapter 2] such that $\widetilde{G}(\mathbb{Q})$ is nonempty. Thus $\widetilde{G}$ is a left-G-torsor equipped with a map

$$\mathrm{Ad} : \widetilde{G} \to \mathrm{Aut}(G)$$

which for $x \in G, \delta \in \widetilde{G}$ satisfies,

$$\mathrm{Ad}(x\delta) = \mathrm{Ad}(x) \circ \mathrm{Ad}(\delta).$$

We will assume that G and $\widetilde{G}$ are the components of a reductive algebraic group. This is somewhat more restrictive than [LW13] but suffices for most applications of the twisted trace formula. All algebraic subgroups of G will be implicitly assumed to be defined over the rational numbers. We define a right-G-action on $\widetilde{G}$ via

$$\delta x = \theta(x)\delta, \qquad \text{with } \theta = \mathrm{Ad}(\delta).$$

Any element $y \in \widetilde{G}(\mathbb{A})$ can be written (not uniquely) as $y = x\delta$ with $x \in G(\mathbb{A})$ and $\delta \in \widetilde{G}(\mathbb{Q})$.

Throughout this paper, we fix a minimal parabolic subgroup P_0 of G defined over $\mathbb{Q}$ with Levi decomposition $P_0 = M_0 \ltimes N_0$ and a maximal compact subgroup $\mathbf{K}_G = \mathbf{K} = \mathbf{K}_\infty \mathbf{K}_f$ which is admissible relative to M_0 in the sense of [Art81, §1]. Thus we have the Iwasawa decomposition

$$G(\mathbb{A}) = P_0(\mathbb{A})\mathbf{K}_G.$$

We will denote the (finite) Weyl group of (G, T_0) by W^G or W. Note that M_0 is the centralizer of a maximal torus which we denote by T_0. The Lie algebra of G

will be denoted by $\mathfrak{g}$ and the universal enveloping algebra of it's complexification by $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$.

Let $\mathcal{L}^G(M_0) = \mathcal{L}^G = \mathcal{L}$ denote the set of Levi subgroups containing M_0, i.e., the (finite) set of centralizers of subtori of T_0. For $M \in \mathcal{L}$, we have the following notation.

- We shall denote by $\mathcal{L}^G(M), \mathcal{F}^G(M), \mathcal{P}^G(M)$, the (finite) set of Levi subgroups containing M, parabolic subgroups containing M and parabolic subgroups with Levi part M respectively. Whenever clear from the context, we shall ignore the superscript G or replace it with a reductive subgroup of G. The notations $\mathcal{L}, \mathcal{F}$ (and $\mathcal{P}$) will denote the Levi (resp. parabolic) subgroups containing M_0 (resp. P_0).

- The Weyl group $W^G(M) = W(M) = N_{G(\mathbb{Q})}(M)/M$ can be identified with a subgroup of W.

- T_M is the split part of the identity component of the center of M and $A_M = A_0 \cap T_M(\mathbb{R})$.

- The real vector space $\mathfrak{a}_M^*$ is spanned by the lattice $X^*(M)$ of rational characters of M and $\mathfrak{a}_{M,\mathbb{C}}^*$ is it's complexification. The dual space $\mathfrak{a}_M$ spanned by the co-characters of T_M is the Lie algebra of A_M. Denote by a_M the dimension of each.

- The map $H_M : M(\mathbb{A}) \to \mathfrak{a}_M$ is the homomorphism given by $\langle \chi, H_M(m) \rangle = \log |\chi(m)|_{\mathbb{A}^*}$, for any $\chi \in X^*(M)$. It's kernel is $M(\mathbb{A})^1$.

- $X_M = A_M M(\mathbb{Q}) \setminus M(\mathbb{A})$.

- The Weyl group $W(M)$ acts on $\mathcal{P}(M)$ and $\mathcal{F}(M)$ by conjugation; $w.P = n_w P n_w^{-1}$.

- $\mathcal{R}_M$ is the set of reduced roots of T_M on $\mathfrak{g}$ and for every root $\alpha \in \mathcal{R}_M$, $\alpha^\vee$ denotes the corresponding co-root. It will be abbreviated $\mathcal{R}_0$ when $M = M_0$.

- For $w \in W, Q(w)$ denotes the smallest standard parabolic subgroup containing a representative n_w of w.

In particular, we have the above notation for $M = G$. For $P \in \mathcal{P}(M)$, we use the following additional notation.

- N_P is the unipotent radical of P and M_P is the unique element $L \in \mathcal{L}(M)$ with $P \in \mathcal{P}(L)$.

- $A_P = A_{M_P};\ \mathfrak{a}_P = \mathfrak{a}_M;\ a_P = \dim \mathfrak{a}_P$.

- For a point $Z \in \mathfrak{a}_0, Z_P$ denotes the projection of Z onto $\mathfrak{a}_P$.

- The map $H_P : G(\mathbb{A}) \to \mathfrak{a}_P$ is the extension of H_M to a left $N_P(\mathbb{A})-$ and right $\mathbf{K}$-invariant map.

- Δ_P (resp. $\hat{\Delta}_P$) is the subset of simple roots (resp. simple weights) of P, which is a basis for $(\mathfrak{a}_P^G)^*$.

- $X_P = A_P N_P(\mathbb{Q}) M_P(\mathbb{A}) \setminus G(\mathbb{A}),\ Y_P = A_G P(\mathbb{Q}) \setminus G(\mathbb{A})$.

- Denote by ξ_P the sum of roots in Δ_0^P. More generally if Q is a parabolic subgroup containing P then denote by ξ_P^Q the sum of roots in $\Delta_0^Q \setminus \Delta_0^P$.

- For $X \in \mathfrak{a}_P, d_P(X) = \inf_{\alpha \in \Delta_P} \alpha(X)$. Indeed, it denotes the distance of X from the 'walls'.

- The Killing form induces an invariant inner product and a Euclidean structure on $\mathfrak{a}_P^G$. $\mathrm{Vol}(\Delta_P^\vee)$ is the volume of the parallelopiped in $\mathfrak{a}_P^G$ whose sides are roots in $\Delta_P^\vee$. For $\Lambda \in i\mathfrak{a}_M^*$ regular, define [1]

$$\epsilon_P(\Lambda) = \mathrm{Vol}(\Delta_P^\vee) \prod_{\alpha \in \Delta_P} \left(\langle \Lambda, \alpha^\vee \rangle \right)^{-1}.$$

We define $\hat{\epsilon}_P(\Lambda)$ by replacing Δ_P by $\hat{\Delta}_P$ and similar sets $\epsilon_P^Q, \hat{\epsilon}_P^Q$ whenever $P \subseteq Q$.

[1]Arthur uses $\theta_P(\Lambda)$ for the inverse of $\epsilon_P(\Lambda)$ but following [LW13], we shall reserve the symbol θ for the automorphism on G.

- $\mathcal{A}(X_P)$ is the space of automorphic forms on X_P (cf. [MW95, §I.2.17] and [BJ79, §4]). For an automorphic representation σ of M, the space $\mathcal{A}(X_P, \sigma)$ is the space of automorphic forms Φ over X_P such that for every $x \in G(\mathbb{A})$, the function

$$m \mapsto \Phi(mx), \quad \text{for } m \in M(\mathbb{A})$$

is an automorphic form in the $\sigma-$isotypical space of $L^2_{\text{disc}}(X_M)$.

- The spaces $\mathcal{A}(X_P), \mathcal{A}_{\text{disc}}(X_P)$ and $\mathcal{A}_{\text{cusp}}(X_P)$ (resp. square-integrable and cuspidal forms) are pre-Hilbert spaces with respect to the inner product

$$\langle \Phi, \Psi \rangle_P = \int_{X_P} \Phi(x)\Psi(x)\mathrm{d}x.$$

We denote by $\overline{\mathcal{A}}(X_P)$ the Hilbert space completion of $\mathcal{A}(X_P)$.

Now we define some objects related to $\widetilde{G}$. Some of the above notation needs to be modified appropriately for such objects.

- We fix once and for all an element $\delta_0 \in \widetilde{G}(\mathbb{Q})$ such that the automorphism $\theta_0 = \mathrm{Ad}(\delta_0)$ preserves P_0 and M_0. Such an element is uniquely determined modulo conjugation by $M_0(\mathbb{Q})$.

- A parabolic subset $\widetilde{P}$ of $\widetilde{G}$ is the normalizer in $\widetilde{G}$ of a parabolic subgroup P of G such that $\widetilde{P}(\mathbb{Q}) \neq \emptyset$.

- $\widetilde{M_P}$ is the Levi subset of $\widetilde{P}$ if there is a Levi decomposition $\widetilde{P} = \widetilde{M_P} N_P$, where N_P is the unipotent radical of P, which is invariant under $\mathrm{Ad}(\delta)$, $\delta \in \widetilde{P}(\mathbb{Q})$.

- $\widetilde{P_0} = P_0.\delta_0$ and any parabolic subset containing $\widetilde{P_0}$ is called standard, denoted by $\mathcal{P}^{\widetilde{G}}(M_0)$ and abbreviated as $\widetilde{\mathcal{P}}(M_0)$ or simply $\widetilde{\mathcal{P}}$. The sets $\mathcal{L}^{\widetilde{G}}(M)$ and $\mathcal{F}^{\widetilde{G}}(M)$ are defined similarly for any $M \in \mathcal{L}$.

- When $P = P_0$, we extend the map $H_0 = H_{P_0}$ to $\widetilde{G}(\mathbb{A})$ by $H_0(x\delta_0) = H_0(x)$. This is well-defined because $G(\mathbb{Q})$ is in the kernel of H_P.

- The automorphism θ_0 on G induces a linear map, also denoted by θ_0, on $\mathfrak{a}_M$ via the action on co-characters. The space $\mathfrak{a}_{\widetilde{P}} = \mathfrak{a}_{\widetilde{M}}$ is the set of vectors of $\mathfrak{a}_M$ fixed under this automorphism. In particular, we can identify $\mathfrak{a}_{\widetilde{P}}$ as a subset of $\mathfrak{a}_P$.

- As before, $a_{\widetilde{P}}$ will denote the dimension of $\mathfrak{a}_{\widetilde{P}}$.

- An inclusion $\widetilde{P} \subset \widetilde{Q}$ of parabolic subsets gives $\mathfrak{a}_{\widetilde{Q}} \subset \mathfrak{a}_{\widetilde{P}}$ and a canonical decomposition

$$\mathfrak{a}_{\widetilde{P}} = \mathfrak{a}_{\widetilde{Q}} \oplus \mathfrak{a}_{\widetilde{P}}^{\widetilde{Q}}.$$

- A root $\alpha \in \Delta_P^Q$ induces a linear form $\widetilde{\alpha}$ on $\mathfrak{a}_{\widetilde{P}}^{\widetilde{Q}}$ by averaging

$$\frac{1}{l}\sum_{r=0}^{l-1} \theta_0^r(\alpha),$$

where l is the order of the automorphism θ_0. Denote by $\Delta_{\widetilde{P}}^{\widetilde{Q}}$ the set of such orbits. Analogously we define the set $\hat{\Delta}_{\widetilde{P}}^{\widetilde{Q}}$ of orbits of weights $\varpi \in \hat{\Delta}_P^Q$.

- The (inclusion-reversing) bijection $P \mapsto \Delta_{P_0}^P$ between standard parabolic subgroups and subsets of the simple roots, in the twisted case becomes a bijection $\widetilde{P} \mapsto \Delta_{\widetilde{P}}^{\widetilde{P}_0}$ of corresponding sets.

- For a standard parabolic subgroup P, we define subgroups $P^- \subseteq P \subseteq P^+$ as follows. P^- is the standard parabolic subgroup whose Levi has, for simple roots, those $\alpha \in \Delta_0^P$ such that the orbit of α under θ_0 is contained in Δ_0^P. Likewise P^+ is the standard parabolic subgroup whose Levi has, for simple roots, elements of orbits of Δ_0^P under θ_0.

- Similar to the non-twisted case, we define for $\Lambda \in i\mathfrak{a}_{\widetilde{M}}^*$ regular,

$$\epsilon_{\widetilde{P}}(\Lambda) = \mathrm{Vol}(\Delta_{\widetilde{P}}^\vee) \prod_{\alpha\in\Delta_{\widetilde{P}}} \langle \Lambda, \alpha^\vee\rangle^{-1}.$$

- The Weyl set $W^{\widetilde{G}} = \widetilde{W}$ is the quotient of the normalizer of M_0 in $\widetilde{G}$ by M_0. Indeed,

$$\widetilde{W} = W \rtimes \theta_0$$

and the representatives of the Weyl set can be chosen as $n_w \delta_0$ where n_w are representatives of W.

- Throughout the paper, we shall fix a unitary character ω of $G(\mathbb{A})$ which is trivial on $A_G G(\mathbb{Q})$.

- A (twisted) representation of $(\widetilde{G}, \omega)$ is a representation π of G on a vector space V along with an invertible endomorphism

$$\widetilde{\pi}(\delta, \omega) \in \mathrm{GL}(V), \quad \delta \in \widetilde{G}(\mathbb{Q})$$

satisfying for every $x, y \in G$ and $\delta \in \widetilde{G}(\mathbb{Q})$,

$$\widetilde{\pi}(x\delta y, \omega) = \pi(x)\ \widetilde{\pi}(\delta, \omega)\ (\pi \otimes \omega)(y).$$

- Since G is unimodular, the measure on $G(\mathbb{A})$ induces a measure on $\widetilde{G}(\mathbb{A})$ via

$$\int_{\widetilde{G}(\mathbb{A})} h(y) \mathrm{d}y = \int_{G(\mathbb{A})} h(x\delta) \mathrm{d}x, \quad \delta \in G(\mathbb{Q}).$$

- For an integrable function f on $\widetilde{G}(\mathbb{A})$, define

$$f^1(y) = \int_{A_G} f(zy) \mathrm{d}z.$$

Remark 2.0.1 *Although all results below are for groups defined over $\mathbb{Q}$, they hold for groups defined over all number fields. The field of rational numbers makes the notations easier, for instance there is only one Archimedean place.*

3. PRELIMINARIES

There is an action of $\widetilde{G}(\mathbb{A})$ on the homogeneous space $X_G = A_G G(\mathbb{Q}) \setminus G(\mathbb{A})$ given by

$$(y, \dot{x}) \mapsto \dot{x} * y = \delta^{-1} x y$$

where $y \in \widetilde{G}(\mathbb{A}), x \in G(\mathbb{A})$ is a representative of $\dot{x}$ and δ is any element of $\widetilde{G}(\mathbb{Q})$. When there is no confusion, we shall denote $\dot{x}$ by it's representative x.

3.1 THE SPACE $\mathcal{C}(\widetilde{G}(\mathbb{A}), K)$

Fix a compact open subgroup K of $G(\mathbb{A}_f)$ in $\mathbf{K}_f$, where $\mathbb{A}_f$ is the ring of finite adeles. The right action of $G(\mathbb{A})$ on $\widetilde{G}(\mathbb{A})$ restricts to that of K. For a smooth function h on $\widetilde{G}(\mathbb{R})$ and $X \in \mathcal{U}(\mathfrak{g})$, we define the smooth function $h * X$ on $\widetilde{G}(\mathbb{R})$ by

$$(h * X)(y) = \frac{\mathrm{d}}{\mathrm{d}t}\, h(y \exp tX)\Big|_{t=0}.$$

We extend this action to smooth functions on $\widetilde{G}(\mathbb{A})$ by ignoring the non-Archimedean component.

Define $\mathcal{C}(\widetilde{G}(\mathbb{A}), K)$ to be the space of smooth functions h on $\widetilde{G}(\mathbb{A})$ which are right K-invariant and which satisfy

$$\|h * X\|_{L^1(\widetilde{G}(\mathbb{A}))} < \infty,$$

for any $X \in \mathcal{U}(\mathfrak{g})$. The topology induced from the seminorms $\|f * X\|_{L^1(\widetilde{G}(\mathbb{A}))}$ makes $\mathcal{C}(\widetilde{G}(\mathbb{A}), K)$ into a Frechet space (i.e., complete, metrizable and locally convex), see [Trè67, Chapter 10]. Sometimes we will abbreviate $\|h\|_1$ for the L^1-norm of h. The following lemma is proved in [Trè67, Proposition 7.7].

Lemma 3.1.1 *A linear form J on a locally convex space E is continuous if and only if there is a continuous seminorm μ on E such that for every $f \in E$,*

$$J(f) \leq \mu(f).$$

Note that in [FLM11, FL16], Finis, Lapid and Müller prove the continuity of the usual (non-twisted) trace formula with the analogous space $\mathcal{C}(G(\mathbb{A}), K)$. Indeed, we have a correspondence between the two spaces:

Lemma 3.1.2 *For $f \in \mathcal{C}(G(\mathbb{A}), K)$ and $\delta \in \widetilde{G}(\mathbb{Q})$ define the function $(L_\delta f)(y) = f(\delta^{-1}y)$ on $\mathcal{C}(\widetilde{G}(\mathbb{A}), K)$ then this map is a bijection between the two spaces with inverse $L_{\delta^{-1}}$. Moreover, $\|f\| = \|L_\delta f\|$ and $L_\delta(f * X) = L_\delta(f) * X$ for any $X \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$.*

The bijection is obvious. The equality of L^1 norms is a consequence of the definition of measure on the twisted space $\widetilde{G}(\mathbb{A})$.

3.2 REDUCTION THEORY

For $Q \in \mathcal{P}$, $T_1, T \in \mathfrak{a}_0$, we define the Siegel set $\mathcal{S}^Q_{P_0}(T_1, T)$ consisting of $x = pak \in G(\mathbb{A})$ such that $k \in \mathbf{K}, p \in \omega$, a fixed compact subset of $M_0(\mathbb{A})^1 N_0(\mathbb{A})$ and $a \in A_0$ satisfying

$$\tau^Q_{P_0}(H_0(a) - T_1) = 1; \quad \hat{\tau}^Q_{P_0}(T - H_0(a)) = 1.$$

If so, we have the partition lemma of Langlands that for any $x \in G(\mathbb{A})$ and $-T_1, T$ sufficiently regular, i.e., $d_0(T) \geq c, d_0(-T_1) \geq c_1$ for fixed positive constants c, c_1,

$$\sum_{\substack{Q: \\ P_0 \subseteq Q \subseteq P}} \sum_{\delta \in P(\mathbb{Q}) \backslash Q(\mathbb{Q})} F^Q_{P_0}(\delta x, T) \tau^P_Q(H_0(\delta x) - T) = 1,$$

where $F^Q_{P_0}(\circ, T)$ is the characteristic function of the set $Q(\mathbb{Q})\mathcal{S}^Q_{P_0}(T_1, T)$. In particular, for $P = G$ we have

$$G(\mathbb{Q})\ \mathcal{S}^G_{P_0}(T_1, T) = G(\mathbb{A}).$$

Throughout the paper we fix such $T_1 \in \mathfrak{a}_0$.

3.3 THE OPERATOR ρ

The usual right regular action $\rho = \rho_G$ of $G(\mathbb{A})$ on $L^2(X_G)$, which is given by

$$(\rho(g)\Phi)(x) = \Phi(xg)$$

extends to a twisted representation $\widetilde{\rho}$ of $\widetilde{G}(\mathbb{A})$:

$$(\widetilde{\rho}(y,\omega)\Phi)(x) = (\omega\Phi)(\dot{x} * y) = (\omega\Phi)(\delta^{-1}xy) = \omega(\delta^{-1}xy).\Phi(\delta^{-1}xy),$$

for any element δ of $\widetilde{G}(\mathbb{Q})$. The representation of $G(\mathbb{A})$ on $L^2(X_G)$ decomposes into a discrete spectrum and a continuous spectrum:

$$L^2(X_G) = L^2_{\text{disc}}(X_G) \oplus L^2_{\text{cont}}(X_G).$$

We shall denote by $\Pi_{\text{disc}}(\widetilde{G},\omega)$ the equivalence classes of automorphic representations $\pi \in \Pi_{\text{disc}}(G)$ which extend to a twisted representation $\widetilde{\pi}$ of $\widetilde{G}(\mathbb{A})$. They are precisely those satisfying $\pi \sim \pi \circ \theta$ [LW13, Lemme 2.3.2].

Fix $P \in \mathcal{P}(M)$. The induced representation of $G(\mathbb{A})$ on $\mathcal{A}(X_P)$ is given by

$$(\rho_{P,\nu}(g)\Phi)(x) = \Phi(xg)\ \exp\langle \nu + \rho_P, H_P(xg) - H_P(x)\rangle .$$

It is isomorphic to $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \rho_{M,\text{disc}} \otimes \exp\langle \nu, H_M(\circ)\rangle$, $\rho_{M,\text{disc}}$ where $\rho_{M,\text{disc}}$ is the restriction of ρ_M to $L^2_{\text{disc}}(X_M)$.

A compactly supported smooth $\mathbf{K}$-invariant function h on $G(\mathbb{A})$ defines an operator $\rho_{P,\nu}(h)$ on $\overline{A}(X_P)$ by

$$\rho_{P,\nu}(h)(\Phi) = \iint_{X_P} h(x)\rho_{P,\nu}(x)(\Phi)\mathrm{d}x$$

whose image lies in the subspace of smooth $\mathbf{K}$-invariant functions.

We now define the twisted analog of ρ. Assume $P \in \mathcal{P}$ and $\delta \in \widetilde{G}(\mathbb{Q})$. Denote by Q the parabolic subgroup obtained by conjugation by δ, i.e., $Q = \delta P \delta^{-1} = \theta(P)$ where $\theta = \mathrm{Ad}(\delta)$. Let σ be an automorphic representation of M.

An element $y \in \widetilde{G}(\mathbb{A})$ defines an operator for $\nu \in \mathfrak{a}^*_{P,\mathbb{C}}$,

$$\widetilde{\rho}_{P,\sigma,\nu}(\delta, y, \omega) : \overline{\mathcal{A}}(X_P, \sigma) \to \overline{\mathcal{A}}(X_Q, \sigma \circ \theta^{-1})$$

by

$$(\widetilde{\rho}_{P,\sigma,\nu}(\delta, y, \omega)\Phi)(x) = \exp \langle \theta(\nu + \rho_P), H_Q(x) \rangle \, (\omega\Phi)(\delta^{-1}xy) \exp \;\; \nu + \rho_P, H_P(\delta^{-1}xy) \;\; .$$

Likewise, by integrating against a smooth function $f \in \mathcal{C}(\widetilde{G}(\mathbb{A}), K)$, we define the operator

$$\widetilde{\rho}_{P,\sigma,\nu}(\delta, f, \omega) = \int_{\widetilde{G}(\mathbb{A})} f(y) \widetilde{\rho}_{P,\sigma,\nu}(\delta, y, \omega) \mathrm{d}y$$

from the space $\mathcal{A}(X_P)$ to $\mathcal{A}(X_Q)$. Hopefully the notations of $\rho_{P,\nu}$ and $\widetilde{\rho}_{P,\sigma,\nu}$ for the induced representations of $G(\mathbb{A})$ and $\widetilde{G}(\mathbb{A})$ will not be confused with ρ_P, which is half of the sum of positive roots of P. [1]

[1] Our notation differs from [LW13] wherein $\widetilde{\rho}(f, \omega)$ stands for $\rho(\delta, f, \omega)$ for some $\delta \in \widetilde{G}(\mathbb{Q})$.

4. THE GEOMETRIC SIDE

4.1 STATEMENT OF THE GEOMETRIC CONTINUITY

Assume that $f \in \mathcal{C}_c^\infty(\widetilde{G}(\mathbb{A}))$ and denote

$$f^1(y) = \int_{A_G} f(yz)\mathrm{d}z$$

where as usual, A_G is the set of real points of the maximal $\mathbb{Q}$-split torus in the center of G or equivalently, the kernel of the map $H_G : G(\mathbb{A}) \to \mathfrak{a}_G$.

An element $\delta \in \widetilde{G}(\mathbb{Q})$ has a Jordan decomposition

$$\delta = s_\delta n_\delta = n_\delta s_\delta$$

where n_δ is a unipotent element of $G(\mathbb{Q})$ and $s_\delta \in \widetilde{G}(\mathbb{Q})$ is quasi-semisimple in that the automorphism $\mathrm{Ad}(s_\delta)$ induced on the derived group G_{der} is semisimple. Two elements of $\widetilde{G}(\mathbb{Q})$ are called coarse-conjugate if their quasi-semisimple parts are conjugate (in $G(\mathbb{Q})$). Denoting the set of equivalence classes by $\mathcal{O}$, the geometric side will be an expansion

$$J^{\widetilde{G},T}(f) = \sum_{\mathcal{O}\in\mathcal{O}} J^{\widetilde{G},T}_{\mathcal{O}}(f)$$

which we shall define and extend to the class $\mathcal{C}(\widetilde{G}(\mathbb{A}), K)$.

Following [CLL84, Lecture 1, 2, 9] we define the "basic identity" for $T \in \mathfrak{a}_0$ with $d_0(T) > d_0$ as

$$k^T_{\mathrm{geom}}(x) = k^{\widetilde{G},T}_{\mathrm{geom}}(x) = \sum_{\widetilde{P}\supseteq\widetilde{P}_0} (-1)^{a_{\widetilde{P}}-a_{\widetilde{G}}} \sum_{\xi\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \hat{\tau}_{\widetilde{P}}(H_0(\xi x) - T) k_{\widetilde{P}}(\xi x)$$

where

$$k_{\widetilde{P}}(x) = \int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} \sum_{\delta\in\widetilde{P}(\mathbb{Q})} \omega(x) f^1(x^{-1}\delta n x)\mathrm{d}n.$$

We can decompose $k^T_{\mathrm{geom}}(x)$ according to coarse conjugacy classes:

$$k^T_{\mathrm{geom}}(x) = \sum_{\mathfrak{o}\in\mathcal{O}} k^T_{\mathfrak{o}}(x)$$

where

$$k^T_{\mathfrak{o}(x)} = \sum_{\widetilde{P}\supseteq\widetilde{P}_0} (-1)^{a_{\widetilde{P}}-a_{\widetilde{G}}} \sum_{\xi\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \hat{\tau}_{\widetilde{P}}(H_0(\xi x)-T)k_{\widetilde{P},\mathfrak{o}}(\xi x),$$

$$k_{\widetilde{P},\mathfrak{o}}(x) = \int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} \sum_{\delta\in\widetilde{P}(\mathbb{Q})\cap\mathfrak{o}} \omega(x)f^1(x^{-1}\delta n x)\mathrm{d}n.$$

The last equality follows from a basic fact [Art78, p. 923] that

$$\widetilde{P}(\mathbb{Q})\cap\mathfrak{o} = (\widetilde{M}_P(\mathbb{Q})\cap\mathfrak{o})N_P(\mathbb{Q}).$$

We also set

$$k_{\mathrm{geom}}(x) = \sum_{\gamma\in\widetilde{G}(\mathbb{Q})} \omega(x)f^1(x^{-1}\gamma x); \quad k_{\mathfrak{o}}(x) = \sum_{\gamma\in\mathfrak{o}} \omega(x)f^1(x^{-1}\gamma x).$$

Following [CLL84], Labesse and Waldspurger [LW13] show that in the expression

$$J^T(f) = \sum_{\mathfrak{o}\in\mathcal{O}} \int_{X_G} k^T_{\mathfrak{o}}(x)\mathrm{d}x,$$

only finitely many coarse conjugacy classes $\mathfrak{o}\in\mathcal{O}$ give a nonzero contribution depending on the support of f and the integral is absolutely convergent for all $T\in\mathfrak{a}_0$ with $d_0(T)$ large enough.

By the partition lemma of Langlands and Arthur,

$$\sum_{P_0\subseteq Q\subseteq P} \sum_{\xi\in Q(\mathbb{Q})\backslash P(\mathbb{Q})} F^Q_{P_0}(\xi x, T)\tau^P_Q(H_0(\xi x)-T) = 1, \quad \forall x\in G(\mathbb{A}),$$

we obtain

$$k^T_{\mathfrak{o}}(x) = \sum_{\substack{\widetilde{P},Q:\\ P_0\subseteq Q\subseteq P}} \sum_{\xi\in Q(\mathbb{Q})\backslash P(\mathbb{Q})} (-1)^{a_{\widetilde{P}}-a_{\widetilde{G}}} F^Q_{P_0}(\xi x,T)\tau^P_Q(H_0(\xi x)-T)\hat{\tau}_{\widetilde{P}}(H_0(\xi x)-T)k_{\widetilde{P},\mathfrak{o}}(\xi x)$$

By a combinatorial identity of Langlands [LW13, Lemme 2.11.5], we have

$$\sum_{R\supseteq P} \sigma^R_P = \tau^P_Q\hat{\tau}_{\widetilde{P}}.$$

So making a change of variables, we can write

$$k_{\mathcal{O}}^T(x) = \sum_{Q \subseteq R} \sum_{\xi \in Q(\mathbb{Q}) \backslash G(\mathbb{Q})} F_{P_0}^Q(\xi x, T) \widetilde{\sigma}_Q^R(H_0(\xi x) - T) k_{\mathcal{O},Q,R}(\xi x)$$

where

$$k_{\mathcal{O},Q,R}(x) = \sum_{\widetilde{P} \supseteq \widetilde{P}_0 : Q \subseteq P \subseteq R} (-1)^{a_{\widetilde{P}} - a_{\widetilde{G}}} k_{\widetilde{P},\mathcal{O}}(x).$$

The twisted version of Finis-Lapid's extension of the geometric side is as follows.

Theorem 4.1.1 *Assume the Root Cone Lemma (Lemma 4.3.2) holds for the pair (G, θ).*

1. *For any $f \in \mathcal{C}(\widetilde{G}(\mathbb{A}), K), \mathfrak{o} \in \mathcal{O}$ and any $T \in \mathfrak{a}_0$ suitably large multiple of the sum of positive coroots (see Theorem 4.3.5), the integrals*

$$J_{geom}^T(f) = \int_{X_G} k_{geom}^T(x) dx \text{ and } \quad J_{\mathcal{O}}^T(f) = \int_{X_G} k_{\mathcal{O}}^T(x) dx$$

are absolutely convergent.

2. *$J_{geom}^T(f)$ and $J_{\mathcal{O}}^T(f)$ are polynomials in T of degree $\leq a_{\widetilde{P}_0} - a_{\widetilde{G}}$ whose coefficients are continuous linear forms in f.*

3. *There is $r \geq 0$ and a continuous seminorm μ on $\mathcal{C}(\widetilde{G}(\mathbb{A}), K)$ such that*

$$\sum_{\mathcal{O} \in \mathcal{O}} \int_{X_G^T} k_{\mathcal{O}}(x) - J_{\mathcal{O}}^T(f) \ \leq \sum_{\mathcal{O} \in \mathcal{O}} \int_{X_G} F^G(x,T) k_{\mathcal{O}}(x) dx - k_{\mathcal{O}}^T(x) \ dx$$

$$\leq \mu(f)(1 + \|T\|)^r \exp(-d_0(T)),$$

for any $f \in \mathcal{C}(\widetilde{G}(\mathbb{A}), K)$ and any $T \in \mathfrak{a}_0$ "suitably large" .

4. *$J_{geom}^T(f) = \sum_{\mathcal{O} \in \mathcal{O}} J_{\mathcal{O}}^T(f)$.*

In addition, the upper bound on the seminorm μ and the coefficients in part (2) depends on the level of K in the same way as in the non-twisted case [FL16, Theorem 5.1].

We defer the proof of this theorem to Section 4.5.

4.2 A FEW TECHNICAL RESULTS

In this section, we review some definitions and lemmas that will go into the proof of the geometric side.

The modulus character

For $w \in W$ let δ_w denote the modulus function of $M_0(\mathbb{A})^1$ on $N_w(\mathbb{A}) \backslash N_0(\mathbb{A})$ where $N_w = N_0 \cap wN_0w^{-1}$. In particular, if $w = w_0$ is the long element in W, we denote δ_{w_0} by δ_0. Denote also by $\delta_{M_0,N}$ the modulus function of $M_0(\mathbb{A})$ on the unipotent radical N of any parabolic subgroup of G. The following lemma is easy to prove, cf. [Sha10, §4.1].

Lemma 4.2.1 *1.* $\delta_w = \frac{1}{2} \sum_{\substack{\alpha>0, \\ w^{-1}\alpha>0}} \alpha, \ \delta_{M_0,N_w} = \dfrac{1}{2} \sum_{\substack{\alpha>0 \\ w^{-1}\alpha<0}} \alpha.$

2. $\delta_0 = \delta_w . \delta_{M_0,N_w}$.

Lemma 4.2.2

$$\delta_0(a^{-1}n_w b n_w^{-1})\delta_{w^{-1}}(a^2) = \delta_{w^{-1}}(ab^{-1})\delta_{M_0,N_{w^{-1}}}(a^{-1}b),$$

for any representative n_w of $w \in W$ and any $a, b \in A_0$.

Proof Using Lemma 4.2.1, we have

$$\begin{aligned}\delta_0(a^{-1})\delta_{w^{-1}}(a^2) &= \delta_{w^{-1}}(a^{-1})\delta_{M_0,N_{w^{-1}}}(a^{-1})\delta_{w^{-1}}(a^2) \\ &= \delta_{w^{-1}}(a)\delta_{M_0,N_{w^{-1}}}(a^{-1})\end{aligned}$$

Following the proof of lemma 2.1 in [FL11a] we can write

$$\begin{aligned}\delta_0(n_w b n_w^{-1}) &= \delta_{M_0,N_{w^{-1}}}(b)\delta_w(n_w b n_w^{-1}) \\ &= \delta_{M_0,N_{w^{-1}}}(b)\delta_{w^{-1}}(b^{-1}).\end{aligned}$$

Multiplying the two gives the desired equality. ■

Twisted Bruhat decomposition

The Bruhat decomposition in the twisted case is similar to the usual case. Since the minimal parabolic P_0 is chosen to be θ_0-stable, any element $\widetilde{w} = w\delta_0$ of the Weyl set $\widetilde{W} = W^{\widetilde{G}}$ gives a twisted Bruhat cell

$$C(\widetilde{w}) = P_0\widetilde{w}P_0 = P_0(w\delta_0)P_0 = (P_0wP_0)\delta_0;$$

and $\widetilde{G}$ is the union of such cells $C(\widetilde{w})$.

Note that subsets of Δ_{P_0} are not always in bijection with standard parabolic subsets but one needs to consider θ_0-stable subsets of Δ_{P_0}. If so, one gets for $Q \in \mathcal{P}$, parabolic subgroups $Q^- \subseteq Q \subseteq Q^+$ and corresponding to the subsets $\Delta_{P_0}^{Q^-}, \Delta_{P_0}^{Q^+}$ introduced in Chapter 2 which are stable under θ_0. Indeed, standard θ_0-stable parabolic subsets are the right parabolic subsets one needs to consider to get the alternating sum in the kernel of the twisted trace formula. Following the notation of [LW13, p. 133], for $Q \in \mathcal{P}$ define

$$\widetilde{G}(Q,G) := \widetilde{G}(\mathbb{Q}) \setminus \bigcup_{\widetilde{Q}^+ \subseteq \widetilde{P}' \subsetneq \widetilde{G}} \widetilde{P}'(\mathbb{Q}).$$

Being a bi-$Q(\mathbb{Q})$-invariant set $\widetilde{G}(Q,G)$ is a finite disjoint union of twisted Bruhat cells $C(\widetilde{w})$ over $\widetilde{w} \in \widetilde{W}(Q,G)$ or equivalently, over $w \in W$ satisfying $Q.Q(w) = G$, where $Q(w)$ is the smallest standard parabolic subgroup containing w.

Mellin transform

For a function $F \in \mathcal{C}_c^\infty(G(\mathbb{A}))$, we recall the definition of Mellin transform on A_0 and inversion formula:

$$\phi(\lambda)(g) = \int_{A_0} F(ag)a^{\lambda+\rho_0}\mathrm{d}a$$

The function can be recovered by the inverse-Mellin transform

$$F(ag) = \int_{\mathrm{Re}\,\lambda=\lambda_0} \phi(\lambda)(g)\delta_0(a)^{\frac{1}{2}}a^\lambda \mathrm{d}\lambda,$$

where $\lambda_0 \in \mathfrak{a}_0^*$ and for convenience, we have denoted $\exp(\langle\lambda, H_0(a)\rangle)$ by a^λ.

Intertwining operators

We briefly recall the properties of intertwining operators; following [FL16], we will need it for principal series representations only. Later in the analysis of the spectral side we will define them more generally for any two associated parabolic subgroups. The space of representations parabolically induced from $P_0(\mathbb{A})$ is defined by

$$I(\lambda) = \{\varphi : G(\mathbb{A}) \to \mathbb{C} \text{ smooth} \mid \phi(pg) = \exp\langle \lambda + \rho_0, H_0(p)\rangle \phi(g) \quad \text{for every } p \in P_0(\mathbb{A}), g \in G(\mathbb{A}).\}$$

The intertwining operator is a map

$$M(w,\lambda) : I(\lambda) \to I(w\lambda)$$

given by

$$M(w,\lambda)\phi(g) = \int_{N_w(\mathbb{A})\backslash N_0(\mathbb{A})} \phi(n_w^{-1}ng)\mathrm{d}n.$$

It is well-known that the integral over λ is a product of local integrals and converges for λ in the positive Weyl chamber "sufficiently away from the origin". It extends meromorphically to $\mathfrak{a}^*_{0,\mathbb{C}}$ with only simple poles which occur on the root hyperplanes [MW95, IV.1]. Moreover,

$$M(w,\lambda)\phi = m(w,\lambda)\phi$$

where

$$m(w,\lambda) = \prod_{\substack{\alpha\in\mathcal{R}_0 \\ w^{-1}\alpha<0}} m_\alpha(\langle \lambda, \alpha^\vee\rangle)$$

and if λ_0 is in the positive Weyl chamber of $\mathfrak{a}_0^*$ with $\langle \lambda_0 - \rho_0, \varpi^\vee\rangle > 0$ for every $\varpi^\vee \in \hat{\Delta}_0^\vee$, the function

$$\lambda \mapsto \prod_{\substack{\alpha\in\Delta_0 \\ w^{-1}\alpha<0}} \langle \lambda, \alpha^\vee\rangle\, m(w,\lambda) \tag{4.1}$$

is holomorphic and of moderate growth on $\|\operatorname{Re}\lambda - \lambda_0\| < \epsilon$ for some $\epsilon > 0$ sufficiently small. See [MW95, IV.1.11] and [HC68, Lemma 101] for details.

4.3 ROOT CONE LEMMA

The Root Cone Lemma 4.3.2 will be used to prove the finiteness of derivatives of $\phi_{T,Q,\ell}(\lambda)$ in Theorem 5.1.1. We will prove this for various pairs (G, θ_0) in Chapter 6 including all cases when G is split semisimple. Note that this lemma depends only on the semisimple part of G. The continuity of the geometric side for groups G which are quasisplit but not split is conditional on proving this lemma which we assume to hold in this section.

Lemma 4.3.1 *1. If λ is any vector in $\mathfrak{a}_0^*$ then*

$$\sum_{\varpi^\vee \in \hat{\Delta}_0^\vee} \left\langle (1-\theta_0^{-1})\lambda, \varpi^\vee \right\rangle = 0.$$

2. Suppose $w \in W, w \neq 1$ and λ is in the (open) positive Weyl chamber of $\mathfrak{a}_0^*$ then

$$\sum_{\beta^\vee \in \Delta_0^\vee} \left\langle \lambda - \theta_0^{-1} w^{-1} \lambda, \varpi_\beta^\vee \right\rangle > 0.$$

Proof The first statement follows because θ_0^{-1} is a permutation on the set Δ_0 or equivalently on $\hat{\Delta}_0^\vee$. The two parts of the lemma estimate the sum of coefficients of respective vectors expressed in the basis $\{\beta \in \Delta_0\}$ of roots. We write $\lambda - \theta_0^{-1} w^{-1}\lambda$ as a sum of $\lambda - w^{-1}\lambda$ and $(1-\theta_0^{-1})(w^{-1}\lambda)$. If $w \neq 1$ then $\Delta_0^{Q(w)}$ is nonempty so by [Bou02, Ch. VI §1.6 Proposition 18] and the choice of λ, the inner product $\sum_{\beta^\vee \in \Delta_0^\vee} \left\langle \lambda - w^{-1}\lambda, \varpi_\beta^\vee \right\rangle$ is positive. The other inner product sum $\sum_{\beta^\vee \in \Delta_0^\vee} \left\langle (1-\theta_0^{-1})(w^{-1}\lambda), \varpi_\beta^\vee \right\rangle$ vanishes using Part 1. ■

Recall that $\Delta_0^{Q(w)}$ is the subset of Δ_0 corresponding to the smallest standard parabolic subgroup $Q(w)$ containing a representative of $w \in W$. For $\lambda, \gamma \in \mathfrak{a}_0^*$ set

$$\gamma(\lambda, w, \theta_0) := \lambda - \theta_0^{-1} w^{-1}\lambda - \gamma(w, \theta_0).$$

Lemma 4.3.2 (Root Cone Lemma) *For $w \in W, w \neq 1$, there exists an open cone Ω_0 inside the positive Weyl chamber $(\mathfrak{a}_0^*)^+$ in $\mathfrak{a}_0^*$ such that for every $\lambda \in \Omega_0$ and every $\beta \in \Delta_0^{Q(w)}$,*

$$\lambda - \theta_0^{-1}w^{-1}\lambda, \varpi_\beta^\vee \quad > 0. \tag{4.2}$$

Remark 4.3.3 *By choosing $\lambda \in (\mathfrak{a}_0^*)^+$ suitably away from the origin we can ensure for fixed $\gamma \in \mathfrak{a}_0^*$ that*

$$\lambda - \theta_0^{-1}w^{-1}\lambda - \gamma, \varpi_\beta^\vee \quad > 0$$

for all $\beta \in \Delta_0^{Q(w)}$. Throughout this section fix the open subset Ω_γ of points λ satisfying this condition. We need the RCL to get the two estimates below.

Lemma 4.3.4 *Assume $\gamma \in \mathfrak{a}_0^*, Q \in \mathcal{P}$ and $w \in \tilde{W}(Q,G)$. Then for every $\lambda \in \mathfrak{a}_0^*$ with $\mathrm{Re}(\lambda) \in \Omega_\gamma$, the integral*

$$\psi_{T,Q,l}(\lambda) := \int_{\mathfrak{a}_Q} \exp \quad X_Q, -(\lambda - \theta_0^{-1}w^{-1}\lambda) + \gamma \quad \tau_Q(X_Q - T) \sum_{\alpha \in \Delta_Q} \langle \alpha, X_Q - T \rangle^l \, dX_Q$$

converges absolutely.

Proof By the condition on Q, we have $\Delta_0^Q \cup \Delta_0^{Q(w)} = \Delta_0$. We can apply the Root Cone Lemma 4.3.2 to obtain $\lambda \in (\mathfrak{a}_0^*)^+$ so that the right hand term in the inner product above is positive. Since elements of Δ_Q are restrictions of elements in $\Delta_0 \setminus \Delta_0^Q \subseteq \Delta_0^{Q(w)}$ to $\mathfrak{a}_Q$, the exponential term is negative whenever X_Q is in the cone defined by τ_Q. It thus dominates the polynomial term, giving the required absolute convergence. ■

Lemma 4.3.5 *There is an unbounded subset of the line $\mathbb{R}(\sum_{\hat\varpi^\vee \in \hat\Delta_0^\vee} \hat\varpi^\vee)$ in $\mathfrak{a}_0$ independent of $w \in W$ such that if we set $\gamma(\lambda, w, \theta_0) = \lambda - \theta_0^{-1}w^{-1}\lambda - \gamma(w, \theta_0)$ where $\lambda \in \Omega_\gamma$ is chosen satisfying Theorem 4.3.3 then $\langle \gamma(\lambda, w, \theta_0), T \rangle > 0$ whenever T belongs to this set.*

Proof Up to a positive number, the above inner product is the sum of coordinates of $\gamma(\lambda, w, \theta_0)$ in the basis Δ_0 of roots and the estimate follows by applying Part 1 (respectively Part 2) of Theorem 4.3.1 to the vector $\gamma(w, \theta_0)$ (resp. $\lambda - \theta_0^{-1}w^{-1}\lambda$). The independence on w is also from Part 2. ■

4.4 TWO THEOREMS

We now state two theorems which will give crucial estimates towards proving the main result on the geometric side and they will be proven in Chapter 5.

Theorem 4.4.1 *There exists an integer $r \geq 0$, a vector $\xi(Q) \in \mathfrak{a}_0^*$ with $\langle \xi(Q), \beta \rangle > 0$ for every $\beta \in \Delta_Q$ and a seminorm μ on $\mathcal{C}(\widetilde{G}(\mathbb{A}), K)$ such that for every $Q \in \mathcal{P}$ and $l \geq 0$,*

$$\int_{Y_Q} F^Q(x,T)\tau_Q(H_Q(x) - T)\|H_Q(x) - T_Q\|^l \sum_{\gamma \in \widetilde{G}(Q,G)} |f^1(x^{-1}\gamma x)|dx$$
$$\ll (1 + \|T\|)^r \exp - \langle \xi(Q), T \rangle \mu(f), \quad (4.3)$$

holds for every $f \in \mathcal{C}(\widetilde{G}(\mathbb{A}), K)$ and $T \in \mathfrak{a}_0$ suitably large multiple of the sum of positive coroots (see Theorem 4.3.5). Moreover, μ satisfies the same bound as in the non-twisted case.

Remark 4.4.2 *In the above sum, recall that*

$$\widetilde{G}(Q,G) := \widetilde{G}(\mathbb{Q}) \setminus \bigcup_{\widetilde{Q}^+ \subseteq \widetilde{P}' \subsetneq \widetilde{G}} \widetilde{P}'(\mathbb{Q}).$$

If $Q = G$ then the inequality reduces to

$$\int_{X_G} F^G(x,T) \sum_{\gamma \in \widetilde{G}(\mathbb{Q})} |f^1(x^{-1}\gamma x)|dx \leq \mu(f)(1 + \|T\|)^r \exp\left(-d_0(T)\right)$$

and the LHS is just $\int_{X_G^T} \sum_{\mathcal{O} \in \mathcal{O}} |k_{\mathcal{O}}(x)|dx$.

Theorem 4.4.3 *Let $\widetilde{Q}$ be a standard parabolic subset and $\widetilde{w} = w\delta_0 \in \widetilde{W}(Q,G)$. Then there is an integer $r \geq 0$, a vector $\xi(Q) \in \mathfrak{a}_0^*$ with $\langle \xi(Q), \beta \rangle > 0$ for all $\beta \in \Delta_Q$ and a seminorm μ on $\mathcal{C}(\widetilde{G}(\mathbb{A}), K)$ such that*

$$\int_{N_w(\mathbb{A}) \setminus N_0(\mathbb{A})} \int_{A_0} \int_{N_0(\mathbb{A})} \int_{M_0(\mathbb{A})^1} |f^1(a^{-1}n^{-1}n_{\widetilde{w}}aum)|\chi(a)\, dm\, du\, da\, dn$$
$$\ll_{K,l} \mu(f)(1 + \|T\|)^r \exp - \langle \xi(Q), T \rangle,$$

holds for every $l \geq 0$, $f \in \mathcal{C}(\widetilde{G}(\mathbb{A}), K)$ and $T \in \mathfrak{a}_0$ a suitably large multiple of the sum of positive coroots (see Theorem 4.3.5). Here, $N_w = N_0 \cap n_w N_0 n_w^{-1}$ and

$$\chi(a) = \chi_{T,Q,l}(a) = \tau_Q(H_Q(a) - T)\, \hat{\tau}_{P_0}^Q(T - H_0(a))\tau_{P_0}^Q(H_0(a) - T_1) \sum_{\alpha \in \Delta_Q} \langle H_Q(a) - T_Q, \alpha \rangle^l .$$

4.5 CONTINUITY OF THE GEOMETRIC SIDE

In this section, we prove Theorem 4.1.1 that the distribution $J_{\text{geom}}^{\widetilde{G}}(f)$ initially defined for $f \in \mathcal{C}_c^\infty(\widetilde{G}(\mathbb{A}))$ extends to $\mathcal{C}(\widetilde{G}(\mathbb{A}), K)$. As in the non-twisted case, the proof involves modifying Arthur's (or rather, Labesse-Waldspurger's) proof in the compactly supported setting to our case. We imitate the method of Finis-Lapid [FL16] whenever possible.

Proof [of Theorem 4.1.1] For $\widetilde{P} \supseteq \widetilde{P}_0$ and $f \in \mathcal{C}(\widetilde{G}(\mathbb{A}), K)$ we could replace the sum over $\widetilde{M}_P(\mathbb{Q})$ in the definition of

$$k_{\widetilde{P}}(x) = \int_{N_P(\mathbb{A})} \sum_{\gamma \in \widetilde{M}_P(\mathbb{Q})} \omega(x) f^1(x^{-1}\gamma n x) \mathrm{d}n$$

by an integral over $\widetilde{M}_P(\mathbb{A})$ of a finite sum of derivatives of f, following Theorem 3.1.2 and [FL16, Lemma 2.1(1)]. Thus each $k_{\widetilde{P}}(x)$ hence $k_{\text{geom}}^T(x)$ is well-defined. We can formally write

$$\sum_{\mathcal{O} \in \mathcal{O}} J_{\mathcal{O}}^T(f) \leq \sum_{\mathcal{O} \in \mathcal{O}} \left| J_{\mathcal{O}}^T(f) - \int_{X_G} k_{\mathcal{O}}^T(x) \mathrm{d}x \right| + \sum_{\mathcal{O} \in \mathcal{O}} \int_{X_G} |k_{\mathcal{O}}^T(x)| \mathrm{d}x.$$

By Theorem 4.4.2, it follows that to prove $J_{\text{geom}}^T(f)$ and $J_{\mathcal{O}}^T(f)$ exist and the relation $J_{\text{geom}}^T(f) = \sum_{\mathcal{O} \in \mathcal{O}} J_{\mathcal{O}}^T(f)$, it suffices to prove part (3). Part (2) is a formal property which holds whenever $J_{\mathcal{O}}^T(f)$ is absolutely convergent, cf. [LW13, Théorème 11.1.1]. We now prove part (3).

The first inequality is obvious. Recall that

$$k_{\mathcal{O}}^T(x) = \sum_{Q \subseteq R} \sum_{\xi \in Q(\mathbb{Q}) \backslash P(\mathbb{Q})} F_{P_0}^Q(\xi x, T) \widetilde{\sigma}_Q^R(H_0(\xi x) - T) k_{\mathcal{O},Q,R}(\xi x).$$

Using twisted Levi decomposition we can write

$$k_{\mathcal{O},Q,R}(x) = \sum_{\substack{S:\\ Q\subseteq S\subseteq R}} k_{\mathcal{O},Q,R,S}(x)$$

where

$$k_{\mathcal{O},Q,R,S}(x) = \sum_{\eta\in \widetilde{M}_S(Q,S)} \sum_{\substack{\widetilde{P}:\\ \widetilde{S}\subseteq\widetilde{P}\subseteq\widetilde{R}^-}} \sum_{\nu\in N_P^S(\mathbb{Q})} \int_{N_P(\mathbb{A})} \omega(x) f^1(x^{-1}\eta\nu n x)\mathrm{d}n,$$

$$\widetilde{M}_S(Q,S) = \widetilde{M}_S(\mathbb{Q}) \setminus \bigcup_{\substack{\widetilde{P}':\\ \widetilde{Q}^+\subseteq\widetilde{P}'\subsetneq\widetilde{S}}} \widetilde{P}'(\mathbb{Q}).$$

Here we are using that if $\widetilde{P} \supseteq \widetilde{P}_0$ is such that $Q \subseteq P \subseteq R$ then $\widetilde{Q}^+ \subseteq \widetilde{P} \subsetneq \widetilde{R}^-$. Note that if $Q = R$ then $\widetilde{\sigma}_Q^R$ vanishes unless $Q = R = G$ in which case it is 1. Hence the contribution from $Q = R$ is precisely

$$F^G(x,T)k_{\mathcal{O},G,G}(x) = F_{P_0}^G(x,T)k_{\mathcal{O}}(x).$$

Thus,

$$F_{P_0}^G(x,T)k_{\mathcal{O}}(x) - k_{\mathcal{O}}^T(x) = \sum_{Q\subsetneq R}\sum_{\xi\in Q(\mathbb{Q})\setminus G(\mathbb{Q})} F_{P_0}^Q(\xi x,T)\widetilde{\sigma}_Q^R(H_0(\xi x)-T)k_{\mathcal{O},Q,R}(\xi x).$$

Making a change of variables,

$$\sum_{\mathcal{O}\in\mathcal{O}} \int_{X_G} F(x,T)k_{\mathcal{O}}(x) - k_{\mathcal{O}}^T(x)\ \mathrm{d}x = \sum_{\mathcal{O}\in\mathcal{O}}\sum_{Q\subsetneq R} \int_{Y_Q} F_{P_0}^Q(x,T)\widetilde{\sigma}_Q^R(H_0(x)-T)|k_{\mathcal{O},Q,R}(x)|\mathrm{d}x.$$

Fixing $Q \subseteq S \subseteq R$ with $Q \neq R$, we need to estimate

$$\sum_{\mathcal{O}\in\mathcal{O}} \int_{Y_Q} F^Q(x,T)\widetilde{\sigma}_Q^R(H_0(x)-T)|k_{\mathcal{O},Q,R,S}(x)|\mathrm{d}x.$$

We can assume that $\widetilde{\sigma}_Q^R(H_Q(x)-T)$ is 1 and so is $\tau_Q^R(H_Q(x)-T)$. Now invoke [FL16, Corollary 4.7], the hypotheses that $F^Q(x,T)\tau_Q^R(H_Q(x)-T) = 1$ being satisfied. (Since the dependence of μ on the level of K in the regular (non-twisted) case is only via this Corollary whose twisted equivalent remains same, the corresponding bound in the twisted case remains the same.) The aforementioned Corollary tracks Arthur's proof in [Art78, Art81] by applying Iwasawa decomposition to x which is justified since the expression above is left invariant under $Q(\mathbb{Q})N_R(\mathbb{A})$. Up to a constant whose dependence on K is tracked in [FL16, Corollary 4.7],

$$|k_{\mathcal{O},Q,R,S}(x)| \ll_K \exp - \left(\langle \xi_S^R, T \ + \ (\xi_S^R)_Q, H_0(x) - T\rangle\right) \times \sum_{\eta \in \widetilde{M}_S(Q,S)} \left(\int_{N_S(\mathbb{A})} |f^1(x^{-1}\eta n x)| \mathrm{d}n \mathrm{d}x \right).$$

The remaining steps to reduce this to Theorem 4.4.1 are the same as in [FL16, p. 21]. By using Theorem 3.1.2, we can invoke [FL11a, Lemma 3.4] to assume $f \geq 0$ and $K = \mathbf{K}_f$. Further, we can apply Iwasawa decomposition with respect to S and use lemma 4.8 of [FL16] as is. The bound now follows by Theorem 4.4.1.

■

5. PROOFS OF THEOREMS 4.4.1 AND 4.4.3

We will apply the twisted Bruhat decomposition to Theorem 4.4.1 to reduce it to Theorem 4.4.3. The proof of Theorem 4.4.3 involves estimating the integrals over twisted Bruhat cells. We will apply the Mellin transform and use the slow growth of intertwining operators with the estimate of [FL16, Proposition 3.4] to get the required estimate.

5.1 REDUCTION OF THEOREM 4.4.1 to THEOREM 4.4.3

The estimate below for $Q \in \mathcal{P}$ and any left $Q(\mathbb{Q})-$invariant measurable function f on $G(\mathbb{A})^1$,

$$\int_{Y_Q} |f(x)|\mathrm{d}x \leq \int_{\mathbf{K}} \int_{N_0(\mathbb{Q})\backslash N_0(\mathbb{A})} \int_{A_0} \int_{X_M} |f(namk)|\delta_0(a)^{-1}\tau_{P_0}^{Q}(H_0(a)-T_1)\mathrm{d}m\mathrm{d}a\mathrm{d}u\mathrm{d}k$$

which occurs in [FL16, Equation 2] remains true in the twisted case. Indeed, averaging over $\widetilde{G}(Q,G)$ makes the integrand on the left hand side of Equation (4.3) bi-$Q(\mathbb{Q})$-invariant. Applying this estimate we need to bound

$$\int_{\mathbf{K}} \int_{N_0(\mathbb{Q})\backslash N_0(\mathbb{A})} \int_{A_0} \int_{X_M} \sum_{\gamma \in \widetilde{G}(Q,G)} \left|f^1\left((namk)^{-1}\gamma(namk)\right)\right| \chi(a)\delta_0(a)^{-1}\mathrm{d}m\mathrm{d}a\mathrm{d}u\mathrm{d}k.$$

Here,

$$\chi(a) = F^Q(a,T)\ \tau_Q(H_Q(a)-T)\ \tau_{P_0}^{Q}(H_0(a)-T) \sum_{\alpha \in \Delta_Q} \langle H_Q(a)-T_Q, \alpha\rangle^l .$$

Ignoring the integration on the compact sets $\mathbf{K}$ and X_{M_0} by [FL16, Prop. 2.1(2)] we are reduced to bounding the integral

$$\int_{N_0(\mathbb{Q})\backslash N_0(\mathbb{A})} \int_{A_0} \sum_{\gamma \in \widetilde{G}(Q,G)} \left|f^1\left((na)^{-1}\gamma na\right)\right| \chi(a)\delta_0(a)^{-1}\mathrm{d}a\mathrm{d}n.$$

Applying the twisted Bruhat decomposition to $\widetilde{G}(Q,G)$ by Section 4.2, we need to estimate for fixed $\widetilde{w} = w\delta_0 \in \widetilde{G}(Q,G)$, the integral over $N_0(\mathbb{Q}) \setminus N_0(\mathbb{A})$ and A_0 of the sum

$$\sum_{u_2 \in N_w(\mathbb{Q})\setminus N_0(\mathbb{Q})} \sum_{u_1 \in N_0(\mathbb{Q})} \sum_{m \in M_0(\mathbb{Q})} |f^1(a^{-1}n^{-1}u_2^{-1}mn_{\widetilde{w}}u_1na)|\ \chi(a)\delta_0(a)^{-1}.$$

Applying [FL16, Lemma 3.3] to the translate $g(u_1) = f^1(bu_1)$ with $b = n^{-1}u_2^{-1}mn_{\widetilde{w}}$ allows us to replace the sum over $u_1 \in N_0(\mathbb{Q})$ by the integral of functions $g * X$ for X ranging over a finite set of differential operators. Replacing g by one such derivative (and that with f^1) reduces to bounding the sum-integral

$$\int_{N_0(\mathbb{Q})\setminus N_0(\mathbb{A})} \int_{A_0} \sum_{u_2 \in N_w(\mathbb{Q})\setminus N_0(\mathbb{Q})} \int_{u_1 \in N_0(\mathbb{A})} \sum_{m \in M_0(\mathbb{Q})} |f^1(a^{-1}n^{-1}u_2^{-1}mn_{\widetilde{w}}u_1a)|\ \chi(a)\delta_0(a)^{-1}\mathrm{d}a\mathrm{d}n$$

$$= \int_{N_0(\mathbb{Q})\setminus N_0(\mathbb{A})} \int_{A_0} \sum_{u_2 \in N_w(\mathbb{Q})\setminus N_0(\mathbb{Q})} \int_{u_1 \in N_0(\mathbb{A})} \sum_{m \in M_0(\mathbb{Q})} |f^1(a^{-1}n^{-1}u_2^{-1}mn_{\widetilde{w}}au_1)|\ \chi(a)\mathrm{d}a\mathrm{d}n.$$

(Combining the sum over u_2 and the integral over n gives),

$$\int_{N_w(\mathbb{Q})\setminus N_0(\mathbb{A})} \int_{A_0} \int_{u_1 \in N_0(\mathbb{A})} \sum_{m \in M_0(\mathbb{Q})} |f^1(a^{-1}n^{-1}mn_{\widetilde{w}}au_1)|\ \chi(a)\mathrm{d}a\mathrm{d}n.$$

Note that as a function of n, the inner integral is left $N_w(\mathbb{A})$-invariant. Hence we can write

$$\int_{N_w(\mathbb{A})\setminus N_0(\mathbb{A})} \int_{A_0} \int_{u_1 \in N_0(\mathbb{A})} \sum_{m \in M_0(\mathbb{Q})} |f^1(a^{-1}n^{-1}mn_{\widetilde{w}}au_1)|\ \chi(a)\mathrm{d}a\mathrm{d}n.$$

Also,

$$\begin{aligned} a^{-1}u^{-1}mn_wau_1 &= a^{-1}u^{-1}(n_wm')au_1 = a^{-1}u^{-1}n_wam'u_1 \\ &= a^{-1}u^{-1}n_wa(m'u_1m'^{-1})m' = a^{-1}u^{-1}n_wau_2m'. \end{aligned}$$

Thus we need to bound

$$\int_{N_w(\mathbb{A})\setminus N_0(\mathbb{A})} \int_{A_0} \int_{u_1 \in N_0(\mathbb{A})} \sum_{m \in M_0(\mathbb{Q})} |f^1(a^{-1}n^{-1}n_{\widetilde{w}}au_1m)|\ \chi(a)\mathrm{d}a\mathrm{d}n,$$

which by [FL16, Lemma 2.1(1)] and Theorem 3.1.2 reduces to a finite sum of derivatives. Replacing f by one such derivative as before, we need to bound

$$\int_{N_w(\mathbb{A})\setminus N_0(\mathbb{A})} \int_{A_0} \int_{u_1 \in N_0(\mathbb{A})} \int_{M_0(\mathbb{A})^1} |f^1(a^{-1}n^{-1}n_{\widetilde{w}}au_1m)|\ \chi(a)\mathrm{d}m\mathrm{d}a\mathrm{d}n,$$

which reduces to Theorem 4.4.3. ■

For a finite dimensional space V, let $\mathcal{D}(V)$ denote the space of invariant diferential operators on V with the standard filtration. The lemma below is a modification of lemma 3.5 of [FL16] in the twisted setting.

Lemma 5.1.1 *Suppose Q be a standard parabolic subgroup of G and $\tilde{w} = w\delta_0 \in \tilde{W}(Q,G)$. Let $\Omega = \Omega_\gamma$ satisfy Theorem 4.3.3 for*

$$\gamma = \gamma(w,\theta_0) = \frac{1}{2}(1-\theta_0^{-1})\left(\left(\sum_{\alpha, w\alpha>0}\left(\alpha - \sum_{\substack{\beta>0\\ w\beta<0}} \beta\right)\right)\right.$$

Then the integral

$$\varphi_{T,Q,l}(\lambda) := \int_{X\in\mathfrak{a}_0} \exp\langle X, -\gamma(\lambda,w,\theta_0)\rangle \tau_Q(X-T)\ \hat{\tau}_{P_0}^Q(T-X)\tau_{P_0}^Q(X-T_1) \sum_{\alpha\in\Delta_Q} \langle X_Q - T_Q, \alpha\rangle^l\, dX,$$

is absolutely convergent for $\mathrm{Re}(\lambda)$ *in compact subsets of Ω and T in the unbounded set of Theorem 4.3.5. Moreover, for fixed $\Lambda_0 \in \Omega$ and any differential operator $D \in \mathcal{D}(\mathfrak{a}_0^*)$ of degree d, there is a vector $\xi(Q)$ such that $\beta(\xi(Q)) > 0$ for all $\beta \in \Delta_Q$ and*

$$|(\varphi_{T,Q,l} * D)(\lambda)| \ll_{D,l} (1+\|T\|)^{d+\mathfrak{a}_0^Q} \exp -\langle T, \xi(Q)\rangle$$

where $\mathrm{Re}(\lambda) = \Lambda_0$.

Proof Similar to [FL16, Lemma 3.5], we can use the decomposition $\mathfrak{a}_0 = \mathfrak{a}_Q \oplus \mathfrak{a}_0^Q$ to write $X = X_Q + X^Q$ and

$$\varphi_{T,Q,l}(\lambda) = \psi_T^Q(\lambda).\psi_{T,Q,l}(\lambda)$$

where

$$\psi_T^Q(\lambda) := \int_{\mathfrak{a}_0^Q} \exp\ X^Q, -\gamma(\lambda,w,\theta_0)\ \tau_0^Q(X^Q - T_1)\hat{\tau}_0^Q(T - X^Q)\mathrm{d}X^Q,$$

and

$$\psi_{T,Q,l}(\lambda) := \int_{\mathfrak{a}_Q} \exp\langle X_Q, -\gamma(\lambda,w,\theta_0)\rangle\, \tau_Q(X_Q - T) \sum_{\alpha\in\Delta_Q} \langle \alpha, X_Q - T\rangle^l\, \mathrm{d}X_Q.$$

The function $X^Q \mapsto \tau_0^Q(X^Q - T_1)\hat{\tau}_0^Q(T - X^Q)$ is the convex hull of the set $\{T_S^Q + T_1^S : P_0 \subseteq S \subseteq Q\}$ [Art81, Section 6], hence the integral $\psi_T^Q(\lambda)$ is it's Fourier transform evaluated at $-\gamma(\lambda, w, \theta_0)$ which is also the smooth function corresponding to the aforementioned (M_Q, M_0)-orthogonal set. In particular it is compactly supported and using Theorem 4.3.4, we have the convergence of $\varphi_{T,Q,l}(\lambda)$. It remains to estimate the derivatives. Assume $D = D^Q D_Q$ where $D^Q \in \mathcal{D}((\mathfrak{a}_0^Q)^*)$ and $D_Q \in \mathcal{D}(\mathfrak{a}_Q^*)$ and let $d = d^Q + d_Q$ be their degrees. By loc. cit. $\psi_T^Q(\lambda)$ equals

$$\sum_{S:P_0\subseteq S\subseteq Q} \exp\left\langle T_S^Q + T_1^S, -\gamma(\lambda, w, \theta_0)\right\rangle \epsilon_Q(-\gamma(\lambda, w, \theta_0))$$

$$= \sum_{S:P_0\subseteq S\subseteq Q} \mathrm{Vol}((\Delta_S^Q)^\vee) \frac{\exp\left\langle T_S^Q + T_1^S, -\gamma(\lambda, w, \theta_0)\right\rangle}{\prod_{\beta\in\Delta_S^Q}\langle -\gamma(\lambda, w, \theta_0), \beta^\vee\rangle}.$$

Tracking the dependence on $T \in \mathfrak{a}_0$, we have

$$|\psi_T^Q * D^Q(\lambda)| \ll_{D^Q} (1 + \|T\|)^{d^Q + a_0^Q} \sum_{P_0\subseteq S\subseteq Q} \exp\left\langle T_S^Q, -\gamma(\Lambda_0, w, \theta_0)\right\rangle$$

Observe by the conditions on Λ_0 and T, the inner product $\left\langle T_S^Q, -\gamma(\Lambda_0, w, \theta_0)\right\rangle$ is negative and equals $\left\langle T_S^Q, -\Lambda_0 + w^{-1}\Lambda_0\right\rangle$ (using Theorem 4.3.1). Since $\Lambda_0 - w^{-1}\Lambda_0$ is a positive linear combination of the roots in Δ_Q, there exists a $\xi(Q) \in \mathfrak{a}_0^*$ whose coefficients in the basis Δ_0 are nonnegative (depending on Λ_0) such that

$$\left\langle T_S^Q, -\Lambda_0 + w^{-1}\Lambda_0\right\rangle \le - \quad T^Q, \xi(Q) \quad .$$

(By the choice of T), it is also possible to choose $\xi(Q)$ independent of w, for example we can choose $\xi(Q) = N\sum_{\beta\in\Delta_Q}\beta$ for N suitably large. Therefore,

$$|\psi_T^Q * D^Q(\lambda)| \ll_{D^Q} (1 + \|T\|)^{d^Q + a_0^Q} \exp - \quad T^Q, \xi(Q) \quad .$$

The estimate for $\psi_{T,Q,\ell} * D_Q(\lambda)$ is similar to that in lemma 3.5 of [FL16] and we have,

$$|(\psi_{T,Q,\ell} * D_Q)(\lambda)| \ll_{D_Q,\ell} (1 + \|T\|)^{d_Q} \exp - \langle T_Q, \xi(Q)\rangle .$$

∎

5.2 PROOF OF THEOREM 4.4.3

Proof The quantity to estimate is

$$\iint_{N_w(\mathbb{A})\backslash N_0(\mathbb{A})} \int_{A_0} \int_{N_0(\mathbb{A})} \iint_{M_0(\mathbb{A})^1} |f^1(a^{-1}n^{-1}n_{\widetilde{w}}aum)|\chi(a)\mathrm{d}m\mathrm{d}u\mathrm{d}a\mathrm{d}n,$$

where

$$\chi(a) = \chi_{T,Q,l}(a) = \tau_Q(H_Q(a) - T)\ \hat{\tau}^Q_{P_0}(T - H_0(a))\times$$
$$\tau^Q_{P_0}(H_0(a) - T_1) \sum_{\alpha\in\Delta_Q} \left\langle H_Q(a) - T_Q, \alpha\right\rangle^l.$$

We split the integral over $u \in N_0(\mathbb{A})$ as $n'u$ where $u \in N_{w^{-1}}(\mathbb{A}) \setminus\ N_0(\mathbb{A})$ and $n' \in N_{w^{-1}}(\mathbb{A})$. Thus, we want to bound

$$\iint_{n\in N_w(\mathbb{A})\backslash N_0(\mathbb{A})} \int_{A_0} \int_{n'\in N_{w^{-1}}(\mathbb{A})} \iint_{u\in N_{w^{-1}}(\mathbb{A})\backslash N_0(\mathbb{A})} \iint_{M_0(\mathbb{A})^1} |f^1(a^{-1}n^{-1}n_{\widetilde{w}}an'um)|\times$$
$$\chi(a)\ \mathrm{d}m\ \mathrm{d}u\ \mathrm{d}n'\ \mathrm{d}a\ \mathrm{d}n.$$

We can conjugate n' over $n_{\widetilde{w}}a$:

$$\begin{aligned}
n_{\widetilde{w}}an' &= n_w\delta_0an' \\
&= n_w\delta_0(an'a^{-1})\delta_0^{-1}n_w^{-1}n_w\delta_0a \\
&= n_w(\delta_0n''\delta_0^{-1})n_w^{-1}n_w\delta_0a \quad \cdots n'' = an'a^{-1} \in N_{w^{-1}}(\mathbb{A}) \\
&= n_w\theta_0(n'')n_w^{-1}n_{\widetilde{w}}a \\
&= n_wn'''n_w^{-1}n_{\widetilde{w}}a
\end{aligned}$$

where $n''' \in N_{w^{-1}}(\mathbb{A}) = N_0(\mathbb{A}) \cap w^{-1}N_0(\mathbb{A})w$. Therefore $n_1 := n_wn'''n_w^{-1} \in N_w(\mathbb{A})$. Making this change of variable, we are reduced to bounding

$$\iint_{n\in N_w(\mathbb{A})\backslash N_0(\mathbb{A})} \iint_{n_1\in N_w(\mathbb{A})} \int_{A_0} \int_{M_0(\mathbb{A})^1} \iint_{u\in N_{w^{-1}}(\mathbb{A})\backslash N_0(\mathbb{A})} |f^1(a^{-1}n^{-1}n_1n_{\widetilde{w}}aum)|\chi(a)\times$$
$$\delta_{M_0,N_{w^{-1}}}(a)^{-1}\mathrm{d}u\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n_1\,\mathrm{d}n,$$

$$= \int_{N_0(\mathbb{A})} \int_{A_0} \iint_{M_0(\mathbb{A})^1} \iint_{N_{w^{-1}}(\mathbb{A})\backslash N_0(\mathbb{A})} |f^1(a^{-1}nn_{\widetilde{w}}aum)|\chi(a)\delta_{M_0,N_{w^{-1}}}(a)^{-1}$$
$$\mathrm{d}u\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n$$

$$= \int_{N_0(\mathbb{A})} \int_{A_0} \int_{M_0(\mathbb{A})^1} \int_{N_{w^{-1}}(\mathbb{A}) \backslash N_0(\mathbb{A})} |f^1(a^{-1} n n_w \theta_0(aum)\delta_0)| \chi(a) \delta_{M_0, N_{w^{-1}}}(a)^{-1} \, \mathrm{d}u \, \mathrm{d}m \, \mathrm{d}a \, \mathrm{d}n$$

$$= \int_{N_0(\mathbb{A})} \int_{A_0} \int_{M_0(\mathbb{A})^1} \int_{N_{w^{-1}}(\mathbb{A}) \backslash N_0(\mathbb{A})} |h(a^{-1} n n_w \theta_0(aum))| \chi(a) \delta_{M_0, N_{w^{-1}}}(a)^{-1} \, \mathrm{d}u \, \mathrm{d}m \, \mathrm{d}a \, \mathrm{d}n$$

$$= \int_{N_0(\mathbb{A})} \int_{A_0} \int_{M_0(\mathbb{A})^1} \int_{N_{w^{-1}}(\mathbb{A}) \backslash N_0(\mathbb{A})} |h(a^{-1} n \theta_0(m) n_w \theta_0(au))| \chi(a) \delta_{M_0, N_{w^{-1}}}(a)^{-1} \, \mathrm{d}u \, \mathrm{d}m \, \mathrm{d}a \, \mathrm{d}n$$

$$= \int_{N_0(\mathbb{A})} \int_{A_0} \int_{M_0(\mathbb{A})^1} \int_{N_{w^{-1}}(\mathbb{A}) \backslash N_0(\mathbb{A})} |h(n \theta_0(m) a^{-1} n_w \theta_0(au))| \chi(a) \delta_{M_0, N_{w^{-1}}}(a)^{-1} \delta_0(a) \, \mathrm{d}u \, \mathrm{d}m \, \mathrm{d}a \, \mathrm{d}n$$

$$= \int_{P_0(\mathbb{A})^1} \int_{A_0} \int_{N_{w^{-1}}(\mathbb{A}) \backslash N_0(\mathbb{A})} |h(p a^{-1} n_w \theta_0(au))| \chi(a) \delta_{M_0, N_{w^{-1}}}(a)^{-1} \delta_0(a) \mathrm{d}u \, \mathrm{d}a \, \mathrm{d}p.$$

Here we have used the Theorem 3.1.2 on the right to get a function h on $G(\mathbb{A})$ satisfying $h(x) = f^1(x\delta_0)$ and that $M_0(\mathbb{A})^1$ is invariant under θ_0 and that $N_0(\mathbb{A})$ normalizes $M_0(\mathbb{A})^1$. By an application of Theorem 4.2.1, this equals

$$\int_{P_0(\mathbb{A})^1} \int_{A_0} \int_{N_{w^{-1}}(\mathbb{A}) \backslash N_0(\mathbb{A})} |h(p a^{-1} n_w \theta_0(au))| \chi(a) \delta_{w^{-1}}(a) \mathrm{d}u \mathrm{d}a \mathrm{d}p.$$

Recall the definition of the space of principal series representations in [FL11a, §3.3] and in particular, that of F_h: for $h \in \mathcal{C}_c^\infty(G(\mathbb{A}))$,

$$F_h(g) = \int_{P_0(\mathbb{A})^1} h(pg) \mathrm{d}p.$$

Thus we want to consider

$$\int_{A_0} \int_{N_{w^{-1}}(\mathbb{A}) \backslash N_0(\mathbb{A})} |F_h(a^{-1} n_w \theta_0(au))| \chi(a) \delta_{w^{-1}}(a) \mathrm{d}u \mathrm{d}a.$$

As u is integrated over $N_{w^{-1}}(\mathbb{A}) \setminus N_0(\mathbb{A})$, so is $\theta_0(u)$ by a change of variables. (Because $\theta_0 : W^G \to W^G$ maps n_w to another representative of $w \in W$.) So look at

$$\int_{A_0} \int_{N_{w^{-1}}(\mathbb{A}) \backslash N_0(\mathbb{A})} |F_h(a^{-1} n_w \theta_0(a) u)| \chi(a) \delta_{w^{-1}}(a) \mathrm{d}u \mathrm{d}a.$$

By [FL11a, Lemma 3.4] and Theorem 3.1.2, we can assume that f, hence F_h is of compact support. This justifies taking the Mellin transform below,

$$\int_{A_0}\int_{N_{w^{-1}}(\mathbb{A})\backslash N_0(\mathbb{A})}\int_{\mathrm{Re}\,\lambda=\lambda_0}\phi(\lambda)(n_w u)\left(a^{-1}n_w\theta_0(a)n_w^{-1}\right)^{\lambda}\chi(a)\delta_{w^{-1}}(a)\times$$
$$\delta_0(a^{-1}n_w\theta_0(a)n_w^{-1})^{\frac{1}{2}}\mathrm{d}\lambda\mathrm{d}u\mathrm{d}a,$$

which by Theorem 4.2.2 reduces to

$$\int_{A_0}\int_{N_{w^{-1}}(\mathbb{A})\backslash N_0(\mathbb{A})}\int_{\mathrm{Re}\,\lambda=\lambda_0}\phi(\lambda)(n_w u)\left(a^{-1}n_w\theta_0(a)n_w^{-1}\right)^{\lambda}\chi(a)\delta_{w^{-1}}(a\theta_0(a^{-1}))$$
$$\delta_{M_0,N_{w^{-1}}}(a^{-1}\theta_0(a))\mathrm{d}\lambda\mathrm{d}u\mathrm{d}a.$$

Moreover,

$$\begin{aligned}\left(a^{-1}n_w\theta_0(a)n_w^{-1}\right)^{\lambda} &= \exp\left\langle\lambda, H_0(a^{-1}n_w\theta_0(a)n_w^{-1})\right\rangle\\ &= \exp\langle\lambda, -H_0(a)+w.\theta_0(a)\rangle\\ &= \exp-\left\langle(1-\theta_0^{-1}w^{-1})\lambda, H_0(a)\right\rangle.\end{aligned}$$

Writing this integral over $\mathfrak{a}_0$ and using Theorem 4.2.1 gives

$$\int_{\mathfrak{a}_0}\int_{N_{w^{-1}}(\mathbb{A})\backslash N_0(\mathbb{A})}\int_{\mathrm{Re}\,\lambda=\lambda_0}\phi(\lambda)(n_w u)\exp\langle X-w\theta_0(X),\lambda\rangle\,\tau_Q(X-T)\,\hat{\tau}_{P_0}^{Q}(T-X)$$
$$\tau_{P_0}^{Q}(X-T_1)\sum_{\alpha\in\Delta_Q}\langle X_Q-T_Q,\alpha\rangle^{l}\exp\left\langle X-\theta_0(X),\frac{1}{2}\sum_{\alpha,w\alpha>0}\alpha-\frac{1}{2}\sum_{\substack{\beta>0\\ w\beta<0}}\beta\right\rangle\mathrm{d}\lambda\mathrm{d}u\mathrm{d}X.$$

$$=\int_{\mathfrak{a}_0}\int_{N_{w^{-1}}(\mathbb{A})\backslash N_0(\mathbb{A})}\int_{\mathrm{Re}\,\lambda=\lambda_0}\phi(\lambda)(n_w u)\tau_Q(X-T)\,\hat{\tau}_{P_0}^{Q}(T-X)\tau_{P_0}^{Q}(X-T_1)\times$$
$$\exp\left\langle X,-(\lambda-\theta_0^{-1}w^{-1}\lambda)+\frac{1}{2}(1-\theta_0^{-1})(\sum_{\alpha,w\alpha>0}\alpha-\sum_{\substack{\beta>0\\ w\beta<0}}\beta)\right\rangle\times$$
$$\sum_{\alpha\in\Delta_Q}\langle X_Q-T_Q,\alpha\rangle^{l}\,\mathrm{d}\lambda\mathrm{d}u\mathrm{d}X.$$

Following Theorem 5.1.1, we will denote the second term in the inner product by $-\gamma(\lambda,w,\theta_0)$ where $\gamma=-\frac{1}{2}(1-\theta_0^{-1})(\sum_{\alpha,w\alpha>0}\alpha-\sum_{\substack{\beta>0\\ w\beta<0}}\beta)$.

We will eventually prove the absolute convergence of this triple integral which will justify the changing the order of integration. Using Equation (4.1), the function

$$h(\lambda) = m(w^{-1}, \lambda) \prod_{\substack{\alpha\in\Delta_0:\\ w^{-1}\alpha<0}} \langle \lambda, \alpha^\vee \rangle$$

is of moderate growth so we are reduced to proving the absolute convergence of

$$\int_{\mathrm{Re}\,\lambda=\lambda_0} \frac{h(\lambda)\varphi_{T,Q,l}(\lambda)}{\prod_{\substack{\alpha\in\Delta_0:\\ w^{-1}\alpha<0}} \langle \lambda, \alpha^\vee \rangle} \mathrm{d}\lambda,$$

where

$$\varphi_{T,Q,l}(\lambda) := \int_{X\in\mathfrak{a}_0} \exp\left\langle X, -(\lambda - \theta_0^{-1} w^{-1}\lambda) + \frac{1}{2}(1-\theta_0^{-1})(\sum_{\alpha, w\alpha>0} \alpha - \sum_{\substack{\beta>0\\ w\beta<0}} \beta) \right\rangle$$

$$\tau_Q(X-T)\ \hat{\tau}_{P_0}^Q(T-X)\tau_{P_0}^Q(X-T_1) \sum_{\alpha\in\Delta_Q} \langle X_Q - T_Q, \alpha\rangle^l \,\mathrm{d}X,$$

is absolutely convergent by Theorem 5.1.1. Having proven Theorem 5.1.1 which is the twisted equivalent of [FL16, Lemma 3.5], we are in a position to apply [FL16, Proposition 3.4] from which the required estimate follows.

■

6. ROOT CONE LEMMA

This section is devoted to proving Lemma 4.3.2, the Root Cone Lemma in various cases. It is clear that the radical of G plays no role in the statement of the lemma so we may as well assume that G is semisimple. After reducing to the case when G is simple, we do a case-by-case exhaustion in the case when G is connected split simple wherein, automorphisms of G correspond to those of the Dynkin diagram of G. We use the Cartan-Killing classification to enumerate all automorphisms and prove the lemma in each case. The proof for the automorphism of E_6 was done using the software `SageMath` [TSD17]. The general case when G is connected simple but possibly not split eludes a proof.

6.1 REDUCTION TO THE SIMPLE CASE

The cyclic base change is a special case of this. Let E/F be a cyclic extension of number fields of order d with a generator θ of the Galois group. Let H be a connected reductive group over E and consider the group $G = \mathrm{Res}_{E/F} H$. We have $G(E) \cong H(E) \times \cdots \times H(E)$ and is equipped with a Galois action which permutes the d copies $H(E)$. Langlands' Functoriality has been established for automorphic representations of $H(\mathbb{A}_F)$ and $H(\mathbb{A}_E)$ when $H = \mathrm{GL}(n)$ by Arthur and Clozel [AC89] by comparing the trace formula for H with the twisted trace formula for G. The lemma below proves RCL for cyclic base change. Moreover using it, we are reduced to proving RCL for automorphisms of connected almost-simple groups.

Lemma 6.1.1 *Let H be a connected reductive group. The Root Cone Lemma 4.3.2 holds when $G \cong H \times \cdots \times H$ (d copies) and θ is a d-cycle that permutes the d-copies of H.*

Proof We will identify d copies of 'objects' of H with the corresponding copies in G. For instance, suppose $w = (w_1, \cdots, w_d) \in W^G$ is given so that $w_i \in W^H$ for $i \in [d]$. We need to show the existence of $\lambda = (\lambda_1, \cdots, \lambda_d) \in (\mathfrak{a}_0^G)^*$ so that $\lambda - \theta^{-1}w^{-1}\lambda$ is a positive linear combination of co-roots $\beta^\vee$ whenever $\beta^\vee \in \Delta_0^{Q^G(w)}$. There is no loss in generality to use θ, w instead of the notationally cumbersome θ^{-1}, w^{-1}. Then,

$$\lambda - \theta w \lambda = (\lambda_1 - w_2\lambda_2, \lambda_2 - w_3\lambda_3, \cdots, \lambda_{d-1} - w_d\lambda_d, \lambda_d - w_1\lambda_1).$$

Choosing coordinates $\lambda_i, \varpi_\beta^\vee$ for each $i \in [d]$ and $\beta \in \Delta_0^H$ will define the vectors $\lambda_1, \cdots, \lambda_d \in ((\mathfrak{a}_0^H)^*)^+$. Fix $\beta \in \Delta_0^H$; we have three cases.

- Suppose there are $i, j \in [d]$ such that $\beta \in \Delta_0^{Q^H(w_j)} \setminus \Delta_0^{Q^H(w_i)}$. Choose

  $$\lambda_{i+1}, \varpi_\beta^\vee \;>\; \lambda_{i+2}, \varpi_\beta^\vee \;> \cdots >\; \lambda_i, \varpi_\beta^\vee \;.$$

  For any $i' \in [d] \setminus \{i\}$, reading $i'+1$ modulo d, we have

  $$\lambda_{i'} - w_{i'+1}\lambda_{i'+1}, \varpi_\beta^\vee \;=\; \lambda_{i'} - \lambda_{i'+1}, \varpi_\beta^\vee \;+\; \lambda_{i'+1} - w_{i'+1}\lambda_{i'+1}, \varpi_\beta^\vee \;> 0.$$

  The former term is positive by the choice above and the latter is non-negative by [Bou02, Ch. VI §1.6 Proposition 18].

- Suppose that $\beta \in \cap_{i=1}^d \Delta_0^{Q^H(w_i)}$ then choose

  $$\lambda_1, \varpi_\beta^\vee \;=\; \lambda_2, \varpi_\beta^\vee \;= \cdots =\; \lambda_d, \varpi_\beta^\vee \;> 0.$$

  Since $\lambda_{i-1} - w_i\lambda_i, \varpi_\beta^\vee \;=\; \lambda_i - w_i\lambda_i, \varpi_\beta^\vee$, positivity follows from lemma 2.2 of [FL11a] (whose proof is an application of loc. cit).

- Finally, if $\beta \notin \cap_{i=1}^d \Delta_0^{Q^H(w_i)}$ then λ_i can be choses such that $\lambda_i, \varpi_\beta^\vee$ is positive.

■

We could now assume that G is almost-simple but since the RCL is a statement about the root system of G, we may assume that G is simple. Additionally when G is split, the statement reduces to the automorphisms of Dynkin diagrams of simple Lie algebras. The Dynkin diagrams for the families B_n and C_n as well as the exceptional ones E_7, E_8, F_4, G_2 have no nontrivial automorphisms. The A-type has a unique automorphism and so does the D-type when $n \neq 4$. D_4 has a nontrivial automorphism of order 3 whereas E_6 has an involution. We handle each of these cases below by explicitly constructing root cones for different elements $w \in W$.

6.2 ROOT CONE LEMMA FOR TYPE A_n

Following [FH91], we can explicitly write a basis for the roots and weights as follows. The space $\mathfrak{a}_0$ of roots is spanned by $L_1, \cdots, L_n$ such that $\sum_{i=1}^{n} L_i = 0$.

$$\Delta_0 = \{\alpha_1, \cdots, \alpha_{n-1} : \alpha_i = L_i - L_{i-1}\}$$

and

$$\hat{\Delta}_0 = \{\varpi_1, \cdots, \varpi_{n-1} : \varpi_i = L_1 + \cdots + L_i\}.$$

A vector

$$\lambda = \sum_{i=1}^{n-1} a_i \varpi_i = \sum_{i=1}^{n-1} \left(\sum_{j=i}^{n-1} a_j \right) L_i =: \sum_{b=1}^{n-1} b_i L_i$$

is in the positive Weyl chamber precisely when $a_1, a_2, \cdots, a_{n-1} > 0$ or equivalently, if

$$b_1 > b_2 > \cdots > b_{n-1} > 0.$$

We can and will write λ as a sum $b_1 L_1 + \cdots + b_{n-1} L_{n-1} + b_n L_n$ with $b_n = 0$.

The action of any $w \in W \cong S_n$ is by permuting the indices and $\theta_0^{-1} = \theta_0$ acts by $\theta_0(L_i) = -L_{n-i}$ which up to a sign, is the action of the long element $w_0 \in W$. Denoting by $\tau \in S_n$ the action of $w_0 w^{-1}$, we see that

$$\theta_0^{-1} w^{-1} \lambda = -w_0 w^{-1} \lambda = -\sum_{i=1}^{n} b_i L_{\tau(i)} = \sum_{i=1}^{n} -b_{\tau^{-1}(i)} L_i.$$

Thus,

$$\lambda - \theta_0^{-1} w^{-1} \lambda = \sum_{i=1}^{n} \left(b_i + b_{\tau^{-1}(i)} \right) L_i = \sum_{i=1}^{n-1} (b_i + b_{\tau^{-1}(i)} - b_{\tau^{-1}(n)}) L_i$$

We now use the Cartan matrix to write this vector in terms of the roots.

$$Z = CU = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & & & \\ 0 & 1 & -1 & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ & & & & 1 & -1 \\ 1 & 1 & \cdots & & 1 & 2 \end{pmatrix}.$$

Since $Z^{-1}L_i = \alpha_i$ for $i = 1, 2, \cdots, n-1$ we have

$$\lambda - \theta_0^{-1} w^{-1} \lambda = \sum_{i=1}^{n-1} (b_i + b_{\tau^{-1}(i)} - b_{\tau^{-1}(n)}) L_i = \sum_{i-1}^{n-1} c_i \alpha_i$$

where

$$\begin{aligned} c_i &= \frac{1}{n} \left[(n-i) \sum_{j=1}^{i} (b_j + b_{\tau^{-1}(j)} - b_{\tau^{-1}(n)}) - i \sum_{j=i+1}^{n-1} (b_j + b_{\tau^{-1}(j)} - b_{\tau^{-1}(n)}) \right] \\ &= \frac{1}{n} \left[n \sum_{j=1}^{i} (b_j + b_{\tau^{-1}(j)} - b_{\tau^{-1}(n)}) - i \sum_{j=1}^{n-1} (b_j + b_{\tau^{-1}(j)} - b_{\tau^{-1}(n)}) \right]. \end{aligned}$$

Therefore,

$$nc_i = n(b_1 + \cdots + b_i + b_{\tau^{-1}(1)} + \cdots + b_{\tau^{-1}(i)}) - 2i(b_1 + \cdots + b_n).$$

Thus we need to be able to choose coefficients b_i such that c_i above is positive whenever $\alpha_i \in \Delta_0^{Q(w)}$. By [FL11a, p. 787],

$$\Delta_0^{Q(w)} = \{\alpha \in \Delta_0 : w\varpi_\alpha^\vee \neq \varpi_\alpha^\vee\},$$

which for $\mathrm{GL}(n)$ implies that $\alpha_i \in \Delta_0^{Q(w)}$ precisely if the permutation τ above satisfies $\tau([i]) \neq [i]+n-i,$ where $[n] := \{1, 2, \cdots, n\}$, cf. [PM]. This is proven in the following

Lemma 6.2.1 *For $n \geq 2$, fix a permutation $\tau \in S_n, \tau \neq (1, n)(2, n-1)\cdots$, i.e., τ isn't the long element. Let*

$$\Delta(\tau) = \{i \in [n-1] : \tau([i]) \neq \{[i] + n - i\}\}.$$

Then there exist real numbers $b_1 > b_2 > \cdots > b_{n-1} > b_n = 0$ such that the inequalities

$$\frac{b_1 + b_2 + \cdots + b_i + b_{\tau^{-1}(1)} + \cdots + b_{\tau^{-1}(i)}}{2i} > \frac{b_1 + b_2 + \cdots + b_n}{n} \tag{6.1}$$

hold simultaneously for every $i \in \Delta(\tau)$.

Proof For a given $\tau \in S_n$ and for any $j \in [n]$ define the numbers

$$a_j := \chi_\Delta(j) - \chi_\Delta(j-1) - \chi_\Delta(n-j) + \chi_\Delta(n-j+1),$$

where $\chi_\Delta : \mathbb{Z} \to \{0, 1\}$ denotes the characteristic function of the set $\Delta := \Delta(\tau) \subset \mathbb{Z}$. Also, with $c := 5n - a_n$, define

$$b_j := a_j - 5j + c.$$

We may note right away that since $|a_j| \le 2$, the b_j are strictly decreasing, and that $b_n = 0$, by the choice of the constant c.

For any $E \subset [n]$, for simplicity of notation we put

$$\alpha(E) := \sum_{j \in E} a_j, \qquad \beta(E) := \sum_{j \in E} b_j$$

(so we may think α and β as discrete signed measures supported in $[n]$).

For $i \in [n]$, summing over $j = 1, \ldots i$ we have

$$\alpha([i]) = \chi_\Delta(i) - \chi_\Delta(n-i).$$

Incidentally, for any $i \in [n]$ we have $i \in \Delta$ if and only if, by definition, $\tau([i]) \neq [i] + n - i$ thus also, since τ is bijective, if and only if $\tau([i]^c) \neq ([i] + n - i)^c$, that is $\tau([n-i]+i) \neq [n-i]$ or $\tau^{-1}([n-i]) \neq [n-i]+i$, which means $n-i \in \Delta^{-1} := \Delta(\tau^{-1})$. Hence the last formula also writes

$$\alpha([i]) = \chi_\Delta(i) - \chi_{\Delta^{-1}}(i).$$

Also note that, since $n \notin \Delta$

$$\alpha([n]) = 0,$$

and

$$\alpha([i] + n - i) = -\alpha([n-i]) = -\chi_\Delta(n-i) + \chi_\Delta(i) = \alpha([i]).$$

We proceed showing the inequalities on the arithmetic means.

Case I. Assume $i \in \Delta \setminus \Delta^{-1}$. Then by definition of Δ^{-1}, $\tau^{-1}([i]) = [i] + n - i$, so that

$$\frac{\alpha([i]) + \alpha(\tau^{-1}[i])}{2i} = \alpha([i]) + \frac{\alpha([i] + n - i)}{2i} = \frac{\chi_\Delta(i) - \chi_{\Delta^{-1}}(i)}{i} = \frac{1}{i} > 0,$$

and summing the arithmetic means of $-5j + c$ on the same sets we have plainly

$$\frac{\beta([i]) + \beta(\tau^{-1}[i])}{2i} > \frac{\beta([n])}{n}.$$

Case II. Assume $i \in \Delta \cap \Delta^{-1}$. Thus $\tau^{-1}([i]) \neq [i] + n - i$ and, just because b_j are strictly decreasing

$$\frac{\beta([i]) + \beta(\tau^{-1}[i])}{2i} > \frac{\beta([i]) + \beta([i] + n - i)}{2i}$$

and since we have $\alpha([i]) = \alpha([i] + n - i) = \alpha([n]) = 0$ because $\chi_\Delta(i) = \chi_{\Delta^{-1}}(i) = 1$, summing as before the arithmetic means of the affine part of b_j,

$$\frac{\beta([i]) + \beta([i] + n - i)}{2i} = \frac{\beta([n])}{n},$$

concluding the proof.

■

6.3 ROOT CONE LEMMA FOR TYPE D_ℓ

The root system D_ℓ has a unique automorphism when $\ell \neq 4$ which for $\ell = 3$ coincides with the automorphism $\theta_0(x) = {}^t x^{-1}$ of $GL(3)$ (and has been proven in the previous section). The root system D_4 has an automorphism of order 3 which will be considered later.

Following [Bou02] we assume the ambient space is spanned by the vectors $e_1, \cdots, e_\ell$ and the roots are given by $\mathcal{R}_0 = \{\pm e_i \pm e_j : 1 \leq i < j \leq \ell\}$. For a base Δ_0, we choose

$$\Delta_0 = \{\alpha_1 = e_1 - e_2, \cdots, \alpha_{\ell-1} = e_{\ell-1} - e_\ell, \alpha_\ell = e_{\ell-1} - e_\ell\}.$$

The corresponding dual basis of weights is

$$\hat{\Delta}_0 = \{\varpi_i = e_1 + \cdots + e_i : 1 \leq i \leq \ell - 2\} \bigcup \{\varpi_{\ell-1} = \frac{1}{2}(e_1 + \cdots + e_{\ell-1} - e_\ell), \varpi_\ell = \frac{1}{2}(e_1 + \cdots + e_\ell)\}.$$

Since the root system D_ℓ is selfdual, the co-roots and co-weights are defined similarly. The Weyl group W consists of even-signed permutations, i.e., any $w \in W$ is a pair (σ, η) where $\sigma \in S_\ell$ and $\eta = (\eta_1, \cdots, \eta_\ell)$ is an ordered ℓ-tuple of ± 1 with even -1's. The action of w on the basis is given by

$$w.e_i = \eta_i e_{\sigma(i)}.$$

The automorphism θ_0 acts on Δ_0 by permuting the set $\{\alpha_{\ell-1}, \alpha_\ell\}$ and fixing other roots and similarly on the weights in $\hat{\Delta}_0$.

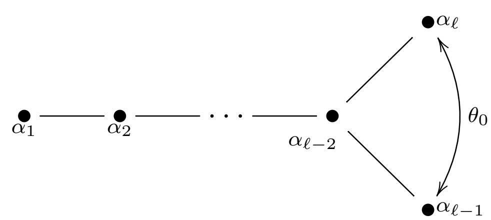

Fig. 6.1. Involution for groups of type D_ℓ

To prove the Root Cone Lemma for D_ℓ, we need to show that for fixed $w = (\sigma, \eta) \in W$ there exists an open cone $\Omega \subseteq (\mathfrak{a}_0^*)^+$ such that if $\lambda \in \Omega$, the inequality

$$\lambda - \theta_0^{-1} w^{-1} \lambda, \varpi_i^\vee \;\; = \langle \lambda, \varpi_i^\vee - w\theta_0 \varpi_i^\vee \rangle > 0 \qquad \text{(Equation (6.2))}(i)$$

holds whenever $w\varpi_i^\vee \neq \varpi_i^\vee$. Assume that $\lambda = c_1\varpi_1 + \cdots + c_\ell\varpi_\ell \in (\mathfrak{a}_0^*)^+$ that is to say, $c_i > 0$ for all $i \leq \ell$. Observe that

$$\begin{aligned}
\lambda = c_1e_1 + c_2(e_1+e_2) + \cdots + c_{\ell-1}\left(e_1 + \cdots + e_{\ell-1} + \frac{e_{\ell-1}+e_\ell}{2}\right) \\
+ c_\ell\left(e_1 + \cdots + e_{\ell-2} + \frac{-e_{\ell-1}+e_\ell}{2}\right) \\
= (c_1 + \cdots + c_{\ell-2})e_1 + \cdots + \left(c_{\ell-2} + \frac{c_{\ell-1}+c_\ell}{2}\right)e_{\ell-2} \\
+ \left(\frac{c_{\ell-1}+c_\ell}{2}\right)e_{\ell-1} + \left(\frac{-c_{\ell-1}+c_\ell}{2}\right)e_\ell, \quad (6.3)
\end{aligned}$$

so if $c_1, \cdots, c_\ell > 0$ then the only possible negative coefficient above is that of e_ℓ. If θ_0 fixes $\varpi_i^\vee$ then every $\lambda \in (\mathfrak{a}_0^*)^+$ satisfies the Inequality (Equation (6.2))(i), as can be seen from [FL11a, Lemma 2.2]. It suffices to prove these inequalities for $i = \ell-1, \ell$. We first analyze the case $i = \ell - 1$ and set $I(w) = \{\sigma(i) : 1 \leq i \leq \ell,\ \eta_i = -1\}$.

$$\begin{aligned}
\varpi_{\ell-1}^\vee - w\theta_0\varpi_{\ell-1}^\vee &= \varpi_{\ell-1}^\vee - w\varpi_\ell^\vee \\
&= \frac{1}{2}(e_1^\vee + \cdots + e_{\ell-1}^\vee - e_\ell^\vee) - \frac{1}{2}(\eta_1 e_{\sigma(1)}^\vee + \cdots + \eta_\ell e_{\sigma(\ell)}^\vee) \\
&= \left(\sum_{i\in I(w)} e_{\sigma(i)}^\vee\right) - e_\ell^\vee.
\end{aligned}$$

- If $I(w) = \emptyset$ then $\eta = (1, \cdots, 1)$ and $\varpi_{\ell-1}^\vee - w\varpi_\ell^\vee = -e_\ell^\vee$ so to ensure the quantity $\lambda, \varpi_{\ell-1}^\vee - w\varpi_\ell^\vee$ is positive, by Equation (6.3) above we can take $c_{\ell-1} > c_\ell$.

- If $I(w) \neq \emptyset$ and $\inf I(w) < \ell - 1$, we can choose $c_i \gg c_{\ell-1},\ c_i \gg c_\ell$ for some $i \in I(w) \setminus \{\ell-1, \ell\}$ to get the positivity condition.

- Finally if $\inf I(w) \geq \ell - 1$ then η must have exactly two sign changes and $I(w) = \{\ell-1, \ell\}$ in which case $\varpi_{\ell-1}^\vee - w\varpi_\ell^\vee = e_{\ell-1}^\vee$ so every $\lambda \in (\mathfrak{a}_0^*)^+$ satisfies Inequality (Equation (6.2))($l-1$).

The case $i = \ell$ is slightly more involved. Observe that if $w\varpi_\ell^\vee \neq \varpi_\ell^\vee$ then $\eta \neq (1, \cdots, 1)$ where $w = (\sigma, \eta)$. Thus we can assume $I(w) \neq \emptyset$.

$$
\begin{aligned}
\varpi_\ell^\vee - w\theta_0\varpi_\ell^\vee &= \varpi_\ell^\vee - w\varpi_{\ell-1}^\vee \\
&= \frac{1}{2}(e_1^\vee + \cdots + e_\ell^\vee) - \frac{1}{2}(\eta_1 e_{\sigma(1)} + \cdots + \eta_{\ell-1}e_{\sigma(\ell-1)}^\vee - \eta_\ell e_{\sigma(\ell)}^\vee) \\
&= \frac{1}{2}\left(e_1^\vee + \cdots + e_\ell^\vee - \eta_1 e_{\sigma(1)}^\vee - \cdots - \eta_\ell e_{\sigma(\ell)}^\vee\right) + \eta_\ell e_{\sigma(\ell)}^\vee \\
&= \eta_\ell e_{\sigma(\ell)} + \sum_{i\in I(w)} e_i^\vee.
\end{aligned}
$$

Thus $\varpi_\ell^\vee - w\theta_0\varpi_\ell^\vee$ is a positive linear combination of $\{e_i\}$ whenever $\eta_\ell \neq -1$. If $\eta_\ell = -1$ then $\eta_{i_0} = -1$ for some $i_0 \neq \ell$;

$$
\begin{aligned}
\varpi_\ell^\vee - w\varpi_{\ell-1}^\vee &= -e_{\sigma(\ell)}^\vee + e_{\sigma(\ell)}^\vee + e_{\sigma(i_0)}^\vee + \sum_{i\in I(w)\setminus\{i_0,\ell\}} e_i^\vee \\
&= e_{i_0}^\vee + \sum_{i\in I(w)\setminus\{i_0,\ell\}} e_i^\vee,
\end{aligned}
$$

which again is a positive linear combination of $\{e_i\}$'s. In either case, choosing $c_{\ell-1} < c_\ell$ ensures that Inequality (Equation (6.2))(l) holds. Observe that this choice is consistent with the choices in Inequality (Equation (6.2))($l-1$).

6.4 ROOT CONE LEMMA FOR THE TRIALITY AUTOMORPHISM OF D_4

Assume that θ_0 is the automorphism of D_4 permuting the set $\{\alpha_1, \alpha_3, \alpha_4\}$ cyclically as shown in 6.4.

We need to find conditions on $c_1, c_2, c_3, c_4 > 0$ where

$$
\lambda = \left(c_1 + c_2 + \frac{c_3+c_4}{2}\right)e_1 + \left(c_2 + \frac{c_3+c_4}{2}\right)e_2 + \left(\frac{c_3+c_4}{2}\right)e_3 + \left(\frac{-c_3+c_4}{2}\right)e_4
$$

so that

(a) $\langle\lambda, \varpi_4^\vee - w\varpi_1^\vee\rangle > 0$ whenever $w\varpi_4^\vee \neq \varpi_4^\vee$;

(b) $\langle\lambda, \varpi_3^\vee - w\varpi_4^\vee\rangle > 0$ whenever $w\varpi_3^\vee \neq \varpi_3^\vee$; and

(c) $\langle\lambda, \varpi_1^\vee - w\varpi_3^\vee\rangle > 0$ whenever $w\varpi_1^\vee \neq \varpi_1^\vee$.

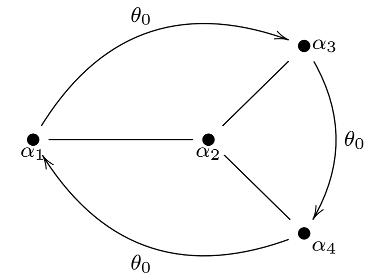

Fig. 6.2. Triality automorphism of D_4

Let us analyze the inequality in each case.

(a) Observe that

$$\varpi_4^\vee - w\varpi_1^\vee = \frac{1}{2}(e_1^\vee + e_2^\vee + e_3^\vee + e_4^\vee) - \eta_1 e_{\sigma(1)}^\vee.$$

- If $\eta_1 = -1$ then this vector is a positive combination of the $e_i^\vee$'s so the condition $\langle \lambda, \varpi_4^\vee - w\varpi_1^\vee \rangle > 0$ can be ensured by choosing $c_3 < c_4$.
- If $\eta_1 = 1$ and $\sigma(1) \in \{2,3,4\}$ then choosing $c_1 \gg c_3, c_4$ guarantees Inequality (a) holds.
- Finally if $\eta_1 = 1$ and $\sigma(1) = 1$ then

$$\langle \lambda, \varpi_4^\vee - w\varpi_1^\vee \rangle = \left\langle \lambda, \frac{1}{2}(-e_1^\vee + e_2^\vee + e_3^\vee + e_4^\vee) \right\rangle = -c_1 + c_4.$$

  Pick $c_1 < c_4$. Observe that if $\eta_1 = 1$ and $\sigma(1) = 1$ then $w\varpi_1^\vee = \varpi_1^\vee$ so Inequality (c) need not be verified.

(b) Writing $\langle \lambda, \varpi_3^\vee - w\varpi_4^\vee \rangle$ as a sum of $\langle \lambda, \varpi_4^\vee - w\varpi_4^\vee \rangle$ (which is positive by [FL11a, Lemma 2.2]) and $\langle \lambda, \varpi_3^\vee - \varpi_4^\vee \rangle = \langle \lambda, -e_4^\vee \rangle = \frac{-c_3+c_4}{2}$, we can ensure Inequality (b) holds if $c_3 < c_4$.

(c) The Inequality (c) is proven similarly;

$$\langle \lambda, \varpi_1^\vee - w\varpi_3^\vee \rangle = \left\langle \lambda, \frac{1}{2}(e_1^\vee - e_2^\vee - e_3^\vee + e_4^\vee) \right\rangle = \frac{c_1 - c_3}{2},$$

so choose $c_1 > c_3$.

It is easy to verify that choices made above are consistent with each other.

6.5 ROOT CONE LEMMA FOR TYPE E_6

Following notations of [Bou02], the automorphism of E_6 is shown in the figure below.

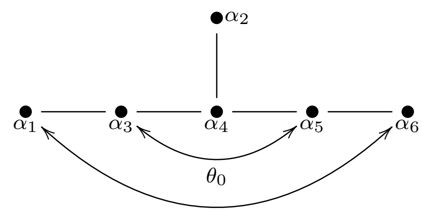

Fig. 6.3. Automorphism of groups of type E_6

We have written a `SageMath` code which proves the Root Cone Lemma for E_6. More specifically, the program loops over every element in the Weyl group and tries to find a point in the root cone satisfying the required inequalities. Since Lemma 4.3.2 is an open condition, existence of a point ensures there is a nonempty open cone for that element. The program outputs the number of elements for which a point is found, which turns out to be 51840, the size of the Weyl group of E_6. The program code is given in Appendix A.

7. THE SPECTRAL SIDE

We begin by reviewing the spectral side of the twisted trace formula.

7.1 TWISTED (G, M)-FAMILIES

Recall the definition of a (G, M)-family [Art81, §6]. A collection

$$c(\Lambda, P),\ \ P \in \mathcal{P}(M), \Lambda \in i\mathfrak{a}_M^*$$

of smooth functions is called a (G, M)-family if

$$c(\Lambda, P) = c(\Lambda, P')$$

for any pair $P, P' \in \mathcal{P}(M)$ of adjacent groups and any point Λ in the hyperplane spanned by the common wall of the chambers corresponding to P and P'. It is well-known that a (G, M)-family $c(\Lambda, P)$ gives naturally a smooth function of Λ as

$$c_M(\Lambda) = c_M^G(\Lambda) := \sum_{P \in \mathcal{P}(M)} \epsilon_P(\Lambda) c(\Lambda, P).$$

A (G, M)-family $c(\Lambda, P)$ also gives a $(\widetilde{G}, \widetilde{M})$-family

$$\{c(\Lambda, \widetilde{P}) :\ \Lambda \in i\mathfrak{a}_{\widetilde{P}}^*,\ \ \widetilde{P} \in \mathcal{P}(\widetilde{M})\}$$

by restricting $\Lambda \in i\mathfrak{a}_P^*$ to the subspace $i\mathfrak{a}_{\widetilde{P}}^*$. The corresponding smooth function is given by

$$c_{\widetilde{M}}(\Lambda) = \sum_{\widetilde{P} \in \mathcal{P}(\widetilde{M})} \epsilon_{\widetilde{P}}(\Lambda) c(\Lambda, \widetilde{P}).$$

7.2 INTERTWINING OPERATORS

Let P and Q be associated parabolic subgroups. This means that the set $W(\mathfrak{a}_P, \mathfrak{a}_Q)$ of isomorphisms from $\mathfrak{a}_P$ to $\mathfrak{a}_Q$ arising from restrictions of elements of W is non-empty. Fix $w \in W(\mathfrak{a}_P, \mathfrak{a}_Q)$ and $\nu \in \mathfrak{a}^*_{P,\mathbb{C}}$. The intertwining operator $M_{Q|P}(w, \nu)$ is defined by

$$M_{Q|P}(w,\nu)\Phi(x) = \exp - \langle w\nu + \rho_Q, H_Q(x)\rangle \times \int_{N_{w,P,Q}(\mathbb{A})} \Phi(n_w^{-1}nx) \exp \langle \nu + \rho_P, H_P(n_w^{-1}nx)\rangle \, dn,$$

where $N_{w,P,Q} = N_Q \cap n_w N_P n_w^{-1} \setminus N_Q$. It is an operator from $\mathcal{A}(X_P)$ to $\mathcal{A}(X_Q)$ which maps the subspace $\mathcal{A}(X_P, \sigma)$ to $\mathcal{A}(X_Q, \theta \circ \sigma)$. The intertwining operator $M(w, \lambda)$ defined in Section 4.2 corresponds to $P = Q = P_0$ is just a special case of this one. As remarked before, the integral converges only for $\mathrm{Re}(\nu)$ in a certain chamber but $M_{P|Q}(w, \nu)$ can be analytically continued to a meromorphic function of $\nu \in \mathfrak{a}^*_{P,\mathbb{C}}$. Set $M_{Q|P}(\nu) = M_{Q|P}(1, \nu)$.

For $P, P_1 \in \mathcal{P}(M)$ and $\nu, \Lambda \in i\mathfrak{a}^*_P$, the collection

$$\mathfrak{M}(P, \nu; \Lambda, P_1) := M_{P_1|P}(\nu)^{-1} M_{P_1|P}(\nu + \Lambda), \quad P_1 \in \mathcal{P}(M), \Lambda \in i\mathfrak{a}^*_M$$

is a (G, M)-family with corresponding smooth function of Λ given by

$$\mathfrak{M}_M(P, \nu; \Lambda) = \mathfrak{M}^G_M(P, \nu; \Lambda) := \sum_{P_1 \in \mathcal{P}(M)} \epsilon_{P_1}(\Lambda) \mathfrak{M}(P, \nu; \Lambda, P_1).$$

As discussed above, we have the associated $(\widetilde{G}, \widetilde{M})$-family $\mathfrak{M}(P, \nu; \Lambda, \widetilde{P_1})$, $\widetilde{P_1} \in \mathcal{P}(\widetilde{M})$ and the smooth function $\mathfrak{M}_M(P, \nu; \Lambda)$ where Λ belongs to the subspace $i\mathfrak{a}^*_{\widetilde{M}}$ of $i\mathfrak{a}^*_M$. Set

$$\mathfrak{M}_{\widetilde{M}}(P, \nu) = \mathfrak{M}_{\widetilde{M}}(P, \nu; 0).$$

It is one of the basic properties of (G, M)-families that the limit of $\mathfrak{M}_M(P, \nu; \Lambda)$ as $\Lambda \to 0$ exists. We are now ready to state the twisted trace formula for a test function $h \in \mathcal{C}^\infty_c(\widetilde{G}(\mathbb{A}))$ as in [LW13, Theorème 14.3.3].

$$J^{\widetilde{G}}(h, \omega) = \sum_{\widetilde{L} \in \mathcal{L}^{\widetilde{G}}} \frac{|\widetilde{W}^L|}{|\widetilde{W}^G|} \frac{1}{|\det(\theta_L - 1|_{\mathfrak{a}^G_M/\mathfrak{a}^{\widetilde{G}}_{\widetilde{L}}})|} J^{\widetilde{G}}_{\widetilde{L}}(h, \omega), \tag{7.1}$$

where

$$J_{\widetilde{L}}^{\widetilde{G}}(f,\omega)=\sum_{M\in\mathcal{L}^L}\frac{|W^M|}{|W^L|}\sum_{\widetilde{w}\in W^{\widetilde{L}}(M)_{\mathrm{reg}}}\frac{1}{|\det(\widetilde{w}-1|_{\mathfrak{a}_M^L})|}J_{M,\widetilde{w}}^{\widetilde{G}}(h,\omega),$$

where finally,

$$J_{M,\widetilde{w}}^{\widetilde{G}}(h,\omega)=\int_{i(\mathfrak{a}_{\widetilde{L}}^{\widetilde{G}})^*}\operatorname{trace}\left(\mathfrak{M}_{\widetilde{L}}^{\widetilde{G}}(P,\nu)M_{P|\widetilde{w}P}(0)\rho_{P,\mathrm{disc},\nu}(\widetilde{w},h,\omega)\right)d\nu.$$

The previous expression can also be written as

$$J_{M,\widetilde{w}}^{\widetilde{G}}(h,\omega)=\sum_{\sigma\in\Pi_{\mathrm{disc}}(M)}\int_{i(\mathfrak{a}_{\widetilde{L}}^{\widetilde{G}})^*}\operatorname{trace}\left(\mathfrak{M}_{\widetilde{L}}^{\widetilde{G}}(P,\nu)M_{P|\widetilde{w}P}(0)\rho_{P,\sigma,\nu}(\widetilde{w},h,\omega)\right)d\nu.$$

The absolute convergence of the spectral side would imply that the distribution $J^{\widetilde{G}}(f,\omega)$ extends continuously to $\mathcal{C}(\widetilde{G}(\mathbb{A}),K)$. We prove it by adopting the method of [FL11b] and [FLM11] for the twisted trace formula. In the regular (non-twisted) trace formula, $\mathfrak{M}_L(P,\nu), M_{P|w.P}(0)$ and $\rho_{P,\nu}(h)$ are operators on the space $\mathcal{A}(X_P)$ and the trace of their composition is integrated over the parameter ν. However, the twisted regular representation $\rho_{P,\mathrm{disc},\nu}(\widetilde{w},f,\omega)$ maps vectors in $\mathcal{A}(X_P)$ into those in $\mathcal{A}(X_Q)$ where $Q=\theta(P)=\mathrm{Ad}(\widetilde{w})(P)$. The intertwining operator $M_{P|Q}(\nu)$ does the opposite, hence $M_{P|Q}(\nu)\circ\rho_{P,\mathrm{disc},\nu}(\widetilde{w},f,\omega)$ is an operator on $\mathcal{A}(X_P)$. We define a unitary operator $\mathcal{S}=\mathcal{S}_{P,\nu}(\widetilde{w})$ which satisfies the lemma below.

$$\mathcal{S}_{P,\nu}(\widetilde{w}):\ \mathcal{A}(X_P)\to\mathcal{A}(X_Q)$$

$$(\mathcal{S}_{P,\nu}(\widetilde{w})\Phi)(x)=\exp-(\langle\widetilde{w}\nu+\rho_Q,H_Q(x)\rangle)\Phi(n_{\widetilde{w}}^{-1}xn_{\widetilde{w}})\exp\left(\ \nu+\rho_P,H_P(n_{\widetilde{w}}^{-1}xn_{\widetilde{w}})\ \right).$$

Recall above that $n_{\widetilde{w}}$ is the representative of $\widetilde{w}$ in $\widetilde{G}(\mathbb{Q})$.

Lemma 7.2.1 *1. For any $y\in\widetilde{G}(\mathbb{A}), y=n_{\widetilde{w}}g$ with $g\in G(\mathbb{A})$, we have*

$$\rho_{P,disc,\nu}(\widetilde{w},y,\omega)=\underline{\omega}\ \mathcal{S}_{P,\nu}(\widetilde{w})\ \rho_{P,\nu}(g).$$

Here, $\underline{\omega}$ is the operator $(\underline{\omega}\Phi)(x)=w(x)\Phi(x)$.

2.

$$\rho_{P,disc,\nu}(\widetilde{w},f,\omega)=\underline{\omega}\ \mathcal{S}_{P,\nu}(\widetilde{w})\ \rho_{P,\nu}(h),$$

where $h=L_{w^{-1}}f$.

3. The operator $\mathcal{S}_{P,\nu}(\widetilde{w})$ is unitary and invertible.

Proof This follows from the definitions and Theorem 3.1.2. ■

Theorem 7.2.2 *For any $f \in \mathcal{C}_c^\infty(\widetilde{G}(\mathbb{A}))$, the spectral side of the twisted trace formula is given by* (7.1). *In this equation the sums are finite (except the one over $\sigma \in \Pi_{disc}(M)$) and the integrals are absolutely convergent with respect to the trace norm, hence define distributions on $\mathcal{C}(\widetilde{G}(\mathbb{A}), K)$.*

Proof We fix $\widetilde{L} \in \mathcal{L}^{\widetilde{G}}, M \in \mathcal{L}^L, \widetilde{w} \in W^{\widetilde{L}}(M)_{\text{reg}}$ and show that the integral

$$\int_{i(\mathfrak{a}_{\widetilde{L}}^{\widetilde{G}})^*} \|\mathfrak{M}_{\widetilde{L}}^{\widetilde{G}}(P,\nu) M_{P|\widetilde{w}P}(0)\rho_{P,\text{disc},\nu}(\widetilde{w},h,\omega)\| \mathrm{d}\nu$$

converges, where $\| \circ \|$ denotes the trace norm of the operator on the space $\overline{\mathcal{A}}(X_P)$. Since the operator $\mathfrak{M}_{\widetilde{L}}^{\widetilde{G}}(P,\nu)$ equals $\mathfrak{M}_L^G(P,\nu)$ on the subspace $i\mathfrak{a}_{\widetilde{L}}$, hence it decomposes into finite sums of the composition of intertwining operators and their first-order derivatives. Referring to notations therein, we recall

Theorem 7.2.3 [FLM11, Theorem 2] *Let $M \in \mathcal{L}, P \in \mathcal{P}(M), L \in \mathcal{L}(M), m = a_L^G$ and $\underline{\mu} \in (\mathfrak{a}_M^*)^m$ be in general position. Then we have*

$$\mathfrak{M}_L(P,\lambda) = \sum_{\underline{\beta} \in \mathcal{B}_{P,L}} \Delta_{\mathcal{X}_{L,\underline{\mu}}(\underline{\beta})}(P,\lambda).$$

Since the sum is over a finite set, it suffices to prove the convergence, for a fixed m-tuple $\mathcal{X}$ of parabolic subgroups, of

$$\int_{i(\mathfrak{a}_{\widetilde{L}}^{\widetilde{G}})^*} \|\Delta_{\mathcal{X}}(P,\nu) M_{P|\widetilde{w}P}(0)\rho_{P,\text{disc},\nu}(\widetilde{w},h,\omega)\| \mathrm{d}\nu.$$

By Theorem 7.2.1, we can write this as

$$\int_{i(\mathfrak{a}_{\widetilde{L}}^{\widetilde{G}})^*} \|\Delta_{\mathcal{X}}(P,\nu) M_{P|\widetilde{w}P}(0) \circ \underline{\omega}\ \mathcal{S}_{P,\nu}(\widetilde{w})\rho_{P,\nu}(h)\| \mathrm{d}\nu.$$

If we denote the composite operator $M_{P|\widetilde{w}P}(0)\,\underline{\omega}\ \mathcal{S}_{P,\nu}(\widetilde{w})$ by $\mathcal{U}$ for convenience, we see that the resulting expression

$$\int_{i(\mathfrak{a}_{\widetilde{L}}^{\widetilde{G}})^*} \|\Delta_{\mathcal{X}}(P,\nu)\ \mathcal{U}\ \rho_{P,\nu}(h)\| \mathrm{d}\nu$$

resembles that in [FLM11, Theorem 3]. The theorem follows by imitating the proof of reducing [FLM11, Theorem 3] to [FLM11, Proposition 1] and observing that $\mathcal{U}$ restricts to a unitary operator on the finite dimensional space $\overline{\mathcal{A}}(X_P,\sigma)^{\tau,K}$ of K-fixed vectors in the σ-isotypical subspace of $\overline{\mathcal{A}}(X_P)$ which transform according to $\tau \in \hat{\mathbf{K}}_\infty$ under $\mathbf{K}_\infty$. ∎

Remark 7.2.4 *Let us elaborate the last step in the above proof.*

We have the algebraic decomposition

$$\mathcal{A}(X_P) = \bigoplus_{\sigma\in\Pi_{disc}(M(\mathbb{A}))} \mathcal{A}(X_P,\sigma)$$

where $\mathcal{A}(X_P,\sigma)$ *is the* K*-finite part of* $\overline{\mathcal{A}}(X_P,\sigma)$*. We further decompose*

$$\mathcal{A}(X_P,\sigma) = \oplus_{\tau\in\hat{\boldsymbol{K}}_\infty}\mathcal{A}(X_P,\sigma)^\tau$$

according to the isotypical subspaces for the action of $\boldsymbol{K}_\infty$*. Let* $\mathcal{A}(X_P,\sigma)^K$ *be the subspace of* K*-invariant functions in* $\mathcal{A}(X_P)$*, and similarly for* $\mathcal{A}(X_P)^{\tau,K}$ *for any* $\tau\in\hat{\boldsymbol{K}}_\infty$*. The integral reduces to*

$$\sum_{\sigma\in\Pi_{disc}(M(\mathbb{A}))}\sum_{\tau\in\hat{\boldsymbol{K}}_\infty}\int_{i(\mathfrak{a}_{\tilde{L}}^{\tilde{G}})^*}\|\Delta_{\mathcal{X}}(P,\nu)\,\mathcal{U}\,\rho_{P,\nu}(h)\|_{\mathcal{A}(X_P,\sigma)^{\tau,K}}\,d\nu.$$

The operator norm of the composition of operators is controlled by the norms of the operators. Using this trick in [FLM11, §5.1]*, we can replace the test function* h *by a high enough exponent of the operator* $\Delta = \mathrm{Id} - \Omega + 2\Omega_{\boldsymbol{K}_\infty}$*. Replacing* $\Delta_{\mathcal{X}}(P,\nu)$ *with it's expansion, the integrals equals a constant multiple of*

$$\sum_{\sigma\in\Pi_{disc}(M(\mathbb{A}))}\sum_{\tau\in\hat{\boldsymbol{K}}_\infty}\int_{i(\mathfrak{a}_{\tilde{L}}^{\tilde{G}})^*}\|M_{P_1|P}(\nu)^{-1}\delta_{P_1|P_1'}(\nu)M_{P_1'|P_2}(\nu)$$
$$\dots\delta_{P_{m-1}|P_{m-1}'}(\nu)M_{P_{m-1}'|P_m}(\nu)\delta_{P_m|P_m'}(\nu)M_{P_m'|P}(\nu)\,\mathcal{U}\,\rho_{P,\nu}(\Delta^{2k})\|_{\mathcal{A}(X_P,\sigma)^{\tau,K}}\,d\nu$$

which can be simplified to

$$\sum_{\tau\in\hat{\boldsymbol{K}}_\infty}\sum_{\sigma\in\Pi_{disc}(M(\mathbb{A}))}\dim(\mathcal{A}(X_P,\sigma)^{\tau,K})\int_{i(\mathfrak{a}_{\tilde{L}}^{\tilde{G}})^*}|\mu(\sigma,\nu,\tau)|^{-2k}\prod_{i=1}^{m}\|\delta_{P_i|P_i'}(\nu)\|_{\mathcal{A}(X_P,\sigma)^{\tau,K}}\,d\nu.$$

Now we can proceed according to [FLM11, §5.1]*.*

8. AN APPLICATION

In this section we discuss an application of the convergence of the spectral side to the finiteness of residues of poles of certain Rankin-Selberg L-functions.

Suppose E/F is a Galois extension of number fields and $G = GL(n)$. Assume that the Galois group Γ is generated by two elements θ, σ (which for proving functoriality for base change can be done without loss in generality, cf. [Get12, §7]). If w is finite, let $K_w = G(\mathcal{O}_{E_w})$ and $K_w = O(n, E_w)$ otherwise. Then $\mathbf{K} = \prod_w K_w$ is a "good" maximal compact subgroup in the sense of [Art81]. Denote the set of Archimedean places of E by ∞. For each $w \in \infty$, let ϕ_w be a smooth function on $\widetilde{G}(E_w)$ of compact support and bi-invariant under K_w, and set $\phi_\infty = \prod_{w\in\infty} \phi_w$. A cuspidal automorphic representation π of $GL(n, \mathbb{A}_E)$ decomposes as $\pi = \otimes_w \pi_w$ and we can form the partial Rankin-Selberg L-function $L^\infty(s, \pi \times \widetilde{\pi}^\sigma)$ where $\widetilde{\pi}$ is the contragradient of π and $\pi^\sigma(g) = \pi(g^\sigma)$.

Theorem 8.0.1 *With notations as above, we have that the sum of residues*

$$\sum_{\pi \simeq \pi^\Gamma} \prod_{w|\infty} \operatorname{trace} \pi_w(\phi_w) \operatorname{Res}_{s=1} L^\infty(s, \pi \times \widetilde{\pi}^\sigma) \tag{8.1}$$

is finite. The above sum is taken over cuspidal automorphic representations of $GL(n, \mathbb{A}_E)$ *invariant under* Γ *which are unramified at every finite place.*

Remark 8.0.2 *The functorial transfer of automorphic representations attached to the below L-homomorphism*

$$b_{E/F} : {}^L\,\mathrm{GL}(n)_F \to {}^L\,\mathrm{Res}_{E/F}\,\mathrm{GL}(n)_E$$

when $\mathrm{Gal}(E/F)$ *is cyclic of prime order is understood completely thanks to the work of Arthur and Clozel* [AC89]*. Following remarks in §7 of* [Get12]*, to study functoriality* *for* $b_{E/F}$ *it suffices "in principle" to assume* $\mathrm{Gal}(E/F)$ *is a universal perfect central extension of a simple nonabelian group, hence a quasi-simple group. By* [GK00, Corollary], *we can assume that* $\mathrm{Gal}(E/F) = \langle\sigma, \theta\rangle$. *For conjectural applications to Langlands' "Beyond Endoscopy" program to approach Functoriality, it is important to understand residues of* L*-functions coming from various* L*-homomorphisms. The result at hand gives a geometric expression for such a sum of residues.*

Proof The convergence of the expression

$$\sum_{\pi\simeq\pi^{\Gamma}}\Bigg(\operatorname{trace}\pi_\infty(\phi_\infty)L^\infty(s,\pi\times\widetilde{\pi}^\sigma)$$

for $\mathrm{Re}(s) \gg 0$ would follow by applying the convergence of the spectral side of the twisted trace formula to a certain basic function. However taking the residue at $s = 1$ is a delicate question to prove which, will require the main result from [Get15] about spherical Fourier transforms. Let F' be the fixed field of θ in E and consider the group $G = \mathrm{Res}_{E/F'}\, GL(n)_E$. The automorphism θ acts on

$$G(E) \cong GL(n, E) \times \cdots \times GL(n, E)$$

(the number of copies being the order of θ, say ℓ) and we can form the semidirect product $G \rtimes \langle\theta\rangle$ of which $G \rtimes \theta$ is a coset. Since this automorphism θ preserves the Borel subgroup and torus, we can identify the group G with the coset $G \rtimes \theta$ using the map $x \mapsto x \rtimes \theta$ and also identify functions on both cosets. We will construct the test function in $\mathcal{C}(\widetilde{G}(\mathbb{A}_E), K)$ which by the above identification and Theorem 3.1.2 can be considered on the space $\mathcal{C}(G(\mathbb{A}_E), K)$. We will apply the spectral side convergence result to this function.

It is known that for every non-Archimedean place w, there exists a unique smooth function $\phi_{w,s}$ on $G(E_w)$, the basic function, such that

$$\operatorname{trace}\pi_w(\phi_{w,s}) = L_w(s, \pi \times \widetilde{\pi}^\sigma) \tag{8.2}$$

holds for every irreducible admissible unramified representation π_w of $G(E_w)$ and $\mathrm{Re}(s) \gg 0$. (See [Ngô14] and [Get12] for details.) Using the decomposition $\widetilde{G}(\mathbb{A}_E) =$

$G(\mathbb{A}^\infty)\widetilde{G}(\mathbb{A}_\infty)$ define the function $\phi_s = \phi_\infty \prod_{w<\infty} \phi_{w,s}$. Then for $\mathrm{Re}(s)$ large enough depending on n, the function

$$f_s(y) = \int_{A_G} \phi_s(ay)\mathrm{d}a, \qquad y \in \widetilde{G}(\mathbb{A}_E)$$

as well as it's Archimedean (spherical) Fourier transform (as defined in [Get15])

$$\hat{f}_s(y) = \int_{A_G} (\mathcal{F}_{r,\psi}(\phi_\infty) \prod_{w<\infty} \phi_{w,s})(ay)\mathrm{d}a$$

are elements of $\mathcal{C}(\widetilde{G}(\mathbb{A}_E), K)$. Here ψ is the additive character used to define the Fourier transform and for the case at hand, r can be taken to be the standard representation. Note that the Fourier transform $\mathcal{F}_{r,\psi}(\phi_\infty)$ of the Archimedean component is in the space $\mathcal{S}^p(G(E_\infty)//K_\infty)$ of [Get15, §3.3] for $p \in (0,1]$ so in particular satisfies the condition at the Archimedean component of our class $\mathcal{C}(\widetilde{G}(\mathbb{A}), K)$. If π is a cuspidal automorphic representation of $G(\mathbb{A}_E)$ then it's contribution to the spectral side of the trace formula for $\widetilde{G}$ will be nonzero precisely if it is invariant under θ and thanks to the choice of the test function, would then equal

$$m(\pi)L^S(s, \pi \times \widetilde{\pi}^\sigma) \prod_{w\in S} \mathrm{trace}\, \pi_w(\phi_w).$$

Although Equation (8.2) is valid for $\mathrm{Re}(s) \gg 0$, the completed Rankin-Selberg L-function $L(s, \pi \times \widetilde{\pi}^\sigma)$ is known to be entire when $\pi \not\simeq \pi^\sigma$. Clearly the residue at $s = 1$ in this case is zero. However if $\pi \simeq \pi^\sigma$ then $L(s, \pi \times \tilde{\pi}^\sigma)$ has meromorphic continuation to $\mathbb{C}$, satisfies a functional equation, has simple poles at $s = 0, 1$ and no other poles [JPSS83]. Additionally by the multiplicity one theorem for $GL(n)$ [Sha74], $m(\pi) = 1$. By [Get15, Proposition 5.3], the sum

$$\sum_\pi \mathrm{trace}\, \pi_\infty(\phi_\infty). \operatorname{Res}_{s=1} L^\infty(s, \pi \times \tilde{\pi}^\sigma)$$

equals

$$\frac{1}{2\pi i} \sum_\pi \int_{\mathrm{Re}(s)=\sigma} \mathrm{trace}\, \pi(f_s)\ \mathrm{d}s - \frac{1}{2\pi i} \sum_\pi \int_{\mathrm{Re}(s)=\sigma} \mathrm{trace}\, \pi(\hat{f}_s)\ \mathrm{d}s$$

This sum is over cuspidal automorphic representations π invariant under the Galois group $\mathrm{Gal}(E/F)$ and unramified at every finite place. Although this depends on [ibid, Conjecture 5.2], the functional equation is known for Rankin-Selberg L-functions. As explained above the two terms in the above difference are both bounded by the discrete part of the twisted trace formula with test functions satisfying Equation (8.2) for $\mathrm{Re}(s) = \sigma$ sufficiently large. Thus their difference is finite. $\blacksquare$

APPENDICES

A. SAGEMATH CODE FOR E_6

```
# Proof of the root cone conjecture for the (unique) automorphism of
the Dynkin Diagram of E_6 using "SageMath".

R = RootSystem([‘E’, 6]);
X = R.root_space()
W = X.weyl_group()
alpha = X.basis()
alphacheck = X.coroot_space().basis()
varpi = X.fundamental_weights_from_simple_roots()
varpicheck = X.coroot_space().fundamental_weights_from_simple_roots()

def theta(vector):
    sigma = PermutationGroupElement(‘(1,6)(3,5)(2)(4)’)
    for i in range(1, len(varpi)+1):
        if vector == varpi[i]:
            break
    return varpi[sigma(i)]

def delta(w):
    list = []
    for item in varpi:
        if (w.action(item)).to_ambient() != item.to_ambient():
            list.append(item)
    return list
```

```
def rhs(w, vector):
    return (vector - w.action(theta(vector))).to_ambient()


def isPositive(Lambda, w):
    for vector in delta(w):
        if (Lambda.to_ambient()).dot_product(rhs(w, vector)) <= 0:
            return False
    return True


def isSuccess(w):
    for x_1 in range(1,4):
        for x_2 in range(1,4):
            for x_3 in range(1,4):
                for x_4 in range(1,4):
                    for x_5 in range(1,4):
                        for x_6 in range(1,4):
                            Lambda = x_1 * varpicheck[1] + x_2 *
                            varpicheck[2] + x_3 * varpicheck[3] +
                            x_4 * varpicheck[4] + x_5 * varpicheck[5] +
                            x_6 * varpicheck[6]
                            if isPositive(Lambda, w):
                                return (Lambda, True)
    Lambda = 0 * varpicheck[1]
    return (Lambda, False)


count = 0
for w in W:
    if isSuccess(w)[1] == True:
        count = count + 1
```

```
    else:
        print "Root Cone Conjecture fails for w = ", w
print "Number of successful elements = ", count
print "(Order of W = ", W.cardinality(), ")"
```

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June 2010	M. Sc. in Mathematics, **Indian Institute of Technology Bombay**, Mumbai, India
May 2007	B. Tech. in Electronics and Communication Engineering, **Visvesvaraya National Institute of Technology**, Nagpur, India

Invited Talks

Oct 2017	**Paul J. Sally Jr. Midwest Representation Theory Conference** honoring the 70th birthday of Professor Freydoon Shahidi, *Purdue University*, West Lafayette, IN
Sep 2017	**Duke Number Theory Seminar**, *Duke University*, Durham, NC
Sep 2017	**AMS Sectional Meeting, Special Session in Automorphic Forms**, *University of Buffalo*, Buffalo, NY
Mar 2017	**Automorphic Forms Workshop**, *East Tenessee State University*, Johnson City, TN
Nov 2016	**Mathematics Seminar**, *Rose-Hulman Institute of Technology*, Terre Haute, IN

Awards / Honors

2017	AMS Travel Grant
2015-16	Supported by NSF Grant DMS-1162299 (P. I. Prof. Shahidi)
2014-15	Purdue Research Fellowship
2012	Shyama-Prasad Mukharjee Fellowship
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2009-10	National Board for Higher Mathematics' M. Sc. Scholarship

Service

2017	**Teaching Assistant for REU**, Department of Mathematics, *Purdue University*
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